A transfer principle from Wiener to Poisson space and applications [1]

Nicolas Privault

Equipe d'Analyse et Probabilités, Université d'Evry-Val d'Essonne
Boulevard des Coquibus, 91025 Evry Cedex, France

Abstract

The aim of this work is to construct the stochastic calculus of variations on Poisson space and some of its applications via the stochastic analysis on Wiener space. We define a new gradient operator on Wiener space, whose adjoint extends the Poisson stochastic integral. This yields a new decomposition of the Ornstein-Uhlenbeck operator and a substructure of the standard Dirichlet structure on Wiener space, with applications to stochastic analysis on Poisson space and infinite-dimensional analysis for the exponential density.

1 Introduction

The stochastic calculus of variations on the Wiener space, cf. [9], [11], makes use of the following ingredients: a gradient operator, its adjoint the divergence operator, and the Ornstein-Uhlenbeck operator which is obtained as the composition of the divergence with the gradient. The Ornstein-Uhlenbeck operator is a number operator on the Wiener chaotic decomposition and it allows to define Sobolev spaces and distributions on the Wiener space, cf. [19]. On the other hand, the connection with the Itô calculus is obtained via the divergence operator which extends the Itô integral, cf. [6]. An important tool in this analysis is the Meyer inequalities, cf. [10] which give an equivalence between the norms defined with the gradient and the norms defined on Sobolev spaces with the Ornstein-Uhlenbeck operator. The question whether an analogous formalism exists on Poisson space has been investigated in e.g. [2], [12], [13]. In [12], a Fock space isomorphism using the Poisson and Wiener multiple stochastic integral is considered. This leads to a gradient defined by finite differences, which is not a derivation operator, and whose adjoint coincides with the compensated

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Poisson stochastic integral on square-integrable predictable processes. However, this isomorphism is not an isometry for the L^p norm, except for $p = 2$, and apparently it does not allow to transpose to the Poisson space case the analysis constructed on the Wiener space, in particular for $p \neq 2$. Another approach, initiated in [2] is to define a gradient by shifting the jump times of a standard Poisson process on the positive real line. The adjoint of this operator also extends the compensated Poisson stochastic integral. It has been shown in [1], [13] that there is a discrete chaotic decomposition on Poisson space on which the composition of this gradient with its adjoint acts as a number operator noted $\mathcal{L}$. In this approach, the trajectories of the Poisson process are considered as sequences of independent identically distributed exponential interjump times.

In this work, we consider the L^p space of Poisson functionals as a subspace of the L^p space of Wiener functionals, and show that the above gradient can be extended to Wiener functionals. It turns out that the composition of this gradient with its adjoint gives a new decomposition for the Ornstein-Uhlenbeck operator on the Wiener space. From the point of view of Dirichlet forms, this yields a substructure of the standard Dirichlet structure on the Wiener space. As a consequence we obtain several results in stochastic analysis for Poisson functionals that can be interpreted in infinite dimensional analysis for the exponential density. The connection with the Poisson process is made through the fact that the adjoint of this new gradient coincides under certain conditions with the compensated Poisson stochastic integral.

We proceed in the following way. In Sect. 2, we consider the σ-algebra $\mathcal{F}$ generated by a countable collection of independent identically distributed exponential random variables on the Wiener space, and call a Poisson functional any Wiener functional which is measurable with respect to $\mathcal{F}$. The Ornstein-Uhlenbeck operator on the Wiener space appears to be an extension of the number operator $\mathcal{L}$ defined in [13] for Poisson functionals. We deduce results in infinite dimensional analysis for the exponential density, such as the hypercontractivity of the semigroup associated to $\mathcal{L}$, the construction of distributions, and an algebra of test functions on the Poisson space. We introduce in Sect. 3 a random unitary operator $\mathcal{X}$ of the Cameron-Martin space which allows to define a new gradient on Wiener space by composition with the Gross-Sobolev derivative. This gradient is related to the conditional gradient given $\mathcal{F}$ on the Wiener space and to the derivative obtained by shifting the Poisson process jump times, and its adjoint extends the compensated Poisson stochastic integral. In Sect. 4, several results in Malliavin calculus concerning the existence and smoothness of densities of Poisson functionals, as well as the Meyer inequalities, are derived on the Poisson space using the operators that are defined above. Sect. 5 is devoted to the extension to higher orders of differentiation of the equivalence of norms obtained in Sect. 4. We obtain in this way the continuity of the gradient and divergence operators on Sobolev spaces of Poisson functionals. In the sixth section, we deal with the independence of Poisson functionals. From the existing criterion on Wiener space, cf. [18], we deduce necessary and sufficient conditions for the independence of discrete multiple Poisson stochastic integrals. Those integrals are defined with the Laguerre polynomials as stochastic integrals of deterministic discrete time kernels, and the conditions for independence are expressed in terms of the supports of those kernels. In the last section, we study the infinite-dimensional diffusion process associated to the operator $\mathcal{L}$ and show that it gives another example of a process whose hitting probabilities can be estimated in terms of capacities.

Notation. The following definitions can be found in [19]. Let $(W, L^2(\mathbb{R}_+), \mu)$ be the classical Wiener space, and let $(h_k)_{k\geq 0}$ be an orthonormal basis of $L^2(\mathbb{R}_+)$, which will remain fixed throughout this work. We note respectively $\hat{D}$ and $\hat{\delta}$ the Gross-Sobolev derivative and its adjoint on the Wiener space. Recall that $(\hat{\delta}(h_k))_{k\in\mathbb{N}}$ is a system of independent gaussian normal random variables, and for $F = f(\hat{\delta}(h_0), \ldots, \hat{\delta}(h_n))$, $f \in \mathcal{C}_c^\infty(\mathbb{R}^{n+1})$, $\hat{D}F \in L^2(W) \otimes L^2(\mathbb{R}_+)$ is defined as

$$\hat{D}F = \sum_{k=0}^{k=n} \partial_k f(\hat{\delta}(h_0), \ldots, \hat{\delta}(h_n))h_k.$$

The Ornstein-Uhlenbeck operator on the Wiener space is denoted by $-\hat{\mathcal{L}}/2$. It is self-adjoint with respect to μ and satisfies to $\hat{\mathcal{L}} = \hat{\delta}\hat{D}$. Let $\hat{I}_n(g_n)$ represent the Wiener multiple stochastic integral of a symmetric function in the completed symmetric tensor product $L^2(\mathbb{R}_+)^{\circ n}$. We have $\hat{\mathcal{L}}\hat{I}_n(g_n) = n\hat{I}_n(g_n) \quad n \in \mathbb{N}$, and any square-integrable functional F on (W, μ) can be decomposed as a series

$$F = \sum_{n=0}^{\infty} \hat{I}_n(g_n) \quad g_k \in L^2(\mathbb{R}_+)^{\circ k}, \quad k \in \mathbb{N}.$$

Let $\mathcal{P}$ denote the algebra of polynomials in $(\hat{\delta}(h_k))_{k\geq 0}$, which is dense in $L^2(W, \mu)$. For $k \in \mathbb{N}$ and $p > 1$, let $\mathbb{D}_{p,k}$ be the completion of $\mathcal{P}$ under the norm $\| F \|_{p,k} = \| (I + \hat{\mathcal{L}})^{k/2} F \|_{L^p(W,\mu)}$, and let $\mathbb{D}_{p,-k}$ be the dual space of $\mathbb{D}_{p,k}$. Let $\mathbb{D}_\infty = \bigcap_{p,k} \mathbb{D}_{p,k}$. The dual of $\mathbb{D}_\infty$ is $\mathbb{D}_{-\infty} = \bigcup_{p,k} \mathbb{D}_{p,k}$.

To end this introduction, we shortly describe the method that will be used in the next sections. Let us write down the usual integration by parts formula on Wiener space:

$$E[F\hat{\delta}(u)] = E[(\hat{D}F, u)_{L^2(\mathbb{R}_+)}],$$

for $u \in Dom(\hat{\delta})$ and $F \in Dom(\hat{D})$. Consider also a random operator

$$\chi : L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+)$$

such that χ is unitary, μ-a.s. This operator can be viewed as an isometry from $L^2(W) \otimes L^2(\mathbb{R}_+)$ to $L^2(W) \otimes L^2(\mathbb{R}_+)$. Let us apply the above integration by parts to F and χu, with $F \in Dom(\hat{D})$ and $u \in L^2(W) \otimes L^2(\mathbb{R}_+)$ such that $\chi u \in Dom(\hat{\delta})$. We have from the properties of χ:

$$\begin{aligned} E[F\hat{\delta} \circ \chi(u)] &= E[(\hat{D}F, \chi u)_{L^2(\mathbb{R}_+)}] \\ &= E[(\chi^* \circ \hat{D}F, u)_{L^2(\mathbb{R}_+)}], \end{aligned}$$

χ^* being the adjoint of χ. We will show that it is possible to choose χ such that $\hat{\delta} \circ \chi$ extends the stochastic integral with respect to a compensated Poisson process defined on the Wiener space. It will appear that $\chi^* \circ \hat{D}$ is closely related to a gradient defined on Poisson space by shifts of the Poisson process jump times, cf. [2], [13]. Moreover, we have

$$\left(\hat{\delta} \circ \chi\right) \circ \left(\chi^* \circ \hat{D}\right) = \hat{\delta}\hat{D} = \hat{\mathcal{L}},$$

and

$$\| \chi^* \circ \hat{D}F \|_{L^2(\mathbb{R}_+)} = \| \hat{D}F \|_{L^2(\mathbb{R}_+)} \quad \mu - a.s.$$

from the fact that χ is a.s. unitary. As a consequence, any result in Malliavin calculus that involves the norm of the gradient $\hat{D}$ or the Ornstein-Uhlenbeck operator $\hat{\mathcal{L}}$ will be valid on Poisson space and interpreted in terms of the stochastic calculus of variations for the Poisson process, using the compensated Poisson stochastic integral and the derivation with respect to shifts of the jump times.

