Blowup estimates for a family of semilinear SPDEs with time-dependent coefficients

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Abstract

We investigate the blowup and stability of semilinear stochastic partial differential equations with time-dependent coefficients using stopping times of exponential functionals of Brownian martingales and a non-homogeneous heat semigroup. In particular we derive lower bounds for the probability of blowup in finite time, and we provide sufficient conditions for the existence of global positive solutions.

Key words: stochastic partial differential equations; blowup of semilinear equations; weak and mild solutions; exponential functionals of Brownian motion.
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1 Introduction

In [5], Dozzi and López-Mimbela have estimated the probability of finite-time blowup of positive solutions and the probability of existence of non-trivial positive global solutions of the SPDE with constant coefficients

$$\begin{cases} du(t,x) = \big(\Delta u(t,x) + u^{1+\beta}(t,x)\big)dt + \kappa u(t,x)dW_t, \\ u(0,x) = f(x) \geq 0, \quad x \in D, \\ u(t,x) = 0, \quad t > 0, \quad x \in \partial D, \end{cases} \tag{1.1}$$

where $\beta > 0$ and $\kappa \geq 0$ are constants, D is a bounded domain of $\mathbb{R}^d$ with smooth boundary ∂D, the initial condition $f : D \to \mathbb{R}_+$ is of class C^2 with $f \not\equiv 0$ and $(W_t)_{t\in\mathbb{R}_+}$ is a standard one-dimensional Brownian motion on a stochastic basis $\big(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}\big)$. In order to explain briefly the type of results obtained in [5] let us consider initial values of the form $f(x) = m\psi(x)$, $x \in D$ (where $m > 0$ is a parameter and ψ is the eigenfunction corresponding to the first eigenvalue of the Laplacian on D normalized by $\int_D \psi(x)\,dx = 1$), and let $u(t,x)$ be a weak solution of (1.1). Then, using essentially the transformation $v(t,x) = e^{-\kappa W_t} u(t,x)$ and Itô's formula one can deduce that the explosion time ϱ of (1.1) fulfills $\sigma_* \le \varrho \le \sigma^*$ almost surely, where

$$\begin{aligned} \sigma_* &= \inf\left\{ t \ge 0 : \int_0^t \exp(\kappa\beta W_r - \beta(\lambda_1 + \kappa^2/2)r)\,dr \ge 1/(\beta m^\beta \|\psi\|_\infty^\beta) \right\}, \\ \sigma^* &= \inf\left\{ t \ge 0 : \int_0^t \exp(\kappa\beta W_r - \beta(\lambda_1 + \kappa^2/2)r)\,dr \ge 1/\left(\beta m^\beta \left(\int \psi^2(x)\,dx\right)^\beta\right) \right\}, \end{aligned}$$

and λ_1 is the first eigenvalue of the Dirichlet problem for the Laplacian Δ on D. Therefore, almost surely the weak solution of (1.1) is global on the event $\{\sigma_* = \infty\}$, and blows up in finite time on $\{\sigma^* < \infty\}$. Moreover we have

$$\mathbb{P}(\sigma^* \le t) \le \mathbb{P}(\varrho \le t) \le \mathbb{P}(\sigma_* \le t)$$

for any $t > 0$, where the random variables σ_* and σ^* are given in terms of one and the same Brownian functional $\int_0^t \exp(\kappa\beta W_r - \beta(\Lambda + \kappa^2/2)r)\,dr$, whose density can be obtained for every $t > 0$ from Yor's formula (see [21], Ch. 4).

Similar results are valid for systems of semilinear SPDEs of the form

$$\begin{aligned} du_1(t,x) &= ((\Delta + V_1)u_1(t,x) + u_2^p(t,x))\,dt + \kappa_1 u_1(t,x)\,dW_t \\ du_2(t,x) &= ((\Delta + V_2)u_2(t,x) + u_1^q(t,x))\,dt + \kappa_2 u_2(t,x)\,dW_t, \quad x \in D, \end{aligned} \tag{1.2}$$

with Dirichlet boundary conditions (where $p \ge q > 1$, and $V_i > 0$ and $\kappa_i \ne 0$ are given constants, $i = 1, 2$). In [4] it is shown that for this kind of systems, the probability distributions of suitable lower and upper bounds for the blow-up time of (1.2) are determined, respectively, by Brownian functionals of the form $\int_0^t e^{aW_r} \wedge e^{bW_r}\,dr$ and $\int_0^t e^{aW_r} \vee e^{bW_r}\,dr$ for appropriate constants a, b.

In this paper we extend the above type of results to the time-dependent semilinear SPDE

$$du(t,x) = \left(\frac{1}{2}k^2(t)\Delta u(t,x) + h(t)G\big(u(t,x)\big)\right)dt + \kappa(t)u(t,x)dW_t \tag{1.3}$$

with the same Dirichlet conditions as in (1.1) above, where h, k, κ are continuous functions from $\mathbb{R}_+$ into $(0,\infty)$ and $G : \mathbb{R} \to \mathbb{R}_+$ is a locally Lipschitz function.

Nonautonomous equations of the above type have been investigated by many authors, see e.g. [2, 8, 13, 14, 17, 16, 20, 19]. Existence and uniqueness of local mild solutions, space-time regularity properties, as well as conditions for existence of a unique global solution have been investigated in [17] and [19] for a wide class of evolution equations on Banach spaces perturbed by Hilbert space-valued Brownian motions. Several notions of solutions for a class of semilinear nonautonomous SPDEs on bounded, convex, smooth Euclidean domains are discussed in [16], which in many cases turn out to be equivalent, see also [19]. Since the equations we study here use a simple time-dependence of coefficients and a noise component which is a scalar multiple of Brownian motion, the existence and uniqueness of their solutions follows from an easy adaptation of the approach used in [10, Sect. 3], where well-posedness of a nonautonomous system of semilinear PDEs is investigated.

Let us explain in brief our main results and the methods we use to prove them. In the sequel we let $\lambda_1 > 0$ denote the first eigenvalue of the Dirichlet problem for the Laplacian Δ on D with associated eigenfunction ψ normalized by $\int_D \psi(x)dx = 1$, i.e. $\Delta\psi + \lambda_1\psi = 0$ on D and $\psi|_{\partial D} = 0$.

Let u be a weak solution of (1.3). Imposing the conditions $G(z) \geq Cz^{1+\beta}$ for $z > 0$, and

$$\frac{h(t)}{\kappa^2(t)} \exp\left(-\frac{\beta\lambda_1}{2}\int_0^t k^2(s)ds + \frac{\beta\lambda_1}{\nu}\int_0^t \kappa^2(s)ds\right) \geq c \tag{1.4}$$

where $C, \beta, c > 0$ are constants and $\nu \in (0,\infty]$ is a parameter, we show in Theorem 3.1 below that the probability of blowup of u in finite time is lower bounded by $\gamma(\mu,\theta)/\Gamma(\mu)$, where $\gamma(\mu,\theta)$ is the lower incomplete gamma function relative to

$$\theta := \frac{2c}{\beta}\left(\int_D f(x)\psi(x)dx\right)^\beta$$

with parameter $\mu := (2\lambda_1 + \nu)/(\nu\beta)$. This is achieved by using the transformation

$v(t,x) = e^{-\int_0^t \kappa(s)\,dW_s} u(t,x)$ and then exploiting the random PDE solved by v.

