Stein normal approximation for multidimensional Poisson random measures by third cumulant expansions

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Abstract

We derive normal approximation bounds by the Stein method for stochastic integrals with respect to a Poisson random measure over $\mathbb{R}^d$, $d \geq 2$. This approach relies on third cumulant Edgeworth-type expansions based on derivation operators defined by the Malliavin calculus for Poisson random measures. The use of third cumulants can exhibit faster convergence rates than the standard Berry-Esseen rate for some sequences of Poisson stochastic integrals.

Key words: Stein approximation; multidimensional Poisson random measures; Poisson stochastic integrals; cumulants; Malliavin calculus; Edgeworth expansions.
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1 Introduction

Normal approximation bounds for stochastic integrals with respect to a Poisson random measure have been obtained by the Stein method in [15], using finite difference operators on the Poisson space. Recent results in this direction include the proof of a fourth moment theorem [8], [9], as an extension of the result of [14] to the setting of

Poisson point processes.

In this paper we derive related bounds for compensated Poisson stochastic integrals $\delta(u) := \int_{\mathbb{R}^d} u_x(\gamma(dx) - \lambda(dx))$ of processes $(u_x)_{x\in\mathbb{R}^d}$ with compact support in $\mathbb{R}^d$, with respect to a Poisson random measure $\gamma(dx)$ with intensity the Lebesgue measure $\lambda(dx)$ on $\mathbb{R}^d$, $d \geq 2$. In contrast with [15], our approach is based on derivation operators and Edgeworth-type expansions that involve the third cumulant of Poisson stochastic integrals, and can result into faster convergence rates, see e.g. (1.5) below.

Edgeworth-type expansions have been obtained on the Wiener space in [11], [5], by a construction of cumulant operators based on the inverse L^{-1} of the Ornstein-Uhlenbeck operator, extending the results of [12] on Stein approximation and Berry-Esseen bounds.

In Proposition 4.1 we derive Edgeworth-type expansions of the form

$$\mathbb{E}\left[\delta(u)g(\delta(u))\right] = \mathbb{E}\left[\|u\|^2_{L^2(\mathbb{R}^d)} g'(\delta(u))\right] + \sum_{k=2}^{n} \mathbb{E}\left[g^{(k)}(\delta(u))\Gamma^u_{k+1}\mathbf{1}\right] + \mathbb{E}\left[g^{(n+1)}(\delta(u))R^u_n\right] \tag{1.1}$$

when the random field $(u_x)_{x\in\mathbb{R}^d}$ is predictable with respect to a given total order on $\mathbb{R}^d$, where Γ^u_k is a cumulant-type operator and R^u_n is a remainder term, defined using the derivation operators of the Malliavin calculus on the Poisson space. In comparison with the results of [15], our bounds apply to a different stochastic integral representation of random variables, and they allow for random integrands $(u_x)_{x\in\mathbb{R}^d}$. In particular, this allows us to deal with random variables $\delta(u)$ having infinite chaos expansions.

Based on (1.1), in Corollary 5.2 we deduce Stein approximation bounds of the form

$$\begin{aligned} d_W(\delta(u), \mathcal{N}) \leq \; & |1 - \mathrm{Var}[\delta(u)]| + \sqrt{\mathrm{Var}\left[\|u\|^2_{L^2(\mathbb{R}^d)}\right]} \\ & + \mathbb{E}\left[\left|\int_{\mathbb{R}^d} u_x^3\,\lambda(dx) + \left\langle u, D\int_{\mathbb{R}^d} u_x^2\,\lambda(dx)\right\rangle_{L^2(\mathbb{R}^d)}\right|\right] + \mathbb{E}\left[|R^u_1|\right], \end{aligned}$$

where D is a gradient operator acting on Poisson functionals, and $\mathcal{N} \simeq \mathcal{N}(0,1)$ is a standard Gaussian random variable, see also Proposition 5.1. Here,

$$d_W(F,G) := \sup_{h\in\mathcal{L}} |\,\mathbb{E}[h(F)] - \mathbb{E}[h(G)]|$$

is the Wasserstein distance between the laws of two random variables F and G, where $\mathcal{L}$ denotes the class of 1-Lipschitz functions on $\mathbb{R}$.

In particular, when f is a differentiable deterministic function on the closed centered ball $B(R) := B(0;R)$ in $\mathbb{R}^d$ with radius $R > 0$, vanishing on the sphere $S(0;R) := \{x \in \mathbb{R}^d \ : \ |x| = R\}$, we obtain bounds of the form

$$\begin{aligned} d_W\left(\int_{\mathbb{R}^d} f(x)(\gamma(dx) - \lambda(dx)), \mathcal{N}\right) \ &\leq \ \left|1 - \|f\|^2_{L^2(\mathbb{R}^d)}\right| + \left|\int_{\mathbb{R}^d} f^3(x)\,\lambda(dx)\right| \\ &\quad + 8(K_d v_d R)^2 \|f\|_{L^2(\mathbb{R}^d)} \|\nabla^{\mathbb{R}^d} f\|^2_{L^\infty(\mathbb{R}^d;\mathbb{R}^d)}, \end{aligned} \tag{1.2}$$

where v_d denotes the volume of the unit ball in $\mathbb{R}^d$ and $K_d > 0$ is a constant depending only on $d \geq 2$. The bound (1.2) can be compared to the classical Stein bound

$$d_W\left(\int_{\mathbb{R}^d} f(x)(\gamma(dx) - \lambda(dx)), \mathcal{N}\right) \leq \left|1 - \|f\|^2_{L^2(\mathbb{R}^d)}\right| + \int_{\mathbb{R}^d} |f^3(x)|\,\lambda(dx), \tag{1.3}$$

for compensated Poisson stochastic integrals, see Corollary 3.4 of [15], which involves the $L^3(\mathbb{R}^d)$ norm of f instead of third cumulant $\kappa_3^f = \int_{\mathbb{R}^d} f^3(x)\lambda(dx)$ of $\int_{\mathbb{R}^d} f(x)(\gamma(dx) - \lambda(dx))$, and relies on the use of finite difference operators, see Theorem 3.1 of [15] and § 4.2 of [4].

For example when f_k, $k \geq 1$, is a radial function given on $B(k^{1/d}R)$ by

$$f_k(x) := \frac{1}{C\sqrt{k}} g\left(\frac{|x|_{\mathbb{R}^d}}{k^{1/d}}\right), \qquad x \in B(k^{1/d}R),$$

where $g \in \mathcal{C}^1([0,R])$ is continuously differentiable on $[0,R]$ with $g(R) = 0$, and

$$C^2 := \int_0^R g^2(r) r^{d-1} dr < \infty,$$

so that $\|f_k\|_{L^2(B(k^{1/d}R))} = 1$, the bound (1.3) yields the standard Berry-Esseen convergence rate

$$d_W\left(\int_{B(k^{1/d}R)} f_k(x)(\gamma(dx) - \lambda(dx)), \mathcal{N}\right) \leq \frac{v_d}{C^3\sqrt{k}} \int_0^R |g(r)|^3 r^{d-1} dr, \quad k \geq 1, \tag{1.4}$$

as k tends to infinity. While (1.2) does not improve on (1.3) when the function f has constant sign, if g satisfies the condition

$$\int_0^R g^3(r) r^{d-1} dr = 0,$$

then the third cumulant bound (1.2) yields the $O(1/k)$ convergence rate

$$d_W \left(\int_{B(k^{1/d}R)} f_k(x)(\gamma(dx) - \lambda(dx)), \mathcal{N} \right) \leq \frac{2(2K_d v_d R)^2 d}{kC^2} \|g'\|_\infty^2, \qquad k \geq 1, \qquad (1.5)$$

which improves on the standard Berry-Esseen rate, see Section 5 for more examples.

In Sections 2 and 3 we recall some background material on the Malliavin calculus and differential geometry on the Poisson space, by revisiting the approach of [16], [17] using the recent constructions of [1] and references therein on the solution of the divergence problem. In Section 4 we derive Edgeworth-type expansions for the compensated Poisson stochastic integral $\delta(u)$, based on a family of cumulant operators that are associated to the random field $(u_x)_{x \in \mathbb{R}^d}$. In Section 5 we obtain Stein-type approximation bounds for stochastic integrals using deterministic examples of integrands.

The d-dimensional setting of this paper requires $d \geq 2$ and a bounded domain in $\mathbb{R}^d$ in order to construct a gradient operator D for Poisson functionals by kernel inversion of the divergence operator on $\mathbb{R}^d$ using results of [1] and references therein. Consequently it does not cover the case $d = 1$ of the standard Poisson process on the half line $\mathbb{R}_+$, which requires a significantly different treatment, see [18]. In particular, the one-dimensional case is technically easier as it does not require Laplace inversion for the construction of the gradient operator D, while stronger conditions on the integrands f in Poisson stochastic integrals have to be imposed in the case $d \geq 2$ through the norm $\|\nabla^{\mathbb{R}^d} f\|_{L^\infty(\mathbb{R}^d;\mathbb{R}^d)}$.

