

Weitzenböck formulas on Poisson probability spaces

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Abstract

This paper surveys and compares some recent approaches to stochastic infinite-dimensional geometry on the space Γ of configurations (i.e. locally finite subsets) of a Riemannian manifold M under Poisson measures. In particular, different approaches to Bochner-Weitzenböck formulas are considered. A unitary transform is also introduced by mapping functions of n configuration points to their multiple stochastic integral.

Key words: Poisson probability measure, Bochner-Weitzenböck formulas, stochastic analysis.

Classification: 60H07, 60J65, 58J65, 58A10

1 Weitzenböck formula under a measure

Let M be a Riemannian manifold with volume measure dx , covariant derivative ∇ , and exterior derivative d . Let ∇_μ^* and d_μ^* denote the adjoints of ∇ and d under a measure μ on M of the form $\mu(dx) = e^{\phi(x)}dx$. The classical Weitzenböck formula under the measure μ states that

$$d_\mu^*d + dd_\mu^* = \nabla_\mu^*\nabla + R - \text{Hess } \phi,$$

where R denotes the Ricci tensor on M . In terms of the de Rham Laplacian $H_R = d_\mu^*d + dd_\mu^*$ and of the Bochner Laplacian $H_B = d_\mu^*d + dd_\mu^*$ we have

$$H_R = H_B + R - \text{Hess } \phi.$$

In particular the term $\text{Hess } \phi$ plays the role of a curvature under the measure μ .

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2 Probability: Poisson space

In this section we recall some facts on random functionals on Poisson space. The Poisson probability measure on \mathbb{N} can be introduced by considering N independent $\{0, 1\}$ -valued Bernoulli random variables X_1, \dots, X_N , with parameter λ/N , $\lambda > 0$. Then $X_1 + \dots + X_N$ has a binomial law, and

$$P(X_1 + \dots + X_N = k) = \binom{N}{k} \left(\frac{\lambda}{N}\right)^k \left(1 - \frac{\lambda}{N}\right)^{N-k}$$

converges to $\frac{\lambda^k}{k!}e^{-\lambda}$ as N goes to infinity. This defines a probability measure π_λ on \mathbb{N} as

$$\pi_\lambda(\{k\}) = \frac{\lambda^k}{k!}e^{-\lambda}.$$

Let X be a metric space with a σ -finite Borel measure σ . The measure π_λ has the convolution property $\pi_\lambda \star \pi_\mu = \pi_{\lambda+\mu}$, which allows to construct the Poisson measure π_σ with intensity σ on

$$\Gamma = \left\{ \gamma = \sum_{k=1}^{k=n} \delta_{x_k}; \quad x_1, \dots, x_n \in X, \quad n \in \mathbb{N} \cup \{\infty\} \right\}$$

by letting

$$\pi_\sigma(\{\gamma \in \Gamma; \quad \gamma(A_1) = k_1, \dots, \gamma(A_n) = k_n\}) = \frac{\sigma(A_1)^{k_1}}{k_1!}e^{-\sigma(A_1)} \dots \frac{\sigma(A_n)^{k_n}}{k_n!}e^{-\sigma(A_n)},$$

where A_1, \dots, A_n are disjoint compact subsets of X . This measure is characterized by its Fourier transform

$$\int_{\Gamma} e^{i \int_X f(x) d\gamma(x)} d\pi(\gamma) = \exp \left(\int_X (e^{if(x)} - 1) d\sigma(x) \right).$$

If $\gamma \in \Gamma$ is finite with cardinal $|\gamma| = n$ we write

$$\gamma = \sum_{i=1}^{i=n} \delta_{x_i}.$$

Given Λ a compact subset we consider $F : \Gamma \rightarrow \mathbb{R}$ such that $F(\gamma) = F(\gamma \cap \Lambda)$, and written as

$$F(\gamma) = F(\gamma \cap \Lambda) = e^{\sigma(\Lambda)/2} \sum_{n=0}^{\infty} 1_{\{|\gamma \cap \Lambda|=n\}} n! f_n(x_1, \dots, x_n) = \sum_{n=0}^{\infty} J_n(f_n)$$

where f_n is a symmetric function with support in Λ^n , with

$$J_n(f_n)(\gamma) = J_n(f_n)(\gamma \cap \Lambda) = n! 1_{\{|\gamma \cap \Lambda|=n\}} e^{\sigma(\Lambda)/2} f_n(x_1, \dots, x_n), \quad n \geq 1.$$

