A probabilistic interpretation to the symmetries of a discrete heat equation

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Summary. A probabilistic interpretation is constructed for the symmetry group G of the finite difference-differential equation $\partial_t \eta(x,t) = \eta(x,t) - \eta(x+1,t)$ using the Doob transform for Markov (jump) processes. While the first three generators of G correspond to the identity and to space and time shifts, we show that in this interpretation the fourth generator of G is associated to time dilations and is linked to a creation operator on the Poisson space.

Key words: Finite difference equations, symmetries, jump processes, Doob transform.
Classification: 39A12, 34C14, 60J25, 81S25.

1 Introduction

Symmetry groups of partial differential equations have been extensively studied, see e.g. [9], [4] for the heat equation, and [8] for finite difference equations. Recently, probabilistic interpretations of the symmetries of the classical heat equation

$$\frac{\partial u}{\partial t}(x,t) = \frac{1}{2}\Delta u(x,t), \qquad x \in \mathbf{R}^d, \quad t \in \mathbf{R}_+, \tag{1}$$

have been provided in [6], [7], using a family of reversible diffusion processes.

In this paper we study in a similar way the symmetry group G of the simple finite difference-differential equation

$$L\eta(k,t) := \frac{\partial \eta}{\partial t}(k,t) + \eta(k+1,t) - \eta(k,t) = 0, \qquad k \in \mathbf{R}, \quad t \in \mathbf{R}_+. \tag{2}$$

Let P and S denote the creation and right shift operators on the Poisson space, defined as

$$P\eta(k,t) = k\eta(k-1,t) - t\eta(k,t) \quad \text{and} \quad S\eta(k,t) = \eta(k+1,t).$$

The operator P is a finite dimensional creation operator, due to its action on multiple Poisson stochastic integrals.

Our main results can be summarized as follows.

a) We show that the symmetry group of (2) has 4 generators denoted by $(N_i)_{i=1,\dots,4}$, where

- $N_1 = I$ the identity,
- $N_2 = \frac{\partial}{\partial t}$ which generates the group of time shifts,
- $N_3 = \frac{\partial}{\partial k}$ which generates the group of space shifts ,

and N_4 is written as $N_4 = PS$, i.e. any element N of the Lie algebra associated to (2) can be written as

$$N = \alpha_1 I + \alpha_2 \frac{\partial}{\partial t} + \alpha_3 \frac{\partial}{\partial k} + \alpha_4 PS, \qquad \alpha_1,\ \alpha_2,\ \alpha_3,\ \alpha_4 \in \mathbf{R}.$$

b) We provide a probabilistic interpretation for G using Markovian Bernstein processes, with a particular attention given to N_4 which is shown to generate time dilations in the following sense. Given η a strictly positive solution of $L\eta = 0$, let $(Z_\eta(t))_{t\in\mathrm{R}_+}$ denote the Markov process with generator $\mathcal{L}_\eta$ defined by the Doob transformation

$$\mathcal{L}_\eta f(k,t) := \frac{1}{\eta(k,t)} L(\eta f)(k,t).$$

Then $(Z_\eta(t))_{t\in\mathrm{R}_+}$ and $(Z_{e^{\beta N_4}\eta}(t))_{t\in\mathrm{R}_+}$ are linked by the relation in distribution

$$Z_{e^{\beta N_4}\eta}(t) \simeq Z_\eta(e^\beta t), \qquad t \in \mathbf{R}_+.$$

Dual versions of *a*) and *b*) are obtained by time reversal.

We proceed as follows. The Doob transformation, which defines the time reversible Markov processes on which our probabilistic interpretation is based, is recalled in Section 2. In Section 3 we consider the symmetries of the discrete heat equation (2), and in Section 4 their probabilistic interpretation is constructed. In Section 5 we state the corresponding Girsanov theorem. In the appendix (Section 6) we recall some elements of normal martingale theory and quantum stochastic calculus to prove, in a general framework, the commutation relations satisfied by P and L.

