Numerical computation of Theta in a jump-diffusion model by integration by parts*

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Abstract

Using the Malliavin calculus in time inhomogeneous jump-diffusion models, we obtain an expression for the sensitivity Theta of an option price (with respect to maturity) as the expectation of the option payoff multiplied by a stochastic weight. This expression is used to design efficient numerical algorithms that are compared to traditional finite difference methods for the computation of Theta. Our proof can be viewed as a generalization of Dupire's integration by parts [6] to arbitrary and possibly non-smooth payoff functions. In the time homogeneous case Theta admits an expression from the Black-Scholes PDE in terms of Delta and Gamma but the representation formula obtained in this way is different from ours. Numerical simulations are presented in order to compare the efficiency of the finite difference and Malliavin methods.

Key words: Greeks, Theta, sensitivity analysis, jump-diffusion models, Malliavin calculus.
Classification: 91B28, 60H07.

1 Introduction

Sensitivity analysis in finance using the Malliavin calculus has been developed by several authors, starting with [7], to design fast Monte Carlo algorithms for the computation of Greeks such as Delta, Gamma, Vega, Rho, which represent the sensitivity of option prices to spot price, volatility and interest rate, respectively.

In this paper we aim at applying similar methods to the computation of sensitivities defined as

$$\text{Theta}_t = \frac{\partial C}{\partial t}(x,t,T), \quad \text{and} \quad \text{Theta}_T = \frac{\partial C}{\partial T}(x,t,T),$$

where $C(x,t,T)$ denote the price at time t of an option with spot price x and maturity T. Theta_t is used for European options for which T is a fixed date, whereas Theta_T (the

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sensitivity with respect to maturity) can be used in case T is a free parameter, e.g. for the choice of the exercise date of a European option, or for American type contracts. When the underlying price process $(S_t)_{t\in[0,T]}$ is time homogeneous, the price C becomes a function of the remaining time $\tau := T - t$ until exercise and we have

$$\mathrm{Theta}_T = -\mathrm{Theta}_t = \frac{\partial C}{\partial \tau}(x, t, t+\tau),$$

which will be simply denoted by Theta.

Here we compute Theta_T in a time inhomogeneous setting, using Itô calculus and integration by parts on the Wiener space. Our method actually extends the argument of the Dupire PDE to arbitrary payoff functions in jump-diffusion models. We present a Malliavin type formula for Theta_T which avoids the use of finite differences, and allows us to consider digital and European options as it does not require any smoothness on the payoff function ϕ. The value of Theta for European and digital options in a geometric Brownian model can be computed analytically, cf. e.g. [8], but such expressions are not available in general (jump) diffusion models, for which our formulas can be used in numerical simulations.

We proceed as follows. Section 2 contains a summary of stochastic calculus for jump-diffusion processes and Malliavin calculus on the Wiener space. In Section 3, using the Malliavin calculus, we obtain an expression of Theta_T in a jump-diffusion model with arbitrary payoff functions, using a random weight $\Lambda(u, v, w)$ depending on three functional parameters $u, v, w \in L^2([0, T])$. In Section 4 we determine the parameters which yield the best numerical performance by minimization of the variance of the weight $\Lambda(u, v, w)$, and find that this minimum is attained when u, v, w are constant functions. A localization argument is also applied to further reduce the variance of our Monte Carlo estimators. Monte Carlo simulations for digital and European options are presented in Section 5 to compare the performance of the finite difference method to that of our Malliavin calculus approach, and the values of Theta_t and Theta_T.

2 Malliavin calculus and jump-diffusion processes

In this section we recall some facts and notation on the Malliavin calculus on the Wiener space, cf. e.g. [10], [4], on jump-diffusion models, and on stochastic calculus with jumps, see e.g. [3] for a recent introduction with references.

Consider a standard Brownian motion $(W_t)_{t\in\mathbb{R}_+}$ on (Ω_W, P_W) and a compound Poisson process $(X_t)_{t\in\mathbb{R}_+}$ on (Ω_X, P_X) with Lévy measure $\mu(dy)$ and finite intensity

$$\lambda = \int_{-\infty}^{\infty} y\mu(dy)$$

which can be represented as

$$X_t = \sum_{k=1}^{N_t} U_k, \qquad t \in \mathbb{R}_+, \tag{2.1}$$

where $(N_t)_{t\in\mathbb{R}_+}$ is a standard Poisson process with intensity λ and $(U_k)_{k\geq 1}$ is an i.i.d. sequence of random variables with probability distribution $\nu(dx) := \lambda^{-1}\mu(dx)$. The processes $(W_t)_{t\in\mathbb{R}_+}$ and $(X_t)_{t\in\mathbb{R}_+}$ are assumed to be independent and are constructed on the product probability space

$$(\Omega, P) = (\Omega_W \times \Omega_X, P_W \otimes P_X).$$

The filtration generated by $(W_t, X_t)_{t\in\mathbb{R}_+}$ is denoted by $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$.

We consider the gradient and divergence operators D and δ acting on the continuous component of jump-diffusion random functionals. Let $D : L^2(\Omega) \to L^2(\Omega \times \mathbb{R}_+)$ denote the (unbounded) Malliavin gradient D on the Wiener space, i.e.

$$D_t F(\omega_W, \omega_X) := \sum_{k=1}^{n} \mathbf{1}_{[0,t_k]}(t)\partial_k f(W_{t_1}, \ldots, W_{t_n}, \omega_X)$$

for F a random variable of the form

$$F(\omega_W, \omega_X) = f(W_{t_1}, \ldots, W_{t_n}, \omega_X),$$

where $f(\cdot, \omega_X) \in \mathcal{C}_b^\infty(\mathbb{R}^n)$, $P_X(d\omega_X)$-a.s., is uniformly bounded on $\mathbb{R}^n \times \Omega_X$. Denote by $\langle \cdot, \cdot \rangle_{L^2(\mathbb{R}_+)}$ and $\| \cdot \|$ the scalar product and norm in $L^2(\mathbb{R}_+)$, and define

$$D_u F := \langle u, DF \rangle, \quad u \in L^2(\Omega \times \mathbb{R}_+),$$

by abuse of notation. Given a symmetric function $g_n \in L^2(\mathbb{R}_+^n \times \Omega_X)$, let

$$I_n(g_n)(\omega_W, \omega_X) = n! \int_0^\infty \int_0^{t_n} \cdots \int_0^{t_2} g_n(t_1, \ldots, t_n, \omega_X) dW_{t_1} \cdots dW_{t_n}$$

denote the multiple stochastic integral of g_n with respect to Brownian motion $(W_t)_{t\in\mathbb{R}_+}$, with the isometry formula

$$\mathbb{E}[I_n(g_n)I_m(g_m)] = n!\mathbf{1}_{\{n=m\}}\mathbb{E}[\langle g_n, g_m \rangle_{L^2(\mathbb{R}_+^n)}], \tag{2.2}$$

$g_n \in L^2(\mathbb{R}_+^n \times \Omega_X)$, $g_m \in L^2(\mathbb{R}_+^m \times \Omega_X)$. The (unbounded) divergence operator $\delta : L^2(\Omega \times \mathbb{R}_+) \to L^2(\Omega)$ adjoint of D, also called the Skorohod integral, satisfies the duality relation

$$\mathbb{E}[\langle DF, u\rangle] = \mathbb{E}[F\delta(u)], \qquad u \in \mathrm{Dom}\,(\delta), \quad F \in \mathrm{Dom}\,(D),$$

where $\mathrm{Dom}\,(D)$ and $\mathrm{Dom}\,(\delta)$ denote the respective closed domains of D and δ.

Recall also the following lemma, cf. Proposition 1.3.3 of [10].

Lemma 2.1 *Let $u \in \mathrm{Dom}\,(\delta)$ and $F \in \mathrm{Dom}\,(D)$ be such that $uF \in L^2(\Omega \times \mathbb{R}_+)$ and $F\delta(u) - \langle u, DF\rangle \in L^2(\Omega)$. Then $uF \in \mathrm{Dom}\,(\delta)$ and we have the divergence formula*

$$\delta(uF) = F\delta(u) - \langle u, DF\rangle \tag{2.3}$$

Recall also that δ coincides with Itô's stochastic integral on square-integrable adapted processes, in particular

$$\delta(u) = \int_0^\infty u_t dW_t$$

for all adapted and square-integrable process $(u_t)_{t\in\mathbb{R}_+}$, and $\delta(u) = I_1(u)$, $u \in L^2(\mathbb{R}_+)$, cf. e.g. [10].