On the other hand, the existence of global solutions of (1.3) is analyzed using the heat semigroup $(S_{r,t})_{0\le r\le t\in \mathbb{R}_+}$ of the process $\left(\int_r^t k(s)d\tilde{W}_s\right)_{t\in[r,\infty)}$ where $\tilde{W}_t$ is a d-dimensional Brownian motion killed on leaving D, under the assumptions that G satisfies $G(0) = 0$ and

$$G(z) \le \Lambda z^{1+\beta}, \qquad z > 0,$$

for some constant $\Lambda > 0$, and $z \mapsto G(z)/z$ is increasing. In Corollary 4.3 below we show that if the initial condition f satisfies

$$\Lambda\beta \int_0^\infty h(r) \exp\left(\beta \int_0^r \kappa(s)dW_s - \frac{\beta}{2}\int_0^r \kappa^2(s)ds\right) \|S_{0,r}f\|_\infty^\beta dr < 1 \tag{1.5}$$

then (1.3) has a global positive solution. In the special setting $C = \Lambda = 1$ and $f(x) = p\psi$, $x \in D$, where $p > 0$ is some parameter and under Condition (1.4) we provide in Corollary 4.5 an upper bound for the probability that the solution of (1.3) is global. In case D is a connected bounded $C^{1,\alpha}$-domain in $\mathbb{R}^d$ where $\alpha > 0$, we show in Theorem 4.7 that (1.5) is satisfied under Conditions (4.13)–(4.14). In case of constant coefficients $h(t) = 1$, $k(t) = \sqrt{2}$ and $\kappa(t) = \kappa > 0$ we recover in particular the blowup and existence results of Dozzi and López-Mimbela [5].

It also follows from our analysis that the presence of noise does not trigger blow-up, and a higher level of noise modeled by the function $\kappa(t)$ can actually increase the regularity of the solution, as can be seen in (1.4) and (1.5). This is in contrast with the analysis of [11] in which the presence of a centered stochastic term does not prevent blow-up of the solution.

Our treatment of (1.3) renders as well an explosion condition for its deterministic version. That is, if we consider the PDE

$$\frac{\partial u}{\partial t}(t,x) = \frac{1}{2}k^2(t)\Delta u(t,x) + h(t)G(u(t,x))$$

with the same initial and boundary conditions, and the same assumptions on h, k and G as for SPDE (1.3), a criterion for explosion of positive solutions of this equation is given in Section 2, which extends partially the classical result of H. Fujita [6]; see Remark 2.3 below.

We remark that, although our approach is based on the methods used in [5], the nonautonomous equation (1.3) is qualitatively different from its autonomous counterpart, owing to the fact that it allows for richer dynamics and behaviors that are not present in the autonomous equation: on the one hand, the coefficients $h(t)$ and $\kappa(t)$ allow one to model time-dependent scale variations in the reaction and noise terms. On the other hand, the linear operators $\frac{1}{2}k^2(t)\Delta$ represent anomalous, time-inhomogeneous diffusions capable of modifying qualitatively the development of (1.3). As a matter of fact, the factor $k^2(t)$ in front of Δ can drastically alter the asymptotic behavior of (1.3), a phenomenon that might be better understood in the deterministic setting $\kappa \equiv 0$: taking for simplicity $h(t) \equiv 1$ and $D = \mathbb{R}^d$ one can show, as in [9], that integrability of k^2 already excludes existence of global positive solutions of

$$\frac{\partial u(t,x)}{\partial t} = \frac{1}{2}k^2(t)\Delta u(t,x) + u^{1+\beta}(t,x), \quad x \in D, \tag{1.6}$$

for all nontrivial initial values $u(0) \geq 0$. From our analysis of Eq. (1.3) one can easily deduce that *also in the case of a bounded smooth domain D*, the solution of (1.6) blows up in finite time whenever k^2 is integrable and $u(0) \geq 0$ is not identically 0, see Condition (2.17) below.

As in [5], our method to study the finite-time blowup and existence of global positive solutions of (1.3) is to rewrite this SPDE into the random PDE (2.2) given in Proposition 2.1 below. Upper and lower bounds for the blowup times of Eq. (2.2), which are stopping times for exponential functionals of the form

$$Y(t) := \int_0^t m(r) \exp\left(-n(r) + \beta \int_0^r \kappa(s)\, dW_s\right) dr, \quad t \geq 0,$$

are obtained for suitable nonnegative functions $m(r)$ and $n(r)$. The asymptotic behavior of (1.3), i.e. finite-time blowup versus existence, globally in time, of its positive solutions is then studied by exploiting assumption (1.4) and classical results about the law of $Y(\infty)$.

We proceed as follows. In Section 2 we recall some background definitions, cf. also [5] for details, and we prove some preliminary results on weak solutions and blowup times. Section 3 deals with the probability of blowup in finite time, while existence results are presented in Section 4 based on semigroup arguments.

2 Preliminaries

In this section we identify a random stopping time τ which is a blowup time for (1.3), i.e. any solution $u(t,x)$ of (1.3) blows up before time τ almost surely in the sense that

$$\lim_{t\nearrow\tau}\sup_{x\in D}|u(t,x)| = +\infty, \quad \text{a.s. on } \{\tau < +\infty\}.$$

In the sequel we will use the process $(M_t)_{t\in\mathbb{R}_+}$ defined by

$$M_t := \int_0^t \kappa(s)dW_s.$$

The next proposition shows that $u(t,x)$ can be transformed into the solution of a random PDE. We refer to § 1 of [5] for the notions of weak and mild solutions of a random PDE.

Proposition 2.1 *Let τ be a given stopping time such that* (1.3) *possesses a weak solution u on the interval $(0,\tau)$. Then the function v defined by*

$$v(t,x) := e^{-M_t}u(t,x), \quad t\in[0,\tau), \quad x\in D, \tag{2.1}$$

is a weak solution of the random PDE

$$\begin{cases} \dfrac{\partial v}{\partial t}(t,x) = \dfrac{1}{2}k^2(t)\Delta v(t,x) - \dfrac{\kappa^2(t)}{2}v(t,x) + h(t)e^{-M_t}G\big(e^{M_t}v(t,x)\big), \\ v(0,x) = f(x) \geq 0, \quad x\in D, \\ v(t,x) = 0, \quad t\in\mathbb{R}_+, \quad x\in\partial D. \end{cases} \tag{2.2}$$

Proof. We proceed as in [5]. Using Itô's formula we write the semimartingale expansion

$$e^{-M_t} = 1 - \int_0^t \kappa(s) e^{-M_s} dW_s + \frac{1}{2}\int_0^t \kappa^2(s) e^{-M_s} ds, \qquad t \in \mathbb{R}_+.$$

Then we have

$$\begin{aligned}\int_D u(t,x)\varphi(x)\,dx \;&=\; \int_D f(x)\varphi(x)\,dx \\ &\quad + \int_0^t \int_D \left(\frac{1}{2}k^2(s)u(s,x)\Delta\varphi(x) + h(s)G\big(u(s,x)\big)\varphi(x)\right)\,dx\,ds \\ &\quad + \int_0^t \int_D \kappa(s)u(s,x)\varphi(x)\,dx\,dW_s, \end{aligned} \tag{2.3}$$

