Density estimation of functionals of spatial point processes with application to wireless networks

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Abstract

In this paper we provide a Monte Carlo algorithm for the density estimation of functionals of spatial point processes on Lipschitz domains with random marks, using the Malliavin calculus. Our method allows us to compute explicitly the Malliavin weight and is applied to density estimation of the interference in a wireless ad hoc network model. This extends and makes more precise some recent results of Privault and Wei [26] who dealt with the particular case of the half line and under stronger assumptions.

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1 Introduction

Following the seminal papers by Fournié et al. [15] and [14], much work on numerical applications of the Malliavin calculus has been carried out. In particular, the Malliavin calculus has been applied to sensitivity analysis in continuous and discontinuous financial markets and in insurance; see e.g. El Khatib and Privault [17], Davis and Johansson [9], Bavouzet-Morel and Messaoud [4], Privault and Wei [25], Bally, Bavouzet-Morel and Messaoud [3], Forster, Lütkebohmert and Teichmann [13]. To the best of our knowledge, the Malliavin calculus has not yet been applied to the sensitivity analysis of signal to interference plus noise ratios, or to density estimation of interferences, in the context of wireless networks

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(we refer the reader to the book by Tse and Viswanath [28] for an introduction to wireless communication).

While the above cited works deal with Poisson random measures, a Monte Carlo method for density estimation of functionals of finite point processes on the half-line has been recently proposed in Privault and Wei [26], see Section 5 therein. More specifically, let $f_0 \in \mathbb{R}$ be a constant, $f_n : \mathbb{R}^n \to \mathbb{R}$ measurable functions, $(T_n)_{n \geq 1}$, $T_0 = 0$, the jump times of a point process on $[0, \infty)$, and $N(T)$ the number of points of the process on $(0, T]$. Privault and Wei [26] considered functionals of the form

$$F = f_0 \mathbb{1}_{\{N(T)=0\}} + \sum_{n=1}^{\infty} \mathbb{1}_{\{N(T)=n\}} f_n(T_1, \ldots, T_n), \qquad f_0 \in \mathbb{R},$$

and claimed that, under some smoothness and integrability assumptions, there exists a positive integer n_0 such that the conditional law of F given $A = \{N(T) \geq n_0\}$ is absolutely continuous with respect to (w.r.t.) the Lebesgue measure with density

$$\varphi_{F \mid A}(x) = \mathrm{E}[W \mathbb{1}_{\{F \geq x\}} \mid A], \qquad x \in \mathbb{R}. \tag{1.1}$$

Here W is a random variable, called the Malliavin weight, which depends on the gradient of the functional F. By definition, the gradient operator depends in turn on a weight function w, which is assumed to be continuously differentiable on $[0, T]$ and such that $w(0) = w(T) = 0$.

In Privault and Wei [26] it is assumed that the form functions f_n are symmetric and continuously differentiable on $[0, T]^n$, and that the point process is specified by continuously differentiable Janossy densities, in addition to various integrability conditions on the functional F and its gradient. Such conditions are not always practical for applications, for example the Janossy densities may be only weakly differentiable.

In this paper, we extend the results in Privault and Wei [26] to the setting of spatial point processes with random marks, and we provide an application in the context of wireless networks. In particular we develop a framework that allows us to treat point processes in the more general setting of multidimensional domains with Lipschitz boundaries. In addition, our arguments are more precise and direct and allow us to relax a number of smoothness conditions while fixing some gaps in the proofs of Privault and Wei [26]. We provide sufficient conditions for the explicit computation of the Malliavin weight that appears in the density estimator of random functionals, and we only assume the weak differentiability of the weight function and the Janossy densities. Our proof of the density estimation formula is again based on a suitable duality relation between the gradient and the divergence operator, however it differs from the one of Privault and Wei [26]. More precisely, formula (1.1) above has been obtained in [26] as a consequence of a result on sensitivities, whereas here we use a more direct argument, cf. the proof of Proposition 5.1. As already mentioned we apply our theoretical result to provide a Monte Carlo estimator for the density estimation of the interference in a wireless ad hoc network model introduced by Baccelli and Błaszczyszyn [2].

The paper is organized as follows. In Section 2 we give some preliminaries on point processes, Sobolev spaces, and closability of linear operators. In Section 3 we introduce the gradient and the divergence operators, and provide the product and chain rules for differentiation. In Section 4 we prove a duality relation for functionals of finite spatial point processes, with random marks taking values on a general measurable space, between the gradient and the divergence operator. Similar formulas on the Poisson space may be found in Albeverio, Kondratiev and Röckner [1] and Decreusefond [10]; see also Privault [24] for a review. In Section 5 we give a theoretical Monte Carlo algorithm for the density estimation of functionals of finite spatial point processes with random marks (see Proposition 5.1). Our formula depends on the Malliavin weight, whose analytical expression is in general not known in closed form. We discuss the main differences between the classical kernel estimator and the Malliavin estimator and, to solve a related variance reduction problem, we provide a modified Malliavin estimator. In Section 6 we give sufficient conditions which lead to a closed form expression of the Malliavin weight. Finally, in Section 7 we apply the result proved in Section 6 to the density estimation of the interference in a wireless ad hoc network model, where nodes' locations are specified by finite point processes whose law is absolutely continuous w.r.t. the law of a homogeneous Poisson process. In particular, the nodes may be distributed according to homogeneous Poisson processes and, more generally, according to suitable pairwise interaction point processes. The first situation is standard in wireless networks, even if it is often too simplistic. Indeed, statistics show that the pattern of nodes exhibits more clustering effects. Usually, to avoid collisions between the packets one introduces in the network scheduling mechanisms for channel allocation, which ensure that nearby nodes do not transmit on the same channel, or power control algorithms, which ensure that no link asymmetry is introduced in the network (see e.g. Mhatre, Papagiannaki and Baccelli [20]). Such algorithms create repulsion in the pattern of nodes allowed to access simultaneously to the channel, and this raises questions on the analysis of networks with a repulsive nature. For this reason, we shall provide examples concerning networks with nodes disributed according to repulsive pairwise interaction point processes, including an example where the Janossy densities are only weakly differentiable. Our results are backed by numerical simulations and an error analysis which show that the Malliavin estimator generally performs better than the finite difference estimator as it is not sensitive to bandwidth selection, cf. Figure 2 in particular.

2 Preliminaries

2.1 Finite point processes

The standard reference for point process theory is the two-volume book by Daley and Vere-Jones [7], [8]. Let B be a Borel subset of $\mathbb{R}^d$, where $d \geq 1$ is an integer, with finite Lebesgue measure $\ell(B)$. For any subset $C \subseteq B$, let $\sharp(C)$ denote the cardinality of C, setting $\sharp(C) = \infty$ if C is not finite. Denote by N_f the set of finite point configurations of B:

$$\mathrm{N}_f := \{C \subseteq B : \sharp(C) < \infty\}.$$

We equip the set of finite point configurations with the σ-field

$$\mathcal{N}_f := \sigma(\{C \in \mathrm{N}_f \ : \ \sharp(C) = m\}, \quad m \geq 0).$$

A finite point process $\mathbf{X}$ on B is a measurable mapping defined on some probability space $(\Omega, \mathcal{F}, P)$ and taking values on $(\mathrm{N}_f, \mathcal{N}_f)$. We denote by $N(B)$ the number of points of $\mathbf{X}$ on B, and by $X_1, \dots, X_n$ the points of $\mathbf{X}$ in B given that $\{N(B) = n\}$.

In this paper we consider finite point processes $\mathbf{X}$ specified by the distribution

$$P(N(B) = n), \qquad n \in \mathbb{N},$$

of the number of points in B and by the family $(j_n(\cdot))_{n \geq 1}$ of symmetric probability densities, called *Janossy densities*, and defined by

$$P(\mathbf{X}_n \in C \,|\, N(B) = n) = \int_C j_n(\mathbf{x}_n)\, \mathrm{d}\mathbf{x}_n \quad \text{for Borel sets } C \subseteq B^n$$

where $\mathbf{x}_n := (x_1, \dots, x_n) \in \mathbb{R}^{dn}$ and $\mathrm{d}\mathbf{x}_n := \mathrm{d}x_1 \dots \mathrm{d}x_n$.

In addition we shall consider marked finite point processes: given $\{N(B) = n\}$, to each point X_k of $\mathbf{X}$ we attach a random variable Z_k, called a *mark*, defined again on the probability space $(\Omega, \mathcal{F}, P)$ and with values in some measurable space $(\mathrm{M}, \mathcal{M})$. Typically, in the applications, the random mark Z_k describes some characteristic of the point X_k. In the following we assume that, given $\{N(B) = n\}$, the mark sequence $(Z_k)_{k=1,\dots,n}$ is independent of the point sequence $(X_k)_{k=1,\dots,n}$, and we denote by $\mu_{\mathbf{Z}_n|N(B)=n}$ the conditional law of $\mathbf{Z}_n$ given $\{N(B) = n\}$.

For later purposes we mention that except in the Poisson case, the distribution of the number of points and the Janossy densities are only known up to normalizing constants. To be more specific, for a finite point process $\mathbf{X}$ on B specified by its density w.r.t. a Poisson process with rate $\lambda > 0$ we have

$$P(N(B) = n) = \frac{c_n}{n!}\, \mathrm{e}^{-\lambda \ell(B)} \quad \text{and} \quad j_n(\mathbf{x}_n) = c_n^{-1}\, \Phi_n(\mathbf{x}_n), \tag{2.1}$$

see e.g. van Lieshout [29] p. 27 and Møller and Waagepetersen [21] pp. 82-83, where

$$c_n := \int_{B^n} \Phi_n(\mathbf{x}_n)\, \mathrm{d}\mathbf{x}_n, \qquad n \geq 0,$$

are unknown normalizing constants, ℓ denotes the Lebesgue measure, and $\Phi_n : B^n \to [0, \infty)$ are known symmetric functions. Finally we introduce some notation. Let $\mathcal{F}_B$ be the σ-field on Ω generated by the points of $\mathbf{X}$ on B and their marks. We denote by $\mathrm{L}^r(B)$, $1 \leq r < \infty$, the space of real-valued random variables Y defined on the probability space $(\Omega, \mathcal{F}_B, P)$ and such that $\|Y\|_r := (\mathrm{E}[|Y|^r])^{1/r} < \infty$. Throughout this paper we adopt the conventions $0/0 := 0$ and $C/0 := +\infty$, for any positive constant $C > 0$.

2.2 Sobolev spaces, Lipschitz boundaries, and the trace theorem

For convenience of notation, we introduce some functional spaces. Let $B \subseteq \mathbb{R}^d$ be a Borel set, $1 \leq r < \infty$, and h a non-negative Borel function defined on B. We denote by $L^r(B, h)$ the space of measurable functions $f : B \to \mathbb{R}$ such that

$$\|f\|_{L^r(B,h)} := \left(\int_B |f(x)|^r\, h(x)\, \mathrm{d}x \right)^{1/r} < \infty.$$

When $h \equiv 1$ we simply write $L^r(B)$ in place of $L^r(B, 1)$. We denote by $L^\infty(B)$ the space of measurable functions $f : B \to \mathbb{R}$ such that

$$\|f\|_\infty := \operatorname{ess\,sup}_{x \in B} |f(x)| < \infty$$

where the essential supremum is w.r.t. the Lebesgue measure ℓ.

For $m \in \{0,1,2,\dots\} \cup \{\infty\}$, we denote by $\mathcal{C}^m(B)$ the space of functions $f : B \to \mathbb{R}$ which are m-times continuously differentiable on B. Here $\mathcal{C}^0(B) = \mathcal{C}(B)$ is the space of continuous functions on B. We denote by $\mathcal{C}_c^m(B)$ the space of functions which have compact support contained in B and belong to $\mathcal{C}^m(B)$, and by $\mathcal{C}_b^m(B)$ the space of functions which belong to $\mathcal{C}^m(B)$ and are uniformly bounded along with all their derivatives up to the order m.

Set $x := (x^{(1)}, \dots, x^{(d)}) \in \mathbb{R}^d$ and recall that if $f : B \to \mathbb{R}$ is integrable on the bounded subsets of B, one says that $\partial_{x^{(i)}} f$ is the *weak partial derivative* of f w.r.t. $x^{(i)}$ if $\partial_{x^{(i)}} f$ is integrable on the bounded subsets of B and

$$\int_B f(x) \partial_{x^{(i)}} \varphi(x) \, \mathrm{d}x = - \int_B \varphi(x) \partial_{x^{(i)}} f(x) \, \mathrm{d}x, \qquad \varphi \in \mathcal{C}_c^1(B^o),$$

where B^o denotes the interior of B. We shall consider the gradient operator $\nabla_x := (\partial_{x^{(1)}}, \dots, \partial_{x^{(d)}})$ and the divergence operator $\mathrm{div}_x := \sum_{i=1}^d \partial_{x^{(i)}}$, $x \in \mathbb{R}^d$, where $\partial_{x^{(i)}}$ denotes the weak partial derivative w.r.t. $x^{(i)}$. When it is clear from the context we simply write ∇ and div in place of ∇_x and div_x. We shall also use the Sobolev space

$$\mathcal{W}^{1,r}(B) := \{ f \in L^r(B) : \partial_{x^{(i)}} f \in L^r(B), \quad i = 1, \dots, d \}, \qquad 1 \le r < \infty,$$

equipped with the norm

$$\|f\|_{\mathcal{W}^{1,r}(B)} := \left(\int_B (|f(x)|^r + \|\nabla f(x)\|^r) \, \mathrm{d}x \right)^{1/r},$$

where the derivatives are taken in the weak sense and $\| \cdot \|$ denotes the Euclidean norm in $\mathbb{R}^d$, cf. for instance, Evans and Gariepy [12] pp. 120-121.

This paper is written in the framework of *bounded Lipschitz domains* (see, for instance, Evans and Gariepy [12] p. 127), which is not a significant restriction for applications. Denote by ∂B the boundary of a Borel set $B \subseteq \mathbb{R}^d$. A bounded open set $\mathrm{S} \subset \mathbb{R}^d$, $d \ge 2$, is said to be a bounded Lipschitz domain if, for each $x \in \partial\mathrm{S}$, there exist a positive constant $c > 0$ and a Lipschitz mapping $\gamma : \mathbb{R}^{d-1} \to \mathbb{R}$ such that (after rotating and relabelling the coordinate axes if necessary) we have

$$\mathrm{S} \cap Q(x,c) = \{ y \in \mathbb{R}^d : \gamma(y^{(1)}, \dots, y^{(d-1)}) < y^{(d)} \} \cap Q(x,c)$$

where $Q(x,c) := \{ y \in \mathbb{R}^d : |y^{(i)} - x^{(i)}| < c, \ i = 1, \dots, d \}$. In other words, a bounded open set S is a bounded Lipschitz domain if, near $x \in \partial\mathrm{S}$, $\partial\mathrm{S}$ is the graph of a Lipschitz function.

From now on, for $d = 1$, S will be a finite union of bounded open intervals, and for $d \geq 2$ it will denote a bounded Lipschitz domain.

Finally we recall the *trace theorem* (see e.g. Evans and Gariepy [12] Theorem 1 p. 133), which extends the classical integration by parts formula. Let the symbol "$\cdot$" denote the inner product and let $\overline{B}$ denote the closure of the Borel set B in $\mathbb{R}^d$. The following formula holds:

$$\int_{\mathrm{S}} \phi(x)\nabla_x \cdot \psi(x)\,\mathrm{d}x = -\int_{\mathrm{S}} \nabla\phi(x)\cdot\psi(x)\,\mathrm{d}x + \int_{\partial\mathrm{S}} \phi(x)\psi(x)\cdot\nu(x)\,\mathcal{H}^{d-1}(\mathrm{d}x). \tag{2.2}$$

Here $\mathcal{H}^{d-1}$ is the $(d-1)$-dimensional Hausdorff measure on $\mathbb{R}^d$; $\nu(x)$ is the unit outer normal to $\partial\mathrm{S}$; $\psi = (\psi^{(1)}, \dots, \psi^{(d)}) \in (\mathcal{C}^1(\overline{\mathrm{S}}))^d$ and $\phi \in \mathcal{W}^{1,r}(\mathrm{S}) \cap \mathcal{C}(\overline{\mathrm{S}})$, $1 \leq r < \infty$.

2.3 Closed and closable linear operators

In this subsection we recall the notion of closed and closable linear operator. We refer to Rudin [27] for details. Let X and Y be two Banach spaces, and A a linear operator defined on a subspace $\mathrm{Dom}(A)$ of X and taking values in Y. The operator $A : \mathrm{Dom}(A) \to Y$ is said to be *closed* if, for any sequence $(x_n)_{n\geq 1} \subset \mathrm{Dom}(A)$, such that $x_n \to x$ in X and $Ax_n \to y$ in Y we have $x \in \mathrm{Dom}(A)$ and $y = Ax$, i.e. the graph of A is closed w.r.t. the product topology on $X \times Y$. A linear operator $A : \mathrm{Dom}(A) \to Y$ is said *closable* if, for any sequence $(x_n)_{n\geq 1} \subset \mathrm{Dom}(A)$ such that $x_n \to 0$ in X and $Ax_n \to y$ in Y it holds $y = 0$. In other words, A is closable if admits a closed extension. The minimal closed extension of the closable operator A is the closed operator $\overline{A}$ whose graph is the closure in $X \times Y$ of the graph of A. It turns out that the domain of $\overline{A}$ is

$$\begin{aligned}&\mathrm{Dom}(\overline{A})\\ &\quad = \{x \in X : \exists\, (x_n)_{n\geq 1} \subset \mathrm{Dom}(A) : x_n \to x \text{ in } X \text{ and } (Ax_n)_{n\geq 1} \text{ converges in } Y\}\end{aligned}$$

and

$$\overline{A}x = \lim_{n\to\infty} Ax_n, \qquad x \in \mathrm{Dom}(\overline{A}),$$

where the limit is in Y and $(x_n)_{n\geq 1}$ is some sequence in $\mathrm{Dom}(\overline{A})$ such that $x_n \to x$ in X and $(Ax_n)_{n\geq 1}$ converges in Y.

3 Differential operators and differentiation rules

Throughout this paper we consider functionals of the form

$$F = f_0 \mathbb{1}_{\{N(\mathrm{S})=0\}} + \sum_{n=1}^{\infty} \mathbb{1}_{\{N(\mathrm{S})=n\}} f_n(\mathbf{X}_n, \mathbf{Z}_n) \tag{3.1}$$

where $f_0 \in \mathbb{R}$ is a constant and $f_n : \mathrm{S}^n \times \mathrm{M}^n \to \mathbb{R}$ are measurable functions. In the following we refer to the f_n's as *form functions* of the functional F. Let $w : \overline{\mathrm{S}} \to \mathbb{R}$ be a measurable function which is referred to as the *weight function*.

In the sequel we let $p_n = P(N(\mathrm{S}) = n)$, $n \geq 0$, denote the distribution of the number of points, and we let $(j_n(\cdot))_{n\geq 1}$ denote the Janossy densities of $\mathbf{X}$ on S.

