Variance-GGC asset price models and their sensitivity analysis*

Nicolas Privault and Dichuan Yang

Abstract This paper reviews the variance-gamma asset price model as well as its symmetric and non-symmetric extensions based on generalized gamma convolutions (GGC). In particular we compute the basic characteristics and decomposition of the variance-GGC model, and we consider its sensitivity analysis based on the approach of [8].

Key words: variance-gamma model, variance-GGC model, sensitivity analysis.
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1 Introduction

Lévy processes play an important role in the modeling of risky asset prices with jumps. In addition to the Black-Scholes model based on geometric Brownian motion, pure jump and jump-diffusion processes have been used by Cox and Ross [5] and Merton [13] for the modeling of asset prices. More recently, Brownian motions time-changed by non-decreasing Lévy processes (i.e. subordinators) have become popular, in particular the Normal Inverse Gaussian (NIG) model [1]), the variance-gamma (VG) model [12] and [11], and the CGMY/KoBol models [4], [3].

The normal inverse Gaussian (NIG) process [1] can constructed as a Brownian motion time-changed by a Lévy process with the inverse Gaussian distribution,

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whose marginal at time t is identical in law to the first hitting time of the positive level t by a drifted Brownian motion.

The variance-gamma process [12], [11] is built on the time change of a Brownian motion by a gamma process, and has been successful in modeling asset prices with jumps and in addressing the issue of slowly decreasing probability tails found in real market data.

The CGMY/KoBol models [4], [3] are extensions of the variance-gamma model by a more flexible choice of Lévy measures. However, this extension loses some nice properties of variance-gamma model, for example variance-gamma processes can be decomposed into the difference of two gamma processes, whereas this property does not hold in general in the CGMY/KoBol models.

In [6] the variance-gamma model has been extended into a symmetric variance-GGC model, based on generalized gamma convolutions (GGCs), see [2] for details and a driftless Brownian motion. In this paper we review this model and propose an extension to non-symmetric case using a drifted Brownian motion.

GGC random variables can be constructed by limits in distribution of sums of independent gamma random variables with varying shape parameters. As a result, the variance-GGC model allows for more flexibility than standard variance-gamma models, while retaining some of their properties. The skewness and kurtosis of variance-GGC processes can be computed in closed form, including the relations between skewness and kurtosis of the GGC process and of the corresponding variance-GGC process. In addition, variance-GGC processes can be represented as the difference of two GGC processes.

On the other hand, the sensitivity analysis of stochastic models is an important topic in financial engineering applications. The sensitivity analysis of time-changed Brownian motion processes has been developed and the Greek formulas have been obtained by following the approach in [8]. In addition, the sentivity analysis of the variance-gamma, stable and tempered stable processes has been performed in [9] and [10] respectively. As an extension of the variance-gamma process, we study the corresponding sensitivity analysis of the variance-GGC model along the lines of [9].

In the remaining of this section we review some facts on generalized gamma convolutions, (GGCs) including their variance, skewness and kurtosis . We also discuss an asset price model based on GGCs and its sensitivity analysis.

Wiener-gamma integrals

Consider a gamma process $(\gamma_t)_{t\in\mathbb{R}_+}$, i.e. $(\gamma_t)_{t\in\mathbb{R}_+}$ is a process with independent and stationary increments such that γ_t at time $t > 0$ has a gamma distribution with shape parameter t and probability density function $e^{-x}x^{t-1}/\Gamma(t)$, $x>0$. We denote by

$$\int_0^\infty g(t)d\gamma_t, \tag{1}$$

the Wiener-gamma stochastic integral of a deterministic function

$$g:\mathbb{R}_+ \longrightarrow \mathbb{R}_+$$

with respect to the standard gamma process $(\gamma_t)_{t\in\mathbb{R}_+}$, provided g satisfies the condition

$$\int_0^\infty \log(1+g(t))dt < \infty, \tag{2}$$

which ensures the finiteness of (1), cf. § 1.2 , page 350 of [7] for details. In particular, there is a one-to-one correspondence between GGC random variables and Wiener-gamma integrals, Proposition 1.1, page 352 of [7].

Generalized gamma convolutions

A random variable Z is a generalized gamma convolution if its Laplace transform admits the representation

$$\mathbb{E}[e^{-uZ}] = \exp\left(-t\int_0^\infty \log\left(1+\frac{u}{s}\right)\mu(ds)\right), \qquad u\ge 0$$

where $\mu(ds)$ is called the Thorin measure and should satisfy the conditions

$$\int_{(0,1]} |\log s|\mu(ds) < \infty \quad \text{and} \quad \int_{(1,\infty)} s^{-1}\mu(ds) < \infty.$$

Generalized gamma convolutions (GGC) can be defined as the limits of independent sums of gamma random variables with various shape parameters, cf. [2] for details.

In particular, the density of the Lévy measure of a GGC random variable is a completely monotone function. From the Laplace transform of Z we find

$$\mathbb{E}[Z] = \int_0^\infty t^{-1}\mu(dt),$$

and the first central moments of Z can be computed as

$$\begin{cases} \mathbb{E}[(Z-\mathbb{E}[Z])^2] = \displaystyle\int_0^\infty t^{-2}\mu(dt), \\ \mathbb{E}[(Z-\mathbb{E}[Z])^3] = 2\displaystyle\int_0^\infty t^{-3}\mu(dt), \\ \mathbb{E}[(Z-\mathbb{E}[Z])^4] = 3\,(\mathrm{Var}[Z])^2 + 6\displaystyle\int_0^\infty t^{-4}\mu(dt). \end{cases} \tag{3}$$

As a consequence we can compute the

$$\mathrm{Skewness}[Z] = \frac{\mathbb{E}[(Z-\mathbb{E}[Z])^3]}{(\mathrm{Var}[Z])^{3/2}} = \frac{2\int_0^\infty t^{-3}\mu(dt)}{(\mathrm{Var}[Z])^{3/2}},$$

and

$$\mathrm{Kurtosis}[Z] = \frac{\mathbb{E}[(Z-\mathbb{E}[Z])^4]}{(\mathrm{Var}[Z])^2} = 3 + 6\frac{\int_0^\infty t^{-4}\mu(dt)}{(\mathrm{Var}[Z])^2}$$

of Z. We refer the reader to Proposition 1.1 of [7] for the relation between the integrand in a Wiener-gamma representation and the cumulative distribution function of the associated generalized gamma convolution.

Market model and sensitivity analysis

As an extension of the model of [9] to GGC random variables we consider an asset price process S_T defined by the exponent

$$S_T = S_0 \exp\left(\theta\int_0^\infty g(s)d\gamma_s + \tau\sqrt{T}\Theta + Z_T + c(\theta,\tau)T\right),$$

of a variance-GGC process, i.e. $\int_0^\infty g(s)d\gamma_s$ is a GGC random variable represented as a Wiener-gamma integral, Θ is an independent Gaussian random variable, $(Z_t)_{t\in\mathbb{R}_+}$ is another GGC-Lévy process, and $\theta\in\mathbb{R}$, $\tau\geq 0$, $T>0$.

In section 3 the sensitivity $\dfrac{\partial}{\partial S_0}\mathbb{E}[\Phi(S_T)]$ of an option with payoff Φ with respect to the initial value S_0 in a variance-GGC model is shown to satisfy

$$\frac{\partial}{\partial S_0}\mathbb{E}[\Phi(S_T)] = \frac{1}{S_0}\mathbb{E}[\Phi(S_T)L_T],$$

where

$$L_T := \frac{2\theta\int_0^\infty g(s)f^2(s)d\gamma_s}{(\theta\int_0^\infty g(s)f(s)d\gamma_s + \tau\sqrt{T}\eta)^2} + \frac{\int_0^\infty f(s)d\gamma_s - T\int_0^\infty f(s)ds + \eta\Theta}{\theta\int_0^\infty g(s)f(s)d\gamma_s + \tau\sqrt{T}\eta}.$$

for any positive function $f : \mathbb{R}_+ \to (0,a)$ and $\eta > 0$. In Theorem 1 we will compute this sensitivity as well as orther Greeks based on the model parameters θ and τ.

The remaining of this paper is organized as follows. In section 2 we introduce a model for Brownian motion time-changed by a GGC subordinator. The variance, skewness and kurtosis of variance-GGC processes are calculated in relation with the corresponding parameters of GGC processes, and several example of variance-GGC models are considered. A Girsanov transform of GGC processes is also stated. The sensitivity analysis with respect to S_0, θ and τ is conducted in section 3.

