A constructive approach to existence of equilibria in time-inconsistent stochastic control problems

Jiang Yu Nguwi* Nicolas Privault†

Division of Mathematical Sciences

School of Physical and Mathematical Sciences

Nanyang Technological University

21 Nanyang Link, Singapore 637371

September 30, 2021

Abstract

We extend the construction of equilibria for linear-quadratic and mean-variance portfolio problems available in the literature to a large class of mean-field time-inconsistent stochastic control problems in continuous time. Our approach relies on a time discretization of the control problem via n-person games, which are characterized via the maximum principle using Backward Stochastic Differential Equations (BSDEs). The existence of equilibria is proved by applying weak convergence arguments to the solutions of n-person games. A numerical implementation is provided by approximating n-person games using finite Markov chains.

Keywords: Stochastic control; time inconsistency; maximum principle; n-person games; BSDE; Markov chain approximation.

Mathematics Subject Classification (2020): 93E20, 91G80, 91B70.

1 Introduction

Stochastic control theory aims at optimizing a time-dependent functional parameterized by a controlled random state process, with applications to numerous problems in physics, biology, finance, economics, etc. For this, the most commonly used approaches rely on Pontryagin's

*nguw0003@e.ntu.edu.sg

†nprivault@ntu.edu.sg

maximum principle and on Hamilton-Jacobi-Bellman (HJB) equations, see e.g. Yong and Zhou (1999) and Fleming and Soner (2006) for classical results on stochastic control theory. This approach deals with time-consistent stochastic control problems, in which an optimal strategy today remains optimal in the future.

However, many stochastic control problems are time-inconsistent in the sense that an optimal strategy today may not be optimal in the future. This is the case for example in the framework of a production economy with time-varying preferences, or in the nonlinear setting of mean-variance portfolio optimization, which cannot be directly treated using the dynamic programming principle and HJB equations. Such problems have recently been the object of increased attention, see e.g. Björk and Murgoci (2014) and Björk et al. (2017).

There are two common formulations for time-inconsistent problems. The first approach is to fix an initial time, to solve the problem given this initial time, and to stick to this pre-committed optimal policy for the remaining time. See for example Zhou and Li (2000) for the solution of mean-variance portfolio selection problem using pre-committed strategies.

The second approach, introduced by Ekeland and Lazrak (2006) in the deterministic setting, is to formulate time-inconsistent problems in a game-theoretic setting using equilibrium controls. This approach, which uses an HJB-type equation to characterize the equilibrium controls, has been extended in Björk and Murgoci (2014) and Björk et al. (2017) to stochastic mean-field control problems in both discrete and continuous time. In Hu et al. (2012; 2017), a related characterization has been proposed by the maximum principle in a linear-quadratic model, where the SDE is linear and the mean-field objective functional is quadratic. This characterization argument has been later extended to general mean-field objective functionals in Djehiche and Huang (2016).

However, no general results are available on the existence of equilibrium controls, except in special cases such as the linear-quadratic model of Hu et al. (2012). In addition, no numerical construction of equilibrium controls has been provided so far, except in mean-variance portfolio selection, see Wang and Forsyth (2011).

In this paper, we present a constructive approach to the existence of equilibrium controls for a class of mean-field time-inconsistent control problems, together with its numerical implementation. Let $(W_t)_{t\in[0,T]}$ denote a standard Brownian motion generating the filtration

$(\mathcal{F}_t)_{t\in[0,T]}$. Our results apply to the class of cost functionals of the form

$$J(t,\xi,\mu)=\mathbb{E}_t\left[g\big(X^{\xi,\mu}_{t,T},\mathbb{E}_t\big[\Psi\big(X^{\xi,\mu}_{t,T}\big)\big]\big)+\int_t^T\int_U h\big(s,X^{\xi,\mu}_{t,s},\mathbb{E}_t\big[\Phi\big(X^{\xi,\mu}_{t,s}\big)\big],v\big)\mu_s(dv)ds\right], \quad (1.1)$$

where μ is a relaxed control, ξ is an $\mathcal{F}_t$-measurable $\mathbb{R}$-valued random variable, $\mathbb{E}_t[X]=\mathbb{E}[X|\mathcal{F}_t]$ is the conditional expectation given $\mathcal{F}_t$, $t\in[0,T]$, and $(X^{\xi,\mu}_{t,s})_{s\in[t,T]}$ is the non-linear controlled diffusion given by

$$\begin{cases} dX^{\xi,\mu}_{t,s}=\displaystyle\int_U b\big(s,X^{\xi,\mu}_{t,s},v\big)\mu_s(dv)ds+\sigma\big(s,X^{\xi,\mu}_{t,s}\big)dW_s, \quad 0\le t<s\le T,\\ X^{\xi,\mu}_{t,t}=\xi. \end{cases} \quad (1.2)$$

Our approach to the existence of equilibrium controls relies on a time discretization of the control problem using n-person games, and on a variation of Pontryagin's maximum principle for the characterization of n-person games, see Theorem 2.1. In Corollary 2.5, we prove the existence of an equilibrium control in the sense of Definition 1.3 for the time-inconsistent mean-field control problem (1.1)-(1.2), based on a formulation of equilibrium controls as weak limits of the sequence of solutions to n-person games, see Theorem 2.4. The proof of Theorem 2.4 uses BSDE convergence arguments and the characterization Theorem 2.1.

The numerical construction of equilibrium controls is achieved by approximating n-person games using finite Markov chains by adapting the method of Kushner (1990a) to our setting, see Theorem 3.4. Precisely, the argument therein applies only to posed inf problems, as it requires comparing the optimal control μ^* to any other control μ via the inequality $J(t,x,\mu^*)\le J(t,x,\mu)$. Here, the control problem (1.1)-(1.2) is not posed inf, instead it is formulated in the game-theoretic setting of equilibrium controls in the sense of Definition 1.3 below. Hence, no such comparison of equilibrium controls is possible as in (1.5), nevertheless we are able to apply the comparison argument to n-person games since they are posed inf. In Section 3.2, the numerical scheme is implemented using a trinomial tree, first on a linear-quadratic model which admits an analytic solution, and then on a linear-quartic model which does not have analytic solution.

The particular case of mean-variance portfolio selection, where the cost functional $J(t,\xi,\mu)$ in (1.1) is given by

$$J(t,\xi,\mu)=-\mathbb{E}_t\big[X^{\xi,\mu}_{t,T}\big]+\frac{\gamma}{2}\mathbb{E}_t\Big[\big(X^{\xi,\mu}_{t,T}-\mathbb{E}_t\big[X^{\xi,\mu}_{t,T}\big]\big)^2\Big]$$

where $\gamma > 0$ has been treated in Czichowsky (2013) using semimartingale theory for the convergence of equilibrium controls from discrete to continuous time. See also Huang and Zhou (2018) in the case where $\left(X_{t,s}^{\xi,\mu}\right)_{s\in[t,T]}$ is a finite Markov chains, for time-inconsistent control problems with infinite horizon.

This paper is organized as follows. After stating the necessary preliminaries on equilibrium and relaxed controls, in Section 2.1 we present a characterization of n-person games using the maximum principle. In Section 2.2 we show the convergence of the solutions of n-person games to an equilibrium control, and we obtain in turn the existence of an equilibrium control in Corollary 2.5. In Section 3.1 we deal with the convergence properties of the Markov chain approximation for the SDE of n-person games, see Theorem 3.4. In Section 3.2 we present a numerical application of the convergence results obtained in Sections 2.2 and 3.1. The proofs of the main Theorems 2.1, Corollary 2.5, Theorems 2.4 and 3.4 rely on technical lemmas presented in appendix.

Preliminaries

Let $T > 0$ be a fixed time horizon and $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ be a filtered probability space satisfying the usual conditions, where $(\mathcal{F}_t)_{t\in[0,T]}$ is the filtration generated by a standard Brownian motion $(W_t)_{t\in[0,T]}$. In the sequel, we let U denote a compact subset of $\mathbb{R}$, and we denote by $\mathcal{B}([0,T]\times U)$ and $\mathcal{B}(U)$ the Borel σ-algebra of $[0,T]\times U$ and U, respectively.

Definition 1.1. *The space Λ of deterministic relaxed controls is the set of nonnegative measures λ on $\mathcal{B}([0,T]\times U)$ such that*

$$\lambda([0,t]\times U) = t, \qquad t \in [0,T]. \tag{1.3}$$

We also denote by $\lambda_t(\cdot)$ the density such that $\lambda(dt, dv) = \lambda_t(dv)dt$, $t \in [0,T]$, whose existence follows from (1.3).

Definition 1.2. *i) The space $\mathcal{U}([0,T])$ of strict controls over $[0,T]$ is the set of $(\mathcal{F}_t)_{t\in[0,T]}$-adapted U-valued processes.*

ii) The space $\mathcal{R}([0,T])$ of relaxed controls over $[0,T]$ is the set of Λ-valued random variables λ such that $\lambda([0,t]\times B)$ is $\mathcal{F}_t$-measurable for all $t \in [0,T]$ and $B \in \mathcal{B}(U)$.

We now turn to the definition of equilibrium controls in a game-theoretic setting, see Ekeland and Lazrak (2006). Given $\mu, \nu \in \mathcal{R}([0,T])$ two relaxed controls and $\varepsilon \in (0, T-t]$, we let

$\nu \otimes_{t,\varepsilon} \mu$ denote the local spike variation of μ, defined as

$$(\nu \otimes_{t,\varepsilon} \mu)_s = \begin{cases} \nu_s, & 0 \leq t \leq s \leq t+\varepsilon, \\ \mu_s, & s \in [0,T] \backslash [t, t+\varepsilon]. \end{cases}$$

As in e.g. Kushner and Dupuis (2001), Buckdahn et al. (2011), Djehiche and Huang (2016) Bahlali et al. (2018), we assume the following boundedness and smoothness conditions on the coefficients and cost functions of the problem (1.1)-(1.2).

Assumption 1. *i) The functions $b, \sigma, h, g, \Phi, \Psi$ are uniformly continuous and bounded.*

ii) The functions $b(t,x,u)$, $\sigma(t,x)$, $\Phi(x)$, $\Psi(x)$ are differentiable with respect to x for all $(t,u) \in [0,T] \times U$, and their first order (partial) derivatives $\partial_x b(t,x,u)$, $\partial_x \sigma(t,x)$, $\Phi'(x)$, $\Psi'(x)$ are differentiable with respect to x for all $(t,u) \in [0,T] \times U$, are uniformly continuous and bounded.

iii) The functions $h(t,x,y,u), g(x,y)$ are differentiable with respect to (x,y) for all $(t,u) \in [0,T] \times U$, and their first order partial derivatives $\partial_x h(t,x,y,u)$, $\partial_y h(t,x,y,u)$, $\partial_x g(x,y)$, $\partial_y g(x,y)$ are uniformly continuous and bounded.

iv) There is a constant $\sigma_0 > 0$ such that $\sigma(t,x) \geq \sigma_0$ for all $(t,x) \in [0,T] \times \mathbb{R}$.

We note that the functions $b(t,\cdot,u)$, $\sigma(t,\cdot)$, $h(t,\cdot,\cdot,u)$, $g(\cdot,\cdot)$, $\Phi(\cdot)$, $\Psi(\cdot)$ are globally Lipschitz continuous for all $(t,u) \in [0,T] \times U$ since they have bounded derivatives. In the sequel, we fix an initial condition $x_0 \in \mathbb{R}$, and given $\mu \in \mathcal{R}([0,T])$ we let $X_t^\mu := X_{0,t}^{x_0,\mu}$, $t \in [0,T]$, denote the solution of the SDE

$$\begin{cases} dX_t^\mu = \displaystyle\int_U b\big(t, X_t^\mu, v\big) \mu_t(dv) dt + \sigma\big(t, X_t^\mu\big) dW_t, & 0 < t \leq T, \\ X_0^\mu = x_0. \end{cases} \tag{1.4}$$

The next definition of equilibrium controls is an extension of Definition 2.1 in Hu et al. (2012) using the space $\mathcal{R}([0,T])$ of relaxed controls instead of the space $\mathcal{U}([0,T])$ of strict controls.

Definition 1.3. *We say that a relaxed control $\mu^* \in \mathcal{R}([0,T])$ is an equilibrium control for the time-inconsistent mean-field control problem (1.1)-(1.2) if*

$$\lim_{h \downarrow 0} \frac{J\big(t, X_t^{\mu^*}, \mu^*\big) - J\big(t, X_t^{\mu^*}, \mu \otimes_{t,h} \mu^*\big)}{h} \leq 0, \quad \mu \in \mathcal{R}([0,T]), \ \text{a.e. } t \in [0,T], \ \mathbb{P}\text{-a.s.}, \tag{1.5}$$

where the equilibrium dynamics $\big(X_t^{\mu^*}\big)_{t \in [0,T]}$ is the solution of (1.4).

In the literature, Definition 1.3 is usually stated in the space $\mathcal{U}([0,T])$ of strict controls instead of using the space $\mathcal{R}([0,T])$ of relaxed controls, see Definition 1.2. The relaxed representation of a strict control $u \in \mathcal{U}([0,T])$ is denoted by

$$\mu(dt, dv) = \mu_t(dv)dt = \delta_{u_t}(dv)dt, \tag{1.6}$$

where $\delta_x(dv)$ denotes the Dirac measure at $x \in U$.

As the proof of our existence result Corollary 2.5 requires the compactness of the control space, we choose to work with the space $\mathcal{R}([0,T])$ of relaxed controls because it is compact when endowed with the weak topology. Examples of control problems which do not admit strict equilibrium controls can be constructed based on the non compactness of the space $\mathcal{U}([0,T])$ of strict controls, see e.g. the Rademacher function example in § 1 of Valadier (1994). In Hu et al. (2012; 2017), the existence of equilibrium controls is proved without requiring the compactness of the control space, however this is for the special case of a linear-quadratic structure on the SDE and cost functional.

For convenience, we introduce the following notation. Given $\mu \in \mathcal{R}([0,T])$ a relaxed control of interest, for example μ^* in Theorem 1.4 or μ^{*n} in Theorem 2.1 below, for $\varphi = b, \sigma, h, g$ and $\gamma = \Phi$, resp. Ψ when $\varphi = h$, resp. g, we set the notation

$$\partial_x \varphi^\mu_{t,s} = \int_U \partial_x \varphi(s, X^\mu_s, \mathbb{E}_t[\gamma(X^\mu_s)], v)\mu_s(dv), \tag{1.7a}$$

$$\partial_y \varphi^\mu_{t,s} = \gamma'(X^\mu_s)\mathbb{E}_t\left[\int_U \partial_y \varphi(s, X^\mu_s, \mathbb{E}_t[\gamma(X^\mu_s)], v)\mu_s(dv)\right], \tag{1.7b}$$

where $t \leq s \leq T$ and X^μ is defined in (1.4). Next, we now introduce the Hamiltonian function

$$H(t, x, y, \mu, p) = p\int_U b(t, x, v)\mu(dv) - \int_U h(t, x, y, v)\mu(dv), \tag{1.8}$$

where $(t, x, y, p) \in [0,T] \times \mathbb{R}^3$, and μ is in the collection $\mathbb{P}(U)$ of all probability measures on U. By abuse of notation, we also denote

$$H(t, x, y, u, p) = pb(t, x, u) - h(t, x, y, u)$$

when the fourth variable in (1.8) is $u \in U$. The next theorem is a direct extension to relaxed controls of the characterization of strict equilibrium controls proved in Theorem 1 of Djehiche and Huang (2016) using the maximum principle, therefore its proof is omitted.

Theorem 1.4. *Let $\mu^* \in \mathcal{R}([0,T])$ denote a relaxed control. Consider $\partial.b^{\mu^*}_{t,s}$, $\partial.\sigma^{\mu^*}_{t,s}$, $\partial.h^{\mu^*}_{t,s}$, $\partial.g^{\mu^*}_{t,T}$ given by (1.7a)-(1.7b), and let $\left(p^{\mu^*}_{t,s}, q^{\mu^*}_{t,s}\right)_{s\in[t,T]}$ be the solution of the first order adjoint equation*

$$\begin{cases} dp^{\mu^*}_{t,s} = -\left(p^{\mu^*}_{t,s}\partial_x b^{\mu^*}_{t,s} + q^{\mu^*}_{t,s}\partial_x \sigma^{\mu^*}_{t,s} - \partial_x h^{\mu^*}_{t,s} - \partial_y h^{\mu^*}_{t,s}\right)ds + q^{\mu^*}_{t,s} dW_s, \quad 0 \le t \le s \le T, \\ p^{\mu^*}_{t,T} = -\partial_x g^{\mu^*}_{t,T} - \partial_y g^{\mu^*}_{t,T}. \end{cases} \tag{1.9}$$

Then, μ^* is an equilibrium control for the problem (1.1)-(1.2) if and only if there exists a pair $\left(p^{\mu*}_{t,s}, q^{\mu*}_{t,s}\right)_{s\in[t,T]}$ of $(\mathcal{F}_s)_{s\in[t,T]}$-adapted process satisfying (1.9), and such that

$$H\left(t, X^{\mu^*}_t, \Phi(X^{\mu^*}_t), \nu, p^{\mu^*}_{t,t}\right) \le H\left(t, X^{\mu^*}_t, \Phi(X^{\mu^*}_t), \mu^*_t, p^{\mu^*}_{t,t}\right), \quad \nu \in \mathbb{P}(U), \ \text{a.e. } t \in [0,T], \ \mathbb{P}\text{-a.s.}$$

In the sequel, $C > 0$ represents a generic constant which may change from line to line.

2 Existence of equilibrium controls

2.1 Maximum principle characterization of n-person games

In this section, we consider n-person games for the construction of an equilibrium control later in Section 2.2. In Yong (2012), equilibrium HJB equations have been used for the characterization of equilibria via n-person games in control problems without mean-field terms. Since the extension of this PDE approach to the mean-field case may not be straightforward, we propose instead to use the maximum principle for the construction of equilibrium controls.

Given $n \ge 1$, we consider the sequence $\{t_k = kT/n, \ k = 0, 1, \ldots, n\}$ with step size $\Delta_n := T/n$. Theorem 2.1 is a characterization of the solution of the discretization of the time-inconsistent mean-field control problem (1.1)-(1.2) into an n-person game, for use in the proofs of Corollary 2.5 and Theorem 2.4.