We assume that w is weakly differentiable on S that j_n is weakly differentiable on S^n, for all $n \geq 1$, and that $f_n(\cdot, \mathbf{z}_n)$ is weakly differentiable on S^n for all $n \geq 1$ and $\mu_{\mathbf{Z}_n|N(\mathrm{S})=n}$-almost all $\mathbf{z}_n$. We define the *gradient* and the *divergence* of F, respectively, by

$$D_w F := -\sum_{n=1}^{\infty} \mathbb{1}_{\{N(\mathrm{S})=n\}} \sum_{k=1}^{n} w(X_k) \mathrm{div}_{x_k} f_n(\mathbf{X}_n, \mathbf{Z}_n)$$

and

$$D_w^* F := F \sum_{k=1}^{N(\mathrm{S})} (\mathrm{div}\, w(X_k) + w(X_k) R_{k,N(\mathrm{S})}(\mathbf{X}_{N(\mathrm{S})})) - D_w F.$$

Here $R_{k,n}$ is the real-valued function defined by

$$R_{k,n}(\mathbf{x}_n) := \frac{\mathrm{div}_{x_k} j_n(\mathbf{x}_n)}{j_n(\mathbf{x}_n)} \quad k = 1, \ldots, n.$$

Throughout this paper, p and q are fixed conjugate exponents, i.e.

$$p \geq q > 1 \quad \text{and} \quad 1/p + 1/q = 1$$

moreover p' and q' are fixed constants such that

$$p' \geq q' > q \quad \text{and} \quad q/q' + q/p' = 1.$$

Definition 1 *For $r > 1$ we denote by $\mathcal{R}_{\mathrm{S}}(r)$ the class of functionals F with form functions f_n such that*

- $f_0 \in \mathbb{R}$ *and* $f_n(\cdot, \mathbf{z}_n)$ *belongs to* $\mathcal{W}^{1,r}(\mathrm{S}^n)$, $n \geq 1$, *and* $\mu_{\mathbf{Z}_n|N(\mathrm{S})=n}$*-almost all* $\mathbf{z}_n$
- $\mathbb{1}_{\{N(\mathrm{S})=n\}} f_n(\mathbf{X}_n, \mathbf{Z}_n), \mathbb{1}_{\{N(\mathrm{S})=n\}} \partial_{x_k^{(i)}} f_n(\mathbf{X}_n, \mathbf{Z}_n) \in \mathrm{L}^r(\mathrm{S})$, $n \geq 1$, $k = 1, \ldots, n$, $i = 1, \ldots, d$.

We also define the set $\mathcal{R}_{\mathrm{S}}^{\mathrm{f}}(r) \subset \mathcal{R}_{\mathrm{S}}(r)$ by

$$\mathcal{R}_{\mathrm{S}}^{\mathrm{f}}(r) = \{F \in \mathcal{R}_{\mathrm{S}}(r) : \text{the sum in (3.1) is over } n \in \{1, \ldots, m\} \text{ for some integer } m < \infty\},$$

In the following we refer to the positive integer m as the *length* of the functional $F \in \mathcal{R}_{\mathrm{S}}^{\mathrm{f}}(r)$.

Remark 3.1 *Clearly,* $\mathcal{R}^{\mathrm{f}}_{\mathrm{S}}(r)$ *is a subspace of* $\mathrm{L}^r(\mathrm{S})$ *and the operators* D_w *and* D^*_w *are linear on* $\mathcal{R}^{\mathrm{f}}_{\mathrm{S}}(r)$*. Finally, note that if* $1 < r \leq r'$ *then* $\mathcal{R}^{\mathrm{f}}_{\mathrm{S}}(r') \subseteq \mathcal{R}^{\mathrm{f}}_{\mathrm{S}}(r)$*.*

Denote by $j_n^{(1)}$ the one-dimensional marginal density of j_n (note that, for a fixed $n \geq 1$, by the symmetry of j_n the one-dimensional marginal densities are all equal). The next proposition provides sufficient conditions which ensure that $D_w(\mathcal{R}^{\mathrm{f}}_{\mathrm{S}}(r)) \subset \mathrm{L}^r(\mathrm{S})$ and $D^*_w(\mathcal{R}^{\mathrm{f}}_{\mathrm{S}}(q')) \subset \mathrm{L}^q(\mathrm{S})$.

Proposition 3.2 (i) *The inclusion* $D_w(\mathcal{R}^{\mathrm{f}}_{\mathrm{S}}(r)) \subset \mathrm{L}^r(\mathrm{S})$ *holds for any* $r > 1$*, provided*

$$w \in L^\infty(\mathrm{S}). \tag{3.2}$$

(ii) *Under Condition* (3.2)*, the inclusion* $D^*_w(\mathcal{R}^{\mathrm{f}}_{\mathrm{S}}(q')) \subset \mathrm{L}^q(\mathrm{S})$ *holds provided*

$$\partial_{x^{(i)}} w \in L^{p'}(\mathrm{S}, j_n^{(1)}) \tag{3.3}$$

and

$$\frac{\partial_{x_k^{(i)}} j_n}{j_n} \in L^{p'}(\mathrm{S}^n, j_n), \tag{3.4}$$

$n \geq 1$*,* $k = 1, \ldots, n$*,* $i = 1, \ldots, d$*.*

Proof of (i). For any $F \in \mathcal{R}^{\mathrm{f}}_{\mathrm{S}}(r)$ with length m and form functions f_n, by Minkowski's inequality we deduce:

$$\|D_w F\|_r \leq \|w\|_\infty \sum_{n=1}^m \sum_{k=1}^n \sum_{i=1}^d \left\| \mathbb{1}_{\{N(\mathrm{S})=n\}} \partial_{x_k^{(i)}} f_n(\mathbf{X}_n, \mathbf{Z}_n) \right\|_r < \infty.$$

$\square$

Proof of (ii). Let $G \in \mathcal{R}^{\mathrm{f}}_{\mathrm{S}}(q')$ be a functional with length m and form functions g_n, then a straightforward computation shows

$$D^*_w G = \sum_{n=1}^m \mathbb{1}_{\{N(\mathrm{S})=n\}} \widetilde{g}_n(\mathbf{X}_n, \mathbf{Z}_n)$$

where

$$\begin{aligned} &\widetilde{g}_n(\mathbf{X}_n, \mathbf{Z}_n) \\ &\quad = \sum_{k=1}^n [(\operatorname{div} w(X_k) + w(X_k) R_{k,n}(\mathbf{X}_n)) g_n(\mathbf{X}_n, \mathbf{Z}_n) + w(X_k) \operatorname{div}_{x_k} g_n(\mathbf{X}_n, \mathbf{Z}_n)]. \end{aligned}$$

So the claim follows if we prove

$$\|\mathbb{1}_{\{N(\mathrm{S})=n\}} \widetilde{g}_n(\mathbf{X}_n, \mathbf{Z}_n)\|_q < \infty, \qquad n \in \{1, \ldots, m\}. \tag{3.5}$$

For any fixed $n \in \{1, \dots, m\}$, $k \in \{1, \dots, n\}$ and $i \in \{1, \dots, d\}$, define the random variables

$$\begin{aligned}
h^{(1)}_{n,k,i}(\mathbf{X}_n, \mathbf{Z}_n) &:= \mathbb{1}_{\{N(\mathrm{S})=n\}} g_n(\mathbf{X}_n, \mathbf{Z}_n) \partial_{x^{(i)}} w(X_k), \\
h^{(2)}_{n,k,i}(\mathbf{X}_n, \mathbf{Z}_n) &:= \mathbb{1}_{\{N(\mathrm{S})=n\}} w(X_k) \partial_{x^{(i)}_k} g_n(\mathbf{X}_n, \mathbf{Z}_n), \\
h^{(3)}_{n,k,i}(\mathbf{X}_n, \mathbf{Z}_n) &:= \mathbb{1}_{\{N(\mathrm{S})=n\}} w(X_k) g_n(\mathbf{X}_n, \mathbf{Z}_n) \frac{\partial_{x^{(i)}_k} j_n(\mathbf{X}_n)}{j_n(\mathbf{X}_n)}.
\end{aligned}$$

Using Minkowski's inequality, one can easily realize that (3.5) holds if, for any $n \in \{1, \dots, m\}$, $k \in \{1, \dots, n\}$ and $i \in \{1, \dots, d\}$ we have

$$\|h^{(1)}_{n,k,i}(\mathbf{X}_n, \mathbf{Z}_n)\|_q < \infty, \tag{3.6}$$

$$\|h^{(2)}_{n,k,i}(\mathbf{X}_n, \mathbf{Z}_n)\|_q < \infty, \tag{3.7}$$

$$\|h^{(3)}_{n,k,i}(\mathbf{X}_n, \mathbf{Z}_n)\|_q < \infty. \tag{3.8}$$

For the inequality (3.6), note that by Hölder's inequality with conjugate exponents q'/q and p'/q we have:

$$\|h^{(1)}_{n,k,i}(\mathbf{X}_n, \mathbf{Z}_n)\|_q^q \leq p_n^{q/p'} \|\mathbb{1}_{\{N(\mathrm{S})=n\}} g_n(\mathbf{X}_n, \mathbf{Z}_n)\|_{q'}^q \|\partial_{x^{(i)}} w\|^q_{L^{p'}(\mathrm{S}, j_n^{(1)})} < \infty.$$

A similar computation shows that the inequality (3.7) is a consequence of (3.2) and

$$\mathbb{1}_{\{N(\mathrm{S})=n\}} \partial_{x^{(i)}_k} g_n(\mathbf{X}_n, \mathbf{Z}_n) \in \mathrm{L}^{q'}(\mathrm{S}).$$

Finally, the inequality (3.8) can be proved using again Hölder's inequality, which yields:

$$\|h^{(3)}_{n,k,i}(\mathbf{X}_n, \mathbf{Z}_n)\|_q^q \leq p_n^{q/p'} \|w\|_\infty^q \|\mathbb{1}_{\{N(\mathrm{S})=n\}} g_n(\mathbf{X}_n, \mathbf{Z}_n)\|_{q'}^q \left\| \frac{\partial_{x^{(i)}_k} j_n}{j_n} \right\|^q_{L^{p'}(\mathrm{S}^n, j_n)} < \infty.$$

$\square$

In the proof of Proposition 3.2 we never used that the form functions $f_n(\cdot, \mathbf{z}_n)$ of a functional $F \in \mathcal{R}^{\mathrm{f}}_{\mathrm{S}}(r)$ belong to $\mathcal{W}^{1,r}(\mathrm{S}^n)$ for $\mu_{\mathbf{Z}_n|N(\mathrm{S})=n}$-almost all $\mathbf{z}_n$. This condition is crucial to prove Lemmas 3.3 and 3.4 below, which provide, respectively, the *product rule* and the *chain rule* for the differentiation, w.r.t. the gradient operator, of functionals of finite point processes with random marks. Here we denote by g' the first order derivative of g.

Lemma 3.3 *For all $F \in \mathcal{R}^{\mathrm{f}}_{\mathrm{S}}(p')$ and $G \in \mathcal{R}^{\mathrm{f}}_{\mathrm{S}}(q')$, we have $FG \in \mathcal{R}^{\mathrm{f}}_{\mathrm{S}}(q)$ and*

$$D_w(FG) = F D_w G + G D_w F.$$

Lemma 3.4 *For all $F \in \mathcal{R}^{\mathrm{f}}_{\mathrm{S}}(r)$ and $g \in \mathcal{C}^1_b(\mathbb{R})$, we have $g(F) \in \mathcal{R}^{\mathrm{f}}_{\mathrm{S}}(r)$ and*

$$D_w g(F) = g'(F) D_w F.$$

Proof of Lemma 3.3. Let $F \in \mathcal{R}^{\mathrm{f}}_{\mathrm{S}}(p')$ and $G \in \mathcal{R}^{\mathrm{f}}_{\mathrm{S}}(q')$ be functionals with length m_1 and m_2, and form functions f_n and g_n, respectively. Letting $a \wedge b$ denote the minimum between $a, b \in \mathbb{R}$, we deduce:

$$FG = f_0 g_0 \mathbb{1}_{\{N(\mathrm{S})=0\}} + \sum_{n=1}^{m_1 \wedge m_2} \mathbb{1}_{\{N(\mathrm{S})=n\}} f_n(\mathbf{X}_n, \mathbf{Z}_n) g_n(\mathbf{X}_n, \mathbf{Z}_n).$$

We first check that $FG \in \mathcal{R}^{\mathrm{f}}_{\mathrm{S}}(q)$. For ease of notation set $h_n := f_n g_n$. By assumption $f_n(\cdot, \mathbf{z}_n) \in \mathcal{W}^{1,p'}(\mathrm{S}^n)$ and $g_n(\cdot, \mathbf{z}_n) \in \mathcal{W}^{1,q'}(\mathrm{S}^n)$ for $\mu_{\mathbf{Z}_n|N(\mathrm{S})=n}$-almost all $\mathbf{z}_n$. So, for fixed $n \geq 1$ and $\mathbf{z}_n$, by Theorem 3 p. 127 in Evans and Gariepy [12] there exist two sequences $(\phi^{(l)}_{n,\mathbf{z}_n})_{l\geq 1} \subset \mathcal{C}^\infty(\overline{\mathrm{S}}^n)$ and $(\gamma^{(l)}_{n,\mathbf{z}_n})_{l\geq 1} \subset \mathcal{C}^\infty(\overline{\mathrm{S}}^n)$ such that $\phi^{(l)}_{n,\mathbf{z}_n} \to f_n(\cdot, \mathbf{z}_n)$ in $\mathcal{W}^{1,p'}(\mathrm{S}^n)$ and $\gamma^{(l)}_{n,\mathbf{z}_n} \to g_n(\cdot, \mathbf{z}_n)$ in $\mathcal{W}^{1,q'}(\mathrm{S}^n)$. For $\varphi \in \mathcal{C}^1_c(\mathrm{S}^n)$ we have:

$$\int_{\mathrm{S}^n} h_n(\mathbf{x}_n, \mathbf{z}_n) \partial_{x^{(i)}_k} \varphi(\mathbf{x}_n) \, \mathrm{d}\mathbf{x}_n = \lim_{l\to\infty} \int_{\mathrm{S}^n} \phi^{(l)}_{n,\mathbf{z}_n}(\mathbf{x}_n) \gamma^{(l)}_{n,\mathbf{z}_n}(\mathbf{x}_n) \partial_{x^{(i)}_k} \varphi(\mathbf{x}_n) \, \mathrm{d}\mathbf{x}_n \tag{3.9}$$

$$= -\int_{\mathrm{S}^n} (f_n(\mathbf{x}_n, \mathbf{z}_n) \partial_{x^{(i)}_k} g_n(\mathbf{x}_n, \mathbf{z}_n) + g_n(\mathbf{x}_n, \mathbf{z}_n) \partial_{x^{(i)}_k} f_n(\mathbf{x}_n, \mathbf{z}_n)) \varphi(\mathbf{x}_n) \, \mathrm{d}\mathbf{x}_n. \tag{3.10}$$

Here (3.9) follows by combining the convergence of $(\phi^{(l)}_{n,\mathbf{z}_n})_{l\geq 1}$ and $(\gamma^{(l)}_{n,\mathbf{z}_n})_{l\geq 1}$ to $f_n(\cdot, \mathbf{z}_n)$ and $g_n(\cdot, \mathbf{z}_n)$, respectively, with Minkowski's and Hölder's inequalities, indeed:

$$\begin{aligned}\|\phi^{(l)}_{n,\mathbf{z}_n} \gamma^{(l)}_{n,\mathbf{z}_n} - f_n(\cdot, \mathbf{z}_n) g_n(\cdot, \mathbf{z}_n)\|_{L^q(\mathrm{S})} \leq{} & \|\phi^{(l)}_{n,\mathbf{z}_n} - f_n(\cdot, \mathbf{z}_n)\|_{L^{p'}(\mathrm{S})} \|\gamma^{(l)}_{n,\mathbf{z}_n} - g_n(\cdot, \mathbf{z}_n)\|_{L^{q'}(\mathrm{S})} \\ & + \|\phi^{(l)}_{n,\mathbf{z}_n} - f_n(\cdot, \mathbf{z}_n)\|_{L^{p'}(\mathrm{S})} \|g_n(\cdot, \mathbf{z}_n)\|_{L^{q'}(\mathrm{S})} \\ & + \|\gamma^{(l)}_{n,\mathbf{z}_n} - g_n(\cdot, \mathbf{z}_n)\|_{L^{q'}(\mathrm{S})} \|f_n(\cdot, \mathbf{z}_n)\|_{L^{p'}(\mathrm{S})}\end{aligned}$$

and this latter term goes to zero as $l \to +\infty$. The equality (3.10) can be proved similarly. So $h_n(\cdot, \mathbf{z}_n)$ is in $\mathcal{W}^{1,q}(\mathrm{S}^n)$ for $\mu_{\mathbf{Z}_n|N(\mathrm{S})=n}$-almost all $\mathbf{z}_n$, and

$$\partial_{x^{(i)}_k} h_n = f_n \partial_{x^{(i)}_k} g_n + g_n \partial_{x^{(i)}_k} f_n, \qquad n \geq 1, \quad k = 1, \ldots, n, \quad i = 1, \ldots, d, \tag{3.11}$$

up to measurable subsets of $\mathrm{S}^n \times \mathrm{M}^n$ with null product measure (the product measure on $\mathrm{S}^n \times \mathrm{M}^n$ is the product between the Lebesgue measure ℓ on S^n and $\mu_{\mathbf{Z}_n|N(\mathrm{S})=n}$ on M^n). Using again Hölder's inequality with conjugate exponents p'/q and q'/q we have:

$$\|\mathbb{1}_{\{N(\mathrm{S})=n\}} h_n(\mathbf{X}_n, \mathbf{Z}_n)\|_q^q \leq \|\mathbb{1}_{\{N(\mathrm{S})=n\}} f_n(\mathbf{X}_n, \mathbf{Z}_n)\|_{p'}^q \|\mathbb{1}_{\{N(\mathrm{S})=n\}} g_n(\mathbf{X}_n, \mathbf{Z}_n)\|_{q'}^q < \infty.$$

Moreover,

$$
\begin{aligned}
&\|\mathbb{1}_{\{N(\mathrm{S})=n\}}\partial_{x_k^{(i)}}h_n(\mathbf{X}_n,\mathbf{Z}_n)\|_q^q = \mathrm{E}[\mathbb{1}_{\{N(\mathrm{S})=n\}}|\partial_{x_k^{(i)}}h_n(\mathbf{X}_n,\mathbf{Z}_n)|^q] \\
&\le 2^q p_n \int_{\mathrm{S}^n} \mathrm{E}[|f_n(\mathbf{x}_n,\mathbf{Z}_n)\partial_{x_k^{(i)}}g_n(\mathbf{x}_n,\mathbf{Z}_n)|^q|N(\mathrm{S})=n]\, j_n(\mathbf{x}_n)\mathrm{d}\mathbf{x}_n \qquad (3.12)\\
&\quad + 2^q p_n \int_{\mathrm{S}^n} \mathrm{E}[|g_n(\mathbf{x}_n,\mathbf{Z}_n)\partial_{x_k^{(i)}}f_n(\mathbf{x}_n,\mathbf{Z}_n)|^q|N(\mathrm{S})=n]\, j_n(\mathbf{x}_n)\mathrm{d}\mathbf{x}_n \\
&\le 2^q\|\mathbb{1}_{\{N(\mathrm{S})=n\}}f_n(\mathbf{X}_n,\mathbf{Z}_n)\|_{p'}^q\|\mathbb{1}_{\{N(\mathrm{S})=n\}}\partial_{x_k^{(i)}}g_n(\mathbf{X}_n,\mathbf{Z}_n)\|_{q'}^q \qquad (3.13)\\
&\quad + 2^q\|\mathbb{1}_{\{N(\mathrm{S})=n\}}g_n(\mathbf{X}_n,\mathbf{Z}_n)\|_{q'}^q\|\mathbb{1}_{\{N(\mathrm{S})=n\}}|\partial_{x_k^{(i)}}f_n(\mathbf{X}_n,\mathbf{Z}_n)\|_{p'}^q < \infty.
\end{aligned}
$$

