Fast computation of risk measures for variable annuities with additional earnings by conditional moment matching

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October 10, 2017

Abstract

We propose an approximation scheme for the computation of the risk measures of Guaranteed Minimum Maturity Benefits (GMMBs) and Guaranteed Minimum Death Benefits (GMDBs), based on the evaluation of single integrals under conditional moment matching. This procedure is computationally efficient in comparison with standard analytical methods while retaining a high degree of accuracy, and it allows one to deal with the case of additional earnings and the computation of related sensitivities.

Key words: Variable annuity guaranteed benefits; risk measures; value at risk; conditional tail expectation; conditional moment matching; additional earnings.
Mathematics Subject Classification (2010): 91B30, 97M30, 65C30.

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1 Introduction

Variable annuity benefits offered by insurance companies are usually protected via different mechanisms such as Guaranteed Minimum Maturity Benefits (GMMBs) or Guaranteed Minimum Death Benefits (GMDBs). The computation of the corresponding risk measures such as value at risk and conditional tail expectation is an important issue for the practitioners in risk management.

We work in the standard model in which the underlying equity value $(S_t)_{t\in\mathbb{R}_+}$ is modeled as a geometric Brownian motion

$$S_t = S_0 \mathrm{e}^{\mu t+\sigma B_t}, \qquad t \in \mathbb{R}_+, \tag{1.1}$$

with constant drift and volatility parameters μ and σ respectively, where $(B_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion.

Given an insurer continuously charging annualized mortality and expense fees at the rate m from the account of variable annuities, the fund value F_t of the variable annuity is defined as

$$F_t := F_0 \mathrm{e}^{-mt} \frac{S_t}{S_0} = F_0 \mathrm{e}^{(\mu-m)t+\sigma B_t}, \qquad t \in \mathbb{R}_+,$$

and the margin offset income M_t^x is given by

$$M_t^x := m_x F_t = m_x F_0 \mathrm{e}^{(\mu-m)t+\sigma B_t}, \qquad t \in \mathbb{R}_+, \tag{1.2}$$

where m_x is replaced by m_e in the GMMB model, and by m_d in the GMDB model.

The GMMB and GMDB riders provide minimum guarantees to protect the investment account of the policyholder. Namely, denoting by τ_x the future lifetime of a policyholder at the age x, the future payment made by the insurer is

$$(G - F_T)^+ \mathbb{1}_{\{\tau_x > T\}}$$

at maturity T for GMMBs, and

$$(\mathrm{e}^{\delta\tau_x} G - F_{\tau_x})^+ \mathbb{1}_{\{\tau_x \leqslant T\}}$$

at the time of death of the insured for GMDBs, where G is the guarantee level expressed as a fraction of the initial fund value F_0, δ is a roll-up rate according to which the guarantee increases up to the payment time.

Variable Annuities with embedded guarantees can be priced by the Monte-Carlo method or PDE discretization, however those methods are generally computationally demanding and a precise estimation of risk measures is difficult with classical Monte Carlo simulation or grid approximation, cf. e.g. [BKR08] for a general framework. In addition, a high level of precision up the 4th of 5th significant digit can be commonly required. On the other hand, faster computational methods based on analytical expressions have recently been introduced in [FV12], [FV14] for the computation of risk measures of GMDBs and GMMBs.

In this framework, the evaluation of quantile risk measures and conditional tail expectations of the net liabilities

$$L_0 := \mathrm{e}^{-rT}(G - F_T)^+ \mathbb{1}_{\{\tau_x > T\}} - \int_0^{T\wedge\tau_x} \mathrm{e}^{-rs} M_s^e \mathrm{d}s \tag{1.3}$$

of GMMBs relies on the knowledge of the probability density function of the time integral $\displaystyle\int_0^{T\wedge\tau_x} \mathrm{e}^{-rs} M_s^e \mathrm{d}s$ of the geometric Brownian motion (1.2). The joint probability density function of $\left(\int_0^T S_t dt, B_T + \mu T/\sigma\right)$ has been computed in [Yor92] as

$$\begin{aligned} &\mathbb{P}\left(\int_0^T \mathrm{e}^{\mu s+\sigma B_s} ds \in dy,\ B_T + \frac{\mu T}{2\sigma} \in dz\right) \\ &\qquad = \frac{\sigma}{2}\mathrm{e}^{\mu z/\sigma - \mu^2 T/2} \exp\left(-2\frac{1+\mathrm{e}^{\sigma z}}{\sigma^2 y}\right) \theta\left(\frac{4\mathrm{e}^{\sigma z/2}}{\sigma^2 y}, \frac{\sigma^2 T}{4}\right) \frac{dy}{y} dz, \end{aligned} \tag{1.4}$$

$y > 0$, $z \in \mathbb{R}$, where $\theta(v, \tau)$ denotes the function defined as

$$\theta(v,\tau) = \frac{v\mathrm{e}^{\pi^2/(2\tau)}}{\sqrt{2\pi^3\tau}} \int_0^\infty \mathrm{e}^{-\xi^2/(2\tau)} \mathrm{e}^{-v\cosh\xi} \sinh(\xi) \sin(\pi\xi/\tau)\, d\xi, \quad v, \tau > 0.$$

The marginal probability density of $\int_0^T S_t dt$, called the Hartman-Watson distribution, has been used in [FV12] for the evaluation of the risk measures of the net liabilities (1.3) by analytic methods. This approach results into double integral expressions for the cumulative distribution function of the time integral $\int_0^T S_t dt$ using Hartman-Watson densities and spectral expansions on the one hand, and on numerical Laplace transform inversion in relation with Asian option pricing, cf. [CS04], [Yor92]. It also allowed the authors to deal with the risk measures of the net liabilities

$$L_0' := \mathrm{e}^{-r\tau_x}(\mathrm{e}^{\delta\tau_x}G - F_{\tau_x})^+ \mathbb{1}_{\{\tau_x \leqslant T\}} - \int_0^{T\wedge\tau_x} \mathrm{e}^{-rs} M_s^d \mathrm{d}s$$

of GMDBs, also written in discrete time as

$$L_0^{(n)} := \mathrm{e}^{-r\kappa_x^{(n)}}\left(\mathrm{e}^{\delta\kappa_x^{(n)}}G - F_{\kappa_x^{(n)}}\right)^+ \mathbb{1}_{\{\kappa_x^{(n)} \leqslant T\}} - \int_0^{T\wedge\kappa_x^{(n)}} \mathrm{e}^{-rs} M_s^d \mathrm{d}s,$$

when n is large enough, where $\kappa_x^{(n)} := \frac{1}{n}\lceil n\tau_x \rceil$ and $\lceil a \rceil$ is the integer ceiling of $a \geqslant 0$.

More computationally efficient expressions for those risk measures have been presented in [FV14] based on identities in law for the geometric Brownian motion with affine drift

$$S_t + a\int_0^t \frac{S_t}{S_s} ds, \qquad t \in \mathbb{R}_+,$$

where $a > 0$. This approach allowed the authors to replace double integrals by single integrals of Whittaker functions, which significantly reduces computation times. These expressions are also subject to approximations by series instead of integrals, cf. Proposition 3.3 in [FV14], and they can be simplified to closed-form solutions using Green's functions, cf. Proposition 3.4 therein, further reducing computation times.

In this paper we propose to use moment matching for the computation of the risk measures of GMMBs and GMDBs. This allows us to derive single integral approximations which are significantly faster than the double integral expressions of [FV12], while approaching the performance of the single integral and series approximations of [FV14]. Moreover, we show that conditional moment matching can be applied to compute the risk measures of the GMDB and GMMB riders with Additional Earnings (AE), which cannot be treated via the approach of [FV14].

Moment matching in option pricing has been introduced for Asian options in [Lev92], [TW92] based on the lognormal approximation, and conditional moment matching has been used in [Cur94], [DLV04], [DDV10] for Asian and basket options. Here we apply the stratified approximation method of [PY16] to GMDBs and GMMBs, which also allows us to take into account additional earning features as it is based on conditioning with respect to the terminal value of geometric Brownian motion.

We proceed as follows. After recalling the considered model and the relevant risk measures in Sections 2 and 3, we present the conditional moment matching technique in Section 4. This technique is used for the approximations of value at risk and conditional tail expectation presented in Section 7 which presents numerical simulations that illustrate the improvement in speed of the proposed method, and an application to GMMBs and GMDBs with additional earnings. Section 6 is devoted to the computation of sensitivities of the value at risk and conditional tail expectation of GMMBs and GMDBs. The appendices Sections A and B contain the proofs of Propositions 5.1, 5.2 and 5.3, and additional computations for the sensitivities of Section 6.

2 GMMBs with additional earnings

In order to reduce incentives to lapse and reenter of the variable annuities, an Additional Earnings (AE) feature has been added to the basic riders, by increasing the benefit payout by a share ρ of the policyholder's variable annuities earnings, capped by the maximum additional payout C, cf. e.g. [MZ16] for details. Taking $\rho = 0$ recovers the plain GMMB and GMDB riders.

For a GMMB rider with AE feature, an extra payment

$$\min\left(C, \rho(F_T - G)^+\right)$$

will be paid to the GMMB policyholder in addition to the guaranteed benefit, thus the net liability (1.3) of the GMMB rider with AE feature becomes

$$L_0 := \left(\mathrm{e}^{-rT}(G - F_T)^+ + \mathrm{e}^{-rT}\min\left(C, \rho(F_T - G)^+\right)\right)\mathbb{1}_{\{\tau_x > T\}} - \int_0^{T\wedge\tau_x} \mathrm{e}^{-rs} M_s^e \mathrm{d}s.$$

Risk measures on the net liability L_0 can still be expressed in terms of Hartmann-Watson distributions and double integral expressions as in [FV12], using the joint distribution of $\left(S_T, \int_0^T S_t ds\right)$, cf. [Yor92]. However, the closed form expressions of [FV14] do not apply to this setting as they rely on the particular distributional properties of geometric Brownian motion with affine drift. For this reason, we propose to use conditional moment matching in order to deal with additional earnings while significantly improving computation speed in comparison with double integral expressions.

The conditional moment matching method applies more generally to the computation of risk measures for variable annuities whose guarantees depend on the fund value at maturity or at the time of death of the insured, i.e. with liabilities of the form

$$L_0 := f(F_\tau) - \int_0^\tau \mathrm{e}^{-rs} M_s^e \mathrm{d}s,$$

where τ is the maturity time or the death time of the insured, whichever comes first, and F_τ is the stochastic resource of the guarantee benefit function $f(\cdot)$. Such examples include the guaranteed minimum income benefits (GMIB) besides the GMMB and GMDB discussed in this paper. However, they do not include guaranteed minimum withdrawal benefits (GMWBs) whose guaranteed benefit functionals depend on the fund values until maturity.