$\mathbb{P}$-a.s. for all $t \in [0,\tau)$ and for every $\varphi \in C^2(D)$ vanishing on ∂D. Letting

$$u(t,\varphi) := \int_D u(t,x)\varphi(x)\,dx,$$

we rewrite (2.3) as

$$u(t,\varphi) = u(0,\varphi) + \frac{1}{2}\int_0^t k^2(s)u(s,\Delta\varphi)ds + \int_0^t h(s)G(u)(s,\varphi)ds + \int_0^t \kappa(s)u(s,\varphi)dW_s,$$

and for any fixed φ, the process $(u(t,\varphi)\mathbf{1}_{[0,\tau)}(t))_{t\in\mathbb{R}_+}$ is also a semimartingale. Observe that the cross variation between e^{-M_t} and $u(t,\varphi)$ is given by

$$d\langle e^{-M_t}, u(t,\varphi)\rangle = -\kappa^2(t)e^{-M_t}u(t,\varphi)\,dt,$$

which yields

$$\begin{aligned} v(t,\varphi) &= e^{-M_t}u(t,\varphi) \\ &= e^{-M_0}u(0,\varphi) + \int_0^t e^{-M_s} d\big(u(s,\varphi)\big) + \int_0^t u(s,\varphi)d\big(e^{-M_s}\big) + \langle e^{-M_t}, u(t,\varphi)\rangle \\ &= u(0,\varphi) + \frac{1}{2}\int_0^t k^2(s)v(s,\Delta\varphi)ds + \int_0^t h(s)e^{-M_s}G\big(e^{M}v\big)(s,\varphi)ds - \frac{1}{2}\int_0^t \kappa^2(s)v(s,\varphi)ds \\ &= v(0,\varphi) + \frac{1}{2}\int_0^t k^2(s)\Delta v(s,\varphi)ds + \int_0^t h(s)e^{-M_s}G(e^{M}v)(s,\varphi)ds - \frac{1}{2}\int_0^t \kappa^2(s)v(s,\varphi)ds, \end{aligned} \tag{2.4}$$

since we have $u(0,\varphi) = v(0,\varphi)$ by (2.1), and $v(s,\Delta\varphi) = \Delta v(s,\varphi)$ because the Laplace operator is self-adjoint. Since v is differentiable with respect to t up to eventual blowup and by uniqueness of weak solution, equality (2.4) shows that $v \in C^2(D)$ is a weak solution of (2.2). □

In the development of this section we assume that the locally Lipschitz function $G : \mathbb{R} \to \mathbb{R}_+$ satisfies

$$G(z) \geq Cz^{1+\beta}, \qquad z > 0, \tag{2.5}$$

where $C, \beta > 0$ are constants. Without loss of generality we assume, just like in [5], that $C = 1$ in (2.5), i.e.

$$G(z) \geq z^{1+\beta}, \qquad z > 0. \tag{2.6}$$

For further use we also let

$$K(r,t) := \int_r^t k^2(s)\, ds, \qquad K(t) := K(0,t), \tag{2.7}$$

$$J(r,t) := \int_r^t \kappa^2(s)\, ds, \quad \text{and} \quad J(t) := J(0,t), \qquad r,t \geq 0. \tag{2.8}$$

In Proposition 2.2 we construct a random upper bound τ^* for the blowup time of the solution u of (1.3).

Proposition 2.2 *Any solution u of* (1.3) *on the interval* $(0,\tau^*)$ *exhibits finite-time blowup on the event* $\{\tau^* < \infty\}$, *where*

$$\tau^* := \inf\left\{t \geq 0 \;:\; \int_0^t h(s) e^{-\beta(\lambda_1 K(s) + J(s))/2 + \beta M_s}\, ds \geq \frac{1}{\beta} v(0,\psi)^{-\beta}\right\}. \tag{2.9}$$

Proof. Since the Laplacian is self-adjoint, writting $v(s,\psi) := \int_D v(s,x)\psi(x)\, dx$ we have

$$\Delta v(s,\psi) = v(s,\Delta\psi) = \int_D v(s,x)\Delta\psi(x)\, dx = -\lambda_1 \int_D v(s,x)\psi(x)\, dx = -\lambda_1 v(s,\psi). \tag{2.10}$$

Moreover, it is well-known (cf. Corollary 3.3.7 of [3]) that $\psi > 0$ on D. Now, assumption (2.6) and Jensen's inequality give

$$\begin{aligned} G(e^M v)(s,\psi) &:= \int_D G\big(e^{M_s} v(s,x)\big)\psi(x)\, dx \\ &\geq \int_D \big(e^{M_s} v(s,x)\big)^{1+\beta} \psi(x)\, dx \\ &\geq e^{(1+\beta)M_s} \left(\int_D v(s,x)\psi(x)\, dx\right)^{1+\beta} \end{aligned}$$

$$= \quad e^{(1+\beta)M_s} v(s,\psi)^{1+\beta}. \tag{2.11}$$

From (2.4) we obtain, for any $\varepsilon > 0$,

$$\begin{aligned}
&\frac{v(t+\varepsilon,\psi) - v(t,\psi)}{\varepsilon} \\
&= \frac{1}{2\varepsilon}\left(\int_t^{t+\varepsilon} k^2(s) v(s,\Delta\psi)\,ds + 2\int_t^{t+\varepsilon} h(s) e^{-M_s} G(e^M v)(s,\psi)\,ds - \int_t^{t+\varepsilon} \kappa^2(s) v(s,\psi)\,ds\right) \\
&= \frac{1}{2\varepsilon}\left(-\lambda_1 \int_t^{t+\varepsilon} k^2(s) v(s,\psi)\,ds + 2\int_t^{t+\varepsilon} h(s) e^{-M_s} G(e^M v)(s,\psi)\,ds - \int_t^{t+\varepsilon} \kappa^2(s) v(s,\psi)\,ds\right) \\
&\geq \frac{1}{2\varepsilon}\left(-\lambda_1 \int_t^{t+\varepsilon} k^2(s) v(s,\psi)\,ds + 2\int_t^{t+\varepsilon} h(s) e^{\beta M_s} v(s,\psi)^{1+\beta}\,ds - \int_t^{t+\varepsilon} \kappa^2(s) v(s,\psi)\,ds\right),
\end{aligned} \tag{2.12}$$

where the second equality follows from (2.10), and inequality (2.12) arises from (2.11). Letting $\varepsilon \to 0$ in (2.12) shows that

$$\frac{dv}{dt}(t,\psi) \geq -\frac{1}{2}\big(\lambda_1 k^2(t) + \kappa^2(t)\big) v(t,\psi) + h(t) e^{\beta M_t} v(t,\psi)^{1+\beta},$$

and by means of a comparison argument (see e.g. Theorem 1.3 of [18]) we get

$$v(t,\psi) \geq I(t), \qquad t \geq 0, \tag{2.13}$$

where $I(\cdot)$ denotes the solution of the ODE

$$\begin{aligned}
I'(t) &= -\frac{1}{2}\big(\lambda_1 k^2(t) + \kappa^2(t)\big) I(t) + h(t) e^{\beta M_t} I(t)^{1+\beta}, \\
I(0) &= v(0,\psi).
\end{aligned} \tag{2.14}$$

In order to solve (2.14) we use the substitution $w = w(t) := I^{-\beta}(t)$ to obtain

$$-\frac{1}{2}\big(\lambda_1 k^2(t) + \kappa^2(t)\big) I(t) + h(t) e^{\beta M_t} I(t)^{1+\beta} = I'(t) = \frac{dI}{dw}\frac{dw}{dt} = -\frac{1}{\beta} w^{-1-1/\beta} w'(t),$$

which simplifies to the linear ODE

$$\begin{cases} w'(t) - \dfrac{\beta}{2}\big(\lambda_1 k^2(t) + \kappa^2(t)\big) w(t) = -\beta h(t) e^{\beta M_t}, \qquad t \geq 0, \\[2ex] w(0) = v(0,\psi)^{-\beta}. \end{cases} \tag{2.15}$$