Here the inequality (3.12) is consequence of (3.11) and the inequality $(a+b)^q \le 2^q(a^q+b^q)$ for all $a,b\ge 0$; the inequality (3.13) follows by Hölder's inequality. Thus $FG \in \mathcal{R}_{\mathrm{S}}^{\mathrm{f}}(q)$. Finally, note that the definition of D_w and (3.11) yield:

$$
\begin{aligned}
D_w FG = &-\sum_{n=1}^{m_1\wedge m_2}\mathbb{1}_{\{N(\mathrm{S})=n\}}\sum_{k=1}^{n} w(X_k)f_n(\mathbf{X}_n,\mathbf{Z}_n)\mathrm{div}_{x_k}g_n(\mathbf{X}_n,\mathbf{Z}_n)\\
&-\sum_{n=1}^{m_1\wedge m_2}\mathbb{1}_{\{N(\mathrm{S})=n\}}\sum_{k=1}^{n} w(X_k)g_n(\mathbf{X}_n,\mathbf{Z}_n)\mathrm{div}_{x_k}f_n(\mathbf{X}_n,\mathbf{Z}_n)\\
=&\ FD_wG + GD_wF.
\end{aligned}
$$

□

Proof of Lemma 3.4. We first prove that $g(F)\in\mathcal{R}_{\mathrm{S}}^{\mathrm{f}}(r)$. If F has length m and form functions f_n, then the functional $g(F)$ has length m and form functions $g\circ f_n$. By assumption $f_n(\cdot,\mathbf{z}_n)$ is in $\mathcal{W}^{1,r}(\mathrm{S}^n)$ for $\mu_{\mathbf{Z}_n|N(\mathrm{S})=n}$-almost all $\mathbf{z}_n$; moreover, S has finite Lebesgue measure $\ell(\mathrm{S})$. So by Theorem 4 (ii) p. 130 in Evans and Gariepy [12] we get that $g\circ f_n(\cdot,\mathbf{z}_n)$ is in $\mathcal{W}^{1,r}(\mathrm{S}^n)$ for almost all $\mathbf{z}_n$, and

$$
\partial_{x_k^{(i)}}g\circ f_n = (g'\circ f_n)\partial_{x_k^{(i)}}f_n, \qquad k=1,\dots,n, \quad i=1,\dots,d, \tag{3.14}
$$

up to subsets of $\mathrm{S}^n\times\mathrm{M}^n$ with null product measure (here, again, the product measure on $\mathrm{S}^n\times\mathrm{M}^n$ is the product between the Lebesgue measure ℓ on S^n and $\mu_{\mathbf{Z}_n|N(\mathrm{S})=n}$ on M^n). Since g is bounded then $g\circ f_n$ is bounded and so $\mathbb{1}_{\{N(\mathrm{S})=n\}}g\circ f_n(\mathbf{X}_n,\mathbf{Z}_n)\in\mathrm{L}^r(\mathrm{S})$. Moreover, by (3.14) and the boundedness of g' we deduce:

$$
\|\mathbb{1}_{\{N(\mathrm{S})=n\}}\partial_{x_k^{(i)}}g\circ f_n(\mathbf{X}_n,\mathbf{Z}_n)\|_r^r \le \|g'\|_\infty\|\mathbb{1}_{\{N(\mathrm{S})=n\}}\partial_{x_k^{(i)}}f_n(\mathbf{X}_n,\mathbf{Z}_n)\|_r^r < \infty.
$$

Thus $g(F) \in \mathcal{R}^{\mathrm{f}}_{\mathrm{S}}(r)$. We conclude the proof by noticing that, by the definition of the gradient operator and (3.14), we have

$$D_w g(F) = -\sum_{n=1}^{m} \mathbb{1}_{\{N(\mathrm{S})=n\}} \sum_{k=1}^{n} w(X_k) g' \circ f_n(\mathbf{X}_n, \mathbf{Z}_n) \mathrm{div}_{x_k} f_n(\mathbf{X}_n, \mathbf{Z}_n) = g'(F) D_w F.$$

□

We shall also need Lemma 5.2 (cf. Section 5) which extends the chain rule of Lemma 3.4 to functionals in the domain of the minimal closed extension of D_w (when it exists). However, for the sake of clarity and notation, we shall state this lemma later on.

4 A duality relation

In the rest of the paper we suppose that $p' \geq p$, and that the weight function and Janossy densities satisfy

$$w \in \mathcal{W}^{1,p'}(\mathrm{S}) \cap \mathcal{C}(\overline{\mathrm{S}}), \tag{4.1}$$

and

$$j_n \in \mathcal{W}^{1,p}(\mathrm{S}^n) \cap \mathcal{C}(\overline{\mathrm{S}}^n), \qquad n \geq 1. \tag{4.2}$$

Clearly, Conditions (4.1) and (4.2) are stronger than the assumptions (3.2) and (3.3) considered in the previous section. The next proposition provides sufficient conditions which ensure the *closability* of the gradient operator $D_w : \mathcal{R}^{\mathrm{f}}_{\mathrm{S}}(p') \to \mathrm{L}^p(\mathrm{S})$ and the divergence operator $D^*_w : \mathcal{R}^{\mathrm{f}}_{\mathrm{S}}(q') \to \mathrm{L}^q(\mathrm{S})$. When they exist, we denote by $\overline{D}_w$ the minimal closed extension of the gradient operator D_w and by $\overline{D}^*_w$ the minimal closed extension of the divergence operator D^*_w. In Proposition 4.1 below we also prove a *duality relation* between $\overline{D}_w$ and $\overline{D}^*_w$ as a consequence of their closability.

Proposition 4.1 *Assume that Conditions* (3.4), (4.1), *and* (4.2) *hold, and that*

$$w(x) = 0, \qquad x \in \partial\mathrm{S}. \tag{4.3}$$

Then the operators $D_w : \mathcal{R}^{\mathrm{f}}_{\mathrm{S}}(p') \to \mathrm{L}^p(\mathrm{S})$ and $D^*_w : \mathcal{R}^{\mathrm{f}}_{\mathrm{S}}(q') \to \mathrm{L}^q(\mathrm{S})$ are closable and the following duality relation holds:

$$\mathrm{E}[G\overline{D}_w F] = \mathrm{E}[F\overline{D}^*_w G], \qquad F \in \mathrm{Dom}(\overline{D}_w), \quad G \in \mathrm{Dom}(\overline{D}^*_w). \tag{4.4}$$

Proof. We divide the proof in 3 steps: in the first step we show a weak duality relation, i.e. a duality relation for functionals in $\mathcal{R}_{\mathrm{S}}^{\mathrm{f}}(p')$ and $\mathcal{R}_{\mathrm{S}}^{\mathrm{f}}(q')$; in the second step we show the closability of D_w and D_w^*; in the third step we prove the duality relation.

Step 1. Weak duality relation. For any $H \in \mathcal{R}_{\mathrm{S}}^{\mathrm{f}}(q)$ with length m and form functions h_n, it holds

$$\begin{aligned}\mathrm{E}[D_w H] &= -\sum_{n=1}^{m} p_n \sum_{k=1}^{n} \mathrm{E}[w(X_k)\mathrm{div}_{x_k} h_n(\mathbf{X}_n, \mathbf{Z}_n) \,|\, N(\mathrm{S}) = n] \qquad (4.5)\\ &= -\sum_{n=1}^{m} p_n \sum_{k=1}^{n} \int_{\mathrm{S}^{n-1}} \prod_{\substack{i=1\\ i\neq k}}^{n} \mathrm{d}x_i \, \mathrm{E}\left[\int_{\mathrm{S}} j_n(\mathbf{x}_n) w(x_k) \mathrm{div}_{x_k} h_n(\mathbf{x}_n, \mathbf{Z}_n)\, \mathrm{d}x_k \,\middle|\, N(\mathrm{S}) = n\right],\end{aligned}$$

where we exchange the order of integration and the expectation by Fubini's theorem. For fixed $n \geq 1$, $k \in \{1, \dots, n\}$ and $\mathbf{x}_{k,n} := (x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_n) \in \mathrm{S}^{n-1}$, define the function

$$J_{\mathbf{x}_{k,n}}(x) := \widetilde{J}_{\mathbf{x}_{k,n}}(x) w(x) \quad x \in \overline{\mathrm{S}}$$

where

$$\widetilde{J}_{\mathbf{x}_{k,n}}(x) := j_n(x_1, \dots, x_{k-1}, x, x_{k+1}, \dots, x_n) \quad x \in \overline{\mathrm{S}}.$$

By assumption (4.1) we have $w \in \mathcal{W}^{1,q}(\mathrm{S}) \cap \mathcal{C}(\overline{\mathrm{S}})$, and by assumption (4.2) we deduce that, for almost all $\mathbf{x}_{k,n}$ (w.r.t. the Lebesgue measure), the function $\widetilde{J}_{\mathbf{x}_{k,n}}$ belongs to $\mathcal{W}^{1,q}(\mathrm{S}) \cap \mathcal{C}(\overline{\mathrm{S}})$. Thus by Theorem 4 (i) p. 129 in Evans and Gariepy [12], it follows that, for almost all $\mathbf{x}_{k,n}$, the function $J_{\mathbf{x}_{k,n}}$ belongs to $\mathcal{W}^{1,q}(\mathrm{S}) \cap \mathcal{C}(\overline{\mathrm{S}})$ and

$$\partial_{x^{(i)}} J_{\mathbf{x}_{k,n}}(x) = w(x) \partial_{x^{(i)}} \widetilde{J}_{\mathbf{x}_{k,n}}(x) + \widetilde{J}_{\mathbf{x}_{k,n}}(x) \partial_{x^{(i)}} w(x) \qquad (4.6)$$

for $i = 1, \dots, d$, almost all $\mathbf{x}_{k,n}$ and almost all $x \in \mathrm{S}$ (w.r.t. the Lebesgue measure). By assumption, for $\mu_{\mathbf{Z}_n | N(\mathrm{S})=n}$-almost all $\mathbf{z}_n$, $h_n(\cdot, \mathbf{z}_n) \in \mathcal{W}^{1,q}(\mathrm{S}^n)$. Then, for almost all $\mathbf{x}_{k,n}$ and $\mathbf{z}_n$, the function

$$\psi_{\mathbf{x}_{n,k}, \mathbf{z}_n}(x) := h_n(x_1, \dots, x_{k-1}, x, x_{k+1}, \dots, x_n, \mathbf{z}_n) \quad x \in \mathrm{S}$$

belongs to $\mathcal{W}^{1,q}(\mathrm{S})$. So, by Theorem 3 p. 127 in Evans and Gariepy [12] we have that, for almost all $\mathbf{x}_{k,n}$ and $\mathbf{z}_n$, there exists a sequence $(\psi^{(l)}_{\mathbf{x}_{k,n}, \mathbf{z}_n})_{l \geq 1} \subset \mathcal{C}^{\infty}(\overline{\mathrm{S}})$ such that

$$\psi^{(l)}_{\mathbf{x}_{k,n}, \mathbf{z}_n} \to \psi_{\mathbf{x}_{k,n}, \mathbf{z}_n} \quad \text{as } l \to \infty, \text{ in } \mathcal{W}^{1,q}(\mathrm{S}). \qquad (4.7)$$

By formula (2.2) and the fact that $w \equiv 0$ on $\partial \mathrm{S}$, we have, for all $n \geq 1$, $h \geq 1$, $k \in \{1, \dots, n\}$, and almost all $\mathbf{x}_{k,n}$ and $\mathbf{z}_n$,

$$\int_{\mathrm{S}} J_{\mathbf{x}_{k,n}}(x) \mathrm{div}_x \psi^{(l)}_{\mathbf{x}_{k,n}, \mathbf{z}_n}(x)\, \mathrm{d}x = -\int_{\mathrm{S}} \psi^{(l)}_{\mathbf{x}_{k,n}, \mathbf{z}_n}(x) \mathrm{div}_x J_{\mathbf{x}_{k,n}}(x)\, \mathrm{d}x. \qquad (4.8)$$

By passing to the limit as $l \to \infty$ in (4.8) we have

$$\int_{\mathrm{S}} J_{\mathbf{x}_{k,n}}(x_k) \mathrm{div}_{x_k} h_n(\mathbf{x}_n, \mathbf{z}_n)\, \mathrm{d}x_k = - \int_{\mathrm{S}} h_n(\mathbf{x}_n, \mathbf{z}_n) \mathrm{div}_{x_k} J_{\mathbf{x}_{k,n}}(x_k)\, \mathrm{d}x_k \tag{4.9}$$

for all $n \geq 1$, $k \in \{1, \dots, n\}$, ℓ-almost all $\mathbf{x}_{k,n}$ and $\mu_{\mathbf{Z}_n | N(\mathrm{S})=n}$-almost all $\mathbf{z}_n$. To prove identity (4.9) we start showing that

$$\int_{\mathrm{S}} J_{\mathbf{x}_{k,n}}(x_k) \mathrm{div}_{x_k} \psi^{(l)}_{\mathbf{x}_{k,n},\mathbf{z}_n}(x_k)\, \mathrm{d}x_k \to \int_{\mathrm{S}} J_{\mathbf{x}_{k,n}}(x_k) \mathrm{div}_{x_k} h_n(\mathbf{x}_n, \mathbf{z}_n)\, \mathrm{d}x_k.$$

Note that

$$\begin{aligned} &\Big| \int_{\mathrm{S}} J_{\mathbf{x}_{k,n}}(x) \mathrm{div}_x \psi^{(l)}_{\mathbf{x}_{k,n},\mathbf{z}_n}(x)\, \mathrm{d}x - \int_{\mathrm{S}} J_{\mathbf{x}_{k,n}}(x) \mathrm{div}_x \psi_{\mathbf{x}_{k,n},\mathbf{z}_n}(x)\, \mathrm{d}x \Big| \\ &\leq \|J_{\mathbf{x}_{k,n}}\|_\infty \int_{\mathrm{S}} |\mathrm{div}_x \psi^{(l)}_{\mathbf{x}_{k,n},\mathbf{z}_n}(x) - \mathrm{div}_x \psi_{\mathbf{x}_{k,n},\mathbf{z}_n}(x)|\, \mathrm{d}x \end{aligned}$$

and this latter term goes to zero as $l \to \infty$ because $J_{\mathbf{x}_{k,n}} \in L^\infty(\mathrm{S})$ and $\psi^{(l)}_{\mathbf{x}_{k,n},\mathbf{z}_n} \to \psi_{\mathbf{x}_{k,n},\mathbf{z}_n}$ in $\mathcal{W}^{1,q}(\mathrm{S})$, where $q > 1$. We now show that

$$\int_{\mathrm{S}} \psi^{(l)}_{\mathbf{x}_{k,n},\mathbf{z}_n}(x_k) \mathrm{div}_{x_k} J_{\mathbf{x}_{k,n}}(x_k)\, \mathrm{d}x_k \to \int_{\mathrm{S}} h_n(\mathbf{x}_n, \mathbf{z}_n) \mathrm{div}_{x_k} J_{\mathbf{x}_{k,n}}(x_k)\, \mathrm{d}x_k. \tag{4.10}$$

Since $w, \widetilde{J}_{\mathbf{x}_{k,n}} \in L^\infty(\mathrm{S})$, by (4.6) we have that the limit in (4.10) follows if we prove

$$\int_{\mathrm{S}} |\partial_{x^{(i)}} \widetilde{J}_{\mathbf{x}_{k,n}}(x)| |\psi^{(l)}_{\mathbf{x}_{k,n},\mathbf{z}_n}(x) - \psi_{\mathbf{x}_{k,n},\mathbf{z}_n}(x)|\, \mathrm{d}x \to 0 \quad \text{as } h \to 0$$

and

$$\int_{\mathrm{S}} |\partial_{x^{(i)}} w(x)| |\psi^{(l)}_{\mathbf{x}_{k,n},\mathbf{z}_n}(x) - \psi_{\mathbf{x}_{k,n},\mathbf{z}_n}(x)|\, \mathrm{d}x \to 0 \quad \text{as } l \to \infty.$$

Both these limits easily follow by (4.7), applying Hölder's inequality with conjugate exponents p and q and noticing that $w, \widetilde{J}_{\mathbf{x}_{k,n}} \in \mathcal{W}^{1,p}(\mathrm{S})$. This concludes the proof of (4.9). Combining (4.9) with (4.5) and using (4.6), we deduce

$$\mathrm{E}[D_w H] = \mathrm{E}\left[H \sum_{k=1}^{N(\mathrm{S})} (\mathrm{div}\, w(X_k) + w(X_k) R_{k,N(\mathrm{S})}(\mathbf{X}_{N(\mathrm{S})})) \right].$$

Therefore, by Lemma 3.3 we have a weak duality relation:

$$\begin{aligned} \mathrm{E}[G D_w F] &= \mathrm{E}[D_w(FG)] - \mathrm{E}[F D_w G] \\ &= \mathrm{E}\left[F \left(G \sum_{k=1}^{N(\mathrm{S})} (\mathrm{div}\, w(X_k) + w(X_k) R_{k,N(\mathrm{S})}(\mathbf{X}_{N(\mathrm{S})})) - D_w G \right) \right] \end{aligned}$$

$$= \mathrm{E}[F D_w^* G], \qquad F \in \mathcal{R}_{\mathrm{S}}^{\mathrm{f}}(p'), \quad G \in \mathcal{R}_{\mathrm{S}}^{\mathrm{f}}(q').$$

Step 2. Closability. Now we show that $D_w : \mathcal{R}_{\mathrm{S}}^{\mathrm{f}}(p') \to \mathrm{L}^p(\mathrm{S})$ is closable (similarly, one can prove that $D_w^* : \mathcal{R}_{\mathrm{S}}^{\mathrm{f}}(q') \to \mathrm{L}^q(\mathrm{S})$ is closable). Let $(F_n)_{n\geq 1}$ be a sequence in $\mathcal{R}_{\mathrm{S}}^{\mathrm{f}}(p')$ converging to 0 in $\mathrm{L}^p(\mathrm{S})$ and such that $D_w F_n \to U$ in $\mathrm{L}^p(\mathrm{S})$. We need to show that $U = 0$ a.s. We have

$$|\mathrm{E}[GU]| = \lim_{n\to\infty} |\mathrm{E}[G D_w F_n]| = \lim_{n\to\infty} |\mathrm{E}[F_n D_w^* G]| \tag{4.11}$$

$$\leq \|D_w^* G\|_q \lim_{n\to\infty} \|F_n\|_p = 0, \qquad G \in \mathcal{R}_{\mathrm{S}}^{\mathrm{f}}(q'). \tag{4.12}$$

Here the first equality in (4.11) follows by noticing that since $G \in \mathcal{R}_{\mathrm{S}}^{\mathrm{f}}(q') \subset \mathrm{L}^q(\mathrm{S})$ (see Remark 3.1) using Hölder's inequality we deduce:

$$||\mathrm{E}[G D_w F_n]| - |\mathrm{E}[GU]|| \leq |\mathrm{E}[G D_w F_n] - \mathrm{E}[GU]| \leq \|G\|_q \|D_w F_n - U\|_p \to 0,$$

as $n \to \infty$. The second inequality in (4.11) follows by the duality relation for functionals in $\mathcal{R}_{\mathrm{S}}^{\mathrm{f}}(p')$ and $\mathcal{R}_{\mathrm{S}}^{\mathrm{f}}(q')$; (4.12) is consequence of Hölder's inequality and the fact that $\|D_w^* G\|_q < \infty$ (see Proposition 3.2). Finally we show that $\mathrm{E}[GU] = 0$ for all $G \in \mathcal{R}_{\mathrm{S}}^{\mathrm{f}}(q')$ implies $U = 0$ a.s. Since U is $\mathcal{F}_{\mathrm{S}}$-measurable, it is of the form $U = u(N(\mathrm{S}), \mathbf{X}_{N(\mathrm{S})}, \mathbf{Z}_{N(\mathrm{S})})$ for some measurable function u. So, defining $u_n(\cdot,\cdot) := u(n,\cdot,\cdot)$, we have

$$U = u_0 \mathbb{1}_{\{N(\mathrm{S})=0\}} + \sum_{n=1}^{\infty} \mathbb{1}_{\{N(\mathrm{S})=n\}} u_n(\mathbf{X}_n, \mathbf{Z}_n) \quad \text{for some } u_0 \in \mathbb{R}.$$