Preliminaries

Let $d \geq 2$ and $0 < R < R' := 2R$. We let $\mathcal{C}_0^\infty(B(R'))$ denote the space of $\mathcal{C}^\infty$ functions on $B(R')$ which vanish on the sphere $S(0; R') = \{x \in \mathbb{R}^d \ : \ |x| = R'\}$. Given

$\eta \in \mathcal{C}_0^\infty(B(R'))$ such that $\int_{B(R)} \eta(x)dx = 1$, we recall the existence of a $\mathcal{C}^\infty$ kernel function $\mathsf{G}_\eta : B(R') \times B(R') \to \mathbb{R}^d$ defined as

$$\mathsf{G}_\eta(x,y) := \int_0^1 \frac{(x-y)}{s} \eta\left(y + \frac{x-y}{s}\right) \frac{ds}{s^d}, \qquad x, y \in B(R'),$$

see [1], and satisfying the following properties:

i) The kernel $\mathsf{G}_\eta(x,y)$ satisfies the bound

$$|\mathsf{G}_\eta(x,y)|_{\mathbb{R}^d} \le \frac{K_d}{|x-y|_{\mathbb{R}^d}^{d-1}}, \qquad x, y \in B(R'), \tag{1.6}$$

for a constant $K_d > 0$ depending only on d, see Lemma 2.1 of [1], by choosing K_d and the function $\eta \in \mathcal{C}_c^\infty(B(R'))$ therein so that $\|\eta\|_\infty \le (d-1)K_d(R')^{-d}$.

ii) For any $p > 1$ and $g \in L^p(B(R'))$ the function

$$f(x) := \int_{B(R')} \mathsf{G}_\eta(x,y) g(y)\, \lambda(dy), \qquad x \in B(R'),$$

satisfies the bound

$$\|f\|_{L^p(B(R');\mathbb{R}^d)} \le K_d v_d R' \|g\|_{L^p(B(R'))}, \qquad p > 1, \tag{1.7}$$

which follows from Young's inequality and (1.6), cf. Theorem 2.4 in [1].

iii) For any $h \in \mathcal{C}_0^\infty(B(R'))$ we have the relation

$$h(y) - \int_{B(R')\setminus B(R)} h(x)\eta(x)\, \lambda(dx) = \int_{B(R')} \langle \mathsf{G}_\eta(x,y), \nabla_x^{\mathbb{R}^d} h(x)\rangle_{\mathbb{R}^d} \lambda(dx), \quad y \in B(R'), \tag{1.8}$$

cf. Lemma 2.2 in [1], by taking $\eta \in \mathcal{C}_c^\infty(B(R') \setminus B(R))$. In particular, when $h \in \mathcal{C}_0^\infty(B(R))$ we have

$$h(y) = \int_{B(R')} \langle \mathsf{G}_\eta(x,y), \nabla_x^{\mathbb{R}^d} h(x)\rangle_{\mathbb{R}^d} \lambda(dx), \qquad y \in B(R'). \tag{1.9}$$

An extension of the framework of this paper, by replacing $B(R)$ with a compact d-dimensional Riemannian manifold M and $\lambda(dx)$ with the volume element of M, would require the Laplacian $\mathcal{L} = \mathrm{div}^M \nabla^M$ to be invertible on $\mathcal{C}_c^\infty(M)$ with

$$\mathcal{L}^{-1} u(x) = \int_M \mathsf{g}(x,y) u(y)\, \lambda(dy), \qquad x \in M,\ u \in \mathcal{C}_c^\infty(M),$$

where $\mathsf{g}(x,y)$ is the heat kernel on M. In this case we can define $\mathsf{G}_\eta(x,y) \in \mathbb{R}^d$ as

$$\mathsf{G}_\eta(x,y) = \nabla^M_x \mathsf{g}(x,y), \qquad \lambda \otimes \lambda(dx,dy) - a.e.,$$

with the relation

$$\nabla^M_x \mathcal{L}^{-1} u(x) = \int_M u(y) \mathsf{G}_\eta(x,y)\, \lambda(dy) \in T_x M, \qquad x \in M,\ u \in \mathcal{C}^\infty_c(M),$$

from which the divergence inversion relation (1.9) holds by duality.

2 Gradient, divergence and covariance derivative

There exists different notions of gradient and divergence operators for functionals of Poisson random measures. The operators of [2], [19], [7], and their associated integration by parts formula rely on an $\mathbb{R}^d$-valued gradient for random functionals and a divergence operator which is associated to the non-compensated Poisson stochastic integral of the divergence of $\mathbb{R}^d$-valued random fields. This particularity, together with a lack of a suitable commutation relation between gradient and divergence operators on Poisson functionals, makes this framework difficult to use for a direct analysis of Poisson stochastic integrals, while it has found applications to statistical estimation and sensitivity analysis, see [7], [19].

In this paper we use the construction of [16], [17] which relies on real-valued tangent processes and on a divergence operator that directly extends the compensated Poisson stochastic integral. This framework also allows for simple commutation relations between gradient and divergence operators using the deterministic inner product in $L^2(\mathbb{R}^d, \lambda)$, see Proposition 2.6, and it naturally involves the Poisson cumulants, see Definition 3.2 and Relation (3.6).

Gradient operator

In the sequel we consider a Poisson random measure $\gamma(dx)$ on $B(R)$, constructed on a probability space $(\Omega, \mathcal{F}, P)$, and we let $\{X_1, \ldots, X_n\}$ denote the configuration points of $\gamma(dx)$ when $B(R)$ contains n points in the configuration γ, i.e. when $\gamma(B(R)) = n$.

Definition 2.1 *Given A a closed subset of $B(R')$, we let $\mathcal{S}_A$ denote the set of random functionals F_A of the form*

$$F_A = \sum_{n=0}^{\infty} \mathbf{1}_{\{\gamma(B(R))=n\}} f_n\left(X_1, \ldots, X_n\right), \tag{2.1}$$

where $f_0 \in \mathbb{R}$ and $(f_n)_{n\geq 1}$ is a sequence of functions satisfying the following conditions:

- *for all $n \geq 1$, $f_n \in \mathcal{C}_c^\infty(A^n)$ is a symmetric function in n variables,*

- *for all $n \geq 1$ and $i = 1, \ldots, n$ we have the continuity condition*

$$f_n\left(x_1, \ldots, x_n\right) = f_{n-1}\left(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n\right), \tag{2.2}$$

for all $x_1, \ldots, x_n \in B(R')$ such that $|x_i|_{\mathbb{R}^d} \geq R$.

We also let $\mathcal{S}$ denote the union of the sets $\mathcal{S}_A$ over the closed subsets A of $B(R')$.

The gradient operator D is defined on random functionals $F \in \mathcal{S}$ of the form (2.1) as

$$D_y F := \sum_{n=1}^{\infty} \mathbf{1}_{\{\gamma(B(R))=n\}} \sum_{i=1}^{n} \langle \mathsf{G}_\eta(X_i, y), \nabla_{x_i}^{\mathbb{R}^d} f\left(X_1, \ldots, X_n\right)\rangle_{\mathbb{R}^d}, \tag{2.3}$$

$y \in B(R)$. For any $F \in \mathcal{S}$, by (1.6) we have $DF \in L^1(\Omega \times B(R))$ from the bound

$$\begin{aligned}
\mathbb{E}\left[\int_{B(R)} |D_x F|\, \lambda(dx)\right] &\leq \||\nabla^{\mathbb{R}^d} f|_{\mathbb{R}^d}\|_\infty \,\mathbb{E}\left[\int_{B(R)}\int_{B(R)} |\mathsf{G}_\eta(x,y)|_{\mathbb{R}^d} \gamma(dx)\lambda(dy)\right] \\
&= \||\nabla^{\mathbb{R}^d} f|_{\mathbb{R}^d}\|_\infty \int_{B(R)}\int_{B(R)} |\mathsf{G}_\eta(x,y)|_{\mathbb{R}^d} \lambda(dx)\lambda(dy) \\
&= K_d \||\nabla^{\mathbb{R}^d} f|_{\mathbb{R}^d}\|_\infty \int_{B(R)}\int_{B(R)} \frac{1}{|x-y|_{\mathbb{R}^d}^{d-1}} \lambda(dx)\lambda(dy) \\
&\leq K_d v_d^2 R' R^d \||\nabla^{\mathbb{R}^d} f|_{\mathbb{R}^d}\|_\infty \\
&< \infty.
\end{aligned}$$

Poisson-Skorohod integral

We let $\mathcal{U}_0$ denote the space of simple random fields of the form

$$u = \sum_{i=1}^{n} g_i G_i, \qquad n \geq 1, \tag{2.4}$$

with $G_i \in \mathcal{S}_{A_i}$ and $g_i \in \mathcal{C}_0^\infty(B(R))$, $i = 1, \ldots, n$.

Definition 2.2 *We define the Poisson-Skorohod integral $\delta(u)$ of $u \in \mathcal{U}_0$ of the form* (2.4) *as*

$$\delta(u) := \sum_{i=1}^{n} \Big(G_i \int_{B(R)} g_i(x)(\gamma(dx) - \lambda(dx)) - \langle g_i, DG_i \rangle_{L^2(B(R))} \Big). \tag{2.5}$$

In particular, for $h \in \mathcal{C}_0^\infty(B(R))$ we have

$$\delta(h) = \int_{B(R)} h(x)(\gamma(dx) - \lambda(dx)).$$

The proof of the next proposition, cf. Proposition 8.5.1 in [16] and Proposition 5.1 in [17], is given in the appendix.

Proposition 2.3 *The operators D and δ satisfy the duality relation*

$$\mathbb{E}[\langle u, DF \rangle_{L^2(B(R))}] = \mathbb{E}[F\delta(u)], \qquad F \in \mathcal{S}, \quad u \in \mathcal{U}_0. \tag{2.6}$$

As a consequence of Proposition 2.3 and the denseness of $\mathcal{S}$ in $L^1(\Omega)$ and that of $\mathcal{U}_0$ in $L^1(\Omega \times B(R))$, the gradient operator D is closable in the sense that if $(F_n)_{n\in\mathbb{N}} \subset \mathcal{S}$ tends to zero in $L^2(\Omega)$ and $(DF_n)_{n\in\mathbb{N}}$ converges to U in $L^1(\Omega \times B(R))$, then $U = 0$ a.e.. Similarly, the divergence operator δ is closable in the sense that if $(u_n)_{n\in\mathbb{N}} \subset \mathcal{U}_0$ tends to zero in $L^2(\Omega \times B(R))$ and $(\delta(u_n))_{n\in\mathbb{N}}$ converges to G in $L^1(\Omega)$, then $G = 0$ a.s..

The gradient operator D defines the Sobolev space $\mathbb{D}^{1,1}$ with the Sobolev norm

$$\|F\|_{\mathbb{D}_{1,1}} := \|F\|_{L^2(\Omega)} + \|DF\|_{L^1(\Omega\times B(R))}, \qquad F \in \mathcal{S}.$$

In the sequel we fix a total order $\preceq$ on $B(R)$ and consider the space $\mathcal{P}_0 \subset \mathcal{U}_0$ of simple predictable random field of the form

$$u := \sum_{i=1}^{n} g_i F_i, \tag{2.7}$$

such that the supports of $g_1, \ldots, g_n$ satisfy

$$\text{Supp}\,(g_i) \preceq \cdots \preceq \text{Supp}\,(g_n) \quad \text{and} \quad F_i \in \mathcal{S}_{A_i},$$

where $\text{Supp}\,(g_1) \cup \cdots \cup \text{Supp}\,(g_{i-1}) \subset A_i \subset B(R')$ and $A_i \preceq \text{Supp}\,(g_i)$, $i = 1, \ldots, n$.