2 Variance-GGC processes

Given $(W_t)_{t\in\mathbb{R}_+}$ a standard Brownian motion and $\theta \in \mathbb{R}$, $\sigma > 0$, consider the drifted Brownian motion

$$B_t^{\theta,\sigma} := \theta t + \sigma W_t, \qquad t \in \mathbb{R}_+.$$

Next, consider a generalized gamma convolution (GGC) Lévy process $(G_t)_{t\in\mathbb{R}_+}$ such that G_1 is a GGC random variable with Thorin measure $\mu(ds)$ on $\mathbb{R}_+$. We define the variance-GGC process $(Y_t^{\sigma,\theta})_{t\in\mathbb{R}_+}$ as the time-changed Brownian motion

$$Y_t^{\sigma,\theta} := B_{G_t}^{\theta,\sigma}, \qquad t \in \mathbb{R}_+.$$

The probability density function of $Y_t^{\sigma,\theta}$ is given by

$$f_{Y_t^{\sigma,\theta}}(x) = \frac{1}{\sigma\sqrt{2\pi}} \int_0^\infty \exp\left(-\frac{|x-\theta y|^2}{2\sigma^2 y}\right) h_t(y) \frac{dy}{\sqrt{y}}, \qquad x \in \mathbb{R},$$

where $h_t(y)$ is the probability density function of G_t, cf. Relation (6) in [11].

The Laplace transform of $Y_t^{\sigma,\theta}$ is

$$\begin{aligned} \mathbb{E}\left[\exp\left(-uY_t^{\sigma,\theta}\right)\right] &= \int_0^\infty e^{-uy} f_{Y_t}(y)dy \qquad (4) \\ &= \Psi_{G_t}\left(\theta u - \frac{\sigma^2}{2}u^2\right) \\ &= \exp\left(-t\int_0^\infty \log\left(1+\frac{\theta u - \sigma^2 u^2/2}{s}\right)\mu(ds)\right), \end{aligned}$$

where Ψ_{G_t} is the Laplace transform of G_t.

This construction extends the symmetric variance-GGC model constructed in Section 4.4, page 124-126 of [6]. In particular, the next proposition extends to variance-GGC processes Relation (8) in [11], [12], which decomposes the variance-gamma process into the difference of two gamma processes. Here, we are writing Y_t as the difference of two independent GGC processes, i.e. Y_t becomes an Extended Generalized Gamma Convolution (EGGC) in the sense of Chapter 7 of [2], cf. also § 3 of [14].

Proposition 1. *The time-changed process Y_t can be decomposed as*

$$Y_t = U_t - W_t,$$

where U_t and W_t are two independent GGC processes with Thorin measures μ_A and μ_B which are the image measures of $\mu(dt)$ on $\mathbb{R}_+$ respectively, by the mappings

$$s \longmapsto B(s) := \frac{\theta}{\sigma^2} + \frac{1}{\sigma}\sqrt{\frac{\theta^2}{\sigma^2} + 2s}, \qquad s \in \mathbb{R}_+,$$

and

$$s \longmapsto A(s) = -\frac{\theta}{\sigma^2} + \frac{1}{\sigma}\sqrt{\frac{\theta^2}{\sigma^2} + 2s}, \qquad s \in \mathbb{R}_+.$$

Proof. From (4), the Laplace tranform of Y_t can be decomposed as

$$\begin{aligned}
\mathbb{E}\left[\exp\left(-uY_t^{\sigma,\theta}\right)\right] &= \exp\left(-t\int_0^\infty \log\left(1-\frac{u}{B(s)}\right)\left(1+\frac{u}{A(s)}\right)\mu(ds)\right) \\
&= \exp\left(-t\int_0^\infty \log\left(1+\frac{u}{A(s)}\right)\mu(ds) - t\int_0^\infty \log\left(1-\frac{u}{B(s)}\right)\mu(ds)\right) \\
&= \exp\left(-t\int_0^\infty \log\left(1+\frac{u}{s}\right)\mu_A(ds) - t\int_0^\infty \log\left(1-\frac{u}{s}\right)\mu_B(ds)\right) \\
&= \mathbb{E}[e^{-uU_t}]\mathbb{E}[e^{uW_t}].
\end{aligned}$$

□

The Laplace tranform of Y_t can also be decomposed as

$$\begin{aligned}
\mathbb{E}\left[\exp\left(-uY_t^{\sigma,\theta}\right)\right] &= \exp\left(-t\int_0^\infty \log\left(1+\frac{u}{s}\right)\mu_A(ds) - t\int_0^\infty \log\left(1-\frac{u}{s}\right)\mu_B(ds)\right) \\
&= \exp\left(-t\int_{-\infty}^0 \log\left(1+\frac{u}{s}\right)\mu_{-B}(ds) - t\int_0^\infty \log\left(1+\frac{u}{s}\right)\mu_A(ds)\right), \qquad (5)
\end{aligned}$$

where μ_{-B} is the image measure of μ_B by $s \mapsto -s$, and in particular, Y_t is an extended GGC (EGGC) with Thorin measure $\mu_A + \mu_{-B}$ in the sense of Chapter 7 of [2].

In the next proposition we compute the variance, skewness and kurtosis of variance-GGC processes.

Proposition 2. *We have*

(i) $\mathrm{Var}[Y_1] = \theta^2 \mathrm{Var}[G_1] + \sigma^2 \mathbb{E}[G_1]$,

$$(ii)\ \text{Skewness}[Y_1] = -\frac{\mathbb{E}[(G_1-\mathbb{E}[G_1])^3]+2(\sigma/\theta)^2\text{Var}[G_1]}{2(\text{Var}[G_1]+(\sigma/\theta)^2\mathbb{E}[G_1])^{3/2}}$$

$$= -\frac{\theta^3}{2}\text{Skewness}[G_1]\frac{(\text{Var}[G_1])^{3/2}}{(\text{Var}[Y_1])^{3/2}} - \frac{\theta\sigma^2\text{Var}[G_1]}{(\text{Var}[Y_1])^{3/2}}, \tag{6}$$

$$(iii)\ \text{Kurtosis}[Y_1] = 3+3\theta^4\frac{\mathbb{E}[(G_1-\mathbb{E}[G_1])^4]-3(\text{Var}[G_1])^2}{8(\theta^2\text{Var}[G_1]+\sigma^2\mathbb{E}[G_1])^2}$$

$$+3\frac{3\theta^2\sigma^2\mathbb{E}[(G_1-\mathbb{E}[G_1])^3]/4+\sigma^4\text{Var}[G_1]}{(\theta^2\text{Var}[G_1]+\sigma^2\mathbb{E}[G_1])^2} \tag{7}$$

$$= 3+\theta^4\frac{(\text{Kurtosis}[G_1]-3)(\text{Var}[G_1])^2}{16(\text{Var}[Y_1])^2}$$

$$+9\sigma^2\theta^2\frac{\text{Skewness}[G_1](\text{Var}[G_1])^{3/2}}{4(\text{Var}[Y_1])^2}+3\frac{\sigma^4\text{Var}[G_1]}{(\text{Var}[Y_1])^2}.$$

Proof. Using the Thorin measure $\mu_A+\mu_{-B}$ of Y_t and (3) we have

$$\begin{aligned}
\text{Var}[Y_1] &= \int_0^\infty t^{-2}\mu_A(dt)+\int_{-\infty}^0 t^{-2}\mu_{-B}(dt)\\
&= \int_0^\infty \frac{1}{A^2(t)}\mu(dt)+\int_0^\infty \frac{1}{B^2(t)}\mu(dt)\\
&= \int_0^\infty \frac{\theta^2+t\sigma^2}{t^2}\mu(dt)\\
&= \theta^2\text{Var}[G_1]+\sigma^2\mathbb{E}[G_1],
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{E}[(Y_1-\mathbb{E}[Y_1])^3] &= 2\int_0^\infty t^{-3}\mu_A(dt)+2\int_{-\infty}^0 t^{-3}\mu_{-B}(dt)\\
&= \frac{1}{2}\int_0^\infty \frac{\theta^3+\theta\sigma^2\left(\theta^2/\sigma^2+2t\right)}{t^3}\mu(dt)\\
&= \frac{\theta^3}{2}\mathbb{E}[(G_1-\mathbb{E}[G_1])^3]+\theta\sigma^2\text{Var}[G_1],
\end{aligned}$$