Multiplying (2.15) by the integrating factor $e^{-\beta(\lambda_1 K(t) + J(t))/2}$ we obtain

$$\frac{d}{dt}\left(e^{-\beta(\lambda_1 K(t) + J(t))/2} w(t)\right) = -\beta h(t) e^{-\beta(\lambda_1 K(t) + J(t))/2 + \beta M_t},$$

and integrating from 0 to t on both sides of this equality we get

$$e^{-\beta(\lambda_1 K(t)+J(t))/2} w(t) = -\beta \int_0^t h(s) e^{-\beta(\lambda_1 K(s)+J(s))/2+\beta M_s} ds + w(0),$$

and

$$w(t) = e^{\beta(\lambda_1 K(t)+J(t))/2} \left(w(0) - \beta \int_0^t h(s) e^{-\beta(\lambda_1 K(s)+J(s))/2+\beta M_s} ds \right),$$

hence

$$I(t) = e^{-(\lambda_1 K(t)+J(t))/2} \left(v(0,\psi)^{-\beta} - \beta \int_0^t h(s) e^{-\beta(\lambda_1 K(s)+J(s))/2+\beta M_s} ds \right)^{-1/\beta} \tag{2.16}$$

for all $t \in [0, \tau^*)$, where τ^* is given by (2.9). Next, the inequality (2.13), i.e. $I(\cdot) \leq v(\cdot, \psi)$, shows that τ^* is an upper bound for the blowup time of $v(\cdot, \psi)$, hence the function

$$t \mapsto \int_D e^{-M_t} u(t,x) \psi(x)\, dx$$

also explodes in finite time on the event $\{\tau^* < \infty\}$. Since $\int_D \psi(x)\, dx = 1$ and $t \mapsto e^{-M_t}$ is bounded on $[0, \tau^*]$ if τ^* is finite, it follows that the function $t \mapsto \|u(t,\cdot)\|_\infty$ cannot be bounded on $[0, \tau^*]$ if $\tau^* < \infty$. Therefore, u also blows up in finite time on $\{\tau^* < \infty\}$, and τ^* is an upper bound for the blowup time of u. □

Remark 2.3 Setting $\kappa(t) \equiv 0$ for all $t \geq 0$ in (2.9) renders an explosion condition for the deterministic version of (1.3). That is, if we consider the PDE

$$\frac{\partial u}{\partial t}(t,x) = \frac{1}{2} k^2(t) \Delta u(t,x) + h(t) G(u(t,x)),$$

with the same initial and boundary conditions, and the same assumptions on h, k and G as for SPDE (1.3), then the solution u of this deterministic PDE is non-global provided that

$$\int_0^\infty h(s) e^{-\beta \lambda_1 K(s)/2}\, ds > \frac{1}{\beta} v(0,\psi)^{-\beta}. \tag{2.17}$$

Hence, if $h(r) e^{-\beta\lambda_1 K(r)/2} \in L^1([0,\infty))$, u is non-global when

$$\int_D f(x)\psi(x)\, dx > \left(\beta \int_0^\infty h(s) e^{-\beta\lambda_1 K(s)/2}\, ds \right)^{-1/\beta}. \tag{2.18}$$

If $h(r) e^{-\beta\lambda_1 K(r)/2} \notin L^1([0,\infty))$, u is always non-global except in the case of $f \equiv 0$.

If (2.18) holds and in addition $k^2 = h$, then the condition

$$\int_D f(x)\psi(x)\,dx > \left(\frac{\lambda_1}{2}\right)^{1/\beta}\left(1 - e^{-(\beta\lambda_1/2)\int_0^\infty k^2(s)\,ds}\right)^{-1/\beta}$$

also implies finite-time blowup of u.

3 Probability bounds for blowup

Throughout the development of this section we let ψ be an eigenfunction of norm $\int_D \psi(x)\,dx = 1$ associated with the first Dirichlet eigenvalue $\lambda_1 > 0$. That is, we have $\Delta\psi + \lambda_1\psi = 0$ on D and $\psi|_{\partial D} = 0$.

As a consequence of Proposition 2.2, in order to find the probability of finite time blowup of u, we investigate $I(\cdot)$ defined by (2.16) instead. More specifically, we need to investigate the law of the functional

$$\int_0^\infty h(s)e^{-\beta(\lambda_1 K(s)+J(s))/2+\beta M_s}\,ds.$$

In practice we will impose relations on h, k and κ so that the above functional reverts to a functional whose law is known. Theorem 3.1 provides such a relation and computes a lower bound for the probability of blowup of (1.3) in finite time.

Theorem 3.1 *Suppose that $h, k, \kappa > 0$ are continuous functions on $\mathbb{R}_+$ such that*

$$\frac{h(t)}{\kappa^2(t)}e^{-\beta\lambda_1(K(t)/2-J(t)/\nu)} \geq c, \qquad t \in \mathbb{R}_+, \tag{3.1}$$

where $c > 0$ is a constant, $\nu \in (0,\infty]$ is a parameter, and $K(t) = \int_0^t k^2(s)\,ds$. Then the probability that the solution of (1.3) blows up in finite time is lower bounded by

$$P(\mu,\theta) := \frac{\gamma(\mu,\theta)}{\Gamma(\mu)}, \tag{3.2}$$

where $\mu := (2\lambda_1 + \nu)/(\nu\beta)$, $\theta := 2cv(0,\psi)^\beta/\beta$ and

$$\gamma(\mu,\theta) := \int_0^\theta y^{\mu-1}e^{-y}dy$$

is the lower incomplete gamma function.

Proof. Observe that $M_t = \int_0^t \kappa(s)\, dW_s$ is a continuous martingale and so it can be written as a time-changed Brownian motion $M_t = B_{J(t)}$, where $J(t) = \big[M\big](t) = \int_0^t \kappa^2(s)\, ds$ is the quadratic variation of M, cf. [7, page 174]. Denoting $\rho := v(0,\psi)^{-\beta}/\beta$ for simplicity and using asssumption (3.1), we have from (2.9)

$$\mathbb{P}(\tau^* = +\infty) = \mathbb{P}\left(\int_0^t h(s) e^{-\beta(\lambda_1 K(s)+J(s))/2+\beta M_s} ds < \rho, \quad t>0\right) \tag{3.3}$$

$$= \mathbb{P}\left(\int_0^\infty h(s) e^{-\beta(\lambda_1 K(s)+J(s))/2+\beta B_{J(s)}} ds \le \rho\right)$$

$$= \mathbb{P}\left(\int_0^\infty \frac{h\big(J^{-1}(s)\big)}{\kappa^2\big(J^{-1}(s)\big)} e^{-\beta\lambda_1 K(J^{-1}(s))/2-\beta J(J^{-1}(s))/2+\beta B_{J(J^{-1}(s))}} ds \le \rho\right)$$

$$\le \mathbb{P}\left(\int_0^\infty e^{-(\beta\lambda_1/\nu+\beta/2)s+\beta B_s} ds \le \rho/c\right). \tag{3.4}$$

By the change of variables $s \mapsto (\beta/2)^2 s$ the probability (3.4) simplifies to

$$\mathbb{P}(\tau^* = +\infty) \le \mathbb{P}\left(\frac{4}{\beta^2}\int_0^\infty e^{-(4\lambda_1/(\nu\beta)+2/\beta)s+\beta B_{4s/\beta^2}} ds \le \rho/c\right) \tag{3.5}$$

$$= \mathbb{P}\left(\int_0^\infty e^{-2(2\lambda_1+\nu)s/(\nu\beta)+2B_s} ds \le \frac{\beta^2\rho}{4c}\right)$$

$$= \mathbb{P}\left(\int_0^\infty e^{2B_s^{(-\mu)}} ds \le \frac{\beta^2\rho}{4c}\right) \tag{3.6}$$

where $\mu := (2\lambda_1+\nu)/(\nu\beta)$, $B_s^{(-\mu)} := B_s - \mu s$, and to get the second line we used the scaling property of Brownian motion. From [21, Cor. 1.2, pag. 95] and (3.6) we obtain

$$\mathbb{P}(\tau^* = +\infty) \le \mathbb{P}\left(\frac{1}{2Z_\mu} \le \frac{\beta^2\rho}{4c}\right) = 1 - \frac{1}{\Gamma(\mu)}\int_0^{\frac{2c}{\beta^2\rho}} y^{\mu-1}e^{-y}\, dy = 1 - P(\mu,\theta) \tag{3.7}$$

where $P(\mu,\theta) := \gamma(\mu,\theta)/\Gamma(\mu)$ and $\gamma(\mu,\theta)$ is the lower incomplete gamma function relative to $\theta := \frac{2c}{\beta^2\rho} = 2cv(0,\psi)^\beta/\beta$ and with parameter μ, cf. e.g. [1], Eq. 6.5.1. and Eq. 6.5.2. By Proposition 2.2 we have $\tau \le \tau^*$ whenever τ is the blowup time of u. Hence, from (3.7) we have

$$\mathbb{P}(\tau < +\infty) \ge \mathbb{P}(\tau^* < +\infty) = 1 - \mathbb{P}(\tau^* = +\infty) \ge P(\mu,\theta), \tag{3.8}$$

as needed. ∎

In the next corollary we consider uniformly bounded coefficients $h(t)$ and $k(t)$ with constant coefficient $\kappa(t)$.