We will consider Markovian jump-diffusion price processes given as solutions to the equation

$$\begin{cases} dS_t = a_t(S_t)dt + b_t(S_t)dW_t + c_t(S_{t^-})dX_t, \\ \\ S_0 = x, \end{cases}$$

where $a_t(\cdot)$, $b_t(\cdot)$, $c_t(\cdot)$ are $\mathcal{C}^1$ Lipschitz functions, uniformly in $t \in [0,T]$, $T > 0$.

Itô's formula for $(S_t)_{t\in\mathbb{R}_+}$ reads

$$\begin{aligned}\phi(S_t, t) = \phi(S_s, s) &+ \int_s^t \frac{\partial\phi}{\partial u}(S_u, u)du + \int_s^t \frac{\partial\phi}{\partial x}(S_u, u)a_u(S_u)du + \int_s^t \frac{\partial\phi}{\partial x}(S_u, u)b_u(S_u)dW_u \\ &+\frac{1}{2}\int_s^t \frac{\partial^2\phi}{\partial x^2}(S_u, u)b_u^2(S_u)du + \sum_{s<u\le t}\left(\phi(S_{u^-} + c_u(S_{u^-})\Delta X_u, u^-) - \phi(S_{u^-}, u^-)\right),\end{aligned}$$

$0 \le s \le t$. Recall also that since μ is the Lévy measure of $(X_t)_{t\in\mathbb{R}_+}$ we have

$$\begin{aligned}&\mathbb{E}\left[\sum_{s<u\le t}\left(\phi(S_{u^-} + c_u(S_{u^-})\Delta X_u, u^-) - \phi(S_{u^-}, u^-)\right)\right] \\ &\quad = \lambda\mathbb{E}\left[\int_s^t\int_{-\infty}^\infty \left(\phi(S_u + zc_u(S_u), u) - \phi(S_u, u)\right)\nu(dz)ds\right],\end{aligned} \tag{2.4}$$

for sufficiently integrable ϕ, cf. e.g. [3] and references therein.

Consider the option price

$$C(x,t,T) := e^{-\int_t^T r_s ds}\mathbb{E}\left[\phi(S_T)\Big|S_t = x\right].$$

We will assume that the discounted price process $(e^{-\int_0^t r_s ds}S_t)_{t\in\mathbb{R}_+}$ is a martingale, i.e. we work under the no arbitrage condition

$$a_t(y) - r_t y + \lambda c_t(y) = 0, \qquad y \in \mathbb{R}, \tag{2.5}$$

which can be realized in multiple ways, depending on the choice of the triple (a_t, λ, c_t). As is well known, this means that jump-diffusion markets are not complete in general, see e.g. Ch. IX of [3] and references therein. Using the fact that $t \mapsto e^{\int_t^T r_s ds}C(S_t,t,T)$ is a martingale, Itô's formula shows that $C(S_t,t,T)$ satisfies the Black-Scholes PDE

$$\begin{aligned} \text{Theta}_t \;&=\; r_t C(x,t,T) - a_t(x)\frac{\partial C}{\partial x}(x,t,T) - \frac{1}{2}b_t^2(x)\frac{\partial^2 C}{\partial x^2}(x,t,T) \\ &\quad -\lambda\int_{-\infty}^{\infty}(C(x+zc_t(x),t,T)) - C(x,t,T))\nu(dz). \end{aligned} \tag{2.6}$$

By differentiation inside the expectation, Delta and Gamma can be written as

$$\text{Delta} := \frac{\partial C}{\partial x}(x,t,T) = e^{-\int_t^T r_s ds}\mathbb{E}\left[Y_T\phi'(S_T)\Big|S_t = x\right] \tag{2.7}$$

and

$$\text{Gamma} := \frac{\partial^2 C}{\partial x^2}(x,t,T) = e^{-\int_t^T r_s ds}\mathbb{E}\left[Z_T\phi'(S_T) + (Y_T)^2\phi''(S_T)\Big|S_t = x\right], \tag{2.8}$$

where $(Y_t)_{t\in\mathbb{R}_+} := (\partial_x S_t)_{t\in\mathbb{R}_+}$ and $(Z_t)_{t\in\mathbb{R}_+} := (\partial_x^2 S_t)_{t\in\mathbb{R}_+}$ are the so called first and second variation processes. Recall that (2.7) can be rewritten as

$$\text{Delta} = \frac{\partial C}{\partial x}(x,t,T) = \frac{e^{-\int_t^T r_s ds}}{T-t}\mathbb{E}\left[\phi(S_T)\int_t^T \frac{Y_s}{b_s(S_s)}dW_s\Big|S_t = x\right], \tag{2.9}$$

by integration by parts using the Malliavin calculus, and similarly for Gamma, cf. e.g. [7], [1], [4], [5].

However, this Black-Scholes PDE approach does not apply to the computation of Theta_T in time inhomogeneous models, for which we provide a different method in Section 3.

3 Computation of $\mathbf{Theta_T}$

Consider an option with payoff function ϕ and price

$$C(x,t,T) = e^{-\int_t^T r_s ds}\mathbb{E}\left[\phi(S_T)\Big|S_t = x\right].$$

Recall that in the case of European options in a continuous market, i.e. with $c_t(\cdot) = 0$, $a_t(y) = \alpha_t y$, and payoff function $\phi(x) = (x - K)^+$, Dupire's PDE [6] reads

$$\begin{aligned}\mathrm{Theta}_T &= \frac{\partial C}{\partial T}(x,t,T,K) \\ &= (\alpha_T - r_T)C(x,t,T,K) + \frac{K^2 b_T^2(K)}{2}\frac{\partial^2 C}{\partial K^2}(x,t,T,K) - K\alpha_T\frac{\partial C}{\partial K}(x,t,T,K),\end{aligned} \tag{3.1}$$

and can be proved by taking expectations on both sides of Itô's formula applied to $C(x,t,T,K)$, differentiation with respect to T, and finally integrating by parts with respect to the Lebesgue measure on $\mathbb{R}$.

Our computation of Theta_T will follow the same steps, replacing integration by parts on $\mathbb{R}$ with the duality between D and δ on the Wiener space, to extend Dupire's argument to arbitrary payoff functions and jump-diffusion models. Consider $(S_{t,s}^x)_{s\in[t,\infty)}$ given by the jump-diffusion equation

$$\begin{cases} dS_{t,s}^x = a_s(S_{t,s}^x)ds + b_s(S_{t,s}^x)dW_s + c_s(S_{t,s^-}^x)dX_s, \\ \\ S_{t,t}^x = x, \end{cases} \tag{3.2}$$

where in addition, $a_t(x)$ and $b_t(x)$ are respectively $\mathcal{C}^1$ and $\mathcal{C}^2$ in x, for all $t \in \mathbb{R}_+$. The derivative with respect to T can be put inside the expectation if ϕ is differentiable. Using Itô's formula, Relation (2.4), and the fact that the expectation of the stochastic integral with respect to $(W_t)_{t\in\mathbb{R}_+}$ is zero, we have:

$$\begin{aligned} C(x,t,T) &= e^{-\int_t^T r_s ds}\mathbb{E}\left[\phi(S_{t,T}^x)\right] \\ &= \phi(x) - \mathbb{E}\left[\int_t^T r_s e^{-\int_t^s r_p dp}\phi(S_{t,s}^x)ds\right] \\ &\quad +\mathbb{E}\left[\int_t^T e^{-\int_t^s r_p dp}\phi'(S_{t,s}^x)a_s(S_{t,s}^x)ds\right] + \mathbb{E}\left[\int_t^T e^{-\int_t^s r_p dp}\phi'(S_{t,s}^x)b_s(S_{t,s}^x)dW_s\right] \\ &\quad +\frac{1}{2}\mathbb{E}\left[\int_t^T e^{-\int_t^s r_p dp}\phi''(S_{t,s}^x)b_s^2(S_{t,s}^x)ds\right] \\ &\quad +\mathbb{E}\left[\sum_{t<s\le T} e^{-\int_t^s r_p dp}\left(\phi(S_{t,s^-}^x + c_s(S_{t,s^-}^x)\Delta X_s) - \phi(S_{t,s^-}^x)\right)\right] \\ &= \phi(x) - \int_t^T r_s e^{-\int_t^s r_p dp}\mathbb{E}\left[\phi(S_{t,s}^x)\right]ds + \int_t^T e^{-\int_t^s r_p dp}\mathbb{E}\left[\phi'(S_{t,s}^x)a_s(S_{t,s}^x)\right]ds \\ &\quad +\frac{1}{2}\int_t^T e^{-\int_t^s r_p dp}\mathbb{E}\left[\phi''(S_{t,s}^x)b_s^2(S_{t,s}^x)\right]ds \\ &\quad +\lambda\int_t^T e^{-\int_t^s r_p dp}\mathbb{E}\left[\int_{-\infty}^{\infty}(\phi(S_{t,s}^x + zc_s(S_{t,s}^x)) - \phi(S_{t,s}^x))\nu(dz)\right]ds. \end{aligned} \tag{3.3}$$