2 Doob transform and reversible Markov processes

Consider $(X_t)_{t\in\mathrm{R}_+}$ a Markov process whose forward and backward generators H and H^* are assumed to be mutually adjoint, i.e. $H^\dagger = H^*$, with respect to a given reference measure λ. Consider $\eta(k,t)$, resp. $\eta^*(k,t)$, a strictly positive solution of the partial integro-differential equation

$$-\frac{\partial \eta}{\partial t}(k,t) = H\eta(k,t), \tag{3}$$

resp.

$$\frac{\partial \eta^*}{\partial t}(k,t) = H^*\eta^*(k,t). \tag{4}$$

To η and η^* we associate the (non homogeneous) Markov jump processes $(Z_\eta(t))_{t\in\mathrm{R}_+}$ and $(Z^*_{\eta^*}(t))_{t\in\mathrm{R}_+}$ whose respective forward and backward generators $\mathcal{L}_\eta$ and $\mathcal{L}^*_{\eta^*}$ are given by the Doob transforms

$$\mathcal{L}_\eta f(k,t) := \frac{1}{\eta(k,t)} \left(H + \frac{\partial}{\partial t} \right) (\eta f)(k,t) \tag{5}$$

and

$$\mathcal{L}^*_{\eta^*} f(k,t) := \frac{1}{\eta^*(k,t)} \left(H^* - \frac{\partial}{\partial t} \right) (\eta^* f)(k,t). \tag{6}$$

The processes $(Z_\eta(t))_{t\in\mathrm{R}_+}$ and $(Z^*_{\eta^*}(t))_{t\in\mathrm{R}_+}$ are called Bernstein processes [13], and by construction $\mathcal{L}^*_{\eta^*}$ is adjoint of $\mathcal{L}_\eta$ with respect to the measure of density $\eta(k,t)\eta^*(k,t)$ with respect to $\lambda(dk)$. Moreover we have the following proposition.

Proposition 1. *For any* $0 \leq u < v$*, both* $(Z_\eta(t))_{t\in[u,v]}$ *and* $(Z^*_{\eta^*}(t))_{t\in[u,v]}$ *have distribution*

$$\eta(k,t)\eta^*(k,t)\lambda(dk), \qquad t \in [u,v],$$

provided they are respectively started with the initial and terminal distributions

$$\eta(k,u)\eta^*(k,u)\lambda(dk) \qquad \text{and} \qquad \eta(k,v)\eta^*(k,v)\lambda(dk).$$

Proof. Let us show that $Z_\eta(t)$ has density $\eta(\cdot,t)\eta^*(\cdot,t)$ at time $t \in [u,v]$. For all $f \in \mathcal{S}(\mathbf{R})$ we have

$$\begin{aligned}
\frac{d}{dt}\langle \eta\eta^*(\cdot,t), f(\cdot)\rangle_{L^2(d\lambda)} &= \left\langle f(\cdot), \eta^*(\cdot,t)\frac{\partial\eta}{\partial t}(\cdot,t)\right\rangle_{L^2(d\lambda)} + \left\langle f(\cdot), \eta(\cdot,t)\frac{\partial\eta^*}{\partial t}(\cdot,t)\right\rangle_{L^2(d\lambda)} \\
&= \left\langle f(\cdot), \eta^*(\cdot,t)\frac{\partial\eta}{\partial t}(\cdot,t)\right\rangle_{L^2(d\lambda)} + \langle f(\cdot), \eta(\cdot,t)H^*\eta^*(\cdot,t)\rangle_{L^2(d\lambda)} \\
&= \left\langle \eta^*(\cdot,t), f(\cdot)\frac{\partial\eta}{\partial t}(\cdot,t)\right\rangle_{L^2(d\lambda)} + \langle \eta^*(\cdot,t), H(f\eta)(\cdot,t)\rangle_{L^2(d\lambda)} \\
&= \langle \eta\eta^*(\cdot,t), \mathcal{L}_\eta f(\cdot)\rangle_{L^2(d\lambda)}.
\end{aligned}$$

Hence by standard arguments,

$$\begin{aligned}
\langle \eta\eta^*(\cdot,t), f(\cdot)\rangle_{L^2(d\lambda)} &= \sum_{n=0}^{\infty}\frac{(t-u)^n}{n!}\frac{d^n}{dt^n}\langle \eta\eta^*(\cdot,t), f(\cdot)\rangle_{L^2(d\lambda)} \, |_{t=u} \\
&= \sum_{n=0}^{\infty}\frac{(t-u)^n}{n!}\langle \eta\eta^*(\cdot,u), (\mathcal{L}_\eta)^n f(\cdot)\rangle_{L^2(d\lambda)} \\
&= \langle \eta\eta^*(\cdot,u), e^{(t-u)\mathcal{L}_\eta} f(\cdot)\rangle_{L^2(d\lambda)} \\
&= \int e^{(t-u)\mathcal{L}_\eta} f(k)\eta\eta^*(\cdot,u)\lambda(dk), \qquad t \in [u,v].
\end{aligned}$$