Corollary 3.2 *Suppose that $h(t) \geq a, k(t) \leq b$ and $\kappa(t) = \kappa$, where $a, b, \kappa \in \mathbb{R}_+$. Then the probability that the solution of* (1.3) *blows up in finite time is lower bounded by $P\big(\mu, \theta\big) = \gamma\big(\mu, \theta\big)/\Gamma(\mu)$, where $\gamma\big(\mu, \theta\big)$ is the lower incomplete gamma function relative to $\theta := 2av(0, \psi)^\beta/(\kappa^2\beta)$ and with parameter $\mu = (2\lambda_1 + \kappa^2)/(\kappa^2\beta)$.*

Proof. Using the assumptions we get $h(t)e^{-\beta\lambda_1(K(t)/2-J(t)/\nu)} \geq ae^{-\beta\lambda_1(b^2/2-\kappa^2/\nu)t}$. Hence, condition (3.1) in Theorem 3.1 is satisfied with $\nu = 2\kappa^2/b^2$ and $c = a/\kappa^2$. □

Next we recover the main blowup result in [5] for the case of constant coefficients.

Corollary 3.3 (Dozzi and López-Mimbela [5]) *Suppose that $h(t), k(t)$ and $\kappa(t)$ are constant functions with $h(t) = 1$, $k(t) = \sqrt{2}$ and $\kappa(t) = \kappa \in \mathbb{R}_+$ for all $t \geq 0$. Then the probability that the solution of* (1.3) *blows up in finite time is lower bounded by $P\big(\mu, \theta\big) = \gamma\big(\mu, \theta\big)/\Gamma(\mu)$, where $\gamma\big(\mu, \theta\big)$ is the lower incomplete gamma function relative to $\theta := 2v(0, \psi)^\beta/(\kappa^2\beta)$ and with parameter $\mu = (2\lambda_1 + \kappa^2)/(\kappa^2\beta)$.*

Proof. This immediately follows from Corollary 3.2 by setting $a = 1$ and $b = \sqrt{2}$. Note that for this case we have condition (3.1) as an equality. Hence, (3.4), (3.5), (3.7), as well as the rightmost inequality in (3.8) will be equalities. □

4 Existence of positive global solutions

In this section we provide conditions for the existence of global solutions of (1.3) using semigroup techniques. We consider again the random PDE in (2.2) and this time assume that our locally Lipschitz function $G : \mathbb{R}_+ \to \mathbb{R}_+$ satisfies $G(0) = 0$, $z \mapsto G(z)/z$ is increasing, and

$$G(z) \leq \Lambda z^{1+\beta}, \qquad z > 0, \tag{4.1}$$

where $\Lambda, \beta > 0$ are constants. Let $\tilde{W}_t$ be a d-dimensional Brownian motion killed on leaving D, and let us denote by $(S_{r,t})_{0\leq r\leq t\in\mathbb{R}_+}$ the semigroup of the process

$$X_{r,t} := \int_r^t k(s)\, d\tilde{W}_s, \qquad 0 \leq r \leq t.$$

Observe that for $x \in D$ and any $f \in C^2(D)$ we have, by the d-dimensional Itô formula,

$$\begin{aligned} S_{r,t}f(x) &= \mathbb{E}\left[f\big(x + X_{r,t}\big)\right] \\ &= \mathbb{E}\left[f(x) + \int_r^t k(s)\langle \nabla f\big(x + X_{r,s}\big),\, d\tilde{W}_s\rangle + \frac{1}{2}\int_r^t k^2(s)\Delta f\big(x + X_{r,s}\big)\, ds\right] \\ &= f(x) + \frac{1}{2}\int_r^t k^2(s)\,\mathbb{E}\left[\Delta f\big(x + X_{r,s}\big)\right] ds, \end{aligned}$$

and so

$$\frac{\partial}{\partial t} S_{r,t}f(x) = \frac{1}{2}k^2(t)\,\mathbb{E}\left[\Delta f(x + X_{r,t})\right] = \frac{1}{2}k^2(t)\Delta\,\mathbb{E}\left[f(x + X_{r,t})\right] = \frac{1}{2}k^2(t)\Delta S_{r,t}f(x).$$

This implies that $S_{r,t}f$ solves the heat equation $\frac{\partial}{\partial t}u(t,x) = \frac{1}{2}k^2(t)\Delta u(t,x)$ with initial value $u(0,x) = f(x)$ and Dirichlet boundary condition. In addition, the associated transition kernels of this semigroup are given by $(P_{K(r,t)}(x,y))_{0\le r<t\in \mathrm{R}_+}$, where $K(r,t)$ is defined in (2.7) and $(P_s(x,y))_{0<s\in\mathrm{R}_+}$ are the transition densities of $\tilde{W}_t$. Taking $k(t) = \sqrt{2}$ with $r \le t$ in particular, we have a time homogeneous process $X_{r,t} = \sqrt{2}\tilde{W}_{t-r}$ and its semigroup can alternatively be rewritten as $S_{r,t} = S_{0,t-r}$ or simply as S_{t-r}, which in case of $r = 0$ gives the homogeneous semigroup S_t of the d-dimensional Brownian motion with variance parameter 2 that was considered in [5].

Note that from the time-inhomogeneous Markov property we have

$$S_{r,t}S_{0,r} = S_{0,r}S_{r,t} = S_{0,t}, \quad 0 \le r \le t. \tag{4.2}$$

In the next Lemma we restate (2.2) as an integral equation.

Lemma 4.1 *Eq.* (2.2) *can be rewritten as*

$$v(t,x) = e^{-J(t)/2}S_{0,t}f(x) + \int_0^t h(r)e^{-M_r - J(r,t)/2}S_{r,t}G\big(e^{M_r}v(r,\cdot)\big)(x)dr, \tag{4.3}$$

$t > 0$, $x \in D$.