As negative liabilities will not be considered in this paper, we restrict the risk tolerance level α to be greater than the probability ξ_m of non-positive liability, which is defined for GMMBs as

$$\xi_m := \mathbb{P}(L_0 \leqslant 0) = 1 - {}_T p_x \mathbb{P}(L_0 > 0 \mid \tau_x > T) = 1 - {}_T p_x P_\rho(T, G, 0),$$

where ${}_T p_x$ is the probability that a policyholder at age x will survive T units of time, $x, T > 0$, and for $w \geqslant 0$, the key quantity $P_\rho(T, G, w)$ is defined as

$$P_\rho(T, G, w) := \mathbb{P}\left(\mathrm{e}^{-rT}(G - F_T)^+ + \mathrm{e}^{-rT} \min\left(C, \rho(F_T - G)^+\right) - \int_0^T \mathrm{e}^{-rs} M_s^e \mathrm{d}s > w\right). \tag{2.1}$$

In the absence of additional earnings we will use

$$P_0(T, G, w) := \mathbb{P}\left(\mathrm{e}^{-rT}(G - F_T)^+ - \int_0^T \mathrm{e}^{-rs} M_s^e \mathrm{d}s > w\right),$$

cf. Proposition 3.3 of [FV12].

Value at Risk for GMMBs

The Value at Risk (VaR)

$$V_\alpha(L_0) := \inf\left\{y \ : \ \mathbb{P}(L_0 \leqslant y) \geqslant \alpha\right\}$$

with risk tolerance level $\alpha > \xi_m$ for the net liability L_0 of GMMB is determined implicitly from the relation

$$1 - \alpha = {}_T p_x P_\rho(T, G, V_\alpha(L_0)). \tag{2.2}$$

Conditional Tail Expectation for GMMBs

The Conditional Tail Expectation (CTE)

$$\mathrm{CTE}_\alpha(L_0) := \mathbb{E}[L_0 \mid L_0 > V_\alpha(L_0)]$$

at the level of risk tolerance level $\alpha > \xi_m$ for the net liability L_0 of the GMMB with AE feature is given by

$$\mathrm{CTE}_\alpha(L_0) = \frac{{}_T p_x}{1-\alpha} Z_\rho(T, G, V_\alpha(L_0)), \tag{2.3}$$

where

$$Z_\rho(T, G, w) := \mathbb{E}\left[\left(\mathrm{e}^{-rT}(G - F_T)^+ + \mathrm{e}^{-rT}\min(C, \rho(F_T - G)^+) - \int_0^T \mathrm{e}^{-rs} M_s^e \mathrm{d}s\right) \mathbb{1}_{A_T(w,G)}\right], \tag{2.4}$$

$w, T \geqslant 0$, and $\mathbb{1}_{A_T(w,G)}$ is the indicator function of the event

$$A_T(w, G) := \left\{\mathrm{e}^{-rT}(G - F_T)^+ + \mathrm{e}^{-rT}\min(C, \rho(F_T - G)^+) - \int_0^T \mathrm{e}^{-rs} M_s^e \mathrm{d}s > w\right\}.$$

In the absence of additional earnings we will use

$$Z_0(T, G, w) = \mathbb{E}\left[\left(\mathrm{e}^{-rT}(G - F_T)^+ - \int_0^T \mathrm{e}^{-rs} M_s^e \mathrm{d}s\right) \mathbb{1}_{\left\{\mathrm{e}^{-rT}(G-F_T)^+ - \int_0^T \mathrm{e}^{-rs} M_s^e \mathrm{d}s > w\right\}}\right],$$

cf. Proposition 3.4 of [FV12].

3 GMDBs with additional earnings

In the case of GMDBs the extra payment is

$$\min\left(C, \rho(F_{\tau_x} - G\mathrm{e}^{\delta\tau_x})^+\right)$$

and the net liability of the GMDB rider with AE feature becomes

$$L_0' := \mathrm{e}^{-r\tau_x}\left(\left(\mathrm{e}^{\delta\tau_x}G - F_{\tau_x}\right)^+ + \min\left(C, \rho(F_{\tau_x} - G\mathrm{e}^{\delta\tau_x})^+\right)\right)\mathbb{1}_{\{\tau_x \leqslant T\}} - \int_0^{T\wedge\tau_x} \mathrm{e}^{-rs} M_s^d \mathrm{d}s.$$

If the benefits of GMDBs with AE feature are payable on a discrete-time basis, their net liability is

$$\begin{aligned} L_0^{(n)} : &= \mathrm{e}^{-r\kappa_x^{(n)}}\left(\left(\mathrm{e}^{\delta\kappa_x^{(n)}}G - F_{\kappa_x^{(n)}}\right)^+ + \min\left(\rho\left(F_{\kappa_x^{(n)}} - G\mathrm{e}^{\delta\kappa_x^{(n)}}\right)^+, C\right)\right)\mathbb{1}_{\{\kappa_x^{(n)} \leqslant T\}} \\ &\quad - \int_0^{T\wedge\kappa_x^{(n)}} \mathrm{e}^{-rs} M_s^d \mathrm{d}s. \end{aligned}$$

The probability of non-positive liability for GMDB riders with AE feature is given by

$$\begin{aligned} \xi_d &:= \mathbb{P}(L_0^{(n)} \leqslant 0) \\ &= 1 - \sum_{k=1}^{\lceil nT\rceil} \mathbb{P}\big(\kappa_x^{(n)} = k/n\big)\mathbb{P}\big(L_0^{(n)} > 0 \mid \kappa_x^{(n)} = k/n\big) \\ &= 1 - \sum_{k=1}^{\lceil nT\rceil} {}_{(k-1)/n}p_x \; {}_{1/n}q_{x+(k-1)/n} P_\rho(k/n, \mathrm{e}^{\delta k/n}G, 0), \end{aligned}$$

where $P_\rho(k/n, \mathrm{e}^{\delta k/n}G, w)$ is defined in (2.1), and ${}_{1/n}q_{x+(k-1)/n}$ is the probability that a policyholder at age of $x + (k-1)/n$ will die in $1/n$ periods.

Value at Risk for GMDBs

The value at risk $V_\alpha\big(L_0^{(n)}\big)$ with $\alpha > \xi_d$ for the net liability of the GMDB is similarly given implicitly from the relation

$$1 - \alpha = \sum_{k=1}^{\lceil nT\rceil} {}_{(k-1)/n}p_x \; {}_{1/n}q_{x+(k-1)/n} P_\rho\big(k/n, \mathrm{e}^{\delta k/n}G, V_\alpha\big(L_0^{(n)}\big)\big), \tag{3.1}$$

cf. e.g. Proposition 3.9 of [FV12] when $\rho = 0$.

The computation of $P_\rho(T, G, w)$ for any $T > 0$ and $w \in \mathbb{R}$ is essential in order to estimate the risk measures $V_\alpha(L_0)$ and $V_\alpha\big(L_0^{(n)}\big)$.

Conditional Tail Expectation for GMDBs

The conditional tail expectation

$$\mathrm{CTE}_\alpha\big(L_0^{(n)}\big) := \mathbb{E}\left[L_0^{(n)} \mid L_0^{(n)} > V_\alpha\big(L_0^{(n)}\big)\right]$$

with risk tolerance level $\alpha > \xi_d$ for the net liability $L_0^{(n)}$ of the GMDB with AE feature is given by

$$\mathrm{CTE}_\alpha\big(L_0^{(n)}\big) = \frac{1}{1-\alpha} \sum_{k=1}^{\lceil nT \rceil} Z_\rho\big(k/n, G\mathrm{e}^{k\delta/n}, V_\alpha(L_0^{(n)})\big)\mathbb{P}\big(\kappa_x^{(n)} = k/n\big), \tag{3.2}$$

where $Z_\rho\big(k/n, \mathrm{e}^{k\delta/n}G, V_\alpha(L_0^{(n)})\big)$ is defined by (2.4) for any $k, n \geqslant 0$.

4 Conditional moment matching

In this section we propose a conditional moment matching approximation for the estimation of the key quantities $P_\rho(T, G, w)$ and $Z_\rho(T, G, w)$ by approaching the probability density function of the time integral

$$\Lambda_T := \int_0^T \tilde{S}_t \mathrm{d}t = \frac{1}{F_0 m_x} \int_0^T \mathrm{e}^{-rt} M_t^x \mathrm{d}t$$

where $\tilde{S}_t := \mathrm{e}^{(\mu - m - r)t + \sigma B_t}$, $t \in \mathbb{R}_+$, using a gamma or lognormal distribution, conditionally to the terminal value $\tilde{S}_T = z$, as in [PY16].

The basic idea of the lognormal approximation is that, since Λ_T is the time integral of lognormal random variables, it is natural to try approximating it using a lognormal distribution. The gamma approximation provides a possible alternative to the lognormal approximation which is motivated by the similarities between the gamma and lognormal densities.

Conditional gamma approximation

Under the gamma approximation we have

$$f_{\Lambda_T \mid \tilde{S}_T = z}(x; \theta_T^z, \nu_T^z) \approx \frac{1}{(\theta_T^z)^{\nu_T^z}} \frac{x^{-1+\nu_T^z}}{\Gamma_{\nu_T^z}} \mathrm{e}^{-x/\theta_T^z}, \qquad x > 0, \tag{4.1}$$

where

$$\Gamma_v := \int_0^\infty y^{v-1} \mathrm{e}^{-y} \mathrm{d}y, \qquad v > 0,$$

is the gamma function, and θ_T^z, ν_T^z are estimated respectively as

$$\theta_T^z := \frac{2}{\sigma^2} \left(\frac{b_T^z}{a_T^z} - 1 - z \right) - a_T^z, \qquad \nu_T^z := \frac{a_T^z}{\theta_T^z},$$

by matching the first and second conditional moments of Λ_T given $\tilde{S}_T = z$ to those of a gamma distribution, where

$$\begin{cases} a_T^z := \dfrac{1}{\sigma^2 p_T^z} \left(\Phi \left(\dfrac{\log z}{\sqrt{\sigma^2 T}} + \dfrac{1}{2} \sqrt{\sigma^2 T} \right) - \Phi \left(\dfrac{\log z}{\sqrt{\sigma^2 T}} - \dfrac{1}{2} \sqrt{\sigma^2 T} \right) \right), \\ b_T^z := \dfrac{1}{\sigma^2 q_T^z} \left(\Phi \left(\dfrac{\log z}{\sqrt{\sigma^2 T}} + \sqrt{\sigma^2 T} \right) - \Phi \left(\dfrac{\log z}{\sqrt{\sigma^2 T}} - \sqrt{\sigma^2 T} \right) \right), \end{cases}$$

and

$$p_T^z := \frac{1}{\sqrt{2\pi\sigma^2 T}} \mathrm{e}^{-(\sigma^2 T/2 + \log z)^2/(2\sigma^2 T)}, \qquad q_T^z := \frac{1}{\sqrt{2\pi\sigma^2 T}} \mathrm{e}^{-(\sigma^2 T + \log z)^2/(2\sigma^2 T)},$$

cf. Proposition 3.1 of [PY16].