Such random fields are predictable in the sense of e.g. § 5 of [10] and references therein.

We will also assume that the order $\preceq$ is compatible with the kernel G_η in the sense that

$$\mathsf{G}_\eta(x,y) = 0 \quad \text{for all} \quad x,y \in B(R) \text{ such that } x \preceq y. \tag{2.8}$$

Under the compatibility condition (2.8) we have in particular

$$D_y F = 0, \quad y \in B(R), \quad A \preceq y, \quad F \in \mathcal{S}_A.$$

Moreover, if $u \in \mathcal{P}_0$ is a predictable random field of the form (2.7) we note that by (2.3) and the compatibility condition (2.8) we have

$$D_y F_i = 0, \qquad A_i \preceq y, \quad i = 1, \dots, n,$$

hence

$$D_y u_x = 0, \qquad x \preceq y, \quad x,y \in B(R). \tag{2.9}$$

Example. The order $\preceq$ defined by

$$x = (x^{(1)}, \dots, x^{(d)}) \preceq y = (y^{(1)}, \dots, y^{(d)}) \quad \Longleftrightarrow \quad x^{(1)} \leq y^{(1)} \tag{2.10}$$

is compatible with the kernel G_η provided that the support of η is contained in

$$\left\{ x = (x^{(1)}, \dots, x^{(d)}) \in B(R') \setminus B(R) \ : \ x^{(1)} > R \right\}.$$

The proof of the next Proposition 2.4 is given in the appendix.

Proposition 2.4 *The Poisson-Skorohod integral of $u = (u_x)_{x \in B(R)}$ in the space $\mathcal{P}_0$ of simple predictable random fields satisfies the relation*

$$\delta(u) = \int_{B(R)} u_x (\gamma(dx) - \lambda(dx)), \tag{2.11}$$

which extends to the closure of $\mathcal{P}_0$ in $L^2(\Omega \times B(R))$ by density and the isometry relation

$$\mathbb{E}[\delta(u)^2] = \mathbb{E}\left[\int_{B(R)} u_x^2 \, \lambda(dx) \right], \qquad u \in \mathcal{P}_0. \tag{2.12}$$

Covariant derivative

In addition to the gradient operator D, we will also need the following notion of covariant derivative operator $\widetilde{\nabla}$ defined on stochastic processes that are viewed as tangent processes on the Poisson space Ω, see [17].

Definition 2.5 *Let the operator $\widetilde{\nabla}$ be defined on $u \in \mathcal{P}_0$ as*

$$\widetilde{\nabla}_y u_x := D_y u_x + \langle \mathsf{G}_\eta(x,y), \nabla_x^{\mathbb{R}^d} u_x \rangle_{\mathbb{R}^d}, \qquad x, y \in B(R).$$

We note that from the compatibility condition (2.8) and Relation (2.9) we also have

$$\widetilde{\nabla}_y u_x = 0, \qquad x \preceq y, \quad x, y \in B(R). \tag{2.13}$$

From the bound

$$\begin{aligned}
&\mathbb{E}\left[\int_{B(R)\times B(R)} |\widetilde{\nabla}_x u_y| \, \lambda(dx)\lambda(dy)\right] \\
&\quad \leq \|Du\|_{L^1(\Omega\times B(R)\times B(R))} + \mathbb{E}\left[\int_{B(R)\times B(R)} |\langle \mathsf{G}_\eta(x,y), \nabla_x^{\mathbb{R}^d} u_x\rangle_{\mathbb{R}^d}| \, \lambda(dx)\lambda(dy)\right] \\
&\quad \leq \|Du\|_{L^1(\Omega\times B(R)\times B(R))} + K_d \, \mathbb{E}\left[\int_{B(R)\times B(R)} \frac{1}{|x-y|_{\mathbb{R}^d}^{d-1}} |\nabla^{\mathbb{R}^d} u_x|_{\mathbb{R}^d} \lambda(dx)\lambda(dy)\right] \\
&\quad \leq \|Du\|_{L^1(\Omega\times B(R)\times B(R))} + K_d v_d R' \, \mathbb{E}\left[\int_{B(R)} |\nabla_x^{\mathbb{R}^d} u_x|_{\mathbb{R}^d} \lambda(dx)\right] \\
&\quad = \|Du\|_{L^1(\Omega\times B(R)\times B(R))} + K_d v_d R' \|\nabla^{\mathbb{R}^d} u\|_{L^1(\Omega\times B(R);\mathbb{R}^d)},
\end{aligned}$$

we check that $\widetilde{\nabla}$ extends to the Sobolev space $\widetilde{\mathbb{D}}_0^{1,1}$ of *predictable* random fields defined as the completion of $\mathcal{P}_0$ under the Sobolev norm

$$\|u\|_{\widetilde{\mathbb{D}}^{1,1}} := \|u\|_{L^2(\Omega, W_0^{1,1}(B(R)))} + \|Du\|_{L^1(\Omega\times B(R)\times B(R))}, \qquad u \in \mathcal{P}_0,$$

where $W_0^{1,p}(B(R))$ is the first order Sobolev space completion of $\mathcal{C}_0^\infty(B(R))$ under the norm

$$\|f\|_{W^{1,p}(B(R))} := \|f\|_{L^p(B(R))} + \|\nabla^{\mathbb{R}^d} f\|_{L^p(B(R);\mathbb{R}^d)}, \qquad p \geq 1.$$

Commutation relation

In the sequel, we denote by $\widetilde{\mathbb{D}}_0^{1,\infty}$ the set of predictable random fields u in $\widetilde{\mathbb{D}}_0^{1,1}$ that are bounded together with their covariant derivative $\widetilde{\nabla} u$.

Proposition 2.6 *For $u \in \widetilde{\mathbb{D}}_0^{1,\infty}$ a predictable random field, we have the commutation relation*

$$D_y \delta(u) = u(y) + \delta(\widetilde{\nabla}_y u), \qquad y \in B(R). \tag{2.14}$$

Proof. Taking $h \in \mathcal{C}_0^\infty(B(R))$, we have $\delta(h) \in \mathcal{S}$ and

$$\begin{aligned} D_y \delta(h) &= D_y \int_{B(R)} h(y)(\gamma(dx) - \lambda(dx)) \\ &= \int_{B(R)} \langle \mathsf{G}_\eta(x,y), \nabla_x^{\mathbb{R}^d} h(x) \rangle_{\mathbb{R}^d} \gamma(dx) \\ &= \int_{B(R)} \langle \mathsf{G}_\eta(x,y), \nabla_x^{\mathbb{R}^d} h(x) \rangle_{\mathbb{R}^d} \lambda(dx) + \delta(\widetilde{\nabla}_y h) \\ &= h(y) + \delta(\widetilde{\nabla}_y h), \qquad y \in B(R), \end{aligned}$$

where we applied (1.9). Next, taking $u = hF \in \mathcal{P}_0$ a simple predictable random field, we check that $\delta(u) \in \mathcal{S}$, and by (2.5) or (6.3) we have

$$\begin{aligned} D_y \delta(Fh) &= D_y \left(F\delta(h) - \langle h, DF \rangle_{L^2(B(R))} \right) \\ &= D_y \left(F\delta(h) \right) \\ &= \delta(h) D_y F + F D_y \delta(h) \\ &= \delta(h) D_y F + F(h(y) + \delta(\widetilde{\nabla}_y h)) \\ &= Fh(y) + \delta(h D_y F + F \widetilde{\nabla}_y h) \\ &= Fh(y) + \delta(\widetilde{\nabla}_y (Fh)) \\ &= u_y + \delta(\widetilde{\nabla}_y u), \qquad y \in B(R). \end{aligned}$$

We conclude by the denseness of $\mathcal{P}_0$ in $\widetilde{\mathbb{D}}_0^{1,1}$ and by the closability of the operators $\widetilde{\nabla}$, D and δ. □

3 Cumulant operators

In the sequel, given h in the standard Sobolev space $W^{1,p}(B(R))$ on $B(R)$ and $f \in L^q(B(R))$ with $1 = p^{-1} + q^{-1}$, $p, q \in [1, \infty]$, we define

$$(\widetilde{\nabla} h) f_x := \int_{B(R)} f(y) \widetilde{\nabla}_y h(x) \, \lambda(dy) = \int_{B(R)} f(y) \langle \mathsf{G}_\eta(x,y), \nabla_x^{\mathbb{R}^d} h(x) \rangle_{\mathbb{R}^d} \lambda(dy), \tag{3.1}$$

$x \in B(R)$. More generally, given $k \geq 1$ and $u \in \widetilde{\mathbb{D}}_0^{1,1}$ a predictable random field, we let the operator $(\widetilde{\nabla} u)^k$ be defined in the sense of matrix powers with continuous indices, as

$$(\widetilde{\nabla} u)^k f_y = \int_{B(R)} \cdots \int_{B(R)} (\widetilde{\nabla}_{x_k} u_y \widetilde{\nabla}_{x_{k-1}} u_{x_k} \cdots \widetilde{\nabla}_{x_1} u_{x_2}) f_{x_1} \, \lambda(dx_1) \cdots \lambda(dx_k),$$

$y \in B(R)$, $f \in L^2(B(R))$.