2 The Poisson space as a subspace of the Wiener space

A characterization of the standard Poisson process on the positive real line is that it is a jump process with jumps of fixed size 1 and independent identically distributed exponential interjump times. We intend here to construct a Poisson process, or equivalently a countable collection of exponential random variables on the Wiener space. We will make use of the fact that the half sum of two independant normal random variables has a $\mathcal{X}^2$ law with 2 degrees of freedom, i.e. an exponential distribution. Let

$$\tau_k = \frac{\hat{\delta}(h_{2k})^2 + \hat{\delta}(h_{2k+1})^2}{2} \quad k \geq 0,$$

then $(\tau_k)_{k\geq 0}$ is a family of independent exponentially distributed random variables, hence it represents a Poisson process $(N_t)_{t\geq 0}$. This does not require the system $(h_k)_{k\geq 0}$ to be complete in $L^2(\mathbb{R}_+)$. Let $T_k = \sum_{i=0}^{i=k-1} \tau_i$, $k \geq 0$, represent the k-th jump time of $(N_t)_{t\geq 0}$. We have

$$N_t = \sum_{k=1}^{\infty} 1_{[T_k,\infty[}(t), \quad t \in \mathbb{R}_+.$$

Note that this construction does not preserve the filtrations generated by the Poisson and Wiener processes, i.e. the filtrations generated by $(N_t)_{t\geq 0}$ and the Brownian motion $(B_t)_{t\geq 0}$ on (W,μ) are not comparable. We define an application $\Xi : W \to \mathbb{R}^{\mathbb{N}}$ by

$$\Xi(\omega) = (\tau_k)_{k\in\mathbb{N}} \quad \mu - a.s. \tag{1}$$

Denote by B the range of Ξ, endowed with the largest σ-algebra that makes Ξ measurable, and let P be the image measure of μ by Ξ:

$$P = \Xi_* \mu,$$

and define an operator $\Theta : L^p(B,P) \to L^p(W,\mu)$ by

$$\Theta F = F \circ \Xi,$$

where F is a polynomial functional on B, i.e. $F((x_k)_{k\in\mathbb{N}}) = f(x_0,\ldots,x_n)$, $n \in \mathbb{N}$, f polynomial. The operator Θ can be extended as an isometry from $L^p(B,P)$ to $L^p(W,\mu)$, $p > 1$. The dual of $\Theta : L^2(B,P) \to L^2(W,\mu)$ is $\Theta^* : L^2(W,\mu) \to L^2(B,P)$, given by

$$\Theta^* F = \Theta^{-1} E[F \mid \mathcal{F}], \quad F \in L^2(W,\mu).$$

We call $\mathcal{F}$ the σ-algebra on W generated by Ξ. In the sequel, $L^p(W,\mathcal{F},\mu_{|\mathcal{F}})$ will be identified with $L^p(B)$ for $p \geq 1$.

Definition 1 *(Poisson space). The space $(W, \mathcal{F}, \mu_{|\mathcal{F}})$ is called the Poisson space. We call a Poisson functional any random variable on $(W, \mathcal{F}, \mu_{|\mathcal{F}})$. Let*

$$\mathcal{P}_{\mathcal{F}} = \{f(\tau_0, \ldots, \tau_n) \ : \ f \textit{ polynomial}, \ n \in \mathbb{N}\}$$

denote the set of polynomial Poisson functionals.

We recall, cf. [13] that $\mathcal{P}_{\mathcal{F}}$ is dense in $L^2(W, \mathcal{F}, \mu_{|\mathcal{F}})$ and that there exists a discrete chaotic decomposition of the space $L^2(W, \mathcal{F}, \mu_{|\mathcal{F}})$ of square-integrable Poisson functionals. This decomposition uses discrete multiple stochastic integrals defined with the Laguerre polynomials

$$L_k(x) = \sum_{i=0}^{i=k} \binom{k}{i} \frac{(-x)^i}{i!} \quad x \in \mathbb{R}_+, \quad k \in \mathbb{N},$$

which are orthonormal with respect to the exponential density. Let $H = l^2(\mathbb{N})$ be the Hilbert space of square-summable sequences, and let $(e_k)_{k \in \mathbb{N}}$ denote the canonical basis of H. For $n \geq 1$, we define the discrete multiple stochastic integral of a symmetric function f_n on $\mathbb{N}^n$ as a linear mapping $I_n : H^{\circ n} \to L^2(W, \mathcal{F}, \mu_{|\mathcal{F}})$, first on elementary functions:

$$I_n(e_{k_1}^{\circ n_1} \circ \cdots \circ e_{k_d}^{\circ n_d}) = n_1! \cdots n_d! L_{n_1}(\tau_{k_1}) \cdots L_{n_d}(\tau_{k_d})$$

where $n_1 + \cdots + n_d = n$, $k_1 \neq \cdots \neq k_d \in \mathbb{N}$. The mapping I_n is extended to any element of the completed symmetric tensor product $H^{\circ n}$ by density, since the linear functional I_n satisfies to an isometry formula, cf. [13]. Moreover, integrals of different orders are orthogonal. As a result, any F in $L^2(W, \mathcal{F}, \mu_{|\mathcal{F}})$ has the orthogonal decomposition

$$F = \sum_{n=0}^{\infty} I_n(f_n) \quad f_k \in H^{\circ k}, \quad k \in \mathbb{N}$$

with the conventions $H^0 = \mathbb{R}$ and $I_0 = I_{\mathbb{R}}$. The following proposition says that the Poisson random variable $I_n(f_n)$ is a multiple Wiener integral of order $2n$, and gives its expression in the Wiener chaotic decomposition. For simplicity, the development is only written for $f_n = e_k^{\circ n}$. Let $C_n^k = n!/(k!(n-k)!)$, $0 \leq k \leq n$, $n \in \mathbb{N}$.

Proposition 1 *The Wiener chaos expansion of $I_n(e_k^{\circ n})$ is given by*

$$I_n(e_k^{\circ n}) = \frac{(-1)^n}{2^n} \sum_{i=0}^{i=n} \hat{I}_{2n} \left(h_{2k}^{\circ 2i} \circ h_{2k+1}^{\circ (2n-2i)} \right) C_n^i / (C_{2n}^{2i})^{1/2}.$$

Proof. The proof relies on the following relation between the Hermite and Laguerre polynomials, cf. [4], p. 195:

$$n!L_n(\frac{x^2+y^2}{2}) = \frac{(-1)^n}{2^n}\sum_{k=0}^{k=n} C_n^k H_{2k}(x)H_{2n-2k}(y)\sqrt{(2k)!(2n-2k)!}$$

and on the definition of the multiple Wiener integral with the Hermite polynomials, cf. [6]. Here, $H_k(x)$ is the k-th normalized Hermite polynomial, defined by the generating series

$$\sum_{n=0}^{\infty} \gamma^n \frac{H_n(x)}{\sqrt{n!}} = exp(\gamma x - \gamma^2/2) \quad \gamma, x \in \mathbb{R}.$$

□

Denote by $\mathcal{L}$ the number operator on the discrete chaotic decomposition, that is $\mathcal{L}$ is a linear operator with

$$\mathcal{L}I_n(g_n) = nI_n(g_n) \quad g_n \in H^{\circ n},\ n \in \mathbb{N},$$

so that the domain of $\mathcal{L}$ is made the following Poisson functionals:

$$Dom(\mathcal{L}) = \left\{ \sum_{n=0}^{\infty} I_n(f_n) \ : \ \sum_{n=0}^{\infty} n^2 \parallel I_n(f_n) \parallel_2^2 < \infty \right\},$$

and $\mathcal{L}$ leaves invariant the space $\mathcal{P}_{\mathcal{F}}$ of polynomial Poisson functionals. The operator $\mathcal{L}$ is the infinite dimensional generalization of the operator $x\partial_x^2 + (1-x)\partial_x$ on $\mathcal{C}^\infty(\mathbb{R})$, whose eigenvectors are the Laguerre polynomials.

We now define Sobolev spaces of Poisson functionals. We call $\mathbb{D}_{p,k}^{\mathcal{F}}$ the completion of the algebra $\mathcal{P}_{\mathcal{F}}$ of polynomial Poisson functionals under the norm

$$\parallel F \parallel_{\mathbb{D}_{p,k}^{\mathcal{F}}} = \parallel (I+\mathcal{L})^{k/2} F \parallel_{L^p(B)} \quad F \in \mathcal{P}_{\mathcal{F}},$$

$p > 1, k \in \mathbb{Z}$, and let

$$\mathbb{D}_{\infty}^{\mathcal{F}} = \bigcap_{p,k} \mathbb{D}_{p,k}^{\mathcal{F}}, \quad \mathbb{D}_{-\infty}^{\mathcal{F}} = \bigcup_{p,k} \mathbb{D}_{p,k}^{\mathcal{F}}.$$

The next proposition says that the σ-algebra $\mathcal{F}$ generated by the Poisson functionals is $\hat{\mathcal{L}}^{-1}$-stable. We refer to [17] for the notion of $\hat{\mathcal{L}}^{-1}$-stable σ-algebra.

Proposition 2 *The operators $\mathcal{L}$ and $\hat{\mathcal{L}}/2$ commute with the conditional expectation with respect to $\mathcal{F}$:*

$$E[\hat{\mathcal{L}}F \mid \mathcal{F}] = 2\mathcal{L}E[F \mid \mathcal{F}] \quad F \in Dom(\hat{\mathcal{L}}),\ p > 1,\ k \in \mathbb{Z},$$

hence $\hat{\mathcal{L}}/2$ is an extension of $\mathcal{L}$. The norms $\|\cdot\|_{I\!D_{p,k}}$ and $\|\cdot\|_{I\!D^{\mathcal{F}}_{p,k}}$ are equivalent on $\mathcal{P}_{\mathcal{F}}$, and $I\!D^{\mathcal{F}}_\infty$ is an algebra. Moreover,

$$E[\,\cdot \mid \mathcal{F}] : I\!D_{p,k} \to I\!D^{\mathcal{F}}_{p,k} \quad p \geq 1,\ k \in \mathbb{Z},$$

is continuous.

Proof. Prop. 1 gives

$$E[H_i(\hat{\delta}(h_{2k}))H_j(\hat{\delta}(h_{2k+1})) \mid \mathcal{F}] = \begin{cases} (-1/2)^{(i+j)} \frac{(i!j!)^{1/2}}{(i/2)!(j/2)!} L_{(i+j)/2}(\tau_k) & i \text{ and } j \text{ even,} \\ 0 \ \ \text{otherwise.} \end{cases}$$