Theorem 2.1. *Let $n \ge 1$. Under Assumption 1, suppose that the n-person discretized time-inconsistent mean-field control problem*

$$J\left(t_k, X^{\mu^{*n}}_{t_k}, \mu^{*n}\right) = \inf_{\mu\in\mathcal{R}([t_k,t_{k+1}])} J\left(t_k, X^{\mu^{*n}}_{t_k}, \mu \otimes_{t_k,\Delta_n} \mu^{*n}\right), \qquad k = 0, 1, \ldots, n-1, \tag{2.1}$$

admits a solution $\mu^{*n} \in \mathcal{R}([0,T])$ and let $\left(p^{\mu^{*n}}_{t_k,t}, q^{\mu^{*n}}_{t_k,t}\right)_{t\in[t_k,T]}$ be the solution of the first order adjoint equation

$$\begin{cases} dp^{\mu^{*n}}_{t_k,t} = -\left(p^{\mu^{*n}}_{t_k,t}\partial_x b^{\mu^{*n}}_{t_k,t} + q^{\mu^{*n}}_{t_k,t}\partial_x \sigma^{\mu^{*n}}_{t_k,t} - \partial_x h^{\mu^{*n}}_{t_k,t} - \partial_y h^{\mu^{*n}}_{t_k,t}\right)dt + q^{\mu^{*n}}_{t_k,t} dW_t, \quad t_k \le t \le T, \\ p^{\mu^{*n}}_{t_k,T} = -\partial_x g^{\mu^{*n}}_{t_k,T} - \partial_y g^{\mu^{*n}}_{t_k,T}, \qquad k = 0, 1, \ldots, n-1. \end{cases} \tag{2.2}$$

Then we have

$$H\big(t, X_t^{\mu^{*n}}, \mathbb{E}_{t_k}[\Phi(X_t^{\mu^{*n}})], \nu, p_{t_k,t}^{\mu^{*n}}\big) \le H\big(t, X_t^{\mu^{*n}}, \mathbb{E}_{t_k}[\Phi(X_t^{\mu^{*n}})], \mu_t^{*n}, p_{t_k,t}^{\mu^{*n}}\big), \tag{2.3}$$

$\nu \in \mathbb{P}(U)$, a.e. $t \in [t_k, t_{k+1}]$, $\mathbb{P}$-*a.s.*, $k = 0, 1, \ldots, n-1$.

Proof. We fix $k \in \{0, 1, \ldots, n-1\}$ and $t \in [t_k, t_{k+1})$. Given $A \in \mathcal{F}_t$ and $\nu \in \mathbb{P}(U)$, applying Lemma 2.2 below to the deviated control $\mu_s := \nu \mathbb{1}_A + \mu_s^{*n} \mathbb{1}_{\Omega\setminus A}$, $s \in [0,T]$, we have

$$\begin{aligned} & J\big(t_k, X_{t_k}^{\mu^{*n}}, \mu \otimes_{t,\varepsilon} \mu^{*n}\big) - J\big(t_k, X_{t_k}^{\mu^{*n}}, \mu^{*n}\big) \\ & = \mathbb{E}_{t_k}\left[\int_t^{t+\varepsilon} \big(H\big(s, X_s^{\mu^{*n}}, \mathbb{E}_{t_k}[\Phi(X_s^{\mu^{*n}})], \mu_s^{*n}, p_{t_k,s}^{\mu^{*n}}\big) - H\big(s, X_s^{\mu^{*n}}, \mathbb{E}_{t_k}[\Phi(X_s^{\mu^{*n}})], \mu_s, p_{t_k,s}^{\mu^{*n}}\big)\big) ds\right] + o(\varepsilon), \end{aligned} \tag{2.4}$$

as ε tends to zero. Since μ^{*n} is a solution of (2.1), the deviation μ of μ^{*n} in $\mathcal{R}([0,T])$ over any time period within $[t_k, t_{k+1}]$ will be sub-optimal. Therefore, letting ε tend to 0, the Lebesgue Differentiation Theorem applied to (2.4) yields

$$\mathbb{E}_{t_k}\left[\mathbb{1}_A\big(H\big(t, X_t^{\mu^{*n}}, \mathbb{E}_{t_k}[\Phi(X_t^{\mu^{*n}})], \mu_t^{*n}, p_{t_k,t}^{\mu^{*n}}\big) - H\big(t, X_t^{\mu^{*n}}, \mathbb{E}_{t_k}[\Phi(X_t^{\mu^{*n}})], \nu, p_{t_k,t}^{\mu^{*n}}\big)\big)\right] \ge 0, \tag{2.5}$$

a.e. $t \in [0,T]$. Since $A \in \mathcal{F}_t$ is arbitrary, we conclude to (2.3). □

The next lemma, which has been used in the proof of Theorem 2.1, yields an expansion of the cost functional $J\big(t_k, X_{t_k}^{\mu^{*n}}, \mu \otimes_{t,\varepsilon} \mu^{*n}\big)$ in ε. For $\mu, \nu \in \mathcal{R}([0,T])$ and $\varphi = b, \sigma, h, g$ we let

$$\begin{aligned} \delta\varphi_{t,s}^{\nu,\mu} &= \int_U \varphi(s, X_s^\mu, \mathbb{E}_t[\gamma(X_s^\mu)], v)\nu_s(dv) - \int_U \varphi(s, X_s^\mu, \mathbb{E}_t[\gamma(X_s^\mu)], v)\mu_s(dv), \\ \delta\partial_x\varphi_{t,s}^{\nu,\mu} &= \int_U \partial_x\varphi(s, X_s^\mu, \mathbb{E}_t[\gamma(X_s^\mu)], v)\nu_s(dv) - \int_U \partial_x\varphi(s, X_s^\mu, \mathbb{E}_t[\gamma(X_s^\mu)], v)\mu_s(dv), \end{aligned}$$

$t \le s \le T$, where $\gamma = \Phi$, resp. Ψ when $\varphi = h$, resp. g.

Lemma 2.2. *Under the assumptions of Theorem 2.1, fix $t \in [0,T)$ and $k \in \{0, 1, \ldots, n-1\}$ such that $t_k \le t < t_{k+1}$, and let $\mu \in \mathcal{R}([0,T])$. Then, as $\varepsilon > 0$ tends to zero we have the expansion*

$$J\big(t_k, X_{t_k}^{\mu^{*n}}, \mu \otimes_{t,\varepsilon} \mu^{*n}\big) = J\big(t_k, X_{t_k}^{\mu^{*n}}, \mu^{*n}\big) + \mathbb{E}_{t_k}\left[\int_t^{t+\varepsilon} \big(\delta h_{t_k,s}^{\mu,\mu^{*n}} - \delta b_{t_k,s}^{\mu,\mu^{*n}} p_{t_k,s}^{\mu^{*n}}\big) ds\right] + o(\varepsilon).$$

Proof. Let $\big(y_s^{(\varepsilon)}\big)_{s\in[t_k,T]}$ denote the solution of the variational equation

$$\begin{cases} dy_s^{(\varepsilon)} = \big(y_s^{(\varepsilon)}\partial_x b_{t_k,s}^{\mu^{*n}} + \delta b_{t_k,s}^{\mu,\mu^{*n}} \mathbb{1}_{[t,t+\varepsilon]}(s)\big) ds + y_s^{(\varepsilon)}\partial_x\sigma_{t_k,s}^{\mu^{*n}} dW_s, \quad t_k \le s \le T, \\ y_{t_k}^{(\varepsilon)} = 0, \end{cases} \tag{2.7}$$

and, for $(s,u,\theta)\in[t_k,T]\times U\times[0,1]$, let $\mu^\varepsilon_s := (\mu\otimes_{t,\varepsilon}\mu^{*n})_s$, $\xi^{(\varepsilon)}_s := X^{\mu^\varepsilon}_s - X^{\mu^{*n}}_s$, $\eta^{(\varepsilon)}_s := \xi^{(\varepsilon)}_s - y^{(\varepsilon)}_s$, and use the notation

$$\begin{cases}\partial_x h_\theta(s,u) = \partial_x h\big(s,(1-\theta)X^{\mu^{*n}}_s + \theta X^{\mu^\varepsilon}_s, \mathbb{E}_{t_k}\big[(1-\theta)\Phi\big(X^{\mu^{*n}}_s\big)+\theta\Phi\big(X^{\mu^\varepsilon}_s\big)\big],u\big),\\ \partial_x g_\theta = \partial_x g\big((1-\theta)X^{\mu^{*n}}_T + \theta X^{\mu^\varepsilon}_T, \mathbb{E}_{t_k}\big[(1-\theta)\Psi\big(X^{\mu^{*n}}_T\big)+\theta\Psi\big(X^{\mu^\varepsilon}_T\big)\big]\big),\\ \partial_x\Phi_\theta(s) = \partial_x\Phi\big((1-\theta)X^{\mu^{*n}}_s+\theta X^{\mu^\varepsilon}_s\big),\end{cases}$$

and similarly for $\partial_y h_\theta(s,u)$, $\partial_y g_\theta$, $\partial_x\Psi_\theta$. We note that by the flow property $X^{X^{\mu^{*n}}_{t_k},\mu^{*n}}_{t_k,s} = X^{\mu^{*n}}_s$ the cost functional in (1.1) rewrites as

$$J\big(t_k, X^\mu_{t_k},\mu\big) = \mathbb{E}_{t_k}\left[g\big(X^\mu_T,\mathbb{E}_{t_k}\big[\Psi\big(X^\mu_T\big)\big]\big) + \int_{t_k}^T\int_U h\big(s,X^\mu_s,\mathbb{E}_{t_k}\big[\Phi\big(X^\mu_s\big)\big],v\big)\mu_s(dv)ds\right].$$

By the fundamental theorem of calculus on $[0,1]$ we have

$$\begin{aligned}
&J\big(t_k,X^{\mu^{*n}}_{t_k},\mu^\varepsilon\big) - J\big(t_k,X^{\mu^{*n}}_{t_k},\mu^{*n}\big)\\
&= \mathbb{E}_{t_k}\bigg[g\big(X^{\mu^\varepsilon}_T,\mathbb{E}_{t_k}\big[\Psi\big(X^{\mu^\varepsilon}_T\big)\big]\big) - g\big(X^{\mu^{*n}}_T,\mathbb{E}_{t_k}\big[\Psi\big(X^{\mu^{*n}}_T\big)\big]\big)\\
&\quad + \int_{t_k}^T\bigg(\int_U h\big(s,X^{\mu^\varepsilon}_s,\mathbb{E}_{t_k}\big[\Phi(X^{\mu^\varepsilon}_s)\big],v\big)\mu^\varepsilon_s(dv) - \int_U h\big(s,X^{\mu^{*n}}_s,\mathbb{E}_{t_k}\big[\Phi\big(X^{\mu^{*n}}_s\big)\big],v\big)\mu^{*n}_s(dv)\bigg)ds\bigg]\\
&= \mathbb{E}_{t_k}\bigg[\xi^{(\varepsilon)}_T\int_0^1\partial_x g_\theta d\theta + \mathbb{E}_{t_k}\bigg[\int_0^1\xi^{(\varepsilon)}_T\partial_x\Psi_\theta d\theta\bigg]\int_0^1\partial_y g_\theta d\theta + \int_{t_k}^T\delta h^{\mu,\mu^{*n}}_{t_k,s}\mathbb{1}_{[t,t+\varepsilon]}(s)ds\\
&\quad + \int_{t_k}^T\int_U\bigg(\xi^{(\varepsilon)}_s\int_0^1\partial_x h_\theta(s,v)d\theta + \mathbb{E}_{t_k}\bigg[\int_0^1\xi^{(\varepsilon)}_s\partial_x\Phi_\theta(s)d\theta\bigg]\int_0^1\partial_y h_\theta(s,v)d\theta\bigg)\mu^\varepsilon_s(dv)ds\bigg]\\
&= \mathbb{E}_{t_k}\bigg[\xi^{(\varepsilon)}_T\int_0^1(\partial_x g_\theta - \partial_x g_0)d\theta + (\xi^{(\varepsilon)}_T - y^{(\varepsilon)}_T)\partial_x g^{\mu^{*n}}_{t_k,T} + y^{(\varepsilon)}_T\partial_x g^{\mu^{*n}}_{t_k,T}\\
&\quad + \mathbb{E}_{t_k}\bigg[\int_0^1\xi^{(\varepsilon)}_T\partial_x\Psi_\theta d\theta\bigg]\bigg(\int_0^1(\partial_y g_\theta - \partial_y g_0)d\theta\bigg)\\
&\quad + \partial_y g_0\mathbb{E}_{t_k}\bigg[\int_0^1\xi^{(\varepsilon)}_T\big(\partial_x\Psi_\theta - \partial_x\Psi_0\big)d\theta\bigg] + \partial_y g_0\mathbb{E}_{t_k}\big[\big(\xi^{(\varepsilon)}_T - y^{(\varepsilon)}_T\big)\partial_x\Psi_0\big] + y^{(\varepsilon)}_T\partial_y g^{\mu^{*n}}_{t_k,T}\\
&\quad + \int_{t_k}^T\bigg(\xi^{(\varepsilon)}_s\int_U\int_0^1\big(\partial_x h_\theta(s,v) - \partial_x h_0(s,v)\big)d\theta\mu^\varepsilon_s(dv)\\
&\qquad + \xi^{(\varepsilon)}_s\bigg(\int_U\partial_x h_0(s,v)\mu^\varepsilon_s(dv) - \int_U\partial_x h_0(s,v)\mu^{*n}_s(dv)\bigg) + \big(\xi^{(\varepsilon)}_s - y^{(\varepsilon)}_s\big)\partial_x h^{\mu^{*n}}_{t_k,s} + y^{(\varepsilon)}_s\partial_x h^{\mu^{*n}}_{t_k,s}\\
&\qquad + \mathbb{E}_{t_k}\bigg[\int_0^1\xi^{(\varepsilon)}_s\partial_x\Phi_\theta(s)d\theta\bigg]\int_U\int_0^1(\partial_y h_\theta(s,v) - \partial_y h_0(s,v))d\theta\mu^\varepsilon_s(dv)\\
&\qquad + \mathbb{E}_{t_k}\bigg[\int_0^1\xi^{(\varepsilon)}_s\big(\partial_x\Phi_\theta(s) - \partial_x\Phi_0(s)\big)d\theta\bigg]\int_U\partial_y h_0(s,v)\mu^\varepsilon_s(dv)
\end{aligned}$$

$$
\begin{aligned}
&\quad + \mathbb{E}_{t_k}\big[\xi_s^{(\varepsilon)}\partial_x\Phi_0(s)\big]\bigg(\int_U \partial_y h_0(s,v)\mu_s^\varepsilon(dv) - \int_U \partial_y h_0(s,v)\mu_s^{*n}(dv)\bigg)\\
&\quad + \mathbb{E}_{t_k}\big[\big(\xi_s^{(\varepsilon)} - y_s^{(\varepsilon)}\big)\partial_x\Phi_0(s)\big]\int_U \partial_y h_0(s,v)\mu_s^{*n}(dv) + y_s^{(\varepsilon)}\partial_y h_{t_k,s}^{\mu^{*n}} + \delta h_{t_k,s}^{\mu,\mu^{*n}}\mathbb{1}_{[t,t+\varepsilon]}(s)\bigg)\,ds\bigg]\\
&= \mathbb{E}_{t_k}\bigg[y_T^{(\varepsilon)}\partial_x g_{t_k,T}^{\mu^{*n}} + y_T^{(\varepsilon)}\partial_y g_{t_k,T}^{\mu^{*n}} + \int_{t_k}^T \big(y_s^{(\varepsilon)}\partial_x h_{t_k,s}^{\mu^{*n}} + y_s^{(\varepsilon)}\partial_y h_{t_k,s}^{\mu^{*n}} + \delta h_{t_k,s}^{\mu,\mu^{*n}}\mathbb{1}_{[t,t+\varepsilon]}(s)\big)ds\bigg] + o(\varepsilon),
\end{aligned}
\tag{2.8}
$$

$$
= \mathbb{E}_{t_k}\bigg[\int_{t_k}^T \big(y_s^{(\varepsilon)}\partial_x h_{t_k,s}^{\mu^{*n}} + y_s^{(\varepsilon)}\partial_y h_{t_k,s}^{\mu^{*n}} + \delta h_{t_k,s}^{\mu,\mu^{*n}}\mathbb{1}_{[t,t+\varepsilon]}(s)\big)ds\bigg] - \mathbb{E}_{t_k}\big[y_T^{(\varepsilon)}p_{t_k,T}^{\mu^{*n}}\big] + o(\varepsilon), \tag{2.9}
$$

as ε tends to zero, where (2.8) is due to Relations (2.10), (2.12) in Lemma 2.3, the conditional Hölder inequality, Assumption 1, and Lemma A.4, and (2.9) is due to (2.2). We conclude using the identity

$$
\mathbb{E}_{t_k}\big[y_T^{(\varepsilon)}p_{t_k,T}^{\mu^{*n}}\big] = \mathbb{E}_{t_k}\bigg[\int_{t_k}^T \big(y_s^{(\varepsilon)}\partial_x h_{t_k,s}^{\mu^{*n}} + y_s^{(\varepsilon)}\partial_y h_{t_k,s}^{\mu^{*n}} + \delta b_{t_k,s}^{\mu,\mu^{*n}}\mathbb{1}_{[t,t+\varepsilon]}(s)p_{t_k,s}^{\mu^{*n}}\big)ds\bigg],
$$

that follows from Itô's lemma. $\square$

In the next lemma, we derive the order of convergence for the variational equation (2.7), which has been used in the proof of Lemma 2.2.

