Asymptotic estimates for white noise distributions

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Abstract - Let θ be a Young function. Using properties of the Laplace and Legendre transforms, it is shown that white noise measures in the dual of a test function space of θ-exponential growth satisfy an exponential decay property with rate θ. An application to stochastic differential equations is given.

Estimations asymptotiques de distributions sur l'espace du bruit blanc

Résumé - Soit θ une fonction de Young. En utilisant des propriétés des transformées de Laplace et de Legendre, on montre que les mesures sur l'espace du bruit blanc qui sont dans le dual d'un espace de fonctions test à croissance θ-exponentielle satisfont une propriété de décroissance exponentielle de taux θ. Une application aux équations différentielles stochastiques est donnée.

Version française abrégée

Des résultats de déviation pour les lois de Lévy on été obtenus dans [3] en utilisant des identité de covariance appliquées à l'estimation de transformées de Laplace. Le cas de la dimension infinie a été traité par de telles méthodes dans [4], sur les espaces de Wiener et de Poisson. Dans cette Note nous mettons en évidence le lien existant entre ces propriétés de déviation et les espaces de fonctions test et de distributions sur l'espace du bruit blanc construits en utilisant la transformée de Legendre dans [2], [5]. Soit $N = \bigcap_{p\in\mathbb{N}^*} N_p$ un espace nucléaire de Fréchet complexe, par exemple la complexification $N = X + iX$ d'un espace de Fréchet nucléaire X, dont la topologie est définie par une famille $\{|\cdot|_p\,,\, p \in \mathbb{N}^*\}$ croissante de normes hilbertiennes, où N_p est le complété de N par rapport à la norme $|\cdot|_p$. Soit N_{-p} le dual topologique de N_p, et $N' = \bigcup_{p\in\mathbb{N}^*} N_{-p}$ le dual de N. Si $(B, \|\cdot\|)$ est un espace de Banach complexe, soit $H(B)$ l'espace des fonctions entières sur B, i.e. l'espace des fonctions continues de B dans $\mathbb{C}$, dont les restrictions à toutes les droites affines de B sont entières sur $\mathbb{C}$. Etant donnée $\theta : \mathbb{R}_+ \to \mathbb{R}_+$ une fonction de Young, soit $\mathrm{Exp}\,(B, \theta, m)$ l'espace des fonctions entières sur B à croissance exponentielle d'ordre θ et de type fini $m > 0$. L'intersection

$$\mathcal{F}_\theta(N^{'}) = \bigcap_{p\in\mathbb{N}^*, m>0} \mathrm{Exp}\,(N_{-p}, \theta, m), \tag{0.1}$$

resp. la réunion

$$\mathcal{G}_\theta(N) = \bigcup_{p\in\mathbb{N}^*, m>0} \mathrm{Exp}\,(N_p, \theta, m), \tag{0.2}$$

munie de la topologie limite projective, resp. inductive, est appelée espace des fonctions entières sur $N^{'}$, resp. N, à croissance θ-exponentielle et de type minumum, resp. de type (fini) quelconque. Soit $\mathcal{F}_\theta(N')^*$ le dual fort de $\mathcal{F}_\theta(N')$. Rappelons que par [2], la transformée de Laplace définie par

$$\widehat{\phi}(\xi) = \mathcal{L}(\phi)(\xi) = \phi(e^\xi), \qquad \xi \in N, \quad \phi \in \mathcal{F}_\theta(N')^*,$$

où $e^{\xi}(z) = e^{\langle z,\xi\rangle}$, $z \in N'$, induit un isomorphisme topologique

$$\mathcal{L} : \mathcal{F}_\theta(N')^* \to \mathcal{G}_{\theta^*}(N), \tag{0.3}$$

où θ^* désigne la transformée de Legendre de θ. Etant donnés $\xi \in X$ et $x \in \mathbb{R}$, soit

$$A_{\xi,x} = \{u \in X' \ : \ \langle u, \xi\rangle > x\}$$

le demi-plan dans X' associé à ξ, x. Notre résultat principal est le suivant.