The multiple Poisson stochastic integral of f_n is defined as

$$I_n(f_n) = \int_{\{(x_1, \dots, x_n) \in X^n; \quad x_i \neq x_j, i \neq j\}} f_n(x_1, \dots, x_n) (\gamma - \sigma)(dx_1) \cdots (\gamma - \sigma)(dx_n),$$

and extends to $f_n \in L_\sigma^2(X)^{\circ n}$ via the well-known isometry

$$\int_\Gamma I_n(f_n) I_m(g_m) d\pi = n! 1_{\{n=m\}} \langle f_n, g_m \rangle_{L_\sigma^2(X)^{\circ n}}, \quad f_n \in L_\sigma^2(X)^{\circ n}, \quad g_m \in L_\sigma^2(X)^{\circ m}.$$

We introduce a combinatorial transform \tilde{K} which has some similarities with the K -transform, cf. [6] and references therein. The transform \tilde{K} identifies the functional $J_n(f_n)$, which makes sense only in finite volume, to $I_n(f_n)$ which is defined for all square-integrable f_n .

Proposition 2.1 *The operator \tilde{K} defined by*

$$\tilde{K} J_n(f_n) = I_n(f_n), \quad f_n \text{ symmetric in } \mathcal{C}_c(\Lambda^n), \quad n \in \mathbb{N},$$

is unitary on $L_{\pi_\sigma}^2(\Gamma)$. Moreover, \tilde{K} satisfies

$$\tilde{K} F(\gamma) = \sum_{\eta \subset \gamma} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \int_{X^k} F(\eta \cup \{y_1, \dots, y_k\}) \sigma(dy_1) \cdots \sigma(dy_k).$$

Proof. We have

$$\begin{aligned} \int_\Gamma J_n(f_n) J_m(g_m) d\pi &= n! 1_{\{n=m\}} e^{\sigma(\Lambda)} \int_\Gamma 1_{\{|\gamma \cap \Lambda|=n\}} f_n(x_1, \dots, x_n) g_n(x_1, \dots, x_n) d\pi(\gamma) \\ &= \frac{n!^2}{\sigma(\Lambda)^n} e^{\sigma(\Lambda)} \pi_\sigma(|\gamma \cap \Lambda| = n) \langle f_n, g_n \rangle_{L_\sigma^2(X)^{\circ n}} \\ &= n! 1_{\{n=m\}} \langle f_n, g_n \rangle_{L_\sigma^2(X)^{\circ n}}, \end{aligned}$$

which shows the first statement. On the other hand we have

$$\begin{aligned} &\tilde{K} J_n(f_n)(\gamma) \\ &= \sum_{\eta \subset \gamma \cap \Lambda} \sum_{k=0}^{\infty} (-1)^k \frac{n!}{k!} \int_X 1_{\{|\eta \cup \{y_1, \dots, y_k\}|=n\}} f_n(\eta \cup \{y_1, \dots, y_k\}) \sigma(dy_1) \cdots \sigma(dy_k) \end{aligned}$$

$$\begin{aligned}
&= \sum_{k=0}^{k=n} (-1)^k \frac{n!}{k!} \sum_{\substack{\eta \subset \gamma \cap \Lambda \\ |\eta|=n-k}} \int_X f_n(\eta \cup \{y_1, \dots, y_k\}) \sigma(dy_1) \cdots \sigma(dy_k) \\
&= \sum_{k=0}^{k=n} (-1)^k \binom{n}{k} \sum_{\substack{x_1, \dots, x_{n-k} \in \gamma \cap \Lambda \\ x_i \neq x_j, i \neq j}} \int_X f_n(\{x_1, \dots, x_{n-k}, y_1, \dots, y_k\}) \sigma(dy_1) \cdots \sigma(dy_k) \\
&= I_n(f_n)(\gamma),
\end{aligned}$$

the last relation follows from e.g. Prop. 4.1 of [9]. \square

If Λ is compact and $F(\gamma) = F(\gamma \cap \Lambda)$ we have

$$\int_{\Gamma} F(\gamma) d\pi(\gamma) = e^{-\sigma(\Lambda)} \sum_{n=0}^{\infty} \frac{1}{n!} \int_X \cdots \int_X f_n(x_1, \dots, x_n) \sigma(dx_1) \cdots \sigma(dx_n).$$