and

$$\begin{aligned}
&\mathbb{E}[(Y_1-\mathbb{E}[Y_1])^4] = 6\int_{-\infty}^0 t^{-4}\mu_A(dt)+6\int_0^\infty t^{-4}\mu_{-B}(dt)\\
&\quad+3\left(\int_{-\infty}^0 t^{-2}\mu^-(dt)+\int_0^\infty t^{-2}\mu^+(dt)\right)^2\\
&= \frac{3}{4}\int_0^\infty \frac{\theta^4+(\theta\sigma)^2(\sqrt{4\theta^2/\sigma^2+8t}/2)^2+\sigma^4(\sqrt{4\theta^2/\sigma^2+8t})^4/2}{t^4}\mu(dt)
\end{aligned}$$

$$+3\left(\int_0^\infty \frac{\theta^2+t\sigma^2}{t^2}\mu(dt)\right)^2$$
$$=\frac{3}{4}\int_0^\infty \frac{3\theta^4+6\sigma^2\theta^2 t+4\sigma^4 t^2}{t^4}\mu(dt)+3\left(\int_0^\infty \frac{\theta^2+t\sigma^2}{t^2}\mu(dt)\right)^2$$
$$=\frac{3}{8}\theta^4(\mathbb{E}[(G_1-\mathbb{E}[G_1])^4]-3(\mathrm{Var}[G_1])^2)$$
$$+\frac{9}{4}\theta^2\sigma^2\mathbb{E}[(G_1-\mathbb{E}[G_1])^3]+3\sigma^4\mathrm{Var}[G_1]+3(\theta^2\mathrm{Var}[G_1]+\sigma^2\mathbb{E}[G_1])^2,$$

and this yields (6) and (7). □

Girsanov theorem

Consider the probability measure Q_λ defined by the Radon-Nikodym density

$$\frac{dQ_\lambda}{dP}:=\frac{e^{\lambda Y_T}}{\mathbb{E}[e^{\lambda Y_T}]}=(1-\lambda)^{aT}e^{\lambda Y_T}=e^{\lambda Y_T+aT\log(1-\lambda)}, \qquad \lambda<1, \tag{8}$$

cf. e.g. Lemma 2.1 of [9], where Y_T is a gamma random variable with shape and scale parameters $(aT,1)$ under P. Then, under Q_λ, the random variable Y_t has a gamma distribution with parameter $(aT,1/(1-\lambda)$, i.e. the distribution of $Y_t/(1-\lambda)$ under P.

In the next proposition we extend this Girsanov transformation to GGC random variables.

Proposition 3. *Consider the probability measure P_f defined by its Radon-Nikodym derivative*

$$\frac{dP_f}{dP}=\frac{e^{\int_0^\infty f(s)d\gamma_s}}{\mathbb{E}[e^{\int_0^\infty f(s)d\gamma_s}]}=e^{\int_0^\infty f(s)d\gamma_s+\int_0^\infty \log(1-f(s))ds},$$

where $f:\mathbb{R}_+\to(0,1)$ satisfies

$$\int_0^\infty \log\left(\frac{1+f(t)}{1-f(t)}\right)dt<\infty. \tag{9}$$

Assume that $g:\mathbb{R}_+\to\mathbb{R}_+$ satisfies (2)*, and*

$$\int_0^\infty \log(1+ug(s)-f(s))\,ds>-\infty, \qquad u>0.$$

Then, under P_f, the law of $\int_0^\infty g(s)d\gamma_s$ is the GGC distribution of the Wiener-gamma integral

$$\int_0^\infty \frac{g(s)}{1-f(s)}d\gamma_s$$

under P.

Proof. For all $u > 0$, we have

$$\begin{aligned}
&\mathbb{E}_{P_f}\left[\exp\left(-u\int_0^\infty g(s)d\gamma_s\right)\right]\\
&= \mathbb{E}\left[\exp\left(-u\int_0^\infty g(s)d\gamma_s + \int_0^\infty f(s)d\gamma_s + \int_0^\infty \log\left(1-f(s)\right)ds\right)\right]\\
&= \mathbb{E}\left[\exp\left(\int_0^\infty f(s) - ug(s)d\gamma_s\right)\right]\exp\left(\int_0^\infty \log\left(1-f(s)\right)ds\right)\\
&= \exp\left(-\int_0^\infty \log\left(1+ug(s)-f(s)\right)ds\right)\exp\left(\int_0^\infty \log\left(1-f(s)\right)ds\right)\\
&= \exp\left(-\int_0^\infty \log\left(1+\frac{ug(s)}{1-f(s)}\right)ds\right)\\
&= \mathbb{E}\left[\exp\left(-u\int_0^\infty \frac{g(s)}{1-f(s)}d\gamma_s\right)\right].
\end{aligned}$$

□

Note that (8) is recovered by taking $g(s) = \mathbf{1}_{[0,aT]}(s)$ and $f(s) = \lambda\mathbf{1}_{[0,aT]}(s)$ for $\lambda \in (0,1)$, i.e. $G_T = \int_0^\infty g(s)d\gamma_s$ is a gamma random variable with shape parameter aT and we have

$$\mathbb{E}_{P_f}[e^{-uG_T}] = \left(1+\frac{u}{1-\lambda}\right)^{-aT} = \mathbb{E}\left[\exp\left(-\frac{u}{1-\lambda}G_T\right)\right],$$

$u > 0$, $\lambda < 1$. Next we consider several examples and particular cases.

Gamma case

In case the Thorin measure μ is given by

$$\mu(dt) = \gamma\delta_c(dt),$$

where δ_c is the Dirac measure at $c > 0$ we find the variance-gamma model of [12]. Here, G_t, $t > 0$, has the gamma probability density

$$\phi_t(x) = c^{\gamma t}\frac{x^{\gamma t-1}e^{-cx}}{\Gamma(\gamma t)}, \qquad x \in \mathbb{R}_+,$$

with mean and variance $\gamma t/c$ and $\gamma t/c^2$, and G_t becomes a gamma random variable with parameters $(\gamma t, c)$. In this case, the decomposition in Proposition 1 reads

$$\Psi_{Y_t}(u) = \left(1-\frac{\sigma^2u^2}{2c}\right)^{-t\gamma} = \left(1-\frac{\sigma u}{\sqrt{2c}}\right)^{-t\gamma}\left(1+\frac{\sigma u}{\sqrt{2c}}\right)^{-t\gamma},$$

and we have

$$\mu_A(dt) = \mu_B(dt) = \gamma \delta_{\sqrt{2c}/\sigma}(dt),$$

thus $(U_t)_{t\in\mathbb{R}_+}$, $(W_t)_{t\in\mathbb{R}_+}$ become independent gamma processes with parameter $(\gamma t, \sqrt{2c}/\sigma)$. The mean and variance of U_1 are

$$\mathbb{E}[U_1] = \int_0^\infty t^{-1}\mu_A(dt) = \frac{\sigma\gamma}{\sqrt{2c}}$$

and

$$\mathrm{Var}[U_1] = \mathbb{E}[(U_1 - \mathbb{E}[U_1])^2] = \int_0^\infty t^{-2}\mu_A(dt) = \frac{\gamma\sigma^2}{2c}.$$

Symmetric case

When $\theta = 0$ we recover the symmetric variance-GGC process

$$Y_t := B^\sigma(G_t), \qquad t \in \mathbb{R}_+,$$

defined in Section 4.4, page 124-126 of [6], i.e. the time-changed Brownian motion is a symmetric variance-GGC process. Here, Y_t is a centered Gaussian random variable with variance $\sigma^2 G_t$ given G_t, where B_t^σ is a standard Brownian motion with variance σ^2.

The Laplace transform of Y_t in Proposition 1 shows that Y_t decomposes into two independent processes with same GGC increments since μ_A and μ_B are the same image measures of $\mu(dt)$ on $\mathbb{R}_+$, by $s \mapsto \sqrt{2s}/\sigma$.