Proposition 3.1 *For any $n \in \mathbb{N}$, $p > 1$, $r \in [0,1]$, $h \in W^{1,p/(1-r)^{n-1}/r}(B(R))$ and $f \in L^{p/(1-r)^n}(B(R))$ we have the bound*

$$\|(\widetilde{\nabla} h)^n f\|_{L^p(B(R))} \leq (K_d v_d R')^n \|f\|_{L^{p/(1-r)^n}(B(R))} \prod_{j=1}^{n} \|\nabla^{\mathbb{R}^d} h\|_{L^{p/(1-r)^{j-1}/r}(B(R);\mathbb{R}^d)}. \quad (3.2)$$

Proof. For $n = 1$ we have

$$\begin{aligned}
\|(\widetilde{\nabla} h) f\|^p_{L^p(B(R))} &= \int_{B(R)} \left| \int_{B(R)} f(y) \widetilde{\nabla}_y h(x) \, \lambda(dy) \right|^p \lambda(dx) \\
&= \int_{B(R)} \left| \int_{B(R)} f(y) \langle \mathsf{G}_\eta(x,y), \nabla_x^{\mathbb{R}^d} h(x) \rangle_{\mathbb{R}^d} \lambda(dy) \right|^p \lambda(dx) \\
&= \int_{B(R)} \left| \left\langle \int_{B(R)} f(y) \mathsf{G}_\eta(x,y) \, \lambda(dy), \nabla_x^{\mathbb{R}^d} h(x) \right\rangle_{\mathbb{R}^d} \right|^p \lambda(dx) \\
&\leq \int_{B(R)} \left| \int_{B(R)} f(y) \mathsf{G}_\eta(x,y) \, \lambda(dy) \right|^p_{\mathbb{R}^d} |\nabla_x^{\mathbb{R}^d} h(x)|^p_{\mathbb{R}^d} \lambda(dx) \\
&= \left(\int_{B(R)} \left| \int_{B(R)} f(y) \mathsf{G}_\eta(x,y) \, \lambda(dy) \right|^{p/(1-r)}_{\mathbb{R}^d} \lambda(dx) \right)^{1-r} \left(\int_{B(R)} |\nabla_x^{\mathbb{R}^d} h(x)|^{p/r}_{\mathbb{R}^d} \lambda(dx) \right)^r \\
&\leq (K_d v_d R')^p \|f\|^p_{L^{p/(1-r)}(B(R))} \|\nabla^{\mathbb{R}^d} h\|^p_{L^{p/r}(B(R);\mathbb{R}^d)}, \quad (3.3)
\end{aligned}$$

where we used the bound (1.7). Next, assuming that (3.2) holds at the rank $n \geq 1$ and using (3.3), we have

$$\begin{aligned}
\|(\widetilde{\nabla} h)^{n+1} f\|_{L^p(B(R))} &= \|(\widetilde{\nabla} h)^n (\widetilde{\nabla} h) f\|_{L^p(B(R))} \\
&\leq (K_d v_d R')^n \|(\widetilde{\nabla} h) f\|_{L^{p/(1-r)^n}(B(R))} \prod_{j=1}^{n} \|\nabla^{\mathbb{R}^d} h\|_{L^{p/(1-r)^{j-1}/r}(B(R);\mathbb{R}^d)} \\
&\leq (K_d v_d R')^{n+1} \|f\|_{L^{p/(1-r)^{n+1}}(B(R))} \prod_{j=1}^{n+1} \|\nabla^{\mathbb{R}^d} h\|_{L^{p/(1-r)^{j-1}/r}(B(R);\mathbb{R}^d)},
\end{aligned}$$

and we conclude to (3.2) by induction on $n \geq 1$. □

In particular, for $r = 0$, $f \in L^p(B(R))$, $p > 1$, and $h \in W^{1,1}(B(R))$ the argument of Proposition 3.1 shows that

$$\|(\widetilde{\nabla} h)^n f\|_{L^p(B(R))} \leq (K_d v_d R')^n \|f\|_{L^p(B(R))} \|\nabla^{\mathbb{R}^d} h\|^n_{L^\infty(B(R);\mathbb{R}^d)}, \qquad n \in \mathbb{N}.$$

We note that for $u \in \widetilde{\mathbb{D}}_0^{1,\infty}$ a predictable random field, the random field $(\widetilde{\nabla} u)u \in \widetilde{\mathbb{D}}_0^{1,\infty}$ is also predictable from (2.13) and (3.1).

In the next definition we construct a family of cumulant operators which differs from the one introduced in [13] on the Wiener space.

Definition 3.2 *Given $k \geq 2$ and $u \in \widetilde{\mathbb{D}}_0^{1,\infty}$ a predictable random field we define the operators $\Gamma_k^u : \mathbb{D}_{1,1} \longrightarrow L^1(\Omega)$ by*

$$\Gamma_k^u F := F \langle (\widetilde{\nabla} u)^{k-2} u, u \rangle_{L^2(B(R))} + \langle (\widetilde{\nabla} u)^{k-1} u, DF \rangle_{L^2(B(R))}, \quad F \in \mathbb{D}_{1,1}.$$

We note that for h in the space $W^{1,\infty}(B(R))$ of bounded functions in $W^{1,1}(B(R))$, and $f \in L^p(B(R))$, $p > 1$, $m \geq 1$, we have

$$\begin{aligned} \langle h^m, (\widetilde{\nabla} h) f \rangle_{L^2(B(R))} &= \int_{B(R)} h^m(x) \int_{B(R)} f(y) \langle \mathsf{G}_\eta(x,y), \nabla_x^{\mathbb{R}^d} h(x) \rangle_{\mathbb{R}^d} \lambda(dy) \lambda(dx) \\ &= \frac{1}{m+1} \int_{B(R)} \int_{B(R)} f(y) \langle \mathsf{G}_\eta(x,y), \nabla_x^{\mathbb{R}^d} h^{m+1}(x) \rangle_{\mathbb{R}^d} \lambda(dy) \lambda(dx) \\ &= \frac{1}{m+1} \int_{B(R)} f(x) h^{m+1}(x) \, \lambda(dx), \end{aligned}$$

where we applied (1.8), hence

$$\langle h^m, (\widetilde{\nabla} h)^{n+1} f \rangle_{L^2(B(R))} = \frac{1}{m+1} \int_{B(R)} h^{m+1}(x) (\widetilde{\nabla} h)^n f(x) \, \lambda(dx),$$

which implies by induction

$$\langle (\widetilde{\nabla} h)^n f, h^m \rangle_{L^2(B(R))} = \frac{m!}{(m+n)!} \int_{B(R)} h^{m+n}(x) f(x) \, \lambda(dx).$$

In Lemma 3.3 we generalize this identity to h a random field.

Lemma 3.3 *For $n \in \mathbb{N}$, $m \geq 1$, $u \in \widetilde{\mathbb{D}}_0^{1,\infty}$ a predictable random field and $f \in L^p(B(R))$, $p > 1$, we have*

$$\langle (\widetilde{\nabla} u)^n f, u^m \rangle_{L^2(B(R))} = \frac{m!}{(m+n)!} \int_{B(R)} u_x^{m+n} f(x) \, \lambda(dx) \tag{3.4}$$

$$+\sum_{k=1}^{n}\frac{m!}{(m+k)!}\left\langle(\widetilde{\nabla}u)^{n-k}f, D\int_{B(R)}u_x^{m+k}\,\lambda(dx)\right\rangle_{L^2(B(R))}.$$

Proof. Using the adjoint $\widetilde{\nabla}^*u$ of $\widetilde{\nabla}u$ on $L^2(B(R))$ given by

$$(\widetilde{\nabla}^*u)v_y := \int_{B(R)}(\widetilde{\nabla}_y u_x)v_x\,\lambda(dx), \qquad y\in B(R), \quad v\in L^2(B(R)),$$

with the duality relation

$$\langle v,(\widetilde{\nabla}^*u)h\rangle_{L^2(B(R))} = \langle(\widetilde{\nabla}u)v,h\rangle_{L^2(B(R))}, \qquad h,v\in L^2(B(R)),$$

we will show by induction on $k=0,1,\dots,n$ that

$$\begin{aligned}
&(\widetilde{\nabla}^*u)^n u_{x_0}^m = \int_{B(R)}\cdots\int_{B(R)} u_{x_n}^m\widetilde{\nabla}_{x_0}u_{x_1}\widetilde{\nabla}_{x_1}u_{x_2}\cdots\widetilde{\nabla}_{x_{n-1}}u_{x_n}\,\lambda(dx_1)\cdots\lambda(dx_n)\\
&=\sum_{i=1}^{k}\frac{m!}{(m+i)!}\int_{B(R)}\cdots\int_{B(R)}\widetilde{\nabla}_{x_0}u_{x_1}\cdots\widetilde{\nabla}_{x_{n-i-1}}u_{x_{n-i}}D_{x_{n-i}}u_{x_{n+1-i}}^{m+i}\,\lambda(dx_1)\cdots\lambda(dx_{n-i-1})\\
&+\frac{m!}{(m+k)!}\int_{B(R)}\cdots\int_{B(R)}u_{x_{n-k}}^{m+k}\widetilde{\nabla}_{x_0}u_{x_1}\cdots\widetilde{\nabla}_{x_{n-k-1}}u_{x_{n-k}}\,\lambda(dx_1)\cdots\lambda(dx_{n-k}). \qquad (3.5)
\end{aligned}$$