Lemma 2.3. *Under the assumptions of Theorem 2.1, fix $t \in [0,T)$ and $k \in \{0,1,\ldots,n-1\}$ such that $t_k \le t < t_{k+1}$, let $\mu \in \mathcal{R}([0,T])$, $\varepsilon > 0$, $p \ge 1$, and denote $\xi_s^{(\varepsilon)} = X_s^{\mu^\varepsilon} - X_s^{\mu^{*n}}$, $\eta_s^{(\varepsilon)} = \xi_s^{(\varepsilon)} - y_s^{(\varepsilon)}$ as in (2.7), where $\mu^\varepsilon = \mu \otimes_{t,\varepsilon} \mu^{*n}$. Then, as ε tends to zero we have the estimates*

$$
\mathbb{E}_{t_k}\bigg[\sup_{s\in[t_k,T]}\big|\xi_s^{(\varepsilon)}\big|^{2p}\bigg] = O(\varepsilon^{2p}), \tag{2.10}
$$

$$
\mathbb{E}_{t_k}\bigg[\sup_{s\in[t_k,T]}|y_s^{(\varepsilon)}|^{2p}\bigg] = O(\varepsilon^{2p}), \tag{2.11}
$$

$$
\mathbb{E}_{t_k}\bigg[\sup_{s\in[t_k,T]}|\xi_s^{(\varepsilon)} - y_s^{(\varepsilon)}|^{2p}\bigg] = o(\varepsilon^{2p}). \tag{2.12}
$$

Proof. 1) Proof of (2.10)-(2.11). Letting

$$
\begin{cases}
\tilde{b}_s^{(\varepsilon)} = \displaystyle\int_U \big(b(s,X_s^{\mu^\varepsilon},v) - b(s,X_s^{\mu^{*n}},v)\big)\mu_s^\varepsilon(dv) = \int_U\int_0^1 \partial_x b\big(s,(1-\theta)X_s^{\mu^{*n}} + \theta X_s^{\mu^\varepsilon},v\big)d\theta\mu_s^\varepsilon(dv),\\
\tilde{\sigma}_s^{(\varepsilon)} = \sigma\big(s,X_s^{\mu^\varepsilon}\big) - \sigma\big(s,X_s^{\mu^{*n}}\big) = \displaystyle\int_0^1 \partial_x\sigma\big(s,(1-\theta)X_s^{\mu^{*n}} + \theta X_s^{\mu^\varepsilon}\big)d\theta,
\end{cases}
$$

by the fundamental theorem of calculus, the process $\big(\xi_s^{(\varepsilon)}\big)_{s\in[t_k,T]}$ satisfies the SDE

$$\begin{cases} d\xi_s^{(\varepsilon)} = \big(\xi_s^{(\varepsilon)}\tilde{b}_s^{(\varepsilon)} + \delta b_{t_k,s}^{\mu,\mu^{*n}}\mathbb{1}_{[t,t+\varepsilon]}(s)\big)ds + \xi_s^{(\varepsilon)}\tilde{\sigma}_s^{(\varepsilon)}dW_s, \qquad t_k\le s\le T,\\ \xi_{t_k}^{(\varepsilon)} = 0. \end{cases}$$

Next, by the Burkholder-Davis-Gundy inequality, we have

$$\begin{aligned}
\mathbb{E}_{t_k}\Big[\sup_{s\in[t_k,T]}\big|\xi_s^{(\varepsilon)}\big|^{2p}\Big] &= \mathbb{E}_{t_k}\Big[\sup_{s\in[t_k,T]}\Big|\int_{t_k}^s \big(\xi_r^{(\varepsilon)}\tilde{b}_r^{(\varepsilon)} + \delta b_{t_k,r}^{\mu,\mu^{*n}}\mathbb{1}_{[t,t+\varepsilon]}(r)\big)dr + \int_{t_k}^s \xi_r^{(\varepsilon)}\tilde{\sigma}_r^{(\varepsilon)}dW_r\Big|^{2p}\Big]\\
&\le C\mathbb{E}_{t_k}\Big[\int_{t_k}^T \big|\xi_s^{(\varepsilon)}\tilde{b}_s^{(\varepsilon)}\big|^{2p}ds + \Big|\int_{t_k}^T \big|\delta b_{t_k,s}^{\mu,\mu^{*n}}\big|\mathbb{1}_{[t,t+\varepsilon]}(s)ds\Big|^{2p} + \int_{t_k}^T \big|\xi_s^{(\varepsilon)}\tilde{\sigma}_s^{(\varepsilon)}\big|^{2p}ds\Big]\\
&\le C\mathbb{E}_{t_k}\Big[\int_{t_k}^T \big|\xi_s^{(\varepsilon)}\big|^{2p}ds\Big] + C\varepsilon^{2p} \qquad (2.13)\\
&\le C\int_{t_k}^T \mathbb{E}_{t_k}\Big[\sup_{r\in[t_k,s]}\big|\xi_r^{(\varepsilon)}\big|^{2p}\Big]ds + C\varepsilon^{2p},
\end{aligned}$$

where (2.13) is due to the boundedness of b, σ and their derivatives in Assumption 1. The proof of (2.10) is completed using Gronwall's inequality, and (2.11) can be proved similarly.
2) Proof of (2.12). By the fundamental theorem of calculus, the process $\big(\eta_s^{(\varepsilon)}\big)_{s\in[t_k,T]}$ satisfies the SDE

$$\begin{cases} d\eta_s^{(\varepsilon)} = \big(\xi_s^{(\varepsilon)}\tilde{b}_s^{(\varepsilon)} - y_s^{(\varepsilon)}\partial_x b_{t_k,s}^{\mu^{*n}}\big)ds + \big(\xi_s^{(\varepsilon)}\tilde{\sigma}_s^{(\varepsilon)} - y_s^{(\varepsilon)}\partial_x\sigma_{t_k,s}^{\mu^{*n}}\big)dW_s, \quad t_k\le s\le T,\\ \eta_{t_k}^{(\varepsilon)} = 0. \end{cases}$$

As ε tends to zero, we have

$$\begin{aligned}
&\mathbb{E}_{t_k}\Big[\sup_{s\in[t_k,T]}\big|\eta_s^{(\varepsilon)}\big|^{2p}\Big] = \mathbb{E}_{t_k}\Big[\sup_{s\in[t_k,T]}\Big|\int_{t_k}^s\Big(\xi_r^{(\varepsilon)}\Big(\tilde{b}_r^{(\varepsilon)} - \int_U \partial_x b\big(r,X_r^{\mu^{*n}},v\big)\mu_r^{\varepsilon}(dv)\Big)\\
&\quad + \xi_r^{(\varepsilon)}\Big(\int_U \partial_x b\big(r,X_r^{\mu^{*n}},v\big)\mu_r^{\varepsilon}(dv) - \partial_x b_{t_k,r}^{\mu^{*n}}\Big) + \eta_r^{(\varepsilon)}\partial_x b_{t_k,r}^{\mu^{*n}}\Big)dr\\
&\quad + \int_{t_k}^v\Big(\xi_r^{(\varepsilon)}\big(\tilde{\sigma}_r^{(\varepsilon)} - \partial_x\sigma_{t_k,r}^{\mu^{*n}}\big) + \eta_r^{(\varepsilon)}\partial_x\sigma_{t_k,r}^{\mu^{*n}}\Big)dW_r\Big|^{2p}\Big]\\
&\le C\mathbb{E}_{t_k}\Bigg[\int_{t_k}^T\Big(\big|\xi_s^{(\varepsilon)}\big|^{2p}\Big|\tilde{b}_s^{(\varepsilon)} - \int_U \partial_x b\big(s,X_s^{\mu^{*n}},v\big)\mu_s^{\varepsilon}(dv)\Big|^{2p} + \big|\eta_s^{(\varepsilon)}\partial_x b_{t_k,s}^{\mu^{*n}}\big|^{2p} + \big|\eta_s^{(\varepsilon)}\partial_x\sigma_{t_k,s}^{\mu^{*n}}\big|^{2p}\\
&\quad + \big|\xi_s^{(\varepsilon)}\big|^{2p}\big|\tilde{\sigma}_s^{(\varepsilon)} - \partial_x\sigma_{t_k,s}^{\mu^{*n}}\big|^{2p}\Big)ds + \Big|\int_{t_k}^T \xi_s^{(\varepsilon)}|\delta\partial_x b_{t_k,s}^{\mu,\mu^{*n}}|\times\mathbb{1}_{[t,t+\varepsilon]}(s)ds\Big|^{2p}\Bigg]\\
&\le C\Bigg(\mathbb{E}_{t_k}\Big[\int_{t_k}^T\big|\eta_s^{(\varepsilon)}\big|^{2p}ds\Big] + \sup_{s\in[t_k,T]}\sqrt{\mathbb{E}_{t_k}\big[\big|\xi_s^{(\varepsilon)}\big|^{4p}\big]\mathbb{E}_{t_k}\Big[\Big|\tilde{b}_s^{(\varepsilon)} - \int_U \partial_x b(s,X_s^{\mu^{*n}},v)\mu_s^{(\varepsilon)}(dv)\Big|^{4p}\Big]}
\end{aligned}$$

$$+\sup_{s\in[t_k,T]}\sqrt{\mathbb{E}_{t_k}\left[\left|\xi_s^{(\varepsilon)}\right|^{4p}\right]\mathbb{E}_{t_k}\left[\left|\tilde{\sigma}_s^{(\varepsilon)}-\partial_x\sigma_{t_k,s}^{\mu^{*n}}\right|^{4p}\right]+\varepsilon^{4p}}\Bigg) \tag{2.14}$$

$$\leq C\int_{t_k}^{T}\mathbb{E}_{t_k}\left[\sup_{r\in[t_k,s]}\left|\eta_r^{(\varepsilon)}\right|^{2p}\right]ds+o\big(\varepsilon^{2p}\big), \tag{2.15}$$

where (2.14) is due to (2.10) and to the boundedness of $\partial_x b$ and $\partial_x\sigma$, (2.15) is due to the uniform continuity of $\partial_x b$, $\partial_x\sigma$ in Assumption 1, Lemma A.4, and (2.10). The proof of (2.12) is completed by Gronwall's inequality. □

2.2 Construction of equilibrium controls

We equip the space Λ of deterministic relaxed controls with the weak topology generated by the bounded continuous functions on $[0,T]\times U$. The spaces $\mathcal{C}([0,T])$ and $\mathcal{D}([0,T])$ of continuous and càdlàg functions on $[0,T]$ are equipped with the uniform and Skorokhod metrics, respectively. In Theorem 2.4, we construct an equilibrium control for (1.1)-(1.2) as the weak limit of the solution of the n-person game (2.1) as n tends to infinity, and in Corollary 2.5 we prove the existence of an equilibrium control. In the sequel we let $\lfloor T\rfloor_n := T$ and $\lfloor t\rfloor_n := t_k$ if $t_k\leq t<t_{k+1}$, $k=0,1,\ldots,n-1$

Theorem 2.4. *Under Assumption 1, for any $n\geq 1$ there exists a solution μ^{*n} of the n-person game (2.1) in the space of relaxed controls. In addition, the weak limit μ^* of any convergent subsequence of $(\mu^{*n})_{n\geq 1}$ is an equilibrium control for the time-inconsistent mean-field control problem (1.1)-(1.2).*

Proof. We start by constructing a solution μ^{*n} of the n-person game (2.1) using backward induction in the compact space of relaxed controls. By Theorem 2.14 of Bahlali et al. (2018) there exists a mapping $\hat{\mu}_n:\mathbb{R}\to\mathcal{R}([t_{n-1},T])$ such that

$$J\big(t_{n-1},x,\hat{\mu}_n(x)\big)=\inf_{\mu\in\mathcal{R}([t_{n-1},T])}J\big(t_{n-1},x,\mu\big),\qquad x\in\mathbb{R}.$$

Next, applying this argument recursively to $k=n-1,\ldots,2,1$, we obtain a mapping $\mu_k:\mathbb{R}\to\mathcal{R}([t_{k-1},t_k])$ such that

$$\begin{aligned}&J\big(t_{k-1},x,\mu_k(x)\otimes_{t_{k-1},\Delta_n}\hat{\mu}_{k+1}\big(X_{t_{k-1},t_k}^{x,\mu_k(x)}\big)\big)\\&\qquad\qquad=\inf_{\mu\in\mathcal{R}([t_{k-1},t_k])}J\big(t_{k-1},x,\mu\otimes_{t_{k-1},\Delta_n}\hat{\mu}_{k+1}\big(X_{t_{k-1},t_k}^{x,\mu}\big)\big),\end{aligned}$$

and let $\hat{\mu}_k(x):=\mu_k(x)\otimes_{t_{k-1},\Delta_n}\hat{\mu}_{k+1}\big(X_{t_{k-1},t_k}^{x,\mu_k(x)}\big)$, $x\in\mathbb{R}$. Then, $\mu^{*n}:=\hat{\mu}_1(x_0)\in\mathcal{R}([0,T])$ is a solution of the n-person game (2.1).

By abuse of notation, we denote by $(\mu^{*n})_{n\geq 1}$ the extracted convergent subsequence on Λ, and show that its weak limit μ^* is an equilibrium control. We have

$$
\begin{aligned}
&\mathbb{E}\Big[\int_0^T \big|H\big(t,X_t^{\mu^{*n}},\mathbb{E}_{\lfloor t\rfloor_n}\big[\Phi\big(X_t^{\mu^{*n}}\big)\big],\mu_t^{*n},p^{\mu^{*n}}_{\lfloor t\rfloor_n,t}\big)-H\big(t,X_t^{\mu^*},\Phi(X_t^{\mu^*}),\mu_t^*,p^{\mu^*}_{t,t}\big)\big|dt\Big]\\
&=\mathbb{E}\Big[\int_0^T \Big|p^{\mu^{*n}}_{\lfloor t\rfloor_n,t}\int_U b\big(t,X_t^{\mu^{*n}},v\big)\mu_t^{*n}(dv)-\int_U h\big(t,X_t^{\mu^{*n}},\mathbb{E}_{\lfloor t\rfloor_n}\big[\Phi\big(X_t^{\mu^{*n}}\big)\big],v\big)\mu_t^{*n}(dv)\\
&\quad -p^{\mu^*}_{t,t}\int_U b(t,X_t^{\mu^*},v)\mu_t^*(dv)+\int_U h(t,X_t^{\mu^*},\mathbb{E}_t[\Phi(X_t^{\mu^*})],v)\mu_t^*(dv)\Big|\,dt\Big]\\
&\leq C\mathbb{E}\Big[\int_0^T\Big(\big|p^{\mu^{*n}}_{\lfloor t\rfloor_n,t}-p^{\mu^*}_{t,t}\big|\times\Big|\int_U b\big(t,X_t^{\mu^{*n}},v\big)\mu_t^{*n}(dv)\Big|+\big|p^{\mu^*}_{t,t}\big|\int_U\big|b\big(t,X_t^{\mu^{*n}},v\big)-b(t,X_t^{\mu^*},v)\big|\mu_t^{*n}(dv)\\
&\quad +\big|p^{\mu^*}_{t,t}\big|\times\Big|\int_U b(t,X_t^{\mu^*},v)\mu_t^{*n}(dv)-\int_U b(t,X_t^{\mu^*},v)\mu_t^*(dv)\Big|\\
&\quad +\int_U\big|h\big(t,X_t^{\mu^{*n}},\mathbb{E}_{\lfloor t\rfloor_n}\big[\Phi\big(X_t^{\mu^{*n}}\big)\big],v\big)-h(t,X_t^{\mu^*},\mathbb{E}_t[\Phi(X_t^{\mu^*})],v)\big|\mu_t^{*n}(dv)\\
&\quad +\Big|\int_U h(t,X_t^{\mu^*},\mathbb{E}_t[\Phi(X_t^{\mu^*})],v)\mu_t^{*n}(dv)-\int_U h(t,X_t^{\mu^*},\mathbb{E}_t[\Phi(X_t^{\mu^*})],v)\mu_t^*(dv)\Big|\Big)\,dt\Big]\\
&\leq C\int_0^T\mathbb{E}\Big[\Big|\int_U h(t,X_t^{\mu^*},\mathbb{E}_t[\Phi(X_t^{\mu^*})],v)\mu_t^{*n}(dv)-\int_U h(t,X_t^{\mu^*},\mathbb{E}_t[\Phi(X_t^{\mu^*})],v)\mu_t^*(dv)\Big| \qquad (2.16)\\
&\qquad\qquad +\big|p^{\mu^{*n}}_{\lfloor t\rfloor_n,t}-p^{\mu^*}_{t,t}\big|+2\big|X_t^{\mu^{*n}}-X_t^{\mu^*}\big|+\big|\mathbb{E}_{\lfloor t\rfloor_n}[\Phi(X_t^{\mu^*})]-\mathbb{E}_t[\Phi(X_t^{\mu^*})]\big|\Big]dt \qquad (2.17)\\
&+C\Big(\mathbb{E}\Big[\int_0^T\big|p^{\mu^*}_{t,t}\big|^2dt\Big]\mathbb{E}\Big[\int_0^T\Big(\big|X_t^{\mu^{*n}}-X_t^{\mu^*}\big|^2+\Big|\int_U b(t,X_t^{\mu^*},v)\mu_t^{*n}(dv)-\int_U b(t,X_t^{\mu^*},v)\mu_t^*(dv)\Big|^2\Big)dt\Big]\Big)^{1/2}. \qquad (2.18)
\end{aligned}
$$

Since

$$
\Big|\int_U h(t,X_t^{\mu^*},\mathbb{E}_t[\Phi(X_t^{\mu^*})],v)\mu_t^{*n}(dv)-\int_U h(t,X_t^{\mu^*},\mathbb{E}_t[\Phi(X_t^{\mu^*})],v)\mu_t^*(dv)\Big|
$$

is uniformly bounded by some $K>0$ from Assumption 1, (2.16) converges to 0 as $n\to\infty$ by dominated convergence and Lemma A.3. The first term in (2.17) converges to 0 as $n\to\infty$ by Lemma 2.7 and dominated convergence, since by Theorem 4.2.1 in Zhang (2017) there exists $C>0$ such that

$$
\begin{aligned}
\sup_{0\leq t\leq T}\mathbb{E}\Big[\sup_{t\leq s\leq T}\big(\big|p^{\mu^n}_{\lfloor t\rfloor_n,s}\big|^2+\big|p^{\mu}_{t,s}\big|^2\big)\Big]&\leq C\sup_{0\leq t\leq T}\mathbb{E}\Big[\big|\partial_x g^{\mu^n}_{\lfloor t\rfloor_n,T}+\partial_y g^{\mu^n}_{\lfloor t\rfloor_n,T}\big|^2+\big|\partial_x g^{\mu}_{t,T}+\partial_y g^{\mu}_{t,T}\big|^2\\
&\quad+\Big(\int_t^T\big|\partial_x h^{\mu^n}_{\lfloor t\rfloor_n,s}+\partial_y h^{\mu^n}_{\lfloor t\rfloor_n,s}\big|ds\Big)^2+\Big(\int_t^T\big|\partial_x h^{\mu}_{t,s}+\partial_y h^{\mu}_{t,s}\big|ds\Big)^2\Big] \qquad (2.19)
\end{aligned}
$$

which is bounded uniformly in $t \in [0,T]$ by Assumption 1. The second term in (2.17) converges to 0 as $n \to \infty$ by Lemma 2.7. The third term in (2.17) converges to 0 as $n \to \infty$ by Theorem 4 in Fetter (1977) and dominated convergence. Since $\big| \int_U b(t, X_t^{\mu^*}, v)\mu_t^{*n}(dv) - \int_U b(t, X_t^{\mu^*}, v)\mu_t^*(dv)\big|^2$ is uniformly bounded by K^2, (2.18) converges to 0 as n tends to infinity by Lemma 2.7, (2.19) and dominated convergence, and Lemma A.3. Therefore, we have

$$\lim_{n\to\infty} \mathbb{E}\left[\int_0^T \left|H\big(t, X_t^{\mu^{*n}}, \mathbb{E}_{\lfloor t\rfloor_n}\big[\Phi\big(X_t^{\mu^{*n}}\big)\big], \mu_t^{*n}, p_{\lfloor t\rfloor_n,t}^{\mu^{*n}}\big) - H\big(t, X_t^{\mu^*}, \Phi(X_t^{\mu^*}), \mu_t^*, p_{t,t}^{\mu^*}\big)\right| dt\right] = 0,$$