Proof. Using the transformation $w = w(t,x) = e^{J(t)/2}v(t,x)$ where $J(t)$ is as defined in (2.8), we get from (2.2) a non-homogeneous random PDE

$$\frac{\partial w}{\partial t} = \frac{\kappa^2(t)}{2}e^{J(t)/2}v(t,x) + e^{J(t)/2}\frac{\partial v}{\partial t}(t,x)$$

$$= \frac{\kappa^2(t)}{2} e^{J(t)/2} v(t,x) + e^{J(t)/2} \left(\frac{1}{2} k^2(t) \Delta v(t,x) - \frac{\kappa^2(t)}{2} v(t,x) + h(t) e^{-M_t} G\big(e^{M_t} v(t,x)\big) \right)$$
$$= \frac{1}{2} k^2(t) \Delta w(t,x) + h(t) e^{-M_t + J(t)/2} G\big(e^{M_t} v(t,x)\big), \quad t \geq 0,$$

with boundary conditions $w(0,x) = f(x) \geq 0$ for $x \in D$ and $w(t,x) = 0$ for $x \in \partial D$. Consequently, any mild solution v of (2.2) satisfies

$$\begin{aligned} v(t,x) &= e^{-J(t)/2} w(t,x) \\ &= e^{-J(t)/2} \left(S_{0,t} f(x) + \int_0^t S_{r,t} h(r) e^{-M_r + J(r)/2} G\big(e^{M_r} v(r,\cdot)\big)(x) dr \right) \\ &= e^{-J(t)/2} S_{0,t} f(x) + \int_0^t h(r) e^{-M_r - J(r,t)/2} S_{r,t} G\big(e^{M_r} v(r,\cdot)\big)(x) dr, \end{aligned} \tag{4.4}$$

where the second equality is due to [15, Def 2.3, page 106]. □

In the next Theorem we provide a lower bound τ_* for the blowup time of the solution u of (1.3).

Theorem 4.2 *Suppose G satisfies (4.1), $G(0) = 0$ and that $z \mapsto G(z)/z$ is increasing. Let $A(t)$ be defined by*

$$A(t) := \left(1 - \Lambda\beta \int_0^t h(r) e^{\beta(M_r - J(r)/2)} \|S_{0,r} f\|_\infty^\beta dr \right)^{-1/\beta} \tag{4.5}$$

for all $t \in [0, \tau_*)$, where

$$\tau_* := \inf \left\{ t \in \mathbb{R}_+ : \int_0^t h(r) e^{\beta(M_r - J(r)/2)} \|S_{0,r} f\|_\infty^\beta dr \geq \frac{1}{\Lambda\beta} \right\}. \tag{4.6}$$

Then equation (1.3) admits a solution $u(t,x)$ that satisfies

$$0 \leq u(t,x) \leq A(t) e^{M_t - J(t)/2} S_{0,t} f(x), \quad x \in D, \quad t \in [0, \tau_*).$$

Proof. Taking the derivative of (4.5) with respect to t we obtain

$$\frac{dA}{dt}(t) = \Lambda h(t) e^{\beta(M_t - J(t)/2)} \|S_{0,t} f\|_\infty^\beta A(t)^{1+\beta}.$$

Integrating this equality from 0 to t and using the fact that $A(0) = 1$ we get

$$A(t) = 1 + \Lambda \int_0^t h(r) e^{\beta(M_r - J(r)/2)} \|S_{0,r} f\|_\infty^\beta A(r)^{1+\beta} dr. \tag{4.7}$$

Suppose now that $(t,x) \mapsto V_t(x) \geq 0$ is a continuous function such that for any $t \geq 0$ we have $V_t(\cdot) \in C_0(D)$ and

$$e^{-J(t)/2} S_{0,t} f(x) \leq V_t(x) \leq A(t) e^{-J(t)/2} S_{0,t} f(x), \quad t \geq 0, \ x \in D. \tag{4.8}$$

The last inequality in (4.8) implies that

$$V_t(x) = |V_t(x)| \leq \|V_t\|_\infty \leq A(t) e^{-J(t)/2} \|S_{0,t} f\|_\infty. \tag{4.9}$$

Now let

$$R(V)(t,x) := e^{-J(t)/2} S_{0,t} f(x) + \int_0^t h(r) e^{-M_r - J(r,t)/2} S_{r,t} \left(G\big(e^{M_r} V_r(\cdot)\big)\right)(x) dr. \tag{4.10}$$

Using (4.9) and the fact that $z \mapsto G(z)/z$ is increasing, followed by inequality (4.1), we obtain

$$\begin{aligned}
&R(V)(t,x) \\
&= e^{-J(t)/2} S_{0,t} f(x) + \int_0^t h(r) e^{-J(r,t)/2} S_{r,t} \left(\frac{G\big(e^{M_r} V_r(\cdot)\big)}{e^{M_r} V_r(\cdot)} V_r(\cdot) \right)(x) dr. \\
&\leq e^{-J(t)/2} S_{0,t} f(x) + \int_0^t h(r) e^{-J(r,t)/2} S_{r,t} \left(\frac{G\big(e^{M_r} A(r) e^{-J(r)/2} \|S_{0,r} f\|_\infty\big)}{e^{M_r} A(r) e^{-J(r)/2} \|S_{0,r} f\|_\infty} V_r(\cdot) \right)(x) dr \\
&\leq e^{-J(t)/2} S_{0,t} f(x) + \int_0^t h(r) e^{-J(r,t)/2} S_{r,t} \left(\Lambda \big(e^{M_r} A(r) e^{-J(r)/2} \|S_{0,r} f\|_\infty\big)^\beta V_r(\cdot) \right)(x) dr.
\end{aligned}$$

Moreover, (4.8) together with equalities (4.2) and (4.7) give

$$\begin{aligned}
R(V)(t,x) &\leq e^{-J(t)/2} S_{0,t} f(x) + \\
&\qquad \Lambda \int_0^t h(r) e^{\beta\left(M_r - J(r)/2\right)} A(r)^\beta \|S_{0,r} f\|_\infty^\beta e^{-J(r,t)/2} S_{r,t} \big(A(r) e^{-J(r)/2} S_{0,r} f\big)(x) dr \\
&= e^{-J(t)/2} S_{0,t} f(x) + \Lambda \int_0^t h(r) e^{\beta\left(M_r - J(r)/2\right)} A(r)^{1+\beta} \|S_{0,r} f\|_\infty^\beta e^{-J(t)/2} S_{0,t} f(x) dr \\
&= e^{-J(t)/2} S_{0,t} f(x) \left[1 + \Lambda \int_0^t h(r) e^{\beta\left(M_r - J(r)/2\right)} A(r)^{1+\beta} \|S_{0,r} f\|_\infty^\beta dr \right] \\
&= e^{-J(t)/2} S_{0,t} f(x) A(t).
\end{aligned} \tag{4.11}$$

Let us now construct a sequence $\left\{v_t^n(x)\right\}_{n=0}^\infty$ of positive functions in the following way:

Set $\quad v_t^0(x) := e^{-J(t)/2} S_{0,t} f(x) \qquad$ and $\qquad v_t^{n+1}(x) = R(v^n)(t,x) \quad$ for $n = 0, 1, 2, \ldots$.

Clearly, $v_t^0(x) \leq R(v^0)(t,x) = v_t^1(x)$. Assume now that $v_t^{n-1}(x) \leq v_t^n(x)$ for all $n \geq 1$, $t \geq 0$ and $x \in D$. Since $z \mapsto G(z)/z$ is increasing we have

$$G\big(e^{M_r}v_r^{n-1}(\cdot)\big) = \frac{G\big(e^{M_r}v_r^{n-1}(\cdot)\big)}{e^{M_r}v_r^{n-1}(\cdot)}e^{M_r}v_r^{n-1}(\cdot) \;\leq\; \frac{G\big(e^{M_r}v_r^{n}(\cdot)\big)}{e^{M_r}v_r^{n}(\cdot)}e^{M_r}v_r^{n}(\cdot) = G\big(e^{M_r}v_r^{n}(\cdot)\big).$$

Consequently, from the construction above and by definition (4.10) we get

$$v_t^n(x) = R(v^{n-1})(t,x) \;\leq\; R(v^n)(t,x) = v_t^{n+1}(x), \quad t \in \mathbb{R}_+,\ x \in D.$$

This shows, by induction, that the constructed sequence is monotonically increasing. Furthermore, it is bounded above as shown in (4.11). Hence, by the monotone convergence theorem the sequence $\big\{v_t^n(x)\big\}_{n=0}^{\infty}$ has a limit which we denote by $v(t,x)$, and the equality $v(t,x) = R(v)(t,x)$ holds for all $x \in D$ and $t \in (0,\tau_*)$, so that v satisfies (4.3) on $(0,\tau_*)$. Also, this limit satisfies

$$0 \leq \lim_{n\to\infty} v_t^n(x) = v(t,x) \leq A(t)e^{-J(t)/2}S_{0,t}f(x) < \infty,$$

for all $t \in [0,\tau_*)$. We conclude by applying Lemma 4.1 and (2.1). □

By virtue of Theorem 4.2 we now give a condition under which we have a global positive solution of (1.3) with probability 1.