By (3.1), this relation holds for $k=0$. Next, assuming that the identity (3.5) holds for some $k\in\{0,1,\dots,n-1\}$, and using the relation

$$\widetilde{\nabla}_{x_{n-k-1}}u_{x_{n-k}} = D_{x_{n-k-1}}u_{x_{n-k}}+\langle\mathsf{G}_\eta(x_{n-k},x_{n-k-1}),\widetilde{\nabla}_{x_{n-k}}u_{x_{n-k}}\rangle_{\mathbb{R}^d}, \quad x_{n-k-1},x_{n-k}\in B(R),$$

we have

$$\begin{aligned}
&(\widetilde{\nabla}^*u)^n u_{x_0}\\
&=\sum_{i=1}^{k}\frac{m!}{(m+i)!}\int_{B(R)}\cdots\int_{B(R)}\widetilde{\nabla}_{x_0}u_{x_1}\cdots\widetilde{\nabla}_{x_{n-i-1}}u_{x_{n-i}}D_{x_{n-i}}u_{x_{n+1-i}}^{m+i}\,\lambda(dx_1)\cdots\lambda(x_{n+1-i})\\
&+\frac{m!}{(m+k)!}\int_{B(R)}\cdots\int_{B(R)}u_{x_{n-k}}^{m+k}\widetilde{\nabla}_{x_0}u_{x_1}\cdots\widetilde{\nabla}_{x_{n-k-1}}u_{x_{n-k}}\,\lambda(dx_1)\cdots\lambda(dx_{n-k})\\
&=\sum_{i=1}^{k}\frac{m!}{(m+i)!}\int_{B(R)}\cdots\int_{B(R)}\widetilde{\nabla}_{x_0}u_{x_1}\cdots\widetilde{\nabla}_{x_{n-i-1}}u_{x_{n-i}}D_{x_{n-i}}u_{x_{n+1-i}}^{m+i}\,\lambda(dx_1)\cdots\lambda(dx_{n+1-i})\\
&+\frac{m!}{(m+k)!}\int_{B(R)}\cdots\int_{B(R)}u_{x_{n-k}}^{m+k}\widetilde{\nabla}_{x_0}u_{x_1}\cdots\widetilde{\nabla}_{x_{n-k-2}}u_{x_{n-k-1}}D_{x_{n-k-1}}u_{x_{n-k}}\,\lambda(dx_1)\cdots\lambda(dx_{n-k})\\
&+\frac{m!}{(m+k)!}\int_{B(R)}\cdots\int_{B(R)}\langle\mathsf{G}_\eta(x_{n-k},x_{n-k-1}),\widetilde{\nabla}_{x_{n-k}}u_{x_{n-k}}\rangle_{\mathbb{R}^d}
\end{aligned}$$

$$
\begin{aligned}
&\times u_{x_{n-k}}^{m+k-2}\widetilde{\nabla}_{x_0}u_{x_1}\cdots\widetilde{\nabla}_{x_{n-2-k}}u_{x_{n-k-1}}\,\lambda(dx_1)\cdots\lambda(dx_{n-k})\\
&=\sum_{i=1}^{k}\frac{m!}{(m+i)!}\int_{B(R)}\cdots\int_{B(R)}\widetilde{\nabla}_{x_0}u_{x_1}\cdots\widetilde{\nabla}_{x_{n-i-1}}u_{x_{n-i}}D_{x_{n-i}}u_{x_{n+1-i}}^{m+i}\,\lambda(dx_1)\cdots\lambda(dx_{n+1-i})\\
&+\frac{m!}{(m+k+1)!}\int_{B(R)}\cdots\int_{B(R)}\widetilde{\nabla}_{x_0}u_{x_1}\cdots\widetilde{\nabla}_{x_{n-k}}u_{x_{n-k-1}}D_{x_{n-k-1}}u_{x_{n-k}}^{m+k+1}\,\lambda(dx_1)\cdots\lambda(dx_{n-k})\\
&+\frac{m!}{(m+k+1)!}\int_{B(R)}\cdots\int_{B(R)}\widetilde{\nabla}_{x_0}u_{x_1}\cdots\widetilde{\nabla}_{x_{n-k-2}}u_{x_{n-k-1}}\\
&\qquad\times\int_{B(R)}\langle\mathsf{G}_\eta(x,x_{n-k-1}),\nabla_x^{\mathbb{R}^d}u_x^{m+k+1}\rangle_{\mathbb{R}^d}\lambda(dx)\lambda(dx_1)\cdots\lambda(dx_{n-k-1})\\
&=\sum_{i=1}^{k+1}\frac{m!}{(m+i)!}\int_{B(R)}\cdots\int_{B(R)}\widetilde{\nabla}_{x_0}u_{x_1}\cdots\widetilde{\nabla}_{x_{n-i-1}}u_{x_{n-i}}D_{x_{n-i}}u_{x_{n+1-i}}^{m+i}\,\lambda(dx_1)\cdots\lambda(dx_{n+1-i})\\
&+\frac{m!}{(m+k+1)!}\int_{B(R)}\cdots\int_{B(R)}u_{x_{n-k-1}}^{m+k+1}\widetilde{\nabla}_{x_0}u_{x_1}\cdots\widetilde{\nabla}_{x_{n-k-2}}u_{x_{n-k-1}}\,\lambda(dx_1)\cdots\lambda(dx_{n-k-1})\\
&=\sum_{i=1}^{k+1}\frac{m!}{(m+i)!}(\widetilde{\nabla}^*u)^{n-i}D_{x_0}\int_{B(R)}u_s^{m+i}\,\lambda(ds)+\frac{m!}{(m+k+1)!}(\widetilde{\nabla}^*u)^{n-k-1}u_{x_0}^{m+k+1},
\end{aligned}
$$

which shows by induction that (3.5) holds at the rank $k=n$, in particular we have

$$
(\widetilde{\nabla}^*u)^n u_x^m=\frac{m!}{(m+k)!}u_x^{m+n}+\sum_{i=2}^{n+1}\frac{m!}{(m+i-1)!}(\widetilde{\nabla}^*u)^{n+1-i}D_x\int_{B(R)}u_y^{m+i-1}\,\lambda(dy),
$$

$x\in B(R)$, which yields (3.4) by integration with respect to $x\in B(R)$ and duality.

□

As a consequence of Lemma 3.3 we have

$$
\Gamma_k^u\mathbf{1}=\int_{B(R)}\frac{u_x^k}{(k-1)!}\,\lambda(dx)+\sum_{i=2}^{k-1}\frac{1}{i!}\left\langle(\widetilde{\nabla}u)^{k-1-i}u,D\int_{B(R)}u_x^i\,\lambda(dx)\right\rangle_{L^2(B(R))},
$$

$k\geq 2$. Hence when $h\in W^{1,p}(B(R))$, $p>1$, is a deterministic function such that $\|\nabla^{\mathbb{R}^d}h\|_\infty<\infty$, we find the relation

$$
\Gamma_k^h\mathbf{1}=\frac{1}{(k-1)!}\int_{B(R)}h^k(x)\,\lambda(dx)=\frac{1}{(k-1)!}\kappa_k^h,\qquad k\geq 2,\tag{3.6}
$$

which shows that $\Gamma_k^h\mathbf{1}$ coincides with the cumulant $\kappa_k^h=\int_{B(R)}h^k(x)\,\lambda(dx)$ of order $k\geq 2$ of the Poisson stochastic integral $\int_{B(R)}h(x)(\gamma(dx)-\lambda(dx))$.

4 Edgeworth-type expansions

Classical Edgeworth series provide expansion of the cumulative distribution function $P(F \leq x)$ of a centered random variable F with $\mathbb{E}[F^2] = 1$ around the Gaussian cumulative distribution function $\Phi(x)$, using the cumulants $(\kappa_n)_{n\geq 1}$ of a random variable F and Hermite polynomials. Edgeworth-type expansions of the form

$$\mathbb{E}[Fg(F)] = \sum_{l=1}^{n} \frac{\kappa_{l+1}}{l!} \mathbb{E}[g^{(l)}(F)] + \mathbb{E}[g^{(n+1)}(F)\Gamma_{n+1}F], \quad n \geq 1,$$

for F a centered random variable, have been obtained by the Malliavin calculus in [11], where Γ_{n+1} is a cumulant-type operator on the Wiener space such that $n!\,\mathbb{E}[\Gamma_n F]$ coincides with the cumulant κ_{n+1} of order $n+1$ of F, $n \in \mathbb{N}$, cf. [13], extending the results of [3] to the Wiener space.

In this section we establish an Edgeworth-type expansion of any finite order with an explicit remainder term for the compensated Poisson stochastic integral $\delta(u)$ of a predictable random field $(u_x)_{x\in B(R)}$. In the sequel we let $\langle \cdot, \cdot \rangle$ denote $\langle \cdot, \cdot \rangle_{L^2(B(R))}$.

Before proceeding to the statement of general expansions in Proposition 4.1, we illustrate the method with the derivation of an expansion of order one for a deterministic integrand f. By the duality relation (2.6) between D and δ, the chain rule of derivation for D and the commutation relation (2.14) we get, for $g \in \mathcal{C}_b^2(\mathbb{R})$ and $f \in W_0^{1,1}(B(R))$ such that $\|\nabla^{\mathbb{R}^d} f\|_\infty < \infty$,

$$\begin{aligned}
&\mathbb{E}[\delta(f)g(\delta(f))] = \mathbb{E}[\langle f, D\delta(f)\rangle g'(\delta(f))] \\
&\quad = \mathbb{E}[\langle f, f\rangle g'(\delta(f))] + \mathbb{E}\left[\langle f, \delta(\widetilde{\nabla}^* f)\rangle g'(\delta(f))\right] \\
&\quad = \mathbb{E}[\langle f, f\rangle g'(\delta(f))] + \mathbb{E}\left[\langle \widetilde{\nabla}^* f, D(g'(\delta(f))f)\rangle\right] \\
&\quad = \mathbb{E}[\langle f, f\rangle g'(\delta(f))] + \mathbb{E}\left[\langle (\widetilde{\nabla} f)f, D\delta(f)\rangle g''(\delta(f))\right] \\
&\quad = \mathbb{E}[\langle f, f\rangle g'(\delta(f))] + \frac{1}{2}\int_{B(R)} f^3(x)\,\lambda(dx)\,\mathbb{E}[g''(\delta(f))] + \mathbb{E}\left[\langle (\widetilde{\nabla} f)f, \delta(\widetilde{\nabla}^* f)\rangle g''(\delta(f))\right] \\
&\quad = \kappa_2^f\,\mathbb{E}[g'(\delta(f))] + \frac{1}{2}\kappa_3^f\,\mathbb{E}[g''(\delta(f))] + \mathbb{E}\left[g''(\delta(f))\delta((\widetilde{\nabla} f)^2 f)\right],
\end{aligned}$$

since by Lemma 3.3 we have

$$\langle (\widetilde{\nabla} f) f, f \rangle = \frac{1}{2} \int_{B(R)} f^3(x)\, \lambda(dx) = \frac{1}{2} \kappa_3^f.$$

In the next proposition we derive general Edgeworth-type expansions for predictable integrand processes $(u_x)_{x \in \mathbb{R}^d}$.