similarly, for any $\nu \in \mathbb{P}(U)$ we have

$$\lim_{n\to\infty} \mathbb{E}\left[\int_0^T \left|H\big(t, X_t^{\mu^{*n}}, \mathbb{E}_{\lfloor t\rfloor_n}\big[\Phi\big(X_t^{\mu^{*n}}\big)\big], \nu, p_{\lfloor t\rfloor_n,t}^{\mu^{*n}}\big) - H\big(t, X_t^{\mu^*}, \Phi(X_t^{\mu^*}), \nu, p_{t,t}^{\mu^*}\big)\right| dt\right] = 0,$$

therefore there exists an increasing sequence $(n_i)_{i\geq 1}$ of integers such that

$$\lim_{i\to\infty} H\big(t, X_t^{\mu^{*n_i}}, \mathbb{E}_{\lfloor t\rfloor_{n_i}}\big[\Phi\big(X_t^{\mu^{*n_i}}\big)\big], \mu_t^{*n_i}, p_{\lfloor t\rfloor_{n_i},t}^{\mu^{*n_i}}\big) = H\big(t, X_t^{\mu^*}, \Phi(X_t^{\mu^*}), \mu_t^*, p_{t,t}^{\mu^*}\big),$$

and

$$\lim_{i\to\infty} H\big(t, X_t^{\mu^{*n_i}}, \mathbb{E}_{\lfloor t\rfloor_{n_i}}\big[\Phi\big(X_t^{\mu^{*n_i}}\big)\big], \nu, p_{\lfloor t\rfloor_{n_i},t}^{\mu^{*n_i}}\big) = H\big(t, X_t^{\mu^*}, \Phi(X_t^{\mu^*}), \nu, p_{t,t}^{\mu^*}\big),$$

a.e. $t \in [0,T]$, $\mathbb{P}$-*a.s.*. In addition, by Theorem 2.1 we have

$$H\big(t, X_t^{n_i}, \mathbb{E}_{\lfloor t\rfloor_{n_i}}\big[\Phi\big(X_t^{\mu^{*n_i}}\big)\big], \nu, p_{\lfloor t\rfloor_{n_i},t}^{\mu^{*n_i}}\big) \leq H\big(t, X_t^{n_i}, \mathbb{E}_{\lfloor t\rfloor_{n_i}}\big[\Phi\big(X_t^{\mu^{*n_i}}\big)\big], \mu_t^{*n_i}, p_{\lfloor t\rfloor_{n_i},t}^{\mu^{*n_i}}\big),$$

a.e. $t \in [0,T]$, $\mathbb{P}$-*a.s.* for all $\nu \in \mathbb{P}(U)$, hence as k tends to infinity we find

$$H\big(t, X_t^{\mu^{*n_i}}, \Phi(X_t^{\mu^*}), \nu, p_{t,t}^{\mu^*}\big) \leq H\big(t, X_t^{\mu^*}, \Phi(X_t^{\mu^*}), \mu_t^*, p_{t,t}^{\mu^*}\big), \quad \text{a.e. } t \in [0,T], \ \mathbb{P}\text{-}a.s.,$$

for all $\nu \in \mathbb{P}(U)$, hence the weak limit μ^* of $(\mu^{*n_i})_{i\geq 1}$ on Λ is an equilibrium control by Theorem 1.4. $\square$

Applying Theorem 2.14 of Bahlali et al. (2018) under Assumption 1 and using backward induction, for any $n \geq 1$ we construct a solution μ^{*n} of the n-person game (2.1) by recursively solving Problem (2.1) in the space $\mathcal{R}([t_k, t_{k+1}])$ of relaxed controls, $k = n-1, \ldots, 1, 0$. By the discussion below Definition 2.1 in El Karoui et al. (1987), the vague topology used therein on Λ is equivalent to the weak topology, and Λ is a compact metrizable space since the set $[0,T] \times U$ is compact. Therefore, the sequence $(\mu^{*n})_{n\geq 1}$ of relaxed controls solutions to the n-person game (2.1) is tight, and it admits at least one weakly convergent subsequence, see Theorem 5.1 in Billingsley (1999). As a consequence, we obtain the next existence result from Theorem 2.4.

Corollary 2.5. *Under Assumption 1, the time-inconsistent mean-field control problem (1.1)-(1.2) admits an equilibrium control μ^*.*

Proof. By Theorem 2.4 above, the weak limit μ^* of any weakly convergent subsequence of $(\mu^{*n})_{n\geq 1}$ is an equilibrium control. □

Applying Theorem 2.4 requires to check the weak convergence of a subsequence of $(\mu^{*n})_{n\geq 1}$ in Λ. The next corollary shows that this may not be necessary if only the value function is concerned.

Corollary 2.6. *Under Assumption 1, the sequence $(J(0,x_0,\mu^{*n}))_{n\geq 1}$ admits at least one convergent subsequence. In addition, the limit of any such subsequence can be written as $J(0,x_0,\mu^*)$.*

Proof. Denoting by $(\mu^{*n_i})_{i\geq 1}$ the weakly convergent subsequence of $(\mu^{*n})_{n\geq 1}$, it suffices to note that by Lemmas 2.7 and A.3, the sequence $(J(0,x_0,\mu^{*n_i}))_{i\geq 1}$ converges to $J(0,x_0,\mu^*)$ due to the Lipschitz continuity of h and g in Assumption 1. □

The next lemma contains stability results for the SDE (1.4) and for the backward SDE (2.2), which have been used in the proofs of Theorem 2.4 and Corollary 2.6.

Lemma 2.7. *Let $(\mu^n)_{n\geq 1}\subset \mathcal{R}([0,T])$ be a sequence of Λ-valued relaxed controls converging weakly to $\mu\in\mathcal{R}([0,T])$. Then under Assumption 1, we have*

$$\lim_{n\to\infty}\mathbb{E}\left[\sup_{0\leq t\leq T}\left|X_t^{\mu^n}-X_t^{\mu}\right|^2\right]=0 \quad \textit{and} \quad \lim_{n\to\infty}\mathbb{E}\left[\sup_{t\leq s\leq T}\left|p^{\mu^n}_{\lfloor t\rfloor_n,s}-p^{\mu}_{t,s}\right|^2\right]=0, \qquad t\in[0,T).$$

Proof. (i) Using Assumption 1, we have

$$\begin{aligned}
&\mathbb{E}\left[\sup_{0\leq t\leq T}\left|X_t^{\mu^n}-X_t^{\mu}\right|^2\right]\\
&=\mathbb{E}\left[\sup_{0\leq t\leq T}\left|\int_0^t\int_U b\big(s,X_s^{\mu^n},v\big)\mu_s^n(dv)ds-\int_0^t\int_U b(s,X_s^{\mu},v)\mu_s(dv)ds\right.\right.\\
&\qquad\left.\left.+\int_0^t\big(\sigma\big(s,X_s^{\mu^n}\big)-\sigma(s,X_s^{\mu})\big)dW_s\right|^2\right]\\
&\leq C\mathbb{E}\left[\sup_{0\leq t\leq T}\left(\left|\int_0^t\int_U\big(b\big(s,X_s^{\mu^n},v\big)-b(s,X_s^{\mu},v)\big)\mu_s^n(dv)ds\right|^2\right.\right.\\
&\qquad\left.\left.+\left|\int_0^t\int_U b(s,X_s^{\mu},v)\mu_s^n(dv)ds-\int_0^t\int_U b(s,X_s^{\mu},v)\mu_s(dv)ds\right|^2\right)+\int_0^T\left|\sigma\big(s,X_s^{\mu^n}\big)-\sigma(s,X_s^{\mu})\right|^2ds\right]\\
&\leq C\mathbb{E}\left[\int_0^T|f_s^n|^2ds\right]+C\int_0^T\mathbb{E}\left[\sup_{0\leq s\leq t}\left|X_s^{\mu^n}-X_s^{\mu}\right|^2\right]dt,
\end{aligned}$$

where

$$f_s^n := \int_U b(s, X_s^\mu, v)\mu_s^n(dv) - \int_U b(s, X_s^\mu, v)\mu_s(dv).$$

Hence, by Gronwall's inequality we get

$$\mathbb{E}\left[\sup_{0\le t\le T} \left|X_t^{\mu^n} - X_t^\mu\right|^2\right] \le C\mathrm{e}^{CT}\mathbb{E}\left[\int_0^T |f_s^n|^2 ds\right].$$

By Lemma A.3 we have $\lim_{n\to\infty} |f_s^n|^2 = 0$, a.e. $s \in [0,T]$, $\mathbb{P}$-$a.s$, hence we conclude by dominated convergence as $|f_s^n|^2$ is uniformly bounded by K^2.

(ii) We fix $t \in [0,T]$ and denote

$$\begin{cases} \xi^n := -\partial_x g^{\mu^n}_{\lfloor t\rfloor_n, T} - \partial_y g^{\mu^n}_{\lfloor t\rfloor_n, T}, \\ \xi := -\partial_x g^\mu_{t,T} - \partial_y g^\mu_{t,T}, \\ f^n(s,p,q) = p\partial_x b^{\mu^n}_{\lfloor t\rfloor_n,s} + q\partial_x \sigma^{\mu^n}_{\lfloor t\rfloor_n,s} - \partial_x h^{\mu^n}_{\lfloor t\rfloor_n,s} - \partial_y h^{\mu^n}_{\lfloor t\rfloor_n,s}, \\ f(s,p,q) := p\partial_x b^\mu_{t,s} + q\partial_x\sigma^\mu_{t,s} - \partial_x h^\mu_{t,s} - \partial_y h^\mu_{t,s}. \end{cases}$$

By Theorem 4.4.3 in Zhang (2017), it suffices to show that $\lim\limits_{n\to\infty} \mathbb{E}\big[|\xi^n - \xi|^2\big] = 0$, that

$$\lim_{n\to\infty} \mathbb{E}\bigg[\int_t^T |f^n(s,0,0) - f(s,0,0)|^2 ds\bigg] = 0, \tag{2.20}$$

and that $f^n(s,p,q) - f(s,p,q)$ converges to 0 in $ds \times dP$-measure as n tends to infinity for any fixed (p,q). We note that the latter condition follows from Chebyshev's inequality and

$$\lim_{n\to\infty} \mathbb{E}\left[\int_t^T |f^n(s,p,q) - f(s,p,q)|^2 ds\right] = 0, \quad (p,q) \in \mathbb{R}^2. \tag{2.21}$$

Since the arguments leading to the above conditions are similar, we focus on the limit (2.20). By (1.7b), we have

$$\begin{aligned}
&\mathbb{E}\left[\int_t^T \left|\partial_y h^\mu_{t,s} - \partial_y h^{\mu^n}_{\lfloor t\rfloor_n,s}\right|^2 ds\right] \\
&= \int_t^T \mathbb{E}\Bigg[\bigg|\Phi'(X_s^\mu)\mathbb{E}_t\bigg[\int_U \partial_y h(s, X_s^\mu, \mathbb{E}_t[\Phi(X_s^\mu)], v)\mu_s(dv)\bigg] \\
&\quad - \Phi'\big(X_s^{\mu^n}\big)\mathbb{E}_{\lfloor t\rfloor_n}\bigg[\int_U \partial_y h\big(s, X_s^{\mu^n}, \mathbb{E}_{\lfloor t\rfloor_n}\big[\Phi\big(X_s^{\mu^n}\big)\big], v\big)\mu_s^n(dv)\bigg]\bigg|^2\Bigg] ds \\
&\le C\int_t^T \mathbb{E}\Bigg[\left|\Phi'(X_s^\mu) - \Phi'\big(X_s^{\mu^n}\big)\right|^2 \times \left|\mathbb{E}_t\bigg[\int_U \partial_y h(s, X_s^\mu, \mathbb{E}_t[\Phi(X_s^\mu)], v)\mu_s(dv)\bigg]\right|^2 + \left|\Phi'\big(X_s^{\mu^n}\big)\right|^2
\end{aligned}$$

$$
\begin{aligned}
&\times \Bigg(\bigg|\mathbb{E}_t\Big[\int_U \partial_y h(s, X_s^\mu, \mathbb{E}_t[\Phi(X_s^\mu)], v)\mu_s(dv)\Big] - \mathbb{E}_{\lfloor t\rfloor_n}\Big[\int_U \partial_y h(s, X_s^\mu, \mathbb{E}_t[\Phi(X_s^\mu)], v)\mu_s(dv)\Big]\bigg|^2 \\
&+ \bigg|\mathbb{E}_{\lfloor t\rfloor_n}\Big[\int_U \partial_y h(s, X_s^\mu, \mathbb{E}_t[\Phi(X_s^\mu)], v)\mu_s(dv) - \int_U \partial_y h(s, X_s^\mu, \mathbb{E}_t[\Phi(X_s^\mu)], v)\mu_s^n(dv)\Big]\bigg|^2 \\
&+ \bigg|\mathbb{E}_{\lfloor t\rfloor_n}\Big[\int_U \big(\partial_y h(s, X_s^\mu, \mathbb{E}_t[\Phi(X_s^\mu)], v) - \partial_y h(s, X_s^\mu, \mathbb{E}_{\lfloor t\rfloor_n}[\Phi(X_s^\mu)], v)\big)\mu_s^n(dv)\Big]\bigg|^2 \\
&+ \bigg|\mathbb{E}_{\lfloor t\rfloor_n}\Big[\int_U \big(\partial_y h(s, X_s^\mu, \mathbb{E}_{\lfloor t\rfloor_n}[\Phi(X_s^\mu)], v) - \partial_y h\big(s, X_s^{\mu^n}, \mathbb{E}_{\lfloor t\rfloor_n}\big[\Phi\big(X_s^{\mu^n}\big)\big], v\big)\big)\mu_s^n(dv)\Big]\bigg|^2\Bigg)\Bigg] ds \\
&\le C\int_t^T \mathbb{E}\Big[\big|\Phi'(X_s^\mu) - \Phi'\big(X_s^{\mu^n}\big)\big|^2 \qquad (2.22) \\
&+ \bigg|\mathbb{E}_t\Big[\int_U \partial_y h(s, X_s^\mu, \mathbb{E}_t[\Phi(X_s^\mu)], v)\mu_s(dv)\Big] - \mathbb{E}_{\lfloor t\rfloor_n}\Big[\int_U \partial_y h(s, X_s^\mu, \mathbb{E}_t[\Phi(X_s^\mu)], v)\mu_s(dv)\Big]\bigg|^2 \qquad (2.23) \\
&+ \bigg|\int_U \partial_y h(s, X_s^\mu, \mathbb{E}_t[\Phi(X_s^\mu)], v)\mu_s(dv) - \int_U \partial_y h(s, X_s^\mu, \mathbb{E}_t[\Phi(X_s^\mu)], v)\mu_s^n(dv)\bigg|^2 \qquad (2.24) \\
&+ \bigg|\int_U \big(\partial_y h(s, X_s^\mu, \mathbb{E}_t[\Phi(X_s^\mu)], v) - \partial_y h(s, X_s^\mu, \mathbb{E}_{\lfloor t\rfloor_n}[\Phi(X_s^\mu)], v)\big)\mu_s^n(dv)\bigg|^2 \qquad (2.25) \\
&+ \bigg|\int_U \big(\partial_y h(s, X_s^\mu, \mathbb{E}_{\lfloor t\rfloor_n}[\Phi(X_s^\mu)], v) - \partial_y h\big(s, X_s^{\mu^n}, \mathbb{E}_{\lfloor t\rfloor_n}\big[\Phi\big(X_s^{\mu^n}\big)\big], v\big)\big)\mu_s^n(dv)\bigg|^2\Bigg] ds. \qquad (2.26)
\end{aligned}
$$

The inequality (2.26) is due to Assumption 1, the conditional Jensen's inequality. Fix $s \in [t, T]$. By Lemma 2.7 and Theorem 4 in Fetter (1977),

$$
\lim_{n\to\infty} \mathbb{E}\big[\big|X_s^\mu - X_s^{\mu^n}\big|^2 + |\mathbb{E}_t[\Phi(X_s^\mu)] - \mathbb{E}_{\lfloor t\rfloor_n}[\Phi(X_s^\mu)]|^2\big] = 0,
$$

and (2.22), (2.25), (2.26) converge to zero by Lemma A.4, conditional Jensen's inequality, Assumption 1 and dominated convergence on $[0, T]$. Similarly, (2.23) and (2.24) tend to zero by Theorem 4 in Fetter (1977) and Lemma A.3 respectively. The term in $\partial_x h^{\mu^n}_{\lfloor t\rfloor_n, s} - \partial_x h^\mu_{t,s}$ is treated similarly using (1.7a). □

Remark. We note that the equilibrium control μ^* constructed in Corollary 2.5 using equal partitions may not be unique. Indeed, two sequences $(\Pi_1^n)_{n\ge1}$ and $(\Pi_2^n)_{n\ge1}$ of partitions may yield distinct limiting equilibrium controls μ_1 and μ_2 by Theorem 2.4. However, under Lipschitz conditions on the function

$$
\psi(t, x, p) = \text{argmax } H(t, x, \Phi(x), \cdot, p), \quad t \in [0, T], \quad (x, p) \in \mathbb{R}^2,
$$

and on the coefficient derivatives appearing in Assumption 1, it can be shown by a contraction argument in small time T that the equilibrium control of (1.1)-(1.2) can be represented as in (1.6) from a strict control in $\mathcal{U}([0,T])$ which is unique in $L^1(\Omega \times [0,T])$.

3 Numerical implementation

3.1 Markov chain approximation of n-person games

Using Markov chains as in Kushner (1990a), we construct an approximation for the relaxed control solution μ^{*n} of the n-person game (2.1) used in Theorem 2.4. Then, in Theorem 3.4 we show the convergence of this approximation to μ^{*n}, $n \geq 1$. Let $n, m \geq 1$, $\Delta_{n,m} := T/(nm)$, and $t_k^m = k\Delta_{n,m}$, $k = 0, 1, \ldots, nm$.

Definition 3.1. *For any $n, m \geq 1$, we let $\mathcal{U}^{n,m}([0,T])$ denote the set of admissible discrete-time strict control sequences $(u_k)_{0 \leq k < nm}$ such that u_k is $\mathcal{F}_{t_k^m}$-measurable.*

Given a sequence $(x_k)_{k=0,1,\ldots,nm}$, we let $\bar{x}$ be the step function defined as

$$\bar{x}_t = \sum_{k=0}^{nm-1} x_k \mathbb{1}_{[t_k^m, t_{k+1}^m)}(t) + x^{nm} \mathbb{1}_{\{T\}}(t), \qquad t \in [0,T].$$

We also let $H_{n,m} : \mathbb{R} \to \sqrt{\Delta}_{n,m}\mathbb{Z}$ denote the rounding function on $\sqrt{\Delta}_{n,m}\mathbb{Z}$, where $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ is the set of integers.