Théorème 1 *Soit $\phi \in \mathcal{F}_\theta(N')^*$, telle que ϕ définit une mesure de Radon (positive) μ_ϕ sur X'. Pour tout $\xi \in X$ et $x > 0$ il existe $C > 0$, $m > 0$ et $p \in \mathbb{N}^*$ tels que:*

$$\mu_\phi(A_{\xi,x}) \leq C \exp\left(-\theta\left(\frac{x}{m|\xi|_p}\right)\right).$$

En particulier, par [6], toute distribution positive $\phi \in \mathcal{F}_\theta(N')^*_+$ définit une mesure de Radon positive μ_ϕ sur X'. Nous obtenons aussi un résultat de déviation sous une hypothèse d'intégrabilité exponentielle. Une application aux équations différentielles stochastiques pour le produit de convolution est donnée.

1 Notation and preliminaries

Let N be a complex nuclear Fréchet space, whose topology is defined by a family $\{|\cdot|_p\, , \, p \in \mathbb{N}^*\}$ of increasing hilbertian norms. We have the representation

$$N = \bigcap_{p\in\mathbb{N}^*} N_p$$

where N_p is the completion of N with respect to the norm $|\cdot|_p$. Denote by N_{-p} the topological dual of the space N_p, then the dual N' of N can be written as

$$N' = \bigcup_{p\in\mathbb{N}^*} N_{-p}.$$

Let now $\theta : \mathbb{R}_+ \to \mathbb{R}_+$ be a Young function, i.e. θ is continuous, convex, strictly increasing and verifies:

$$\theta(0) = 0 \qquad \text{and} \qquad \lim_{x\to+\infty} \frac{\theta(x)}{x} = +\infty. \tag{1.1}$$

Denote by θ^* the Legendre transform of θ:

$$\theta^*(x) = \sup_{t\geq 0}\, (tx - \theta(t)), \qquad x \geq 0,$$

which is also a Young function. Given a complex Banach space $(B, \|\cdot\|)$, let $H(B)$ denote the space of entire functions on B, i.e. the space of continuous functions from B to $\mathbb{C}$, whose restrictions to all affine lines of B are entire on $\mathbb{C}$. Let $\mathrm{Exp}\,(B,\theta,m)$ denote the space of all entire functions on B with exponential growth of order θ, and of finite type $m>0$:

$$\mathrm{Exp}\,(B,\theta,m)=\left\{f\in H(B)\ :\ \|f\|_{\theta,m}=\sup_{u\in B}|f(u)|\,e^{-\theta(m\|u\|)}<+\infty\right\}.$$

Let also

$$\|f\|_{\theta,m,p}=\sup_{u\in N_p}|f(u)|\,e^{-\theta(m\|u\|_p)},\qquad f\in \mathrm{Exp}\,(N_p,\theta,m).$$

The intersection

$$\mathcal{F}_\theta(N')=\bigcap_{p\in\mathbb{N}^*,m>0}\mathrm{Exp}\,(N_{-p},\theta,m),\tag{1.2}$$

equipped with the projective limit topology, is called the space of entire functions on N' of θ-exponential growth and minimal type. The union

$$\mathcal{G}_\theta(N)=\bigcup_{p\in\mathbb{N}^*,m>0}\mathrm{Exp}\,(N_p,\theta,m),\tag{1.3}$$

equipped with inductive limit topology, is called the space of entire functions on N of θ-exponential growth and (arbitrarily) finite type. Denote by $\mathcal{F}_\theta(N')^*$ the strong dual of the test function space $\mathcal{F}_\theta(N')$. From the condition (1.1), the exponential function defined as

$$\begin{aligned} e^\xi :&N'\to\mathbb{C}\\ &z\ \mapsto e^\xi(z)=e^{\langle z,\xi\rangle},\end{aligned}$$

$\xi\in N$, belongs to $\mathcal{F}_\theta(N')$. For every $\phi\in\mathcal{F}_\theta(N')^*$ the Laplace transform of ϕ is defined by

$$\widehat{\phi}(\xi)=\mathcal{L}(\phi)(\xi)=\phi(e^\xi),\qquad \xi\in N.$$

Theorem 1 *([2], Th. 1) The Laplace transform of analytical functionals induces a topological isomorphism*

$$\mathcal{L}:\mathcal{F}_\theta(N')^*\to\mathcal{G}_{\theta^*}(N).\tag{1.4}$$

As a consequence, $\phi\in\mathcal{F}_\theta(N')^*$ if and only if the Laplace transform of ϕ satisfies the growth condition

$$|\widehat{\phi}(\xi)|\le Ce^{\theta^*(m|\xi|_p)},\qquad \xi\in N,\tag{1.5}$$

for some $m>0$ and $p\in\mathbb{N}^*$.