Proposition 4.1 *Let $u \in \widetilde{\mathbb{D}}_0^{1,\infty}$ and $n \geq 0$. For all $g \in \mathcal{C}_b^{n+1}(\mathbb{R})$ and bounded $G \in \mathbb{D}_{1,1}$ we have*

$$\begin{aligned}
&\mathbb{E}\left[G\delta(u)g(\delta(u))\right] = \mathbb{E}\left[\langle u, DG\rangle g(\delta(u))\right] + \sum_{k=1}^{n} \mathbb{E}\left[g^{(k)}(\delta(u))\Gamma_{k+1}^u G\right] \\
&+ \mathbb{E}\left[G g^{(n+1)}(\delta(u)) \left(\int_{B(R)} \frac{u_x^{n+2}}{(n+1)!}\, \lambda(dx) + \sum_{k=2}^{n+1} \left\langle (\widetilde{\nabla} u)^{n+1-k} u, D \int_{B(R)} \frac{u_x^k}{k!}\, \lambda(dx) \right\rangle \right)\right] \\
&+ \mathbb{E}\left[G g^{(n+1)}(\delta(u)) \langle (\widetilde{\nabla} u)^n u, \delta(\widetilde{\nabla}^* u)\rangle\right].
\end{aligned}$$

Proof. By the duality relation (2.6) between D and δ, the chain rule of derivation for D and the commutation relation (2.14), we get

$$\begin{aligned}
&\mathbb{E}\left[G\langle (\widetilde{\nabla} u)^k u, D\delta(u)\rangle g(\delta(u))\right] - \mathbb{E}\left[G\langle (\widetilde{\nabla} u)^{k+1} u, D\delta(u)\rangle g'(\delta(u))\right] \\
&= \mathbb{E}\left[G\langle (\widetilde{\nabla} u)^k u, u\rangle g(\delta(u))\right] + \mathbb{E}\left[G\langle (\widetilde{\nabla} u)^k u, \delta(\widetilde{\nabla}^* u)\rangle g(\delta(u))\right] - \mathbb{E}\left[G\langle (\widetilde{\nabla} u)^{k+1} u, D\delta(u)\rangle g'(\delta(u))\right] \\
&= \mathbb{E}\left[G\langle (\widetilde{\nabla} u)^k u, u\rangle g(\delta(u))\right] + \mathbb{E}\left[\langle \widetilde{\nabla}^* u, D(Gg(\delta(u))(\widetilde{\nabla} u)^k u]\rangle\right) - \mathbb{E}\left[G\langle (\widetilde{\nabla} u)^{k+1} u, D\delta(u)\rangle g'(\delta(u))\right] \\
&= \mathbb{E}\left[G\langle (\widetilde{\nabla} u)^k u, u\rangle g(\delta(u))\right] + \mathbb{E}\left[\langle (\widetilde{\nabla} u)^{k+1} u, DG\rangle g(\delta(u))\right] + \mathbb{E}\left[G\langle \widetilde{\nabla}^* u, D((\widetilde{\nabla} u)^k u)\rangle g(\delta(u))\right] \\
&= \mathbb{E}\left[g(\delta(u))\Gamma_{k+2}^u G\right],
\end{aligned}$$

where we used (2.9) and (2.13). Therefore, we have

$$\begin{aligned}
&\mathbb{E}[G\delta(u)g(\delta(u))] = \mathbb{E}[\langle u, D(Gg(\delta(u)))\rangle] \\
&= \mathbb{E}[G\langle u, D\delta(u)\rangle g'(\delta(u))] + \mathbb{E}[\langle u, DG\rangle g(\delta(u))] \\
&= \mathbb{E}[\langle u, DG\rangle g(\delta(u))] + \mathbb{E}\left[G g^{(n+1)}(\delta(u))\langle (\widetilde{\nabla} u)^n u, D\delta(u)\rangle\right] \\
&\quad + \sum_{k=0}^{n-1} \left(\mathbb{E}\left[G g^{(k+1)}(\delta(u))\langle (\widetilde{\nabla} u)^k u, D\delta(u)\rangle\right] - \mathbb{E}\left[G g^{(k+2)}(\delta(u))\langle (\widetilde{\nabla} u)^{k+1} u, D\delta(u)\rangle\right]\right) \\
&= \mathbb{E}[\langle u, DG\rangle g(\delta(u))] + \sum_{k=1}^{n} \mathbb{E}\left[g^{(k)}(\delta(u))\Gamma_{k+1}^u G\right] + \mathbb{E}\left[G g^{(n+1)}(\delta(u))\langle (\widetilde{\nabla} u)^n u, D\delta(u)\rangle\right]
\end{aligned}$$

$$= \mathbb{E}[\langle u, DG\rangle g(\delta(u))] + \sum_{k=1}^{n} \mathbb{E}\left[g^{(k)}(\delta(u))\Gamma_{k+1}^{u} G\right]$$
$$+ \mathbb{E}\left[Gg^{(n+1)}(\delta(u))\langle(\widetilde{\nabla}u)^n u, u\rangle\right] + \mathbb{E}\left[Gg^{(n+1)}(\delta(u))\langle(\widetilde{\nabla}u)^n u, \delta(\widetilde{\nabla}^* u)\rangle\right],$$

and we conclude by Lemma 3.3. □

When $f \in W_0^{1,1}(B(R))$ is a deterministic function such that $\|\nabla^{\mathbb{R}^d} f\|_\infty < \infty$, and $g \in \mathcal{C}_b^\infty(\mathbb{R})$, Proposition 4.1 shows that

$$\begin{aligned}
&\mathbb{E}\left[\delta(f)g(\delta(f))\right] \\
&= \sum_{k=1}^{n+1} \frac{1}{k!}\int_{B(R)} f^{k+1}(x)\,\lambda(dx)\,\mathbb{E}[g^{(k)}(\delta(f))] + \mathbb{E}\left[g^{(n+1)}(\delta(f))\langle(\widetilde{\nabla}f)^n f, \delta(\widetilde{\nabla}^* f)\rangle\right] \\
&= \sum_{k=1}^{n+1} \frac{1}{k!}\kappa_{k+1}^{f}\,\mathbb{E}[g^{(k)}(\delta(f))] + \mathbb{E}\left[g^{(n+1)}(\delta(f))\delta((\widetilde{\nabla}f)^{n+1} f)\right], \qquad n \geq 0,
\end{aligned}$$

with, by Proposition 3.1 applied with $p = 2$ and $r = 0$,

$$\begin{aligned}
\mathbb{E}\left[|\delta((\widetilde{\nabla}f)^{n+1} f)|\right] &\leq \sqrt{\mathbb{E}\left[|\delta((\widetilde{\nabla}f)^{n+1} f)|^2\right]} \\
&= \|(\widetilde{\nabla}f)^{n+1} f\|_{L^2(B(R))} \\
&\leq (K_d v_d R')^{n+1}\|f\|_{L^2(B(R))}\|\widetilde{\nabla}f\|_{L^\infty(B(R);\mathbb{R}^d)}^{n+1}.
\end{aligned}$$

In addition, as n tends to $+\infty$ we have

$$\begin{aligned}
\mathbb{E}\left[\delta(f)g(\delta(f))\right] &= \sum_{k=1}^{\infty} \frac{1}{k!}\int_{B(R)} f^{k+1}(x)\,\lambda(dx)\,\mathbb{E}\left[g^{(k)}(\delta(f))\right] \\
&= \sum_{k=1}^{\infty} \frac{1}{k!}\int_{B(R)} f^{k+1}(x)\,\lambda(dx)\,\mathbb{E}\left[g^{(k)}(\delta(f))\right] \\
&= \mathbb{E}\left[\int_{B(R)} f(x)\big(g(\delta(f) + f(x)) - g(\delta(f))\big)\lambda(dx)\right]
\end{aligned}$$

provided that the derivatives of g decay fast enough, which is a particular instance of the standard integration by parts identity for finite difference operators on the Poisson space, see e.g. Lemma 2.9 in [15] or Lemma 5 in [4].

5 Stein approximation

Applying Proposition 4.1 with $n = 0$ and $G = 1$ to the solution g_x of the Stein equation

$$\mathbf{1}_{(-\infty,x]}(z) - \Phi(z) = g_x'(z) - zg_x(z), \qquad z \in \mathbb{R},$$

and letting $u \in \widetilde{\mathbb{D}}_0^{1,1}$ be a predictable random field, this gives the expansion

$$\begin{aligned} P(\delta(u) \le x) - \Phi(x) &= \mathbb{E}\left[g_x'(\delta(u))\langle u, u\rangle - \delta(u) g_x(\delta(u))\right] \\ &= \mathbb{E}\left[(1 - \langle u, u\rangle) g_x'(\delta(u))\right] + \mathbb{E}\left[\langle u, \delta(\widetilde{\nabla} u)\rangle g_x'(\delta(u))\right], \end{aligned} \tag{5.1}$$

around the Gaussian cumulative distribution function $\Phi(x)$, with $\|g_x\|_\infty \le \sqrt{2\pi}/4$ and $\|g_x'\|_\infty \le 1$, $x \in \mathbb{R}$, by Lemma 2.2-(v) of [6]. The next result follows from the application of Proposition 4.1 with $n = 1$ and $G = 1$.