Conditional lognormal approximation

Here we approximate the conditional probability density of Λ_T given $\tilde{S}_T = z$ by the lognormal density function with parameters $(-\mu_T^z (\sigma_T^z)^2 T/2, (\sigma_T^z)^2 T)$ as

$$f_{\Lambda_T | \tilde{S}_T = z}(x; \mu_T^z, (\sigma_T^z)^2) \approx \frac{1}{x \sigma_T^z \sqrt{2\pi T}} \mathrm{e}^{-(\mu_T^z (\sigma_T^z)^2 T/2 + \log x)^2/(2(\sigma_T^z)^2 T)}, \tag{4.2}$$

where μ_T^z and σ_T^z are also derived by conditional moment matching by taking

$$(\sigma_T^z)^2 := \frac{1}{T} \log \left(\frac{2}{\sigma^2 a_T^z} \left(\frac{b_T^z}{a_T^z} - 1 - z \right) \right) \quad \text{and} \quad \mu_T^z := 1 - \frac{2}{(\sigma_T^z)^2 T} \log a_T^z,$$

cf. Proposition 3.2 of [PY16].

The next Figure 1, plotted with the parameters $S_0 = 4\%$, $\mu - m - r = 0$, and $\sigma = 30\%$, compares the gamma and lognormal density approximations (4.1) and (4.2) to the integral density expression (1.4) of Λ_T. It shows in particular that the lognormal conditional approximation tends to provide a better match of density than the gamma approximation, which can naturally be expected as S_t itself is lognormally distributed.

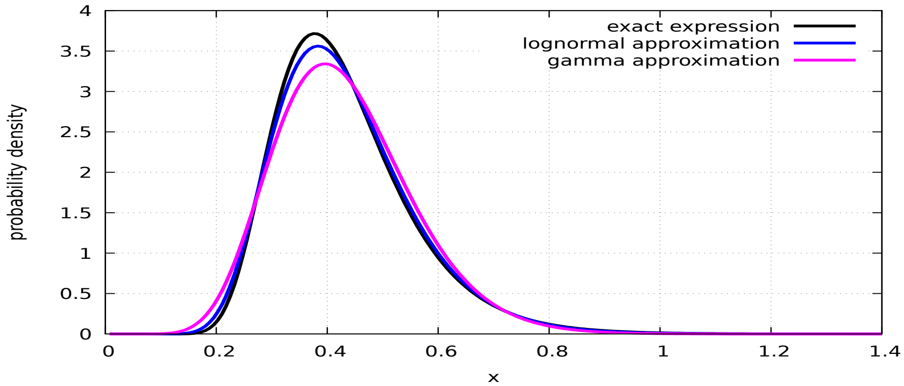

Figure 1: Lognormal *vs* gamma density approximations.

5 Conditional approximations of VaR and CTE

Conditional gamma approximation

Using the gamma approximation (4.1) we will evaluate the key quantities $P_\rho(T, G, w)$ in (2.2) and $Z_\rho(T, G, w)$ in (2.3) by single numerical integrations in Propositions 5.1 and 5.2, which will significantly reduce the computation time of the VaR and CTE of GMMBs and GMDBs with and without AE features.

Proposition 5.1 *Under the conditional gamma approximation, the key quantity $P_\rho(T, G, w)$ in the calculation (2.2) of VaR can be estimated by the single integrals*

$$P_\rho(T,G,w) \approx \int_0^{\frac{\mathrm{e}^{-rT}G-w}{F_0}} \Gamma_{\nu_T^z}\left(\frac{\mathrm{e}^{-rT}G - w - zF_0}{F_0\theta_T^z m_x}\right) f_{\tilde{S}_T}(z)dz \tag{5.1}$$

$$+ \int_{\frac{\rho\mathrm{e}^{-rT}G+w}{\rho F_0}}^{\frac{\mathrm{e}^{-rT}}{\rho F_0}(\rho G+C)} \Gamma_{\nu_T^z}\left(\frac{\rho z F_0 - \mathrm{e}^{-rT}\rho G - w}{F_0\theta_T^z m_x}\right) f_{\tilde{S}_T}(z)dz + \int_{\frac{\mathrm{e}^{-rT}}{\rho F_0}(\rho G+C)}^{\infty} \Gamma_{\nu_T^z}\left(\frac{\mathrm{e}^{-rT}C - w}{F_0\theta_T^z m_x}\right) f_{\tilde{S}_T}(z)dz, \tag{5.2}$$

where

$$f_{\tilde{S}_T}(x) := \frac{1}{x\sigma\sqrt{2\pi T}}\mathrm{e}^{-(-(\mu-m-r)T+\log x)^2/(2\sigma^2 T)}, \qquad x > 0,$$

is the lognormal probability density function of $\tilde{S}_T$, and

$$\Gamma_v(y) := \frac{1}{\Gamma_v}\int_0^y t^{v-1}\mathrm{e}^{-t}\mathrm{d}t, \quad y > 0,$$

is the normalized lower incomplete gamma function.

Proposition 5.1 is proved in the Appendix Section A. Without additional earnings, we replace (5.1) with the approximation

$$P_0(T,G,w) \approx \int_0^{\frac{e^{-rT}G-w}{F_0}} \Gamma_{\nu_T^z}\left(\frac{e^{-rT}G-w-zF_0}{F_0\theta_T^z m_x}\right) f_{\tilde{S}_T}(z)dz.$$

Proposition 5.2 *Under the conditional gamma approximation, the key quantity $Z_\rho(T,G,w)$ in the CTE formula (2.3) can be estimated by the single integrals*

$$\begin{aligned}
&Z_\rho(T,G,w) \approx \qquad (5.3)\\
&F_0\int_0^{\frac{e^{-rT}G-w}{F_0}}\left(\left(\frac{e^{-rT}G}{F_0}-z\right)\Gamma_{\nu_T^z}\left(\frac{e^{-rT}G\text{-}w\text{-}zF_0}{F_0\theta_T^z m_x}\right)-m_x\theta_T^z\nu_T^z\Gamma_{\nu_T^z+1}\left(\frac{e^{-rT}G\text{-}w\text{-}zF_0}{F_0\theta_T^z m_x}\right)\right)f_{\tilde{S}_T}(z)dz\\
&+F_0\int_{\frac{\rho e^{-rT}G+w}{\rho F_0}}^{\frac{e^{-rT}}{\rho F_0}(\rho G+C)}\left(\rho\left(z\text{-}\frac{e^{-rT}G}{F_0}\right)\Gamma_{\nu_T^z}\left(\frac{\rho(z\text{-}\frac{e^{-rT}G}{F_0})\text{-}\frac{w}{F_0}}{\theta_T^z m_x}\right)-m_x\theta_T^z\nu_T^z\Gamma_{\nu_T^z+1}\left(\frac{\rho(z\text{-}\frac{e^{-rT}G}{F_0})\text{-}\frac{w}{F_0}}{\theta_T^z m_x}\right)\right)f_{\tilde{S}_T}(z)dz\\
&+e^{-rT}C\int_{\frac{e^{-rT}}{\rho F_0}(\rho G+C)}^{\infty}\Gamma_{\nu_T^z}\left(\frac{e^{-rT}C\text{-}w}{\theta_T^z m_x F_0}\right)f_{\tilde{S}_T}(z)dz-F_0m_x\int_{\frac{e^{-rT}}{\rho F_0}(\rho G+C)}^{w}\theta_T^z\nu_T^z\Gamma_{\nu_T^z+1}\left(\frac{e^{-rT}C\text{-}w}{\theta_T^z m_x F_0}\right)f_{\tilde{S}_T}(z)dz.
\end{aligned}$$

Proposition 5.2 is proved in the Appendix Section A. In the absence of additional earnings, i.e. when $\rho = 0$, we replace (5.3) with the approximation

$$\begin{aligned}
&Z_0(T,G,w) \approx\\
&F_0\int_0^{\frac{e^{-rT}G-w}{F_0}}\left(\left(\frac{e^{-rT}G}{F_0}-z\right)\Gamma_{\nu_T^z}\left(\frac{e^{-rT}G\text{-}w\text{-}zF_0}{F_0\theta_T^z m_x}\right)-m_x\theta_T^z\nu_T^z\Gamma_{\nu_T^z+1}\left(\frac{e^{-rT}G\text{-}w\text{-}zF_0}{F_0\theta_T^z m_x}\right)\right)f_{\tilde{S}_T}(z)dz.
\end{aligned}$$

Conditional lognormal approximation

In Proposition 5.3 we use the lognormal approximation (4.2) to evaluate the key quantity $P_\rho(T,G,w)$ used in the compuation (2.2) of VaR, by single numerical integrations.

Proposition 5.3 *Under the conditional lognormal approximation the key quantity $P_\rho(T,G,w)$ in the calculation (2.2) of VaR can be estimated by the single integrals*

$$\begin{aligned}
P_\rho(T,G,w) &\approx \int_0^{\frac{e^{-rT}G-w}{F_0}}\Phi\left(\frac{\mu_T^z\frac{(\sigma_T^z)^2T}{2}+\log\frac{e^{-rT}G-w-zF_0}{F_0m_x}}{\sigma_T^z\sqrt{T}}\right)f_{\tilde{S}_T}(z)dz \qquad (5.4)\\
&+\int_{\frac{\rho e^{-rT}G+w}{\rho F_0}}^{\frac{e^{-rT}}{\rho F_0}(\rho G+C)}\Phi\left(\frac{\mu_T^z\frac{(\sigma_T^z)^2T}{2}+\log\frac{\rho zF_0-e^{-rT}\rho G-w}{F_0m_x}}{\sigma_T^z\sqrt{T}}\right)f_{\tilde{S}_T}(z)dz \qquad (5.5)
\end{aligned}$$

$$+\int_{\frac{e^{-rT}}{\rho F_0}(\rho G+C)}^{\infty}\Phi\left(\frac{\mu_T^z\frac{(\sigma_T^z)^2T}{2}+\log\frac{e^{-rT}C-w}{F_0m_x}}{\sigma_T^z\sqrt{T}}\right)f_{\tilde{S}_T}(z)dz. \tag{5.6}$$

Proposition 5.3 is proved in the Appendix Section A. Without additional earnings we will use the approximation

$$P_0(T,G,w)\approx\int_0^{\frac{e^{-rT}G-w}{F_0}}\Phi\left(\frac{\mu_T^z\frac{(\sigma_T^z)^2T}{2}+\log\frac{e^{-rT}G-w-zF_0}{F_0m_x}}{\sigma_T^z\sqrt{T}}\right)f_{\tilde{S}_T}(z)dz.$$

Similarly, applying (A.10) and the approximation

$$\begin{aligned}\int_0^{\eta}yf_{\Lambda_T|\tilde{S}_T=z}(y)\mathrm{d}y &\approx \frac{1}{\sigma_T^z\sqrt{2\pi T}}\int_0^{\log\eta}e^{y-\frac{(\mu_T^z(\sigma_T^z)^2T/2+y)^2}{2(\sigma_T^z)^2T}}\mathrm{d}y\\ &= e^{(1-\mu_T^z)(\sigma_T^z)^2T/2}\Phi\left(\frac{(\mu_T^z-2)\frac{(\sigma_T^z)^2T}{2}+\log\eta}{\sigma_T^z\sqrt{T}}\right),\qquad \eta>0,\end{aligned}$$

to (A.7), (A.8) and (A.9), we get the following approximation result of the key quantity $Z_\rho(T,G,w)$ appearing in the CTE expression (2.3).