Hence

$$\begin{aligned} & E[\hat{\mathcal{L}}\hat{I}_{i+j}(h_{2k}^{\circ i} \circ h_{2k+1}^{\circ j}) \mid \tau_k] \\ = \ & \sqrt{(i+j)!}E[\hat{\mathcal{L}}(H_i(\hat{\delta}(h_{2k}))H_j(\hat{\delta}(h_{2k+1}))) \mid \tau_k] \\ = \ & (i+j)\sqrt{(i+j)!}E[H_i(\hat{\delta}(h_{2k}))H_j(\hat{\delta}(h_{2k+1})) \mid \tau_k] \\ = \ & 2\sqrt{(i+j)!}\mathcal{L}E[H_i(\hat{\delta}(h_{2k}))H_j(\hat{\delta}(h_{2k+1})) \mid \tau_k] \\ = \ & 2\mathcal{L}E[\hat{I}_{i+j}(h_{2k}^{\circ i} \circ h_{2k+1}^{\circ j}) \mid \tau_k] \end{aligned}$$

for any $i, j, k \in \mathbb{N}$. It follows that if $i_1, \ldots, i_d, j_1, \ldots, j_d \in \mathbb{N}$, $k_1 \neq \cdots \neq k_d$ and $F = \prod_{l=1}^{l=d} \hat{I}_{i_l+j_l}(h_{2k_l}^{\circ i_l} \circ h_{2k_l+1}^{\circ j_l})$,

$$\begin{aligned} E\left[\hat{\mathcal{L}}F \mid \mathcal{F}\right] &= \sum_{p=1}^{p=d} E\left[\prod_{l\neq p} \hat{I}_{i_l+j_l}(h_{2k_l}^{\circ i_l} \circ h_{2k_l+1}^{\circ j_l}) E\left[\hat{\mathcal{L}}\hat{I}_{i_p+j_p}(h_{2k_p}^{\circ i_p} \circ h_{2k_p+1}^{\circ j_p}) \mid \tau_p\right] \mid \mathcal{F}\right] \\ &= 2\sum_{p=1}^{p=d} \prod_{l\neq p} E\left[\hat{I}_{i_l+j_l}(h_{2k_l}^{\circ i_l} \circ h_{2k_l+1}^{\circ j_l}) \mid \tau_l\right] \mathcal{L}E\left[\hat{I}_{i_p+j_p}(h_{2k_p}^{\circ i_p} \circ h_{2k_p+1}^{\circ j_p}) \mid \tau_p\right] \\ &= 2\mathcal{L}E\left[F \mid \mathcal{F}\right], \end{aligned}$$

hence

$$E[\hat{\mathcal{L}}\hat{I}_n(g_n) \mid \mathcal{F}] = 2\mathcal{L}E[\hat{I}_n(g_n) \mid \mathcal{F}], \quad g_n \in L^2(\mathbb{R}_+)^{\circ n}.$$

This implies that $2\mathcal{L}I_n(f_n) = \hat{\mathcal{L}}I_n(f_n), \quad f_n \in H^{\circ n}$. The equivalence of norms follows from the L^p-multiplier theorem, with the fact that $I\!D^{\mathcal{F}}_\infty$ is an algebra, since for $p, q, r > 1$ such that $1/r = 1/p + 1/q$ and $k \in \mathbb{Z}$, there exists a constant $C_{p,q,k}$ such that

$$\begin{aligned} & \| (I + \hat{\mathcal{L}}/2)^{k/2}(FG) \|_{L^r(W)} \\ \leq \ & C_{p,q,k} \| (I + \hat{\mathcal{L}}/2)^{k/2} F \|_{L^p(W)} \| (I + \hat{\mathcal{L}}/2)^{k/2} G \|_{L^q(W)} \quad F, G \in \mathcal{P}_{\mathcal{F}}, \end{aligned}$$

cf. [19]. The continuity of $E[\,\cdot \mid \mathcal{F}]$ can be established as follows. For $p > 1$ and $k \in \mathbb{Z}$, there exists a constant $C_{p,k}$ such that

$$\begin{aligned}
\| E[F \mid \mathcal{F}] \|_{\mathbb{D}^{\mathcal{F}}_{p,k}} &\leq C_{p,k} \| E[F \mid \mathcal{F}] \|_{\mathbb{D}_{p,k}} \\
&= C_{p,k} \| (I + \hat{\mathcal{L}})^{k/2} E[F \mid \mathcal{F}] \|_{L^p(W)} \\
&= C_{p,k} \| E[(I + \hat{\mathcal{L}})^{k/2} F \mid \mathcal{F}] \|_{L^p(W)} \\
&\leq C_{p,k} \| F \|_{\mathbb{D}_{p,k}} \qquad F \in \mathcal{P}.
\end{aligned}$$

□

Another consequence of this proposition is that $\mathcal{L}$ is self-adjoint with respect to $\mu_{|\mathcal{F}}$. Being the restriction of $\hat{\mathcal{L}}/2$ to Poisson functionals, $\mathcal{L}$ shares several properties with $\hat{\mathcal{L}}$. The theorem below can be interpreted as a result in infinite-dimensional analysis for the exponential density, since $\mathcal{L}$ is the infinite dimensional generalization of the operator whose eigenvectors are the Laguerre polynomials, which form an orthonormal sequence for the measure $e^{-x}1_{\{x>0\}}dx$.

Theorem 1 *(Hypercontractivity). Let $p > 1$ and $t > 0$. There exists $q > p$ such that*

$$\| exp(-t\mathcal{L})F \|_{L^q(B)} \leq \| F \|_{L^p(B)} \qquad F \in L^q(B).$$

Proof. Since $exp(-t\mathcal{L}) = exp(-t\hat{\mathcal{L}}/2)$ on $L^2(W, \mathcal{F}, \mu_{\mathcal{F}})$, we can apply to Poisson functionals the existing hypercontractivity theorem on Wiener space, which says that for any $t > 0$ there is $q > p$ such that

$$\| exp(-t\hat{\mathcal{L}}/2)F \|_{L^q(W)} \leq \| F \|_{L^p(W)} \qquad F \in L^q(W).$$

□

Example of a generalized Wiener functionals which is a Poisson functional.
From [1], Prop. VI.1.2.2, we have the following Wiener chaos expansion for the distribution

$$T = 2\pi\delta\left(\int_0^\infty h_0(t)dB_t, \int_0^\infty h_1(s)dB_s\right) \in \mathbb{D}_{2,-r},\ r > 1,$$

where δ is the Dirac distribution at 0 in $\mathbb{R}^2$:

$$T = \sum_{n\geq 0} \frac{1}{2n!} \hat{I}_{2n}\left(\left(\frac{-1}{2}\right)^n (2n)! \sum_{k=0}^{k=n} \frac{1}{k!(n-k!)} h_0^{\circ 2k} \circ h_1^{\circ 2n-2k}\right)$$

Hence from Prop. 1, T is the limit in $\mathbb{D}^{\mathcal{F}}_{2,-r}$, $r > 1$, of a sequence of polynomial Poisson functionals.

We end this section with two definitions. In [13], a gradient operator has been defined for Poisson functionals as a directional derivative in the directions of $H = l^2(\mathbb{N})$, or equivalently by shifts of the Poisson process jump times. We recall this definition with a different interpretation.

Definition 2 *We define* $D : L^2(W,\mu) \to L^2(W,\mathcal{F},\mu_{|\mathcal{F}}) \otimes H$ *by*

$$(DF, h)_H = -\lim_{\varepsilon\to 0} \frac{[\Theta^{-1}F](\Xi + \varepsilon h) - F}{\varepsilon} \quad h \in H,\ F \in \mathcal{P}_{\mathcal{F}}.$$

If $F \in \mathcal{P}_{\mathcal{F}}$ with $F = f(\tau_0, \dots, \tau_n)$, then

$$DF = -\sum_{k=0}^{k=n} \partial_k f(\tau_0, \dots, \tau_n) 1_{\{k\}}.$$

The operator $D : L^2(W,\mathcal{F},\mu_{|\mathcal{F}}) \longrightarrow L^2(W,\mathcal{F},\mu_{|\mathcal{F}}) \otimes H$ is closable and its expression in the discrete chaotic decomposition is written as follows, cf. [13]:

$$D_j I_n(f_n) = \sum_{k=0}^{k=n-1} \frac{n!}{k!} I_k(f_n(*, j, \dots, j)) \quad j \in \mathbb{N},\ f_n \in l^2(\mathbb{N})^{\circ n}.$$

Finally, we define for later use an operator i that turns a discrete-time process into a continuous-time process, using the Poisson process itself.

Definition 3 *If* $f : \mathbb{N}^d \longrightarrow \mathbb{R}^n$ *is a function of discrete variable, we define a* d*-parameter process* $i(f)$ *by*

$$i(f)(t_1, \dots, t_d) = f(N_{t_1^-}, \dots, N_{t_d^-}) \quad t_1, \dots, t_d \in \mathbb{R}_+.$$

The operator i is easily extended to stochastic processes of discrete d-dimensional parameter. If $n = d = 1$, let $j : L^2(W) \otimes L^2(\mathbb{R}_+) \to L^2(W) \otimes l^2(\mathbb{N})$ denote the dual of $i : L^2(W) \otimes l^2(\mathbb{N}) \to L^2(W) \otimes L^2(\mathbb{R}_+)$, i.e. j is a random operator such that

$$(i(u), v)_{L^2(\mathbb{R}_+)} = (u, j(v))_{l^2(\mathbb{N})} \quad \mu - a.s.$$

for $u \in L^2(W) \otimes l^2(\mathbb{N})$, $v \in L^2(W) \otimes L^2(\mathbb{R}_+)$. We have explicitely

$$j(v) = \sum_{k\geq 0} 1_{\{k\}} \int_{T_k}^{T_{k+1}} v(s)ds.$$

3 A new gradient operator on Wiener space

In this section, we define an extension to Wiener functionals of the above gradient operator D, taking into account the conditional gradient given $\mathcal{F}$, cf. [1] for this notion. This new gradient has the following properties: its adjoint coincides with the compensated Poisson stochastic integral under certain conditions, and by composition with its adjoint it yields the Ornstein-Uhlenbeck operator on the Wiener space. It is expressed by composition of the Gross-Sobolev derivative $\hat{D}$ on the Wiener space with a random unitary operator which is defined below. The n-th jump time of the Poisson process $(N_t)_{t\in\mathbb{R}_+}$ defined on (W,μ) is denoted by $T_n = \sum_{k=0}^{k=n-1}\tau_k$, $n \geq 0$.

Definition 4 *For μ-a.s. ω, we define an operator $\chi : L^2(\mathbb{R}_+,\mathbb{R}^2) \longrightarrow L^2(\mathbb{R}_+)$ by*

$$\begin{aligned}\chi u = -\frac{1}{\sqrt{2}}\sum_{k\geq 0} & \frac{\hat{\delta}(h_{2k})h_{2k}+\hat{\delta}(h_{2k+1})h_{2k+1}}{\tau_k}\int_{T_k}^{T_{k+1}} u^{(1)}(s)ds \\ & +\frac{\hat{\delta}(h_{2k})h_{2k+1}-\hat{\delta}(h_{2k+1})h_{2k}}{\tau_k}\int_{T_k}^{T_{k+1}} u^{(2)}(s)ds \qquad u=(u^{(1)},u^{(2)})\in\mathcal{C}_c^\infty(\mathbb{R}_+,\mathbb{R}^2).\end{aligned}$$

We are going to show that μ-a.s., χ is unitary from a certain random subspace $\tilde{H}$ of $L^2(\mathbb{R}_+,\mathbb{R}^2)$ into $L^2(\mathbb{R}_+)$.

Definition 5 *For μ-a.s. $\omega\in W$, we define $\tilde{H}$ to be the random subspace of $L^2(\mathbb{R}_+,\mathbb{R}^2)$ of the form*

$$\tilde{H} = \left\{ i((f,g)) \quad : \quad (f,g)\in l^2(\mathbb{N},\mathbb{R}^2)\right\}.$$

The operator i was introduced in Def. 3.