Proposition 5.1 *For any random field $u \in \widetilde{\mathbb{D}}_0^{1,\infty}$ we have*

$$\begin{aligned} & d_W(\delta(u), \mathcal{N}) \\ &\le \mathbb{E}\left[|1 - \langle u, u\rangle - \langle \widetilde{\nabla}^* u, Du\rangle|\right] + \mathbb{E}\left[\left|\int_{B(R)} u_x^3\, \lambda(dx) + \left\langle u, D\int_{B(R)} u_x^2\, \lambda(dx)\right\rangle\right|\right] \\ &\quad + 2\,\mathbb{E}\left[|\langle (\widetilde{\nabla} u) u, \delta(\widetilde{\nabla}^* u)\rangle|\right]. \end{aligned} \tag{5.2}$$

Proof. For $n = 1$ and $G = 1$, Proposition 4.1 shows that

$$\begin{aligned} \mathbb{E}[\delta(u) g(\delta(u))] &= \mathbb{E}[g'(\delta(u))(\langle u, u\rangle + \langle \widetilde{\nabla}^* u, Du\rangle)] \\ &\quad + \frac{1}{2}\mathbb{E}\left[g''(\delta(u))\left(\int_{B(R)} u_x^3\, \lambda(dx) + \left\langle u, D\int_{B(R)} u_x^2\, \lambda(dx)\right\rangle\right)\right] \\ &\quad + \mathbb{E}[g''(\delta(u))\langle (\widetilde{\nabla} u) u, \delta(\widetilde{\nabla} u)\rangle]. \end{aligned}$$

Let $h : \mathbb{R} \to [0, 1]$ be a continuous function with bounded derivative. Using the solution $g_h \in \mathcal{C}_b^1(\mathbb{R})$ of the Stein equation

$$h(z) - \mathbb{E}[h(\mathcal{N})] = g'(z) - z g(z), \qquad z \in \mathbb{R},$$

with the bounds $\|g_h'\|_\infty \le \|h'\|_\infty$ and $\|g_h''\|_\infty \le 2\|h'\|_\infty$, $x \in \mathbb{R}$, cf. Lemma 1.2-(v) of [12] and references therein, we have

$$\begin{aligned} \mathbb{E}[h(\delta(u))] - \mathbb{E}[h(\mathcal{N})] &= \mathbb{E}[\delta(u) g_h(\delta(u)) - g_h'(\delta(u))] \\ &= \mathbb{E}[g_h'(\delta(u))(\langle u, u\rangle + \langle \widetilde{\nabla}^* u, Du\rangle - 1)] \\ &\quad + \frac{1}{2}\mathbb{E}\left[g''(\delta(u))\left(\int_{B(R)} u_x^3\, \lambda(dx) + \left\langle u, D\int_{B(R)} u_x^2\, \lambda(dx)\right\rangle\right)\right] \\ &\quad + 2\,\mathbb{E}[g_h''(\delta(u))\langle (\widetilde{\nabla} u) u, \delta(\widetilde{\nabla}^* u)\rangle], \end{aligned}$$

hence

$$
\begin{aligned}
|\,\mathbb{E}[\delta(u)h(\delta(u))] - \mathbb{E}[h(\mathcal{N})]| \;\; \leq \;\; & \|h'\|_\infty\, \mathbb{E}\left[|1 - \langle u, u\rangle - \langle \widetilde{\nabla}^* u, Du\rangle|\right] \\
& + \|h'\|_\infty\, \mathbb{E}\left[\left|\int_{B(R)} u_x^3\, \lambda(dx) + \left\langle u, D\int_{B(R)} u_x^2\, \lambda(dx)\right\rangle\right|\right] \\
& + 2\|h'\|_\infty\, \mathbb{E}\left[|\langle(\widetilde{\nabla} u)u, \delta(\widetilde{\nabla}^* u)\rangle|\right],
\end{aligned}
$$

which yields (5.2). □

As a consequence of Proposition 5.1 and the Itô isometry (2.12) we have the following corollary.

Corollary 5.2 *For $u \in \widetilde{\mathbb{D}}_0^{1,\infty}$ we have*

$$
\begin{aligned}
d_W(\delta(u), \mathcal{N}) \;\; \leq \;\; & |1 - \mathrm{Var}[\delta(u)]| + \sqrt{\mathrm{Var}\left[\|u\|^2_{L^2(B(R))}\right]} \\
& + \mathbb{E}\left[\left|\int_{B(R)} u_x^3\, \lambda(dx) + \left\langle u, D\int_{B(R)} u_x^2\, \lambda(dx)\right\rangle\right|\right] \\
& + \mathbb{E}[|\langle \widetilde{\nabla}^* u, Du\rangle|] + 2\,\mathbb{E}\left[|\langle(\widetilde{\nabla} u)u, \delta(\widetilde{\nabla}^* u)\rangle|\right].
\end{aligned}
$$

Proof. By the Itô isometry (2.12) we have

$$
\mathrm{Var}[\delta(u)] = \mathbb{E}\left[\left(\int_{B(R)} u_x(\gamma(dx) - \lambda(dx))\right)^2\right] = \mathbb{E}[\langle u, u\rangle],
$$

hence

$$
\begin{aligned}
& \mathbb{E}\left[|1 - \langle u, u\rangle - \langle \widetilde{\nabla}^* u, Du\rangle|\right] \\
& \quad \leq \;\; \mathbb{E}\left[|1 - \mathbb{E}[\langle u, u\rangle]|\right] + \mathbb{E}\left[|\langle u, u\rangle - \mathbb{E}[\langle u, u\rangle]|\right] + \mathbb{E}[|\langle \widetilde{\nabla}^* u, Du\rangle|] \\
& \quad = \;\; |1 - \mathrm{Var}[\delta(u)]| + \sqrt{\mathbb{E}[(\langle u, u\rangle - \mathbb{E}[\langle u, u\rangle])^2]} + \mathbb{E}[|\langle \widetilde{\nabla}^* u, Du\rangle|] \\
& \quad = \;\; |1 - \mathrm{Var}[\delta(u)]| + \sqrt{\mathrm{Var}\left[\|u\|^2_{L^2(B(R))}\right]} + \mathbb{E}[|\langle \widetilde{\nabla}^* u, Du\rangle|].
\end{aligned}
$$

□

In particular, when $\mathrm{Var}[\delta(u)] = 1$, Corollary 5.2 shows that

$$
\begin{aligned}
d_W(\delta(u), \mathcal{N}) \;\; \leq \;\; & \sqrt{\mathrm{Var}\left[\|u\|^2_{L^2(B(R))}\right]} + \mathbb{E}\left[\left|\int_{B(R)} u_x^3\, \lambda(dx) + \left\langle u, D\int_{B(R)} u_x^2\, \lambda(dx)\right\rangle\right|\right] \\
& + \mathbb{E}[|\langle \widetilde{\nabla}^* u, Du\rangle|] + 2\,\mathbb{E}\left[|\langle(\widetilde{\nabla} u)u, \delta(\widetilde{\nabla}^* u)\rangle|\right].
\end{aligned}
$$

When $f \in W_0^{1,\infty}(B(R))$ is a deterministic function we have

$$\mathrm{Var}[\delta(f)] = \mathbb{E}\left[\left(\int_{B(R)} f(x)(\gamma(dx) - \lambda(dx))\right)^2\right] = \int_{B(R)} f^2(x)\,\lambda(dx),$$

and Corollary 5.1 shows that

$$d_W(\delta(f), \mathcal{N}) \le \left|1 - \int_{B(R)} f^2(x)\,\lambda(dx)\right| + \left|\int_{B(R)} f^3(x)\,\lambda(dx)\right| + 2\,\mathbb{E}\left[|\delta((\widetilde{\nabla} f)^2 f)|\right].$$

Given the bound

$$\begin{aligned}
\mathbb{E}\left[|\delta((\widetilde{\nabla} f)^2 f)|\right] &\le \sqrt{\mathbb{E}\left[|\delta((\widetilde{\nabla} f)^2 f)|^2\right]} \\
&= \|(\widetilde{\nabla} f)^2 f\|_{L^2(B(R))} \\
&\le (K_d v_d R')^2 \|f\|_{L^2(B(R))} \|\nabla^{\mathbb{R}^d} f\|^2_{L^\infty(B(R);\mathbb{R}^d)}
\end{aligned}$$

obtained from Proposition 3.1 with $p = 2$ and $r = 0$, $f \in W_0^{1,\infty}(B(R))$, we also have the following corollary.

Corollary 5.3 *For $f \in W_0^{1,\infty}(B(R))$ we have*

$$\begin{aligned}
d_W\left(\int_{B(R)} f(x)(\gamma(dx) - \lambda(dx)), \mathcal{N}\right) &\le \left|1 - \|f\|^2_{L^2(B(R))}\right| + \left|\int_{B(R)} f^3(x)\,\lambda(dx)\right| \\
&\quad + 2(K_d v_d R')^2 \|f\|_{L^2(B(R))} \|\nabla^{\mathbb{R}^d} f\|^2_{L^\infty(B(R);\mathbb{R}^d)}.
\end{aligned}$$

In particular, if $\|f\|_{L^2(B(R))} = 1$ we find

$$d_W\left(\int_{B(R)} f(x)(\gamma(dx) - \lambda(dx)), \mathcal{N}\right) \le \left|\int_{B(R)} f^3(x)\,\lambda(dx)\right| + 2(K_d v_d R')^2 \|\nabla^{\mathbb{R}^d} f\|^2_{L^\infty(B(R);\mathbb{R}^d)}.$$

As an example, consider f_k given on $B(k^{1/d}R)$ by

$$f_k(x) := \frac{1}{C\sqrt{k}} g\left(\frac{|x|_{\mathbb{R}^d}}{k^{1/d}}\right), \qquad x \in B(k^{1/d}R),$$

where $g \in \mathcal{C}^1([0,R])$ is such that $g(R) = 0$, and

$$C^2 := v_d \int_0^R g^2(r) r^{d-1} dr,$$

so that $f_k \in L^2(B(k^{1/d}R))$ with

$$\|f\|^2_{L^2(B(k^{1/d}R))} = \frac{v_d}{C^2 k} \int_0^{k^{1/d}R} g^2\left(\frac{r}{k^{1/d}}\right) r^{d-1} dr = \frac{v_d}{C^2} \int_0^R g^2(r) r^{d-1} dr = 1,$$

and

$$\int_{B(k^{1/d}R)} f_k^3(x)dx = \frac{1}{C^3 k^{3/2}} \int_0^{k^{1/d}R} g^3(rk^{-1/d}) r^{d-1} dr = \frac{1}{C^3\sqrt{k}} \int_0^R g^3(r) r^{d-1} dr,$$

$k \geq 1$. We have

$$\|\nabla^{\mathbb{R}^d} f_k\|^2_{L^\infty(B(R);\mathbb{R}^d)} \leq \frac{\|g'\|_\infty^2 d}{C^2 k^{1+2/d}},$$

hence

$$\begin{aligned} d_W\left(\int_{B(R)} f_k(x)(\gamma(dx)-\lambda(dx)), \mathcal{N}\right) &\leq \left|\int_{B(R)} f_k^3(x)\,\lambda(dx)\right| + \frac{2(K_d v_d k^{1/d} R')^2 d}{k^{1+2/d} C^2} \|g'\|_\infty^2 \\ &\leq \frac{v_d}{C^3\sqrt{k}} \left|\int_0^R g^3(r) r^{d-1} dr\right| + \frac{2(K_d v_d R')^2 d}{kC^2} \|g'\|_\infty^2. \end{aligned}$$