In the particular case $X = \mathbb{R}_+$ with σ the Lebesgue measure, the standard Poisson process is defined as

$$N_t(\gamma) = \gamma([0, t]) = \sum_{k=1}^{\infty} 1_{[T_k, \infty[}(t), \quad t > 0,$$

i.e. every configuration $\gamma \in \Gamma$ can be viewed as the ordered sequence $\gamma = (T_k)_{k \geq 1}$ of jump times of $(N_t)_{t \in \mathbb{R}_+}$ on \mathbb{R}_+ . Let $f_n \in \mathcal{C}_c([0, \lambda]^n)$ be symmetric. Then

$$\begin{aligned}
\int_{\Gamma} f_n(T_1, \dots, T_n) d\pi(\gamma) &= e^{-\lambda} \sum_{k=n}^{\infty} \frac{1}{k!} \int_0^{\lambda} \cdots \int_0^{\lambda} f_n(t_1, \dots, t_n) \sigma(dt_1) \cdots \sigma(dt_k). \\
&= \sum_{k=n}^{\infty} e^{-\lambda} \int_0^{\lambda} \int_0^{t_k} \cdots \int_0^{t_1} f_n(t_1, \dots, t_n) \sigma(dt_1) \cdots \sigma(dt_k). \\
&= \sum_{k=n}^{\infty} e^{-\lambda} \int_0^{\lambda} \frac{(\lambda - t_n)^{k-n}}{(k-n)!} \int_0^{t_n} \cdots \int_0^{t_1} f_n(t_1, \dots, t_n) \sigma(dt_1) \cdots \sigma(dt_n). \\
&= \int_0^{\infty} e^{-t_n} \int_0^{t_n} \cdots \int_0^{t_2} f_n(t_1, \dots, t_n) dt_1 \cdots dt_n.
\end{aligned}$$

This formula extends to f bounded and measurable.

3 Geometry

We recall the construction of [1], [2] in the case of 1-forms, see also [3] for the case of n -forms. We assume that X is a Riemannian manifold. The tangent space at $\gamma \in \Gamma$

is taken to be

$$L^2(X; TX, \gamma) = \oplus_{x \in \gamma} T_x X.$$

A differential form of order n maps $\gamma \in \Gamma$ into the antisymmetric tensor product

$$\wedge^n(T_\gamma \Gamma) = \wedge^n(\oplus_{x \in \gamma} T_x X).$$

Bochner and de Rham Laplacians on differential forms over configuration spaces are then constructed from their counterparts at the level of the manifold X . Let d_x^X be the exterior differential on X , let ∇_x^X, Δ_x^X be the natural covariant derivative and Bochner Laplacian on the bundle $T_{\gamma \setminus \{x\} \cup \{y\}} \Gamma \rightarrow y \in \mathcal{O}_{\gamma, x}$, where $\mathcal{O}_{\gamma, x}$ is an open set in X such that $\bar{\mathcal{O}}_{\gamma, x} \cap (\gamma \setminus \{x\}) = \emptyset$. The covariant derivative of the smooth differential 1-form W is defined as

$$(\nabla_x W_x(\gamma, x))_{x \in \gamma} \in T_\gamma \Gamma \otimes T_\gamma \Gamma,$$

where $W_x(\gamma, y) = W((\gamma \setminus \{x\}) \cup \{y\})$, $x, y \in X$. The Bochner Laplacian H^B on Γ is defined as

$$H^B W(\gamma) = - \sum_{x \in \gamma} \Delta_x^X W_x(\gamma, x).$$

The exterior derivative d^Γ is defined as

$$d^\Gamma W = \sum_{x \in \gamma} \sum_{y \in \gamma} d_x^X W_x(\gamma, x)_y,$$

where $W_x(\gamma, x)_y$ is the component of $W_x(\gamma, x)$ of index $y \in \gamma$, with adjoint

$$d^{\Gamma*} W = \sum_{x \in \gamma} \sum_{y \in \gamma} d_x^{X*} W_x(\gamma, x)_{x, y},$$

where $W_x(\gamma, x)_{x, y}$ is the component of $W_x(\gamma, x)$ of index (x, y) and d_x^{X*} is the adjoint of d_x^X under the volume element σ on X . A Weitzenböck formula is stated in [1], [3] as