Therefore by (4.12) we get, for all integers $n \geq 1$, $g_0 \in \mathbb{R}$ and form functions g_n satisfying the assumptions in the definition of $\mathcal{R}_{\mathrm{S}}^{\mathrm{f}}(q')$:

$$p_0 g_0 u_0 = 0 \quad \text{and} \quad \mathrm{E}[g_n(\mathbf{X}_n, \mathbf{Z}_n) u_n(\mathbf{X}_n, \mathbf{Z}_n) \,|\, N(\mathrm{S}) = n] = 0. \tag{4.13}$$

Clearly, the first equality above yields $u_0 = 0$. We now prove that, for any $n \geq 1$, $u_n(\mathbf{x}_n, \mathbf{z}_n) = 0$ up to sets of null measure w.r.t. the product measure, say π_n, between $j_n(\mathbf{x}_n)\,\mathrm{d}\mathbf{x}_n$ and $\mu_{\mathbf{Z}_n \,|\, N(\mathrm{S})=n}(\mathrm{d}\mathbf{z}_n)$. Denote by u_n^+ and u_n^- the positive and the negative part of u_n, respectively. Clearly (take $g_n \equiv 1$ in (4.13)) we have

$$E_n := \mathrm{E}[u_n^+(\mathbf{X}_n, \mathbf{Z}_n) \,|\, N(\mathrm{S}) = n] = \mathrm{E}[u_n^-(\mathbf{X}_n, \mathbf{Z}_n) \,|\, N(\mathrm{S}) = n].$$

If $E_n = 0$ then $u_n^+ = u_n^- = 0$ π_n-a.s., hence $u_n = 0$ π_n-a.s. If $E_n > 0$, then consider the probability measures on $\mathrm{S}^n \times \mathrm{M}^n$:

$$\pi_n^{\pm}(\mathrm{d}\mathbf{x}_n \mathrm{d}\mathbf{z}_n) := \widehat{u}_n^{\pm}(\mathbf{x}_n, \mathbf{z}_n)\, \pi_n(\mathrm{d}\mathbf{x}_n \mathrm{d}\mathbf{z}_n).$$

where

$$\widehat{u}_n^{\pm}(\mathbf{x}_n, \mathbf{z}_n) := \frac{1}{E_n} u_n^{\pm}(\mathbf{x}_n, \mathbf{z}_n).$$

Let R be a rectangular cell in $\mathbb{R}^n$, let $A \in \mathcal{M}^{\otimes n}$, and take $g_n(\mathbf{x}_n, \mathbf{z}_n) = \varphi_l(\mathbf{x}_n)\mathbb{1}_{\{\mathbf{z}_n \in A\}}$ where φ_l is a sequence in $\mathcal{C}_c^\infty(\mathbb{R}^n)$ such that $\varphi_l(\mathbf{x}_n) \to \mathbb{1}_{\{\mathbf{x}_n \in \mathrm{R}\}}$ as $l \to \infty$, for all $\mathbf{x}_n$. Combining the second equality in (4.13) with the dominated convergence theorem, we have $\pi_n^+((\mathrm{R} \cap \mathrm{S}) \times A) = \pi_n^-((\mathrm{R} \cap \mathrm{S}) \times A)$. Therefore, $\pi_n^+ \equiv \pi_n^-$ on $\mathcal{B}(\mathrm{S}^n) \otimes \mathcal{M}^{\otimes n}$, where $\mathcal{B}(\mathrm{S}^n)$ is the Borel σ-field on S^n. So $u_n^+(\mathbf{x}_n, \mathbf{z}_n) = u_n^-(\mathbf{x}_n, \mathbf{z}_n)$ π_n-a.s., and the claim follows.

Step 3. Duality relation. By Step 2, both the gradient and the divergence operators are closable. Let $\overline{D_w}$ and $\overline{D_w^*}$ be the respective closed minimal extensions. Take $F \in \mathrm{Dom}(\overline{D_w})$ and $G \in \mathrm{Dom}(\overline{D_w^*})$, let $(F_n)_{n\geq 1} \subset \mathcal{R}_\mathrm{S}^\mathrm{f}(p')$ be such that $F_n \to F$ and $D_w F_n \to \overline{D_w} F$ in $\mathrm{L}^p(\mathrm{S})$, and let $(G_n)_{n \geq 1} \subset \mathcal{R}_\mathrm{S}^\mathrm{f}(q')$ be such that $G_n \to G$ and $D_w^* G_n \to \overline{D_w^*} G$ in $\mathrm{L}^q(\mathrm{S})$. By Step 1 the duality relation applies to functionals in $\mathcal{R}_\mathrm{S}^\mathrm{f}(p')$ and $\mathcal{R}_\mathrm{S}^\mathrm{f}(q')$, and therefore $\mathrm{E}[G_n D_w F_n] = \mathrm{E}[F_n D_w^* G_n]$ for all $n \geq 1$. The claim follows if we prove

$$\lim_{n\to\infty} \mathrm{E}[G_n D_w F_n] = \mathrm{E}[G\overline{D_w}F] \quad \text{and} \quad \lim_{n\to\infty} \mathrm{E}[F_n D_w^* G_n] = \mathrm{E}[F\overline{D_w^*}G].$$

We only show the first limit above; the second limit can be proved similarly. The claim is given by the following computations:

$$\begin{aligned} |\mathrm{E}[G_n D_w F_n] - \mathrm{E}[G\overline{D_w}F]| &= |\mathrm{E}[G_n D_w F_n] - \mathrm{E}[G_n\overline{D_w}F] + \mathrm{E}[G_n\overline{D_w}F] - \mathrm{E}[G\overline{D_w}F]| \\ &\leq \|G_n\|_q \|D_w F_n - \overline{D_w}F\|_p + \|G_n - G\|_q \|\overline{D_w}F\|_p \to 0 \qquad (4.14) \end{aligned}$$

where in (4.14) we used Hölder's inequality and that $\|G_n\|_q \to \|G\|_q < \infty$ (this is implied by the convergence of G_n to G in $\mathrm{L}^q(\mathrm{S})$). □

We conclude this section with the following simple remark.

Remark 4.2 *Note that, under the assumptions of Proposition* 4.1*, we have:* $\mathcal{R}_\mathrm{S}^\mathrm{f}(p') \subset \mathrm{Dom}(\overline{D}_w)$*,* $\mathcal{R}_\mathrm{S}^\mathrm{f}(q') \subset \mathrm{Dom}(\overline{D}_w^*)$*,* $\overline{D}_w F = D_w F$ *and* $\overline{D}_w^* G = D_w^* G$*,* $\forall\ F \in \mathcal{R}_\mathrm{S}^\mathrm{f}(p')$ *and* $G \in \mathcal{R}_\mathrm{S}^\mathrm{f}(q')$*.*

5 Density estimation

Let $A \in \mathcal{F}_\mathrm{S}$ be such that $P(A) > 0$. If the conditional law of F given A admits a probability density $\varphi_{F\,|\,A}$ w.r.t. the Lebesgue measure ℓ, then the classical kernel estimator of $\varphi_{F\,|\,A}(x)$

(see Parzen [23]) is defined, at each continuity point of $\varphi_{F\,|\,A}$, by

$$\widehat{c}_n(x) := \frac{1}{nh_n}\sum_{i=1}^{n} K\left(\frac{x - F^{(i)}}{h_n}\right), \tag{5.1}$$

where $F^{(i)}$, $i = 1, \ldots, n$, are n independent samples of F under $P(\cdot\,|\,A)$. Here, $(h_n)_{n\geq 1}$ is a sequence of positive numbers called bandwidths, such that

$$\lim_{n\to\infty} h_n = 0 \tag{5.2}$$

and

$$K \text{ is a bounded probability density w.r.t. } \ell(\mathrm{d}x)\text{, such that } \lim_{x\to+\infty} x|K(x)| = 0. \tag{5.3}$$

Note that the kernel estimator $\widehat{c}_n(x)$ is biased for all fixed $n \geq 1$. In this section we apply the duality relation (4.4) to the construction of unbiased Monte Carlo estimators of $\varphi_{F\,|\,A}$, which are an alternative to kernel estimators. We also provide conditions on the functionals F and events $A \in \mathfrak{F}_{\mathrm{S}}$ that ensure that the conditional law of F given A admits a continuous probability density.

5.1 The Malliavin estimator

The following proposition holds:

Proposition 5.1 *Assume that the assumptions of Proposition* 4.1 *hold and let* $F \in \mathrm{Dom}(\overline{D}_w)$. *Suppose that there exists* $A \in \mathfrak{F}_{\mathrm{S}}$ *such that*

$$P(A) > 0 \quad \textit{and} \quad \overline{D}_w F \neq 0 \textit{ on } A, \tag{5.4}$$

up to a P-null set, and

$$\frac{\mathbb{1}_A}{\overline{D}_w F} \in \mathrm{Dom}(\overline{D}_w^*). \tag{5.5}$$

Then the conditional law of F given A is absolutely continuous w.r.t. the Lebesgue measure, with probability density

$$\varphi_{F\,|\,A}(x) = \mathrm{E}[W \mathbb{1}_{\{F\geq x\}}\,|\,A], \quad x \in \mathbb{R}, \tag{5.6}$$

where the Malliavin weight

$$W := \overline{D}_w^*\left(\frac{\mathbb{1}_A}{\overline{D}_w F}\right) \tag{5.7}$$

is in $\mathrm{L}^q(\mathrm{S})$. *In addition, $\varphi_{F\,|\,A}$ is bounded and Hölder continuous with exponent* $1/p$.

This proposition is proved using Lemma 5.2 below, whose proof follows from a standard regularization argument and will be given at the end of this subsection.

Lemma 5.2 *Assume that the conditions of Proposition* 4.1 *hold and let* $f \in \mathcal{C}_b^1(\mathbb{R})$. *Then*

$$f(F) \in \mathrm{Dom}(\overline{D}_w) \quad \textit{and} \quad \overline{D}_w f(F) = f'(F)\overline{D}_w F, \qquad F \in \mathrm{Dom}(\overline{D}_w).$$

Proof of Proposition 5.1. Note that

$$\mathrm{E}[\mathbb{1}_A f'(F)] = \mathrm{E}\left[\frac{\mathbb{1}_A}{\overline{D}_w F}\overline{D}_w f(F)\right] \tag{5.8}$$

$$= \mathrm{E}\left[\mathbb{1}_A W f(F)\right], \qquad f \in \mathcal{C}_c^1(\mathbb{R}). \tag{5.9}$$

Here (5.8) is consequence of Lemma 5.2; (5.9) follows by the duality relation (Proposition 4.1). A straightforward computation gives

$$\begin{aligned}
\mathrm{E}[\mathbb{1}_A f(F)] &= \mathrm{E}\left[\mathbb{1}_A \int_{-\infty}^{F} f'(x)\,\mathrm{d}x\right] = \mathrm{E}\left[\mathbb{1}_A \int_{-\infty}^{0} f'(y+F)\,\mathrm{d}y\right] \\
&= \int_{-\infty}^{0} \mathrm{E}\left[\mathbb{1}_A f'(y+F)\right]\,\mathrm{d}y = \int_{-\infty}^{0} \mathrm{E}\left[\mathbb{1}_A W f(y+F)\right]\,\mathrm{d}y \\
&= \mathrm{E}\left[\mathbb{1}_A W \int_{-\infty}^{0} f(y+F)\,\mathrm{d}y\right] = \int_{\mathbb{R}} f(y)\mathrm{E}\left[\mathbb{1}_A W \mathbb{1}_{\{F\geq y\}}\right]\,\mathrm{d}y, \qquad f \in \mathcal{C}_c^1(\mathbb{R}),
\end{aligned}$$

where we exchange the integrals and the means by Fubini's theorem, and we use (5.9). The above equality, proved for all $f \in \mathcal{C}_c^1(\mathbb{R})$, easily extends to indicators $f(y) = \mathbb{1}_B(y)$, where B is a Borel subset of $\mathbb{R}$. So in particular,

$$P(F \leq x \,|\, A) = \int_{-\infty}^{x} \varphi_{F\,|\,A}(y)\,\mathrm{d}y, \quad x \in \mathbb{R} \tag{5.10}$$

where $\varphi_{F\,|\,A}(x) := \mathrm{E}\left[W\mathbb{1}_{\{F\geq x\}} \,|\, A\right]$. Since $W \in \mathrm{L}^q(\mathrm{S})$ (indeed, $\overline{D}_w^*(\mathrm{Dom}(\overline{D}_w^*)) \subset \mathrm{L}^q(\mathrm{S})$), it is easily realized that $\varphi_{F\,|\,A}$ is bounded by $\|W\|_1/P(A)$. Furthermore, by (5.6) and Hölder's inequality we have

$$|\varphi_{F|A}(z) - \varphi_{F|A}(y)| \;\leq\; \frac{1}{P^2(A)}\|W\|_q\|W\|_1|z-y|^{1/p}, \qquad y, z \in \mathbb{R}, \quad y \leq z,$$

as in e.g. Proposition 2 of Loisel and Privault [19], hence $\varphi_{F\,|\,A}$ is Hölder continuous with exponent $1/p$. $\square$

Remark 5.3 *Under the assumptions of Proposition* 5.1 *we have*

$$\mathrm{E}[W] = \mathrm{E}[\mathbb{1}_A W] = 0.$$

Proof. Indeed, $\mathbb{1}_A W = W$ almost surely and, since $1 \in \mathrm{Dom}(\overline{D}_w)$ and $\overline{D}_w 1 = 0$, by the duality relation we deduce

$$\mathrm{E}[W] = \mathrm{E}\left[\overline{D}_w^*\left(\frac{\mathbb{1}_A}{\overline{D}_w F}\right)\right] = \mathrm{E}\left[\frac{\mathbb{1}_A}{\overline{D}_w F}\overline{D}_w 1\right] = 0.$$

□

In the following discussion, we compare the classical kernel estimator (5.1) with the Malliavin estimator

$$\widehat{m}_n(x) := \frac{1}{n}\sum_{i=1}^{n}(W\mathbb{1}_{\{F\geq x\}})^{(i)}, \qquad x \in \mathbb{R}, \tag{5.11}$$

where $(W\mathbb{1}_{\{F\geq x\}})^{(i)}$, $i = 1, \ldots, n$, are n independent samples of $W\mathbb{1}_{\{F\geq x\}}$, under $P(\cdot\,|\,A)$. Under the assumptions of Proposition 5.1, by Corollary 1A in Parzen [23], we have that $\widehat{c}_n(x)$ is only asymptotically unbiased, i.e.

$$\lim_{n\to\infty} \mathrm{E}_{P(\cdot\,|\,A)}[\widehat{c}_n(x)] = \varphi_{F\,|\,A}(x), \quad \text{for any } x \in \mathbb{R}.$$

Note that, in contrast to $\widehat{c}_n(x)$, the Malliavin estimator does not depend on bandwidths and it is unbiased for all $n \geq 1$.

Now, suppose that the conditions of Proposition 5.1 are satisfied with $q = p = 2$. Then $W \in \mathrm{L}^2(\mathrm{S})$ and therefore by Theorem 2A in Parzen [23] we easily have

$$\lim_{n\to\infty} \frac{\mathrm{Var}_{P(\cdot\,|\,A)}(\widehat{c}_n(x))}{\mathrm{Var}_{P(\cdot\,|\,A)}(\widehat{m}_n(x))} = \lim_{n\to\infty}\frac{C}{h_n} = +\infty, \qquad x \in \mathbb{R}, \tag{5.12}$$

for some constant $C > 0$, which shows that the Malliavin estimator is better than the classical kernel estimator even in terms of asymptotic variance. In Section 7, a numerical comparison between the above sample errors for fixed n is provided in Figure 2 in function of the discretization step.

Finally, if in addition to the assumptions of Proposition 5.1 and conditions (5.2) and (5.3), we suppose

$$\lim_{n\to\infty} nh_n = \infty,$$

then $\widehat{c}_n(x)$ is consistent in square mean (see Parzen [23] p. 1069), i.e.

$$\lim_{n\to\infty} \mathrm{E}_{P(\cdot\,|\,A)}[|\widehat{c}_n(x) - \varphi_{F\,|\,A}(x)|^2] = 0, \qquad x \in \mathbb{R}.$$

A straightforward computation shows that the same property holds true for the Malliavin estimator if we again assume that the conditions of Proposition 5.1 are satisfied with $q = p = 2$.

As already mentioned, we end this section with the proof of Lemma 5.2.

Proof of Lemma 5.2. Since the operator $\overline{D}_w$ is closed, we have to show that for any fixed $F \in \mathrm{Dom}(\overline{D}_w)$ there exists a sequence $(f(F_{n'})) \subset \mathcal{R}_{\mathrm{S}}^{\mathrm{f}}(p')$ such that $f(F_{n'}) \to f(F)$ and $D_w f(F_{n'}) \to f'(F)\overline{D}_w F$ in $\mathrm{L}^p(\mathrm{S})$. Take $F \in \mathrm{Dom}(\overline{D}_w)$, then there exists a sequence $(F_n)_{n\geq 1} \subset \mathcal{R}_{\mathrm{S}}^{\mathrm{f}}(p')$ such that $F_n \to F$ and $D_w F_n \to \overline{D}_w F$ in $\mathrm{L}^p(\mathrm{S})$. Note that the convergence in $\mathrm{L}^p(\mathrm{S})$ implies the convergence in probability. Thus, we can select a subsequence $\{n'\}$ of $\{n\}$ such that $F_{n'} \to F$ almost surely. By Lemma 3.4 we have $(f(F_{n'})) \subset \mathcal{R}_{\mathrm{S}}^{\mathrm{f}}(p')$ and $D_w f(F_{n'}) = f'(F_{n'})D_w F_{n'}$. By the dominated convergence theorem we have $f(F_{n'}) \to f(F)$ in $\mathrm{L}^p(\mathrm{S})$, moreover we have:

$$\begin{aligned}
\|D_w f(F_{n'}) - f'(F)\overline{D}_w F\|_p &= \|f'(F_{n'})D_w F_{n'} - f'(F)\overline{D}_w F\|_p \\
&= \|f'(F_{n'})D_w F_{n'} - f'(F_{n'})\overline{D}_w F + f'(F_{n'})\overline{D}_w F - f'(F)\overline{D}_w F\|_p \\
&\leq \|f'(F_{n'})(D_w F_{n'} - \overline{D}_w F)\|_p + \|(f'(F_{n'}) - f'(F))\overline{D}_w F\|_p \\
&\leq \|f'\|_\infty \|D_w F_{n'} - \overline{D}_w F\|_p + \|(f'(F_{n'}) - f'(F))\overline{D}_w F\|_p.
\end{aligned}$$

The claim follows noticing that the latter two terms above go to zero as $n' \to \infty$. In particular, the rightmost term tends to zero by the dominated convergence theorem. $\square$

5.2 The modified Malliavin estimator

By (5.6), Remark 5.3 and the monotone convergence theorem we have $\lim_{x\to\pm\infty} \varphi_{F\,|\,A}(x) = 0$ and

$$\lim_{x\to-\infty} \mathrm{Var}_{P(\cdot\,|\,A)}(\mathbb{1}_{\{F\geq x\}}W) = \mathrm{E}[W^2\,|\,A]. \tag{5.13}$$

So, when $\mathrm{E}[W^2\,|\,A]$ is large (this happens e.g. when W is not square integrable or $D_w F$ is close to zero with high probability), the performance of the Malliavin estimator for large negative values of x is poor. In this subsection we tackle this problem by applying the localization method, see Fournié, Lasry, Lebuchoux and Lions [14], Kohatsu-Higa and Petterson [18], Privault and Wei [26].

We start with the following result which follows from Proposition 5.1 by a classical regularization argument.