Hence Theta_T can be expressed as

$$\mathrm{Theta}_T = \frac{\partial}{\partial T}\left(e^{-\int_t^T r_s ds}\mathbb{E}\left[\phi(S_{t,T}^x)\right]\right)$$

$$\begin{aligned} &= -r_T e^{-\int_t^T r_p dp} \mathbb{E}\left[\phi(S^x_{t,T})\right] + e^{-\int_t^T r_p dp} \mathbb{E}\left[\phi'(S^x_{t,T}) a_T(S^x_{t,T})\right] \\ &\quad + \frac{1}{2} e^{-\int_t^T r_p dp} \mathbb{E}\left[\phi''(S^x_{t,T}) b_T^2(S^x_{t,T})\right] \\ &\quad + \lambda e^{-\int_t^T r_s ds} \mathbb{E}\left[\int_{-\infty}^{\infty} (\phi(S^x_{t,T} + z c_T(S^x_{t,T})) - \phi(S^x_{t,T}))\nu(dz)\right]. \end{aligned} \tag{3.4}$$

The above expression fails when ϕ is not twice differentiable, but the derivatives on ϕ will be removed by integration by parts on the Wiener space, using the relation

$$\phi'(S^x_{t,T}) = \frac{D_u \phi(S^x_{t,T})}{D_u S^x_{t,T}}, \qquad u \in L^2([t,T]), \tag{3.5}$$

which follows from the derivation property of D. The jump component in (3.4) can be left untouched since it does not contain any derivative of ϕ.

Proposition 3.1 *Let $u, v, w \in L^2([t,T])$ such that $D_u S^x_{t,T}$, $D_v S^x_{t,T}$, $D_w S^x_{t,T}$ are a.s. non-zero and belong to* Dom (D) *and* $\frac{a_T(S^x_{t,T})}{D_u S^x_{t,T}} u$, $\frac{b_T^2(S^x_{t,T})}{D_v S^x_{t,T}} v$, $\frac{\Gamma_{t,T}(v)}{D_w S^x_{t,T}} w$ *satisfy the hypotheses of Lemma 2.1, where* $\Gamma_{t,T}(v) := \frac{b_T^2(S^x_{t,T})}{D_v S^x_{t,T}} I_1(v) - 2 b_T(S^x_{t,T}) b_T'(S^x_{t,T}) + \frac{b_T^2(S^x_{t,T}) D_v^2 S^x_{t,T}}{|D_v S^x_{t,T}|^2}$. *Then the weight $\Lambda_{t,T}(u,v,w)$, $0 < t < T$, defined by*

$$\begin{aligned} \Lambda_{t,T}(u,v,w) &:= -a_T'(S^x_{t,T}) - r_T + a_T(S^x_{t,T}) \left(\frac{I_1(u)}{D_u S^x_{t,T}} + \frac{D_u^2 S^x_{t,T}}{|D_u S^x_{t,T}|^2} \right) \\ &+ \frac{1}{2} \left(\left(\frac{b_T^2(S^x_{t,T})}{D_v S^x_{t,T}} I_1(v) - 2 b_T(S^x_{t,T}) b_T'(S^x_{t,T}) + \frac{b_T^2(S^x_{t,T}) D_v^2 S^x_{t,T}}{|D_v S^x_{t,T}|^2} \right) \left(\frac{I_1(w)}{D_w S^x_{t,T}} + \frac{D_w^2 S^x_{t,T}}{|D_w S^x_{t,T}|^2} \right) \right. \\ &+ \frac{b_T^2(S^x_{t,T})}{D_w S^x_{t,T} D_v S^x_{t,T}} \left(I_1(v) \frac{D_w D_v S^x_{t,T}}{D_v S^x_{t,T}} - \langle v, w \rangle - \frac{D_w D_v^2 S^x_{t,T}}{D_v S^x_{t,T}} + 2 \frac{D_w D_v S^x_{t,T} D_v^2 S^x_{t,T}}{|D_v S^x_{t,T}|^2} \right) \\ &\left. - \frac{2 b_T'(S^x_{t,T}) b_T(S^x_{t,T})}{D_v S^x_{t,T}} \left(I_1(v) + \frac{D_v^2 S^x_{t,T}}{D_v S^x_{t,T}} \right) \right) + b_T'(S^x_{t,T})^2 + b_T''(S^x_{t,T}) b_T(S^x_{t,T}), \end{aligned}$$

belongs to $L^2(\Omega)$ and for any $\phi : \mathbb{R} \to \mathbb{R}$ such that $|\phi|$ is bounded by a Lipschitz function on $\mathbb{R}$ we have

$$\begin{aligned} &\text{Theta}_T = \\ &e^{-\int_t^T r_s ds} \mathbb{E}\left[\Lambda_{t,T}(u,v,w) \phi(S^x_{t,T}) + \lambda \int_{-\infty}^{\infty} (\phi(S^x_{t,T} + z c_T(S^x_{t,T})) - \phi(S^x_{t,T}))\nu(dz) \right]. \end{aligned} \tag{3.6}$$

Proof. Using (2.3) and (3.5) we get, assuming first that ϕ, g_T are twice continuously differentiable:

$$\mathbb{E}\left[\phi'(S^x_{t,T}) g_T(S^x_{t,T})\right] = \mathbb{E}\left[\frac{g_T(S^x_{t,T})}{D_u S^x_{t,T}} D_u \phi(S^x_{t,T}) \right]$$

$$
\begin{aligned}
&= \mathbb{E}\left[\left\langle D\phi(S^x_{t,T}), \frac{g_T(S^x_{t,T})}{D_u S^x_{t,T}} u\right\rangle\right] \\
&= \mathbb{E}\left[\phi(S^x_{t,T})\delta\left(\frac{g_T(S^x_{t,T})}{D_u S^x_{t,T}} u\right)\right] \\
&= \mathbb{E}\left[\phi(S^x_{t,T})\left(\frac{g_T(S^x_{t,T})}{D_u S^x_{t,T}} I_1(u) - D_u\left(\frac{g_T(S^x_{t,T})}{D_u S^x_{t,T}}\right)\right)\right] \\
&= \mathbb{E}\left[\phi(S^x_{t,T})\left(\frac{g_T(S^x_{t,T})}{D_u S^x_{t,T}} I_1(u) - g'_T(S^x_{t,T}) + \frac{g_T(S^x_{t,T}) D^2_u S^x_{t,T}}{|D_u S^x_{t,T}|^2}\right)\right].
\end{aligned}
$$

With $g_T(\cdot) = a_T(\cdot)$ we obtain

$$
\mathbb{E}\left[\phi'(S^x_{t,T}) a_T(S^x_{t,T})\right] = \mathbb{E}\left[\phi(S^x_{t,T})\left(\frac{a_T(S^x_{t,T})}{D_u S^x_{t,T}} I_1(u) - a'_T(S^x_{t,T}) + \frac{a_T(S^x_{t,T}) D^2_u S^x_{t,T}}{|D_u S^x_{t,T}|^2}\right)\right], \tag{3.7}
$$

while $g_T(\cdot) = b^2_T(\cdot)$ yields

$$
\mathbb{E}\left[\phi''(S^x_{t,T}) b^2_T(S^x_{t,T})\right] = \mathbb{E}\left[\phi'(S^x_{t,T}) \Gamma_{t,T}(v)\right].
$$