Proposition 5.4 *Under the conditional lognormal approximation, the key quantity $Z_\rho(T,G,w)$ in the CTE formula (2.3) can be estimated by the single integrals*

$$\begin{aligned}&Z_\rho(T,G,w)\\ &\approx\int_0^{\frac{e^{-rT}G-w}{F_0}}\left(e^{-rT}G-F_0z\right)\Phi\left(\frac{\mu_T^z\frac{(\sigma_T^z)^2T}{2}+\log\frac{e^{-rT}G-w-zF_0}{F_0m_x}}{\sigma_T^z\sqrt{T}}\right)f_{\tilde{S}_T}(z)dz\\ &\quad-F_0m_x\int_0^{\frac{e^{-rT}G-w}{F_0}}e^{(1-\mu_T^z)(\sigma_T^z)^2T/2}\Phi\left(\frac{(\mu_T^z-2)\frac{(\sigma_T^z)^2T}{2}+\log\frac{e^{-rT}G-w-zF_0}{F_0m_x}}{\sigma_T^z\sqrt{T}}\right)f_{\tilde{S}_T}(z)dz\\ &\quad+\rho\int_{\frac{e^{-rT}G}{F_0}+\frac{w}{\rho F_0}}^{\frac{e^{-rT}}{\rho F_0}(\rho G+C)}\left(F_0z-e^{-rT}G\right)\Phi\left(\frac{\mu_T^z\frac{(\sigma_T^z)^2T}{2}+\log\frac{\rho zF_0-e^{-rT}\rho G-w}{m_xF_0}}{\sigma_T^z\sqrt{T}}\right)f_{\tilde{S}_T}(z)dz\\ &\quad-F_0m_x\int_{\frac{\rho e^{-rT}G+w}{\rho F_0}}^{\frac{e^{-rT}}{\rho F_0}(\rho G+C)}e^{(1-\mu_T^z)(\sigma_T^z)^2T/2}\Phi\left(\frac{(\mu_T^z-2)\frac{(\sigma_T^z)^2T}{2}+\log\frac{\rho zF_0-e^{-rT}\rho G-w}{m_xF_0}}{\sigma_T^z\sqrt{T}}\right)f_{\tilde{S}_T}(z)dz\\ &\quad+e^{-rT}C\int_{\frac{e^{-rT}}{\rho F_0}(\rho G+C)}^{\infty}\Phi\left(\frac{(\mu_T^z-2)\frac{(\sigma_T^z)^2T}{2}+\log\frac{e^{-rT}C-w}{m_xF_0}}{\sigma_T^z\sqrt{T}}\right)f_{\tilde{S}_T}(z)dz\\ &\quad-F_0m_x\int_{\frac{e^{-rT}}{\rho F_0}(\rho G+C)}^{w}e^{(1-\mu_T^z)(\sigma_T^z)^2T/2}\Phi\left(\frac{(\mu_T^z-2)\frac{(\sigma_T^z)^2T}{2}+\log\frac{e^{-rT}C-w}{m_xF_0}}{\sigma_T^z\sqrt{T}}\right)f_{\tilde{S}_T}(z)dz.\end{aligned}$$

Proposition 5.4 is proved in the Appendix Section A. In the absence of additional earnings we will use the approximation

$$
\begin{aligned}
Z_0(T,G,w) \approx & \int_0^{\frac{e^{-rT}G-w}{F_0}} \left(e^{-rT}G - F_0 z\right) \Phi\left(\frac{\mu_T^z\frac{(\sigma_T^z)^2 T}{2} + \log\frac{e^{-rT}G-w-zF_0}{F_0 m_x}}{\sigma_T^z\sqrt{T}}\right) f_{\tilde{S}_T}(z)dz \\
& - F_0 m_x \int_0^{\frac{e^{-rT}G-w}{F_0}} e^{(1-\mu_T^z)(\sigma_T^z)^2 T/2}\Phi\left(\frac{(\mu_T^z-2)\frac{(\sigma_T^z)^2 T}{2} + \log\frac{e^{-rT}G-w-zF_0}{F_0 m_x}}{\sigma_T^z\sqrt{T}}\right) f_{\tilde{S}_T}(z)dz.
\end{aligned}
$$

6 Calculation of sensitivities

In this section we show that the lognormal and gamma approximations can be used for the approximation of sensitivities with respect to the parameters μ, σ, m_x, and r. Such formulas provide more stable alternatives to the use of finite difference approximations.

Sensitivity analysis for GMMBs

The sensitivity of the VaR of GMMBs with respect to μ can then be estimated by differentiating equation (2.2) as

$$
\frac{\partial}{\partial\mu}V_\alpha(L_0) = -\left(\frac{\partial P_\rho}{\partial w}(T,G,V_\alpha(L_0))\right)^{-1}\frac{\partial}{\partial\mu}P_\rho(T,G,w)_{|w=V_\alpha(L_0)}. \tag{6.1}
$$

As for the sensitivity of the CTE of GMMBs with respect to μ, it can be similarly estimated as

$$
\begin{aligned}
\frac{\partial}{\partial\mu}\mathrm{CTE}_\alpha(L_0) &= \frac{{}_Tp_x}{1-\alpha}\left(\frac{\partial}{\partial\mu}Z_\rho(T,G,w)_{|w=V_\alpha(L_0)} + \frac{\partial Z_\rho}{\partial w}(T,G,V_\alpha(L_0))\frac{\partial}{\partial\mu}V_\alpha(L_0)\right) \\
&= \frac{{}_Tp_x}{1-\alpha}\frac{\partial}{\partial\mu}Z_\rho(T,G,w)_{|w=V_\alpha(L_0)} \\
&\quad -\frac{{}_Tp_x}{1-\alpha}\frac{\partial Z_\rho}{\partial w}(T,G,V_\alpha(L_0))\left(\frac{\partial P_\rho}{\partial w}(T,G,V_\alpha(L_0))\right)^{-1}\frac{\partial}{\partial\mu}P_\rho(T,G,w)_{|w=V_\alpha(L_0)}.
\end{aligned} \tag{6.2}
$$

Sensitivity analysis for GMDBs

The sensitivity of the VaR of GMDBs can be estimated by differentiating the equation (3.1), as

$$
\frac{\partial}{\partial\mu}V_\alpha(L_0^{(n)}) = -\left(\sum_{k=1}^{\lceil nT\rceil} {}_{(k-1)/n}p_x \; {}_{1/n}q_{x+(k-1)/n}\frac{\partial P_\rho}{\partial w}\big(k/n, e^{\delta k/n}G, V_\alpha\big(L_0^{(n)}\big)\big)\right)^{-1}
$$

$$\times \sum_{k=1}^{\lceil nT\rceil} {}_{(k-1)/n}p_x \ {}_{1/n}q_{x+(k-1)/n} \frac{\partial}{\partial\mu} P_\rho\big(k/n, \mathrm{e}^{\delta k/n}G, w\big)_{|w=V_\alpha\left(L_0^{(n)}\right)},$$

and the sensitivity of their CTEs can be derived from (3.2) as

$$\frac{\partial}{\partial\mu}\mathrm{CTE}_\alpha\big(L_0^{(n)}\big) = \frac{1}{1-\alpha}\sum_{k=1}^{\lceil nT\rceil}\frac{\partial}{\partial\mu}Z_\rho\big(k/n, G\mathrm{e}^{k\delta/n}, w\big)_{|w=V_\alpha(L_0^{(n)})}\mathbb{P}\big(\kappa_x^{(n)} = k/n\big)$$
$$+\frac{1}{1-\alpha}\sum_{k=1}^{\lceil nT\rceil}\frac{\partial Z_\rho}{\partial w}\big(k/n, G\mathrm{e}^{k\delta/n}, V_\alpha(L_0^{(n)})\big)\frac{\partial}{\partial\mu}V_\alpha(L_0^{(n)})\mathbb{P}\big(\kappa_x^{(n)} = k/n\big).$$

In order to estimate $\frac{\partial}{\partial\mu}P_\rho(T,G,w)$ and $\frac{\partial}{\partial\mu}Z_\rho(T,G,w)$, it suffices to replace the lognormal density $f_{\tilde S_T}(x)$ in Propositions 5.1, 5.2, 5.3 and 5.4 with its derivative with respect to μ, i.e.

$$\frac{1}{x\sigma^3\sqrt{2\pi T}}(-(\mu-m-r)T+\log x)\,\mathrm{e}^{-(-(\mu-m-r)T+\log x)^2/(2\sigma^2T)}, \qquad x>0. \tag{6.3}$$

We refer to the Appendix Section B for the estimation of $\frac{\partial P_\rho}{\partial w}(T,G,w)$ and $\frac{\partial Z_\rho}{\partial w}(T,G,w)$ under the conditional gamma approximation in Propositions 5.1 and 5.2.

The sensitivities with respect to σ and r can be similarly computed as the sensitivity with respect to μ, while the sensitivity with respect to m_x requires to differentiate the incomplete Gamma function or the Gaussian cumulative distribution function. In the absence of additional earnings, by differentiating (5.1) we find, in the conditional gamma approximation,

$$\frac{\partial}{\partial m_x}P_0(T,G,w) \approx$$
$$-\int_0^{\frac{\mathrm{e}^{-rT}G-w}{F_0}}\frac{\mathrm{e}^{-rT}G-w-zF_0}{\Gamma_{\nu_T^z}F_0\theta_T^z m_x^2}\left(\frac{\mathrm{e}^{-rT}G-w-zF_0}{F_0\theta_T^z m_x}\right)^{\nu_T^z-1}\exp\left(-\frac{\mathrm{e}^{-rT}G-w-zF_0}{F_0\theta_T^z m_x}\right)f_{\tilde S_T}(z)dz,$$

and, under the conditional lognormal approximation,

$$\frac{\partial}{\partial m_x}P_0(T,G,w) \approx$$
$$-\frac{1}{m_x\sigma_T^z\sqrt{2\pi T}}\int_0^{\frac{\mathrm{e}^{-rT}G-w}{F_0}}\exp\left(-\frac{1}{2}\left(\frac{\mu_T^z\frac{(\sigma_T^z)^2T}{2}+\log\frac{\mathrm{e}^{-rT}G-w-zF_0}{F_0m_x}}{\sigma_T^z\sqrt{T}}\right)^2\right)f_{\tilde S_T}(z)dz.$$

The derivatives $\frac{\partial}{\partial m_x}P_\rho(T,G,w)$ and $\frac{\partial}{\partial m_x}Z_\rho(T,G,w)$ with respect to m_x can be similarly computed in the case of additional earnings from Propositions 5.1, 5.2 5.3 and 5.4 as above.

7 Numerical examples

In this section we illustrate the efficiency of the stratified approximation method introduced in the previous sections. In order to compare the accuracy and computation time of the stratified approximation with that of the existing methods, we use the same model and products as in [FV12]. For GMMBs, the underlying asset of the variable annuities is assumed to follow (1.1) with $r = 4\%$, $\mu = 9\%$, and $\sigma = 30\%$. The variable annuities with GMMB and GMDB riders are designed for policyholders of age 65 with the product parameters $T = 10$, $m = 1\%$, and $m_e = 0.35\%$. The future life time table is the published by US Social Security Administration (Bell and Miller, 2005) in 2005, cf. Table 1 in [FV12]. The initial account value is set to be $F_0 = 100$, the guarantee level G and the risk measures VaR and CTE are represented in percentages of initial account value.