Assumption 2. *Let $n, m \geq 1$, and $u^{n,m} = (u_k)_{0 \leq k < nm} \in \mathcal{U}^{n,m}([0,T])$ be a sequence of admissible discrete-time strict controls. We assume that there exists a discrete-time Markov chain $\big(X_k^{n,m,u}\big)_{k=0,\ldots,nm}$ on $\sqrt{\Delta}_{n,m}\mathbb{Z}$, such that*

(i) $X_0^{n,m,u} = H_{n,m}(x_0)$,

(ii) $\mathbb{P}\big(X_{k+1}^{n,m,u} = y \mid (X_l^{n,m,u}, u_l^{n,m})_{l=0,1,\ldots,k}\big) = \mathbb{P}\big(X_{k+1}^{n,m,u} = y \mid X_k^{n,m,u},\ u_k^{n,m}\big)$, $\ y \in \sqrt{\Delta}_{n,m}\mathbb{Z}$,

(iii) $\mathbb{E}\big[X_{k+1}^{n,m,u} - X_k^{n,m,u} \mid (X_l^{n,m,u}, u_l^{n,m})_{l=0,1,\ldots,k}\big] = \Delta_{n,m} b\big(t_k^m, X_k^{n,m,u}, u_k\big)$,

(iv) $\mathbb{E}\big[\big(X_{k+1}^{n,m,u} - X_k^{n,m,u} - \Delta_{n,m} b\big(t_k^m, X_k^{n,m,u}, u_k\big)\big)^2 \mid (X_l^{n,m,u}, u_l^{n,m})_{l=0,1,\ldots,k}\big]$
$= \Delta_{n,m}\sigma^2\big(t_k^m, X_k^{n,m,u}\big) + o(\Delta_{n,m})$, $k = 0, 1, \ldots, nm-1$.

(v) There exists $C > 0$ such that $\sup_{0 \leq k < nm} \big|X_{k+1}^{n,m,u} - X_k^{n,m,u}\big| \leq C\sqrt{\Delta}_{n,m}$, $n, m \geq 1$.

Let $\big(\mathcal{F}_t^{n,m}\big)_{t\in[0,T]}$ denote the filtration generated by $\big(\overline{X}_t^{n,m,u}\big)_{t\in[0,T]}$. Given $u^{n,m} = (u_k)_{0\le k<nm} \in \mathcal{U}^{n,m}([0,T])$ an admissible control sequence, we define the cost functional

$$\begin{aligned} J^{n,m}\big(t_k^m, \overline{X}_{t_k^m}^{n,m,u}, \bar{u}^{n,m}\big) \;&=\; \mathbb{E}\bigg[g\big(\overline{X}_T^{n,m,u}, \mathbb{E}\big[\Psi\big(\overline{X}_T^{n,m,u}\big) \mid \mathcal{F}_{t_k^m}^{n,m}\big]\big) \\ &\qquad + \int_{t_k^m}^T h\big(s, \overline{X}_s^{n,m,u}, \mathbb{E}\big[\Phi(\overline{X}_s^{n,m,u}) \mid \mathcal{F}_{t_k^m}^{n,m}\big], \bar{u}_s^{n,m}\big) ds \;\Big|\; \mathcal{F}_{t_k^m}^{n,m}\bigg], \end{aligned}$$

$k = 0, 1, \ldots, nm-1$. Consider the discretization

$$J^{n,m}\big(t_k, \overline{X}_{t_k}^{n,m,u^*}, \bar{u}^{*n,m}\big) = \inf_{u\in\mathcal{U}^{n,m}([0,T])} J^{n,m}\big(t_k, \overline{X}_{t_k}^{n,m,u^*}, \bar{u} \otimes_{t_k,\Delta_n} \bar{u}^{*n,m}\big), \tag{3.1}$$

$k = 0, 1, \ldots, n-1$, of the n-person game (2.1), which admits a solution $\bar{u}^{*n,m}$ due to the compactness of U. Let the sequence $(W_k^{n,m,u})_{k=0,\ldots,nm}$ be defined by $W_0^{n,m,u} := 0$ and

$$W_{k+1}^{n,m,u} - W_k^{n,m,u} := \frac{X_{k+1}^{n,m,u} - X_k^{n,m,u} - \Delta_{n,m} b\big(t_k^m, X_k^{n,m,u}, u_k\big)}{\sigma\big(t_k^m, X_k^{n,m,u}\big)}, \quad k = 0, 1, \ldots, nm-1.$$

By Assumption 2-(iii) we check that $\big(\overline{W}_t^{n,m,u}\big)_{t\in[0,T]}$ is a martingale with respect to its own filtration, which coincides with $\big(\mathcal{F}_t^{n,m}\big)_{t\in[0,T]}$. By the Skorokhod representation Theorem A.1, all processes can be defined on a same probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The next lemma follows from Theorem 4.6 in Kushner (1990a), see also Theorem 10.4.1 in Kushner and Dupuis (2001).

Lemma 3.2. *Under Assumptions 1 and 2, fix $n \ge 1$ and for any $m \ge 1$ let $u^{n,m} = (u_k)_{0\le k<nm} \in \mathcal{U}^{n,m}([0,T])$ be an admissible control sequence. Then, letting $\mu^{n,m}$ denote the relaxed control representation of $\bar{u}^{n,m}$, $m \ge 1$, see (1.6),*

a) the sequence $\big(\overline{X}^{n,m,u}, \mu^{n,m}, \overline{W}^{n,m,u}\big)_{m\ge1}$ is tight on $\mathcal{D}([0,T]) \times \Lambda \times \mathcal{D}([0,T])$,

b) the limit of any weakly converging subsequence of $\big(\overline{X}^{n,m,u}, \mu^{n,m}, \overline{W}^{n,m,u}\big)_{m\ge1}$ takes the form (X^μ, μ, W) on $\mathcal{D}([0,T]) \times \Lambda \times \mathcal{D}([0,T])$, where $W = (W_t)_{t\in[0,T]}$ is a Wiener process and X^μ solves (1.4) with the relaxed control μ.

The following approximation lemma, see e.g. Theorems 3.2.2 and 3.5.2 in Kushner (1990b) and references therein, will be used to approximate relaxed controls $\mu \in \mathcal{R}([0,T])$ using elements an admissible control sequences in $\mathcal{U}^{n,m}([0,T])$.

Lemma 3.3. *[Chattering lemma] Let $n \ge 1$ and $\mu \in \mathcal{R}([0,T])$. Under Assumptions 1 and 2 there exists a sequence $(u^{n,m})_{m\ge1}$ of admissible controls $u^{n,m} \in \mathcal{U}^{n,m}([0,T])$, $m \ge 1$, such* *that the relaxed control representation $(\mu^{n,m})_{m\geq 1}$ of $(\bar{u}^{n,m})_{m\geq 1}$, see (1.6), converges weakly to μ on Λ as m tends to infinity.*

Proof. The sequence $(\bar{u}^{n,m})_{m\geq 1}$ is constructed in the proof of Theorem 3.5.2 in Kushner (1990b) and its relaxed control representation $(\mu^{n,m})_{m\geq 1}$ is shown to converge weakly to μ. $\square$

The next theorem, which is the main result of this section, shows the convergence of the solution of the discretized problem (3.1) to the solution of the n-person game (2.1).

Theorem 3.4. *Under Assumption 1, fix $n \geq 1$ and let $(\bar{u}^{*n,m})_{m\geq 1}$ be a sequence of solutions to Problem (3.1) with relaxed control representation $(\mu^{*n,m})_{m\geq 1}$, see (1.6), and let $(\overline{X}^{n,m,u^*})_{m\geq 1}$ denote the Markov chain defined in Assumption 2. Then,*

a) The sequence $\big(\overline{X}^{n,m,u^*}, \mu^{*n,m}\big)_{m\geq 1}$ is tight on $\mathcal{D}([0,T]) \times \Lambda$.

b) Denoting by $(X^{\mu^{*n}}, \mu^{*n})$ the limit of any weakly converging subsequence of $\big(\overline{X}^{n,m,u^*}, \mu^{*n,m}\big)_{m\geq 1}$ on $\mathcal{D}([0,T]) \times \Lambda$, the process $(X_t^{\mu^{*n}})_{t\in[0,T]}$ solves the SDE (1.4) with relaxed control μ^{*n}.

c) The relaxed control μ^{*n} solves the n-person game (2.1).

Proof. The tightness of $\big(\overline{X}^{n,m,u^*}, \mu^{*n,m}\big)_{m\geq 1}$ and the fact that the weak limit of an extracted subsequence $\big(\overline{X}^{n,m,u^*}\big)_{m\geq 1}$ solves (1.4) with relaxed control μ^{*n} follow from Lemma 3.2. To show (c), it suffices to prove that for all $k = 0, 1, \ldots, n-1$ we have

$$J\big(t_k, X_{t_k}^{\mu^{*n}}, \mu^{*n}\big) = \inf_{\mu\in\mathcal{R}([t_k,t_{k+1}])} J\big(t_k, X_{t_k}^{\mu^{*n}}, \mu \otimes_{t_k,\Delta_n} \mu^{*n}\big). \tag{3.2}$$

Fix any $k \in \{0, 1, \ldots, n-1\}$, and let J_k^* be the infimum in the right-hand side of (3.2). For any $\varepsilon > 0$ there exists $\mu^{(\varepsilon)}$ such that

$$J_k^* + \varepsilon > J\big(t_k, X_{t_k}^{\mu^{*n}}, \mu^{(\varepsilon)} \otimes_{t_k,\Delta_n} \mu^{*n}\big).$$

By Lemma 3.3, we can find an admissible control sequence $u^{n,m,\varepsilon} = (u_k)_{0\leq k<nm} \in \mathcal{U}^{n,m}([0,T])$ such that the relaxed control representation $\mu^{n,m,\varepsilon}$ of $\bar{u}^{n,m,\varepsilon}$ converges weakly to $\mu^{(\varepsilon)}$ on Λ as m tends to infinity. By (b), $(\mu^{*n,m})_{m\geq 1}$ converges weakly to μ^{*n}, and therefore $\big(\mu^{*n,m}\mathbb{1}_{[t_k,t_{k+1})^c} + \mu^{n,m,\varepsilon}\mathbb{1}_{[t_k,t_{k+1})}\big)_{m\geq 1}$ converges weakly to $\mu^{*n}\mathbb{1}_{[t_k,t_{k+1})^c} + \mu^{(\varepsilon)}\mathbb{1}_{[t_k,t_{k+1})}$ on Λ as $m \to \infty$. Then, we have

$$J_k^* + \varepsilon \;>\; J\big(t_k, X_{t_k}^{\mu^{*n}}, \mu^{(\varepsilon)} \otimes_{t_k,\Delta_n} \mu^{*n}\big)$$

$$
\begin{aligned}
&= \lim_{i\to\infty} J^{n,m_i}\big(t_k, \overline{X}^{n,m_i,u^*}_{t_k}, \bar{u}^{n,m_i,\varepsilon} \otimes_{t_k,\Delta_n} \bar{u}^{*n,m_i}\big) && (3.3)\\
&\geq \lim_{i\to\infty} J^{n,m_i}\big(t_k, \overline{X}^{n,m_i,u^*}_{t_k}, \bar{u}^{*n,m_i}\big) && (3.4)\\
&= J\big(t_k, X^{\mu^{*n}}_{t_k}, \mu^{*n}\big), && (3.5)
\end{aligned}
$$

where $(m_i)_{i\geq 1}$ is an increasing sequence of integers. (3.4) is because $\bar{u}^{*n,m_i}$ is solution of Problem (3.1) with Markov chain $\overline{X}^{n,m_i,u^*}$, (3.3) and (3.5) follow from Lemma 3.6, up to extraction of a subsequence to ensure almost sure convergence. Since $\varepsilon > 0$ is arbitrary, we conclude to (3.2). □

Applying Theorem 3.4 requires to check the weak convergence of a subsequence of $(\mu^{*n,m})_{m\geq 1}$ in Λ. As in Corollary 2.6, the next result shows that this may not be necessary if only the value function is concerned.

Corollary 3.5. *Under Assumption 1 and 2, the sequence $(J^{n,m}(0, x_0, \bar{u}^{*n,m}))_{m\geq 1}$ admits at least one convergent subsequence. In addition, the limit of any such subsequence can be written as $J(0, x_0, \mu^{*n})$.*

Proof. By the tightness of $(\mu^{*n,m})_{m\geq 1}$, we can extract a weakly convergent subsequence also denoted by $(\mu^{*n,m})_{m\geq 1}$ whose weak limit, denoted by μ^{*n}, is the solution to the n-person game (2.1) by Theorem 3.4, $n \geq 1$. By Lemma 3.6 below, we conclude that $(J^{n,m}(0, x_0, \bar{u}^{*n,m}))_{m\geq 1}$ converges to $J(0, x_0, \mu^{*n})$. □

The next lemma has been used in the proofs of Theorem 3.4 and Corollary 3.5.

Lemma 3.6. *Under Assumptions 1 and 2, fix $n \geq 1$ and consider a weakly convergent sequence $\big(\overline{X}^{n,m,u}, \mu^{n,m}, \overline{W}^{n,m,u}\big)_{m\geq 1}$, where for $m \geq 1$, $u^{n,m} \in \mathcal{U}^{n,m}([0,T])$. Then, for any $k = 0, 1, \ldots, n-1$, the sequence $\big(J^{n,m}\big(t_k, \overline{X}^{n,m,u}_{t_k}, \bar{u}^{n,m}\big)\big)_{m\geq 1}$ converges to $J(t_k, X^{\mu}_{t_k}, \mu)$ in probability as m tends to infinity.*

Proof. By the Skorokhod representation Theorem A.1, Lemmas 3.2 and A.2, there is a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that as $m \to \infty$, we have

$$
\begin{cases}
\lim\limits_{m\to\infty} \sup\limits_{t\in[0,T]} \big|\overline{X}^{n,m,u}_t - X^{\mu}_t\big| = 0, \\
\lim\limits_{m\to\infty} \left|\int_t^T \int_U f(s,v)\mu^{n,m}(ds,dv) - \int_t^T \int_U f(s,v)\mu(ds,dv)\right| = 0, \quad t \in [0,T], \\
\lim\limits_{m\to\infty} \sup\limits_{t\in[0,T]} \big|\overline{W}^{n,m,u}_t - W_t\big| = 0,
\end{cases}
\tag{3.6}
$$

$\mathbb{P}$-$a.s.$, for f any bounded random function, measurable in $t \in [0,T]$ and continuous in $u \in U$. Since $\big(\overline{W}^{n,m,u}\big)_{t\in[0,T]}$ is an $\big(\mathcal{F}_t^{n,m}\big)_{t\in[0,T]}$-martingale for all $m \geq 1$, by Proposition 3 in Briand et al. (2002), the filtrations $\big(\mathcal{F}_t^{n,m}\big)_{t\in[0,T]}$ converge weakly to $(\mathcal{F}_t)_{t\in[0,T]}$ as m tends to infinity, hence for all $X \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ we have the convergence

$$\lim_{m\to\infty} \sup_{t\in[0,T]} \mathbb{E}[X \mid \mathcal{F}_t^{n,m}] = \sup_{t\in[0,T]} \mathbb{E}[X \mid \mathcal{F}_t], \tag{3.7}$$

in probability. For any $k = 0, 1, \ldots, (n-1)m$, let

$$Z_k^{n,m,u} := g\big(\overline{X}_T^{n,m,u}, \mathbb{E}\big[\Psi(\overline{X}_T^{n,m,u}) \mid \mathcal{F}_{t_k^m}^{n,m}\big]\big) + \int_{t_k^m}^T \int_U h\big(s, \overline{X}_s^{n,m,u}, \mathbb{E}\big[\Phi(\overline{X}_s^{n,m,u}) \mid \mathcal{F}_{t_k^m}^{n,m}\big], v\big)\mu_s^{n,m}(dv)ds,$$

where $\mu^{n,m}$ is the relaxed control representation of $\bar{u}^{n,m}$, see (1.6), and

$$Z_k^\mu := g(X_T^\mu, \mathbb{E}[\Psi(X_T^\mu) \mid \mathcal{F}_{t_k^m}]) + \int_{t_k^m}^T \int_U h(s, X_s^\mu, \mathbb{E}[\Phi(X_s^\mu) \mid \mathcal{F}_{t_k^m}], v)\mu_s(dv)ds,$$

with $J(t_k^m, X_{t_k^m}^\mu, \mu) = \mathbb{E}[Z_k^\mu \mid \mathcal{F}_{t_k^m}]$ and $J^{n,m}\big(t_k^m, \overline{X}_{t_k^m}^{n,m,u}, \bar{u}^{n,m}\big) = \mathbb{E}\big[Z_k^{n,m,u} \mid \mathcal{F}_{t_k^m}^{n,m}\big]$. Since convergence in L^1 implies convergence in probability, it suffices to show that

$$\lim_{m\to\infty} \mathbb{E}\big[\big|\mathbb{E}\big[Z_k^{n,m,u} \mid \mathcal{F}_{t_k^m}^{n,m}\big] - \mathbb{E}[Z_k^\mu \mid \mathcal{F}_{t_k^m}]\big|\big] = 0.$$

By the conditional Jensen's inequality and Assumption 1, we have

$$\begin{aligned}
&\mathbb{E}\big[\big|\mathbb{E}\big[Z_k^{n,m,u} \mid \mathcal{F}_{t_k^m}^{n,m}\big] - \mathbb{E}\big[Z_k^\mu \mid \mathcal{F}_{t_k^m}\big]\big|\big] \leq \mathbb{E}\big[\big|Z_k^{n,m,u} - Z_k^\mu\big| + |\mathbb{E}[Z_k^\mu \mid \mathcal{F}_{t_k^m}^{n,m}] - \mathbb{E}[Z_k^\mu|\mathcal{F}_{t_k^m}]|\big] \\
&\leq C\mathbb{E}\bigg[\int_{t_k^m}^T \big(\big|\overline{X}_s^{n,m,u} - X_s^\mu\big| + |\mathbb{E}[\Phi(X_s^\mu) \mid \mathcal{F}_{t_k^m}^{n,m}] - \mathbb{E}[\Phi(X_s^\mu) \mid \mathcal{F}_{t_k^m}]|\big)ds \\
&\quad + \bigg|\int_{t_k^m}^T \bigg(\int_U h(s, X_s^\mu, \mathbb{E}[\Phi(X_s^\mu) \mid \mathcal{F}_{t_k^m}], v)\mu_s^{n,m}(dv) - \int_U h(s, X_s^\mu, \mathbb{E}[\Phi(X_s^\mu) \mid \mathcal{F}_{t_k^m}], v)\mu_s(dv)\bigg)ds\bigg| \\
&\quad + \big|\overline{X}_T^{n,m,u} - X_T^\mu\big| + \big|\mathbb{E}\big[\Psi(X_T^\mu) \mid \mathcal{F}_{t_k^m}^{n,m}\big] - \mathbb{E}\big[\Psi(X_T^\mu) \mid \mathcal{F}_{t_k^m}\big]\big| + \big|\mathbb{E}\big[Z_k^\mu \mid \mathcal{F}_{t_k^m}^{n,m}\big] - \mathbb{E}\big[Z_k^\mu \mid \mathcal{F}_{t_k^m}\big]\big|\bigg].
\end{aligned}$$

The first, third, and fourth terms in the last inequality converge to 0 by (3.6), and the fifth and sixth terms converges to 0 by (3.7) and uniform boundedness. Similarly, by (3.7) we have

$$\lim_{m\to\infty} \mathbb{E}\big[\big|\mathbb{E}\big[\Phi(X_t^\mu) \mid \mathcal{F}_{t_k^m}^{n,m}\big] - \mathbb{E}\big[\Phi(X_t^\mu) \mid \mathcal{F}_{t_k^m}\big]\big|\big] = 0, \qquad t \in [t_k^m, T],$$

hence the second term tends to zero by the boundedness of Φ and dominated convergence.