Corollary 4.3 *Assume that $f \geq 0$ is such that*

$$\Lambda\beta \int_0^\infty h(r)e^{\beta(M_r - J(r)/2)}\|S_{0,r}f\|_\infty^\beta dr < 1. \tag{4.12}$$

Then equation (1.3) *admits a global solution $u(t,x)$ that satisfies*

$$0 \leq u(t,x) \leq \frac{e^{M_t - J(t)/2}S_{0,t}f(x)}{\Big(1 - \Lambda\beta\int_0^t h(r)e^{\beta(M_r - J(r)/2)}\|S_{0,r}f\|_\infty^\beta\, dr\Big)^{1/\beta}}, \qquad t \geq 0.$$

Proof. This immediately follows from Theorem 4.2 since $\tau_* = \infty$ by (4.6) and (4.12). □

Remark 4.4 Notice that, if

(i) ψ is an eigenfunction associated with the first eigenvalue λ_1 of the Dirichlet eigenvalue problem for the Laplacian Δ on D, i.e. $S_{0,r}\psi(x) = e^{-\frac{\lambda_1}{2}\int_0^r k^2(s)ds}\psi(x)$ where $r \geq 0$, $x \in D$, with $\int_D \psi(x)dx = 1$,

(ii) $C = \Lambda = 1$ and

(iii) $f(x) = p\psi(x)$, $x \in D$, where $p > 0$ is a parameter,

then τ_* defined in (4.6) and τ^* defined in (2.9) are given in terms of the same exponential Brownian functional, namely:

$$\tau_* = \inf\left\{t \geq 0 \ : \ \int_0^t h(s)e^{-\beta(\lambda_1 K(s)+J(s))/2+\beta M_s}ds \geq \frac{p^{-\beta}}{\beta}\|\psi\|_\infty^{-\beta}\right\}$$
$$\tau^* = \inf\left\{t \geq 0 \ : \ \int_0^t h(s)e^{-\beta(\lambda_1 K(s)+J(s))/2+\beta M_s}ds \geq \frac{p^{-\beta}}{\beta}\left(\int_D \psi^2(x)dx\right)^{-\beta}\right\},$$

with $\tau_* \leq \tau^*$ a.s. since $\int_D \psi^2(x)dx \leq \int_D \psi(x)\|\psi\|_\infty dx = \|\psi\|_\infty$.

As a consequence of Theorem 4.2 and in the special setting of Remark 4.4 we now give an estimate for the probability of existence of global positive solution of (1.3).

Corollary 4.5 *Under the conditions* (*i*)-(*iii*) *in Remark 4.4 and assumption* (3.1) *of Theorem 3.1, the probability that a global positive solution of* (1.3) *exists is upper bounded by*

$$Q(\mu, \theta) := \frac{\Gamma(\mu, \theta)}{\Gamma(\mu)},$$

where $\mu := (2\lambda_1 + \nu)/(\nu\beta)$, $\theta := 2c\,(p\|\psi\|_\infty)^\beta/\beta$ *and*

$$\Gamma(\mu, \theta) := \int_\theta^\infty y^{\mu-1}e^{-y}dy$$

is the upper incomplete gamma function.

Proof. Replacing ρ in (3.3) by $p^{-\beta}\|\psi\|_\infty^{-\beta}/\beta$ and arguing as in the proof Theorem 3.1 while working with τ_* defined in (4.6), we get $\mathbb{P}(\tau_* = +\infty) \leq 1 - P(\mu, \theta) = Q(\mu, \theta)$ with Q, μ and θ as above and $P(\mu, \theta)$ defined in (3.2). We conclude using Theorem 4.2. □

For further use in the proof of Theorem 4.7, we quote the following bounds for the transition kernels given by Ouhabaz and Wang [12].

Theorem 4.6 *Let* $\psi > 0$ *be the first Dirichlet eigenfunction of the Laplacian on a connected bounded* $C^{1,\alpha}$*-domain* $D \subset \mathbb{R}^d$, *where* $\alpha > 0$ *and* $d \geq 1$, *and let* $P_t(x, y)$ *be* *the corresponding Dirichlet heat kernel. There exists a constant $\xi > 0$ such that, for any $t > 0$,*

$$\max\left\{1,\ \frac{1}{\xi}t^{-(d+2)/2}\right\} \le e^{\lambda_1 t}\sup_{x,y\in D}\frac{P_t(x,y)}{\psi(x)\psi(y)} \le 1+\xi(1\wedge t)^{-(d+2)/2}e^{-(\lambda_2-\lambda_1)t},$$

where $\lambda_2 > \lambda_1$ are the first two Dirichlet eigenvalues. This estimate is sharp for both short and long times.

We now give sufficient conditions for (4.12) using upper bounds derived from Theorem 4.6 for the transition kernels $(P_{K(r,t)}(x,y))_{0\le r<t\in\mathbb{R}_+}$ of the semigroup $(S_{r,t})_{0\le r\le t\in\mathbb{R}_+}$, and the first Dirichlet eigenvalue $\lambda_1 > 0$ with corresponding eigenfunction ψ.

Theorem 4.7 *Let G satisfy* (4.1), *$G(0) = 0$ and that $z \mapsto G(z)/z$ is increasing, and let D be a connected bounded $C^{1,\alpha}$-domain in $\mathbb{R}^d$, where $\alpha > 0$. If for some fixed η with $K(\eta) \ge 1$ and constant $m > 0$ we have*

$$f(y) \le mS_{0,\eta}\psi(y), \quad y \in D \tag{4.13}$$

and

$$\int_0^\infty h(r)e^{-\beta(\lambda_1 K(r)+J(r))/2+\beta M_r}dr < \frac{e^{\lambda_1\beta K(\eta)}}{\Lambda\beta\left(m(1+\xi)\left(\sup_{x\in D}\psi(x)\right)^2\int_D\psi(y)dy\right)^\beta} \tag{4.14}$$

where $\xi > 0$ is as given in Theorem 4.6, then (4.12) *is satisfied and consequently the solution of equation* (1.3) *is global.*

Proof. Let $f \ge 0$ be chosen so that (4.13) is satisfied. Observe that for any $t > 0$,

$$S_{0,t}f(x) \le mS_{0,t}S_{0,\eta}\psi(x) = m\int_D P_{K(t)+K(\eta)}(x,y)\psi(y)dy.$$