Variance-stable processes

Let $(G_t)_{t\in\mathbb{R}_+}$ be a Lévy stable process with index parameter $\alpha \in (0,1)$ and moment generating function $h(s) = e^{-s^\alpha}$. In this section we consider a non-symmetric extension of the symmetric variance stable process considered in Section 4.5, pages 126-127 of [6]. The Thorin measure of the stable distribution is given by

$$\mu(dt) = \varphi(t)dt = \frac{\alpha}{\pi}\sin(\alpha\pi)t^{\alpha-1}dt,$$

cf. page 35 of [2]. By Proposition 1, Y_t can be decomposed as

$$Y_t = U_t - W_t,$$

where U_t and W_t are processes with independent stable increments and Thorin measures

$$\mu_A(dt) = \varphi_A(t)dt = \frac{\alpha}{\pi}\sin(\alpha\pi)(\sigma^2 t + \theta)\left(\frac{1}{2}(\sigma t - \theta/\sigma)^2 - \frac{\theta^2}{2\sigma^2}\right)^{\alpha-1} dt,$$

and

$$\mu_B(dt) = \varphi_B(t)dt = \frac{\alpha}{\pi}\sin(\alpha\pi)(\sigma^2 t - \theta)\left(\frac{1}{2}(\sigma t - \theta/\sigma)^2 - \frac{\theta^2}{2\sigma^2}\right)^{\alpha-1} dt.$$

In the symmetric case $\theta = 0$ we find

$$\mu_A(dt) = \varphi_A(t)dt = \mu_B(dt) = \varphi_B(t)dt = \sigma^2 t \varphi\left(\frac{\sigma^2 t^2}{2}\right) dt = \frac{\alpha \sin(\alpha\pi)}{2^{\alpha-1}\pi}\sigma^{2\alpha} t^{2\alpha-1} dt,$$

i.e. $\sqrt{2}U_t/\sigma$ and $\sqrt{2}W_t/\sigma$ are stable processes of index 2α. Note that the skewness and kurtosis of G_t and Y_t are undefined. Figure 1 presents a simulation of the variance-stable process.

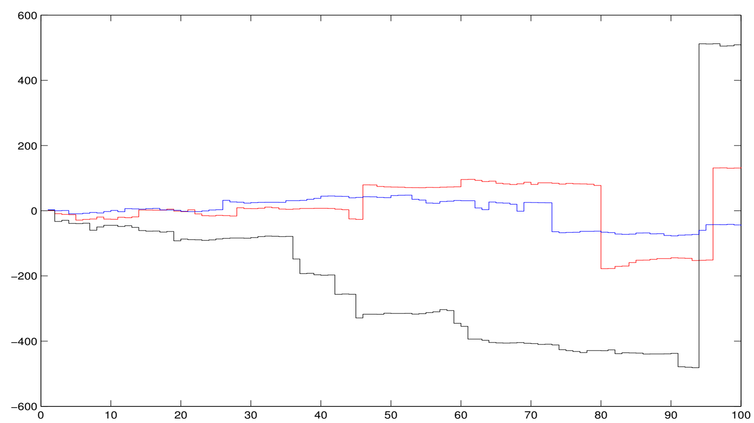

Fig. 1 Sample paths of variance-stable process with $\alpha = 0.99$.

Variance product of stable processes

Here we take $G_1 = Z^{1/\alpha} X_\alpha$ where Z is a $\Gamma(\gamma, 1)$ random variable and X_α is a stable random variable with index $\alpha < 1$. The MGF of G_1 is $h(s) = (1 + s^\alpha)^\gamma$, cf. page 38 of [2], i.e. G_1 is a GGC with Thorin measure

$$\mu(dt) = \varphi(t)dt = \frac{1}{\pi}\frac{\gamma\alpha t^{\alpha-1}\sin(\alpha\pi)}{1 + t^{2\alpha} + 2t^\alpha \cos(\alpha\pi)} dt,$$

and Y_t decomposes as

$$Y_t = U_t - W_t,$$

where U_t and W_t are processes of independent product of stable increment and Thorin measures

$$\mu_A(dt) = \varphi_A(t)dt$$
$$= \frac{1}{\pi}\frac{\gamma\alpha((\sigma t+\theta/\sigma)^2/2-\theta^2/(2\sigma^2))^{\alpha-1}\sin(\alpha\pi)(\sigma^2 t+\theta)}{1+((\sigma t+\theta/\sigma)^2/2-\theta^2/(2\sigma^2))^{2\alpha}+2((\sigma t+\theta/\sigma)^2/2-\theta^2/(2\sigma^2))^{\alpha}\cos(\alpha\pi)}dt,$$

and

$$\mu_B(dt) = \varphi_B(t)dt$$
$$= \frac{1}{\pi}\frac{\gamma\alpha((\sigma t-\theta/\sigma)^2/2-\theta^2/(2\sigma^2))^{\alpha-1}\sin(\alpha\pi)(\sigma^2 t-\theta)}{1+((\sigma t-\theta/\sigma)^2/2-\theta^2/(2\sigma^2))^{2\alpha}+2((\sigma t-\theta/\sigma)^2/2-\theta^2/2\sigma^2)^{\alpha}\cos(\alpha\pi)}dt.$$

In the symmetric case

$$\begin{aligned}\mu_A(dt) &= \varphi_A(t)dt = \mu_B(dt) = \varphi_B(t)dt\\ &= \sigma^2 t\varphi\left(\frac{\sigma^2 t^2}{2}\right)dt = \frac{\gamma\alpha\sigma^{2\alpha}t^{2\alpha-1}\sin(\alpha\pi)}{\pi(2^{\alpha-1}+2^{-\alpha-1}\sigma^{4\alpha}t^{4\alpha}+\sigma^{2\alpha}t^{2\alpha}\cos(\alpha\pi))}dt.\end{aligned}$$

The skewness and kurtosis of G_t and Y_t are undefined. Figure 2 presents the corresponding simulation.

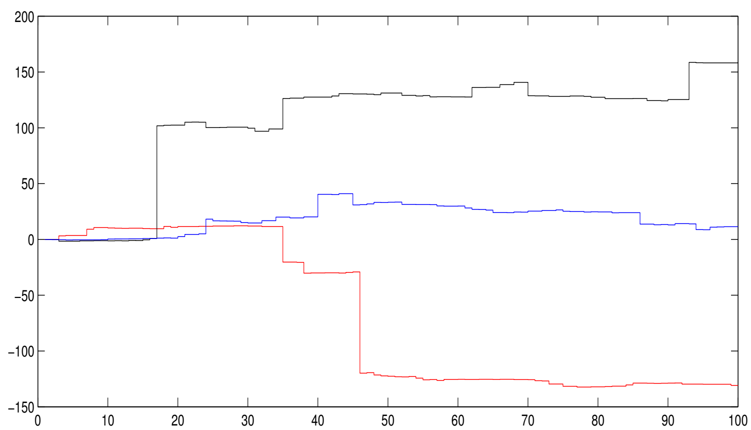

Fig. 2 Sample paths of variance-product of stable process with $\alpha = 0.99$ and $\gamma = 0.2$.

3 Sensitivity analysis

In this section we extend approach of [8] to the sensitivity analysis of variance-GGC models. Consider $(B_t)_{t\in\mathbb{R}_+}$ a standard one-dimensional standard Brownian motion independent of the Lévy process $(Y_t)_{t\in[0,T]}$ generated by

$$Y_T := \int_0^\infty g(s)d\gamma_s.$$

Let Θ be a standard Gaussian random variable independent of $(Y_t)_{t\in[0,T]}$. For each $t \in [0,T]$, we denote by $\mathscr{F}_t$ the filtration generated by Θ and $\sigma(Y_s \ : \ s \in [0,t])$.

Let $(Z_t)_{t\in\mathbb{R}_+}$ be a real-valued stochastic process in $\mathbb{R}$ independent of $(Y_t)_{t\in\mathbb{R}_+}$ and $(B_t)_{t\in\mathbb{R}_+}$. Finally we denote by and let $C_b^n(\mathbb{R}_+;\mathbb{R})$ denote the class of n-time continuously differentiable functions with bounded derivatives, whereas $\mathscr{C}_c(\mathbb{R}_+;\mathbb{R})$ denotes the space of continuous functions with compact support.

Given $\theta \in \mathbb{R}$ and $\tau \in \mathbb{R}_+$ we consider the asset price S_T written as

$$S_T = S_0 \exp\Big(\theta Y_T + \tau\sqrt{T}\Theta + Z_T + T c(\theta,\tau)\Big),$$

where the function $g(s) : \mathbb{R}_+ \to \mathbb{R}_+$ verifies (2).

Remark 1. When $\theta = 0$ the above model reduces to the standard Black-Scholes model, and in case $\theta \neq 0$ we find the variance-GGC model by taking $(Z_t)_{t\in[0,T]}$ to be a GGC process.