$$H^R = H^B + R, \tag{3.1}$$

where H^R is the de Rham Laplacian $H^R = d^\Gamma d^{\Gamma*} + d^{\Gamma*} d^\Gamma$ and the curvature term

$$R(\gamma) = \sum_{x \in \gamma} R(\gamma, x)$$

has the explicit expression

$$R(\gamma, x)(V(\gamma)_y) = 1_{\{x=y\}} \sum_{i,j=1}^d \text{Ric}_{ij}(x) e_i \langle V(\gamma)_x, e_j \rangle_x,$$

where $(e_j)_{j=1}^{j=d}$ is an orthonormal basis of $T_x X$. Formula (3.1) can be viewed as the lifting to Γ of the Weitzenböck formula on X .

Note that in the above construction the curvature term in (3.1) is essentially due to the curvature of X , in particular it vanishes if $X = \mathbb{R}^d$ and no curvature term is induced from the Poisson measure itself.

In this paper we present a different geometry on the infinite-dimensional space Γ , in which the Ricci curvature tensor under the Poisson measure appears to be the identity operator when $X = \mathbb{R}_+$, see [8] when X is a more general Riemannian manifold. .

Lifting of differential structure

Let \mathcal{S} denote the space of cylindrical functionals of the form

$$F(\gamma) = f(T_1, \dots, T_n), \quad f \in C_b^\infty(\mathbb{R}^n). \quad (3.2)$$

Let \mathcal{U} denote the space of smooth processes of the form

$$u(\gamma, x) = \sum_{i=1}^{i=n} F_i(\gamma) h_i(x), \quad (\gamma, x) \in \Gamma \times \mathbb{R}_+, \quad h_i \in C_c^\infty(\mathbb{R}_+), \quad F_i \in \mathcal{S}, \quad i = 1, \dots, n. \quad (3.3)$$

The differential geometric objects to be introduced below have finite dimensional counterparts, and each of them has a stochastic interpretation. The following table describes the correspondence between geometry and probability.

	Geometry	Probability
γ	element of Γ	point measure on \mathbb{R}_+
$\mathcal{C}_c^\infty(\mathbb{R}_+)$	tangent vectors to Γ	test functions on \mathbb{R}_+
σ	Riemannian metric on Γ	Lebesgue measure
d	gradient on Γ	stochastic gradient
\mathcal{U}	vector fields on Γ	stochastic processes
du	exterior derivative of $u \in \mathcal{U}$	two-parameter process
$\{\cdot, \cdot\}$	bracket of vector fields on Γ	bracket on $\mathcal{U} \times \mathcal{U}$
Ω	curvature tensor on Γ	trilinear mapping on \mathcal{U}
d^*	divergence on Γ	stochastic integral operator

Divergence operator

The definition of the following gradient operator goes back to [4].

Definition 3.1 Given $F \in \mathcal{S}$, $F = f(T_1, \dots, T_d)$, let

$$d_t F(\gamma) = - \sum_{k=1}^{k=d} 1_{[0, T_k]}(t) \partial_k f(T_1, \dots, T_d), \quad t \geq 0.$$

The following is a finite-dimensional integration by parts formula for d .

Lemma 3.1 We have for $F = f(T_1, \dots, T_d)$ and $h \in \mathcal{C}_c(\mathbb{R}_+)$:

$$\int_{\Gamma} \langle dF, h \rangle_{L^2(\mathbb{R}_+)} d\pi(\gamma) = \int_{\Gamma} F(\gamma) \left(\sum_{k=1}^{k=d} h(T_k) - \int_0^{T_d} h(t) dt \right) d\pi(\gamma).$$

Proof. All \mathcal{C}^∞ functions on $\Delta_d = \{(t_1, \dots, t_d); 0 \leq t_1 < \dots < t_d\}$ are extended by continuity to the closure of Δ_d . We have

$$\begin{aligned} \int_{\Gamma} \langle dF(\gamma), h \rangle_{L^2(\mathbb{R}_+)} d\pi(\gamma) &= - \sum_{k=1}^{k=d} \int_0^\infty e^{-t_d} \int_0^{t_d} \dots \int_0^{t_2} \int_0^{t_k} h(s) ds \partial_k f(t_1, \dots, t_d) dt_1 \dots dt_d \\ &= \int_0^\infty e^{-t_d} \int_0^{t_d} \dots \int_0^{t_2} h(t_1) f(t_1, \dots, t_d) dt_1 \dots dt_d \\ &\quad - \int_0^\infty e^{-t_d} \int_0^{t_d} \dots \int_0^{t_3} \int_0^{t_2} h(s) ds f(t_2, t_2, \dots, t_d) dt_2 \dots dt_d \end{aligned}$$