By a similar argument we get

$$
\begin{aligned}
&\mathbb{E}\left[\phi''(S^x_{t,s}) b^2_s(S^x_{t,s})\right] = \mathbb{E}\left[\phi'(S^x_{t,T}) \Gamma_{t,T}(v)\right] \\
&\quad = \mathbb{E}\left[\Gamma_{t,T}(v) \frac{D_w \phi(S^x_{t,T})}{D_w(S^x_{t,T})}\right] \\
&\quad = \mathbb{E}\left[\phi(S^x_{t,T})\delta\left(w \frac{\Gamma_{t,T}(v)}{D_w S^x_{t,T}}\right)\right] \\
&\quad = \mathbb{E}\left[\phi(S^x_{t,T})\left(\frac{\Gamma_{t,T}(v)}{D_w S^x_{t,T}} I_1(w) - D_w\left(\frac{\Gamma_{t,T}(v)}{D_w S^x_{t,T}}\right)\right)\right] \\
&\quad = \mathbb{E}\left[\phi(S^x_{t,T})\left(\frac{\Gamma_{t,T}(v)}{D_w S^x_{t,T}}\left(I_1(w) + \frac{D^2_w S^x_{t,T}}{D_w S^x_{t,T}}\right) - \frac{2b'_T(S^x_{t,T}) b_T(S^x_{t,T})}{D_v S^x_{t,T}}\left(I_1(v) + \frac{D^2_v S^x_{t,T}}{D_v S^x_{t,T}}\right)\right.\right. \\
&\qquad + \frac{b^2_T(S^x_{t,T})}{D_w S^x_{t,T} D_v S^x_{t,T}}\left(\frac{I_1(v) D_w D_v S^x_{t,T}}{D_v S^x_{t,T}} - \langle v, w\rangle - \frac{D_w D^2_v S^x_{t,T}}{D_v S^x_{t,T}} + \frac{2 D_w D_v S^x_{t,T} D^2_v S^x_{t,T}}{|D_v S^x_{t,T}|^2}\right) \\
&\qquad \left.\left. + 2b'_T(S^x_{t,T})^2 + 2b''_T(S^x_{t,T}) b_T(S^x_{t,T})\right)\right].
\end{aligned}
$$

Summing (3.7) with the above relation we rewrite (3.3) as

$$
\begin{aligned}
C(x,t,T) &= \phi(x) + \int_t^T e^{-\int_t^s r_p dp} \mathbb{E}\left[\Lambda_{t,s}(u,v,w)\phi(S^x_{t,s})\right] ds \qquad (3.8) \\
&\quad + \lambda \int_t^T e^{-\int_t^s r_p dp} \mathbb{E}\left[\int_{-\infty}^{\infty} (\phi(S^x_{t,s} + z c_s(S^x_{t,s})) - \phi(S^x_{t,s}))\nu(dz)\right] ds.
\end{aligned}
$$

To conclude the proof we extend (3.8) by approximation of ϕ using $\mathcal{C}^2_b$ functions and then differentiate with respect to T in order to obtain (3.6). $\square$

Remarks

a) As noted at the end of Section 2, Theta_t can be computed via the Black-Scholes PDE (2.6) and the values (2.7)-(2.9) of Delta and Gamma. In the time homogeneous linear model (4.1) below this representation coincides with our expression (3.6) of $-\text{Theta}_T$ when using the optimal weights of Section 4. In a general time homogeneous model we still have $\text{Theta}_t = -\text{Theta}_T$ but the two representations may differ.

b) Instead of (3.5) we could have used the pointwise version

$$\phi'(S^x_{t,T}) = \frac{D_s\phi(S^x_{t,T})}{D_s S^x_{t,T}}, \qquad s \in [0,T],$$

as in e.g. [7], but its usefulness relies mainly on the possibility to express $D_s S^x_T$ using the first variation process $(Y_t)_{t\in\mathbb{R}_+}$ in continuous models as

$$D_s S^x_T = b_s(S^x_s)\frac{Y_T}{Y_s}, \qquad s \in [0,T],$$

cf. p. 124 of [10], which leads to

$$\frac{Y_s}{b_s(S_s)} D_s\phi(S_T) = Y_T\phi'(S_T), \qquad s \in [0,T], \tag{3.9}$$

and to (2.9). On the other hand, our expressions do not directly use the first and second variation processes and involve elementary Wiener integrals $I_1(u)$ of deterministic functions (i.e. centered Gaussian random variables) instead of Itô stochastic integrals of adapted processes as in (2.9).

c) The argument of proof of Proposition 3.1 requires to differentiate with respect to the terminal value T and not with respect to the current time t, thus in a time inhomogeneous setting it does not apply to the computation of Theta_t.

To close this section we recall the principle of the localization method introduced in [7] which aims to reduce the variance of Monte Carlo estimators, in a formulation adapted to our setting and in the time homogeneous case. Payoff functions of the form

$$\phi(y) = \mathbf{1}_{[K,\infty)}(y) \qquad \text{and} \qquad \phi(y) = (y-K)^+$$

have a singularity at $y = K$. The idea of localization is to decompose the payoff function ϕ as

$$\phi = g_\eta + h_\eta, \qquad \eta > 0,$$

in such a way that h_η is twice differentiable and g_η contains the singularity of ϕ, see e.g. [9] for digital options and [2] for European options. Applying the Malliavin approach to g_η and using (3.4) for $\phi = h_\eta$ we get

$$\begin{aligned}
\text{Theta} &= e^{-\tau r}\mathbb{E}\left[\left(\Lambda(u,v,w)g_\eta(S_\tau^x) + \lambda\int_{-\infty}^{\infty}(\phi(S_\tau^x + c(S_\tau^x)y) - \phi(S_\tau^x))\nu(dy)\right)\right] \\
&\quad -re^{-\tau r}\mathbb{E}\left[h_\eta(S_\tau^x)\right] + e^{-\tau r}\mathbb{E}\left[h_\eta'(S_\tau^x)a(S_\tau^x)\right] \\
&\quad +\frac{1}{2}e^{-\tau r}\mathbb{E}\left[h_\eta''(S_\tau^x)b^2(S_\tau^x)\right],
\end{aligned}$$

where the integration by parts method has been applied to the first and second derivatives on g_η. In the case of European options we take

$$h_\eta(y) = \frac{1}{4\eta}(y-(K-\eta))^2\mathbf{1}_{[-\eta,\eta]}(y-K) + (y-K)\mathbf{1}_{]\eta,\infty)}(y-K),$$

and for digital options we choose

$$\begin{aligned}
h_\eta(y) &= \frac{1}{2}\left(1+\frac{y-K}{\eta}\right)^2\mathbf{1}_{(-\eta,0]}(y-K) + \left(1-\frac{1}{2}\left(1-\frac{y-k}{\eta}\right)^2\right)\mathbf{1}_{(0,\eta)}(y-K) \\
&+ \mathbf{1}_{[\eta,\infty)}(y-K).
\end{aligned}$$

4 Optimization of convergence

In this section we consider constant interest rate and volatilities r, σ and ζ, i.e. we consider (3.2) in the linear case, with

$$\begin{cases} a(y) = (r-\lambda\zeta)y, \\ b(y) = \sigma y, \\ c(y) = \zeta y, \end{cases} \tag{4.1}$$

and

$$S_t = x\exp\left(\left(r-\lambda\zeta-\frac{\sigma^2}{2}\right)t + \sigma W_t\right)\prod_{0<s\le t}(1+\zeta\Delta X_s), \qquad t\in\mathbb{R}_+,$$

hence the no arbitrage condition (2.5) is satisfied and the discounted price process $(e^{-rt}S_t)_{t\in\mathbb{R}_+}$ is a martingale. We have

$$D_u S_\tau^x = \sigma\int_0^\tau u_s ds S_\tau^x,$$

hence $D_v^2 S_\tau^x / |D_v S_\tau^x|^2 = 1/S_\tau$ and we get

$$\Lambda_\tau(u,v,w) = -r + \frac{\hat{r}}{\sigma}\frac{I_1(u)}{\int_0^\tau u_s ds} - \frac{\sigma}{2}\frac{I_1(w)}{\int_0^\tau w_s ds} + \frac{I_2(v\circ w)}{2\int_0^\tau v_s ds\int_0^\tau w_s ds},$$

where $\hat{r} = r-\lambda\zeta$. Our goal is now to find functions u, v, w which minimize $\mathrm{Var}[\Lambda_\tau(u,v,w)]$.