Proposition 3 *The operator χ is unitary from $\tilde{H}$ into $L^2(\mathbb{R}_+)$:*

$$\chi^*\chi = I_{\tilde{H}} \qquad \text{and} \qquad \chi\chi^* = I_{L^2(\mathbb{R}_+)} \quad \mu-a.s.$$

and its adjoint is $\chi^* : L^2(\mathbb{R}_+) \longrightarrow L^2(\mathbb{R}_+,\mathbb{R}^2)$, given by

$$\begin{aligned}\chi^* v \;=\; -\frac{1}{\sqrt{2}}\sum_{k\geq 0}^{\infty}\frac{1}{\tau_k}1_{[T_k,T_{k+1}[}&((v,h_{2k})_{L^2(\mathbb{R}_+)}\hat{\delta}(h_{2k})+(v,h_{2k+1})_{L^2(\mathbb{R}_+)}\hat{\delta}(h_{2k+1}),\\ &(v,h_{2k+1})_{L^2(\mathbb{R}_+)}\hat{\delta}(h_{2k})-(v,h_{2k})_{L^2(\mathbb{R}_+)}\hat{\delta}(h_{2k+1})) \quad \mu-a.s.\end{aligned}$$

Proof. We have if $u=(u^{(1)},u^{(2)})\in\mathcal{C}_c^\infty(\mathbb{R}_+,\mathbb{R}^2)$ and $v\in\mathcal{C}_c^\infty(\mathbb{R}_+)$:

$$(\chi u, v)_{L^2(\mathbb{R}_+)}$$

$$
\begin{aligned}
&= \frac{1}{\sqrt{2}}\sum_{k\geq 0}\int_{T_k}^{T_{k+1}} u^{(1)}(s)ds\frac{\hat{\delta}(h_{2k})(h_{2k},v)_{L^2(\mathbb{R}_+)}+\hat{\delta}(h_{2k+1})(h_{2k+1},v)_{L^2(\mathbb{R}_+)}}{\tau_k}\\
&\quad + \int_{T_k}^{T_{k+1}} u^{(2)}(s)ds\frac{\hat{\delta}(h_{2k+1})(h_{2k+1},v)_{L^2(\mathbb{R}_+)}-\hat{\delta}(h_{2k+1})(h_{2k},v)_{L^2(\mathbb{R}_+)}}{\tau_k}\\
&= (u,\mathcal{X}^*v)_{L^2(\mathbb{R}_+,\mathbb{R}^2)} \quad \mu - a.s.
\end{aligned}
$$

Hence $\mathcal{X}$ and $\mathcal{X}^*$ are adjoint $\mu - a.s.$ It is easy to check that $\mathcal{X}^*\mathcal{X} = I_{\tilde{H}}$ and $\mathcal{X}\mathcal{X}^* = I_{L^2(\mathbb{R}^+)}$ $\mu - a.s.$, and the fact that $\mathcal{X} : \tilde{H} \to L^2(\mathbb{R}_+)$ is unitary follows.

$\square$

Again, $\mathcal{X}$ is easily extended to two dimensional stochastic processes as an isometry $\mathcal{X} : L^2(W) \otimes L^2(\mathbb{R}_+,\mathbb{R}^2) \to L^2(W) \otimes L^2(\mathbb{R}_+)$, with the properties that

$$\mathcal{X}\mathcal{X}^* = I_{L^2(W)\otimes L^2(\mathbb{R}_+)} \quad and \quad (\mathcal{X}u,v)_{L^2(\mathbb{R}_+)} = (u,\mathcal{X}^*v)_{L^2(\mathbb{R}_+,\mathbb{R}^2)} \ \mu - a.s.,$$

$u \in L^2(W) \otimes L^2(\mathbb{R}_+,\mathbb{R}^2)$, $v \in L^2(W) \otimes L^2(\mathbb{R}_+)$. We now define a gradient $\tilde{D}$ by composition of $\hat{D}$ with $\mathcal{X}^*$.

Definition 6 *We define an operator* $\tilde{D} : L^2(W) \longrightarrow L^2(W) \otimes L^2(\mathbb{R}_+,\mathbb{R}^2)$ *by*

$$\tilde{D}F = \frac{1}{\sqrt{2}}\mathcal{X}^* \circ \hat{D}F \quad F \in \mathcal{P}.$$

Then $\hat{D}F = \sqrt{2}\mathcal{X} \circ \tilde{D}F$, $F \in \mathcal{P}$. According to this definition, $\tilde{D}$ is a derivation operator on the Wiener space.

Proposition 4 *As a direct consequence of the fact that $\mathcal{X}$ is unitary, we have:*

- *The operators $\tilde{D}$ and $\hat{D}$ can be extended to the same domains. More precisely,*

$$2 \parallel \tilde{D}F \parallel^2_{L^2(\mathbb{R}_+,\mathbb{R}^2)} = \parallel \hat{D}F \parallel^2_{L^2(\mathbb{R}_+)} \quad F \in \mathbb{D}_{2,1},\ \mu - a.s.,$$

  hence the operator $\tilde{D}$ is closable and local.

- *Let $-\hat{\mathcal{L}}/2$ denote the Ornstein-Uhlenbeck operator on the Wiener space. We have the following decomposition of $\hat{\mathcal{L}}$:*

$$\hat{\mathcal{L}}/2 = \tilde{\delta}\tilde{D}.$$

Note that the usual decomposition of the Ornstein-Uhlenbeck operator is given by $\hat{\mathcal{L}} = \hat{\delta}\hat{D}$.

We now show that for $F \in Dom(\hat{D})$, the second component $\tilde{D}^{(2)}F$ of $\tilde{D}F$ is related to the conditional gradient of F given $\mathcal{F}$, cf. [1], whereas its first component $\tilde{D}^{(1)}F$ is expressed with the operator D defined in Def. 2 by shifts of the Poisson process jump times. Denote by $\mathcal{H}$ the orthogonal subspace in $L^2(W) \otimes L^2(\mathbb{R}_+)$ of the set

$$\left\{ Z\hat{D}U \ : \ U \in \mathbb{D}^{\mathcal{F}}_{2,1}, \ Z \in L^\infty(W,\mu) \right\}.$$

Let $P^{\mathcal{H}}$ be the orthogonal projection on $\mathcal{H}$ in $L^2(W) \otimes L^2(\mathbb{R}_+)$. Recall that the conditional gradient given $\mathcal{F}$ of $F \in \mathbb{D}_{2,1}$ is defined as $\hat{D}^{\mathcal{F}}F = P^{\mathcal{H}}\hat{D}F$, $F \in \mathbb{D}_{2,1}$, cf. [1], V.5.2.3.

Proposition 5 *The conditional gradient $\hat{D}^{\mathcal{F}}F$ of $F \in \mathbb{D}_{2,1}$ given $\mathcal{F}$ is*

$$\hat{D}^{\mathcal{F}}F = \sqrt{2}\chi(0, \tilde{D}^{(2)}F) \quad \mu - a.s.$$

Let $F \in \mathcal{P}_{\mathcal{F}}$ be a polynomial Poisson functional. We have

$$\tilde{D}F = i((DF, 0)) \quad \mu - a.s.$$

Proof. Let $F = f(\hat{\delta}(h_0), \ldots, \hat{\delta}(h_n))$ with $f \in \mathcal{C}^\infty_c(\mathbb{R}^{n+1})$. We have

$$\hat{D}F - \sqrt{2}\chi(0, \tilde{D}^{(2)}F) = \sqrt{2}\chi(\tilde{D}^{(1)}F, 0),$$

hence $\hat{D}F - \sqrt{2}\chi(0, \tilde{D}^{(2)}F) \in \left\{ Z\hat{D}U \ : \ U \in \mathbb{D}^{\mathcal{F}}_{2,1}, \ Z \in L^\infty(W,\mu) \right\}$. We also have

$$E[Z(\chi(0, \tilde{D}^{(2)}F), \hat{D}U)_{L^2(\mathbb{R}_+)}] = 0 \quad U \in \mathcal{P}_{\mathcal{F}}, \ Z \in L^\infty(W,\mu),$$

hence $\sqrt{2}\chi(0, \tilde{D}^{(2)}F) = P^{\mathcal{H}}\hat{D}F$. The result is obtained by density. For the second part, we notice that the conditional gradient of a Poisson functional given $\mathcal{F}$ is 0 and that a simple calculation yields $\chi^*\hat{D}U = \sqrt{2}i(DU, 0)$, $\quad U \in \mathcal{P}_{\mathcal{F}}$.

□

The following definition gives the adjoint of $\tilde{D}$. Let $\mathcal{V}$ be the class of processes defined by

$$\begin{aligned} \mathcal{V} = \{ & u \in L^2(W) \otimes L^2(\mathbb{R}_+, \mathbb{R}^2) \ : \\ & u = \left(f(\cdot, \hat{\delta}(h_0), \ldots, \hat{\delta}(h_n)), g(\cdot, \hat{\delta}(h_0), \ldots, \hat{\delta}(h_n)) \right), \ f, g \in \mathcal{C}^\infty_c(\mathbb{R}^{n+1}_+), \ n \in \mathbb{N}\}. \end{aligned}$$

Definition 7 *We define the operator $\tilde{\delta} : L^2(W) \otimes L^2(\mathbb{R}_+, \mathbb{R}^2) \longrightarrow L^2(W)$ by*

$$\tilde{\delta}(v) = \frac{1}{\sqrt{2}}\hat{\delta} \circ \chi(v), \quad v \in \mathcal{V}.$$

We have the following commutative diagram:

$$
\begin{array}{ccccccc}
L^2(W) & \xrightarrow{\sqrt{2}\hat{D}} & L^2(W)\otimes L^2(\mathbb{R}_+) & \quad & L^2(W)\otimes L^2(\mathbb{R}_+) & \xrightarrow{\sqrt{2}\hat{\delta}} & L^2(W) \\
\updownarrow & & \downarrow \chi^* & & \chi \uparrow & & \updownarrow \\
L^2(W) & \xrightarrow{\tilde{D}} & L^2(W)\otimes L^2(\mathbb{R}_+,\mathbb{R}^2) & & L^2(W)\otimes L^2(\mathbb{R}_+,\mathbb{R}^2) & \xrightarrow{\tilde{\delta}} & L^2(W).
\end{array}
$$

Proposition 6 *The operator $\tilde{\delta}$ is closable, adjoint of $\tilde{D}$ and satisfies to*

$$\tilde{\delta}(u) = \int_0^\infty u^{(1)}(s)d(N_s - s) - trace(\tilde{D}u), \;\; u \in \mathcal{V},$$

where $trace(\tilde{D}u) = \int_0^\infty \tilde{D}_s^{(1)} u^{(1)}(s)ds + \int_0^\infty \tilde{D}_s^{(2)} u^{(2)}(s)ds$.

Let $Dom(\tilde{\delta})$ denote the domain of the closed extension of $\tilde{\delta}$.