Proposition 5.4 *Assume that the conditions of Proposition* 5.1 *hold. Then for any* $f \in L^p(\mathbb{R})$, $p > 1$, *we have*

$$\frac{\mathrm{d}}{\mathrm{d}y}\mathrm{E}[\mathbb{1}_A f(F-y)] = -\mathrm{E}\left[\mathbb{1}_A W f(F-y)\right], \qquad y \in \mathbb{R}, \tag{5.14}$$

where the Malliavin weight W *is defined in* (5.7).

Proof. Let $(f_n)_{n\geq 1}$ be a sequence in $\mathcal{C}^1_c(\mathbb{R})$ that converges to f in $L^p(\mathbb{R})$. Since the first derivative of f_n is bounded, f_n is a Lipschitz function hence by the dominated convergence theorem we have

$$\frac{\mathrm{d}}{\mathrm{d}y}\mathrm{E}\left[\mathbb{1}_A f_n(F-y)\right] = -\mathrm{E}\left[\mathbb{1}_A f_n'(F-y)\right], \qquad y \in \mathbb{R}.$$

Next we note that, as in the beginning of the proof of Proposition 5.1, by Lemma 5.2 and the duality relation we have

$$\mathrm{E}[\mathbb{1}_A f_n'(F-y)] = \mathrm{E}\left[\mathbb{1}_A W f_n(F-y)\right], \qquad y \in \mathbb{R}.$$

Hence

$$\begin{aligned}
\left|\frac{\mathrm{d}}{\mathrm{d}y}\mathrm{E}\left[\mathbb{1}_A f_n(F-y)\right] + \mathrm{E}\left[\mathbb{1}_A W f(F-y)\right]\right| &= \left|\mathrm{E}[\mathbb{1}_A f_n'(F-y)] - \mathrm{E}\left[\mathbb{1}_A W f(F-y)\right]\right| \\
&= \left|\mathrm{E}\left[\mathbb{1}_A W (f_n(F-y) - f(F-y))\right]\right| \\
&\leq \|\mathbb{1}_A (f_n(F-y) - f(F-y))\|_p \|W\|_q \\
&= \left(P(A)\int_{-\infty}^{\infty} |f_n(x-y) - f(x-y)|^p \varphi_{F\,|\,A}(x)\,\mathrm{d}x\right)^{1/p} \|W\|_q \\
&= \left(P(A)\int_{-\infty}^{\infty} |f_n(x) - f(x)|^p \varphi_{F\,|\,A}(x+y)\,\mathrm{d}x\right)^{1/p} \|W\|_q \\
&\leq CP(A)^{1/p}\|W\|_q \|f_n - f\|_{L^p(\mathbb{R})}, \qquad y \in \mathbb{R},
\end{aligned}$$

for some positive constant $C > 0$, since $\varphi_{F\,|\,A}$ is bounded. Consequently, $\frac{\mathrm{d}}{\mathrm{d}y}\mathrm{E}\left[\mathbb{1}_A f_n(F-y)\right]$ converges to $-\mathrm{E}\left[\mathbb{1}_A W f(F-y)\right]$ uniformly in $y \in \mathbb{R}$, and the claim follows by noticing that $\mathrm{E}[\mathbb{1}_A f_n(F-y)]$ similarly converges to $\mathrm{E}\left[\mathbb{1}_A f(F-y)\right]$ uniformly in $y \in \mathbb{R}$ as a consequence of Hölder's inequality, the boundedness of $\varphi_{F\,|\,A}$ and the convergence of f_n to f in $L^p(\mathbb{R})$.

□

We now construct the modified Malliavin estimator, under the hypotheses of Proposition 5.1. By decomposing the indicator function as

$$\mathbb{1}_{[0,\infty)} = f + g,$$

where $f(x) := \mathbb{1}_{[0,\infty)}(x)\mathrm{e}^{-\theta x}$, $x \in \mathbb{R}$, $\theta > 0$, and $g := \mathbb{1}_{[0,\infty)} - f$, we have

$$\begin{aligned}
\varphi_{F\,|\,A}(x) &= -\frac{\partial}{\partial x}\mathrm{E}\left[\mathbb{1}_{[0,\infty)}\left(\frac{F-x}{\zeta}\right)\,\Big|\,A\right] \\
&= \mathrm{E}\left[Wf\left(\frac{F-x}{\zeta}\right)\,\Big|\,A\right] - \frac{\partial}{\partial x}\mathrm{E}\left[g\left(\frac{F-x}{\zeta}\right)\,\Big|\,A\right] \\
&= \mathrm{E}\left[Wf\left(\frac{F-x}{\zeta}\right)\,\Big|\,A\right] - \frac{\theta}{\zeta}\mathrm{E}\left[\mathbb{1}_{\{F\geq x\}}\mathrm{e}^{-\theta\left(\frac{F-x}{\zeta}\right)}\,\Big|\,A\right], \qquad x \in \mathbb{R},
\end{aligned}$$

where $\zeta > 0$ is a parameter and we applied Proposition 5.4 and the Lebesgue theorem of weak differentiation under the integral sign, cf. e.g. Lemma 3, Chapter 1 in [5]. This leads to the modified Malliavin estimator of $\varphi_{F\,|\,A}(x)$:

$$\widehat{m}_n^{\mathrm{mod}}(x) := \frac{1}{n}\sum_{i=1}^{n}\left(\mathbb{1}_{\{F\geq x\}}\mathrm{e}^{-\theta\left(\frac{F-x}{\zeta}\right)}\left(W - \frac{\theta}{\zeta}\right)\right)^{(i)},$$

where $\left(\mathbb{1}_{\{F\geq x\}}\mathrm{e}^{-\theta\left(\frac{F-x}{\zeta}\right)}\left(W - \frac{\theta}{\zeta}\right)\right)^{(i)}$, $i = 1, \ldots, n$, are n independent samples of $\mathbb{1}_{\{F\geq x\}}\mathrm{e}^{-\theta\left(\frac{F-x}{\zeta}\right)}\left(W - \frac{\theta}{\zeta}\right)$ under $P(\cdot\,|\,A)$.

Finally we note that if W is square integrable then

$$\lim_{x\to-\infty}\mathrm{Var}_{P(\cdot\,|\,A)}\left(\mathbb{1}_{\{F\geq x\}}\mathrm{e}^{-\theta\left(\frac{F-x}{\zeta}\right)}\left(W - \frac{\theta}{\zeta}\right)\right) = 0$$

which, in view of (5.11) and (5.13), shows that $\widehat{m}_n^{\mathrm{mod}}(x)$ performs better than $\widehat{m}_n(x)$ for large negative values of x. Note also that, as a straightforward computation shows, the modified Malliavin estimator has the same properties as the Malliavin estimator, i.e. it is unbiased and, if the Malliavin weight is square integrable, then it is consistent in square mean and its asymptotic variance is smaller than that one of any classical kernel estimator as in (5.12).

6 Computing the Malliavin weight

In this section, for some functionals of finite spatial point processes with random marks, we provide an explicit expression of the Malliavin weight W and, consequently, of the Malliavin estimator for the density. Let w be some weight function and $(f_n)_{n\geq 1}$ form functions of some functional. For ease of notation, for $n \geq 1$, we define

$$\widetilde{f}_n(\mathbf{x}_n, \mathbf{z}_n) := \left(\sum_{k=1}^{n} w(x_k)\mathrm{div}_{x_k} f_n(\mathbf{x}_n, \mathbf{z}_n)\right)^{-1}, \qquad (\mathbf{x}_n, \mathbf{z}_n) \in \mathrm{S}^n \times \mathrm{M}^n.$$

The next proposition provides sufficient conditions for the application of Proposition 5.1 and for the use of the modified Malliavin estimator described in Subsection 5.2.

Proposition 6.1 *Let F be a functional with form functions f_n, and assume that the conditions of Proposition* 4.1 *hold. If in addition:*
(i) $F \in \mathcal{R}_{\mathrm{S}}(p') \cap \mathrm{L}^p(\mathrm{S})$,
(ii) $D_w F \in \mathrm{L}^p(\mathrm{S})$,
(iii) Condition (5.4) *holds for the functional F, with $\overline{D}_w F = D_w F$ and $A = \{N(\mathrm{S}) \geq n_0\}$ for some integer $n_0 \geq 1$,*
(iv) $\widetilde{F} \in \mathcal{R}_{\mathrm{S}}(q') \cap \mathrm{L}^q(\mathrm{S})$, *where*

$$\widetilde{F} := -\sum_{n=n_0}^{\infty} \mathbb{1}_{\{N(\mathrm{S})=n\}} \widetilde{f}_n(\mathbf{X}_n, \mathbf{Z}_n),$$

(v) $D_w^* \widetilde{F} \in \mathrm{L}^q(\mathrm{S})$,
then the conditional law of F given $A = \{N(\mathrm{S}) \geq n_0\}$ is absolutely continuous w.r.t. the Lebesgue measure, with probability density given by (5.6) *with $W = D_w^* \widetilde{F}$, i.e.*

$$W = \widetilde{F} \sum_{k=1}^{N(\mathrm{S})} (\operatorname{div} w(X_k) + w(X_k) R_{k,N(\mathrm{S})}(\mathbf{X}_{N(\mathrm{S})})) - D_w \widetilde{F}.$$

Moreover, the density $\varphi_{F\,|\,A}$ is bounded and Hölder continuous with exponent $1/p$.

Proof. We want to apply Proposition 5.1, and so we need to check the hypotheses therein.
Step 1. $F \in \mathrm{Dom}(\overline{D}_w)$ and Condition (5.4). Consider the truncated functionals $F_m \in \mathcal{R}^{\mathrm{f}}_{\mathrm{S}}(p')$, $m \geq 1$, with form functions f_n and length m. It turns out (see Remark 4.2) that $(F_m)_{m \geq 1} \subset \mathrm{Dom}(\overline{D}_w)$. Note that $F_m \to F$ in $\mathrm{L}^p(\mathrm{S})$ as $m \to \infty$. Indeed

$$\|F_m - F\|_p^p \;=\; \sum_{n=m+1}^{\infty} \mathrm{E}\left[\mathbb{1}_{\{N(\mathrm{S})=n\}} |f_n(\mathbf{X}_n, \mathbf{Z}_n)|^p\right] \to 0, \quad \text{as } m \to \infty \tag{6.1}$$

since $F \in \mathrm{L}^p(\mathrm{S})$. Similarly, using assumption *(ii)* (and Remark 4.2) one has that $\overline{D}_w F_m = D_w F_m \to D_w F$ in $\mathrm{L}^p(\mathrm{S})$. Since the operator $\overline{D}_w$ is closed, we get $F \in \mathrm{Dom}(\overline{D}_w)$ and $\overline{D}_w F = D_w F$. In particular, note that, by this latter identity we deduce that Condition (5.4) corresponds to assumption *(iii)*.
Step 2. Domain Condition (5.5) *and computation of the Malliavin weight.* Consider the truncated functionals

$$\widetilde{F}_m := -\sum_{n=n_0}^{m} \mathbb{1}_{\{N(\mathrm{S})=n\}} \widetilde{f}_n(\mathbf{X}_n, \mathbf{Z}_n), \qquad m \geq n_0.$$

By assumption (iv) we have $\widetilde{F} \in \mathrm{L}^q(\mathrm{S})$, and so, arguing as for (6.1), one has $\widetilde{F}_m \to \widetilde{F}$ in $\mathrm{L}^q(\mathrm{S})$, as $m \to \infty$. Arguing again as for (6.1), it can be checked that

$$D_w^* \widetilde{F}_m \to D_w^* \widetilde{F} \quad \text{in } \mathrm{L}^q(\mathrm{S}), \text{ as } m \to \infty. \tag{6.2}$$

By assumption (iv) we have $\widetilde{F}_m \in \mathcal{R}_{\mathrm{S}}^{\mathrm{f}}(q')$ for all $m \geq n_0$ and so $\overline{D}_w^* \widetilde{F}_m = D_w^* \widetilde{F}_m$, cf. Remark 4.2. Therefore, by (6.2) and the fact that the operator $\overline{D}_w^*$ is closed, we have $\widetilde{F} \in \mathrm{Dom}(\overline{D}_w^*)$ and $\overline{D}_w^* \widetilde{F} = D_w^* \widetilde{F}$. The domain Condition (5.5) is verified since

$$\frac{1}{\overline{D}_w F} \mathbb{1}_A = \frac{1}{D_w F} \mathbb{1}_{\{N(\mathrm{S}) \geq n_0\}} = - \sum_{n \geq n_0} \mathbb{1}_{\{N(\mathrm{S})=n\}} \widetilde{f}_n(\mathbf{X}_n, \mathbf{Z}_n) = \widetilde{F}.$$

The expression of the Malliavin weight is a consequence of the definition of the divergence operator. $\square$

7 Application to a wireless ad hoc network model

In this section we apply the theoretical result of Proposition 6.1 to provide Malliavin estimators for the density of the interference in a wireless ad hoc network model.

7.1 Model description

We consider the following variant of a wireless ad hoc network model introduced in Baccelli and Błaszczyszyn [2]; see also Ganesh and Torrisi [16]. Let $(X_k)_{k \geq 1}$ be the points of a point process on the rectangular cell $\mathrm{S} := (a, b) \times (c, d)$, where $a < b$ and $c < d$. Attach to each X_k a positive random variable P_k. We interpret X_k as the location of node k, and P_k as its transmission power. Assume that a receiver is located at the origin and that a transmitter, with transmission power $P \in (0, \infty)$, is located at $y \in \mathbb{R}^2$. Let ν and τ be positive constants which denote, respectively, the noise power at the receiver, and the threshold signal to interference plus noise ratio needed for successful reception of a message. The physical signal propagation is described by a measurable positive function $L : \mathbb{R}^2 \to (0, \infty)$ which gives the attenuation or path-loss of the signal. In addition, the signal undergoes random fading (due to occluding objects, reflections, multi-path interference, etc). We denote by H_k the random fading between node k and the receiver, and define the random marks $Z_k = P_k H_k$. Similarly, we denote by H the random fading between the transmitter at y and the receiver, and define the random variable $Z = PH$. Thus, the quantities $Z_k L(X_k)$ and $ZL(y)$ are, respectively, the received power at the origin due to the transmission of node k, and the received power at the origin due to the transmitter at y. Within this framework we say that the receiver can decode the signal emitted by the transmitter at y if

$$\frac{ZL(y)}{\nu + F} \geq \tau \qquad \text{where} \qquad F := \sum_{k=1}^{N(\mathrm{S})} Z_k L(X_k).$$

Here the random variable F is the interference at the receiver due to simultaneous transmissions of nodes $1 \leq k \leq N(\mathrm{S})$. The attenuation function is often taken to be isotropic (i.e. rotation invariant) and of the form $L(x) = \|x\|^{-\alpha}$ or $(1 + \|x\|)^{-\alpha}$, where the symbol $\|\cdot\|$ denotes the Euclidean norm. Here $\alpha > 0$ is the path loss exponent which, in practice, is observed between 3 and 6. The first choice of attenuation corresponds to Hertzian propagation and is the one we shall work with.

From now on, we suppose that the random marks are bounded away from zero, i.e. $\mathrm{M} := [\delta, \infty)$, for some positive constant $\delta > 0$, and that the rectangular region S is contained in $\mathrm{T}_1 \cup \mathrm{T}_2$ where

$$\mathrm{T}_1 := \{x : x^{(2)} \geq -x^{(1)} + \eta\} \quad \text{and} \quad \mathrm{T}_2 := \{x : x^{(2)} \leq -x^{(1)} - \eta\}$$

for some positive constant $\eta > 0$.

The above condition $S \subset T_1 \cup T_2$ guarantees that the distance between the receiver at the origin and any point in the region S is bigger than or equal to $\eta/\sqrt{2}$. From the point of view of applications, this choice of placement of S corresponds to a scheduling strategy in which all transmitters within some vicinity and direction of the receiver are forced to remain silent. This can be thought of as a simplistic model of the 802.11 protocol with request-to-send/clear-to-send (RTS/CTS), with the exclusion zone corresponding to the region within which the CTS can be heard.

As in the previous sections, we assume that, given $\{N(\mathrm{S}) = n\}$, the points $(X_k)_{k=1,\dots,n}$ and the marks $(Z_k)_{k=1,\dots,n}$ are independent. We provide a Malliavin estimator for the density of the interference F, under the statistical assumption that the points are located according to suitable Janossy densities.

Note that

$$F = \sum_{k=1}^{N(\mathrm{S})} Z_k \|X_k\|^{-\alpha} = \sum_{n=1}^{\infty} \mathbb{1}_{\{N(\mathrm{S})=n\}} f_n(\mathbf{X}_n, \mathbf{Z}_n)$$

and so the form functions of this functional are given by

$$f_n(\mathbf{x}_n, \mathbf{z}_n) = \sum_{k=1}^{n} z_k \|x_k\|^{-\alpha}, \qquad (\mathbf{x}_n, \mathbf{z}_n) \in \mathrm{S}^n \times \mathrm{M}^n. \tag{7.1}$$

7.2 A suitable family of weight functions

The aim of this paragraph is to introduce a family of weight functions and to check some assumptions in Proposition 6.1 related to the choice of w. Define the function

$$v(x) := (x^{(1)} - a)(b - x^{(1)})(x^{(2)} - c)(d - x^{(2)}) \quad x \in \overline{\mathrm{S}},$$

and consider the family of weight functions defined by

$$w(x) := v(x)^{\beta} \quad x \in \overline{\mathrm{S}}, \text{ where } \beta > 0.$$

The following lemma holds:

Lemma 7.1 *If $\beta > \frac{p'-1}{p'}$, then Conditions (4.1) and (4.3) are satisfied.*

Proof. Clearly $w \in \mathcal{C}(\overline{\mathrm{S}})$ and $w \equiv 0$ on $\partial\mathrm{S}$. So we only need to check that the partial derivatives of w are in $L^{p'}(\mathrm{S})$. In the following we just check the integrability of $|\partial_{x^{(1)}} w|^{p'}$. The integrability of $|\partial_{x^{(2)}} w|^{p'}$ can be proved similarly. Note that

$$|\partial_{x^{(1)}} w(x)|^{p'} \leq 2\beta^{p'}(b-a)(d-c)^2 v(x)^{(\beta-1)p'}, \qquad x \in \mathrm{S}.$$

So we only need to check the integrability of $v^{(\beta-1)p'}$. A straightforward computation gives

$$\int_{\mathrm{S}} v(x)^{(\beta-1)p'} \,\mathrm{d}x = ((b-a)(d-c))^{(\beta-1)p'+1}\, \mathrm{B}^2((\beta-1)p'+1, (\beta-1)p'+1) < \infty$$

where the latter term involves the Euler beta function

$$\mathrm{B}(r,s) := \int_0^1 y^{r-1}(1-y)^{s-1} \,\mathrm{d}y < \infty, \qquad r, s > 0.$$

$\square$

Let $n_0 \geq 1$ be a fixed integer. For $n \geq n_0$, define the functions

$$\widetilde{f}_n(\mathbf{x}_n, \mathbf{z}_n) := \left(-\alpha \sum_{k=1}^{n} w(x_k) z_k \left(\sum_{i=1}^{2} x_k^{(i)} \right) \|x_k\|^{-(\alpha+2)} \right)^{-1}, \qquad (\mathbf{x}_n, \mathbf{z}_n) \in \mathrm{S}^n \times \mathrm{M}^n. \quad (7.2)$$

The following lemma holds:

Lemma 7.2 *If $\beta \geq 1$ and $n_0 > 2q'\beta$, then $\widetilde{f}_n(\cdot, \mathbf{z}_n) \in \mathcal{W}^{1,q'}(\mathrm{S}^n)$ for any $n \geq n_0$ and all $\mathbf{z}_n \in \mathrm{M}^n$.*

Proof. We divide the proof in two steps. In the first step we show that the claim follows if

$$\mathcal{I}(n_0) := \int_{S^{n_0}} \Big| \sum_{k=1}^{n_0} w(x_k) \Big|^{-2q'} \mathrm{d}\mathbf{x}_{n_0} < \infty, \tag{7.3}$$

and in the second step we check the above condition (7.3).