Proposition 4.1 *The infimum on* $\mathrm{Var}[\Lambda_\tau(u,v,w)]$ *is attained for any non-zero constant functions* u, v, w *of the form* $u_s = c_1$, $v_s = c_2$, $w_s = c_3$, $s \in [0,\tau]$, *and is given by*

$$\inf_{u,v,w} \mathrm{Var}[\Lambda_\tau(u,v,w)] = \mathrm{Var}[\Lambda_{\mathrm{opt}}] = \frac{1}{2\tau^2} + \frac{1}{\sigma^2\tau}\left|\hat{r} - \frac{\sigma^2}{2}\right|^2,$$

where $\hat{r} = r - \lambda\zeta$, *with*

$$\Lambda_{\mathrm{opt}} = -r + \frac{W_\tau}{\sigma\tau}\left(\hat{r} - \frac{\sigma^2}{2}\right) + \frac{1}{2\tau}\left(\frac{W_\tau^2}{\tau} - 1\right). \tag{4.2}$$

Proof. For any $u \in L^2([0,\tau])$ such that $\int_0^\tau u_s ds \neq 0$, letting

$$\tilde{u}_t := \frac{u_t}{\int_0^\tau u_s ds}, \qquad t \in [0,\tau],$$

the weight $\Lambda_\tau(u,v,w)$ is expressed as

$$\Lambda_\tau(u,v,w) = -r + \frac{\hat{r}}{\sigma} I_1(\tilde{u}) - \frac{\sigma}{2} I_1(\tilde{w}) + \frac{1}{2} I_2(\tilde{v} \circ \tilde{w}).$$

Recall that the Cauchy-Schwarz inequality yields

$$\|\tilde{u}\|^2 \geq \frac{1}{\tau}, \tag{4.3}$$

with equality if and only if $\tilde{u}_t = 1/\tau$, $t \in [0,\tau]$.

Let

$$\begin{aligned} F(u,v,w) &= \mathrm{Var}[\Lambda_\tau(u,v,w)] \\ &= \frac{\hat{r}^2}{\sigma^2}\|\tilde{u}\|^2 - \hat{r}\langle \tilde{u}, \tilde{w}\rangle + \frac{\sigma^2}{4}\|\tilde{w}\|^2 + \frac{1}{4}\|\tilde{v}\|^2\|\tilde{w}\|^2 + \frac{1}{4}\langle \tilde{v}, \tilde{w}\rangle^2 \\ &= \frac{1}{\sigma^2}\left\|\hat{r}\tilde{u} - \frac{\sigma^2}{2}\tilde{w}\right\|^2 + \frac{1}{4}\|\tilde{v}\|^2\|\tilde{w}\|^2 + \frac{1}{4}\langle \tilde{v}, \tilde{w}\rangle^2, \end{aligned}$$

where we applied the isometry (2.2). The optimal value of (u,v,w) solves

$$\left\{ \begin{array}{l} \dfrac{d}{d\varepsilon} F(u + \varepsilon h, v, w)_{|\varepsilon=0} = 0 \\[2ex] \dfrac{d}{d\varepsilon} F(u, v + \varepsilon h, w)_{|\varepsilon=0} = 0 \\[2ex] \dfrac{d}{d\varepsilon} F(u, v, w + \varepsilon h)_{|\varepsilon=0} = 0, \end{array} \right. \tag{4.4}$$

for all $h \in L^2([0,\tau])$, i.e.

$$\frac{2\hat{r}^2}{\sigma^2}\left(\langle h, \tilde{u}\rangle - \|\tilde{u}\|^2 \int_0^\tau h_s ds\right) - \hat{r}\left(\langle h, \tilde{w}\rangle - \langle \tilde{u}, \tilde{w}\rangle \int_0^\tau h_s ds\right) = 0,$$

$$\frac{1}{2}\|\tilde{w}\|^2\left(\langle h,\tilde{v}\rangle-\|\tilde{v}\|^2\int_0^\tau h_s ds\right)+\frac{1}{2}\left(\langle\tilde{v},\tilde{w}\rangle\langle h,\tilde{w}\rangle-\langle\tilde{v},\tilde{w}\rangle^2\int_0^\tau h_s ds\right)=0,$$

and

$$\begin{aligned}&\frac{\sigma^2}{2}\left(\langle h,\tilde{w}\rangle-\|\tilde{w}\|^2\int_0^\tau h_s ds\right)+\frac{1}{2}\|\tilde{v}\|^2\left(\langle h,\tilde{w}\rangle-\|\tilde{w}\|^2\int_0^\tau h_s ds\right)\\&\quad+\frac{1}{2}\left(\langle\tilde{v},\tilde{w}\rangle\langle h,\tilde{v}\rangle-\langle\tilde{v},\tilde{w}\rangle^2\int_0^\tau h_s ds\right)-\hat{r}\left(\langle h,\tilde{u}\rangle-\langle\tilde{u},\tilde{w}\rangle\int_0^\tau h_s ds\right)=0.\end{aligned}$$

Clearly, for any $c_1,c_2,c_3\neq 0$ the constant functions $u_s=c_1$, $v_s=c_2$, $w_s=c_3$, $s\in[0,\tau]$, are solutions of this problem. Let us show that this solution is unique. For all $h\in L^2([0,\tau])$ such that $\int_0^\tau h_s ds=0$, equation (4.4) yields

$$\begin{cases}\dfrac{2\hat{r}^2}{\sigma^2}\langle h,\tilde{u}\rangle-\hat{r}\langle h,\tilde{w}\rangle=0\\[2ex]\|\tilde{w}\|^2\langle h,\tilde{v}\rangle+\langle\tilde{v},\tilde{w}\rangle\langle h,\tilde{w}\rangle=0\\[2ex]\sigma^2\langle h,\tilde{w}\rangle+\|\tilde{v}\|^2\langle h,\tilde{w}\rangle+\langle\tilde{v},\tilde{w}\rangle\langle h,\tilde{v}\rangle-2\hat{r}\langle h,\tilde{u}\rangle=0.\end{cases}$$

If a solution $(\tilde{u},\tilde{v},\tilde{w})$ different from $(1/\tau,1/\tau,1/\tau)$ exists, then one can find $h\in L^2([0,\tau])$ such that $\int_0^\tau h_s ds=0$ and $(\langle h,\tilde{u}\rangle,\langle h,\tilde{v}\rangle,\langle h,\tilde{w}\rangle)\neq(0,0,0)$, hence the determinant

$$\|\tilde{v}\|^2\|\tilde{w}\|^2-|\langle\tilde{v},\tilde{w}\rangle|^2=0 \tag{4.5}$$

of the above linear system vanishes. From (4.3) and (4.5) we get

$$\begin{aligned}F(u,v,w)&=\frac{1}{\sigma^2}\left\|\hat{r}\tilde{u}-\frac{\sigma^2}{2}\tilde{w}\right\|^2+\frac{1}{4}\|\tilde{v}\|^2\|\tilde{w}\|^2+\frac{1}{4}|\langle\tilde{v},\tilde{w}\rangle|^2\\&=\frac{1}{\sigma^2}\left\|\hat{r}\tilde{u}-\frac{\sigma^2}{2}\tilde{w}\right\|^2+\frac{1}{2}\|\tilde{v}\|^2\|\tilde{w}\|^2\\&\geq\frac{1}{\tau\sigma^2}\left|\int_0^\tau\left(\hat{r}\tilde{u}_s-\frac{\sigma^2}{2}\tilde{w}_s\right)ds\right|^2+\frac{1}{2\tau^2}\\&=\frac{1}{\tau\sigma^2}\left|\hat{r}-\frac{\sigma^2}{2}\right|^2+\frac{1}{2\tau^2},\end{aligned}$$

which is greater than the optimal value found when $\tilde{u}$, $\tilde{v}$, $\tilde{w}$ are constant functions. Moreover, equality occurs only when $\|\tilde{v}\|^2=1/\tau$, $\|\tilde{w}\|^2=1/\tau$, and

$$\left\|\hat{r}\tilde{u}-\frac{\sigma^2}{2}\tilde{w}\right\|^2=\frac{1}{\tau}\left|\hat{r}-\frac{\sigma^2}{2}\right|^2,$$

i.e. when $\hat{r}\tilde{u}-\frac{\sigma^2}{2}\tilde{w}$, $\tilde{v}$, $\tilde{w}$ are constant, which implies that $\tilde{u}$ is also constant, except when $\hat{r}=0$, in which case no constraint is imposed on u.