For example, we can take the Wiener-gamma integral $\int_0^\infty g(s)d\gamma_s$ to be a stable random variable and set Z_T to be another stable random variable, then the exponent of S_t is a variance-stable process. This example will be developed in the next section.

The next theorem deals with the sensitivity analysis of the variance-GGC model with respect to S_0, θ and τ, and is the main result in this section. Define the classes of functions

$$\mathscr{C}_L(\mathbb{R}_+;\mathbb{R}) := \{f \in C(\mathbb{R}_+;\mathbb{R}) \ : \ |f(x)| \leq C(1+|x|) \text{ for some } C>0\},$$

and

$$D(\mathbb{R}_+;\mathbb{R}) := \Big\{f : \mathbb{R}_+ \to \mathbb{R} \ : \ f = \sum_{k=1}^n c_k f_k \mathbf{1}_{A_k},\ n \geq 1,$$
$$c_k \in \mathbb{R},\ f_k \in \mathscr{C}_L(\mathbb{R}_+;\mathbb{R}),\ A_k \text{ intervals of } \mathbb{R}_+\Big\}.$$

Theorem 1. *Let $\Phi \in D(\mathbb{R}_+;\mathbb{R})$. Assume that the law of Z_T is absolutely continuous with respect to the Lebesgue measure, with*

$$\int_0^\infty \log\left(1+\frac{g(s)f^k(s)}{(1-\lambda f(s))^{k+1}}\right)ds<\infty, \qquad k=1,2,3. \tag{10}$$

Then

(i) (Delta - sensitivity with respect to S_0). We have

$$\frac{\partial}{\partial S_0}\mathbb{E}[\Phi(S_T)]=\frac{1}{S_0}\mathbb{E}[\Phi(S_T)L_T],$$

where

$$L_T=\frac{2\theta\int_0^\infty g(s)f^2(s)d\gamma_s}{(\theta\int_0^\infty g(s)f(s)d\gamma_s+\tau\sqrt{T}\eta)^2}+\frac{\int_0^\infty f(s)d\gamma_s-T\int_0^\infty f(s)ds+\eta\Theta}{\theta\int_0^\infty g(s)f(s)d\gamma_s+\tau\sqrt{T}\eta}.$$

(ii) (sensitivity with respect to θ). We have

$$\frac{\partial}{\partial\theta}\mathbb{E}[\Phi(S_T)]=\mathbb{E}\left[\Phi(S_T)\left(L_T\int_0^\infty g(s)d\gamma_s-\frac{1}{H_T}\int_0^\infty g(s)f(s)d\gamma_s\right)\right]$$
$$+TS_0\frac{\partial c}{\partial\theta}(\theta,\tau)\frac{\partial}{\partial S_0}\mathbb{E}[\Phi(S_T)],$$

where $H_T=\theta\int_0^\infty g(s)f(s)d\gamma_s+\tau\sqrt{T}\eta$.

(iii) (Theta - sensitivity with respect to τ). We have

$$\frac{\partial}{\partial\tau}\mathbb{E}[\Phi(S_T)]=\mathbb{E}\left[\Phi(S_T)L_T\sqrt{T}\left(\Theta-\frac{\eta}{H_T}\right)\right]+TS_0\frac{\partial c}{\partial\tau}(\theta,\tau)\frac{\partial}{\partial S_0}\mathbb{E}[\Phi(S_T)].$$

(iv) (Gamma - second derivative with respect to S_0). We have

$$\begin{aligned}&\frac{\partial^2}{\partial S_0^2}\mathbb{E}[\Phi(S_T)]\\&=\frac{1}{S_0^2}\mathbb{E}\left[\Phi(S_T)\left((L_T)^2-\frac{1}{H_T}\left(\frac{I_TH_T-2(K_T)^2}{(H_T)^3}+\frac{N_TH_T-M_TK_T}{(H_T)^2}\right)\right)\right]\\&\quad-\frac{1}{S_0}\frac{\partial}{\partial S_0}\mathbb{E}[\Phi(S_T)],\end{aligned}$$

where

$$K_T=2\theta\int_0^\infty g(s)f^2(s)d\gamma_s,\quad M_T=\int_0^\infty f(s)d\gamma_s-T\int_0^\infty f(s)ds+\eta\Theta,$$

and

$$I_T=6\theta\int_0^\infty g(s)f(s)^3d\gamma_s,\quad N_T=\left(\int_0^\infty f(s)d\gamma_s-T\int_0^\infty f(s)ds+\eta\Theta\right)^2.$$

Next we state two lemmas which are needed for the proof of Theorem 1.

Lemma 1. *Assume that* $\mathbb{E}[e^{2\gamma Z_T}] < \infty$ *for some* $\gamma > 1$*. Let* $f : \mathbb{R} \to (0,a)$ *be a positive function and* $\lambda \in (0,\varepsilon)$ *for* $\varepsilon < 1/a$ *such that* (10) *holds. Fix* $\eta > 0$ *and suppose that one of the following conditions holds:*

(i) The density function of $Y_T = \int_0^\infty g(s)d\gamma_s$ *decays exponentially, or*
(ii) $\mathbb{E}\left[e^{2\gamma(1+\theta\delta)Y_T}\right] < \infty$ *for all* $\delta > 0$*.*

Let also

$$S_T^{(\lambda f)} = S_0 \exp\left(\theta \int_0^\infty \frac{g(s)}{1-\lambda f(s)} d\gamma_s + \tau\sqrt{T}(\Theta + \eta\lambda) + Z_T + c(\theta,\tau)T\right),$$

and

$$H_T^{(\lambda f)} = \frac{\partial}{\partial \lambda} \log S_T^{(\lambda f)} = \theta \int_0^\infty \frac{g(s)f(s)}{(1-\lambda f(s))^2} d\gamma_s + \tau\sqrt{T}\eta, \quad H_T = H_T^{(0)},$$

and

$$K_T^{(\lambda f)} = \frac{\partial}{\partial \lambda} H_T^{(\lambda f)} = 2\theta \int_0^\infty \frac{g(s)f^2(s)}{(1-\lambda f(s))^3} d\gamma_s, \quad K_T = K_T^{(0)}.$$

Then we have the $L^2(\Omega)$*-limits*

$$\lim_{\lambda \to 0} S_T^{(\lambda f)} H_T^{(\lambda f)} = S_T H_T \quad \textit{and} \quad \lim_{\lambda \to 0} \frac{K_T^{(\lambda f)}}{(H_T^{(\lambda f)})^2} = \frac{K_T}{(H_T)^2}.$$

Proof. For any $\lambda \in (0,\varepsilon)$, we have

$$\begin{aligned}
&\sup_{\lambda \in (0,\varepsilon)} \mathbb{E}\left[|S_T^{(\lambda f)} H_T^{(\lambda f)}|^{2\gamma}\right] \le C_1 \mathbb{E}\left[e^{2\gamma\tau\sqrt{T}\Theta}\right] \mathbb{E}\left[e^{2\gamma Z_T}\right] \\
&\quad \times \sup_{\lambda \in (0,\varepsilon)} \mathbb{E}\left[\left(\theta \int_0^\infty \frac{g(s)f(s)}{(1-\lambda f(s))^2} d\gamma_s + \tau\sqrt{T}\eta\right)^{2\gamma} \exp\left(2\gamma \int_0^\infty \frac{g(s)}{1-\lambda f(s)} d\gamma_s\right)\right] \\
&\le C_1 \mathbb{E}[e^{2\gamma\tau\sqrt{T}\Theta}] \mathbb{E}[e^{2\gamma Z_T}] \\
&\quad \times \sup_{\lambda \in (0,\varepsilon)} \left(\frac{a}{(1-\lambda a)^2}\right)^{2\gamma} \mathbb{E}\left[\left(\int_0^\infty g(s)d\gamma_s + \tau\sqrt{T}\eta\right)^{2\gamma} \exp\left(\frac{2\gamma\theta}{1-\lambda a} \int_0^\infty g(s)d\gamma_s\right)\right] \\
&\le C_1 \mathbb{E}[e^{2\gamma\tau\sqrt{T}\Theta}] \mathbb{E}[e^{2\gamma Z_T}] \\
&\quad \times \left(\frac{a}{(1-\varepsilon a)^2}\right)^{2\gamma} \mathbb{E}\left[\left(\int_0^\infty g(s)d\gamma_s + \tau\sqrt{T}\eta\right)^{2\gamma} \exp\left(\frac{2\gamma\theta}{1-\varepsilon a} \int_0^\infty g(s)d\gamma_s\right)\right],
\end{aligned}$$