$$\begin{aligned}
& + \sum_{k=2}^{k=d} \int_0^\infty e^{-t_d} \int_0^{t_d} \cdots \int_0^{t_2} h(t_k) f(t_1, \dots, t_d) dt_1 \cdots dt_d \\
& - \int_0^\infty e^{-t_d} \int_0^{t_d} h(s) ds \int_0^{t_d} \cdots \int_0^{t_2} f(t_1, \dots, t_d) dt_1 \cdots dt_d \\
& - \sum_{k=2}^{k=d-1} \int_0^\infty e^{-t_d} \int_0^{t_d} \cdots \int_0^{t_{k+1}} \int_0^{t_{k-1}} \cdots \int_0^{t_2} \\
& \quad \int_0^{t_{k+1}} h(s) ds f(t_1, \dots, t_{k-1}, t_{k+1}, t_{k+1}, \dots, t_d) dt_1 \cdots dt_{k+1} dt_{k-1} \cdots dt_d \\
& + \sum_{k=2}^{k=d} \int_0^\infty e^{-t_d} \int_0^{t_d} \cdots \int_0^{t_k} \int_0^{t_{k-2}} \cdots \int_0^{t_2} \int_0^{t_k} h(s) ds f(t_1, \dots, t_{k-2}, t_k, t_k, \dots, t_d) dt_1 \cdots dt_d \\
& = \sum_{k=1}^{k=d} \int_0^\infty e^{-t_d} \int_0^{t_d} \cdots \int_0^{t_2} h(t_k) f(t_1, \dots, t_d) dt_1 \cdots dt_d \\
& \quad - \int_0^\infty e^{-t_d} \int_0^{t_d} h(s) ds \int_0^{t_d} \cdots \int_0^{t_2} f(t_1, \dots, t_d) dt_1 \cdots dt_d \\
& = \int_\Gamma F(\gamma) \left(\sum_{k=1}^{k=d} h(T_k) - \int_0^{T_d} h(t) dt \right) d\pi(\gamma).
\end{aligned}$$

□

The following definition of the divergence coincides with the compensated Poisson stochastic integral with respect to $(N_t - t)_{t \in \mathbb{R}_+}$ on the adapted square-integrable processes.

Definition 3.2 We define d_π^* on \mathcal{U} by

$$d_\pi^*(hG) = \int_0^\infty h(t)(\gamma(dt) - dt) - \langle h, dG \rangle_{L^2(\mathbb{R}_+)}, \quad G \in \mathcal{S}, \quad h \in L^2(\mathbb{R}_+).$$

Using this definition, an integration by parts formula can be obtained independently of the dimension.

Proposition 3.1 The divergence operator $d_\pi^* : L^2(\Gamma \times \mathbb{R}_+) \longrightarrow L^2(\Gamma)$ is the closable adjoint of d , i.e.

$$\int_\Gamma F d_\pi^* u \, d\pi(\gamma) = \int_\Gamma \langle dF, u \rangle_{L^2(\mathbb{R}_+)} d\pi(\gamma), \quad F \in \mathcal{S}, \quad u \in \mathcal{U}. \quad (3.4)$$

Proof. Given Lemma 3.1 it suffices to notice that if $k > d$,

$$\begin{aligned}
\int_{\Gamma} F(\gamma) h(T_k) d\pi(\gamma) &= \int_0^{\infty} e^{-t_k} h(t_k) \int_0^{t_k} \cdots \int_0^{t_d} \cdots \int_0^{t_2} f(t_1, \dots, t_d) dt_1 \cdots dt_k \\
&= \int_0^{\infty} e^{-t_k} \int_0^{t_k} h(s) ds \int_0^{t_k} \cdots \int_0^{t_d} \cdots \int_0^{t_2} f(t_1, \dots, t_d) dt_1 \cdots dt_k \\
&\quad - \int_0^{\infty} e^{-t_{k-1}} \int_0^{t_{k-1}} h(s) ds \int_0^{t_{k-1}} \cdots \int_0^{t_d} \cdots \int_0^{t_2} f(t_1, \dots, t_d) dt_1 \cdots dt_{k-1} \\
&= \int_{\Gamma} F(\gamma) \int_{x_{k-1}}^{T_k} h(t) dt d\pi(\gamma),
\end{aligned}$$