We now need to prove that this solution corresponds to the global minimum of F. Since $F(u,v,w) \geq 0$, the infimum exists and we denote it by l. By continuity of F on $L^2([0,\tau])^3$ there exist a sequence $(u_n, v_n, w_n)_{n\in\mathbb{N}}$ such that

$$l = \lim_{n\to\infty} \mathbb{E}[\Lambda_\tau(u_n, v_n, w_n)^2].$$

We can assume that (u_n, v_n, w_n) is bounded: if not, replace it by the bounded sequence

$$\left(\frac{u_n}{\|u_n\|}, \frac{v_n}{\|v_n\|}, \frac{w_n}{\|w_n\|}\right)_{n\in\mathbb{N}},$$

on which F takes the same values as on $(u_n, v_n, w_n)_{n\in\mathbb{N}}$. Under this hypothesis, there exists a subsequence $(u_{n_k}, v_{n_k}, w_{n_k})_{k\in\mathbb{N}}$ converging weakly to (u,v,w) in $L^2([0,\tau])^3$. We have

$$\begin{aligned}
&\mathbb{E}[\Lambda_\tau(u,v,w)\Lambda_\tau(u_{n_k}, v_{n_k}, w_{n_k})] \\
&\qquad = \hat{r}^2 + \frac{\hat{r}^2}{\sigma^2}\langle \tilde{u}, \tilde{u}_{n_k}\rangle - \frac{\hat{r}}{2}\langle \tilde{u}, \tilde{w}_{n_k}\rangle - \frac{\hat{r}}{2}\langle \tilde{u}_{n_k}, \tilde{w}\rangle + \frac{\sigma^2}{4}\langle \tilde{w}, \tilde{w}_{n_k}\rangle \\
&\qquad\quad + \frac{1}{4}\langle \tilde{u}, \tilde{u}_{n_k}\rangle\langle \tilde{w}, \tilde{w}_{n_k}\rangle + \frac{1}{4}\langle \tilde{u}, \tilde{w}_{n_k}\rangle\langle \tilde{w}, \tilde{u}_{n_k}\rangle,
\end{aligned}$$

and by weak convergence of $(u_{n_k}, v_{n_k}, w_{n_k})_{k\in\mathbb{N}}$ to (u,v,w) we get

$$\lim_{n\to\infty} \mathbb{E}[\Lambda_\tau(u,v,w)\Lambda_\tau(u_{n_k}, v_{n_k}, w_{n_k})] = \mathbb{E}[|\Lambda_\tau(u,v,w)|^2].$$

Moreover,

$$\begin{aligned}
0 &\geq l - \mathbb{E}[|\Lambda_\tau(u,v,w)|^2] \\
&= \lim_{n\to\infty} \mathbb{E}[\Lambda_\tau(u_{n_k}, v_{n_k}, w_{n_k})^2] - \mathbb{E}[|\Lambda_\tau(u,v,w)|^2] \\
&\geq \lim_{n\to\infty} \mathbb{E}[|\Lambda_\tau(u,v,w) - \Lambda_\tau(u_{n_k}, v_{n_k}, w_{n_k})|^2] \\
&\quad + 2\mathbb{E}[\Lambda_\tau(u,v,w)\Lambda_\tau(u_{n_k}, v_{n_k}, w_{n_k})] - 2\mathbb{E}[|\Lambda_\tau(u,v,w)|^2] \\
&\geq \lim_{n\to\infty} \mathbb{E}[(\Lambda_\tau(u,v,w) - \Lambda_\tau(u_{n_k}, v_{n_k}, w_{n_k}))^2] \\
&\quad + 2\lim_{n\to\infty} \mathbb{E}[\Lambda_\tau(u,v,w)\Lambda_\tau(u_{n_k}, v_{n_k}, w_{n_k})] \\
&\quad - 2\mathbb{E}[|\Lambda_\tau(u,v,w)|^2] \\
&\geq \lim_{n\to\infty} \mathbb{E}[(\Lambda_\tau(u,v,w) - \Lambda_\tau(u_{n_k}, v_{n_k}, w_{n_k}))^2] \\
&\geq 0,
\end{aligned}$$

hence $\lim_{n\to\infty} \Lambda_\tau(u_{n_k}, v_{n_k}, w_{n_k}) = \Lambda_\tau(u,v,w)$ in $L^2(\Omega)$ and

$$l = \mathbb{E}[|\Lambda_\tau(u,v,w)|^2].$$

Thus the global minimum is attained for $\tilde{u} = \tilde{v} = \tilde{w} = 1/\tau$. □

Note that $\inf_{u,v,w\in L^2([0,\tau])} \mathrm{Var}[\Lambda_\tau(u,v,w)]$ is minimal in terms of σ and r when $(S_t^x)_{t\in\mathbb{R}_+}$ is an exponential Brownian motion, i.e. $\hat{r} = \sigma^2/2$. In this case we have

$$\inf_{u,v,w\in L^2([0,\tau])} \mathrm{Var}[\Lambda_\tau(u,v,w)] = \frac{1}{2\tau^2}.$$

5 Numerical simulations

As in Section 4, we consider the (time homogeneous) linear model (4.1):

$$\begin{cases} dS_s = rS_s ds + \sigma S_s dW_s + \zeta S_{s^-}(dX_s - \lambda ds), \\ \\ S_0 = x, \end{cases} \tag{5.1}$$

where

$$X_t = a_1 N_t^1 + \cdots + a_d N_t^d, \qquad t \in \mathbb{R}_+, \tag{5.2}$$

and $(N_t^k)_{t\in\mathbb{R}_+}$, $k = 1, \ldots, d$, are independent Poisson processes with respective intensities $\lambda_1, \ldots, \lambda_d$, with $\lambda = \lambda_1 + \cdots + \lambda_d$ and

$$\nu(dx) = \frac{\lambda_1}{\lambda}\delta_{a_1}(dx) + \cdots + \frac{\lambda_d}{\lambda}\delta_{a_d}(dx).$$

We have

$$S_t = x \exp\left(\left(r - \lambda\zeta - \frac{\sigma^2}{2}\right) t + \sigma W_t\right) (1 + \zeta a_1)^{N_t^1} \cdots (1 + \zeta a_d)^{N_t^d}, \qquad t \in \mathbb{R}_+,$$

and

$$\text{Theta} = e^{-r\tau}\mathbb{E}\left[\Lambda_\tau(u, v, w)\phi(S_\tau^x) + \sum_{k=1}^{d} \lambda_k(\phi(S_\tau^x(1 + \zeta a_k)) - \phi(S_\tau^x))\right].$$

We apply the Malliavin formula (3.6) with $\tilde{u}_s = \tilde{v}_s = \tilde{w}_s = 1/\tau$, $s \in [0, \tau]$, to compute $\text{Theta}_T = -\text{Theta}_t$ for European and digital options, i.e. with non-smooth payoff functions. In the geometric model with the optimal weight $\Lambda(u, v, w)$, localization yields:

$$\begin{aligned} \text{Theta} \;=\; & -re^{-r\tau}\mathbb{E}\left[\phi(S_\tau^x)\right] + r\frac{e^{-r\tau}}{\sigma\tau}\mathbb{E}\left[g_\eta(S_\tau^x)W_\tau\right] + \frac{e^{-r\tau}}{2\tau}\mathbb{E}\left[g_\eta(S_\tau^x)\left(\frac{W_\tau^2}{\tau} - \sigma W_\tau - 1\right)\right] \\ & + e^{-r\tau}\mathbb{E}\left[\sum_{k=1}^{d} \lambda_k(\phi(S_\tau^x(1 + \zeta a_k)) - \phi(S_\tau^x))\right] \\ & + re^{-r\tau}\mathbb{E}\left[S_\tau^x h_\eta'(S_\tau^x)\right] + \frac{\sigma^2}{2}e^{-r\tau}\mathbb{E}\left[S_\tau^{x2} h_\eta''(S_\tau^x)\right]. \end{aligned}$$

Finite differences approximations for Theta_T are computed using the following formula:

$$\text{Theta}_T = \frac{C(x, t, (1 + \varepsilon)T) - C(x, t, (1 - \varepsilon)T)}{2\varepsilon T}. \tag{5.3}$$

We take x=100, r=0.05, $K = 110$, $t = 0.8$, $T = 1$, $\sigma = 0.15$, $\zeta = 0.3$, $\lambda = 1$, and choose $d = 1$, and $\eta = 10$ for the localization parameter. Figure 5.1 shows the convergence of the Malliavin and finite difference methods as the number of Monte Carlo events increases.