Step 1. For any $n \geq n_0$ and all $\mathbf{z}_n \in \mathrm{M}^n$, we have

$$\begin{aligned}\int_{\mathrm{S}^n} |\widetilde{f}(\mathbf{x}_n, \mathbf{z}_n)|^{q'} \,\mathrm{d}\mathbf{x}_n = \alpha^{-q'} \int_{(\mathrm{S}\cap\mathrm{T}_1)^n} \Big| \sum_{k=1}^{n} w(x_k) z_k \left(\sum_{i=1}^{2} x_k^{(i)} \right) \|x_k\|^{-(\alpha+2)} \Big|^{-q'} \mathrm{d}\mathbf{x}_n \\ + \alpha^{-q'} \int_{(\mathrm{S}\cap\mathrm{T}_2)^n} \Big| \sum_{k=1}^{n} w(x_k) z_k \left(\sum_{i=1}^{2} x_k^{(i)} \right) \|x_k\|^{-(\alpha+2)} \Big|^{-q'} \mathrm{d}\mathbf{x}_n.\end{aligned}$$

Note that, on $\mathrm{S} \cap \mathrm{T}_1$, for all $z \in \mathrm{M}$, we have

$$w(x) z \left(\sum_{i=1}^{2} x^{(i)} \right) \|x\|^{-(\alpha+2)} \geq w(x) \delta \eta \min_{y \in \mathrm{S}\cap\mathrm{T}_1} \|y\|^{-(\alpha+2)} > 0 \tag{7.4}$$

and, on $\mathrm{S} \cap \mathrm{T}_2$, for all $z \in \mathrm{M}$, we have

$$w(x) z \left(\sum_{i=1}^{2} x^{(i)} \right) \|x\|^{-(\alpha+2)} \leq w(x) - \eta \delta \min_{y \in \mathrm{S}\cap\mathrm{T}_2} \|y\|^{-(\alpha+2)} < 0. \tag{7.5}$$

Thus

$$\begin{aligned}\int_{\mathrm{S}^n} |\widetilde{f}(\mathbf{x}_n, \mathbf{z}_n)|^{q'} \,\mathrm{d}\mathbf{x}_n \leq K_1 \int_{\mathrm{S}^n} \Big| \sum_{k=1}^{n} w(x_k) \Big|^{-q'} \mathrm{d}\mathbf{x}_n \leq K_2\, \ell(\mathrm{S})^n \int_{\mathrm{S}^{n_0}} \Big| \sum_{k=1}^{n_0} w(x_k) \Big|^{-q'} \mathrm{d}\mathbf{x}_{n_0} \\ \leq \; K_3^{(n)} \sqrt{\mathcal{I}(n_0)} < \infty. \end{aligned} \tag{7.6}$$

Here K_1,K_2,$K_3^{(n)}$ are positive constants (with $K_3^{(n)}$ depending on n) and we used the Cauchy-Schwartz inequality. It remains to check that, under (7.3), the functions $|\partial_{x_k^{(i)}} \widetilde{f}_n(\cdot, \mathbf{z}_n)|^{q'}$ are integrable for any $n \geq n_0$, $k = 1, \ldots, n$, $i = 1, 2$ and all $\mathbf{z}_n \in \mathrm{M}^n$. We have

$$\partial_{x_k^{(i)}} \widetilde{f}_n(\mathbf{x}_n, \mathbf{z}_n) = h_{k,n}^{(i),1}(\mathbf{x}_n, \mathbf{z}_n) + h_{k,n}^{(i),2}(\mathbf{x}_n, \mathbf{z}_n) + h_{k,n}^{(i),3}(\mathbf{x}_n, \mathbf{z}_n) \tag{7.7}$$

where

$$h_{k,n}^{(i),1}(\mathbf{x}_n, \mathbf{z}_n) := \alpha^{-1} \frac{z_k \left(\sum_{j=1}^{2} x_k^{(j)} \right) \|x_k\|^{-(\alpha+2)} \partial_{x_k^{(i)}} w(x_k)}{\left(\sum_{k=1}^{n} w(x_k) z_k \left(\sum_{i=1}^{2} x_k^{(i)} \right) \|x_k\|^{-(\alpha+2)} \right)^2},$$

$$h_{k,n}^{(i),2}(\mathbf{x}_n, \mathbf{z}_n) := \alpha^{-1} \frac{z_k \|x_k\|^{-(\alpha+2)} w(x_k)}{\left(\sum_{k=1}^{n} w(x_k) z_k \left(\sum_{i=1}^{2} x_k^{(i)} \right) \|x_k\|^{-(\alpha+2)} \right)^2},$$

$$h^{(i),3}_{k,n}(\mathbf{x}_n,\mathbf{z}_n) := -(\alpha+2)\alpha^{-1}\frac{z_k x^{(i)}_k\left(\sum_{i=1}^2 x^{(i)}_k\right)\|x_k\|^{-(\alpha+4)}w(x_k)}{\left(\sum_{k=1}^n w(x_k)z_k\left(\sum_{i=1}^2 x^{(i)}_k\right)\|x_k\|^{-(\alpha+2)}\right)^2}.$$

Note that, for all fixed $\mathbf{z}_n \in \mathrm{M}^n$, the numerators of these ratios are bounded functions (in particular $\|\partial_{x^{(i)}_k} w\|_\infty < \infty$ since $\beta \geq 1$). So, using Minkowski's inequality, (7.4) and (7.5), one can easily realize that the claim holds if

$$\int_{\mathrm{S}^n}\Big|\sum_{k=1}^n w(x_k)\Big|^{-2q'}\mathrm{d}\mathbf{x}_n < \infty, \qquad n \geq n_0.$$

This in turn follows by (7.3).

Step 2. For simplicity of notation we set $n = n_0$. We have

$$\mathcal{I}(n) = \int_{\mathrm{S}}\mathrm{d}x_1\ldots\int_{\mathrm{S}}\mathrm{d}x_{n-1}\int_{\mathrm{S}}\Big|\sum_{k=1}^n w(x_k)\Big|^{-2q'}\mathrm{d}x_n.$$

Now let $c_1, c_2 > 0$ be two positive constants, and consider the function

$$\phi(y) := (c_1((y-a)(b-y))^\beta + c_2)^{-2q'} \quad y \in (a,b).$$

It is easily seen that ϕ is symmetric w.r.t. the line $y = (a+b)/2$. Therefore, since

$$\Big|\sum_{k=1}^n w(x_k)\Big|^{-2q'} = \big(c_1((x^{(1)}_n - a)(b - x^{(1)}_n))^\beta + c_2\big)^{-2q'}$$

where $c_1 := ((x^{(2)}_n - c)(d - x^{(2)}_n))^\beta$ and $c_2 := \sum_{k=1}^{n-1}((x^{(1)}_k - a)(b - x^{(1)}_k)(x^{(2)}_k - c)(d - x^{(2)}_k))^\beta$, we have

$$\int_a^b\Big|\sum_{k=1}^n w(x_k)\Big|^{-2q'}\mathrm{d}x^{(1)}_n = 2\int_a^{(a+b)/2}\Big|\sum_{k=1}^n w(x_k)\Big|^{-2q'}\mathrm{d}x^{(1)}_n$$

and similarly

$$\int_c^d\Big|\sum_{k=1}^n w(x_k)\Big|^{-2q'}\mathrm{d}x^{(2)}_n = 2\int_c^{(c+d)/2}\Big|\sum_{k=1}^n w(x_k)\Big|^{-2q'}\mathrm{d}x^{(2)}_n.$$

So

$$\int_{\mathrm{S}}\Big|\sum_{k=1}^n w(x_k)\Big|^{-2q'}\mathrm{d}x_k = 4\int_{(a,(a+b)/2)\times(c,(c+d)/2)}\Big|\sum_{k=1}^n w(x_k)\Big|^{-2q'}\mathrm{d}x_k$$

and thus

$$\mathcal{I}(n) = 2^{2n}\int_{(a,(a+b)/2)\times(c,(c+d)/2)}\ldots\int_{(a,(a+b)/2)\times(c,(c+d)/2)}\Big|\sum_{k=1}^n w(x_k)\Big|^{-2q'}\mathrm{d}\mathbf{x}_n.$$

Now note that $y < (a+b)/2$ is equivalent to $b - y > (b-a)/2$ (and the same holds with c and d in place of a and b). Therefore

$$\mathcal{I}(n) \leq K_4\, 2^{2n} \int_{((a,(a+b)/2)\times(c,(c+d)/2))^n} \Big| \sum_{k=1}^{n} ((x_k^{(1)} - a)(x_k^{(2)} - c))^{\beta} \Big|^{-2q'} \mathrm{d}\mathbf{x}_n$$

for some positive constant $K_4 > 0$. Now, using the change of variables:

$$t_k^{(1)} := \frac{x_k^{(1)} - a}{b - a} \qquad t_k^{(2)} := \frac{x_k^{(2)} - c}{d - c}$$

we have

$$\mathcal{I}(n) \leq K_5^{(n)}\, 2^{2n} \int_{(0,1/2)\times(0,1/2)} \cdots \int_{(0,1/2)\times(0,1/2)} \Big| \sum_{k=1}^{n} (t_k^{(1)} t_k^{(2)})^{\beta} \Big|^{-2q'} \mathrm{d}\mathbf{t}_n$$

for some positive constant $K_5^{(n)} > 0$ (depending on n). By the arithmetic-geometric inequality we have

$$\left(\sum_{k=1}^{n} (t_k^{(1)} t_k^{(2)})^{\beta} \right)^{2q'} \geq n^{2q'} \left(\prod_{k=1}^{n} (t_k^{(1)} t_k^{(2)})^{\beta} \right)^{2q'/n}.$$

So

$$\begin{aligned} \mathcal{I}(n) &\leq K_5^{(n)}\, 2^{2n}\, n^{-2q'} \int_{(0,1/2)\times(0,1/2)} \cdots \int_{(0,1/2)\times(0,1/2)} \left(\prod_{k=1}^{n} (t_k^{(1)} t_k^{(2)})^{\beta} \right)^{-2q'/n} \mathrm{d}\mathbf{t}_n \\ &= K_5^{(n)}\, 2^{2n}\, n^{-2q'} \left(\int_0^{1/2} y^{-2\beta q'/n}\, \mathrm{d}y \right)^{2n}. \end{aligned}$$

This latter integral is finite since $n = n_0 > 2\beta q'$. □

In the rest of the paper, we do the following choice of the parameters:

$$q \in (1,2], \quad \beta \geq 1, \quad n_0 \geq 3 \quad \text{and} \quad n_0 > 2\beta q$$

$$q < q' < n_0/(2\beta) \quad \text{and} \quad q' \leq \max(2q, q/(2-q))$$

$$p' = (1/q - 1/q')^{-1} \quad \text{and} \quad p = \frac{q}{q-1}.$$

A straightforward computation shows that $p \geq q$, p and q are conjugate exponents, $p' \geq q'$, $p' \geq p$, and $q/q' + q/p' = 1$. In the next paragraph, we provide Malliavin estimators for the density of the interference, by applying Proposition 6.1.

7.3 Dominated networks

In the following we suppose that the nodes are located on the rectangular cell S according to a finite point process $\mathbf{X}$ whose law is absolutely continuous w.r.t. the law of a homogeneous Poisson process with intensity $\lambda > 0$. Recall that in such a case the distribution of $N(\mathrm{S})$ and the Janossy densities are given by (2.1) with S in place of B.

Let n_0, β, p, q, p' and q' be as at the end of the Paragraph 7.2, and F and $\widetilde{F}$ specified by the form functions (7.1) and (7.2), respectively. Moreover, set $A := \{N(\mathrm{S}) \geq n_0\}$.

The following proposition provides sufficient conditions for the application of Proposition 6.1 and for the use of the modified Malliavin estimator described in Subsection 5.2.

Proposition 7.3 *Assume* (4.2) *and in addition that*

(i) there exists a sequence $(\gamma_n)_{n\geq 1}$ *such that*

$$\left\| \frac{\partial_{x_k^{(i)}} \Phi_n}{\Phi_n} \right\|_{L^{p'}(\mathrm{S}^n, j_n)} \leq \gamma_n, \qquad n \geq 1, \quad k = 1, \dots, n, \quad i = 1, 2, \tag{7.8}$$

and

$$\sum_{n=n_0}^{\infty} n^p (1 + \gamma_n)^q \left(\frac{\ell(\mathrm{S})^n}{n!} \|\Phi_n\|_\infty \right)^{q/q'} < \infty, \tag{7.9}$$

(ii) there exists a constant $\gamma > 0$ *such that*

$$\mathrm{E}[Z_k^{p'} | N(\mathrm{S}) = n] \leq \gamma, \qquad n \geq 1, \quad k = 1, \dots, n. \tag{7.10}$$

Then, the conditional law of F given A is absolutely continuous w.r.t. the Lebesgue measure with probability density given by (5.6) *with*

$$W = \widetilde{F} \sum_{k=1}^{N(\mathrm{S})} (\operatorname{div} w(X_k) + w(X_k) R_{k,N(S)}(\mathbf{X}_{N(S)})) - D_w \widetilde{F}$$

where $R_{k,n}(\mathbf{x}_n) := \operatorname{div}_{x_k} \Phi_n(\mathbf{x}_n)/\Phi_n(\mathbf{x}_n)$. *Moreover, the density* $\varphi_{F\,|\,A}$ *is bounded and Hölder continuous with exponent* $1/p$.

The proof of Proposition 7.3 is based on the following technical lemma.

Lemma 7.4 *Under the assumptions of Proposition* 7.3*, Condition* (3.4) *holds with* $d = 2$*, and we have:*

$$\mathrm{E}\left[\left(\sum_{k=1}^{n} Z_k \right)^{p'} \Big| N(\mathrm{S}) = n \right] < \infty, \qquad n \geq 1, \tag{7.11}$$

$$\mathrm{E}\left[\left(\sum_{k=1}^{N(\mathrm{S})} Z_k\right)^p\right] < \infty, \tag{7.12}$$

$$\sum_{n=n_0}^{\infty} \frac{\ell(\mathrm{S})^n}{n!}\|\Phi_n\|_\infty < \infty, \tag{7.13}$$

$$\sum_{n=n_0}^{\infty} \frac{\ell(\mathrm{S})^n}{n!}\|\Phi_n\|_\infty \,\mathrm{E}\left[\left(\sum_{k=1}^{n} Z_k\right)^q \Big| N(\mathrm{S}) = n\right] < \infty, \tag{7.14}$$

and

$$\sum_{n=n_0}^{\infty}\left(\frac{\ell(\mathrm{S})^n}{n!}\|\Phi_n\|_\infty\right)^{q/q'}\left(\sum_{k=1}^{n}\sum_{i=1}^{2}\left(\|\partial_{x^{(i)}} w\|_\infty + \|w\|_\infty \left\|\frac{\partial_{x_k^{(i)}}\Phi_n}{\Phi_n}\right\|_{L^{p'}(\mathrm{S}^n, j_n)}\right)\right)^q < \infty. \tag{7.15}$$

Proof of Proposition 7.3. Let us check the assumptions of Proposition 6.1. By Lemma 7.4 we have (3.4) with $d = 2$. So by Lemma 7.1 it follows that the assumptions of Proposition 4.1 are satisfied. In the next steps we verify the remaining hypotheses.

Step 1. Condition (i). Note that, for some positive constant $K_1 > 0$,

$$f_n(\mathbf{x}_n, \mathbf{z}_n) \le K_1 \sum_{k=1}^{n} z_k, \qquad n \ge 1, \quad \mathbf{x}_n \in \mathrm{S}^n,$$

and $\mu_{\mathbf{Z}_n|N(\mathrm{S})=n}$-almost all $\mathbf{z}_n$. So $f_n(\cdot, \mathbf{z}_n)$ belongs to $L^{p'}(\mathrm{S}^n)$ $\forall\, n \ge 1$ and $\mu_{\mathbf{Z}_n|N(\mathrm{S})=n}$-almost all $\mathbf{z}_n$. Since

$$\partial_{x_k^{(i)}} f_n(\mathbf{x}_n, \mathbf{z}_n) = -\alpha z_k x_k^{(i)} \|x_k\|^{-(\alpha+2)} \tag{7.16}$$

we have, for a suitable positive constant $K_2 > 0$,

$$|\partial_{x_k^{(i)}} f_n(\mathbf{x}_n, \mathbf{z}_n)| \le K_2 z_k, \qquad n \ge 1, \quad \mathbf{x}_n \in \mathrm{S}^n,$$

and $\mu_{\mathbf{Z}_n|N(\mathrm{S})=n}$-almost all $\mathbf{z}_n$. So $f_n(\cdot, \mathbf{z}_n)$ belongs to $\mathcal{W}^{1,p'}(\mathrm{S}^n)$, $\forall\, n \ge 1$ and $\mu_{\mathbf{Z}_n|N(\mathrm{S})=n}$-almost all $\mathbf{z}_n$. Note that, for some positive constant $K_3 > 0$, by (7.11) in Lemma 7.4, we have

$$\mathrm{E}[\mathbb{1}_{\{N(\mathrm{S})=n\}} f_n^{p'}(\mathbf{X}_n, \mathbf{Z}_n)] \le K_3 \mathrm{E}\left[\left(\sum_{k=1}^{n} Z_k\right)^{p'} \Big| N(\mathrm{S}) = n\right] < \infty, \qquad n \ge 1.$$

Similarly, using (7.16), one can check that

$$\mathbb{1}_{\{N(\mathrm{S})=n\}} \partial_{x_k^{(i)}} f_n(\mathbf{X}_n, \mathbf{Z}_n) \in \mathrm{L}^{p'}(\mathrm{S}), \qquad n \ge 1, \quad k = 1, \dots, n, \quad i = 1, 2.$$

So $F \in \mathcal{R}_\mathrm{S}(p')$. Finally, note that $F \in \mathrm{L}^p(\mathrm{S})$ follows by (7.12) in Lemma 7.4.

Step 2. Condition (ii). By (7.16) we have

$$D_w F = \alpha \mathbb{1}_{\{N(\mathrm{S}) \ge 1\}} \sum_{k=1}^{N(\mathrm{S})} w(X_k) Z_k \left(\sum_{i=1}^{2} X_k^{(i)}\right) \|X_k\|^{-(\alpha+2)}.$$

So, by (7.12) in Lemma 7.4, we deduce

$$\mathrm{E}[|D_w F|^p] \le K_4 \mathrm{E}\left[\left(\sum_{k=1}^{N(\mathrm{S})} Z_k\right)^p\right] < \infty,$$

where $K_4 > 0$ is a constant.