in other terms the discrete-time process

$$\left(\sum_{k=1}^{k=n} h(T_k) - \int_0^{T_k} h(t) dt \right)_{k \geq 1} = \left(\int_0^{T_k} h(t) d(N_t - t) \right)_{k \geq 1}$$

is a martingale. Hence Relation (3.4) also implies that for $F, G \in \mathcal{S}$,

$$\int_{\Gamma} \langle dF, hG \rangle_{L^2(\mathbb{R}_+)} d\pi(\gamma) = \int_{\Gamma} \langle d(FG), h \rangle_{L^2(\mathbb{R}_+)} - F \langle dG, h \rangle_{L^2(\mathbb{R}_+)} d\pi(\gamma) \quad (3.5)$$

$$\begin{aligned}
&= \int_{\Gamma} F \left(G \int_0^{\infty} h(t) (\gamma(dt) - dt) - \langle h, dG \rangle_{L^2(\mathbb{R}_+)} \right) d\pi(\gamma) \\
&= \int_{\Gamma} F d_{\pi}^*(hG) d\pi(\gamma). \quad (3.6)
\end{aligned}$$

□

Covariant derivative

Given $u \in \mathcal{U}$ we define the covariant derivative $\nabla_u v$ in the direction $u \in L^2(\mathbb{R}_+)$ of the vector field $v = \sum_{i=1}^{i=n} F_i h_i \in \mathcal{U}$ as

$$\nabla_u v(t) = \sum_{i=1}^{i=n} h_i(t) d_u F_i - F_i \dot{h}_i(t) \int_0^t u(s) ds, \quad t \in \mathbb{R}_+, \quad (3.7)$$

where

$$d_u F = \langle dF, u \rangle_{L^2(\mathbb{R}_+)}, \quad F \in \mathcal{S}.$$

We have

$$\nabla_{uF}(vG) = F v d_u G + FG \nabla_u v, \quad u, v \in \mathcal{C}_c^{\infty}(\mathbb{R}_+), \quad F, G \in \mathcal{S}. \quad (3.8)$$

We also let

$$\nabla_s v(t) = \sum_{i=1}^{i=n} h_i(t) d_s F_i - F_i \dot{h}_i(t) 1_{[0,t]}(s), \quad s, t \in \mathbb{R}_+,$$

in order to write

$$\nabla_u v(t) = \int_0^\infty u(s) \nabla_s v(t) ds, \quad t \in \mathbb{R}_+, \quad u, v \in \mathcal{U}.$$

Lie-Poisson bracket

Definition 3.3 *The Lie bracket $\{u, v\}$ of $u, v \in \mathcal{C}_c^\infty(\mathbb{R}_+)$, is defined to be the unique element of $\mathcal{C}_c^\infty(\mathbb{R}_+)$ satisfying $(d_u d_v - d_v d_u)F = d_w F$, $F \in \mathcal{S}$.*

The bracket $\{u, v\}$ is defined for $u, v \in \mathcal{U}$ with

$$\{Fu, Gv\}(x) = FG\{u, v\}(x) + v(x)F d_u G - u(x)G d_v F, \quad x \in \mathbb{R}_+,$$

$u, v \in \mathcal{C}_c^\infty(\mathbb{R}_+)$, $F, G \in \mathcal{S}$.

Vanishing of torsion

Proposition 3.2 *The Lie bracket $\{u, v\}$ of $u, v \in \mathcal{U}$ satisfies*

$$\{u, v\} = \nabla_u v - \nabla_v u,$$

i.e. the connection defined by ∇ has a vanishing torsion.