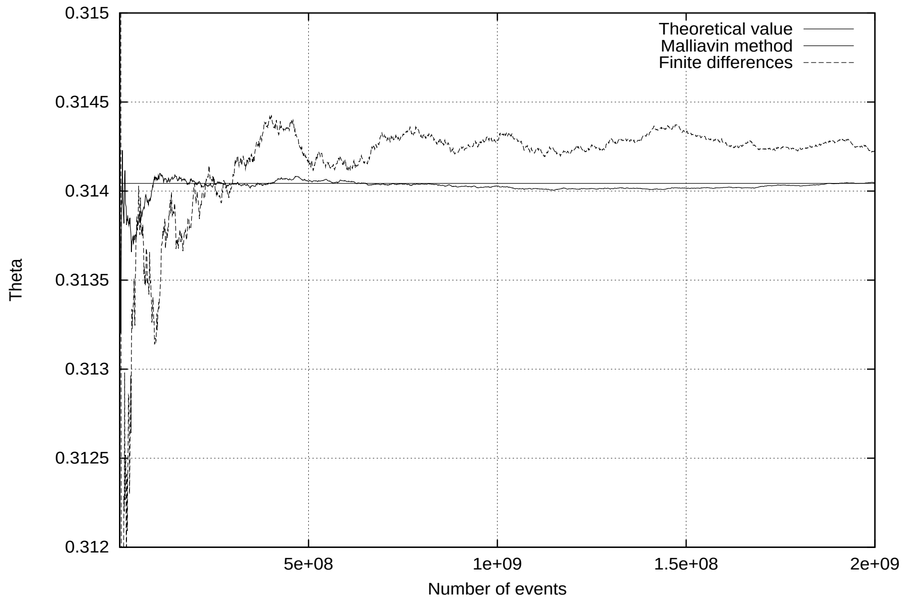

Figure 5.1: Estimation of Theta vs number of events

Digital options

The next graphs allow us to compare the Monte Carlo simulations of Theta as a function of K obtained by finite differences and by the Malliavin method in a jump model, with $\varepsilon = 10^{-3}$. The main interest of the Malliavin method is to be independent of the choice of the parameter ε and to perform better or at least comparably to the finite differences method, including when ε is adjusted to its optimal value, see also Figures 5.5 and 5.6 below.

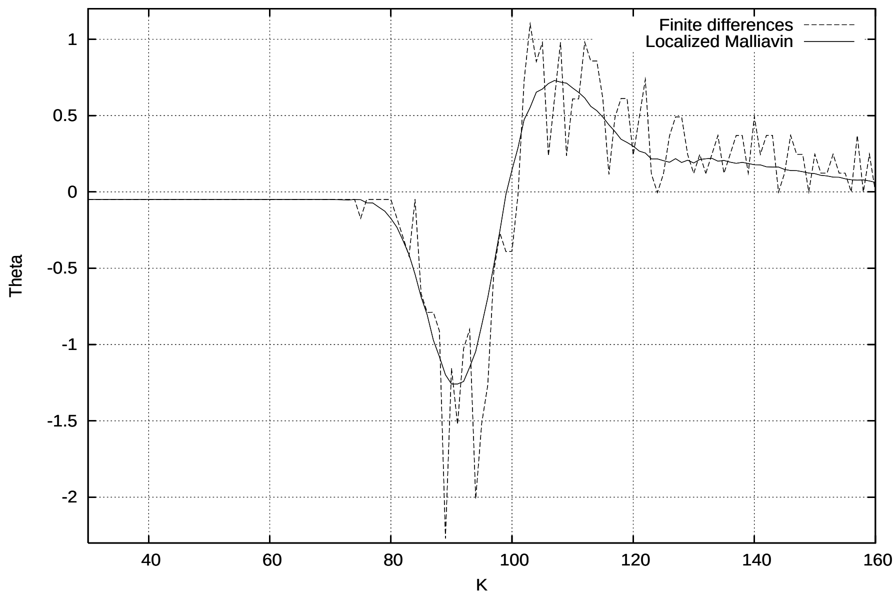

Figure 5.2: Finite differences vs localized Malliavin; digital option with jumps (20000 samples)

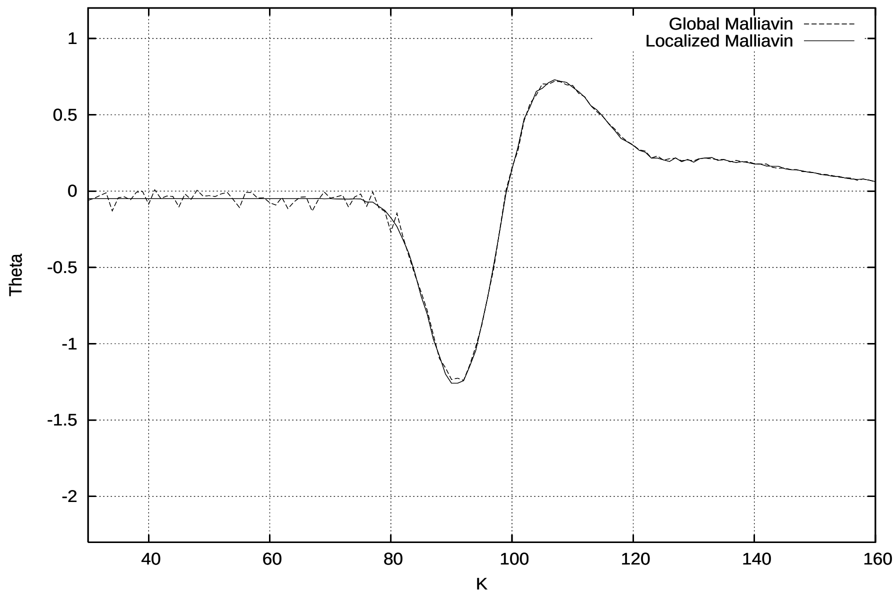

Figure 5.3: Localized vs global Malliavin; digital option in a jump model (20000 samples)

The localized Malliavin method appears to perform best, while the finite differences yields the worse results. Figure 5.4 allows one to compare the graphs of Theta for European options in continuous and jump models, using the localized Malliavin method. In this figure as well as in Figures 5.7 and 5.8 below we take $\sigma = 0.05$, $\zeta = 0.224$, and $\lambda = 2$.

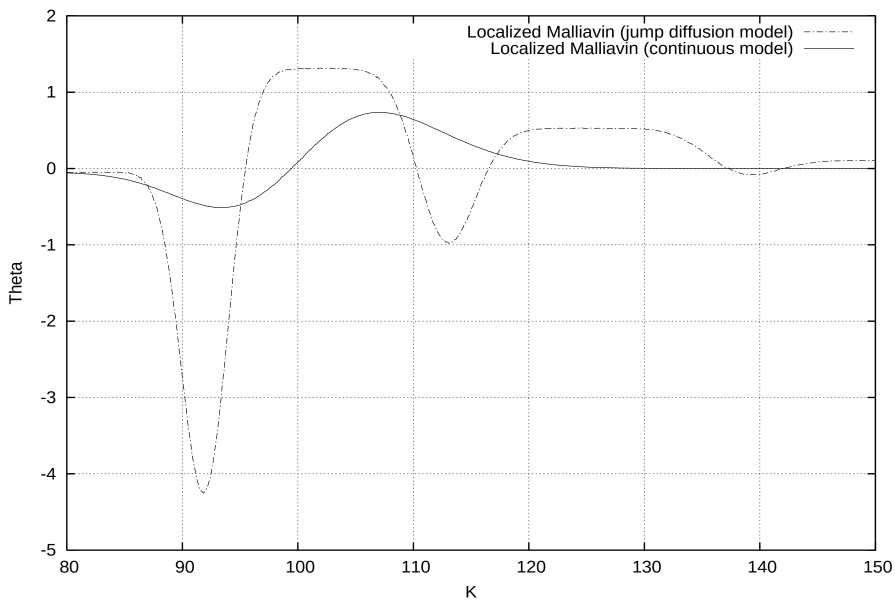

Figure 5.4: Comparison of Theta in continuous and jump models for digital options (200000 samples)

In continuous models, analytic formulas are available for the computation of Theta for digital and European options, cf. e.g. [8].

European options

In the next graph we have chosen $\varepsilon = 0.3$, for which the finite differences method showed the best performance.