Proof. Recall that by definition, cf. [6], [19], if $v = \sum_{i=0}^{i=n} h_i f_i(\hat{\delta}(h_0), \ldots, \hat{\delta}(h_n))$ with $f_i \in \mathcal{C}_c^\infty(\mathbb{R}^{n+1})$, $i = 0, \ldots, n$, then

$$\hat{\delta}(v) = \sum_{i=0}^{i=n} \hat{\delta}(h_i) f_i(\hat{\delta}(h_0), \ldots, \hat{\delta}(h_n)) - \partial_i f_i(\hat{\delta}(h_0), \ldots, \hat{\delta}(h_n)).$$

Applying the above formula to χu, $u \in \mathcal{V}$, we obtain:

$$
\begin{aligned}
\tilde{\delta}(u) &= \hat{\delta}(\chi u) \\
&= \sum_{k\geq 0} u^{(1)}(T_{k+1}) \frac{\hat{\delta}(h_{2k})^2 + \hat{\delta}(h_{2k+1})^2}{2\tau_k} + u^{(2)}(T_{k+1}) \frac{\hat{\delta}(h_{2k})\hat{\delta}(h_{2k+1}) - \hat{\delta}(h_{2k+1})\hat{\delta}(h_{2k})}{2\tau_k} \\
&\quad + \frac{1}{2\tau_k} \int_{T_k}^{T_{k+1}} \Big((\hat{D}u^{(1)}(s), h_{2k})\hat{\delta}(h_{2k}) + (\hat{D}u^{(2)}(s), h_{2k+1})\hat{\delta}(h_{2k+1}) \Big) ds \\
&\quad + \frac{1}{2\tau_k} \int_{T_k}^{T_{k+1}} \Big((\hat{D}u^{(2)}(s), h_{2k+1})\hat{\delta}(h_{2k+1}) - (\hat{D}u^{(2)}(s), h_{2k})\hat{\delta}(h_{2k}) \Big) ds \\
&\quad + \frac{1}{\tau_k} \int_{T_k}^{T_{k+1}} u^{(1)}(s)ds - \int_{T_k}^{T_{k+1}} u^{(1)}(s) \frac{\hat{\delta}(h_{2k})^2 + \hat{\delta}(h_{2k+1})^2}{2\tau_k^2} - \int_{T_k}^{T_{k+1}} u^{(1)}(s)ds \\
&= \int_0^\infty u^{(1)}(s)d(N_s - s) - \int_0^\infty \tilde{D}_2^{(1)} u^{(1)}(s)ds - \int_0^\infty \tilde{D}_s^{(2)} u^{(2)}(s)ds.
\end{aligned}
$$

The operator $\tilde{\delta}$ is adjoint of $\tilde{D}$ and closable since χ and χ^* are adjoint and the domain of $\tilde{D}$ is dense in $L^2(W)$.

□

Let $(\mathcal{F}_t)_{t\geq 0}$ be the filtration generated by $(N_t)_{t\geq 0}$ on (W, μ).

Corollary 1 *If $u = (u^{(1)}, u^{(2)}) \in L^2(W) \otimes L^2(\mathbb{R}_+, \mathbb{R}^2)$ is $(\mathcal{F}_t)$-predictable, then $\tilde{\delta}(u)$ coincides with the compensated Poisson stochastic integral of $u^{(1)}$:*

$$\tilde{\delta}(u) = \int_0^\infty u^{(1)}(s)d(N_s - s),$$

and any Poisson stochastic integral has a representation as an anticipative Wiener-Skorohod integral:

$$\int_0^\infty u^{(1)}(s)d(N_s - s) = \hat{\delta}(\chi(u^{(1)}, 0))/\sqrt{2}.$$

Proof. The conditional gradient given $\mathcal{F}$ of a Poisson functional is 0, cf. Prop. 5, hence from Prop. 6 the first part of this statement is identical to the Poisson space result that can be found in [1], [2], [13]. The representation property comes from the relation $\tilde{\delta} = \frac{1}{\sqrt{2}}\hat{\delta} \circ \chi$.

□

The above coincidence can occur under weaker conditions, for instance without predictability requirements. For example, it is sufficient to have $(u^{(1)}, 0) \in \mathcal{V}$ with $u^{(1)} \in L^2(W, \mathcal{F}, \mu_{|\mathcal{F}}) \otimes L^2(\mathbb{R}_+)$ and

$$u^{(2)}(t) = -\sum_{k\geq 0} 1_{]T_k, T_{k+1}]}(t) \arctan\left(\frac{\hat{\delta}(h_{2k})}{\hat{\delta}(h_{2k+1})}\right) \tilde{D}_t^{(1)} u^{(1)}(t) \quad t \in \mathbb{R}_+.$$

In this case, $\tilde{D}^{(1)}u^{(1)} + \tilde{D}^{(2)}u^{(2)} = 0$ $\mu\otimes dt$-a.e., and the trace term in (6) vanishes. The representation property for Poisson stochastic integrals as Wiener-Skorohod integrals also extends to anticipative integrands in $Dom(\tilde{\delta})$. This result differs from the result obtained via the Clark formula, cf. [3], in that the process $\frac{1}{\sqrt{2}}\chi u$ that we obtain is not adapted and its expression is easier to compute.

4 Meyer inequalities on Poisson space and applications

The first consequence of the above propositions is that the Meyer inequalities on Poisson space hold for the operators $\tilde{D}$ and $\mathcal{L}$, given that they are verified for $\hat{D}$ and $\hat{\mathcal{L}}$. The spaces $L^p(B, P)$ and $L^p(W, \mathcal{F}, \mu_{|\mathcal{F}})$ are identified via the operator Θ for $p \geq 1$.

Theorem 2 *For any $p > 1$, there exist $A_p, B_p > 0$ such that for any Poisson polynomial functional $F \in \mathcal{P}_{\mathcal{F}}$,*

$$A_p \parallel \tilde{D}F \parallel_{L^p(B,L^2(\mathbb{R}_+))}$$
$$\leq \ \parallel (I+\mathcal{L})^{1/2}F \parallel_{L^p(B)} \leq B_p(\parallel \tilde{D}F \parallel_{L^p(B,L^2(\mathbb{R}_+))} + \parallel F \parallel_{L^p(B)}).$$

Proof. We write the Meyer inequalities, cf. [10], on the Wiener space and make use of the facts that χ is unitary from $\tilde{H}$ to $L^2(\mathbb{R}_+)$, $\mu - a.s.$ and $\hat{\mathcal{L}}$ is an extension of $2\mathcal{L}$.

□

The difference between this result and the Meyer inequalities on the Wiener space comes from the fact that on Poisson functionals, $\tilde{D}$ is defined by shifting the jump times of the Poisson process, and its adjoint extends the compensated Poisson stochastic integral, whereas $\hat{D}$ is defined by shifts of the Wiener process trajectories and its adjoint extends the Itô-Wiener stochastic integral.

We can also define the composition of a Schwartz distribution with a Poisson functional as a distribution in $\mathbb{D}^{\mathcal{F}}_{-\infty}$. Let $\mathcal{S}_{2k}$, $k \in \mathbb{Z}$, be the completion of the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ under the norm $\parallel \phi \parallel_{\mathcal{S}_{2k}} = \parallel (1+ \mid x \mid^2 + \Delta)^k \phi \parallel_{\infty}$.

Theorem 3 *Let $F_1, \ldots, F_d \in \mathbb{D}^{\mathcal{F}}_{\infty}$ such that $det(((\tilde{D}F_i, \tilde{D}F_j)_{L^2(\mathbb{R}_+,\mathbb{R}^2)})_{1\leq i,j\leq d})^{-1} \in \bigcap_{p>1} L^p(B,P)$. Then for $k \in \mathbb{Z}$ and $p > 1$, there exists $C_{p,k} > 0$ such that*

$$\parallel \phi \circ F \parallel_{p,2k} \leq C_{p,k} \parallel \phi \parallel_{\mathcal{S}_{2k}} \qquad \phi \in \mathcal{S}(\mathbb{R}^d).$$

This implies that if $T \in \mathcal{S}_{2k}$, $T \circ F$ is well defined in $\mathbb{D}_{p,k}$, $p > 1$, $k \in \mathbb{Z}$.

The proof relies again on the fact that χ is unitary and $\mathbb{D}^{\mathcal{F}}_{\infty} \subset \mathbb{D}_{\infty}$, given the Wiener space result in [19]. In the same way, we obtain:

Theorem 4 *Under the hypothesis of the preceding theorem, the Poisson functional $F = (F_1, \ldots, F_d)$ has a $\mathcal{C}^{\infty}$ density on $\mathbb{R}^d$.*

The hypothesis is expressed by perturbations of the Poisson process trajectories. The following exponential integrability criterion comes from [5] and [16] for the gaussian case. It is proved in the same way as Th. 2 and 3.

Theorem 5 *If $F \in \mathbb{D}^{\mathcal{F}}_{p,1}$, $p > 1$, is such that $\parallel \tilde{D}F \parallel_{L^{\infty}(W,L^2(\mathbb{R}_+,\mathbb{R}^2))} < \infty$, then there exists $\lambda > 0$ such that*

$$E[exp(\lambda F^2)] < \infty.$$

Denote by $(W, \mu, \mathbb{D}_{2,1}, \epsilon)$ the standard Dirichlet structure on Wiener space, cf. [1]. The Dirichlet form ϵ is defined as $\epsilon(F,G) = -\frac{1}{2}E[F\hat{\mathcal{L}}G]$, $F,G \in Dom(\hat{D})$. It admits a carré du champ operator Γ defined by $\Gamma(F,G) = (\hat{D}F, \hat{D}G)_{L^2(\mathbb{R}_+)}$. Prop. 4 shows that this structure admits $\sqrt{2}\tilde{D}$ as well as $\hat{D}$ as a gradient, i.e. $\Gamma(F,G) = 2(\tilde{D}F, \tilde{D}G)_{L^2(\mathbb{R}_+)}$. Moreover, $(W, \mathcal{F}, \mu_{|\mathcal{F}}, \mathbb{D}^{\mathcal{F}}_{2,1}, \epsilon \mid \mathbb{D}^{\mathcal{F}}_{2,1})$ is the Dirichlet substructure generated by $(\tau_k)_{k\in\mathbb{N}}$, cf. [1], V.5.1.1. As a substructure, $(W, \mathcal{F}, \mu_{|\mathcal{F}}, \mathbb{D}^{\mathcal{F}}_{2,1}, \epsilon \mid \mathbb{D}^{\mathcal{F}}_{2,1})$ is local, admits a carré du champ operator, and satisfies the energy image density property:

Theorem 6 *If $F_1, \ldots, F_d \in \mathbb{D}^{\mathcal{F}}_{2,1}$ with $det(((\tilde{D}F_i, \tilde{D}F_j)_{L^2(\mathbb{R}_+,\mathbb{R}^2)})_{1\leq i,j\leq d}) > 0$ μ-a.s., then the law of $F = (F_1, \ldots, F_d)$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^d$.*

We obtained a criterion for the law of a Poisson functional to have a density, which directly involves the stochastic calculus of variations by perturbations of the Poisson process trajectories.