Proof. We have $F(\gamma) = T_n$. If $u, v \in \mathcal{C}_c^\infty(\mathbb{R}_+)$ we have

$$\begin{aligned} (d_u d_v - d_v d_u)T_n &= -d_u \int_0^{T_n} v(s) ds + d_v \int_0^{T_n} u(s) ds \\ &= v(T_n) \int_0^{T_n} u(s) ds - u(T_n) \int_0^{T_n} v(s) ds \\ &= \int_0^{T_n} \left(\dot{v}(t) \int_0^t u(s) ds - \dot{u}(t) \int_0^t v(s) ds \right) dt \\ &= d_{\nabla_u v - \nabla_v u} T_n. \end{aligned}$$

Since d is a derivation, this shows that

$$d_u d_v - d_v d_u = d_{\nabla_u v - \nabla_v u}, \quad u, v \in \mathcal{U}.$$

The extension to $u, v \in \mathcal{U}$ follows from (3.8). □

Vanishing of curvature

Proposition 3.3 *The curvature tensor Ω of ∇ vanishes on \mathcal{U} , i.e.*

$$\Omega(u, v)h := [\nabla_u, \nabla_v]h - \nabla_{\{u, v\}}h = 0, \quad u, v, h \in \mathcal{U},$$

and \mathcal{U} is a Lie algebra under the bracket $\{\cdot, \cdot\}$.

Proof. We have, letting $\tilde{u}(t) = -\int_0^t u(s)ds$:

$$[\nabla_u, \nabla_v]h = \overbrace{\tilde{u}\nabla_v h} - \overbrace{\tilde{v}\nabla_u h} = \overbrace{\tilde{u}\tilde{v}\dot{h}} - \overbrace{\tilde{v}\tilde{u}\dot{h}} = -\tilde{u}\tilde{v}\dot{h} + \tilde{v}\tilde{u}\dot{h},$$

and

$$\nabla_{\{u, v\}}h = \nabla_{\tilde{u}\tilde{v} - \tilde{v}\tilde{u}}h = (\widetilde{\tilde{u}\tilde{v} - \tilde{v}\tilde{u}})\dot{h} = (u\tilde{v} - v\tilde{u})\dot{h},$$

hence $\Omega(u, v)h = 0$, $h, u, v \in \mathcal{C}_c^\infty(\mathbb{R}_+)$. The extension of the result to \mathcal{U} follows again from (3.8). The Lie algebra property follows from the vanishing of Ω . \square

Exterior derivative

The exterior derivative du of a smooth vector field $u \in \mathcal{U}$ is defined from

$$\langle du, h_1 \wedge h_2 \rangle_{L^2(\mathbb{R}_+) \wedge L^2(\mathbb{R}_+)} = \langle \nabla_{h_1} u, h_2 \rangle_{L^2(\mathbb{R}_+)} - \langle \nabla_{h_2} u, h_1 \rangle_{L^2(\mathbb{R}_+)},$$

$h_1, h_2 \in \mathcal{U}$. We have

$$\|du\|_{L^2(\mathbb{R}_+) \wedge L^2(\mathbb{R}_+)}^2 = 2 \int_0^\infty \int_0^\infty (du(s, t))^2 ds dt, \quad (3.9)$$

where

$$du(s, t) = \frac{1}{2}(\nabla_s u(t) - \nabla_t u(s)), \quad s, t \in \mathbb{R}_+, \quad u \in \mathcal{U}.$$

Isometry formula

Lemma 3.2 *We have for $u \in \mathcal{U}$:*

$$\int_\Gamma (d_\pi^* u)^2 d\pi(\gamma) = \int_\Gamma \|u\|_{L^2(\mathbb{R}_+)}^2 d\pi(\gamma) + \int_\Gamma \int_0^\infty \int_0^\infty \nabla_s u(t) \nabla_t u(s) ds dt d\pi(\gamma). \quad (3.10)$$

Proof. (cf. [8], [7] and the proof of [5] for path spaces over Lie groups). Given $u = \sum_{i=1}^n h_i F_i \in \mathcal{U}$ we have