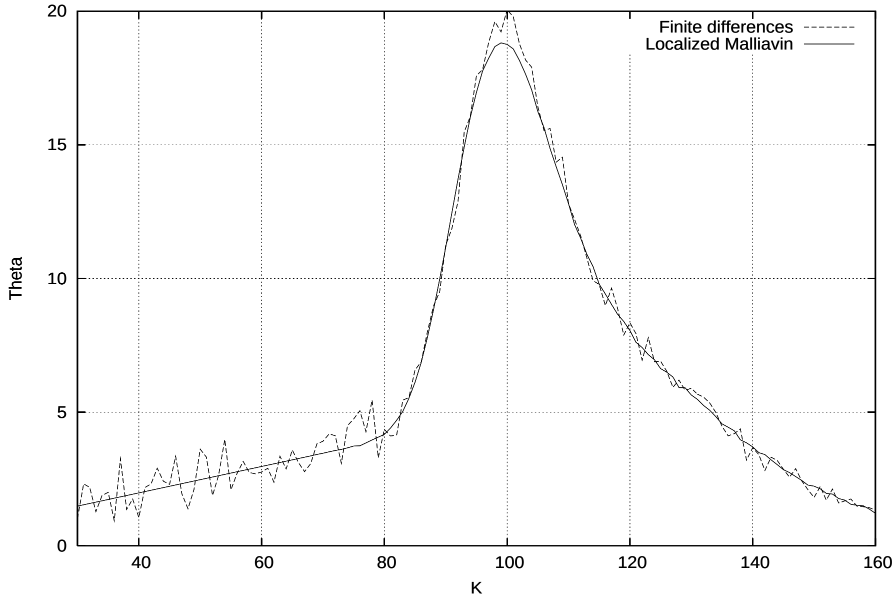

Figure 5.5: Finite differences vs localized Malliavin; European option with jumps (20000 samples)

In this case the Malliavin and finite differences method appear to give comparable levels of precision, but the localized Malliavin method still improves on both methods. In particular it corrects the lack of precision of the Malliavin method for smaller values of K, as shown in Figure 5.6.

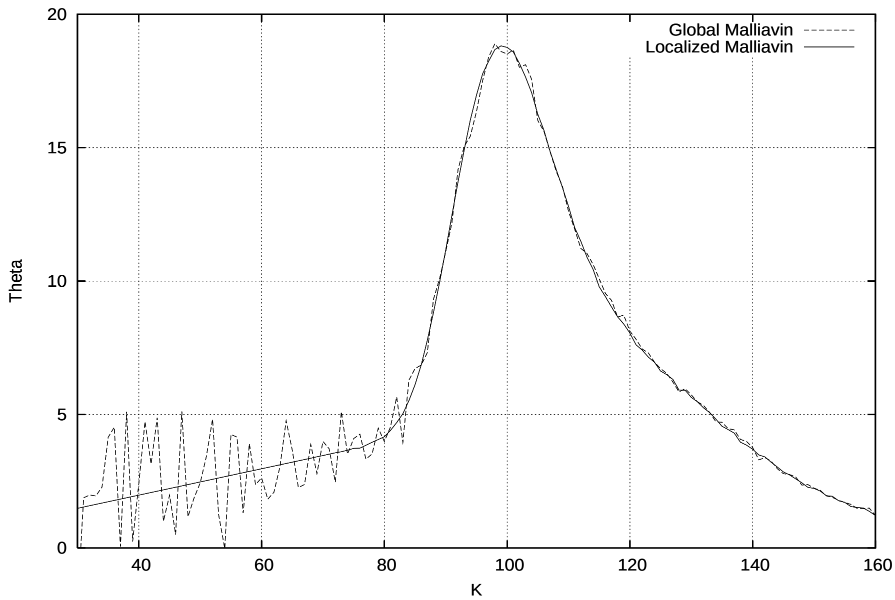

Figure 5.6: Localized vs global Malliavin; European option with jumps (20000 samples)

The next graph allows one to compare the simulation of Theta in continuous and jump model for European options using the localized Malliavin method.

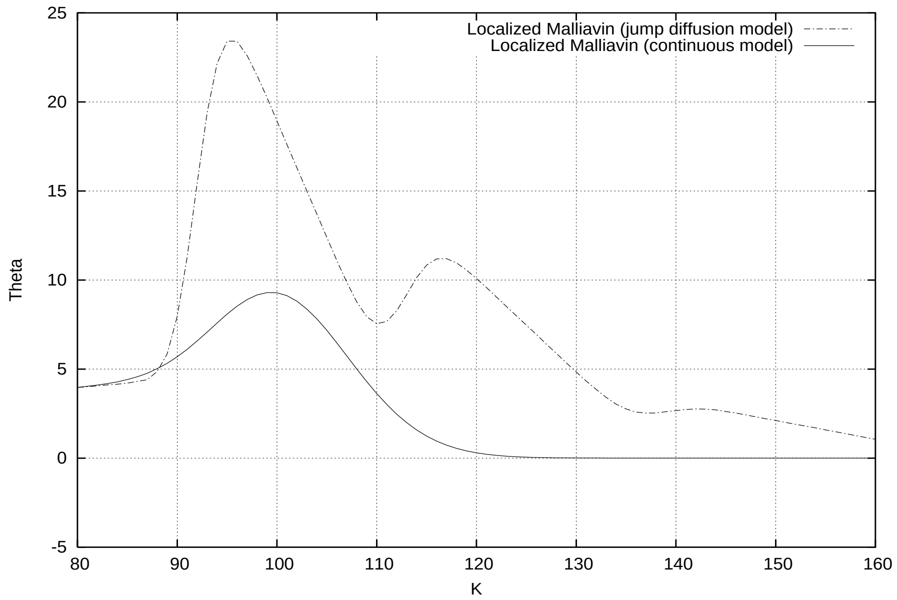

Figure 5.7: Comparison of continuous and jump models for European options (200000 samples)

Finally in Figure 5.8 below we compare Theta_t and $-\text{Theta}_T$ in a simple time inhomogeneous jump-diffusion model for digital options with $c(y) = \zeta e^{\beta(T_0-t)}y$ and $a(y) = (r - \lambda\zeta e^{\beta(T_0-t)})y$ using the localized Malliavin method, with $\beta = 4$ and $T_0 = 1$.

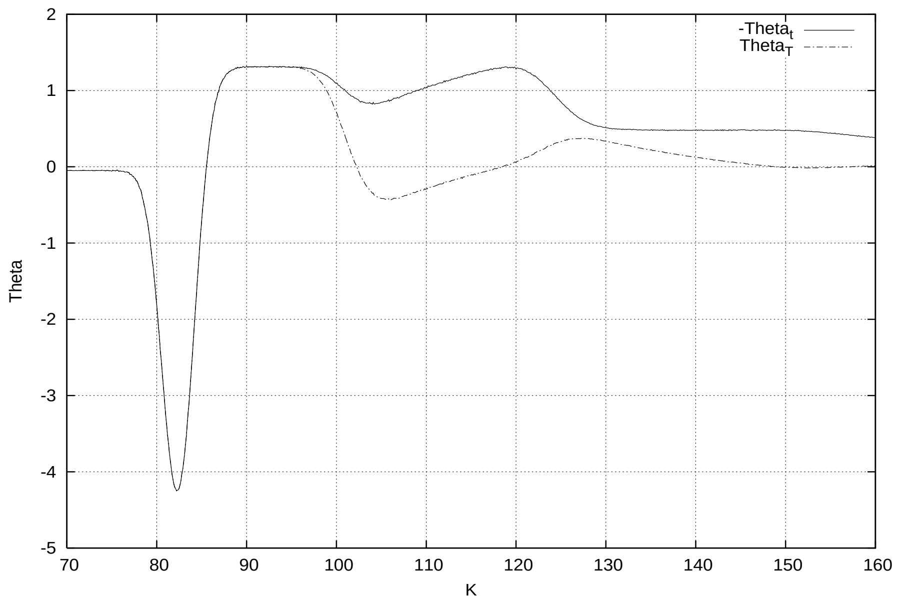

Figure 5.8: Comparison of -Theta_t and Theta_T in a time inhomogeneous model (200000 samples)

Conclusion

The Malliavin method provides an expression for Theta in time inhomogeneous models, which is independent of the parameter ε of the finite differences method. In time homogeneous models this representation is different from the one obtained from the Black-Scholes PDE, which does not apply to the time inhomogeneous case. The numerical performances of the Malliavin and finite differences method are comparable when the window parameter ε of the finite differences method is adjusted to its optimal value, but the localized Malliavin method appears to improve on both the finite differences and global Malliavin methods.

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