$G/F_0 = 75\%$	[FV12]†	[FV14]‡	lognormal	gamma
$V_{80\%}/F_0$	0^*	0^*	0^*	0^*
$V_{90\%}/F_0$	0^*	0^*	0^*	0^*
$V_{95\%}/F_0$	12.177731	12.177734	12.177230	12.177232
$\text{CTE}_{80\%}/F_0$	6.911066*	6.911064*	6.911050*	6.911062*
$\text{CTE}_{90\%}/F_0$	13.822132*	13.822127*	13.822099*	13.822124*
$\text{CTE}_{95\%}/F_0$	23.283511	23.283517	23.283757	23.283801

$G/F_0 = 100\%$	[FV12]†	[FV14]‡	lognormal	gamma
$V_{80\%}/F_0$	0^*	0^*	0^*	0^*
$V_{90\%}/F_0$	12.550369	12.550367	12.550349	12.550352
$V_{95\%}/F_0$	28.935733	28.935735	28.935231	28.935233
$\text{CTE}_{80\%}/F_0$	16.208562*	16.429038*	16.429031*	16.429049*
$\text{CTE}_{90\%}/F_0$	30.296490	30.296486	30.296445	30.296471
$\text{CTE}_{95\%}/F_0$	40.041515	40.041519	40.041758	40.041802

$G/F_0 = 120\%$	[FV12]†	[FV14]‡	lognormal	gamma
$V_{80\%}/F_0$	0^*	0^*	0^*	0^*
$V_{90\%}/F_0$	25.956765	25.956768	25.956747	25.956752
$V_{95\%}/F_0$	42.342135	42.342136	42.341631	42.341633
$\text{CTE}_{80\%}/F_0$	27.545146*	27.333610*	27.333606*	27.333617*
$\text{CTE}_{90\%}/F_0$	43.702883	43.702887	43.702841	43.702872
$\text{CTE}_{95\%}/F_0$	53.447918	53.447919	53.448157	53.448202

Table 1: Risk measure estimates in % for the GMMB rider with different levels of risk tolerance α.

Table 1 presents the computation of VaR and CTE for the GMMB rider with different of risk tolerance levels α, by the conditional lognormal and gamma approximations of Propositions 5.1 and 5.3. We note that the stratified lognormal and gamma approximations yield the same results up to 4 decimal places, and that they agree with the results of [FV12] and [FV14].

The algorithms are implemented in C++ with the PNL Library for special functions and numerical integration routines, while the original implementations of [FV12] and [FV14] for the inverse Laplace and Green function methods are using Maple. We applied the Newton-Raphson method with precision of 5 decimal places for the root search procedure to solve equations (2.2) and (3.1) for the computation of VaR for GMMBs and GMDBs. The conditional tail expectations of net liabilities $\mathrm{CTE}_\alpha(L_0)$ for GMMBs and $\mathrm{CTE}_\alpha(L^{(n)})$ for GMDBs are computed from

$$\mathrm{CTE}_\alpha(L_0) := \frac{\mathbb{E}[L_0 \mathbb{1}_{\{L_0>0\}}]}{1-\alpha} = \frac{(1-\xi_m)\,\mathbb{E}[L_0 \mathbb{1}_{\{L_0>0\}}]}{1-\alpha} = \frac{(1-\xi_m)\mathrm{CTE}_{\xi_m}(L_0)}{1-\alpha}$$

as in [FV12].

In Table 2 we compare the computation times of the stratified approximations for the GMMB rider with the double integral approach of [FV12] and with the Green function method in [FV14]. The method of [FV14] is the fastest known analytical method, however it does not cover the case of additional earnings considered in this paper.

Method	[FV12]†	[FV14]‡	lognormal	gamma
$V_{90\%}/F_0$	2.6226s	0.0023s	0.0119s	0.0336s
$\mathrm{CTE}_{90\%}/F_0$	0.1282s	0.00016s	0.0082s	0.0064s

Table 2: Time comparison in seconds between the different methods using C.

The computation times are based on an implementation in C on an Intel Corel i5 CPU (1.7GHz) and 4GB of RAM.

†Inverse Laplace method (implemented in C).
‡Green function method (implemented in C).
*This value has been computed using $L_0^* := \max(L_0, 0)$ when L_0 yields a negative risk measure.
†Inverse Laplace method (implemented in C).
‡Green function method (implemented in C).

The computation of risk measures for the GMDB rider is presented in Table 3. The parameters of the products and the underlying asset (1.1) are the same as for GMMBs except that here $r = 7\%$, and the roll-up rate per annum is $\delta = 6\%$. We take $n = 1$, but one can also take $n \geqslant 2$ and apply the fractional age assumption in order to consider payments more frequent than yearly payments.

$G/F_0 = 75\%$	[FV12]†	[FV14]‡	lognormal	gamma
$V_{80\%}/F_0$	0*	0*	0*	0*
$V_{90\%}/F_0$	0*	0*	0*	0*
$V_{95\%}/F_0$	8.198224	8.198239	8.198215	8.194312
$\mathrm{CTE}_{80\%}/F_0$	7.018565*	7.018559*	7.018555*	7.118478*
$\mathrm{CTE}_{90\%}/F_0$	14.037130*	14.037118*	14.037111*	14.236956*
$\mathrm{CTE}_{95\%}/F_0$	26.965800	26.965792	26.965780	27.261278*

$G/F_0 = 100\%$	[FV12]†	[FV14]‡	lognormal	gamma
$V_{80\%}/F_0$	0*	0*	0*	0*
$V_{90\%}/F_0$	2.135087	2.135188	2.135182	2.069297
$V_{95\%}/F_0$	31.825680	31.825697	31.825660	31.821012
$\mathrm{CTE}_{80\%}/F_0$	16.871263*	16.871439*	16.871434*	17.048815*
$\mathrm{CTE}_{90\%}/F_0$	33.706317	33.706297	33.706289	34.048215
$\mathrm{CTE}_{95\%}/F_0$	50.390319	50.3903583	50.390345	50.687882

$G/F_0 = 120\%$	[FV12]†	[FV14]‡	lognormal	gamma
$V_{80\%}/F_0$	0*	0*	0*	0*
$V_{90\%}/F_0$	21.144542	21.144667	21.144658	21.076596
$V_{95\%}/F_0$	50.732692	50.732711	50.732661	50.727330
$\mathrm{CTE}_{80\%}/F_0$	27.981319*	27.978583*	27.981355*	28.216016*
$\mathrm{CTE}_{90\%}/F_0$	52.568651	52.568633	52.568625	52.909990
$\mathrm{CTE}_{95\%}/F_0$	69.140613	69.140653	69.140640	69.439727

Table 3: Risk measure estimates in % for the GMDB rider with different levels of risk tolerance α.

The lognormal approximation appears the most precise and consistent when compared with other methods, while the gamma approximation is not as accurate.

Table 4 presents the computation of VaR and CTE of net liabilities for GMMBs with

†Inverse Laplace method (implemented in C).

‡Green function method (implemented in C).

*This value has been computed using $L_0^{(n)*} := \max\left(L_0^{(n)}, 0\right)$ when $L_0^{(n)}$ yields a negative risk measure.

AE feature. The VaR $V_\alpha(L_0)$ is computed from (2.2) given $P_\rho(T, G, V_\alpha(L_0))$ approximated by (5.1) under the gamma approximation, and by (5.4) under the lognormal approximation. The CTE is similarly computed from (2.3) given $Z_\rho(T, G, w)$ evaluated as in Propositions 5.2 and 5.4. We take the risk tolerance level $\alpha = 90\%$, $G/F_0 = 100\%$, and $C/F_0 = 100\%, 200\%, 250\%$ as in [MZ16], the other model and product parameters being the same as above. The computation time for VaR and CTE by stratified approximation is around 0.01 and 0.004 seconds respectively.

	$\rho = 0.1$		$\rho = 0.2$		$\rho = 0.3$	
$C/F_0 = 100\%$	lognormal	gamma	lognormal	gamma	lognormal	gamma
$V_{90\%}/F_0$	36.1990	36.2035	53.5788	53.5398	58.1323	58.0785
$\mathrm{CTE}_{90\%}/F_0$	46.9541	46.9517	57.5319	57.5290	60.1738	60.1956

	$\rho = 0.1$		$\rho = 0.2$		$\rho = 0.3$	
$C/F_0 = 200\%$	lognormal	gamma	lognormal	gamma	lognormal	gamma
$V_{90\%}/F_0$	36.4298	36.4299	64.1508	64.1511	99.9247	99.9373
$\mathrm{CTE}_{90\%}/F_0$	57.7870	57.7875	97.6804	97.6804	118.4403	119.8467

	$\rho = 0.1$		$\rho = 0.2$		$\rho = 0.3$	
$C/F_0 = 250\%$	lognormal	gamma	lognormal	gamma	lognormal	gamma
$V_{90\%}/F_0$	36.4301	36.4302	64.1603	64.1604	100.4536	100.4544
$\mathrm{CTE}_{90\%}/F_0$	59.4663	59.4668	106.9436	106.9436	138.5511	138.5772

Table 4: Risk measure estimates in % for the GMMB rider with AE feature and level of risk tolerance $\alpha = 90\%$.

The VaR $V_\alpha(L_0^{(n)})$ and CTE of the net liabilities can be similarly calculated implicitly from (3.1) for GMDBs.

	$\rho = 0.1$		$\rho = 0.2$		$\rho = 0.3$	
$C/F_0 = 100\%$	lognormal	gamma	lognormal	gamma	lognormal	gamma
$V_{90\%}/F_0$	14.732510	14.718029	22.554267	22.546765	28.058254	28.054762
$\mathrm{CTE}_{90\%}/F_0$	37.527729	37.786991	42.585388	42.792351	46.538218	46.708692

	$\rho = 0.1$		$\rho = 0.2$		$\rho = 0.3$	
$C/F_0 = 200\%$	lognormal	gamma	lognormal	gamma	lognormal	gamma
$V_{90\%}/F_0$	14.735675	14.721054	22.566120	22.558268	28.094065	28.089785
$\mathrm{CTE}_{90\%}/F_0$	38.180667	38.439814	45.741347	45.948158	53.113941	53.284557

	$\rho = 0.1$		$\rho = 0.2$		$\rho = 0.3$	
$C/F_0 = 250\%$	lognormal	gamma	lognormal	gamma	lognormal	gamma
$V_{90\%}/F_0$	14.735688	14.721067	22.566146	22.558296	28.094109	28.089834
$\mathrm{CTE}_{90\%}/F_0$	38.268264	38.527405	46.325110	46.532120	54.554886	54.725914

Table 5: Risk measure estimates in % for the GMDB rider with AE feature and level of risk tolerance $\alpha = 90\%$.

In Table 6 we present the numerical computation of the sensitivity of VaR based on the estimates of Section 6 with $G/F_0 = 100\%$, the other model and product parameters being the same as in Table 1, with $\rho = 0.1$ and $C/F_0 = 100\%$ in the case of AEs.