where C_1 is a positive constant. Under condition (i) or (ii) above we have

$$\mathbb{E}\left[Y_T^{2\gamma} \exp\left(\frac{2\gamma\theta}{1-\varepsilon a} Y_T\right)\right] \le \mathbb{E}\left[\exp\left(2\gamma\left(1 + \frac{\theta}{1-\varepsilon a}\right) Y_T\right)\right] < \infty,$$

and similarly we have $\mathbb{E}\left[e^{\frac{2\gamma\theta}{1-\varepsilon a}Y_T}\right]<\infty$. Finally, we have $\mathbb{E}[e^{2\gamma Z_T}]<\infty$ by assumption, and it is clear that $\mathbb{E}[e^{2\gamma\tau\sqrt{T}\Theta}]<\infty$. Then $|S_T^{(\lambda f)}H_T^{(\lambda f)}|$ is $L^{2\gamma}(\Omega)$-integrable, hence $(S_T^{(\lambda f)}H_T^{(\lambda f)})^2$ is uniformly-integrable since $\gamma>1$. Therefore, we have proved that $S_T^{(\lambda f)}H_T^{(\lambda f)}$ converges to S_TH_T in $L^2(\Omega)$ as $\lambda\to 0$.

Next, for any $\lambda\in(0,\varepsilon)$ we have

$$\begin{aligned}
\sup_{\lambda\in(0,\varepsilon)}\mathbb{E}[|K_T^{(\lambda f)}/(H_T^{(\lambda f)})^2|^{2\gamma}] &\le \sup_{\lambda\in(0,\varepsilon)}\mathbb{E}\left[\left(\left(\frac{2\theta}{\tau\sqrt{T}\eta}\right)\int_0^\infty \frac{g(s)f^2(s)}{(1-\lambda f(s))^3}d\gamma_s\right)^{2\gamma}\right]\\
&\le \left(\frac{a^2}{(1-\lambda a)^3}\right)^{2\gamma}\mathbb{E}\left[\left(\int_0^\infty g(s)d\gamma_s\right)^{2\gamma}\right]\sup_{\lambda\in(0,\varepsilon)}\left|\frac{2\theta}{\tau\sqrt{T}\eta}\right|^{2\gamma}\\
&\le \left(\frac{a^2}{(1-\varepsilon a)^3}\right)^{2\gamma}\mathbb{E}\left[\left(\int_0^\infty g(s)d\gamma_s\right)^{2\gamma}\right]\left|\frac{2\theta}{\tau\sqrt{T}\eta}\right|^{2\gamma},
\end{aligned}$$

since $\mathbb{E}\left[(\int_0^\infty g(s)d\gamma_s)^{2\gamma}\right]$ is finite under Condition (i) or (ii) above. Therefore $(K_T^{(\lambda f)}/(H_T^{(\lambda f)})^2)^2$ is uniformly-integrable since $\gamma>1$, and this shows that $K_T^{(\lambda f)}/(H_T^{(\lambda f)})^2$ converges to $K_T/(H_T)^2$ as $\lambda\to 0$ in $L^2(\Omega)$. □

Lemma 2. *Assume that $\mathbb{E}[e^{2\gamma Z_T}]<\infty$ for some $\gamma>1$ and that* (10) *holds. Suppose in addition that one of the following conditions holds:*

1. *The density function of $\int_0^\infty g(s)d\gamma_s$ decays exponentially;*
2. $\mathbb{E}\left[\left|e^{2\gamma(1+\theta\delta)Y_T}\right|\right]<\infty$ *for all $\delta>0$, where $Y_T=\int_0^\infty g(s)d\gamma_s$;*

then for $\Phi\in\mathscr{C}_b^1(\mathbb{R}_+,\mathbb{R})$ it holds that

(i) $\mathbb{E}\left[\Phi'(S_T)S_TH_T\right]=\mathbb{E}\left[\left(\int_0^\infty f(s)d\gamma_s-T\int_0^\infty f(s)ds+\eta\Theta\right)\Phi(S_T)\right]$,

(ii) $\mathbb{E}[\Phi'(S_T)S_T]=\mathbb{E}[\Phi(S_T)L_T]$,

(iii) $\mathbb{E}\left[\Phi'(S_T)S_T\int_0^\infty g(s)d\gamma_s\right]=\mathbb{E}\left[\Phi(S_T)\left(L_T\int_0^\infty g(s)d\gamma_s-\frac{1}{H_T}\int_0^\infty g(s)f(s)d\gamma_s\right)\right]$,

(iv) $\mathbb{E}[\Phi'(S_T)S_TB_T]=\sqrt{T}\mathbb{E}\left[\Phi(S_T)L_T\left(\Theta-\frac{\eta}{H_T}\right)\right]$,

(v) If in addition $\Phi\in\mathscr{C}_b^2(\mathbb{R}_+,\mathbb{R})$ and (10) *is satisfied then we have*

$$\begin{aligned}
&\mathbb{E}[\Phi''(S_T)(S_T)^2]+\mathbb{E}[\Phi'(S_T)S_T]\\
&=\mathbb{E}\left[\Phi(S_T)\left((L_T)^2-\frac{1}{H_T}\left(\frac{I_TH_T-2(K_T)^2}{(H_T)^3}+\frac{N_TH_T-M_TK_T}{(H_T)^2}\right)\right)\right].
\end{aligned}$$

Proof. We have

$$\mathbb{E}[(\Phi(S_T))^2]\le 2\mathbb{E}[(\Phi(S_T)-\Phi(S_0))^2]+2\mathbb{E}[(\Phi(S_0))^2]$$

$$\leq 2\mathbb{E}[(\Phi(S_0))^2] + 2\int_0^1 \mathbb{E}[(\Phi'(rS_T + (1-r)S_0))^2 (S_T - S_0)^2] dr$$
$$< \infty,$$

since $\Phi \in C_b^1(\mathbb{R}_+;\mathbb{R})$. As for (i) we have

$$\mathbb{E}\left[\Phi(S_T^{(\lambda f)})\right] = \mathbb{E}\left[\frac{dP_{\lambda f}}{dP}_{|\mathscr{F}_T} \Phi(S_T)\right], \tag{11}$$

where we define the probability measure $P_{\lambda f}$ via its Radon-Nikodym derivative

$$\frac{dP_{\lambda f}}{dP}_{|\mathscr{F}_T} = \frac{e^{\lambda \int_0^\infty f(s) d\gamma_s}}{\mathbb{E}[e^{\lambda \int_0^\infty f(s) d\gamma_s}]} \frac{e^{\lambda \eta \Theta}}{\mathbb{E}[e^{\lambda \eta \Theta}]} = e^{\lambda \int_0^\infty f(s) d\gamma_s + T \int_0^\infty \log(1-\lambda f(s)) ds + \lambda \eta \Theta - \lambda^2 \eta^2/2},$$

where $f : \mathbb{R} \to (0,a)$ and $\lambda \in (0,\varepsilon)$. In this way the GGC random variable $\int_0^\infty g(s) d\gamma_s$ and the Gaussian random variable Θ under $P_{\lambda f}$ are transformed to $\int_0^\infty \frac{g(s)}{1-\lambda f(s)} d\gamma_s$ and $\Theta + \eta\lambda$ under P.