From the inequality above, Theorem 4.6 then gives

$$\begin{aligned}
&S_{0,t}f(x)\\
&\le m\int_D e^{\lambda_1\left(K(t)+K(\eta)\right)}\frac{P_{K(t)+K(\eta)}(x,y)}{\psi(x)\psi(y)}e^{-\lambda_1\left(K(t)+K(\eta)\right)}\psi(x)\psi(y)\psi(y)dy\\
&\le m\left(\sup_{x\in D}\psi(x)\right)^2\int_D e^{\lambda_1\left(K(t)+K(\eta)\right)}\sup_{x,y\in D}\frac{P_{K(t)+K(\eta)}(x,y)}{\psi(x)\psi(y)}e^{-\lambda_1\left(K(t)+K(\eta)\right)}\psi(y)dy
\end{aligned}$$

$$
\begin{aligned}
&\le m\left(\sup_{x\in D}\psi(x)\right)^2 \times \\
&\qquad \int_D \left(1+\xi\big[1\wedge\big(K(t)+K(\eta)\big)\big]^{-(d+2)/2} e^{-(\lambda_2-\lambda_1)\big(K(t)+K(\eta)\big)}\right) e^{-\lambda_1\big(K(t)+K(\eta)\big)}\psi(y)dy \\
&= m\left(\sup_{x\in D}\psi(x)\right)^2 \int_D \left(e^{-\lambda_1\big(K(t)+K(\eta)\big)} + \xi e^{-\lambda_2\big(K(t)+K(\eta)\big)}\right)\psi(y)dy \\
&\le m\left(\sup_{x\in D}\psi(x)\right)^2 \left(e^{-\lambda_1\big(K(t)+K(\eta)\big)} + \xi e^{-\lambda_1\big(K(t)+K(\eta)\big)}\right)\int_D \psi(y)dy \\
&= m\big(1+\xi\big)e^{-\lambda_1 K(\eta)}\left(\sup_{x\in D}\psi(x)\right)^2 e^{-\lambda_1 K(t)}\int_D \psi(y)dy, \qquad (4.15)
\end{aligned}
$$

which is independent of x. Inequality (4.15) implies that the map $(t,x)\mapsto S_{0,t}f(x)$ is uniformly bounded in x and so

$$
\|S_{0,t}f\|_\infty \le m\big(1+\xi\big)e^{-\lambda_1 K(\eta)}\left(\sup_{x\in D}\psi(x)\right)^2 e^{-\lambda_1 K(t)}\int_D \psi(y)dy. \qquad (4.16)
$$

Multiplying both sides of (4.16) by $e^{-J(t)/2}$ we would have

$$
\begin{aligned}
e^{-J(t)/2}\|S_{0,t}f\|_\infty &\le m\big(1+\xi\big)e^{-\lambda_1 K(\eta)}\left(\sup_{x\in D}\psi(x)\right)^2 e^{-\lambda_1 K(t)-J(t)/2}\int_D \psi(y)dy \\
&\le m\big(1+\xi\big)e^{-\lambda_1 K(\eta)}\left(\sup_{x\in D}\psi(x)\right)^2 e^{-\big(\lambda_1 K(t)+J(t)\big)/2}\int_D \psi(y)dy \qquad (4.17)
\end{aligned}
$$

Now from (4.14) and the last inequality in (4.17) we get

$$
\begin{aligned}
1 &> \Lambda\beta e^{-\lambda_1\beta K(\eta)}\left(m(1+\xi)\left(\sup_{x\in D}\psi(x)\right)^2\int_D \psi(y)dy\right)^\beta \int_0^\infty h(r)e^{-\beta(\lambda_1 K(r)+J(r))/2+\beta M_r}dr \\
&= \Lambda\beta\int_0^\infty h(r)e^{\beta M_r}\left(m(1+\xi)e^{-\lambda_1 K(\eta)}\left(\sup_{x\in D}\psi(x)\right)^2 e^{-\big(\lambda_1 K(r)+J(r)\big)/2}\int_D \psi(y)dy\right)^\beta dr \\
&\ge \Lambda\beta\int_0^\infty h(r)e^{\beta(M_r-J(r)/2)}\|S_{0,r}f\|_\infty^\beta dr,
\end{aligned}
$$

which is precisely (4.12). Global existence of solution to (1.3) now immediately follows from Corollary 4.3. □

Remark 4.8 *Assuming both* (2.5) *and* (4.1) *with* $C=\Lambda=1$*, i.e.* $G(z)=z^{1+\beta}$ *and if* D *is a connected bounded* $C^{1,\alpha}$*-domain in* $\mathbb{R}^d$*, the results of both Sections 3 and 4 can be applied to the solution of SPDE* (1.3) *by virtue of Proposition 2.1.*

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References

[1] M. Abramowitz and I.A. Stegun. *Handbook of mathematical functions with formulas, graphs, and mathematical tables*, volume 55. Dover Publications, New York, 1972. 9th Edition.

[2] V.V. Baklan. On the existence of solutions of stochastic equations in Hilbert space. *Dopovidi Akad. Nauk Ukraïn. RSR.*, 1963:1299–1303, 1963.

[3] E.B. Davies. *Heat kernels and spectral theory*, volume 92 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1989.

[4] M. Dozzi, E.T. Kolkovska, and J.A. López-Mimbela. Exponential functionals of Brownian motion and explosion times of a system of semilinear SPDEs. *Stochastic Anal. Appl.*, 31(6):975–991, 2013.

[5] M. Dozzi and J.A. López-Mimbela. Finite-time blowup and existence of global positive solutions of a semi-linear SPDE. *Stochastic Process. Appl.*, 120(6):767–776, 2010.

[6] H. Fujita. On some nonexistence and nonuniqueness theorems for nonlinear parabolic equations. In *Nonlinear Functional Analysis (Chicago, Ill., 1968)*, volume XVIII, Part 1 of *Proc. Sympos. Pure Math.*, pages 105–113, Providence, RI, 1970. Amer. Math. Soc.

[7] I. Karatzas and S. Shreve. *Brownian motion and stochastic calculus*, volume 113 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1991.

[8] N.V. Krylov and B.L. Rozovskii. Stochastic evolution equations. *Journal of Soviet Mathematics*, 14:1233–1277, 1981.

[9] J.A. López-Mimbela and A. Pérez. Finite time blow up and stability of a semilinear equation with a time dependent Lévy generator. *Stoch. Models*, 22(4):735–752, 2006.

[10] J.A. López-Mimbela and A. Pérez. Global and nonglobal solutions of a system of nonautonomous semilinear equations with ultracontractive Lévy generators. *J. Math. Anal. Appl.*, 423:720–733, 2015.

[11] G. Lv and J. Duan. Impacts of noise on a class of partial differential equations. *J. Differential Equations*, 258(6):2196–2220, 2015.

[12] E.M. Ouhabaz and F.Y. Wang. Sharp estimates for intrinsic ultracontractivity on $C^{1,\alpha}$ -domains. *Manuscripta Math*, 122(2):229–244, 2007.

[13] É. Pardoux. Équations aux dérivées partielles stochastiques nonlinéares monotones: étude de solutions fortes de type Itô. PhD thesis, Université Paris-Orsay, 1975.

[14] N.H. Pavel. *Nonlinear evolution operators and semigroups. Applications to partial differential equations*, volume 1260 of *Lecture Notes in Math.* Springer-Verlag, Berlin, 1987.

[15] A. Pazy. *Semigroups of linear operators and applications to partial differential equations*, volume 44 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1983.

[16] M. Sanz-Solé and P.-A. Vuillermot. Equivalence and Hölder-Sobolev regularity of solutions for a class of non-autonomous stochastic partial differential equations. *Ann. Inst. H. Poincaré Probab. Statist.*, 39(4):703–742, 2003.

[17] J. Seidler. Da prato-Zabczyk's maximal inequality revisited. *Math. Bohem.*, 118(1):67–106, 1993.

[18] G. Teschl. *Ordinary Differential Equations and Dynamical Systems*, volume 40 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2012.

[19] M. Veraar. Non-autonomous stochastic evolution equations and applications to stochastic partial differential equations. *J. Evol. Equ.*, 10(1):85–127, 2010.

[20] M. Veraar and J. Zimmerschied. Non-autonomous stochastic Cauchy problems in Banach spaces. *Studia. Math.*, 185:1–34, 2008.

[21] M. Yor. *Exponential functionals of Brownian motion and related processes*. Springer Finance. Springer-Verlag, Berlin, 2001.

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