□

Remark. In addition to the dependence of $h(s, X_s^\mu, \mathbb{E}_t[\Phi(X_s^\mu)], u)$ and $g(X_T^\mu, \mathbb{E}_t[\Psi(X_T^\mu])$ on the mean-field term, time inconsistency of a control problem can also be caused by the dependence of h and g on initial time and initial state t and X_t^μ, i.e.

$$J(t, X_t^\mu, \mu) = \mathbb{E}_t\left[g\big(t, X_t^\mu, X_T^\mu, \mathbb{E}_t[\Psi(X_T^\mu)]\big) + \int_t^T \int_U h\big(t, s, X_t^\mu, X_s^\mu, \mathbb{E}_t[\Phi(X_s^\mu)], v\big)\mu_s(dv)ds\right],$$

which admits the discretization

$$\begin{aligned} J^{n,m}\big(t, \overline{X}_t^{n,m,u}, \bar{u}^{n,m}\big) \;=\; & \mathbb{E}_t^{n,m}\Big[g\big(t, \overline{X}_t^{n,m,u}, \overline{X}_T^{n,m,u}, \mathbb{E}_t^{n,m}\big[\Psi\big(\overline{X}_T^{n,m,u}\big)\big]\big) \\ & + \int_t^T h\big(t, s, \overline{X}_t^{n,m,u}, \overline{X}_s^{n,m,u}, \mathbb{E}_t^{n,m}\big[\Phi\big(\overline{X}_s^{n,m,u}\big)\big], \bar{u}_s^{n,m}\big)ds\Big], \end{aligned} \tag{3.8}$$

where $\mathbb{E}_t^{n,m}[\cdot] = \mathbb{E}\big[\cdot\,|\,\mathcal{F}_t^{n,m}\big]$. We note that under additional uniform continuity and Lipschitz continuity assumptions on $h(t, s, \xi, x, y, u)$, and $g(t, \xi, x, y)$ in initial time t and initial state ξ respectively, the analysis of Theorem 2.4, Corollary 2.5 and Theorem 3.4 can be extended to the setting of (3.8), by replacing (1.8) with the Hamiltonian

$$H(t, s, \xi, x, y, \mu, p) = p\int_U b(s, x, v)\mu(dv) - \int_U h(t, s, \xi, x, y, v)\mu(dv).$$

The proofs of Section 2.1 remain unchanged because the spike perturbation $\mu \otimes_{t,\varepsilon} \mu^{*n}$ does not affect the initial state $X_{t_k^m}^{\mu^{*n}}$. The main changes to Section 2.2 are in Theorem 2.4, where the bound (2.18) on

$$\mathbb{E}\left[\int_0^T \big|H\big(\lfloor t\rfloor_n, t, X_{\lfloor t\rfloor_n}^{\mu^n}, X_t^{\mu^n}, \mathbb{E}_{\lfloor t\rfloor_n}\big[\Phi\big(X_t^{\mu^n}\big)\big], \mu_t^{*n}, p_{\lfloor t\rfloor_n, t}^{\mu^{*n}}\big) - H\big(t, t, X_t, X_t, \Phi(X_t), \mu_t^*, p_{t,t}^{\mu^*}\big)\big|dt\right]$$

now contains two additional terms

$$\int_0^T \mathbb{E}\Big[|X_{\lfloor t\rfloor_n} - X_t| + \int_U \big|h\big(\lfloor t\rfloor_n, t, X_t, X_t, \mathbb{E}_t[\Phi(X_t)], v\big) - h\big(t, t, X_t, X_t, \mathbb{E}_t[\Phi(X_t)], v\big)\big|\mu_t^{*n}(dv)\Big]dt,$$

which converge to 0 by noting the uniform continuity of h on initial time and the continuity property of SDE. The proofs in Section 3.1, particularly Lemma 3.6, can be modified similarly.

3.2 Numerical results

In this section we present numerical illustrations based on Theorem 3.4. Assume that K is the bounding constant in Assumption 1, and let $p^{n,m}(y; t_k^m, x, u)$ denote the transition probability of $(X_{k+1}^{n,m,u})_{0\le k<nm}$, $x \in \sqrt{\Delta}_{n,m}\mathbb{Z}$, $u \in \mathcal{U}^{n,m}([0,T])$. As in § 4 of Fischer and Reiss (2007), Assumption 2 is satisfied using a trinomial tree constructed as

$$p^{n,m}(y;t_k^m,x,u) = \begin{cases} \dfrac{\sqrt{\Delta}_{n,m}}{2K} b(t_k^m,x,u_k) + \dfrac{1}{2K^2}\sigma^2(t_k^m,x), & y = x + K\sqrt{\Delta}_{n,m}, \\ -\dfrac{\sqrt{\Delta}_{n,m}}{2K} b(t_k^m,x,u_k) + \dfrac{1}{2K^2}\sigma^2(t_k^m,x), & y = x - K\sqrt{\Delta}_{n,m}, \\ 1 - \dfrac{1}{K^2}\sigma^2(t_k^m,x), & y = x, \\ 0, & \text{otherwise.} \end{cases}$$

We consider the following numerical implementation of Theorem 3.4.

(i) For each time t_k^m, initialize the nodes $\mathcal{Y}_k := \{H_{n,m}(x_0) + jK\sqrt{\Delta}_{n,m} \ : \ -k \le j \le k\}$.

(ii) Starting from $t_{(n-1)m}^m$, solve Problem (3.1) for every initial value $x \in \mathcal{Y}_{(n-1)m}$ at time $t_{(n-1)m}^m$.

(iii) Repeat (ii) recursively at times $t_{(n-2)m}^m, \dots, t_m^m, t_0^m$.

However, solving Problem (3.1) can still be computationally expensive for large m because we need to optimize $1 + 3 + 3^2 + \cdots + 3^{m-1} = (3^m - 1)/2$ controls at each node $x \in \mathcal{Y}_{t_k}$, $k = n-1, n-2, \dots, 0$. If the function (3.8) does not depend on a mean-field term then for each node $x \in \mathcal{Y}_{t_k}$, $k = n-1, n-2, \dots, 0$, the optimization problem

$$\inf_{u\in\mathcal{U}^{n,m}([0,T])} J^{n,m}(t_k, x, \bar{u} \otimes_{t_k,t_{k+1}} \bar{u}^{n,m}) \tag{3.9}$$

can be solved using dynamic programming, which reduces the number of parameters to be optimized from exponential $(3^m - 1)/2$ to polynomial $1 + 3 + 5 + \cdots + (2(m-1)+1) = m^2$ at every node $x \in \mathcal{Y}_{t_k}$, $k = n-1, n-2, \dots, 0$.

To solve (3.9) using dynamic programming at each time t_l^m with $t_k \le t_l^m < t_{k+1}$, we need to access the optimal control on $[t_{l+1}^m, T]$ and calculate $J^{n,m}(t_l^m, x, u)$, which involves a calculation from time t_l^m to time T. The complexity of the algorithm can be reduced in case (3.8) takes the particular form

$$\begin{aligned} J^{n,m}\big(t, \overline{X}_t^{n,m,u}, \bar{u}^{n,m}\big) &= g\big(t, \overline{X}_t^{n,m,u}, \mathbb{E}_t^{n,m}[g_1(\overline{X}_T^{n,m,u})], \dots, \mathbb{E}_t^{n,m}[g_p(\overline{X}_T^{n,m,u})]\big) \\ &+ \int_t^T h\big(t, s, \overline{X}_t^{n,m,u}, \mathbb{E}_t^{n,m}[h_1(\overline{X}_s^{n,m,u}, u_s)], \dots, \mathbb{E}_t^{n,m}[h_q(\overline{X}_s^{n,m,u}, u_s)]\big) ds, \end{aligned} \tag{3.10}$$

from which we have

$$
\begin{aligned}
J^{n,m}(t_l^m, x, \bar{u}^{n,m}) &= g\big(t_l^m, x, \mathbb{E}_{t_l^m}^{n,m}\big[\mathbb{E}_{t_{l+1}^m}^{n,m}\big[g_1\big(\overline{X}_T^{n,m,u}\big)\big]\big], \ldots, \mathbb{E}_{t_l^m}^{n,m}\big[\mathbb{E}_{t_{l+1}^m}^{n,m}\big[g_p\big(\overline{X}_T^{n,m,u}\big)\big]\big]\big) \\
&+ \int_{t_l^m}^{t_{l+1}^m} h\big(t_l^m, s, x, \mathbb{E}_{t_l^m}^{n,m}\big[h_1\big(\overline{X}_s^{n,m,u}, \bar{u}_s^{n,m}\big)\big], \ldots, \mathbb{E}_{t_l^m}^{n,m}\big[h_q\big(\overline{X}_s^{n,m,u}, \bar{u}_s^{n,m}\big)\big]\big) ds \\
&+ \int_{t_{l+1}^m}^{T} h\big(t_l^m, s, x, \mathbb{E}_{t_l^m}^{n,m}\big[\mathbb{E}_{t_{l+1}^m}^{n,m}\big[h_1\big(\overline{X}_s^{n,m,u}, \bar{u}_s^{n,m}\big)\big]\big], \ldots, \mathbb{E}_{t_l^m}^{n,m}\big[\mathbb{E}_{t_{l+1}^m}^{n,m}\big[h_q\big(\overline{X}_s^{n,m,u}, \bar{u}_s^{n,m}\big)\big]\big]\big) ds.
\end{aligned}
\tag{3.11}
$$

In this case it suffices to maintain an array for the values of $h_1, \ldots, h_q, g_1, \ldots, g_p$ at $\overline{X}_s^{n,m,u}, \bar{u}_s^{n,m}$ in order to solve (3.11) at time t_l^m, which involves calculations from time t_l^m to time t_{l+1}^m, instead of from t_l^m to T. This method is applied to the quadratic and quartic cost functions examples $\mathbb{E}_t^{n,m}\big[\big(\overline{X}_T^{n,m,u} - \overline{X}_t^{n,m,u}\big)^2\big]$ and $\mathbb{E}_t^{n,m}\big[\big(\overline{X}_T^{n,m,u} - \overline{X}_t^{n,m,u}\big)^4\big]$ below, however not all cost functions satisfy (3.10), e.g. $\mathbb{E}_t^{n,m}[1/(\overline{X}_T^{n,m,u} + \overline{X}_t^{n,m,u})]$ cannot be written in that form.

3.2.1 Linear-quadratic control problem

We first check the numerical application of Theorems 2.4 and 3.4 to a linear-quadratic control problem which admits an analytic solution, see Björk and Murgoci (2010) and Djehiche and Huang (2016), allowing us to evaluate the performance of our numerical scheme. Here, the state of the system is driven by the SDE

$$
\begin{cases}
dX_{t,s}^{\xi,\mu} = \left(aX_{t,s}^{\xi,\mu} + c\displaystyle\int_U v\mu_s(dv)\right) ds + \sigma dW_s, \quad t \le s \le T, \\
X_{t,t}^{\xi,\mu} = \xi,
\end{cases}
\tag{3.12}
$$

where $a, c, \sigma \in \mathbb{R}$, with the cost functional

$$
J(t, \xi, \mu) = \frac{\gamma}{2}\mathbb{E}\big[\big(X_{t,T}^{\xi,\mu} - \xi\big)^2\big] + \frac{1}{2}\mathbb{E}\left[\int_t^T \int_U v^2 \mu_s(dv) ds\right],
\tag{3.13}
$$

where $\gamma > 0$, $g(t, \xi, x, y) = \gamma(x-\xi)^2/2$ and $h(t, s, \xi, x, y, u) = u^2/2$, in the framework of (3.8). Extending the solution technique of Djehiche and Huang (2016) from the strict control space to the relaxed control space by replacing Theorem 1 therein with Theorem 1.4 above, it can be shown that (3.13) admits a strict equilibrium control represented as

$$
\mu_t^*(dv) = \delta_{u_t^*}(dv) := \delta_{c(\beta(t)-\alpha(t))X_t^{\mu^*}}(dv),
\tag{3.14}
$$

where

$$
X_t^{\mu^*} := X_{0,t}^{x_0,\mu^*} = x_0\Gamma(0,t) + \sigma\int_0^t \Gamma(s,t) dW_s, \quad \Gamma(t,s) = \exp\left(\int_t^s \big(a + c^2(\beta(r) - \alpha(r))\big) dr\right),
$$

and the functions $\alpha(t)$, $\beta(t)$ are defined by

$$\beta(t) = \gamma e^{a(T-t)}, \quad \alpha(t) = \gamma \frac{\exp\left(2a(T-t) + c^2 \int_t^T \beta(s)ds\right)}{1 + \gamma c^2 \int_t^T \exp\left(2a(T-s) + c^2 \int_s^T \beta(r)dr\right)ds}.$$

Proposition 3.7. *Let $k \in \{1, \ldots, n\}$. The solution of the n-person game*

$$J\big(t_{k-1}, X_{t_{k-1}}^{\mu^{*n}}, \mu^{*n}\big) = \inf_{\mu \in \mathcal{R}([t_{k-1}, t_k])} J\big(t_{k-1}, X_{t_{k-1}}^{\mu^{*n}}, \mu \otimes_{t_{k-1}, \Delta_n} \mu^{*n}\big)$$

is given by the strict equilibrium control represented as

$$\mu_t^{*n}(dv) = \delta_{u_t^{*n}}(dv) := \delta_{c\beta_n(t) X_{t_{k-1}}^{\mu^{*n}} - c\alpha_n(t) X_t^{\mu^{*n}}}(dv), \qquad t_{k-1} < t \le t_k, \tag{3.15}$$

where

$$\begin{cases} \alpha_n(t) &= \dfrac{2a\alpha_n(t_k)e^{-2a(t-t_k)}}{2a + c^2\alpha_n(t_k)(e^{-2a(t-t_k)} - 1)}, \\ \beta_n(t) &= \beta_n(t_k) \exp\left(a(t_k - t) - c^2 \displaystyle\int_t^{t_k} \alpha_n(s)ds\right), \qquad t_{k-1} < t \le t_k, \end{cases} \tag{3.16}$$

with the terminal conditions

$$\alpha_n(t_k) = \gamma e^{a(T-t_k)} \prod_{l=k}^{n-1} \left(\Gamma_n(t_l, t_{l+1}) + c \int_{t_l}^{t_{l+1}} \Gamma_n(s, t_{l+1})\beta_n(s)ds\right), \quad \beta_n(t_k) = \gamma e^{a(T-t_k)}, \tag{3.17}$$

where $\Gamma_n(t,s) = \exp\left(\int_t^s (a - c^2\alpha_n(r))dr\right)$.

Proof. We work by backward induction, starting from $k = n$. In addition to proving (3.15)-(3.17), we also show that

$$\mathbb{E}_{t_k}\big[X_T^{\mu^{*n}}\big] = X_{t_k}^{\mu^{*n}} \prod_{l=k}^{n-1} \left(\Gamma_n(t_l, t_{l+1}) + c \int_{t_l}^{t_{l+1}} \Gamma_n(s, t_{l+1})\beta_n(s)ds\right). \tag{3.18}$$

The corresponding adjoint equation (2.2) can be written as

$$\begin{cases} dp_{t_{k-1},t}^{k-1,\mu^{*n}} &= -ap_{t_{k-1},t}^{k-1,\mu^{*n}} dt + q_{t_{k-1},t}^{k-1,\mu^{*n}} dW_t, \qquad t_{k-1} \le t \le T, \\ p_{t_{k-1},T}^{k-1,\mu^{*n}} &= \gamma\big(X_{t_{k-1}}^{\mu^{*n}} - X_T^{\mu^{*n}}\big), \end{cases} \tag{3.19}$$

with solution given by

$$p_{t_{k-1},t}^{k-1,\mu^{*n}} = \gamma e^{a(T-t)}\big(X_{t_{k-1}}^{\mu^{*n}} - \mathbb{E}_t\big[X_T^{\mu^{*n}}\big]\big),$$

hence by (3.18) we have

$$p^{k-1,\mu^{*n}}_{t_{k-1},t_k} = \gamma e^{a(T-t_k)} X^{\mu^{*n}}_{t_{k-1}} - \gamma e^{a(T-t_k)} X^{\mu^{*n}}_{t_k} \prod_{l=k}^{n-1} \left(\Gamma_n(t_l, t_{l+1}) + c \int_{t_l}^{t_{l+1}} \Gamma_n(s, t_{l+1}) \beta_n(s) ds \right). \tag{3.20}$$

Next, we look for the solution of the form

$$p^{k-1,\mu^{*n}}_{t_{k-1},t} = \beta_n(t) X^{\mu^{*n}}_{t_{k-1}} - \alpha_n(t) X^{\mu^{*n}}_t, \qquad t_{k-1} < t \leq t_k. \tag{3.21}$$

By Itô's lemma and (3.12), we have

$$\begin{aligned} dp^{k-1,\mu^{*n}}_{t_{k-1},t} &= \beta_n'(t) X^{\mu^{*n}}_{t_{k-1}} dt - \alpha_n'(t) X^{\mu^{*n}}_t dt - \alpha_n(t) dX^{\mu^{*n}}_t \\ &= \beta_n'(t) X^{\mu^{*n}}_{t_{k-1}} dt - \alpha_n'(t) X^{\mu^{*n}}_t dt - \alpha_n(t) \left(a X^{\mu^{*n}}_t + c \int_U v \mu^{*n}_t(dv) \right) dt - \sigma \alpha_n(t) dW_t, \end{aligned}$$

and comparing the resulting coefficients in 'dt' and 'dW_t' with (3.19), we obtain

$$(\alpha_n'(t) + 2a\alpha_n(t)) X^{\mu^{*n}}_t + c\alpha_n(t) \int_U v \mu^{*n}_t(dv) dt = (\beta_n'(t) + a\beta_n(t)) X^{\mu^{*n}}_{t_{k-1}}, \quad t_{k-1} < t \leq t_k, \tag{3.22}$$

and $q^{k-1,\mu^{*n}}_{t_{k-1},t} = -\sigma \alpha_n(t)$. By (1.8), the Hamiltonian of this system is

$$H(t, x, \mu, p) := axp + cp \int_U v \mu(dv) - \frac{1}{2} \int_U v^2 \mu(dv).$$