5 Extensions

In this section, we give a version of the Meyer inequalities for higher orders of differentiation, and extend the operators $j \circ i \circ D$ and $\tilde{\delta} \circ i$ to Sobolev spaces of H-valued functionals. Let $\mathcal{P}^*_{\mathcal{F}}$ denote the set of functions $u : \mathbb{N} \to \mathcal{P}$ such that u has a finite support in $\mathbb{N}$. This set is dense in $L^2(B) \otimes l^2(\mathbb{N})$.

Lemma 1 *Define the operator $P^{(1)}_t : L^2(B) \otimes l^2(\mathbb{N}) \to L^2(B) \otimes l^2(\mathbb{N})$, $t \in \mathbb{R}_+$ by $P^{(1)}_t u = ((P^{(1)}_t u)_k)_{k\geq 0}$, where*

$$(P^{(1)}_t u)_k = (e^{-t} - 1)D_k P_t u_k + e^{-t} P_t u_k, \ k \in \mathbb{N}, \ u \in \mathcal{P}^*_{\mathcal{F}}.$$

Then $(P^{(1)}_t)_{t\in\mathbb{R}_+}$ is a semi-group, and we have the relation

$$P_t F = P^{(1)}_t DF \quad F \in \mathcal{P}_{\mathcal{F}}, \ t \in \mathbb{R}_+.$$

Proof. Let $F = I_n(f_n)$, $n \geq 1$ and $f_n \in l^2(\mathbb{N})^{\circ n}$. We have from the expression of D as an annihilation operator, cf. [13]:

$$D_k P_t F = e^{-nt} \sum_{l=0}^{l=n-1} I_l(f_n(*, k, \ldots, k))$$

and

$$e^{-t}P_tD_kF = e^{-t}\sum_{l=0}^{l=n-1} e^{-lt}I_l(f_n(*,k,\ldots,k)).$$

Hence

$$\begin{aligned}(e^{-t}-1)D_kP_tD_kF &= \sum_{p=1}^{p=n-1}(e^{-(p+1)t}-e^{-pt})\sum_{l=0}^{l=p-1} I_l(f_n(*,k,\ldots,k))\\ &= D_kP_tF - e^{-t}P_tD_kF,\end{aligned}$$

or $(P_t^{(1)}DF)_k = D_kP_tF \quad F \in \mathcal{P}$. From the following equalities, $(P_t^{(1)})_{t\in\mathbb{R}_+}$ is a semigroup. Let $u \in \mathcal{P}_{\mathcal{F}}^*$, $k \in \mathbb{N}$, and choose $F_k \in \mathcal{P}$ such that $u_k = D_kF_k$. We have for $s,t > 0$:

$$\begin{aligned}(P_{t+s}^{(1)}u)_k &= (P_{t+s}^{(1)}DF_k)_k = D_kP_{t+s}F_kD_kP_tP_sF_k\\ &= (P_t^{(1)}DP_sF_k)_k = (P_t^{(1)}P_s^{(1)}DF_k)_k = (P_t^{(1)}P_s^{(1)}u)_k \quad k \in \mathbb{N}.\end{aligned}$$

Hence $P_{t+s}^{(1)} = P_t^{(1)}P_s^{(1)}$, for $s,t > 0$.

□

Proposition 7 *Let $\mathcal{L}^{(1)}$ denote the generator of $(P_t^{(1)})_{t\geq 0}$. For $u \in \mathcal{P}_{\mathcal{F}}^*$, we have $\mathcal{L}^{(1)}u = \big((\mathcal{L}^{(1)}u)_k\big)_{k\in\mathbb{N}}$ with*

$$(\mathcal{L}^{(1)}u)_k = (\mathcal{L} + I + D_k)u_k, \quad k \in \mathbb{N}.$$

The duality relation

$$(i(u), i(\mathcal{L}^{(1)}v))_{L^2(B)\otimes L^2(\mathbb{R}_+)} = (i(\mathcal{L}^{(1)}u), i(v))_{L^2(B)\otimes L^2(\mathbb{R}_+)} \quad u,v \in \mathcal{P}_{\mathcal{F}}^*,$$

holds, and we have the commutation relation

$$\mathcal{L}^{(1)}D = D\mathcal{L} \ \text{ on } \mathcal{P}_{\mathcal{F}}.$$

Proof. This is a consequence of the above proposition. The duality relation comes from the equality

$$E[\tau_ku_k(\mathcal{L}^{(1)}v)_k] = E[\tau_k(\mathcal{L}^{(1)}u)_kv_k] \ \ u,v \in \mathcal{P}_{\mathcal{F}}^*,\ k \in \mathbb{N},$$

that can be checked using the explicit expression of $\mathcal{L}^{(1)}$:

$$\begin{aligned}
& E[\tau_k u_k(\mathcal{L}+I+D_k)v_k] \\
&= E[v_k\mathcal{L}(\tau_k u_k)+\tau_k u_k v_k+\tau_k u_k D_k v_k] \\
&= E[v_k\tau_k\mathcal{L}u_k+v_k u_k\mathcal{L}\tau_k-2(\tilde{D}\tau_k,\tilde{D}u_k)+\tau_k u_k v_k+\tau_k u_k D_k v_k] \\
&= E[v_k\tau_k\mathcal{L}u_k-v_k u_k+\tau_k v_k u_k+2v_k\tau_k D_k u_k+u_k D_k(\tau_k v_k)] \\
&= E[\tau_k v_k\mathcal{L}u_k+v_k\tau_k D_k u_k+\tau_k u_k v_k] \\
&= E[\tau_k v_k(\mathcal{L}+I+D_k)u_k], \quad u,v\in\mathcal{P}_{\mathcal{F}}^*,\ k\in\mathbb{N}.
\end{aligned}$$

We used here the relation $\mathcal{L}(FG)=F\mathcal{L}G+G\mathcal{L}F-2(\tilde{D}F,\tilde{D}G)_{L^2(\mathrm{R}_+,\mathrm{R}^2)}$ $F,G\in\mathcal{P}_{\mathcal{F}}$, cf. [13], and the fact that $I+D_k$ is adjoint of D_k, $k\in\mathbb{N}$ with respect to P.

□

We now aim to construct Sobolev spaces of H-valued functionals, in order to extend the Poisson gradient and divergence operators to distributions.

Definition 8 *We define the norm* $\|\cdot\|_{\mathbb{D}_{p,k}(H)}$ *on* $\mathcal{P}_{\mathcal{F}}^*$ *by*

$$\| u \|_{\mathbb{D}_{p,k}(H)}=\| i((I_H+\mathcal{L}^{(1)})^{k/2}u) \|_{L^p(B,L^2(\mathrm{R}_+))} .$$

The space $\mathbb{D}_{p,k}(H)$ *is defined to be the completion of* $\mathcal{P}_{\mathcal{F}}^*$ *with respect to the norm* $\|\cdot\|_{\mathbb{D}_{p,k}(H)}$.

The following extension of Th. 2 holds:

Theorem 7 *For* $p>1$ *and* $k\in\mathbb{Z}$, *there exists two constants* $A_{p,k},B_{p,k}>0$ *such that for any Poisson polynomial functional* $F\in\mathcal{P}_{\mathcal{F}}$:

$$A_{p,k}\| DF \|_{\mathbb{D}_{p,k}(H)}\leq\| F \|_{\mathbb{D}^{\mathcal{F}}_{p,k+1}}\leq B_{p,k}(\| DF \|_{\mathbb{D}_{p,k}(H)}+\| F \|_{L^p(B)}).$$

Proof. We have $(I_H+\mathcal{L}^{(1)})^{k/2}DF=D(I+\mathcal{L})^{k/2}F$, $F\in\mathcal{P}_{\mathcal{F}}$. Hence

$$\begin{aligned}
\| DF \|_{\mathbb{D}_{p,k}(H)} &= \| i((I_H+\mathcal{L}^{(1)})^{k/2}DF) \|_{L^p(B,L^2(\mathrm{R}_+))} \\
&= \| i(D(I+\mathcal{L})^{k/2}F) \|_{L^p(B,L^2(\mathrm{R}_+))} \\
&= \| D(I+\mathcal{L})^{k/2}F \|_{\mathbb{D}_{p,0}(H)} \\
&= \| \tilde{D}(I+\mathcal{L})^{k/2}F \|_{L^p(B,L^2(\mathrm{R}_+))} \quad k\in\mathbb{Z},\ p>1.
\end{aligned}$$

It remains to apply Th. 2 to $(I+\mathcal{L})^{k/2}F$.

□

Corollary 2 *The operator $j \circ i \circ D$ can be extended as a continuous operator*

$$j \circ i \circ D : \mathbb{D}^{\mathcal{F}}_{p,k} \longrightarrow \mathbb{D}_{p,k-1}(H) \quad k \in \mathbb{Z}, \quad p > 1.$$

The operator $\tilde{\delta} \circ i(\cdot, 0)$ can be extended as a continuous operator

$$\tilde{\delta} \circ i \circ (\cdot, 0) : \mathbb{D}_{p,k}(H) \longrightarrow \mathbb{D}^{\mathcal{F}}_{p,k-1} \quad k \in \mathbb{Z}, \quad p > 1.$$

Proof. We have for $u \in \mathcal{P}^*_{\mathcal{F}}$ and $F \in \mathcal{P}_{\mathcal{F}}$:

$$\begin{aligned}
\mid E\left[F\tilde{\delta} \circ i \circ (u, 0)\right] \mid &= \mid E\left[(i(u, 0), \tilde{D}F)_{L^2(\mathrm{R}_+, \mathrm{R}^2)}\right] \mid \\
&= \mid E\left[(i(u), i(DF))_{L^2(\mathrm{R}_+)}\right] \mid \\
&= \mid E\left[(i((I_H + \mathcal{L}^{(1)})^{k/2}u), i((I_H + \mathcal{L}^{(1)})^{-k/2}DF))_{L^2(\mathrm{R}_+)}\right] \mid \\
&\leq \parallel u \parallel_{\mathbb{D}_{p,k}(H)} \parallel DF \parallel_{\mathbb{D}_{q,-k}(H)} \\
&\leq C_{p,k} \parallel u \parallel_{\mathbb{D}_{p,k}(H)} \parallel F \parallel_{\mathbb{D}_{q,-k+1}}
\end{aligned}$$

from Th. 7 and Prop. 7, where $p, q > 1$ are such that $1/p + 1/q = 1$ and $C_{p,k}$ is a constant. Hence $\parallel \tilde{\delta} \circ i(u) \parallel_{\mathbb{D}^{\mathcal{F}}_{p,k-1}} \leq C_{p,k} \parallel u \parallel_{\mathbb{D}_{p,k}(H)}$. For the second relation, we have

$$\begin{aligned}
(j \circ i \circ DF, u)_{L^2(B) \otimes l^2(\mathrm{N})} &= E\left[F\tilde{\delta} \circ i(u)\right] \\
&= E\left[(I + \mathcal{L})^{k/2}F(I + \mathcal{L})^{-k/2}\tilde{\delta} \circ i(u)\right] \\
&\leq \parallel F \parallel_{\mathbb{D}^{\mathcal{F}}_{p,k}} \parallel \tilde{\delta} \circ i(u) \parallel_{\mathbb{D}^{\mathcal{F}}_{q,-k}} \\
&\leq C_{p,k} \parallel F \parallel_{\mathbb{D}^{\mathcal{F}}_{p,k}} \parallel u \parallel_{\mathbb{D}^{\mathcal{F}}_{q,-k+1}} \qquad u \in \mathcal{P}^*_{\mathcal{F}}, \; F \in \mathcal{P}_{\mathcal{F}}.
\end{aligned}$$

Hence $\parallel j \circ i \circ DF \parallel_{\mathbb{D}^{\mathcal{F}}_{p,k-1}} \leq C_{p,k} \parallel F \parallel_{\mathbb{D}^{\mathcal{F}}_{p,k}}$, $F \in \mathcal{P}_{\mathcal{F}}$.