Step 3. Condition (iii). The claim follows noticing that the random variables

$$w(X_k) Z_k \left(\sum_{i=1}^{2} X_k^{(i)}\right) \|X_k\|^{-(\alpha+2)}, \qquad 1 \le k \le N(\mathrm{S}),$$

are different from zero a.s. (see the inequalities (7.4) and (7.5)), and by (2.1)

$$P(A) \ge P(N(\mathrm{S}) = n_0) = \frac{c_{n_0}}{n_0!} \mathrm{e}^{-\lambda \ell(\mathrm{S})} > 0.$$

Step 4. Condition (iv). By Lemma 7.2 we have that $\widetilde{f}_n(\cdot, \mathbf{z}_n)$ belongs to $\mathcal{W}^{1,q'}(\mathrm{S}^n)$ $\forall\ n \ge n_0$ and all $\mathbf{z}_n \in \mathrm{M}^n$. For a suitable positive constant $K_5 > 0$ and $\forall\ n \ge n_0$, by (2.1) and (7.6) we deduce

$$\mathrm{E}[\mathbb{1}_{\{N(\mathrm{S})=n\}} |\widetilde{f}_n(\mathbf{X}_n, \mathbf{Z}_n)|^{q'}] \le K_5 \left(\int_{\mathrm{S}^{n_0}} \left|\sum_{k=1}^{n_0} w(x_k)\right|^{-q'} \mathrm{d}\mathbf{x}_{n_0}\right) \frac{\ell(\mathrm{S})^n}{n!} \|\Phi_n\|_\infty < \infty, \tag{7.17}$$

where the finiteness of this latter term follows by (4.2) and (7.3). Similarly, using (7.7) and Minkowski's inequality, by (7.11) in Lemma 7.4 and $p' \ge q'$, we have

$$\mathrm{E}[\mathbb{1}_{\{N(\mathrm{S})=n\}} |\partial_{x_k^{(i)}} \widetilde{f}_n(\mathbf{X}_n, \mathbf{Z}_n)|^{q'}] < \infty, \qquad n \ge n_0, \quad k = 1, \ldots, n, \quad i = 1, 2.$$

So $\widetilde{F} \in \mathcal{R}_{\mathrm{S}}(q')$. Finally, for a some positive constant $K_6 > 0$, arguing as for (7.17), we have

$$\mathrm{E}[|\widetilde{F}|^q] \le K_6 \left(\int_{\mathrm{S}^{n_0}} \left|\sum_{k=1}^{n_0} w(x_k)\right|^{-q} \mathrm{d}\mathbf{x}_{n_0}\right) \sum_{n=n_0}^{\infty} \frac{\ell(\mathrm{S})^n}{n!} \|\Phi_n\|_\infty < \infty$$

where the finiteness of this latter term follows by $q < q'$, (7.3) and (7.13) in Lemma 7.4.

Step 5. Condition (v). We need to show

$$\left\|\widetilde{F} \sum_{k=1}^{N(\mathrm{S})} (\operatorname{div} w(X_k) + w(X_k) R_{k,N(\mathrm{S})}(\mathbf{X}_{N(\mathrm{S})}))\right\|_q < \infty \quad \text{and} \quad \|D_w \widetilde{F}\|_q < \infty. \tag{7.18}$$

For the first inequality note that

$$\widetilde{F} \sum_{k=1}^{N(\mathrm{S})} (\operatorname{div} w(X_k) + w(X_k) R_{k,N(\mathrm{S})}(\mathbf{X}_{N(\mathrm{S})}))$$

$$= \sum_{n=n_0}^{\infty} \mathbb{1}_{\{N(\mathrm{S})=n\}} \widetilde{f}_n(\mathbf{X}_n, \mathbf{Z}_n) \sum_{k=1}^{n} \sum_{i=1}^{2} \left(\partial_{x^{(i)}} w(X_k) + w(X_k) \frac{\partial_{x_k^{(i)}} \Phi_n(\mathbf{X}_n)}{\Phi_n(\mathbf{X}_n)} \right).$$

Therefore, using Hölder's inequality with conjugate exponents q'/q and p'/q, (7.17) and Minkowski's inequality, for a suitable positive constant $K_7 > 0$, we obtain

$$\begin{aligned}
&\left\| \widetilde{F} \sum_{k=1}^{N(\mathrm{S})} (\operatorname{div} w(X_k) + w(X_k) R_{k,N(\mathrm{S})}(\mathbf{X}_{N(\mathrm{S})})) \right\|_q^q \\
&\leq K_7 \sum_{n=n_0}^{\infty} \left(\frac{\ell(\mathrm{S})^n}{n!} \|\Phi_n\|_\infty \right)^{q/q'} \\
&\qquad \times \left(\sum_{k=1}^{n} \sum_{i=1}^{2} \left(\|\partial_{x^{(i)}} w\|_\infty + \|w\|_\infty \left\| \frac{\partial_{x_k^{(i)}} \Phi_n}{\Phi_n} \right\|_{L^{p'}(\mathrm{S}^n, j_n)} \right) \right)^q
\end{aligned}$$

and this latter infinite sum is finite due to (7.15) in Lemma 7.4. For the second inequality in (7.18), note that

$$\|D_w \widetilde{F}\|_q^q \leq \|w\|_\infty^q \sum_{n=n_0}^{\infty} \mathrm{E} \left[\mathbb{1}_{\{N(\mathrm{S})=n\}} \left(\sum_{k=1}^{n} \sum_{i=1}^{2} |\partial_{x_k^{(i)}} \widetilde{f}_n(\mathbf{X}_n, \mathbf{Z}_n)| \right)^q \right].$$

Now, by (7.7) and inequalities (7.4), (7.5), for positive constants $K_8, K_9 > 0$, we have

$$\begin{aligned}
&\mathrm{E} \left[\mathbb{1}_{\{N(\mathrm{S})=n\}} \left(\sum_{k=1}^{n} \sum_{i=1}^{2} |\partial_{x_k^{(i)}} \widetilde{f}_n(\mathbf{X}_n, \mathbf{Z}_n)| \right)^q \right] \\
&\leq K_8 \, P(N(\mathrm{S}) = n) \int_{\mathrm{S}^n} \left| \sum_{k=1}^{n_0} w(x_k) \right|^{-2q} j_n(\mathbf{x}_n) \, \mathrm{d}\mathbf{x}_n \int_{\mathrm{M}^n} \left(\sum_{k=1}^{n} z_k \right)^q \mu_{\mathbf{Z}_n | N(\mathrm{S})=n}(\mathrm{d}\mathbf{z}_n) \\
&\leq K_9 \left(\int_{\mathrm{S}^{n_0}} \left| \sum_{k=1}^{n_0} w(x_k) \right|^{-2q} \mathrm{d}\mathbf{x}_{n_0} \right) \frac{\ell(\mathrm{S})^n}{n!} \|\Phi_n\|_\infty \, \mathrm{E} \left[\left(\sum_{k=1}^{n} Z_k \right)^q \middle| N(\mathrm{S}) = n \right].
\end{aligned}$$

The claim follows by (7.14) in Lemma 7.4, $q < q'$ and (7.3). $\square$

Remark 7.5 *A close look at the proof of Proposition* 7.3 *shows that the claim therein still holds if, more generally, we assume that* (3.4) *with* $d = 2$ *and in addition that* (4.2), (7.11), (7.12), (7.13), (7.14) *and* (7.15) *hold. However, these conditions are somewhat technical and less practical than those of Proposition* 7.3.

Proof of Lemma 7.4. Condition (3.4) with $d = 2$ follows by (7.8). As a consequence of Hölder's inequality (see e.g. Chow and Teicher [6] p.107) we have

$$\sum_{k=1}^{n} Z_k \leq n^{1-1/p'} \left(\sum_{k=1}^{n} Z_k^{p'} \right)^{1/p'}$$

and so, using (7.10),

$$\mathrm{E}\left[\left(\sum_{k=1}^{n} Z_k\right)^{p'} \Big| N(\mathrm{S})=n\right] \leq \gamma n^{p'}, \qquad n\geq 1, \tag{7.19}$$

which implies (7.11). By (7.9) we deduce

$$\sum_{n=n_0}^{\infty} n^p \left(\frac{\ell(\mathrm{S})^n}{n!}\|\Phi_n\|_\infty\right)^{q/q'} < \infty,$$

hence

$$\sum_{n=n_0}^{\infty} n^p \,\frac{\ell(\mathrm{S})^n}{n!}\|\Phi_n\|_\infty < \infty. \tag{7.20}$$

since $q<q'$. Note that $c_n \leq \ell(\mathrm{S})^n\,\|\Phi_n\|_\infty$, therefore by (2.1) and (7.20) we deduce $\mathrm{E}[N(\mathrm{S})^p] < \infty$. Relation (7.12) is consequence of the following inequality

$$\mathrm{E}\left[\left(\sum_{k=1}^{N(\mathrm{S})} Z_k\right)^{p}\right] \leq \gamma^{p/p'}\,\mathrm{E}[N(\mathrm{S})^p] < \infty,$$

which can be obtained using Hölder's inequality and (7.19). The inequality (7.13) is an easy consequence of (7.20). By Hölder's inequality with conjugate exponents p'/q, $p'/(p'-q)$ and (7.19), we have, $\forall\ n\geq 1$,

$$\mathrm{E}\left[\left(\sum_{k=1}^{n} Z_k\right)^{q} \Big| N(\mathrm{S})=n\right] \leq \left(\mathrm{E}\left[\left(\sum_{k=1}^{n} Z_k\right)^{p'} \Big| N(\mathrm{S})=n\right]\right)^{q/q'} \leq \gamma^{q/p'}\,n^q.$$

Thus relation (7.14) follows by (7.20) and $p\geq q$. It remains to check (7.15). For a suitable positive constant K_1, by assumption (7.8) we deduce:

$$\begin{aligned}&\sum_{n=n_0}^{\infty}\left(\frac{\ell(\mathrm{S})^n}{n!}\|\Phi_n\|_\infty\right)^{q/q'}\left(\sum_{k=1}^{n}\sum_{i=1}^{2}\left(\|\partial_{x^{(i)}}w\|_\infty + \|w\|_\infty\left\|\frac{\partial_{x_k^{(i)}}\Phi_n}{\Phi_n}\right\|_{L^{p'}(\mathrm{S}^n,j_n)}\right)\right)^q\\ &\leq K_1\sum_{n=n_0}^{\infty} n^q\,(1+\gamma_n)^q\left(\frac{\ell(\mathrm{S})^n}{n!}\|\Phi_n\|_\infty\right)^{q/q'},\end{aligned}$$

and this latter term is finite due to Condition (7.9). □

7.4 Pairwise interaction networks

A finite point process $\mathbf{X}$ on the rectangular cell S is said *pairwise interaction* if it has Janossy densities as in (2.1) with

$$\Phi_n(\mathbf{x}_n) := \prod_{k=1}^{n}\phi_1(x_k)\prod_{\{x_h,x_j\}}^{1,n}\phi_2(\{x_h,x_j\}).$$

Here the symbol $\prod_{\{x_h,x_j\}}^{1,n}$ means that the product is taken over all the subsets $\{x_h, x_j\}$ of cardinality 2 of the configuration $\{x_1, \ldots, x_n\}$; ϕ_1 and ϕ_2 are two non-negative measurable functions called *intensity* and *interaction* function, respectively.

In the following examples we shall consider three different models of pairwise interaction point processes and, for each of them, we shall compare numerically the finite difference estimator with the modified Malliavin estimator. The finite difference estimator is a classical kernel estimator of the form (5.1) with $K(x) := \frac{1}{2}\mathbb{1}_{[-1,1]}(x)$, i.e.

$$\widehat{c}_n(x) := \frac{1}{2nh}\sum_{k=1}^{n} \mathbb{1}_{[x-h,x+h]}(F^{(i)}),$$

where h is a small positive bandwidth and $F^{(i)}$, $i = 1, \ldots, n$, are n independent replications of F, under $P(\cdot \mid A)$. On the other hand, the modified Malliavin estimator is

$$\widehat{m}_n^{\mathrm{mod}}(x) := \frac{1}{n}\sum_{i=1}^{n}\left(\mathbb{1}_{\{F\geq x\}}\mathrm{e}^{-\theta\left(\frac{F-x}{\zeta}\right)}\left(W - \frac{\theta}{\zeta}\right)\right)^{(i)},$$

where $\left(\mathbb{1}_{\{F\geq x\}}\mathrm{e}^{-\theta\left(\frac{F-x}{\zeta}\right)}\left(W - \frac{\theta}{\zeta}\right)\right)^{(i)}$, $i = 1, \ldots, n$, are n independent samples of $\mathbb{1}_{\{F\geq x\}}\mathrm{e}^{-\theta\left(\frac{F-x}{\zeta}\right)}\left(W - \frac{\theta}{\zeta}\right)$, under $P(\cdot \mid A)$.

In the next numerical illustrations we compare the performances of kernel and Malliavin estimators for the density of the interference

$$F = \sum_{k=1}^{N(\mathrm{S})} Z_k \|X_k\|^{-\alpha}$$

in a wireless network, using the common parameters $\alpha = 3$, $\theta = 1$, $\zeta = 10$, $a = 1$, $b = 2$, $c = 1$, $d = 2$, $\delta = 1$, $\lambda = 1$, $\eta = 0.5$, $\beta = 1$, and $n_0 = 5$. Here, we simulate from the conditional distribution of $\mathbf{X}$ given $A = \{N(\mathrm{S}) \geq n_0\}$, using a simple rejection sampling: we repeatedly simulate $\mathbf{X}$ on S, until there are at least n_0 points.

Example 1: Homogeneous Poisson networks

Taking $\phi_2 \equiv 1$ yields a Poisson process. If in addition $\phi_1 \equiv \lambda > 0$, then $\mathbf{X}$ is a homogeneous Poisson process on S with intensity λ. In this case, the Janossy densities are equal to $\ell(\mathrm{S})^{-n}$ and $\Phi_n(\mathbf{x}) = \lambda^n\ \forall\ n \geq 1$ and $\mathbf{x} \in \mathrm{S}^n$. So (4.2) is satisfied, and (7.8) holds with $\gamma_n \equiv 0$.

Condition (7.9) is readily checked since the infinite sum therein reads

$$\sum_{n=n_0}^{\infty} n^p \left(\frac{(\lambda \ell(\mathrm{S}))^n}{n!} \right)^{q/q'}$$

which is a convergent series. So, assuming Condition (7.10) (which is satisfied if, for instance, either the marks are bounded above or they are independent, identically distributed, independent of $\mathbf{X}$ and have finite moment of order p'), by Proposition 7.3 we deduce that the conditional law of F given A has a bounded and continuous probability density equal to (5.6) with the Malliavin weight

$$W = \widetilde{F} \sum_{k=1}^{N(\mathrm{S})} \operatorname{div} w(X_k) - D_w \widetilde{F}.$$

In Figure 1 we display the finite difference and the modified Malliavin estimators of the density. The number of replications is set to $N = 2 \times 10^5$, the discretization step is $h = 0.001$, and the marks are distributed as $1 + \mathrm{EXP}(1)$, where $\mathrm{EXP}(1)$ is an exponentially distributed random variable with mean 1. It shows that the modified Malliavin estimator yields a more precise estimation with the same number of random samples.

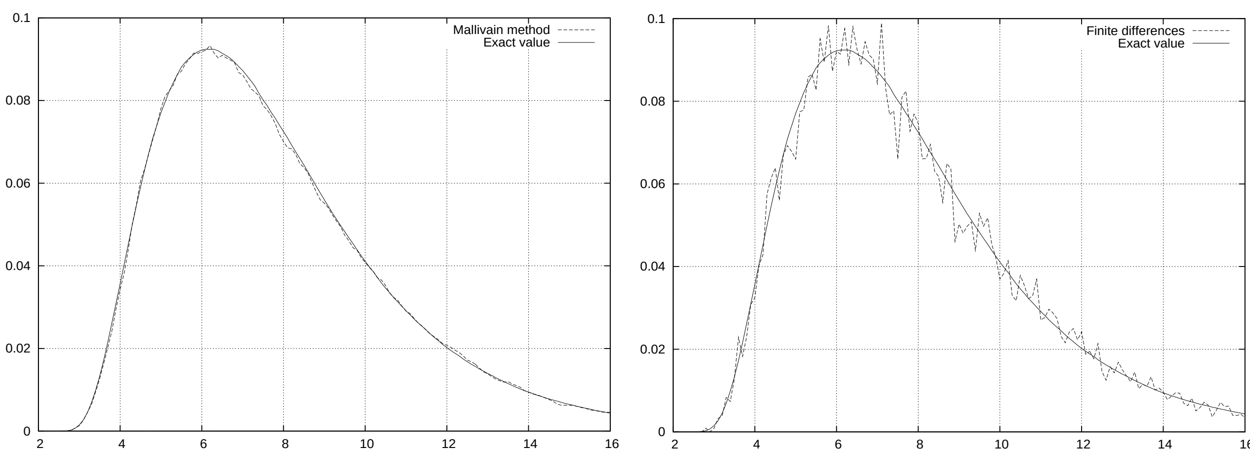

Figure 1: Malliavin method vs finite differences.

Figure 2 provides a comparison of the sample $L^2([0,16])$-error between the numerical estimation and the exact value for both methods, as a function of the discretization step in base 10 logarithmic scale with 2×10^4 (left) and 2×10^6 (right) samples.

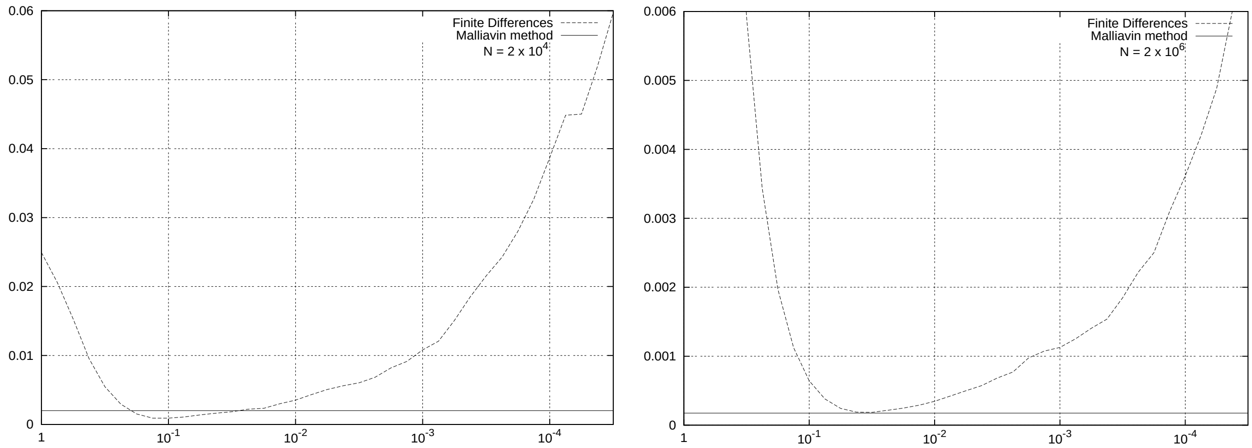

Figure 2: Error comparison between the Malliavin and finite difference methods.

Figure 2 also shows that with 2×10^4 samples, the comparative performance of the Malliavin and finite difference methods depends on the size of the discretization step, while the Malliavin method always perform better with 2×10^6 samples and above.

We conclude this section with two examples of networks whose nodes are distributed according to a locally stable and repulsive pairwise interaction point process.

Example 2: Networks with very soft core nodes

For the reasons explained in the introduction, *repulsive* pairwise interaction point processes are of particular interest in the context of wireless networks. In mathematical terms the inter-point repulsion is described by the inequality $\phi_2(\{x_1, x_2\}) \leq 1$ for any $x_1, x_2 \in \mathrm{S}$. Indeed, intuitively, this condition means that the conditional probability of $\{y \in \mathbf{X}\}$ given $\{\mathbf{X} \setminus \{y\} = \{x_1, \ldots, x_n\}\}$ (where $y, x_1, \ldots, x_n \in \mathrm{S}$) is a decreasing function of the configuration $\{x_1, \ldots, x_n\}$ (see Møller and Waagepetersen [21] pp. 83-85). In order to perfectly simulate the point process over the finite window S one should require the local stability, i.e. $\phi_1 \in L^1(\mathrm{S})$. Indeed, in such a case, one may use the coupling from the past or the clan of ancestors perfect simulation algorithms to generate $\mathbf{X}$. Alternatively, in the simulations of this example and the next one we shall use the so called birth-and-death Metropolis Hastings algorithm, which is faster than the perfect simulation routine, to generate the considered point process. The interested reader can find a detailed description of these algorithms in van Lieshout [29] pp. 93-95 and Møller and Waagepetersen [21] pp. 112-113 and pp. 227-233.