$G/F_0 = 100\%$	without AE feature			with AE feature		
	lognormal	gamma	FD†	lognormal	gamma	FD†
$\partial V_{90\%}/\partial\mu$	-5.296026	-5.296026	-5.296029	1.072569	1.073818	1.072572
$\partial V_{95\%}/\partial\mu$	-3.673600	-3.673600	-3.673601	1.177017	1.160743	1.177016

Table 6: Sensitivities of VaR with respect to μ for the GMMB rider with different levels of risk tolerance α.

We note that sensitivities are negative without AEs, due to the negativity of (6.3) in the integral representations of $\frac{\partial}{\partial\mu}P_\rho(T,G,w)_{|w=V_\alpha(L_0)}$ used in (6.1). On the other hand, with AE we have $\rho > 0$ and the additional terms (5.2) and (5.5)-(5.6) result into positive sensitivities.

8 Conclusion

We have derived single integral approximations for the computation of the risk measures of GMMBs and GMDBs under Black-Scholes framework using conditional moment matching. The implementation of these expressions is significantly faster than

†Finite Difference Method.

the double integral and inverse Laplace transform algorithms [FV12], and they also match the results obtained in [FV14] by single integral and series approximations using Green functions. In general the lognormal approximation yields the most precise and consistent results, in agreement with the intuition given by Figure 1, while the gamma approximation is less precise in the case of GMDBs. Our approximations also apply to guaranteed benefits with additional earnings which have not been treated via other methods. The pricing of variable annuities has been extended to Guaranteed Minimum Withdrawal Benefits (GMWBs) with stochastic interest rate, stochastic volatility and stochastic mortality via Monte Carlo and PDE arguments in e.g. [DYL15], [GMZ16] and references therein. An extensions of our method to such settings would basically require the computation of conditional moments in multi factor models and would involve additional analytical difficulties.

A Appendix

Proof of Proposition 5.1. We have $P_\rho(T,G,w) = \mathbb{P}(A_T(w,G))$, where $A_T(w,G)$ is partitioned into

$$A_T(w,G) \cap \{F_T < G\} = \left\{ \tilde{S}_T + m_x \Lambda_T < (\mathrm{e}^{-rT}G - w)/F_0 \right\},$$

$$A_T(w,G) \cap \{G \leqslant F_T < (G + C/\rho)\} = \left\{ \frac{F_0 m_x \Lambda_T + w + \rho \mathrm{e}^{-rT} G}{\rho F_0} \leqslant \tilde{S}_T < \frac{\mathrm{e}^{-rT}}{\rho F_0}(\rho G + C) \right\},$$

and

$$A_T(w,G) \cap \{F_T \geqslant (G + C/\rho)\} = \left\{ m_x \Lambda_T < \frac{\mathrm{e}^{-rT} C - w}{F_0}, \quad \tilde{S}_T \geqslant \frac{\mathrm{e}^{-rT}}{\rho F_0}(\rho G + C) \right\},$$

which yields the decomposition

$$P_\rho(T,G,w) = Q_0(T,G,w) + Q_1(T,G,w) + Q_2(T,G,w), \tag{A.1}$$

where

$$\begin{aligned} Q_0(T,G,w) &= \mathbb{P}\left(\tilde{S}_T + m_x \Lambda_T < \frac{\mathrm{e}^{-rT} G - w}{F_0} \right) \qquad\qquad (A.2) \\ &= \int_0^{(\mathrm{e}^{-rT}G - w)/F_0} \mathbb{P}\left(\Lambda_T < \frac{\mathrm{e}^{-rT} G - w - z F_0}{F_0 m_x} \Big| \tilde{S}_T = z \right) f_{\tilde{S}_T}(z) dz \end{aligned}$$

$$= \int_0^{(\mathrm{e}^{-rT}G-w)/F_0} \int_0^{(\mathrm{e}^{-rT}G-w-zF_0)/(F_0 m_x)} f_{\Lambda_T|\tilde{S}_T=z}(y)\mathrm{d}y f_{\tilde{S}_T}(z)dz,$$

$$Q_1(T,G,w) = \mathbb{P}\left(\frac{F_0 m_x \Lambda_T + w + \rho \mathrm{e}^{-rT}G}{\rho F_0} \leqslant \tilde{S}_T < \frac{\mathrm{e}^{-rT}}{\rho F_0}(\rho G + C)\right) \tag{A.3}$$
$$= \int_{\frac{\rho \mathrm{e}^{-rT}G+w}{\rho F_0}}^{\frac{\mathrm{e}^{-rT}}{\rho F_0}(\rho G+C)} \int_0^{(\rho z F_0 - \mathrm{e}^{-rT}\rho G - w)/(F_0 m_x)} f_{\Lambda_T|\tilde{S}_T=z}(y)\mathrm{d}y f_{\tilde{S}_T}(z)dz,$$

and

$$Q_2(T,G,w) = \mathbb{P}\left(m_x\Lambda_T < \frac{\mathrm{e}^{-rT}C - w}{F_0}, \quad \tilde{S}_T \geqslant \frac{\mathrm{e}^{-rT}}{\rho F_0}(\rho G + C)\right) \tag{A.4}$$
$$= \int_{\frac{\mathrm{e}^{-rT}}{\rho F_0}(\rho G+C)}^{\infty} \int_0^{\frac{\mathrm{e}^{-rT}C-w}{F_0 m_x}} f_{\Lambda_T|\tilde{S}_T=z}(y)\mathrm{d}y f_{\tilde{S}_T}(z)dz.$$

Finally we use the estimate

$$\int_0^\eta f_{\Lambda_T|\tilde{S}_T=z}(y)\mathrm{d}y \approx \frac{1}{(\theta_T^z)^{\nu_T^z}\Gamma_{\nu_T^z}} \int_0^\eta \mathrm{e}^{-y/\theta_T^z} y^{-1+\nu_T^z}\mathrm{d}y = \Gamma_{\nu_T^z}(\eta/\theta_T^z), \qquad \eta > 0, \tag{A.5}$$

which is based on the conditional gamma approximation (4.1). □

Proof of Proposition 5.2. Expressing $Z_\rho(T,G,w)$ in term of $\tilde{S}_T$ and Λ_T, we have

$$Z_\rho(T,G,w) = \mathbb{E}\left[\left(\mathrm{e}^{-rT}(G-F_T)^+ + \mathrm{e}^{-rT}\min\left(C, \rho(F_T-G)^+\right) - \int_0^T \mathrm{e}^{-rs}M_s^e \mathrm{d}s\right)\mathbb{1}_{A_T(w)}\right] \tag{A.6}$$
$$= \mathbb{E}\left[\left(\mathrm{e}^{-rT}G - F_0\tilde{S}_T - F_0 m_x\Lambda_T\right)\mathbb{1}_{\left\{\tilde{S}_T + m_x\Lambda_T < (\mathrm{e}^{-rT}G-w)/F_0\right\}}\right]$$
$$+ \mathbb{E}\left[\left(\rho(F_0\tilde{S}_T - \mathrm{e}^{-rT}G) - F_0 m_x\Lambda_T\right)\mathbb{1}_{\left\{\frac{F_0 m_x\Lambda_T + w + \rho\mathrm{e}^{-rT}G}{\rho F_0} \leqslant \tilde{S}_T < \frac{\mathrm{e}^{-rT}}{\rho F_0}(\rho G+C)\right\}}\right]$$
$$+ \mathbb{E}\left[\left(\mathrm{e}^{-rT}C - F_0 m_x\Lambda_T\right)\mathbb{1}_{\left\{m_x\Lambda_T < \frac{\mathrm{e}^{-rT}C-w}{F_0}, \quad \tilde{S}_T \geqslant \frac{\mathrm{e}^{-rT}}{\rho F_0}(\rho G+C)\right\}}\right]$$
$$= \mathrm{e}^{-rT}GQ_0(T,G,w) - F_0W_0(T,G,w) - \rho\mathrm{e}^{-rT}GQ_1(T,G,w) + F_0W_1(T,G,w)$$
$$+ \mathrm{e}^{-rT}CQ_2(T,G,w) - F_0W_2(T,G,w),$$

where

$$W_0(T,G,w) = \mathbb{E}\left[\left(\tilde{S}_T + m_x\Lambda_T\right)\mathbb{1}_{\left\{\tilde{S}_T + m_x\Lambda_T < \frac{\mathrm{e}^{-rT}G-w}{F_0}\right\}}\right] \tag{A.7}$$

$$
\begin{aligned}
&= \int_0^{(\mathrm{e}^{-rT}G-w)/F_0} \mathbb{E}\left[\left(\tilde{S}_T + m_x\Lambda_T\right)\mathbb{1}_{\left\{\tilde{S}_T+m_x\Lambda_T<\frac{\mathrm{e}^{-rT}G-w}{F_0}\right\}}\middle|\tilde{S}_T=z\right] f_{\tilde{S}_T}(z)dz \\
&= \int_0^{(\mathrm{e}^{-rT}G-w)/F_0} z\int_0^{(\mathrm{e}^{-rT}G-w-zF_0)/(F_0m_x)} f_{\Lambda_T|\tilde{S}_T=z}(x)\mathrm{d}x f_{\tilde{S}_T}(z)dz \\
&\quad +m_x\int_0^{(\mathrm{e}^{-rT}G-w)/F_0}\int_0^{(\mathrm{e}^{-rT}G-w-zF_0)/(F_0m_x)} xf_{\Lambda_T|\tilde{S}_T=z}(x)\mathrm{d}x f_{\tilde{S}_T}(z)dz,
\end{aligned}
$$

$$
\begin{aligned}
W_1(T,G,w) &:= \mathbb{E}\left[\left(\rho\tilde{S}_T - m_x\Lambda_T\right)\mathbb{1}_{\left\{\frac{F_0m_x\Lambda_T+w+\rho\mathrm{e}^{-rT}G}{\rho F_0}\leqslant\tilde{S}_T<\frac{\mathrm{e}^{-rT}}{\rho F_0}(\rho G+C)\right\}}\right] \quad (A.8)\\
&= \int_{\frac{\rho\mathrm{e}^{-rT}G-w}{\rho F_0}}^{\frac{\mathrm{e}^{-rT}}{\rho F_0}(\rho G+C)} \mathbb{E}\left[\left(\rho\tilde{S}_T - m_x\Lambda_T\right)\mathbb{1}_{\left\{\frac{\rho\mathrm{e}^{-rT}G+F_0m_x\Lambda_T+w}{\rho F_0}\leqslant\tilde{S}_T)\right\}}\middle|\tilde{S}_T=z\right] f_{\tilde{S}_T}(z)dz \\
&= \int_{\frac{\rho\mathrm{e}^{-rT}G-w}{\rho F_0}}^{\frac{\mathrm{e}^{-rT}}{\rho F_0}(\rho G+C)}\int_0^{\frac{\rho F_0z-\rho\mathrm{e}^{-rT}G-w}{m_xF_0}} (\rho z-m_xx)f_{\Lambda_T|\tilde{S}_T=z}(x)\mathrm{d}x f_{\tilde{S}_T}(z)dz,
\end{aligned}
$$

and

$$
\begin{aligned}
W_2(T,G,w) &:= \mathbb{E}\left[m_x\Lambda_T\mathbb{1}_{\left\{m_x\Lambda_T<\frac{\mathrm{e}^{-rT}C-w}{F_0},\ \tilde{S}_T\geqslant\frac{\mathrm{e}^{-rT}}{\rho F_0}(\rho G+C)\right\}}\right] \quad (A.9)\\
&= \int_{\frac{\mathrm{e}^{-rT}}{\rho F_0}(\rho G+C)}^{\infty} \mathbb{E}\left[m_x\Lambda_T\mathbb{1}_{\left\{m_x\Lambda_T<\frac{\mathrm{e}^{-rT}C-w}{F_0}\right\}}\middle|\tilde{S}_T=z\right] f_{\tilde{S}_T}(z)dz \\
&= m_x\int_{\frac{\mathrm{e}^{-rT}}{\rho F_0}(\rho G+C)}^{w}\int_0^{\frac{\mathrm{e}^{-rT}C-w}{m_xF_0}} xf_{\Lambda_T|\tilde{S}_T=z}(x)\mathrm{d}x f_{\tilde{S}_T}(z)dz.
\end{aligned}
$$