□

The main problem that we encounter in the extension of the Meyer inequalities to the case of higher derivatives lies with the definition of the iterated gradient $\tilde{D}\tilde{D}F$. In fact, even for $F \in \mathcal{P}_{\mathcal{F}}$, e.g. $F = \tau_0$, $\tilde{D}F$ is a random indicator function and $\tilde{D}\tilde{D}F$ can not make sense as a random variable. To circumvent this difficulty, we choose to take

$$\parallel i \circ D^k F \parallel_{L^2(B) \otimes L^2(\mathrm{R}^k_+)},$$

where $D^k : L^2(B) \to L^2(B) \otimes H^{\circ k}$ is the k-th iteration of D, for the norm of the iterated gradient of $F \in \mathcal{P}_{\mathcal{F}}$. We are going to give an equivalence of norms between the norm $\parallel \cdot \parallel_{\mathbb{D}^{\mathcal{F}}_{2,k}}$ and the norm defined with $i \circ D^k$, for $p = 2$ and $k \geq 0$.

Proposition 8 *For $k \in \mathbb{N}$, there exists $A_k, B_k > 0$ such that for any Poisson polynomial functional $F \in \mathcal{P}_{\mathcal{F}}$,*

$$\begin{aligned} & A_k \parallel i \circ D^k F \parallel^2_{L^2(B)\otimes L^2(\mathrm{R}^{+k})} \\ \leq & \parallel F \parallel^2_{\mathbb{D}^{\mathcal{F}}_{2,k}} \leq B_k \left(\parallel i \circ D^k F \parallel^2_{L^2(B)\otimes L^2(\mathrm{R}^{+k})} + \parallel F \parallel^2_{L^2(B)} \right). \end{aligned}$$

Proof. We need the following lemma, which is a generalization of Eq. (2).

Lemma 2 *Let $F \in \mathcal{P}_{\mathcal{F}}$. We have for $n \geq 1$ and $k_1, \dots, k_n \in \mathbb{N}$:*

$$\begin{aligned} D_{k_1} \cdots D_{k_n} P_t F &= e^{-nt} P_t D_{k_1} \cdots D_{k_n} F \\ &\quad + (e^{-t} - 1) \sum_{j=1}^{j=n} e^{-jt} D_{k_1} \cdots D_{k_{n-j}} P_t D_{k_{n-j+1}} \cdots D_{k_n} D_{k_j} F. \end{aligned}$$

Proof. By induction. From Lemma 1, the result is true for $n = 1$. Assume that the relation is verified at the order $n \geq 1$. We have for $k_1, \dots, k_{n+1} \in \mathbb{N}$:

$$\begin{aligned} D_{k_1} \cdots D_{k_{n+1}} F &= e^{-nt} D_{k_1} P_t D_{k_2} \cdots D_{k_n} F \\ &\quad + (e^{-t} - 1) D_{k_1} \sum_{j=2}^{j=n} e^{-jt} D_{k_2} \cdots D_{k_{n-j-1}} P_t D_{k_{n-j}} \cdots D_{k_{n+1}} D_{k_j} F \\ &= e^{-nt} D_{k_1} P_t D_{k_2} \cdots D_{k_n} F + (e^{-t} - 1) e^{-nt} D_{k_1} P_t D_{k_1} \cdots D_{k_{n+1}} F \\ &\quad + (e^{-t} - 1) \sum_{j=2}^{j=n} e^{-jt} D_{k_1} \cdots D_{k_{n-j-1}} P_t D_{k_{n-j}} \cdots D_{k_{n+1}} D_{k_j} F \\ &= e^{-nt} D_{k_1} P_t D_{k_2} \cdots D_{k_{n+1}} F \\ &\quad + (e^{-t} - 1) \sum_{j=1}^{j=n} e^{-jt} D_{k_1} \cdots D_{k_{n-j-1}} P_t D_{k_{n-j}} \cdots D_{k_{n+1}} D_{k_j} F. \end{aligned}$$

This shows that the equality is satisfied for any $n \geq 1$.

□

Proof of Prop. 8. Let us write the discrete chaotic decomposition of F:

$$F = \sum_{n \geq 0} I_n(f_n),$$

which gives

$$\parallel F \parallel^2_{\mathbb{D}^{\mathcal{F}}_{2,k}} = E[F(I + \mathcal{L})^k F] = \sum_{n \geq 0} (1 + n)^k \parallel I_n(f_n) \parallel^2_2 .$$

Taking $A_k = 1/((k+1)^k)$ and $B_k = 1$, we have

$$A_k(n+1)^k \leq 1 + n(n-1)\cdots(n-k) \leq B_k(1+n)^k \quad n > k.$$

Hence $\| F \|_{\mathbb{D}^{\mathcal{F}}_{2,k}}$ is equivalent to

$$F \to \left(E[F\mathcal{L}(\mathcal{L}-I)\cdots(\mathcal{L}-(k-1)I)(\mathcal{L}-kI)F] + E[F^2]\right)^{\frac{1}{2}}.$$

It remains to show that

$$E\left[F\mathcal{L}(\mathcal{L}-I)\cdots(\mathcal{L}-nI)F\right] = \| i \circ D^{n+1}F \|^2_{L^2(B)\otimes L^2(\mathbb{R}^{n+1}_+)}, \quad n \geq 0.$$

We know that this statement is true for $n = 0$. Suppose that it is true at the rank n, and let us show that then it is also true at the rank $n+1$.

$$\begin{aligned}
&E[F\mathcal{L}(\mathcal{L}-I)\cdots(\mathcal{L}-(n+1)I)F] \\
&= E\left[(i\circ D^n F, i\circ D^n(\mathcal{L}-(n+1)I)F)_{L^2(\mathbb{R}^n_+)}\right] \\
&= E\left[\sum_{k_1,\ldots,k_n} \tau_{k_1}\cdots\tau_{k_n} D_{k_1}\cdots D_{k_n} F D_{k_1}\cdots D_{k_n}(\mathcal{L}-(n+1)I)F\right] \\
&= E\left[\sum_{k_1,\ldots,k_n} \tau_{k_1}\cdots\tau_{k_n} D_{k_1}\cdots D_{k_n} F \left(\sum_{i=1}^{i=n} D_{k_i} + \mathcal{L}\right) D_{k_1}\cdots D_{k_n} F\right] \\
&= E\left[\sum_{k_1,\ldots,k_n} \tau_{k_1}\cdots\tau_{k_n} D_{k_1}\cdots D_{k_n} F \left(\sum_{i=1}^{i=n} D_{k_i}\right) D_{k_1}\cdots D_{k_n} F\right] \\
&\quad + E\left[\sum_{k_1,\ldots,k_n} \tau_{k_{n+1}} D_{k_{n+1}} \tau_{k_1}\cdots\tau_{k_n} D_{k_1}\cdots D_{k_n} F D_{k_1}\cdots D_{k_{n+1}} F\right] \\
&= E\left[\sum_{k_1,\ldots,k_{n+1}} \tau_{k_1}\cdots\tau_{k_{n+1}} D_{k_1}\cdots D_{k_{n+1}} F D_{k_1}\cdots D_{k_{n+1}} F\right] \\
&= E\left[(i\circ D^{n+1}F, i\circ D^{n+1}F)_{L^2(\mathbb{R}^{n+1}_+)}\right] \quad F \in \mathcal{P}_{\mathcal{F}},
\end{aligned}$$

where we used the relation

$$D_{k_1}\cdots D_{k_n}(\mathcal{L}-(n+1)I)F = \left(\sum_{j=0}^{j=n} D_{k_i} + \mathcal{L}\right) D_{k_1}\cdots\cdots D_{k_n} F,$$

obtained by differentiating the result of Lemma 2.

□

6 Independence of Poisson functionals

In this section, we apply the criterion given in [18] for the independence of multiple Wiener integral in order to obtain similar results for discrete multiple Poisson stochastic integrals of the type $I_n(f_n)$, $f_n \in l^2(\mathbb{N})^{\circ n}$. The following result allows to characterize the independence of discrete multiple Poisson stochastic integrals in terms of the supports of their discrete time kernels.

Theorem 8 *Let $f_n \in H^{\circ n}$ and $g_m \in H^{\circ m}$, $m \geq n$. The Poisson functionals $I_n(f_n)$ and $I_m(g_m)$ are independent if and only if*

$$f_n(k_1, \ldots, k_n) g_m(k_1, k_{n+1}, \ldots, k_{n+m-1}) = 0, \quad \forall k_1, \ldots, k_{n+m-1} \in \mathbb{N}.$$

Proof. We have the following orthogonal decompositions for f_n and g_m:

$$f_n = \sum_{\substack{k_1 \neq \cdots \neq k_d \\ n_1 + \cdots + n_d = n}} \alpha_{k_1,\ldots,k_d}^{n_1,\ldots,n_d} e_{k_1}^{\circ n_1} \circ \cdots \circ e_{k_d}^{\circ n_d}$$

$$g_m = \sum_{\substack{l_1 \neq \cdots \neq l_p \\ m_1 + \cdots + m_p = m}} \beta_{l_1,\ldots,l_p}^{m_1,\ldots,m_p} e_{l_1}^{\circ m_1} \circ \cdots \circ e_{l_p}^{\circ m_p}.$$

From Prop. 1, the random variables $I_n(f_n)$ and $I_m(g_m)$ belong respectively to the $2n$-th and $2m$-th Wiener chaos. Denote by $\hat{f}_{2n}$ and $\hat{g}_{2m}$ the corresponding kernels. We have $I_n(f_n) = \hat{I}_{2n}(\hat{f}_{2n})$ and $I_m(g_m) = \hat{I}_{2m}(\hat{g}_{2m})$, i.e.