$$\begin{aligned}
\int_{\Gamma} d_{\pi}^*(h_i F_i) d_{\pi}^*(h_j F_j) d\pi(\gamma) &= \int_{\Gamma} F_i d_{h_i} d_{\pi}^*(h_j F_j) d\pi(\gamma) \\
&= \int_{\Gamma} F_i d_{h_i} (F_j d_{\pi}^*(h_j) - d_{h_j} F_j) d\pi(\gamma) \\
&= \int_{\Gamma} (F_i F_j d_{h_i} d_{\pi}^* h_j + F_i d_{\pi}^*(h_j) d_{h_i} F_j - F_i d_{h_i} d_{h_j} F_j) d\pi(\gamma) \\
&= \int_{\Gamma} (F_i F_j \langle h_i, h_j \rangle_{L^2(\mathbb{R}_+)} + F_i F_j d_{\pi}^*(\nabla_{h_i} h_j) + F_i d_{\pi}^*(h_j) d_{h_i} F_j - F_i d_{h_i} d_{h_j} F_j) d\pi(\gamma) \\
&= \int_{\Gamma} (F_i F_j \langle h_i, h_j \rangle_{L^2(\mathbb{R}_+)} + d_{\nabla_{h_i} h_j} (F_i F_j) + d_{h_j} (F_i d_{h_i} F_j) - F_i d_{h_i} d_{h_j} F_j) d\pi(\gamma) \\
&= \int_{\Gamma} (F_i F_j \langle h_i, h_j \rangle_{L^2(\mathbb{R}_+)} + d_{\nabla_{h_i} h_j} (F_i F_j) + d_{h_j} F_i d_{h_i} F_j + F_i (d_{h_j} d_{h_i} F_j - d_{h_i} d_{h_j} F_j)) d\pi(\gamma) \\
&= \int_{\Gamma} (F_i F_j \langle h_i, h_j \rangle_{L^2(\mathbb{R}_+)} + d_{\nabla_{h_i} h_j} (F_i F_j) + d_{h_j} F_i d_{h_i} F_j + F_i d_{\nabla_{h_j} h_i - \nabla_{h_i} h_j} F_j) d\pi(\gamma) \\
&= \int_{\Gamma} (F_i F_j \langle h_i, h_j \rangle_{L^2(\mathbb{R}_+)} + F_j d_{\nabla_{h_i} h_j} F_i + F_i d_{\nabla_{h_j} h_i} F_j + d_{h_j} F_i d_{h_i} F_j) d\pi(\gamma) \\
&= \int_{\Gamma} \left(F_i F_j \langle h_i, h_j \rangle_{L^2(\mathbb{R}_+)} + F_j \int_0^{\infty} d_s F_i \int_0^{\infty} \nabla_t h_j(s) h_i(t) dt ds \right. \\
&\quad \left. + F_i \int_0^{\infty} d_t F_j \int_0^{\infty} \nabla_s h_i(t) h_j(s) ds dt + \int_0^{\infty} h_i(t) d_t F_j \int_0^{\infty} h_j(s) d_s F_i ds dt \right) d\pi(\gamma),
\end{aligned}$$

where we used the commutation relation satisfied by the gradient d :

$$d_u d_{\pi}^* v = d_{\pi}^* \nabla_u v + \langle u, v \rangle_{L^2(\mathbb{R}_+)}, \quad u, v \in \mathcal{C}_c^{\infty}(\mathbb{R}_+), \quad (3.11)$$

which can be proved as follows:

$$\begin{aligned}
d_u d_{\pi}^* v &= - \sum_{k=1}^{\infty} \dot{v}(T_k) \int_0^{T_k} u(s) ds = -d_{\pi}^* \left(v(\cdot) \int_0^{\cdot} u(s) ds \right) - \int_0^{\infty} \dot{v}(t) \int_0^t u(s) ds dt \\
&= d_{\pi}^* (\nabla_u v) + \langle u, v \rangle_{L^2(\mathbb{R}_+)}.
\end{aligned}$$

□

Finally we state a Weitzenböck type identity on configuration space, that can be read as

$$dd_{\pi}^* + d^* d = \nabla^* \nabla + \text{Id}_{L^2(\mathbb{R}_+)},$$

i.e. the Ricci tensor under the Poisson measure is the identity $\text{Id}_{L^2(\mathbb{R}_+)}$ on $L^2(\mathbb{R}_+)$.

Theorem 3.1 *We have for $u \in \mathcal{U}$:*

$$\begin{aligned} & \int_{\Gamma} (d_{\pi}^* u)^2 d\pi(\gamma) + \int_{\Gamma} \|du\|_{L^2(\mathbb{R}_+) \wedge L^2(\mathbb{R}_+)}^2 d\pi(\gamma) \\ &= \int_{\Gamma} \|u\|_{L^2(\mathbb{R}_+)}^2 d\pi(\gamma) + \int_{\Gamma} \|\nabla u\|_{L^2(\mathbb{R}_+) \otimes L^2(\mathbb{R}_+)}^2 d\pi(\gamma). \end{aligned} \quad (3.12)$$

Proof. Relation (3.12) for $u = \sum_{i=1}^n h_i F_i \in \mathcal{U}$ follows from (3.9) and Lemma 3.2. □

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