In particular, if g satisfies the condition

$$\int_0^R g^3(r) r^{d-1} dr = 0,$$

then we find the $O(1/k)$ convergence rate

$$d_W\left(\int_{B(R)} f_k(x)(\gamma(dx)-\lambda(dx)), \mathcal{N}\right) \leq \frac{2(K_d v_d R')^2 d}{kC^2} \|g'\|_\infty^2, \qquad k \geq 1.$$

For example, taking

$$f_k(x) := \frac{1}{C\sqrt{k}} g\left(\frac{|x|_{\mathbb{R}^d}}{k^{1/d}}\right) = \frac{1}{C\sqrt{k}}\left(h_1\left(\frac{|x|_{\mathbb{R}^d}}{k^{1/d}}\right) - a h_2\left(\frac{|x|_{\mathbb{R}^d}}{k^{1/d}}\right)\right), \quad x \in B(k^{1/d}R),$$

with $a \in \mathbb{R}$, $h_1, h_2 \in \mathcal{C}^1([0,R])$ such that $h_1(R) = h_2(R) = 0$, and

$$C^2 := \int_0^R (h_1(r) - a h_2(r))^2 r^{d-1} dr > 0,$$

we can choose $a \in \mathbb{R}$ satisfying the cubic equation

$$\begin{aligned} &\int_{B(R)} g^3(r) r^{d-1} dr \\ &= a^3 \int_0^R h_2^3(r) r^{d-1} dr + 3a^2 \int_0^R h_1(r) h_2^2(r) r^{d-1} dr - 3a \int_0^R h_1^2(r) h_2(r) r^{d-1} dr + \int_0^R h_1^3(r) r^{d-1} dr \\ &= 0, \end{aligned}$$

which yields the bound

$$d_W\left(\int_{B(k^{1/d}R)} f_k(x)(\gamma(dx)-\lambda(dx)), \mathcal{N}\right) \leq \frac{c(a,d,h_1,h_2)}{k}, \qquad k \geq 1,$$

from (1.5), where $c(a,d,h_1,h_2)$ depends only on $a \in \mathbb{R}$, $d \geq 2$ and $h_1, h_2 \in \mathcal{C}^1([0,R])$, whereas (1.3) can only yield the standard Berry-Esseen convergence rate (1.4) as $\int_0^R |g(r)|^3 r^{d-1} dr > 0$.

6 Appendix

Proof of Proposition 2.3.

As a consequence of (1.8) and (2.2) we have

$$\begin{aligned}
&f_n(x_1,\ldots,x_{i-1},y,x_{i+1},\ldots x_n) - f_{n-1}(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n) \\
&= f_n(x_1,\ldots,x_{i-1},y,x_{i+1},\ldots x_n) - f_{n-1}(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n)\int_{B(R')\setminus B(R)} \eta(x)\,\lambda(dx) \\
&= f_n(x_1,\ldots,x_{i-1},y,x_{i+1},\ldots x_n) - \int_{B(R')\setminus B(R)} \eta(x) f_n(x_1,\ldots,x_{i-1},x,x_{i+1},\ldots,x_n)\,\lambda(dx) \\
&= \int_{B(R')} \langle G(x_i,y), \nabla^{\mathbb{R}^d}_{x_i} f_n(x_1,\ldots,x_n)\rangle_{\mathbb{R}^d}\lambda(dx_i) \\
&= \int_{B(R)} \langle G(x_i,y), \nabla^{\mathbb{R}^d}_{x_i} f_n(x_1,\ldots,x_n)\rangle_{\mathbb{R}^d}\lambda(dx_i), \qquad (6.1)
\end{aligned}$$

$x_1,\ldots,x_{i-1},y,x_{i+1},\ldots,x_n \in B(R')$. Recall that for all $F \in \mathcal{S}$ of the form (2.1) we have

$$\mathbb{E}[F] = e^{-\lambda(B(R))}\sum_{n=0}^{\infty}\frac{1}{n!}\int_{B(R)}\cdots\int_{B(R)} f_n(x_1,\ldots,x_n)\,\lambda(dx_1)\cdots\lambda(dx_n).$$

Hence, using (6.1), for $g \in \mathcal{C}_0^1(B(R))$ and F of the form (2.1) we have

$$\begin{aligned}
&\mathbb{E}\left[\int_{B(R)} g(y) D_y F\,\lambda(dy)\right] \\
&= \mathbb{E}\left[\sum_{n=1}^{\infty}\mathbf{1}_{\{\gamma(B(R))=n\}}\sum_{i=1}^{n}\int_{B(R)} g(y)\langle \mathsf{G}_\eta(X_i,y), \nabla^{\mathbb{R}^d}_{X_i} f(X_1,\ldots,X_n)\rangle_{\mathbb{R}^d}\lambda(dy)\right] \qquad (6.2) \\
&= e^{-\lambda(B(R))}\sum_{n=1}^{\infty}\frac{1}{n!}\int_{B(R)}\cdots\int_{B(R)}\sum_{i=1}^{n}\int_{B(R)} g(y)\langle \mathsf{G}_\eta(x_i,y), \nabla^{\mathbb{R}^d}_{x_i} f_n(x_1,\ldots,x_n)\rangle_{\mathbb{R}^d}\lambda(dy)\lambda(dx_1)\cdots\lambda(dx_n) \\
&= e^{-\lambda(B(R))}\sum_{n=1}^{\infty}\frac{1}{n!}\int_{B(R)}\cdots \\
&\quad \cdots\int_{B(R)}\sum_{i=1}^{n}\int_{B(R)} g(y) f_n(x_1,\ldots,x_{i-1},y,x_{i+1},\ldots,x_n)\,\lambda(dx_1)\cdots\lambda(dy)\cdots\lambda(dx_n) \\
&\quad - e^{-\lambda(B(R))}\sum_{n=1}^{\infty}\frac{1}{n!}\int_{B(R)}\cdots\int_{B(R)}\sum_{i=1}^{n}\int_{B(R)} g(y)\,\lambda(dy) f_{n-1}(x_1,\ldots,x_{n-1})\,\lambda(dx_1)\cdots\lambda(dx_{n-1}) \\
&= e^{-\lambda(B(R))}\sum_{n=1}^{\infty}\frac{1}{n!}\int_{B(R)}\cdots\int_{B(R)}\left(\sum_{i=1}^{n} g(x_i) - \int_{B(R)} g(y)\,\lambda(dy)\right) f_n(x_1,\ldots,x_n)\,\lambda(dx_1)\cdots\lambda(dx_n)
\end{aligned}$$

$$= \mathbb{E}\left[F \left(\int_{B(R)} g(x)(\gamma(dx) - \lambda(dx)) \right) \right].$$

Next, for u of the form (2.4), we check by a standard argument that

$$\begin{aligned}
\mathbb{E}[\langle u, DF \rangle] &= \sum_{i=1}^{n} \mathbb{E}[G_i \langle g_i, DF \rangle] \\
&= \sum_{i=1}^{n} (\mathbb{E}[\langle g_i, D(FG_i) \rangle - F \langle g_i, DG_i \rangle]) \\
&= \mathbb{E}\left[F \sum_{i=1}^{n} \left(G_i \int_{B(R)} g_i(x)(\gamma(dx) - \lambda(dx)) - \langle g_i, DG_i \rangle \right) \right] \\
&= \mathbb{E}[F \delta(u)].
\end{aligned}$$

□

Proof of Proposition 2.4. Taking $u \in \mathcal{P}_0$ a predictable random field of the form (2.7) we note that by (2.3) and the compatibility condition (2.10) we have

$$g_i(y) D_y F_i = 0, \qquad y \in B(R), \quad i = 1, \ldots, n,$$

hence by (2.5) we have

$$\begin{aligned}
\delta(u) &= \delta\left(\sum_{i=1}^{n} F_i g_i \right) = \sum_{i=1}^{n} F_i \delta(g_i) \qquad (6.3) \\
&= \sum_{i=1}^{n} F_i \int_{B(R)} g_i(x)(\gamma(dx) - \lambda(dx)) \\
&= \int_{B(R)} u_x(\gamma(dx) - \lambda(dx)),
\end{aligned}$$

showing that $\delta(u)$ coincides with the Poisson stochastic integral of $(u_x)_{x \in B(R)}$. Regarding the isometry relation (2.12), we have

$$\begin{aligned}
\mathbb{E}[\delta(u)^2] &= \mathbb{E}\left[\left(\sum_{i=1}^{n} F_i \int_{B(R)} g_i(x)(\gamma(dx) - \lambda(dx)) \right)^2 \right] \\
&= \mathbb{E}\left[\sum_{i,j=1}^{n} F_i F_j \int_{B(R)} g_i(x)(\gamma(dx) - \lambda(dx)) \int_{B(R)} g_j(x)(\gamma(dx) - \lambda(dx)) \right]
\end{aligned}$$

$$
\begin{aligned}
&= 2\,\mathbb{E}\left[\sum_{1\le i<j\le n} F_i \int_{B(R)} g_i(x)(\gamma(dx)-\lambda(dx)) F_j \int_{B(R)} g_j(x)(\gamma(dx)-\lambda(dx))\right] \\
&\quad + \mathbb{E}\left[\sum_{i=1}^n F_i^2 \left(\int_{B(R)} g_i(x)(\gamma(dx)-\lambda(dx))\right)^2\right] \\
&= \mathbb{E}\left[\sum_{i=1}^n F_i^2 \int_{B(R)} g_i^2(x)\lambda(dx)\right] \\
&= \mathbb{E}\left[\int_{B(R)} u^2(x)\,\lambda(dx)\right],
\end{aligned}
$$

which shows that (2.11) extends to the closure of $\mathcal{P}_0$ in $L^2(\Omega \times B(R))$ by density and a Cauchy sequence argument. □

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