$$\hat{f}_{2n} = \sum_{\substack{k_1 \neq \cdots \neq k_d \\ n_1 + \cdots + n_d = n}} \alpha_{k_1,\ldots,k_d}^{n_1,\ldots,n_d} f_{k_1,\ldots,k_d}^{n_1,\ldots,n_d}$$

and

$$\hat{g}_{2m} = \sum_{\substack{l_1 \neq \cdots \neq l_p \\ m_1 + \cdots + m_d = m}} \beta_{l_1,\ldots,l_p}^{m_1,\ldots,m_p} g_{l_1,\ldots,l_p}^{m_1,\ldots,m_p},$$

with

$$\hat{I}_{2n}(f_{k_1,\ldots,k_d}^{n_1,\ldots,n_d}) = I_n(e_{k_1}^{\circ n_1} \circ \cdots \circ e_{k_d}^{\circ n_d}), \qquad \hat{I}_{2m}(g_{l_1,\ldots,l_p}^{m_1,\ldots,m_p}) = I_n(e_{l_1}^{\circ m_1} \circ \cdots \circ e_{l_p}^{\circ m_p}).$$

From Prop. 1, we find explicitely

$$f_{k_1,\ldots,k_d}^{n_1,\ldots,n_d} = \sum_{\substack{0 \leq i_1 \leq n_1 \\ \cdots \\ 0 \leq i_d \leq n_d}} \frac{(-1)^n C_{n_1}^{i_1} \cdots C_{n_d}^{i_d}}{2^n (C_{2n_1}^{2i_1} \cdots C_{2n_d}^{2i_d})^{1/2}} h_{2k_1}^{\circ 2i_1} \circ h_{2k_1+1}^{\circ 2n_1 - 2i_1} \circ \cdots \circ h_{2k_d}^{\circ 2i_d} \circ h_{2k_d+1}^{\circ 2n_d - 2i_d}$$

and

$$g^{m_1,\dots,m_p}_{l_1,\dots,l_p} = \sum_{\substack{0\le j_1\le m_1\\ \cdots \\ 0\le j_p\le m_p}} \frac{(-1)^m C^{j_1}_{m_1}\cdots C^{j_p}_{m_p}}{2^m (C^{2j_1}_{2m_1}\cdots C^{2j_p}_{2m_p})^{1/2}} h^{\circ 2j_1}_{2l_1}\circ h^{\circ 2m_1-2j_1}_{2l_1+1}\circ\cdots\circ h^{\circ 2j_p}_{2l_p}\circ h^{\circ 2m_p-2j_p}_{2l_p+1}.$$

From [18], $I_n(f_n)$ is independent of $I_m(g_m)$ if and only if $\hat{f}_{2n}\otimes_1 \hat{g}_{2m}=0$ a.s., i.e.

$$\sum \alpha^{n_1,\dots,n_d}_{k_1,\dots,k_d}\beta^{m_1,\dots,m_p}_{l_1,\dots,l_p} f^{n_1,\dots,n_d}_{k_1,\dots,k_d}\otimes_1 g^{m_1,\dots,m_p}_{l_1,\dots,l_p}=0,$$

which means

$$\alpha^{n_1,\dots,n_d}_{k_1,\dots,k_d}\beta^{m_1,\dots,m_p}_{l_1,\dots,l_p} f^{n_1,\dots,n_d}_{k_1,\dots,k_d}\otimes_1 g^{m_1,\dots,m_p}_{l_1,\dots,l_p}=0$$

for $k_1\neq\cdots\neq k_d$ and $l_1\neq\cdots\neq l_p$, since

$$\left\{ f^{n_1,\dots,n_d}_{k_1,\dots,k_d}\otimes_1 g^{m_1,\dots,m_p}_{l_1,\dots,l_p} \ :\ k_1\neq\cdots\neq k_d \ \ and\ \ l_1\neq\cdots\neq l_p\right\}$$

is orthogonal in $L^2(\mathbb{R}_+)^{\circ n+m-2}$, due to the particular form of $f^{n_1,\dots,n_d}_{k_1,\dots,k_d}$ and $g^{m_1,\dots,m_p}_{l_1,\dots,l_p}$. This condition is equivalent to $\alpha^{n_1,\dots,n_d}_{k_1,\dots,k_d}\beta^{m_1,\dots,m_p}_{l_1,\dots,l_p}=0$ if $\{k_1,\dots,k_d\}\bigcap\{l_1,\dots,l_p\}\neq\emptyset$, or

$$f_n(k_1,\dots,k_n)g_m(k_1,k_{n+1},\dots,k_{n+m-1})=0 \quad \forall k_1,\dots,k_{n+m-1}\in\mathbb{N}.$$

□

7 Diffusion process and capacities

In this section, we study the diffusion process associated with $-\mathcal{L}$, and show that it gives another example of a process whose hitting probabilities of open sets can be estimateds in terms of capacities, cf. [8], [15]. We start by introducing capacities on the Poisson space. The space B is endowed with the largest topology that makes $O\subset B$ open in B if $\Xi^{-1}(O)$ is open in W. We can define the capacities $c_{r,p}$ on B as follows:

$$c_{r,p}(O)=\inf\left\{\parallel u\parallel_{\mathbb{D}^{\mathcal{F}}_{p,r}} \ :\ \Theta^{-1}u\geq 1_{\{O\}}\ P-a.s.\right\}$$

for O open in B, and

$$c_{r,p}(A)=\inf\left\{c_{r,p}(O)\ :\ O\ open\ and\ A\subset O\right\}$$

for any subset A of B.

Let $(X_t^{(n)})_{t\in\mathbb{R}_+^n}$ denote the W-valued n-parameter Ornstein-Uhlenbeck process, i.e.

$$X_t^{(n)} = e^{-(t_1+\cdots+t_n)/2} W^{(n+1)}_{e^{t_1},\ldots,e^{t_n},\cdot},$$

where $W^{(n+1)}$ is the $(n+1)$-parameter Brownian sheet defined on a probability space $(\Omega, \mathcal{A}, Q)$, cf. [7].

Proposition 9 *Let $Y_t^{(n)} = \Xi(X_t^{(n)})$, $t \in \mathbb{R}_+^n$. The process $Y^{(n)}$ is a B-valued P-symmetric n-parameter process with continuous paths. Its transition semi-groups are given by*

$$P_t^i = \Theta^{-1} exp(-t\mathcal{L})\Theta, \quad t \in \mathbb{R}_+, \ i = 1, \ldots, n.$$

Proof. We refer to the definitions in [8]. We know that $X^{(n)}$ is a μ-symmetric n-parameter process. Let $(\mathcal{F}_t^i)_{t\in\mathbb{R}_+}$, $i = 1, \ldots, n$ denote its associated filtrations. We have

(1) For any $t \in \mathbb{R}_+^n$, $Y_t^{(n)} \in \bigcup_{1\le i\le n} \mathcal{F}_{t_i}^i$ since $X_t^{(n)} \in \bigcup_{1\le i\le n} \mathcal{F}_{t_i}^i$, and the law of $Y_t^{(n)}$ is P since $P = \Xi_*\mu$ and $X_t^{(n)}$ has law μ.

(2) For any $1 \le i \le n$ and $F \in L^2(B, P)$, we have for $u \in \mathbb{R}_+^n$ and $a \in \mathbb{R}_+$:

$$\begin{aligned} E[F(Y^{(n)}_{u_1,\ldots,u_i+a,\ldots,u_n}) \mid \mathcal{F}_{u_i}^i] &= E[\Theta F(X^{(n)}_{u_1,\ldots,u_i+a,\ldots,u_n}) \mid \mathcal{F}_{u_i}^i] \\ &= e^{-a\hat{\mathcal{L}}}\Theta F(X_u^{(n)}) \\ &= \Theta^{-1} e^{-a\mathcal{L}} \Theta F(Y_u^{(n)}). \end{aligned}$$

□

Applying the result of [8], [15], we obtain that the process $(Y_t^{(n)})_{t\in\mathbb{R}_+^n}$ is another example of a process whose hitting probabilities can be estimated in terms of capacities:

Theorem 9 *Let O be an open set in B. For $t \in \mathbb{R}_+^n$, there exists two constants $K_1, K_2 > 0$ depending only on t and $n \in \mathbb{N}$ such that*

$$K_1 c_{n,2}(O) \le Q(\exists s \in [0,t] \ : \ Y_s^{(n)} \in O) \le K_2 c_{n,2}(O).$$

Proof. From [15], there exists $\hat{K}1, \hat{K}2 > 0$ such that

$$\hat{K}_1 \hat{c}_{n,2}(\Xi^{-1}(O)) \le Q(\exists s \in [0,t] \ : \ Y_s^{(n)} \in O) \le \hat{K}_2 \hat{c}_{n,2}(\Xi^{-1}(O)),$$

where $\hat{c}_{n,2}(\Xi^{-1}(O))$ is the usual capacity on Wiener space, defined as

$$\hat{c}_{n,2}(\Xi^{-1}(O)) = \inf\left\{\| u \|_{\mathbb{D}_{2,n}} : \ u \geq 1_O \circ \Xi \ \ \mu - a.s.\right\}.$$

We need to show that $c_{n,2}(O)$ can be estimated in terms of $\hat{c}_{n,2}(\Xi^{-1}(O))$. We have

$$\begin{aligned} c_{n,2}(O) &\geq \inf\left\{\| u \|_{\mathbb{D}_{2,n}} : \ u \geq 1_O \circ \Xi \ \ \mu - a.s.\right\} \\ &\geq \hat{c}_{n,2}(\Xi^{-1}(O)) \\ &\geq \inf\left\{\| u \|_{\mathbb{D}_{2,n}} : \ E[u \mid \mathcal{F}] \geq 1_O \circ \Xi \ \ \mu - a.s.\right\} \\ &\geq K \inf\left\{\| E[u \mid \mathcal{F}] \|_{\mathbb{D}^{\mathcal{F}}_{2,n}} : u \in \mathbb{D}_{2,n} \ and \ E[u \mid \mathcal{F}] \geq 1_O \circ \Xi \ \ \mu - a.s.\right\} \\ &= K c_{n,2}(O), \end{aligned}$$

with $K > 0$. The last inequality comes from the continuity of $E[\,\cdot \mid \mathcal{F}]$ from $\mathbb{D}_{p,k}$ to $\mathbb{D}^{\mathcal{F}}_{p,k}$, cf. Prop. 2.

□

For $n = 1$, Prop. 9 shows that the diffusion process associated to $-\mathcal{L}$ is the B-valued process $Y = (\Xi(X^{(1)}_t))_{t\geq 0}$. The coordinates of $(Y_t)_{t\in\mathbf{R}_+}$ are the square norms of independent two-dimensional Ornstein-Uhlenbeck processes, hence they satisfy the stochastic differential equation

$$dV_t = \sqrt{2V_t}dW_t + (1 - V_t)dt,$$

where $(W_t)_{t\in\mathbf{R}_+}$ is a brownian motion. In the usual Poisson space interpretation, the trajectories of $(Y_t)_{t\geq 0}$ take their values in a space of step functions whose interjump times move according to the square norms of independent 2-dimensional Ornstein-Uhlenbeck processes.

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