The following model of locally stable and repulsive pairwise interaction point process is a simple modification of the model introduced by Ogata and Tanemura [22] (see also Møller and Waagepetersen [21] p. 88). The modification consists in allowing the interaction function to be bounded away from zero. Consider a constant intensity function $\phi_1 \equiv \lambda > 0$ and an interaction function of the form

$$\phi_2(\{x_h, x_k\}) = 1 - \exp\left(-\frac{\|x_h - x_k\|^2 + \varepsilon}{\rho}\right) \quad \text{where } \varepsilon, \rho > 0 \text{ are positive parameters.}$$

We start checking (4.2). By the definition of ϕ_1 we have

$$\Phi_n(\mathbf{x}_n) = \lambda^n \prod_{s=1}^{n-1} \prod_{r=s}^{n-1} \phi_2(\{x_s, x_{r+1}\}) \tag{7.21}$$

and therefore $\Phi_n \in \mathcal{C}(\overline{\mathrm{S}}^n)$. Note that for a fixed $k \in \{1, \dots, n\}$ we have

$$\Phi_n(\mathbf{x}_n) = \Psi_n(x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_n) \prod_{\substack{h=1 \\ h \neq k}}^{n} \phi_2(\{x_h, x_k\}) \tag{7.22}$$

where

$$\Psi_n(x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_n) := \lambda^n \frac{\prod_{s=1}^{n-1} \prod_{r=s}^{n-1} \phi_2(\{x_s, x_{r+1}\})}{\prod_{\substack{h=1 \\ h \neq k}}^{n} \phi_2(\{x_h, x_k\})}.$$

A straightforward computation gives:

$$\begin{aligned}
\partial_{x_k^{(i)}} \Phi_n(\mathbf{x}_n) &= \Psi_n(x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_n) \sum_{\substack{j=1 \\ j \neq k}}^{n} \partial_{x_k^{(i)}} \phi_2(\{x_j, x_k\}) \prod_{\substack{h=1 \\ h \neq j,k}}^{n} \phi_2(\{x_h, x_k\}) \\
&= \frac{2}{\rho} \Psi_n(x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_n) \sum_{\substack{j=1 \\ j \neq k}}^{n} (x_k^{(i)} - x_j^{(i)}) \exp\left(-\frac{\|x_k - x_j\|^2 + \varepsilon}{\rho}\right) \\
&\qquad \times \prod_{\substack{h=1 \\ h \neq j,k}}^{n} \phi_2(\{x_h, x_k\}).
\end{aligned} \tag{7.23}$$

This partial derivative is continuous on $\overline{\mathrm{S}}^n$. So $\Phi_n \in \mathcal{W}^{1,p}(\mathrm{S}^n)$, and (4.2) is checked. By the above computations it follows

$$\begin{aligned}
\frac{\partial_{x_k^{(i)}} \Phi_n(\mathbf{x}_n)}{\Phi_n(\mathbf{x}_n)} &= \sum_{\substack{j=1 \\ j \neq k}}^{n} \frac{\partial_{x_k^{(i)}} \phi_2(\{x_j, x_k\})}{\phi_2(\{x_j, x_k\})} \\
&= \frac{2}{\rho} \sum_{\substack{j=1 \\ j \neq k}}^{n} (x_k^{(i)} - x_j^{(i)}) \frac{\exp\left(-(\|x_k - x_j\|^2 + \varepsilon)/\rho\right)}{\phi_2(\{x_j, x_k\})}
\end{aligned} \tag{7.24}$$

and so, for a suitable positive constant K_1,

$$\left|\frac{\partial_{x_k^{(i)}}\Phi_n(\mathbf{x}_n)}{\Phi_n(\mathbf{x}_n)}\right| \le K_1\, n, \qquad n \ge 1, \quad k = 1, \ldots, n, \quad i = 1, 2, \quad \mathbf{x}_n \in \mathrm{S}^n.$$

As a consequence, (7.8) holds with $\gamma_n = K_1\, n$. Since

$$\Phi_n(\mathbf{x}) \le \lambda^n, \qquad \mathbf{x} \in \mathrm{S}^n, \tag{7.25}$$

Condition (7.9) holds if

$$\sum_{n=n_0}^{\infty} n^p (n+1)^q \left(\frac{(\lambda \ell(\mathrm{S}))^n}{n!}\right)^{q/q'} < \infty.$$

This claim is true, as can be easily realized applying e.g. the ratio criterion. So, assuming (7.10), by Proposition 7.3 we have that the conditional law of F given A is absolutely continuous w.r.t. the Lebesgue measure with a bounded and continuous probability density given by (5.6), with the Malliavin weight described in the statement of the proposition, properly modified.

In the next graph we display the finite difference and the modified Malliavin estimators of the density with $\varepsilon = 0.4$, $\rho = 0.5$, constant marks all equal to 1, $h = 0.001$, and $N = 10^5$ random samples.

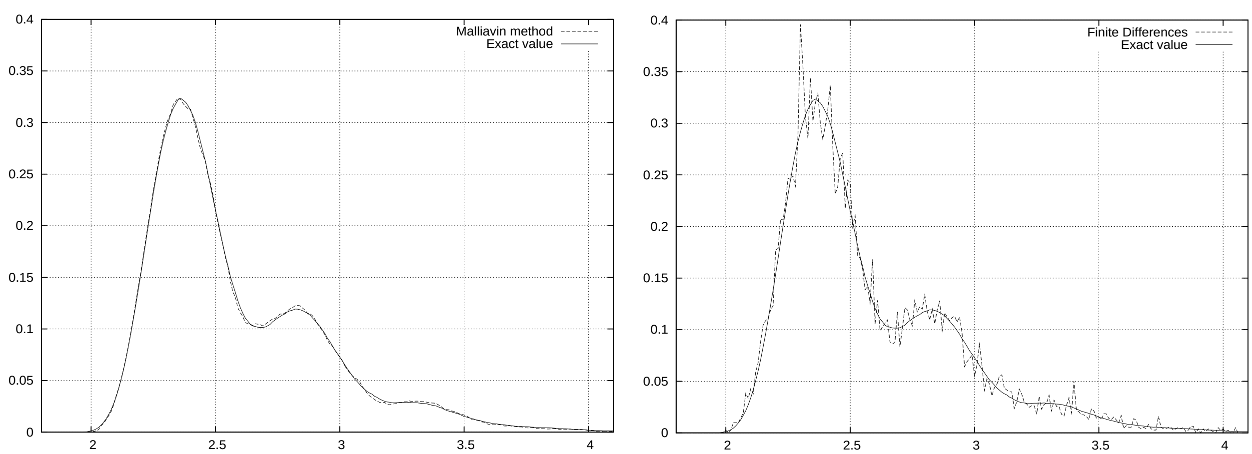

Figure 3: Malliavin method vs finite differences.

Again, Figure 3 shows that the modified Malliavin estimator performs better than the finite difference estimator, with the same number of random samples.

Example 3: Networks with hard core nodes

The following model of locally stable and repulsive pairwise interaction point process is a simple modification of the well-known hard core model (see, for instance, van Lieshout [29] p. 51). Here the point process is modified in such a way that the interaction function is continuous and bounded away from zero. Consider a constant intensity function $\phi_1 \equiv \lambda > 0$ and an interaction function of the form

$$\phi_2(\{x_h, x_k\}) = \varepsilon \mathbb{1}_{\{\|x_h - x_k\| \leq R\}} + \left(\frac{1-\varepsilon}{\varepsilon}(\|x_h - x_k\| - R) + \varepsilon\right) \mathbb{1}_{\{\|x_h - x_k\| \in (R, R+\varepsilon]\}} \\ + \mathbb{1}_{\{\|x_h - x_k\| > R+\varepsilon\}}$$

where $\varepsilon \in (0,1)$ and $R \in (0,\infty)$ are such that $R + \varepsilon < \sqrt{(b-a)^2 + (d-c)^2}$. Here again, we start checking (4.2). Writing Φ_n as in (7.21), due to the continuity of the mapping $(x_h, x_k) \mapsto \phi_2(\{x_h, x_k\})$ on $\overline{\mathrm{S}}^2$, we deduce $\Phi_n \in \mathcal{C}(\overline{\mathrm{S}}^n)$. Note that the functions $x_k \mapsto \phi_2(\{x_h, x_k\})$ belong to $L^\infty(\mathrm{S})$. We shall check later that these functions belong even to $\mathcal{W}^{1,p}(\mathrm{S})$. So by Theorem 4 (i) p. 129 in Evans and Gariepy [12] and the expression of Φ_n in (7.22), we have that, for fixed $k \in \{1, \ldots, n\}$ and $i \in \{1, 2\}$, the functions

$$x_k \mapsto \Phi_n(x_1, \ldots, x_{k-1}, x_k, x_{k+1}, \ldots, x_n)$$

are in $\mathcal{W}^{1,p}(\mathrm{S})$ and $\partial_{x_k^{(i)}} \Phi_n$ is equal to the term in the right-hand side of (7.23). Now we compute the weak partial derivative $\partial_{x_k^{(i)}} \phi_2(\{x_h, x_k\})$. Recall that, for a constant K_1, $\partial_y \mathbb{1}_{\{y \geq K_1\}} = \partial_y \mathbb{1}_{\{y > K_1\}} = \delta_{K_1}(y)$, $y \in \mathbb{R}$, where δ_{K_1} is the Dirac delta function at K_1. We deduce:

$$\begin{aligned} &\partial_{x_k^{(1)}} \mathbb{1}_{\{\|x_h - x_k\| \leq R\}} \\ &= \mathbb{1}_{\{|x_h^{(2)} - x_k^{(2)}| \leq R\}} \left(\delta_{x_h^{(1)} - \sqrt{R^2 - (x_h^{(2)} - x_k^{(2)})^2}}(x_k^{(1)}) - \delta_{x_h^{(1)} + \sqrt{R^2 - (x_h^{(2)} - x_k^{(2)})^2}}(x_k^{(1)})\right) \\ &=: H(R, x_h, x_k), \end{aligned}$$

and

$$\partial_{x_k^{(1)}} \mathbb{1}_{\{\|x_h - x_k\| \in (R, R+\varepsilon]\}} = H(R+\varepsilon, x_h, x_k) - H(R, x_h, x_k).$$

So, using again Theorem 4 (i) p. 129 in Evans and Gariepy [12], and the relation $\delta_a(x) f(x) = \delta_a(x) f(a)$, for $a \in \mathbb{R}$ and any measurable function f, we obtain

$$\partial_{x_k^{(1)}} \phi_2(\{x_h, x_k\}) = \frac{(1-\varepsilon)(x_k^{(1)} - x_h^{(1)})}{\varepsilon \|x_h - x_k\|} \mathbb{1}_{\{\|x_h - x_k\| \in (R, R+\varepsilon]\}}.$$

The weak partial derivative $\partial_{x_k^{(2)}}\phi_2(\{x_h, x_k\})$ can be computed similarly, and it is given by

$$\partial_{x_k^{(2)}}\phi_2(\{x_h, x_k\}) = (1-\varepsilon)\frac{x_k^{(2)} - x_h^{(2)}}{\varepsilon\|x_h - x_k\|}\mathbb{1}_{\{\|x_h - x_k\| \in (R, R+\varepsilon]\}}.$$

So

$$|\partial_{x_k^{(i)}}\phi_2(\{x_h, x_k\})| \leq K_2 \quad \text{for any } i = 1, 2 \text{ and } x_h \in \mathrm{S}, \tag{7.26}$$

where $K_2 > 0$ is a constant. Therefore the functions $x_k \mapsto \phi_2(\{x_h, x_k\})$ belong to $\mathcal{W}^{1,p}(\mathrm{S})$. Since $\varepsilon \leq \phi_2 \leq 1$ on S^2, by (7.23) and the upper bound (7.26) we get $\Phi_n \in \mathcal{W}^{1,p}(\mathrm{S}^n)$, and (4.2) is checked. The ratio $\partial_{x_k^{(i)}}\Phi_n/\Phi_n$ equals the right-hand side of (7.24). So, by (7.26) and $\phi_2 \geq \varepsilon$ on S^2, Condition (7.8) holds with $\gamma_n = K_3\, n$, for some positive constant $K_3 > 0$. Here again, (7.25) holds, and so Condition (7.9) can be checked exactly as in the Example 2. Hence, if the conditional law of the marks given $\{N(\mathrm{S}) = n\}$ satisfies the moment Condition (7.10), then (5.6) holds due to Proposition 7.3.

In the following picture we display finite difference and modified Malliavin estimates of the density in Example 3 with $\varepsilon = 0.9$, $R = 0.257$, $h = 0.001$, $N = 2 \times 10^6$ random samples, and constant marks all equal to 1.

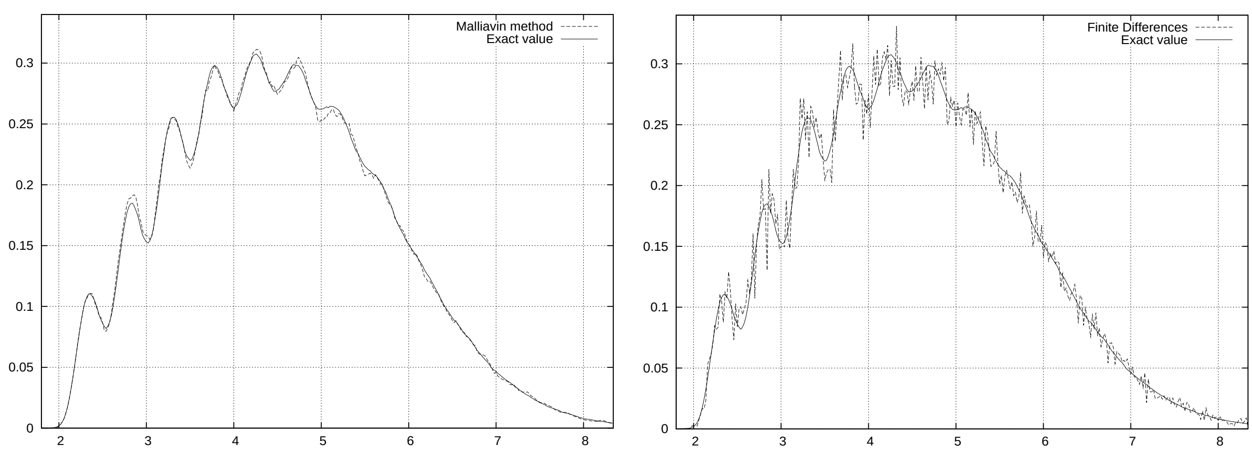

Figure 4: Malliavin method vs finite differences.

Figure 4 shows that the modified Malliavin estimator performs better than the finite difference estimator, with the same number of random samples.

Conclusion

In this paper we constructed statistical estimators for the density of functionals of spatial point processes, with marks on a general measurable space, using a Malliavin integration by parts formula. We applied our theoretical result to the estimation of the density of the interference in a wireless ad hoc network model. In comparison with kernel estimators, the proposed estimator is unbiased and asymptotically more efficient, as confirmed by numerical simulations.

References

[1] S. Albeverio, Yu. G. Kondratiev, and M. Röckner. Analysis and geometry on configuration spaces. *J. Funct. Anal.*, 154(2):444–500, 1998.

[2] F. Baccelli and B. Błaszczyszyn. On a coverage process ranging from the Boolean model to the Poisson-Voronoi tessellation with applications to wireless communications. *Adv. in Appl. Probab.*, 33(2):293–323, 2001.

[3] V. Bally, M.-P. Bavouzet-Morel, and M. Messaoud. Integration by parts formula for locally smooth laws and applications to sensitivity computations. *Ann. Appl. Probab.*, 17(1):33–66, 2007.

[4] M.-P. Bavouzet-Morel and M. Messaoud. Computation of Greeks using Malliavin's calculus in jump type market models. *Electron. J. Probab.*, 11:no. 10, 276–300 (electronic), 2006.

[5] V.I. Burenkov. *Sobolev spaces on domains*, volume 137 of *Teubner Texts in Mathematics*. B. G. Teubner Verlagsgesellschaft mbH, Stuttgart, 1998.

[6] Y.S. Chow and H. Teicher. *Probability theory*. Springer-Verlag, New York, 1978. Independence, interchangeability, martingales.

[7] D. J. Daley and D. Vere-Jones. *An introduction to the theory of point processes. Vol. I*. Probability and its Applications. Springer-Verlag, New York, 2003.

[8] D. J. Daley and D. Vere-Jones. *An introduction to the theory of point processes. Vol. II*. Probability and its Applications (New York). Springer, New York, second edition, 2008.

[9] M.H.A. Davis and M.P. Johansson. Malliavin Monte Carlo Greeks for jump diffusions. *Stochastic Processes and their Applications*, 116(1):101–129, 2006.

[10] L. Decreusefond. Perturbation analysis and Malliavin calculus. *Ann. Appl. Probab.*, 8(2):496–523, 1998.

[11] O. Dousse, F. Baccelli, and P. Thiran. Impact of interferences on connectivity in ad hoc networks. *IEEE/ACM Trans. Networking*, 13:425–436, 2005.

[12] L.C. Evans and R. F. Gariepy. *Measure theory and fine properties of functions*. Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1992.

[13] B. Forster, E. Lütkebohmert, and J. Teichmann. Absolutely continuous laws of jump-diffusions in finite and infinite dimensions with applications to mathematical finance. *SIAM J. Math. Anal.*, 40(5):2132–2153, 2008/09.

[14] E. Fournié, J.M. Lasry, J. Lebuchoux, and P.L. Lions. Applications of Malliavin calculus to Monte-Carlo methods in finance. II. *Finance and Stochastics*, 5(2):201–236, 2001.

[15] E. Fournié, J.M. Lasry, J. Lebuchoux, P.L. Lions, and N. Touzi. Applications of Malliavin calculus to Monte Carlo methods in finance. *Finance and Stochastics*, 3(4):391–412, 1999.

[16] A. J. Ganesh and G.L. Torrisi. Large deviations of the interference in a wireless communication model. *IEEE Trans. Inform. Theory*, 54(8):3505–3517, 2008.

[17] Y. El Khatib and N. Privault. Computations of Greeks in markets with jumps via the Malliavin calculus. *Finance and Stochastics*, 4(2):161–179, 2004.

[18] A. Kohatsu-Higa and R. Pettersson. Variance reduction methods for simulation of densities on Wiener space. *SIAM J. Numer. Anal.*, 40(2):431–450, 2002.

[19] S. Loisel and N. Privault. Sensitivity analysis and density estimation for finite-time ruin probabilities. *J. Comput. Appl. Math.*, 230(1):107–120, 2009.

[20] P. Mhatre, K. Papagiannaki, and F. Baccelli. Interference mitigation through power control in high density 802.11 WLANs. INFOCOM, 2007.

[21] J. Møller and R.P. Waagepetersen. *Statistical inference and simulation for spatial point processes*, volume 100 of *Monographs on Statistics and Applied Probability*. Chapman & Hall/CRC, Boca Raton, FL, 2004.

[22] Y. Ogata and M. Tanemura. Likelihood analysis of spatial point patterns. *J. Roy. Statist. Soc. Ser. B*, 46(3):496–518, 1984.

[23] E. Parzen. On estimation of a probability density function and mode. *Ann. Math. Statist.*, 33:1065–1076, 1962.

[24] N. Privault. *Stochastic Analysis in Discrete and Continuous Settings*, volume 1982 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2009.

[25] N. Privault and X. Wei. A Malliavin calculus approach to sensitivity analysis in insurance. *Insurance Math. Econom.*, 35(3):679–690, 2004.

[26] N. Privault and X. Wei. Integration by parts for point processes and Monte Carlo estimation. *J. Appl. Probab.*, 44:806–823, 2007.

[27] W. Rudin. *Functional Analysis*. Mc graw-Hill, 1991.

[28] D. Tse and P. Viswanath. *Fundamentals of Wireless Communication*. Cambridge University Press, Cambridge, 2005.

[29] M. N. M. van Lieshout. *Markov point processes and their applications*. Imperial College Press, London, 2000.