Remark 1 *Although $\theta(x)=x$ is not a Young function, $\mathcal{L}$ also realizes a topological isomorphism between $\mathcal{F}_\theta(N')^*$, denoted here by $\mathcal{F}_x(N')^*$, and the space $\mathrm{Hol}_0(N)$ of holomorphic functions on a neighborhood of zero in N (see e.g. Lemma 2 in [1]).*

In the sequel we take $N = X + iX$ the complexification of a nuclear Frechet space X. Let $\mathcal{F}_\theta(N')_+$ denote the cone of positive test functions, i.e. $f \in \mathcal{F}_\theta(N')_+$ if $f(y+i0) \geq 0$ for all y in the topological dual X' of X.

Definition 1 *The space $\mathcal{F}_\theta(N')^*_+$ of positive distributions is defined as the space of $\phi \in \mathcal{F}_\theta(N')^*$ such that*

$$\langle \phi, f \rangle \geq 0, \qquad f \in \mathcal{F}_\theta(N')_+.$$

We recall the following results on the representation of positive distributions.

Theorem 2 *([6], Th. 1) Let $\phi \in \mathcal{F}_\theta(N')^*_+$. There exists a unique Radon measure μ_ϕ on X', such that*

$$\phi(f) = \int_{X'} f(y+i0) d\mu_\phi(y), \qquad f \in \mathcal{F}_\theta(N').$$

Theorem 3 *([6], Th. 2) Let μ be a finite, positive Borel measure on X'. Then μ represents a positive distribution in $\mathcal{F}_\theta(N')^*_+$ if and only if μ is supported by some X_{-p}, $p \in \mathbb{N}^*$, and there exists some $m > 0$ such that*

$$\int_{X_{-p}} e^{\theta(m|y|_{-p})} d\mu(y) < \infty.$$

2 Tail estimates

Given $\xi \in X$ and $x \in \mathbb{R}$, let

$$A_{\xi,x} = \{y \in X' \ : \ \langle y, \xi \rangle > x\},$$

denote the half-plane in X' associated to ξ, x.

Theorem 4 *Let $\phi \in \mathcal{F}_\theta(N')^*$ such that ϕ defines a (positive) Radon measure μ_ϕ on X'. For all $\xi \in X$ and $x > 0$ there exists $m > 0$ and $p \in \mathbb{N}^*$ such that:*

$$\mu_\phi(A_{\xi,x}) \leq C \exp\left(-\theta\left(\frac{x}{m|\xi|_p} \right) \right),$$

where $C = \|\widehat{\phi}\|_{\theta,m,p}$.

Proof. From Theorem 1, the Laplace transform of $\phi \in \mathcal{F}_\theta(N')^*$ verifies the growth condition

$$|\widehat{\phi}(\xi)| \leq C e^{\theta^*(m|\xi|_p)}, \qquad \xi \in X,$$

for some $m > 0$ and $p \in \mathbb{N}^*$. For all $t \geq 0$ we have the Chernoff type inequality:

$$e^{tx}\mu_\phi(A_{\xi,x}) \;\; = \;\; \int_{X'} e^{tx} 1_{A_{\xi,x}} d\mu_\phi \leq \int_{X'} e^{t\langle y,\xi\rangle} 1_{A_{\xi,x}}(y) d\mu_\phi(y)$$

$$\leq \int_{X'} e^{t\langle y,\xi\rangle} d\mu_\phi(y) = \phi(e^{t\xi}) \leq |\widehat{\phi}(t\xi)| \leq C e^{\theta^*(mt|\xi|_p)},$$

hence

$$\mu_\phi(A_{\xi,x}) \leq C e^{-(tx-\theta^*(mt|\xi|_p))}, \qquad t \geq 0.$$

Minimizing in $t \geq 0$ we get, since $(\theta^*)^* = \theta$:

$$\mu_\phi(A_{\xi,x} > x) \leq C \exp\left(-\theta\left(\frac{x}{m|\xi|_p}\right)\right), \qquad x > 0.$$