Due to the concavity of $H(t, x, \mu, p)$, the optimality necessary condition (2.3) in Theorem 2.1 becomes sufficient, see i.e. Theorem 3.5.2 in Yong and Zhou (1999) and Theorem 4.1 in Andersson and Djehiche (2011), and it yields $\mu^{*n}_t(dv) = \delta_{cp^{k-1,\mu^{*n}}_{t_{k-1},t}}(dv)$ on $(t_{k-1}, t_k]$ after maximizing $H(t, X^{\mu^{*n}}_t, \cdot, p^{k-1,\mu^{*n}}_{t_{k-1},t})$, which shows (3.15). Next, plugging (3.15) into (3.22) and identifying the coefficients in '$X^{\mu^{*n}}_{t_{k-1}}$' and '$X^{\mu^{*n}}_t$', we obtain

$$\begin{cases} \alpha_n'(t) + 2a\alpha_n(t) - c^2(\alpha_n(t))^2 = 0, \\ \beta_n'(t) + (a - c^2\alpha_n(t))\beta_n(t) = 0, \qquad t_{k-1} < t \leq t_k, \end{cases}$$

which yields (3.16), while the terminal conditions (3.17) are obtained by a comparison of (3.20) and (3.21). Regarding (3.18), we have

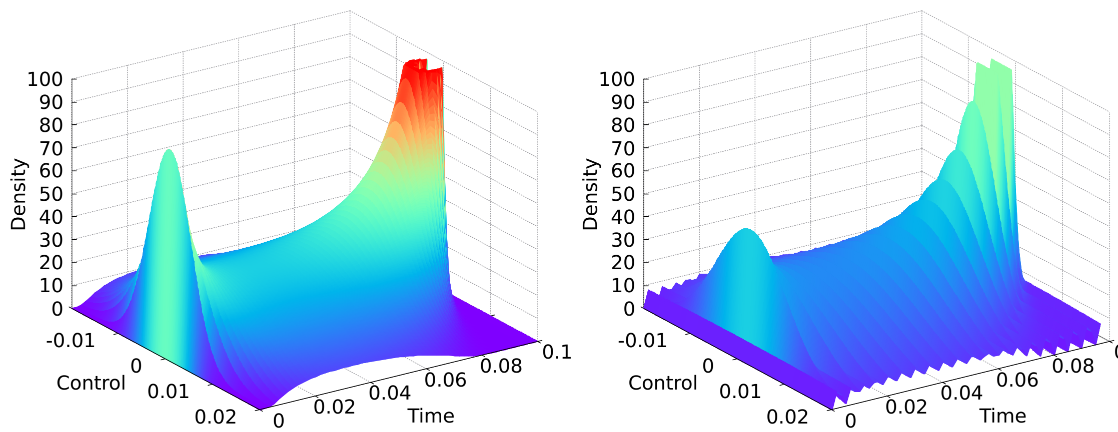
$$\begin{aligned} \mathbb{E}_{t_{k-1}}\left[X^{\mu^{*n}}_T\right] &= \mathbb{E}_{t_{k-1}}\left[\mathbb{E}_{t_k}\left[X^{\mu^{*n}}_T\right]\right] \\ &= \mathbb{E}_{t_{k-1}}\left[X^{\mu^{*n}}_{t_k} \prod_{l=k}^{n-1} \left(\Gamma_n(t_l, t_{l+1}) + c \int_{t_l}^{t_{l+1}} \Gamma_n(s, t_{l+1}) \beta_n(s) ds \right) \right] \end{aligned}$$

$$
= \mathbb{E}_{t_{k-1}}\left[X_{t_k}^{\mu^{*n}}\right] \prod_{l=k}^{n-1} \left(\Gamma_n(t_l, t_{l+1}) + c \int_{t_l}^{t_{l+1}} \Gamma_n(s, t_{l+1}) \beta_n(s) ds \right)
$$

$$
= X_{t_{k-1}}^{\mu^{*n}} \prod_{l=k-1}^{n-1} \left(\Gamma_n(t_l, t_{l+1}) + c \int_{t_l}^{t_{l+1}} \Gamma_n(s, t_{l+1}) \beta_n(s) ds \right),
$$

where the last equality is obtained by solving the linear SDE (3.12) using (3.15). Finally, assuming that (3.15)-(3.18) hold at the rank k, we repeat the above argument to show that they hold at the rank $k-1$. □

In Figure 1 we compare the actual probability density of the equilibrium control u_t^* given by (3.14) to the 20-person game solution u_t^{*20} obtained from (3.15) for $t \in [0, T]$ with $T = 0.1$.



(a) Equilibrium controls. (b) Solution of 20-person game.

Figure 1: Comparison between the equilibrium control and the 20-person game solution.

In Figure 2 we check the convergence in distribution of u_t^{*n} in (3.15) to u_t^* in (3.14) by comparing the CDFs of u_t^* and u_t^{*n} with $n = 20$ at times $t = 0.02, 0.04, 0.06, 0.08$, with $a = c = \sigma = \gamma = 1$ and $x_0 = 0$.

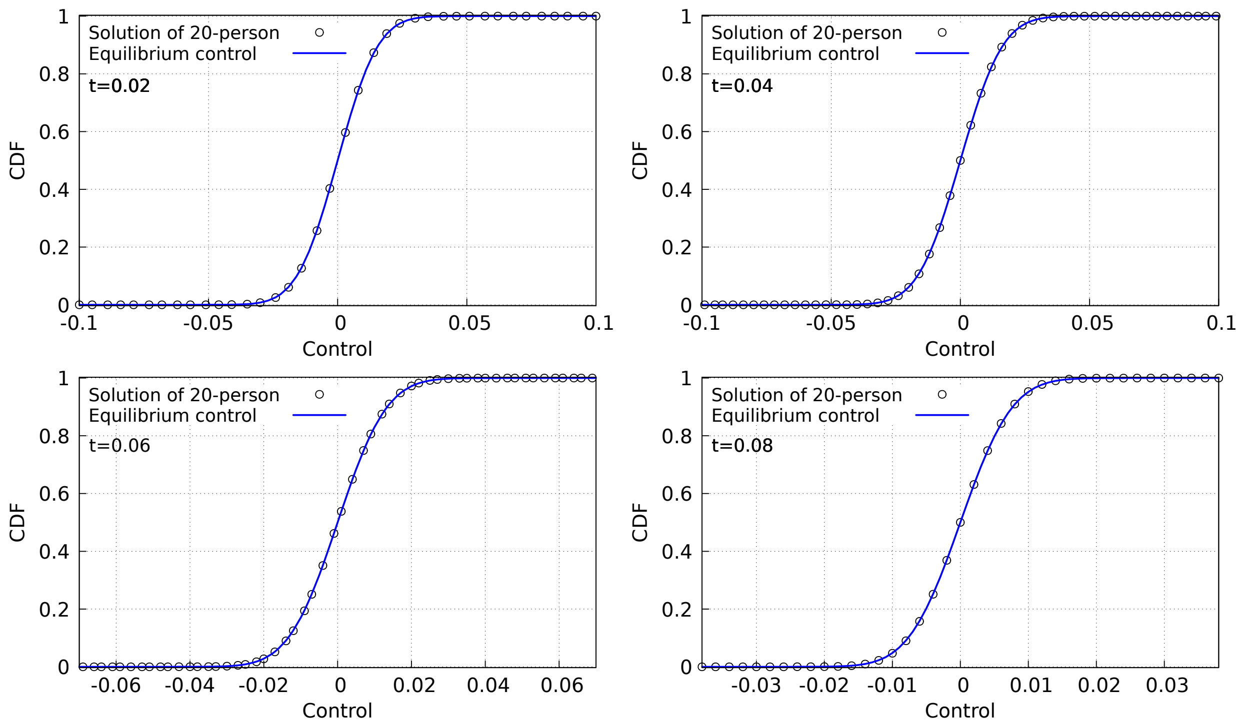


Figure 2: CDF comparison between μ^* and the 20-person game solution.

Numerical approximation of the n-person game solution

To assess the weak convergence of controls stated in Theorem 2.4 and 3.4, in Figure 3 we compare the closed form CDFs of u_t^{*n} obtained from (3.15) to the numerical solution $\bar{u}_t^{*n,m}$ of Problem (3.1) with $n = m = 20$ at times $t = 0.02, 0.04, 0.06, 0.08$, and $U = [-10, 10]$, by truncating $b(t, x, u)$, $h(t, x, y, u)$, $g(x, y)$ up to K.

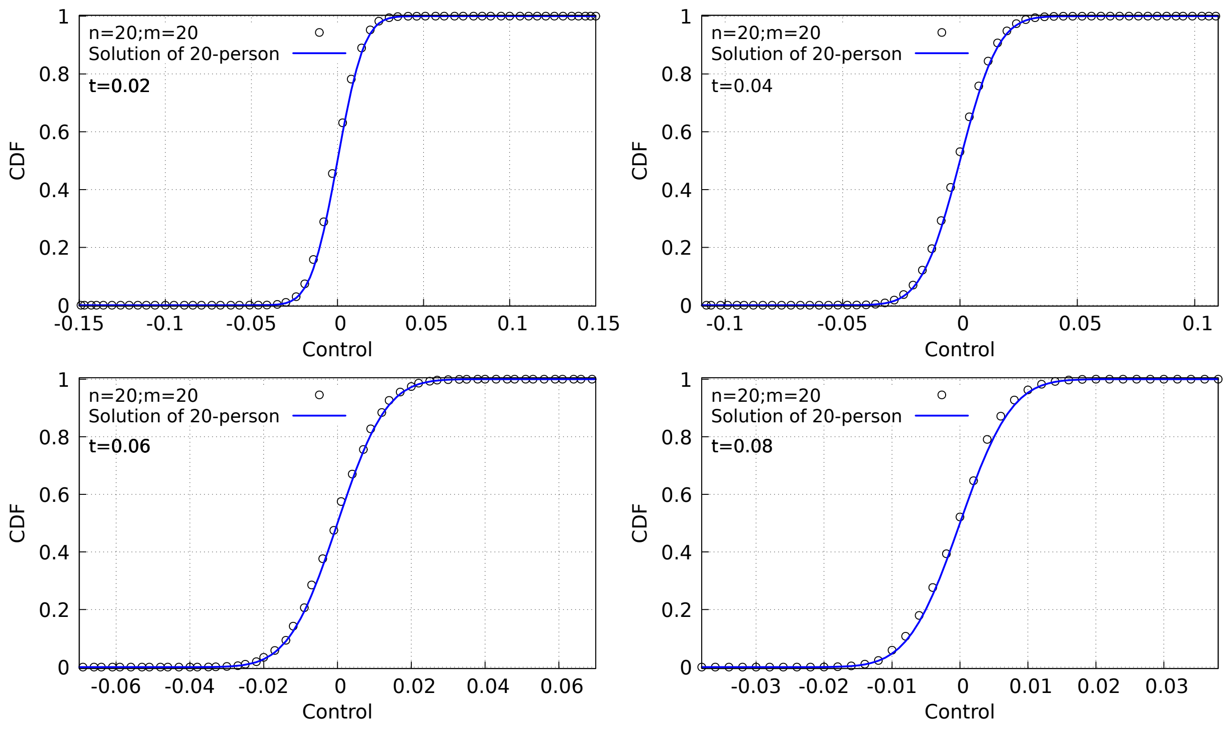


Figure 3: CDF Comparison between the 20-person game solution and the numerical solution.

In Figure 4, we compare the value functions $J^{n,m}(0, x_0, \bar{u}^{*n,m})$ with $n = 5, 10, 15, 20$ and $m \in \{1, \dots, 20\}$.

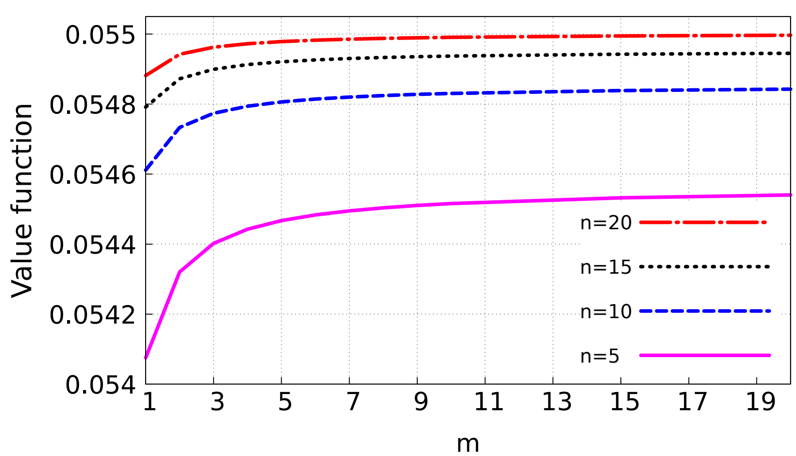


Figure 4: Comparison of value functions.

In Figure 5, we compare the relative errors of the value function $J^{n,m}(0, x_0, \bar{u}^{*n,m})$ with respect to $J(0, x_0, \mu^*)$.

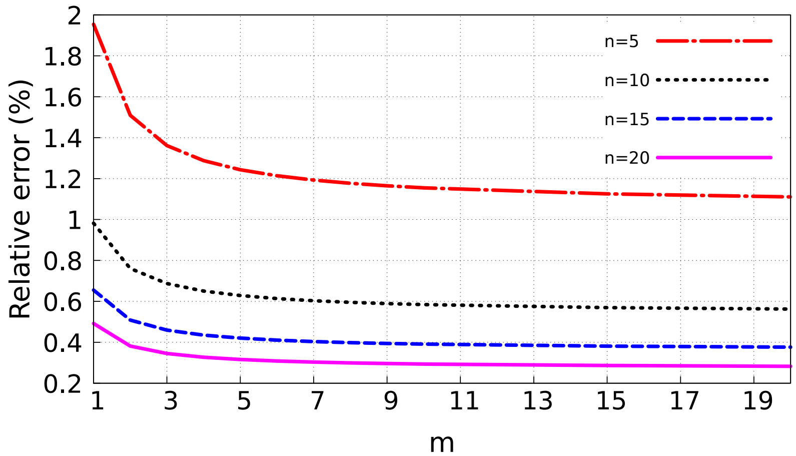


Figure 5: Comparison of relative errors in percentage.

3.2.2 Linear-quartic control problem

Here, we apply our solution algorithm to the problem

$$J(t,\xi,\mu) = \frac{\gamma}{2}\mathbb{E}[(X_{t,T}^{\xi,\mu}-\xi)^4] + \frac{1}{2}\mathbb{E}\left[\int_t^T \int_U v^2 \mu_s(dv)ds\right],$$

where $\gamma > 0$, $a = b = \sigma = \gamma = 1$, $T = 0.1$, $x_0 = 0$, and $g(t,\xi,x,y) = \gamma(x-\xi)^4/2$, $h(t,s,\xi,x,y,u) = u^2/2$, in the framework of (3.8). To the best of our knowledge, this problem admits no analytic solution, hence we construct a numerical approximation of its equilibrium control based on Theorems 2.4 and 3.4 and the numerical solution $\bar{u}^{*n,m}$ of Problem (3.1). In Figure 6, we plot the value functions $J^{n,m}(0,x_0,\bar{u}^{*n,m})$ for $n = 5, 10, 15, 20$ and $m \in \{1,\ldots,20\}$.

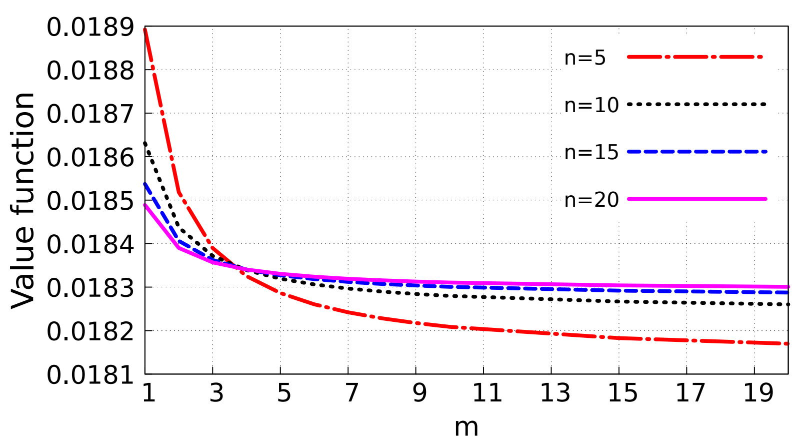


Figure 6: Comparison of value functions.

In Figure 7, we present the CDFs of $\bar{u}_t^{*n,m}$ with $n = m = 20$ at times $t = 0.02, 0.04, 0.06, 0.08$.

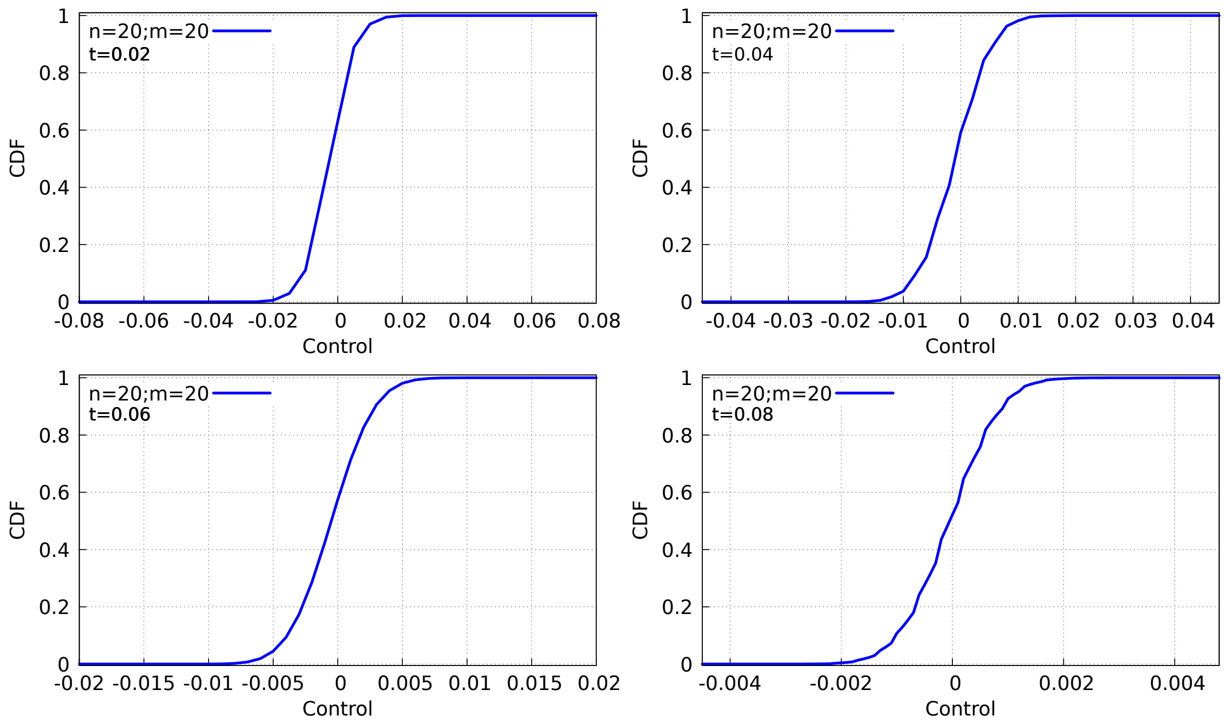


Figure 7: CDFs at different times.