We conclude by the approximation

$$
\int_0^\eta yf_{\Lambda_T|\tilde{S}_T=z}(y)\mathrm{d}y \approx \frac{1}{\Gamma_{\nu_T^z}}\int_0^\eta \mathrm{e}^{-y/\theta_T^z}(y/\theta_T^z)^{\nu_T^z}\mathrm{d}y = \theta_T^z\nu_T^z\Gamma_{\nu_T^z+1}(\eta/\theta_T^z), \qquad \eta>0.
$$

□

Proof of Propositions 5.3 and 5.4. We replace (A.5) with the approximation

$$
\begin{aligned}
\int_0^\eta f_{\Lambda_T|\tilde{S}_T=z}(y)\mathrm{d}y &\approx \frac{1}{\sigma_T^z\sqrt{2\pi T}}\int_0^\eta \mathrm{e}^{-(\mu_T^z(\sigma_T^z)^2T/2+\log y)^2/(2(\sigma_T^z)^2T)}\frac{\mathrm{d}y}{y} \\
&= \Phi\left(\frac{\mu_T^z(\sigma_T^z)^2T/2+\log\eta}{\sigma_T^z\sqrt{T}}\right), \qquad \eta>0, \quad (A.10)
\end{aligned}
$$

that follows from (4.2), and apply it to the estimation of (A.2), (A.3) and (A.4).

□

B Appendix

Under the conditional gamma approximation in Propositions 5.1 and 5.2, $\frac{\partial P_\rho}{\partial w}(T,G,w)$ and $\frac{\partial Z_\rho}{\partial w}(T,G,w)$ can be estimated respectively as

$$
\begin{aligned}
\frac{\partial P_\rho}{\partial w}(T,G,w) \approx & \int_0^{\frac{\mathrm{e}^{-rT}G-w}{F_0}} \frac{\partial}{\partial w}\Gamma_{\nu_T^z}\left(\frac{\mathrm{e}^{-rT}G-w-zF_0}{F_0\theta_T^z m_x}\right) f_{\tilde{S}_T}(z)dz \\
&+\int_{\frac{\rho\mathrm{e}^{-rT}G+w}{\rho F_0}}^{\frac{\mathrm{e}^{-rT}}{\rho F_0}(\rho G+C)} \frac{\partial}{\partial w}\Gamma_{\nu_T^z}\left(\frac{\rho z F_0-\mathrm{e}^{-rT}\rho G-w}{F_0\theta_T^z m_x}\right) f_{\tilde{S}_T}(z)dz \\
&-\int_{\frac{\mathrm{e}^{-rT}}{\rho F_0}(\rho G+C)}^{\infty} \frac{\partial}{\partial w}\Gamma_{\nu_T^z}\left(\frac{\mathrm{e}^{-rT}C-w}{F_0\theta_T^z m_x}\right) f_{\tilde{S}_T}(z)dz \\
=& \int_0^{\frac{\mathrm{e}^{-rT}G-w}{F_0}} \frac{1}{\Gamma_{\nu_T^z}}\left(\frac{\mathrm{e}^{-rT}G-w-zF_0}{F_0\theta_T^z m_x}\right)^{\nu_T^z-1} \mathrm{e}^{-\frac{\mathrm{e}^{-rT}G-w-zF_0}{F_0\theta_T^z m_x}} \frac{-1}{F_0\theta_T^z m_x} f_{\tilde{S}_T}(z)dz \\
&+\int_{\frac{\rho\mathrm{e}^{-rT}G+w}{\rho F_0}}^{\frac{\mathrm{e}^{-rT}}{\rho F_0}(\rho G+C)} \frac{1}{\Gamma_{\nu_T^z}}\left(\frac{\rho z F_0-\mathrm{e}^{-rT}\rho G-w}{F_0\theta_T^z m_x}\right)^{\nu_T^z-1} \mathrm{e}^{-\frac{\rho z F_0-\mathrm{e}^{-rT}\rho G-w}{F_0\theta_T^z m_x}} \frac{-1}{F_0\theta_T^z m_x} f_{\tilde{S}_T}(z)dz \\
&+\int_{\frac{\mathrm{e}^{-rT}}{\rho F_0}(\rho G+C)}^{\infty} \frac{1}{\Gamma_{\nu_T^z}}\left(\frac{\mathrm{e}^{-rT}C-w}{F_0\theta_T^z m_x}\right)^{\nu_T^z-1} \mathrm{e}^{-\frac{\mathrm{e}^{-rT}C-w}{F_0\theta_T^z m_x}} \frac{-1}{F_0\theta_T^z m_x} f_{\tilde{S}_T}(z)dz,
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{\partial Z_\rho}{\partial w}(T,G,w) \approx \\
&F_0\int_0^{\frac{\mathrm{e}^{-rT}G-w}{F_0}}\left(\left(\frac{\mathrm{e}^{-rT}G}{F_0}-z\right)\frac{\partial\Gamma_{\nu_T^z}}{\partial w}\left(\frac{\mathrm{e}^{-rT}G\text{-}w\text{-}zF_0}{F_0\theta_T^z m_x}\right) - m_x\theta_T^z\nu_T^z\frac{\partial\Gamma_{\nu_T^z+1}}{\partial w}\left(\frac{\mathrm{e}^{-rT}G\text{-}w\text{-}zF_0}{F_0\theta_T^z m_x}\right)\right) f_{\tilde{S}_T}(z)dz \\
&+F_0\int_{\frac{\rho\mathrm{e}^{-rT}G+w}{\rho F_0}}^{\frac{\mathrm{e}^{-rT}}{\rho F_0}(\rho G+C)}\left(\rho\left(z\text{-}\frac{\mathrm{e}^{-rT}G}{F_0}\right)\frac{\partial\Gamma_{\nu_T^z}}{\partial w}\left(\frac{\rho(z\text{-}\frac{\mathrm{e}^{-rT}G}{F_0})\text{-}\frac{w}{F_0}}{\theta_T^z m_x}\right) - m_x\theta_T^z\nu_T^z\frac{\partial\Gamma_{\nu_T^z+1}}{\partial w}\left(\frac{\rho(z\text{-}\frac{\mathrm{e}^{-rT}G}{F_0})\text{-}\frac{w}{F_0}}{\theta_T^z m_x}\right)\right) f_{\tilde{S}_T}(z)dz \\
&+\mathrm{e}^{-rT}C\int_{\frac{\mathrm{e}^{-rT}}{\rho F_0}(\rho G+C)}^{\infty}\frac{\partial\Gamma_{\nu_T^z}}{\partial w}\left(\frac{\mathrm{e}^{-rT}C\text{-}w}{\theta_T^z m_x F_0}\right) f_{\tilde{S}_T}(z)dz - F_0 m_x\int_{\frac{\mathrm{e}^{-rT}}{\rho F_0}(\rho G+C)}^{w}\theta_T^z\nu_T^z\frac{\partial\Gamma_{\nu_T^z+1}}{\partial w}\left(\frac{\mathrm{e}^{-rT}C\text{-}w}{\theta_T^z m_x F_0}\right) f_{\tilde{S}_T}(z)dz \\
&=\int_0^{\frac{\mathrm{e}^{-rT}G-w}{F_0}}\left(\frac{\mathrm{e}^{-rT}G}{F_0}-z\right)\frac{-1}{\Gamma_{\nu_T^z}\theta_T^z m_x}\left(\frac{\mathrm{e}^{-rT}G\text{-}w\text{-}zF_0}{F_0\theta_T^z m_x}\right)^{\nu_T^z-1}\mathrm{e}^{-\frac{\mathrm{e}^{-rT}G\text{-}w\text{-}zF_0}{F_0\theta_T^z m_x}} f_{\tilde{S}_T}(z)dz \\
&+\int_0^{\frac{\mathrm{e}^{-rT}G-w}{F_0}}\frac{\nu_T^z}{\Gamma_{\nu_T^z+1}}\left(\frac{\mathrm{e}^{-rT}G\text{-}w\text{-}zF_0}{F_0\theta_T^z m_x}\right)^{\nu_T^z}\mathrm{e}^{-\frac{\mathrm{e}^{-rT}G\text{-}w\text{-}zF_0}{F_0\theta_T^z m_x}} f_{\tilde{S}_T}(z)dz \\
&-\int_{\frac{\rho\mathrm{e}^{-rT}G+w}{\rho F_0}}^{\frac{\mathrm{e}^{-rT}}{\rho F_0}(\rho G+C)}\rho\left(z\text{-}\frac{\mathrm{e}^{-rT}G}{F_0}\right)\frac{1}{\Gamma_{\nu_T^z}\theta_T^z m_x}\left(\frac{\rho(z\text{-}\frac{\mathrm{e}^{-rT}G}{F_0})\text{-}\frac{w}{F_0}}{\theta_T^z m_x}\right)^{\nu_T^z-1}\mathrm{e}^{-\frac{\rho(z\text{-}\frac{\mathrm{e}^{-rT}G}{F_0})\text{-}\frac{w}{F_0}}{\theta_T^z m_x}} f_{\tilde{S}_T}(z)dz
\end{aligned}
$$

$$+\int_{\frac{\rho e^{-rT}G+w}{\rho F_0}}^{\frac{e^{-rT}}{\rho F_0}(\rho G+C)} \frac{\nu_T^z}{\Gamma_{\nu_T^z+1}} \left(\frac{\rho(z-\frac{e^{-rT}G}{F_0})-\frac{w}{F_0}}{\theta_T^z m_x}\right)^{\nu_T^z} e^{-\frac{\rho(z-\frac{e^{-rT}G}{F_0})-\frac{w}{F_0}}{\theta_T^z m_x}} f_{\tilde{S}_T}(z)dz$$

$$-e^{-rT}C\int_{\frac{e^{-rT}}{\rho F_0}(\rho G+C)}^{\infty} \frac{1}{\Gamma_{\nu_T^z}\theta_T^z m_x F_0}\left(\frac{e^{-rT}C-w}{\theta_T^z m_x F_0}\right)^{\nu_T^z-1} e^{-\frac{e^{-rT}C-w}{\theta_T^z m_x F_0}} f_{\tilde{S}_T}(z)dz$$

$$+\int_{\frac{e^{-rT}}{\rho F_0}(\rho G+C)}^{w} \frac{\nu_T^z}{\Gamma_{\nu_T^z+1}}\left(\frac{e^{-rT}C-w}{\theta_T^z m_x F_0}\right)^{\nu_T^z} e^{\frac{e^{-rT}C-w}{\theta_T^z m_x F_0}} f_{\tilde{S}_T}(z)dz.$$