First we prove that $\frac{\partial}{\partial \lambda}\mathbb{E}\left[\Phi(S_T^{(\lambda f)})\right]$ exists and equals the left hand side of (i). For every $\varepsilon \in (-\lambda,\lambda)$ we have

$$\frac{\Phi(S_T^{(\varepsilon f)}) - \Phi(S_T)}{\varepsilon} = \int_0^1 \Phi'(S_T^{(r\varepsilon f)}) S_T^{(r\varepsilon f)} H_T^{(r\varepsilon f)} dr,$$

and by the Cauchy-Schwarz inequality we get

$$\begin{aligned}
&\mathbb{E}\left[\left|\frac{1}{\varepsilon}(\Phi(S_T^{(\varepsilon f)}) - \Phi(S_T)) - \Phi'(S_T) S_T H_T\right|\right] \qquad (12)\\
&\leq \int_0^1 \mathbb{E}[|\Phi'(S_T^{(r\varepsilon f)}) S_T^{(r\varepsilon f)} H_T^{(r\varepsilon f)} - \Phi'(S_T) S_T H_T] dr \\
&\leq \int_0^1 \sqrt{\mathbb{E}[(\Phi'(S_T^{(r\varepsilon f)}))^2]} \sqrt{\mathbb{E}[(S_T^{(r\varepsilon f)} H_T^{(r\varepsilon f)} - S_T H_T)^2]} dr \\
&\quad + \int_0^1 \sqrt{\mathbb{E}[(\Phi'(S_T^{(r\varepsilon f)}) - \Phi'(S_T))^2]} \sqrt{\mathbb{E}[(S_T H_T)^2]} dr.
\end{aligned}$$

From the boundedness and continuity of $\Phi'(S_T^{(\varepsilon f)})$ with respect to ε in $L^2(\Omega)$, we have

$$\mathbb{E}[(\Phi'(S_T^{(\varepsilon f)}))^2] < \infty \quad \text{and} \quad \lim_{\varepsilon \to 0} \mathbb{E}[(\Phi'(S_T^{(\varepsilon f)}) - \Phi'(S_T))^2] = 0.$$

By Lemma 1 we get that $S_T^{(\lambda f)} H_T^{(\lambda f)}$ converges in $L^2(\Omega)$. Finally, we take the limit on both sides of (12) as $\varepsilon \to 0$. Next we prove that $\frac{\partial}{\partial \lambda}\mathbb{E}\left[\frac{dP_{\lambda f}}{dP}_{|\mathscr{F}_T} \Phi(S_T)\right]$ exists and equals the right hand side of (i).

For every $\varepsilon \in (-\lambda, \lambda)$ the Cauchy-Schwarz inequality yields

$$\begin{aligned}
&\mathbb{E}\left[\left|\frac{1}{\varepsilon}\left(\frac{dP_{\varepsilon f}}{dP}_{|\mathscr{F}_T} - \frac{dP_0}{dP}_{|\mathscr{F}_T}\right)\Phi(S_T) - \left(\int_0^\infty f(s)d\gamma_s - T\int_0^\infty f(s)ds + \eta\Theta\right)\Phi(S_T)\right|\right] \\
&\leq \sqrt{\mathbb{E}[(\Phi(S_T))^2]} \\
&\quad \mathbb{E}\sqrt{\left[\left(\frac{1}{\varepsilon}\left(\frac{dP_{\varepsilon f}}{dP}_{|\mathscr{F}_T} - \frac{dP_0}{dP}_{|\mathscr{F}_T}\right) - \left(\int_0^\infty f(s)d\gamma_s - T\int_0^\infty f(s)ds + \eta\Theta\right)\right)^2\right]}.
\end{aligned}$$

It is then straightforward to check that $\mathbb{E}[|\Phi(S_T)|^2] < \infty$ and

$$\frac{1}{\lambda}\left(\exp\left(\lambda\int_0^\infty f(s)d\gamma_s + T\int_0^\infty \log(1-\lambda f(s))\,ds + \lambda\eta\Theta - \lambda^2\eta^2/2\right) - 1\right)$$

converges to

$$\int_0^\infty f(s)d\gamma_s - T\int_0^\infty f(s)ds + \eta\Theta$$

in $L^2(\Omega)$ as λ tends to 0 since $\lambda^{-1}(e^{\lambda\int_0^\infty f(s)d\gamma_s} - 1)$ converges to $\int_0^\infty f(s)d\gamma_s$ in $L^2(\Omega)$ as $\lambda \to 0$. We conclude by taking the limit on both sides as $\lambda \to 0$.

For (ii) we start with the identity

$$\mathbb{E}\left[\frac{\Phi(S_T^{(\lambda f)})}{H_T^{(\lambda f)}}\right] = \mathbb{E}\left[\frac{dP_{\lambda f}}{dP}_{|\mathscr{F}_T}\frac{\Phi(S_T)}{H_T}\right].$$

First we prove that $\dfrac{\partial}{\partial\lambda}\mathbb{E}\left[\dfrac{\Phi(S_T^{(\lambda f)})}{H_T^{(\lambda f)}}\right]$ exists and equals the left hand side of (ii). For every $\varepsilon \in [-\lambda, \lambda]$ we have

$$\frac{1}{\varepsilon}\left(\frac{\Phi(S_T^{(\varepsilon f)})}{H_T^{(\varepsilon f)}} - \frac{\Phi(S_T^{(0)})}{H_T}\right) = \int_0^1 \frac{\Phi'(S_T^{(r\varepsilon f)})S_T^{(r\varepsilon f)}(H_T^{(r\varepsilon f)})^2 - \Phi(S_T^{(r\varepsilon f)})K_T^{(r\varepsilon f)}}{(H_T^{(r\varepsilon f)})^2}dr,$$

and by the Cauchy-Schwarz inequality we get

$$\begin{aligned}
&\mathbb{E}\left[\left|\frac{1}{\varepsilon}\left(\frac{\Phi(S_T^{(\varepsilon f)})}{H_T^{(\varepsilon f)}} - \frac{\Phi(S_T)}{H_T}\right) - \frac{\Phi'(S_T)S_T(H_T)^2 - \Phi(S_T)K_T}{(H_T)^2}\right|\right] \qquad (13) \\
&\leq \int_0^1 \mathbb{E}\left[\left|\frac{\Phi'(S_T^{(r\varepsilon f)})S_T^{(r\varepsilon f)}(H_T^{(r\varepsilon f)})^2 - \Phi(S_T^{(r\varepsilon f)})K_T^{(r\varepsilon f)}}{(H_T^{(r\varepsilon f)})^2} - \frac{\Phi'(S_T)S_T^{(0)}(H_T)^2 - \Phi(S_T)K_T}{(H_T)^2}\right|\right]dr
\end{aligned}$$

$$\leq \int_0^1 \sqrt{\mathbb{E}[(\Phi'(S_T^{(r\varepsilon f)}))^2]}\sqrt{\mathbb{E}[(S_T^{(r\varepsilon f)} - S_T)^2]}dr$$
$$+ \int_0^1 \sqrt{\mathbb{E}[(\Phi'(S_T^{(r\varepsilon f)}) - \Phi'(S_T))^2]}\sqrt{\mathbb{E}[(S_T)^2]}dr$$
$$+ \int_0^1 \sqrt{\mathbb{E}[(\Phi(S_T^{(r\varepsilon f)}))^2]}\sqrt{\mathbb{E}[(K_T^{(r\varepsilon f)}/(H_T^{(r\varepsilon f)})^2 - K_T/(H_T)^2)^2]}dr$$
$$+ \int_0^1 \sqrt{\mathbb{E}[(\Phi(S_T^{(r\varepsilon f)}) - \Phi(S_T))^2]}\sqrt{\mathbb{E}[(K_T/(H_T)^2)^2]}dr.$$

We have shown $\mathbb{E}[(\Phi(S_T))^2] < \infty$ in the proof of (i). Then

$$\mathbb{E}[(\Phi(S_T^{(\varepsilon f)}))^2] \leq 2\mathbb{E}[(\Phi(S_T^{(\varepsilon f)}) - S_T)^2] + 2\mathbb{E}[(\Phi(S_T))^2]$$
$$\leq 2\varepsilon^2 \int_0^1 \mathbb{E}[(\Phi'(S_T^{(r\varepsilon f)})S_T^{(r\varepsilon f)}H_T^{(r\varepsilon f)})^2]dr + 2\mathbb{E}[(\Phi(S_T))^2]$$
$$\leq 2\varepsilon^2 \sup_{x\in\mathbb{R}} |\Phi'(x)|^2 \sup_{|\varepsilon|\leq\lambda} \mathbb{E}[(S_T^{(\varepsilon f)}H_T^{(\varepsilon f)})^2] + 2\mathbb{E}[(\Phi(S_T))^2] < \infty,$$

where the Cauchy-Schwarz inequality and the Fubini theorem have been used for the second inequality. The convergence of $S_T^{(\varepsilon f)}H_T^{(\varepsilon f)}$ as $\varepsilon \to 0$ in $L^2(\Omega)$ has been proved in Lemma 1. Note that $\mathbb{E}[(\Phi(S_T^{(\varepsilon f)}))^2] < \infty$ also implies

$$\mathbb{E}[(\Phi(S_T^{(\varepsilon f)}) - \Phi(S_T))^2] \to 0 \qquad as \qquad \varepsilon \to 0.$$

By Lemma 1, we get $K_T^{(\varepsilon f)}/(H_T^{(\varepsilon f)})^2$ converges to $K_T/(H_T)^2$ as $\varepsilon \to 0$ in $L^2(\Omega)$. Taking the limit on both sides of (13) as $\varepsilon \to 0$.