A Appendix

The proof of Theorem 2.4 uses Lemma A.3 below, which requires the Skorokhod representation theorem in order to construct all random variables on a single underlying probability space as in Kushner (1990a).

Theorem A.1 (Skorokhod representation theorem, see Theorem 6.7 in Billingsley (1999))**.** *Let $(\mathbb{P}_n)_{n\geq 1}$ and $\mathbb{P}$ be probability measures on a metric space S such that $(\mathbb{P}_n)_{n\geq 1}$ converges weakly to $\mathbb{P}$ on S and the support of $\mathbb{P}$ is separable. Then there exist a random variable X and a sequence $(X_n)_{n\geq 1}$ of random variables defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$, such that $\mathcal{L}(X_n) = \mathbb{P}_n$, $\mathcal{L}(X) = \mathbb{P}$, and $(X_n)_{n\geq 1}$ converges to X, $\mathbb{P}$-a.s. on S.*

The following lemma was proved in Lemma 2.4 in Jacod and Mémin (1981) and Theorem 3 in Valadier (1994), and is included for completeness. Stable convergence of measures, see Rényi (1963), is defined using the test function space of bounded measurable functions

$f : [0,T] \times U \to \mathbb{R}$ such that $f(t,\cdot)$ is continuous in $u \in U$ for all $t \in [0,T]$. We respectively denote by $\mathcal{C}_b(\mathbb{R})$ and $\mathcal{C}_b(U)$ the spaces of bounded continuous functions on $\mathbb{R}$ and U.

Lemma A.2. *Consider a family $(\lambda^n)_{n\geq 1} \subset \Lambda$ and $\lambda \in \Lambda$. The following are equivalent:*

i) The sequence $(\lambda^n)_{n\geq 1}$ converges stably to $\lambda \in \Lambda$.

ii) $\lambda^n\left(\sum_{l=1}^{m} \mathbb{1}_{A_l}(t) g_l(u)\right) \xrightarrow[n\to\infty]{} \lambda\left(\sum_{l=1}^{m} \mathbb{1}_{A_l}(t) g_l(u)\right)$, for any $m \geq 1$, any finite $\mathcal{B}([0,T])$-partition $\{A_1, A_2, \ldots, A_m\}$ of $[0,T]$ and $g_1, \ldots, g_m \in \mathcal{C}_b(\mathbb{R})$,

iii) The sequence $(\lambda^n)_{n\geq 1}$ converges weakly to $\lambda \in \Lambda$.

Proof. As $(i) \Rightarrow (ii)$ and $(i) \Rightarrow (iii)$ are straightforward, we only show the following.
$(ii) \Rightarrow (i)$: Let f be a bounded measurable function $f(t,u)$ such that $f(t,\cdot)$ is continuous in $u \in U$ for all $t \in [0,T]$. By the Riesz Theorem, see § 12.3 page 251 of Royden and Fitzpatrick (2010), the space $\mathcal{C}_b(U)$ is separable. Denoting by $(c_l)_{l\geq 0}$ a countable dense subset of $\mathcal{C}(U)$ with respect to $\|\cdot\|_\infty$, with $c_0 \equiv 0$ and letting

$$D'_{l,m} = \left\{ t \in [0,T] \ : \ \|f(t,\cdot) - c_l\|_\infty = \min_{0\leq k\leq m} \|f(t,\cdot) - c_k\|_\infty \right\}, \quad m \geq 0, \quad l = 0,1,\ldots m,$$

we partition $[0,T]$ into the measurable sets

$$D_{l,m} = D'_{l,m} \setminus \bigcup_{k=0}^{l-1} D'_{k,m}$$

made of $t \in [0,T]$ such that $l \in \{0,1,\ldots,m\}$ is the smallest integer satisfying $\|f(t,\cdot) - c_l\|_\infty = \min_{0\leq k\leq m} \|f(t,\cdot) - c_k\|_\infty$. Letting $f_m(t,u) := \sum_{l=0}^{m} \mathbb{1}_{D_{l,m}}(t) c_l(u)$, by the denseness of $(c_l)_{l\geq 1}$ in $\mathcal{C}(U)$ we have

$$\lim_{m\to\infty} \|f(t,\cdot) - f_m(t,\cdot)\|_\infty = 0, \qquad t \in [0,T].$$

Since $c_0 \equiv 0$ we have $\min_{0\leq l\leq m} \|f(t,\cdot) - c_l\|_\infty \leq \|f(t,\cdot)\|_\infty$, and $\|f_m(t,\cdot)\|_\infty \leq 2\|f(t,\cdot)\|_\infty$, $t \in [0,T]$. By the uniform boundedness of f and f_m, $m \geq 0$, we have

$$\begin{aligned}
\sup_{\lambda\in\Lambda} |\lambda(f) - \lambda(f_m)| &= \sup_{\lambda\in\Lambda} \left| \int_0^T \int_U f(t,v) - f_m(t,v) \lambda_t(dv) dt \right| \\
&\leq \int_0^T \|f(t,\cdot) - f_m(t,\cdot)\|_\infty \, dt \xrightarrow[m\to\infty]{} 0.
\end{aligned}$$

Therefore, for any $\varepsilon > 0$, picking m such that

$$\sup_{\lambda\in\Lambda} |\lambda(f) - \lambda(f_m)| < \frac{\varepsilon}{3},$$

and N such that for all $n > N$ by (ii), we have

$$\big|\lambda^n(f_m) - \lambda(f_m)\big| < \frac{\varepsilon}{3},$$

hence

$$\big|\lambda^n(f) - \lambda(f)\big| \leq \big|\lambda^n(f) - \lambda^n(f_m)\big| + \big|\lambda^n(f_m) - \lambda(f_m)\big| + |\lambda(f_m) - \lambda(f)| < \varepsilon, \tag{A.1}$$

which shows (i).

$(iii) \Rightarrow (ii)$: Let $f(t,u) = \sum_{l=1}^{m} \mathbb{1}_{A_l}(t) g_l(u)$ be given as in (ii). Reasoning as in (A.1), it suffices to show that for any given $\varepsilon > 0$, we can find bounded functions $f^{(\varepsilon)}(t,u)$ continuous in both $t \in [0,T]$ and $u \in U$, and such that

$$\sup_{\lambda \in \Lambda} |\lambda(f) - \lambda(f^{(\varepsilon)})| < \varepsilon.$$

Denoting by K the bounding constant on $g_1, \ldots, g_m$, by Lusin's Theorem, see e.g. Exercise 2.44 in Folland (1999), for each $\mathbb{1}_{A_l}(t)$ we can find a closed set $F_l^{(\varepsilon)}$ such that $[0,T] \setminus F_l^{(\varepsilon)}$ has Lebesgue measure $\mathrm{Leb}(F_l^{(\varepsilon)}) \leq \varepsilon/(2mK)$ and $\mathbb{1}_{A_l}(t)$ is continuous on $F_l^{(\varepsilon)}$. By Tietze's extension theorem, see Theorem 4.16 in Folland (1999), we can find a continuous extension $f_l^{(\varepsilon)}(t)$ of $\mathbb{1}_{A_l}(t)$ from $F_l^{(\varepsilon)}$ to $[0,T]$ such that $|f_l|$ is bounded by 1, $l = 1, \ldots, m$. Letting $f^{(\varepsilon)}(t,u) := \sum_{l=1}^{m} f_l^{(\varepsilon)}(t) g_l(u)$, we have

$$\begin{aligned}
\sup_{\lambda \in \Lambda} |\lambda(f) - \lambda(f^{(\varepsilon)})| &\leq K \sup_{\lambda \in \Lambda} \int_0^T \int_U \sum_{l=1}^{m} |\mathbb{1}_{A_l}(t) - f_l(t)| \lambda_t(dv) dt \\
&= K \sum_{l=1}^{m} \int_0^T |\mathbb{1}_{A_l}(t) - f_l(t)| dt \\
&\leq 2K \sum_{l=1}^{m} \mathrm{Leb}(F_l^{(\varepsilon)}) = \varepsilon.
\end{aligned}$$

□

The following technical lemma has been used in the proofs of Theorem 2.4, Corollary 2.6, and Lemma 2.7.

Lemma A.3. *Let $(\mu^n)_{n \geq 1} \subset \mathcal{R}([0,T])$ be a sequence of Λ-valued relaxed controls converging weakly to $\mu \in \mathcal{R}([0,T])$. Then, for any bounded random function $f : [0,T] \times U \times \Omega \to \mathbb{R}$ such that $f(t,\cdot,\omega)$ is continuous for all $(t,\omega) \in [0,T] \times \Omega$, we have*

$$\lim_{n \to \infty} \int_U f(t,v,\omega) \mu_t^n(dv) = \int_U f(t,v,\omega) \mu_t(dv), \quad a.e.\ t \in [0,T], \quad \mathbb{P}\text{-}a.s. \tag{A.2}$$

Proof. Since $(\mu^n)_{n\geq 1}$ is a sequence of random measures converging weakly to μ^*, by the Skorokhod representation Theorem A.1 there exists $\widetilde{\Omega} \in \mathcal{F}$ with $\mathbb{P}(\widetilde{\Omega}) = 1$, such that for all $\omega \in \widetilde{\Omega}$, $(\mu^{*n}(\omega))_{n\geq 1}$ is a sequence of deterministic measures converging weakly to $\mu^*(\omega)$. Since the function $\mathbb{1}_A(t)f(t,u,\omega)$ is bounded, measurable in t and continuous in u for all $A \in \mathcal{B}([0,T])$, by Lemma A.2 we have

$$\lim_{n\to\infty} \int_A \int_U f(t,v,\omega)\mu_t^n(\omega)(dv)dt = \int_A \int_U f(t,v,\omega)\mu_t(\omega)(dv)dt, \qquad A \in \mathcal{B}([0,T]),$$

hence

$$\lim_{n\to\infty} \int_U f(t,v,\omega)\mu_t^n(\omega)(dv) = \int_U f(t,v,\omega)\mu_t(\omega)(dv), \qquad \text{a.e. } t \in [0,T], \quad \mathbb{P}\text{-}a.s.$$

□

The following lemma, which has been used in the proofs of Theorems 2.1 and 2.4, can be proved from the almost Lipschitz property of uniformly continuous functions.

Lemma A.4. *Let X be a real-valued stochastic process and let $(X^{(\varepsilon)})_{\varepsilon\geq 0}$ be a family of real-valued stochastic processes such that for any $p \geq 1$, we have*

$$\lim_{\varepsilon\downarrow 0} \sup_{t\in[0,T]} \mathbb{E}\big[\big|X_t^{(\varepsilon)} - X_t\big|^{2p}\big] = 0.$$

Then, for any uniformly continuous function $f : \mathbb{R} \to \mathbb{R}$ and any $p \geq 1$, we have

$$\lim_{\varepsilon\downarrow 0} \sup_{t\in[0,T]} \mathbb{E}\big[\big|f\big(X_t^{(\varepsilon)}\big) - f(X_t)\big|^{2p}\big] = 0.$$

Proof. We shall prove that for any $\varepsilon > 0$,

$$\lim_{\varepsilon\downarrow 0} \sup_{t\in[0,T]} \mathbb{E}\big[\big|f\big(X_t^{(\varepsilon)}\big) - f(X_t)\big|^{2p}\big] \leq \varepsilon.$$

Since f is uniformly continuous, for any $\rho > 0$, we can pick $K_\rho > 0$ such that for all $x, y \in \mathbb{R}$, we have

$$|f(x) - f(y)| \leq \rho + K_\rho |x - y|,$$

which implies

$$\lim_{\varepsilon\downarrow 0} \sup_{t\in[0,T]} \mathbb{E}\big[\big|f\big(X_t^{(\varepsilon)}\big) - f(X_t)\big|^{2p}\big] \leq 2^{2p}\big(\rho^{2p} + K_\rho^{2p} \lim_{\varepsilon\downarrow 0} \sup_{t\in[0,T]} \mathbb{E}\big[\big|X_t^{(\varepsilon)} - X_t\big|^{2p}\big]\big) = |2\rho|^{2p}.$$

We conclude by taking $\rho = \varepsilon^{1/(2p)}/2$. □

References

D. Andersson and B. Djehiche. A maximum principle for SDEs of mean-field type. *Appl. Math. Optim.*, 63 (3):341–356, 2011.

K. Bahlali, M. Mezerdi, and B. Mezerdi. On the relaxed mean-field stochastic control problem. *Stochastics and Dynamics*, 18(03):1850024, 2018.

P. Billingsley. *Convergence of Probability Measures.* Wiley series in Probability and Statistics. Wiley-Interscience, 2nd edition, 1999.

T. Björk and A. Murgoci. A general theory of Markovian time-inconsistent stochastic control problems. Available at SSRN: https://ssrn.com/abstract=1694759 or http://dx.doi.org/10.2139/ssrn.1694759, 2010.

T. Björk and A. Murgoci. A theory of Markovian time-inconsistent stochastic control in discrete time. *Finance and Stochastics*, 18(3):545–592, 2014.

T. Björk, M. Khapko, and A. Murgoci. On time-inconsistent stochastic control in continuous time. *Finance and Stochastics*, 21(2):331–360, 2017.

P. Briand, B. Delyon, and J. Mémin. On the robustness of backward stochastic differential equations. *Stochastic Processes and their Applications*, 97(2):229–253, 2002.

R. Buckdahn, B. Djehiche, and J. Li. A general stochastic maximum principle for SDEs of mean-field type. *Applied Mathematics & Optimization*, 64(2):197–216, 2011.

C. Czichowsky. Time-consistent mean-variance portfolio selection in discrete and continuous time. *Finance and Stochastics*, 17(2):227–271, 2013.

B. Djehiche and M. Huang. A characterization of sub-game perfect equilibria for SDEs of mean-field type. *Dynamic Games and Applications*, 6(1):55–81, 2016.

I. Ekeland and A. Lazrak. Being serious about non-commitment: subgame perfect equilibrium in continuous time. *Preprint arXiv:math/0604264*, 2006.

N. El Karoui, H.D. Nguyen, and M. Jeanblanc-Picqué. Compactification methods in the control of degenerate diffusions: existence of an optimal control. *Stochastics*, 20(3):169–219, 1987.

H. Fetter. On the continuity of conditional expectations. *J. Math. Anal. Appl.*, 61(1):227–231, 1977.

M. Fischer and M. Reiss. Discretisation of stochastic control problems for continuous time dynamics with delay. *J. Comput. Appl. Math.*, 205(2):969–981, 2007.

W.H. Fleming and H.M. Soner. *Controlled Markov processes and viscosity solutions*, volume 25 of *Stochastic Modelling and Applied Probability.* Springer, New York, second edition, 2006.

G.B. Folland. *Real Analysis: Modern Techniques and Their Applications.* John Wiley & Sons, Inc., Canada, 2nd edition, 1999.

Y. Hu, H. Jin, and X.Y. Zhou. Time-inconsistent stochastic linear-quadratic control. *SIAM journal on Control and Optimization*, 50(3):1548–1572, 2012.

Y. Hu, H. Jin, and X.Y. Zhou. Time-inconsistent stochastic linear-quadratic control: characterization and uniqueness of equilibrium. *SIAM Journal on Control and Optimization*, 55(2):1261–1279, 2017.

Y.-J. Huang and Z. Zhou. Strong and weak equilibria for time-inconsistent stochastic control in continuous time. *Preprint arXiv:1809.09243*, 2018.

J. Jacod and J. Mémin. Sur un type de convergence intermédiaire entre la convergence en loi et la convergence en probabilité. In *Seminar on Probability, XV (Univ. Strasbourg, Strasbourg, 1979/1980) (French)*, volume 850 of *Lecture Notes in Math.*, pages 529–546. Springer, Berlin-New York, 1981.

H.J. Kushner. Numerical methods for stochastic control problems in continuous time. *SIAM Journal on control and optimization*, 28(5):999–1048, 1990a.

H.J. Kushner. *Weak convergence methods and singularly perturbed stochastic control and filtering problems*, volume 3 of *Systems & Control: Foundations & Applications*. Birkhäuser Boston, Inc., Boston, MA, 1990b.

H.J. Kushner and P. Dupuis. *Numerical methods for stochastic control problems in continuous time*, volume 24 of *Applications of Mathematics*. Springer-Verlag, New York, second edition, 2001.

A. Rényi. On stable sequences of events. *Sankhyā Ser. A*, 25:293 302, 1963.

H.L. Royden and P.M. Fitzpatrick. *Real analysis*. Pearson, 4th edition, 2010.

M. Valadier. A course on Young measures. Prepublication, Département des Sciences Mathématiques, Université des Sciences et Techniques du Languedoc Montpellier UM2, 33 pages, 1994.

J. Wang and P.A. Forsyth. Continuous time mean variance asset allocation: A time-consistent strategy. *European Journal of Operational Research*, 209(2):184–201, 2011.

J. Yong. Time-inconsistent optimal control problems and the equilibrium HJB equation. *Mathematical Control and Related Fields*, 2(3):271–329, 2012.

J. Yong and X.Y. Zhou. *Stochastic controls*, volume 43 of *Applications of Mathematics (New York)*. Springer-Verlag, New York, 1999.

J. Zhang. *Backward stochastic differential equations*, volume 86 of *Probability Theory and Stochastic Modelling*. Springer, New York, 2017.

X.Y. Zhou and D. Li. Continuous-time mean-variance portfolio selection: A stochastic LQ framework. *Appl. Math. Optim.*, 42(1):19–33, 2000.