□

We also have

$$\mu_\phi(|\langle \xi, \cdot\rangle| > x) = \mu_\phi(A_{\xi,x}) + \mu_\phi(A_{-\xi,x}) \leq 2C \exp\left(-\theta\left(\frac{x}{m|\xi|_p}\right)\right), \qquad x > 0.$$

From Theorem 2, the result of Theorem 4 holds in particular for every positive distribution $\phi \in \mathcal{F}_\theta(N')^*_+$. Applying Theorem 3 and Theorem 4 we obtain a deviation result under an exponential integrability assumption.

Corollary 1 *Let μ be a finite, positive Borel measure on X' supported by some X_{-p}, $p \in \mathbb{N}^*$. Assume that for some $m > 0$,*

$$\int_{X_{-p}} e^{\theta(m|y|_{-p})} d\mu(y) < \infty.$$

Then for all $\xi \in X$ we have:

$$\mu(A_{\xi,x}) \leq C \exp\left(-\theta\left(\frac{x}{m|\xi|_p}\right)\right), \qquad x > 0.$$

Note that if $X = \mathbb{R}$, Corollary 1 follows directly from the inequality

$$|\widehat{\mu}(\xi)| = \int_{\mathbb{R}} e^{|\xi y|} d\mu(y) \leq e^{\theta^*(|\xi|)} \int_{\mathbb{R}} e^{\theta(|y|)} d\mu(y), \qquad \xi \in \mathbb{R},$$

associated to the proof of Theorem 4. Before turning to applications to stochastic differential equations in the next section, we treat some particular cases.
Gaussian case. Let $X = \mathcal{S}(\mathbb{R})$, $\theta(t) = \sigma^2 t^2/2$, $\theta^*(x) = x^2/(2\sigma^2)$, and

$$\hat{\gamma}(\xi) = \int_{X'} e^{\langle y,\xi\rangle} d\gamma(y) = e^{\frac{\sigma^2}{2}\|\xi\|^2_{L^2(\mathbb{R})}}, \qquad \xi \in X. \tag{2.1}$$

Hence the Gaussian measure γ on X' with variance σ^2 belongs to $\mathcal{F}_\theta(N')^*$ and we recover the classical deviation bound

$$\gamma(A_{\xi,x}) \leq e^{-\frac{x^2}{2\sigma^2}}, \qquad x > 0.$$

Poisson case. Let $X = \mathbb{R}$ and $\theta(t) = \lambda(e^t - 1)$, $\lambda > 0$. We have $\theta^*(x) = -x + \lambda + x \log \frac{x}{\lambda}$, moreover the Poisson measure π_λ of intensity λ defined by

$$\widehat{\pi}_\lambda(t) = \int_{\mathbb{R}} e^{ty} d\pi_\lambda(y) = e^{-\theta(t)}, \qquad t \in \mathbb{R},$$

belongs to $\mathcal{F}_{\theta^*}(N')^*_+$, and we have

$$\pi_\lambda(Z > x) \leq e^{x - \lambda - x \log \frac{x}{\lambda}}, \qquad x > 0,$$

where Z is a Poisson random variable with intensity λ. Note however that

$$\int_{\mathbb{R}} e^{\theta^*(m|y|)} d\pi_\lambda(y) < \infty$$

only if $m < 1$, hence the statement of Corollary 1 is not an equivalence.
Gamma case. Let $X = \mathbb{R}$ and $\theta(t) = \beta \log(1 - t)$, $\beta > 0$. From Remark 1, the gamma measure μ defined by

$$\widehat{\mu}(t) = \int_{\mathbb{R}} e^{ty} d\mu(y) = e^{-\theta(t)}, \qquad t \in (-1, 1),$$

belongs to $\mathcal{F}_x(N')^*$, and if Z is a gamma random variable, we get the classical deviation bound

$$\mu(Z > x) \leq e^{-mx}, \qquad x > 0,$$

for some $m > 0$.