Next, we prove that $\frac{\partial}{\partial\lambda}\mathbb{E}\left[\frac{dP_{\lambda f}}{dP}\Big|_{\mathscr{F}_T}\frac{\Phi(S_T)}{H_T}\right]$ exists and is equal to the right hand side of (ii). For all $p > 0$ we have

$$\mathbb{E}[(H_T^{(\lambda f)})^{-2p}] = \int_0^\infty \left(\theta \int_0^\infty \frac{g(s)f(s)}{(1-\lambda f(s))^2}d\gamma_s + \tau\sqrt{T}\eta\right)^{-2p} f_1(y)dy < (\tau\sqrt{T}\eta)^{-2p},$$

where f_1 is the density function of $\int_0^\infty \frac{g(s)f(s)}{(1-\lambda f(s))^2}d\gamma_s$. Therefore, the moment is uniformly bounded.

We conclude as in the second part of proof of (i). The proof of $(iii) - (iv)$ is similar to that of (ii). As for (iii) we have

$$\mathbb{E}\left[\frac{\Phi(S_T^{(\lambda f)})}{H_T^{(\lambda f)}}\int_0^\infty \frac{g(s)}{1-\lambda f(s)}d\gamma_s\right] = \mathbb{E}\left[\frac{dP_{\lambda f}}{dP}\Big|_{\mathscr{F}_T}\frac{\Phi(S_T)}{H_T}\int_0^\infty g(s)d\gamma_s\right].$$

For the first part, the existence of the derivative can be obtained as

$$\begin{aligned}\mathbb{E}\left[\left(\int_0^\infty \frac{g(s)}{1-\lambda f(s)}d\gamma_s - \int_0^\infty g(s)d\gamma_s\right)^2\right] &\le \mathbb{E}\left[\left(\lambda \int_0^\infty g(s)\frac{f(s)}{1-\lambda f(s)}d\gamma_s\right)^2\right]\\ &\le \mathbb{E}\left[\left(\lambda \frac{a}{1-\lambda a}\int_0^\infty g(s)d\gamma_s\right)^2\right]\\ &\le \infty.\end{aligned}$$

Similarly, $\displaystyle\int_0^\infty \frac{g(s)f(s)}{(1-\lambda f(s))^2}d\gamma_s$ converges to $\int_0^\infty g(s)f(s)d\gamma_s$ in $L^2(\Omega)$ as $\lambda \to 0$. The second part is almost the same as (i) by uniform boundedness of $H_T^{(\lambda f)}$.

For (iv) we have

$$(\Theta + \eta\lambda)\mathbb{E}\left[\frac{\Phi(S_T^{(\lambda f)})}{H_T^{(\lambda f)}}\right] = \Theta\mathbb{E}\left[\frac{dP_{\lambda f}}{dP}_{|\mathscr{F}_T}\frac{\Phi(S_T)}{H_T}\right].$$

For the first part, the existence of the derivative follows from the fact that Θ has a Gaussian distribution. The second part is proved similarly.

Finally, for (v), define $\Psi(x) = \Phi'(x)x$, and by the result of (ii) we have

$$\mathbb{E}[\Phi''(S_T)(S_T)^2] = \mathbb{E}[(\Psi'(S_T) - \Phi'(S_T))S_T] = \mathbb{E}[\Psi(S_T)L_T] - \mathbb{E}[\Phi'(S_T)S_T]$$

Hence, we obtain the desired equation by differentiating

$$\mathbb{E}\left[\Phi(S_T^{(\lambda f)})\frac{L_T^{(\lambda f)}}{H_T^{(\lambda f)}}\right] = \mathbb{E}\left[\frac{dP_{\lambda f}}{dP}_{|\mathscr{F}_T}\Phi(S_T)\frac{L_T}{H_T}\right]$$

at $\lambda = 0$. □

Now we can prove Theorem 1.

Proof. The proof of Theorem 1 uses the same argument as in the proof of Corollary 3.6 of [9]. The only difference is that S_T is a variance-gamma process in the proof of Corollary 3.6 of [9], while S_T is a variance-GGC process in this proof.

When $\Phi \in \mathscr{C}_b^2(\mathbb{R}_+,\mathbb{R})$, all four formulas in Theorem 1 are direct consequences of $(ii)-(v)$ in lemma 2, and we now extend this result to the class $D(\mathbb{R}_+;\mathbb{R})$. In general, in order to obtain an extension to Φ in a class $\mathfrak{R}_1$ of functions based on an approximating sequence $(\Phi_n)_{n\in\mathbb{N}}$ in a class $\mathfrak{R}_2 \subset \mathfrak{R}_1$, it suffices to show that for each compact set $K \subset \mathbb{R}$ we have

$$\sup_{S_0\in K} |\mathbb{E}[\Phi_n(S_T)] - \mathbb{E}[\Phi(S_T)]| \to 0 \qquad as \qquad n \to \infty, \tag{14}$$

and

$$\lim_{n\to\infty}\sup_{S_0\in K}\left|\frac{\partial}{\partial S_0}\mathbb{E}[\Phi_n(S_T)]-\frac{1}{S_0}\mathbb{E}[\Phi(S_T)L_T]\right|=0. \tag{15}$$

The extension is then based on the above steps, first from $\mathscr{C}_b^2(\mathbb{R}_+,\mathbb{R})$ to $\mathscr{C}_c(\mathbb{R}_+,\mathbb{R})$, then to $\mathscr{C}_b(\mathbb{R}_+,\mathbb{R})$ and to the class of finite linear combinations of indicator functions on an interval of $\mathbb{R}$. Finally the result is extended to the class of functions Φ of the form $\Phi=\Psi\times\mathbf{1}_A$ where $\Psi\in\mathscr{C}_L(\mathbb{R}_+,\mathbb{R})$ and A is an interval of $\mathbb{R}_+$. This shows that (14) and (15) are satisfied, and the details of each step are the same as in the proof of Corollary 3.6 of [9]. □

References

1. O.E. Barndorff-Nielsen. Processes of normal inverse Gaussian type. *Finance and Stochastics*, 2(1):41–68, 1998.
2. L. Bondesson. *Generalized gamma convolutions and related classes of distributions and densities*, volume 76 of *Lecture Notes in Statistics*. Springer-Verlag, New York, 1992.
3. S.I. Boyarchenko and S.Z. Levendorskiĭ. *Non-Gaussian Merton-Black-Scholes theory*, volume 9 of *Advanced Series on Statistical Science & Applied Probability*. World Scientific Publishing Co., Inc., River Edge, NJ, 2002.
4. P.P. Carr, H. Geman, D.B. Madan, and M. Yor. The fine structure of asset returns: an empirical investigation. *J. Business*, 75(2), 2002.
5. J.C. Cox and S.A. Ross. The valuation of options for alternative stochastic processes. *Journal of Financial Economics*, 3:145–166, 1976.
6. H. Geman, D.B. Madan, and M. Yor. Asset prices are Brownian motion: only in business time. In *Quantitative analysis in financial markets*, pages 103–146. World Sci. Publ., River Edge, NJ, 2001.
7. L.F. James, B. Roynette, and M. Yor. Generalized gamma convolutions, Dirichlet means, Thorin measures, with explicit examples. *Probab. Surv.*, 5:346–415, 2008.
8. R. Kawai and A. Kohatsu-Higa. Computation of Greeks and multidimensional density estimation for asset price models with time-changed Brownian motion. *Applied Mathematical Finance*, 17(4):301–321, 2010.
9. R. Kawai and A. Takeuchi. Greeks formulas for an asset price model with gamma processes. *Math. Finance*, 21(4):723–742, 2011.
10. R. Kawai and A. Takeuchi. Computation of Greeks for asset price dynamics driven by stable and tempered stable processes. *Quantitative Finance*, 13(8):1303–1316, 2013.
11. D.B. Madan, P.P. Carr, and E.C. Chang. The variance gamma process and option pricing. *European Finance Review*, 2:79–105, 1998.
12. D.B. Madan and E. Seneta. The variance gamma model for share market returns. *J. Business*, 63:511–524, 1990.
13. R. Merton. Option pricing when underlying stock returns are discontinuous. *J. of Financial Economics*, 3, 1976.
14. N. Privault and D.C. Yang. Infinite divisibility of interpolated gamma powers. *J. Math. Anal. Appl.*, 405(2):373–387, 2013.