On the other hand, under the conditional lognormal approximation in Propositions 5.3 and 5.4, $\frac{\partial P_\rho}{\partial w}(T,G,w)$ and $\frac{\partial Z_\rho}{\partial w}(T,G,w)$ can be estimated respectively as

$$\frac{\partial P_\rho}{\partial w}(T,G,w) \approx \int_0^{\frac{e^{-rT}G-w}{F_0}} \frac{\partial}{\partial w}\Phi\left(\frac{\mu_T^z\frac{(\sigma_T^z)^2T}{2}+\log\frac{e^{-rT}G-w-zF_0}{F_0 m_x}}{\sigma_T^z\sqrt{T}}\right) f_{\tilde{S}_T}(z)dz$$

$$+\int_{\frac{\rho e^{-rT}G+w}{\rho F_0}}^{\frac{e^{-rT}}{\rho F_0}(\rho G+C)} \frac{\partial}{\partial w}\Phi\left(\frac{\mu_T^z\frac{(\sigma_T^z)^2T}{2}+\log\frac{\rho zF_0-e^{-rT}\rho G-w}{F_0 m_x}}{\sigma_T^z\sqrt{T}}\right) f_{\tilde{S}_T}(z)dz$$

$$+\int_{\frac{e^{-rT}}{\rho F_0}(\rho G+C)}^{\infty} \frac{\partial}{\partial w}\Phi\left(\frac{\mu_T^z\frac{(\sigma_T^z)^2T}{2}+\log\frac{e^{-rT}C-w}{F_0 m_x}}{\sigma_T^z\sqrt{T}}\right) f_{\tilde{S}_T}(z)dz$$

$$=\int_0^{\frac{e^{-rT}G-w}{F_0}} \frac{-1}{\sqrt{2\pi T}\sigma_T^z(e^{-rT}G-w-zF_0)}\exp\left(-\frac{1}{2}\left(\frac{\mu_T^z\frac{(\sigma_T^z)^2T}{2}+\log\frac{e^{-rT}G-w-zF_0}{F_0 m_x}}{\sigma_T^z\sqrt{T}}\right)^2\right) f_{\tilde{S}_T}(z)dz$$

$$+\int_{\frac{\rho e^{-rT}G+w}{\rho F_0}}^{\frac{e^{-rT}}{\rho F_0}(\rho G+C)} \frac{-1}{\sqrt{2\pi T}\sigma_T^z(\rho zF_0-e^{-rT}\rho G-w)}\exp\left(-\frac{1}{2}\left(\frac{\mu_T^z\frac{(\sigma_T^z)^2T}{2}+\log\frac{\rho zF_0-e^{-rT}\rho G-w}{F_0 m_x}}{\sigma_T^z\sqrt{T}}\right)^2\right) f_{\tilde{S}_T}(z)dz$$

$$+\int_{\frac{e^{-rT}}{\rho F_0}(\rho G+C)}^{\infty} \frac{-1}{\sqrt{2\pi T}\sigma_T^z(e^{-rT}C-w)}\exp\left(-\frac{1}{2}\left(\frac{\mu_T^z\frac{(\sigma_T^z)^2T}{2}+\log\frac{e^{-rT}C-w}{F_0 m_x}}{\sigma_T^z\sqrt{T}}\right)^2\right) f_{\tilde{S}_T}(z)dz,$$

and

$$\frac{\partial Z_\rho}{\partial w}(T,G,w) \approx \int_0^{\frac{e^{-rT}G-w}{F_0}} \left(e^{-rT}G-F_0z\right)\frac{\partial}{\partial w}\Phi\left(\frac{\mu_T^z\frac{(\sigma_T^z)^2T}{2}+\log\frac{e^{-rT}G-w-zF_0}{F_0 m_x}}{\sigma_T^z\sqrt{T}}\right) f_{\tilde{S}_T}(z)dz$$

$$-F_0 m_x\int_0^{\frac{e^{-rT}G-w}{F_0}} e^{(1-\mu_T^z)(\sigma_T^z)^2T/2}\frac{\partial}{\partial w}\Phi\left(\frac{(\mu_T^z-2)\frac{(\sigma_T^z)^2T}{2}+\log\frac{e^{-rT}G-w-zF_0}{F_0 m_x}}{\sigma_T^z\sqrt{T}}\right) f_{\tilde{S}_T}(z)dz$$

$$+\rho\int_{\frac{e^{-rT}G}{F_0}+\frac{w}{\rho F_0}}^{\frac{e^{-rT}}{\rho F_0}(\rho G+C)} \left(F_0z-e^{-rT}G\right)\frac{\partial}{\partial w}\Phi\left(\frac{\mu_T^z\frac{(\sigma_T^z)^2T}{2}+\log\frac{\rho zF_0-e^{-rT}\rho G-w}{m_x F_0}}{\sigma_T^z\sqrt{T}}\right) f_{\tilde{S}_T}(z)dz$$

$$
\begin{aligned}
&- F_0 m_x \int_{\frac{\rho \mathrm{e}^{-rT}G+w}{\rho F_0}}^{\frac{\mathrm{e}^{-rT}}{\rho F_0}(\rho G+C)} \mathrm{e}^{(1-\mu_T^z)(\sigma_T^z)^2T/2} \frac{\partial}{\partial w}\Phi\left(\frac{(\mu_T^z-2)\frac{(\sigma_T^z)^2T}{2}+\log\frac{\rho z F_0-\mathrm{e}^{-rT}\rho G-w}{m_xF_0}}{\sigma_T^z\sqrt{T}}\right) f_{\tilde{S}_T}(z)dz \\
&+ \mathrm{e}^{-rT}C \int_{\frac{\mathrm{e}^{-rT}}{\rho F_0}(\rho G+C)}^{\infty} \frac{\partial}{\partial w}\Phi\left(\frac{(\mu_T^z-2)\frac{(\sigma_T^z)^2T}{2}+\log\frac{\mathrm{e}^{-rT}C-w}{m_xF_0}}{\sigma_T^z\sqrt{T}}\right) f_{\tilde{S}_T}(z)dz \\
&- F_0 m_x \int_{\frac{\mathrm{e}^{-rT}}{\rho F_0}(\rho G+C)}^{w} \mathrm{e}^{(1-\mu_T^z)(\sigma_T^z)^2T/2} \frac{\partial}{\partial w}\Phi\left(\frac{(\mu_T^z-2)\frac{(\sigma_T^z)^2T}{2}+\log\frac{\mathrm{e}^{-rT}C-w}{m_xF_0}}{\sigma_T^z\sqrt{T}}\right) f_{\tilde{S}_T}(z)dz \\
&= \int_0^{\frac{\mathrm{e}^{-rT}G-w}{F_0}} \frac{-(\mathrm{e}^{-rT}G-F_0z)}{\sqrt{2\pi T}\sigma_T^z(\mathrm{e}^{-rT}G-w-zF_0)} \exp\left(-\frac{1}{2}\left(\frac{\mu_T^z\frac{(\sigma_T^z)^2T}{2}+\log\frac{\mathrm{e}^{-rT}G-w-zF_0}{F_0m_x}}{\sigma_T^z\sqrt{T}}\right)^2\right) f_{\tilde{S}_T}(z)dz \\
&- \frac{F_0m_x}{\sqrt{2\pi T}} \int_0^{\frac{\mathrm{e}^{-rT}G-w}{F_0}} \frac{-\mathrm{e}^{(1-\mu_T^z)(\sigma_T^z)^2T/2}}{\sigma_T^z(\mathrm{e}^{-rT}G-w-zF_0)} \exp\left(-\frac{1}{2}\left(\frac{(\mu_T^z-2)\frac{(\sigma_T^z)^2T}{2}+\log\frac{\mathrm{e}^{-rT}G-w-zF_0}{F_0m_x}}{\sigma_T^z\sqrt{T}}\right)^2\right) f_{\tilde{S}_T}(z)dz \\
&+ \frac{\rho}{\sqrt{2\pi T}} \int_{\frac{\mathrm{e}^{-rT}G}{F_0}+\frac{w}{\rho F_0}}^{\frac{\mathrm{e}^{-rT}}{\rho F_0}(\rho G+C)} \frac{-(F_0z-\mathrm{e}^{-rT}G)}{\sigma_T^z(\rho z F_0-\mathrm{e}^{-rT}\rho G-w)} \exp\left(-\frac{1}{2}\left(\frac{\mu_T^z\frac{(\sigma_T^z)^2T}{2}+\log\frac{\rho zF_0-\mathrm{e}^{-rT}\rho G-w}{m_xF_0}}{\sigma_T^z\sqrt{T}}\right)^2\right) f_{\tilde{S}_T}(z)dz \\
&- \frac{F_0m_x}{\sqrt{2\pi T}} \int_{\frac{\rho\mathrm{e}^{-rT}G+w}{\rho F_0}}^{\frac{\mathrm{e}^{-rT}}{\rho F_0}(\rho G+C)} \frac{-\mathrm{e}^{(1-\mu_T^z)(\sigma_T^z)^2T/2}}{\sigma_T^z(\rho zF_0-\mathrm{e}^{-rT}\rho G-w)} \exp\left(-\frac{1}{2}\left(\frac{(\mu_T^z-2)\frac{(\sigma_T^z)^2T}{2}+\log\frac{\rho zF_0-\mathrm{e}^{-rT}\rho G-w}{m_xF_0}}{\sigma_T^z\sqrt{T}}\right)\right)^2 f_{\tilde{S}_T}(z)dz \\
&+ \frac{\mathrm{e}^{-rT}C}{\sqrt{2\pi T}} \int_{\frac{\mathrm{e}^{-rT}}{\rho F_0}(\rho G+C)}^{\infty} \frac{-1}{\sigma_T^z(\mathrm{e}^{-rT}C-w)} \exp\left(-\frac{1}{2}\left(\frac{(\mu_T^z-2)\frac{(\sigma_T^z)^2T}{2}+\log\frac{\mathrm{e}^{-rT}C-w}{m_xF_0}}{\sigma_T^z\sqrt{T}}\right)^2\right) f_{\tilde{S}_T}(z)dz \\
&- \frac{F_0m_x}{\sqrt{2\pi T}} \int_{\frac{\mathrm{e}^{-rT}}{\rho F_0}(\rho G+C)}^{w} \frac{\mathrm{e}^{(1-\mu_T^z)(\sigma_T^z)^2T/2}}{\sigma_T^z(\mathrm{e}^{-rT}C-w)} \exp\left(-\frac{1}{2}\left(\frac{(\mu_T^z-2)\frac{(\sigma_T^z)^2T}{2}+\log\frac{\mathrm{e}^{-rT}C-w}{m_xF_0}}{\sigma_T^z\sqrt{T}}\right)^2\right) f_{\tilde{S}_T}(z)dz.
\end{aligned}
$$

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