3 Application to stochastic differential equations

Given $y \in N'$, define the translation operator τ_{-y} on $\mathcal{F}_\theta(N')$ by

$$\tau_{-y}\varphi(z) = \varphi(y + z), \quad z \in N', \quad \varphi \in \mathcal{F}_\theta(N').$$

The convolution of $\phi \in \mathcal{F}_\theta(N')^*$ with a test function $\varphi \in \mathcal{F}_\theta(N')$ is defined as

$$\phi * \varphi(z) = \langle \phi, \tau_{-z}\varphi \rangle, \qquad z \in N'.$$

The convolution of $\phi_1, \phi_2 \in \mathcal{F}_\theta(N')^*$ is the distribution $\phi_1 * \phi_2$ of $\mathcal{F}_\theta(N')^*$ defined as

$$\langle \phi_1 * \phi_2, f \rangle = [\phi_1 * (\phi_2 * f)](0), \qquad f \in \mathcal{F}_\theta(N').$$

Clearly we have $\phi_1 * \phi_2 \in \mathcal{F}_\theta(N')^*_+$ if $\phi_1, \phi_2 \in \mathcal{F}_{\theta_2}(N')^*_+$. Let $\phi : [0, T] \to \mathcal{F}_\theta(N')^*$ and $M : [0, T] \to \mathcal{F}_\theta(N')^*$ be two continuous generalized processes, and consider the initial value problem

$$\frac{dX_t}{dt} = \phi_t * X_t + M_t, \qquad X_0 \in \mathcal{F}_\theta(N')^*. \tag{3.1}$$

In the particular case where $\phi_t = \alpha\delta_0$, $\alpha \in \mathbb{R}$, $X = \mathcal{S}(\mathbb{R})$ and $(M_t)_{t \in [0,T]}$ is a Gaussian white noise on $[0, T]$, (3.1) is a classical Ornstein-Uhlenbeck equation.

Theorem 5 *([1], Th. 4) The stochastic differential equation* (3.1) *has a unique solution in $\mathcal{F}_{(e^{\theta^*}-1)^*}(N)^*$, given by*

$$X_t = X_0 * e^{* \int_0^t \phi_s ds} + \int_0^t e^{* \int_s^t \phi_u du} * M_s ds.$$

From the relation $\langle \phi_1 * \phi_2, 1 \rangle = \langle \phi_1, 1 \rangle \langle \phi_2, 1 \rangle$, the expectation of X_t satisfies

$$\langle X_t, 1 \rangle = \langle X_0, 1 \rangle e^{\int_0^t \langle \phi_s, 1 \rangle ds} + \int_0^t e^{\int_s^t \langle \phi_u, 1 \rangle du} \langle M_s, 1 \rangle ds, \qquad t > 0.$$

If $\phi_t, M_t \in \mathcal{F}_{\theta_2^*}(N')_+^*$ for all $t \in \mathbb{R}_+$, then $e^{* \int_0^t \phi_s ds}, e^{* \int_0^t \phi_s ds} * M_t \in \mathcal{F}_{\theta_2^*}(N')_+^*$ and we have the following corollary of Theorem 4 and Theorem 5.

Corollary 2 *Let θ_2^* be such that $\theta_2^*(r) \le (e^{\theta^*} - 1)^*(r)$ for all r large enough, and assume that $X_0, \phi_t, M_t \in \mathcal{F}_{\theta_2^*}(N')_+^*$, $t > 0$. Then the solution X_t of* (3.1) *belongs to $\mathcal{F}_{\theta_2^*}(N')_+^*$ and the associated Radon measure (denoted by μ_{X_t}) satisfies*

$$\mu_{X_t}(A_{\xi,x}) \le C_t \exp\left(-\theta_2^* \left(\frac{x}{m_t |\xi|_{p_t}} \right) \right), \qquad x > 0, \quad \xi \in X,$$

for some $C_t, m_t, p_t > 0$, $t \in \mathbb{R}_+$.

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