Similarly we have

$$\begin{aligned}
\frac{d}{dt}\langle \eta\eta^*(\cdot,t), f(\cdot)\rangle_{L^2(d\lambda)} &= \left\langle f(\cdot), \eta(\cdot,t)\frac{\partial\eta^*}{\partial t}(\cdot,t)\right\rangle_{L^2(d\lambda)} + \left\langle f(\cdot), \eta^*(\cdot,t)\frac{\partial\eta}{\partial t}(\cdot,t)\right\rangle_{L^2(d\lambda)} \\
&= \left\langle f(\cdot), \eta(\cdot,t)\frac{\partial\eta^*}{\partial t}(\cdot,t)\right\rangle_{L^2(d\lambda)} - \langle f(\cdot), \eta^*(\cdot,t)H\eta(\cdot,t)\rangle_{L^2(d\lambda)} \\
&= \left\langle \eta(\cdot,t), f(\cdot)\frac{\partial\eta^*}{\partial t}(\cdot,t)\right\rangle_{L^2(d\lambda)} - \langle \eta(\cdot,t), H^*(f\eta^*)(\cdot,t)\rangle_{L^2(d\lambda)} \\
&= -\langle \eta\eta^*(\cdot,t), \mathcal{L}^*_{\eta^*} f(\cdot)\rangle_{L^2(d\lambda)},
\end{aligned}$$

hence

$$\begin{aligned}
\langle \eta\eta^*(\cdot,t), f(\cdot)\rangle_{L^2(d\lambda)} &= \sum_{n=0}^{\infty}\frac{(t-v)^n}{n!}\frac{d^n}{dt^n}\langle \eta\eta^*(\cdot,t), f(\cdot)\rangle_{L^2(d\lambda)} \, |_{t=v} \\
&= \sum_{n=0}^{\infty}\frac{(v-t)^n}{n!}\langle \eta\eta^*(\cdot,v), (\mathcal{L}^*_{\eta^*})^n f(\cdot)\rangle_{L^2(d\lambda)} \\
&= \langle \eta\eta^*(\cdot,v), e^{(v-t)\mathcal{L}^*_{\eta^*}} f(\cdot)\rangle_{L^2(d\lambda)} \\
&= \int e^{(v-t)\mathcal{L}^*_{\eta^*}} f(k)\eta\eta^*(k,v)\lambda(dk), \qquad t \in [u,v].
\end{aligned}$$

□

Recall that a Theorem of Beurling [1] states that any given initial and final probability densities π_u and π_v on $\mathbf{R}^d$ can be written in the product forms $\pi_u(k) = \eta(k,u)\eta^*(k,u)$ and $\pi_v(k) = \eta(k,v)\eta^*(k,v)$ where η, η^* are suitably chosen solutions of (3) and (4). Hence the processes $(Z_\eta(t))_{t\in[u,v]}$ and $(Z^*_{\eta^*}(t))_{t\in[u,v]}$ can be used to construct Markovian bridges having arbitrary prescribed absolutely continuous initial and final distributions. Thus they provide an Euclidean probabilistic interpretation of a given quantum system with Hamiltonian H, in which complex conjugation is replaced with time reversal, cf. [13], [11], [12].

Let now

$$L = \frac{\partial}{\partial t} + H, \quad \text{and} \quad L^* = -\frac{\partial}{\partial t} + H^*.$$

Given $A \in \mathsf{G}$ and $B^* \in \mathsf{G}^*$ in the symmetry groups G, G^* of L and L^*, a natural question is to consider the application

$$\{\eta, \eta^*\} \mapsto \{A\eta, B^*\eta^*\}$$

which acts on respective solutions of $L\eta = 0$ and $L^*\eta^* = 0$, and to determine the associated pathwise transformations which map

$$(Z_\eta(t))_{t\in[u,v]} \quad \text{to} \quad (Z_{A\eta}(t))_{t\in[u,v]}$$

and

$$(Z^*_{\eta^*}(t))_{t\in[u,v]} \quad \text{to} \quad (Z_{B^*\eta^*}(t))_{t\in[u,v]}.$$

In this way we construct a probabilistic representation of the symmetry groups of (3) and (4) using Bernstein jump processes.

In other terms we would like to study the relationship between $\mathcal{L}_\eta$ and $\mathcal{L}_{A\eta}$, resp. $\mathcal{L}^*_{\eta^*}$ and $\mathcal{L}^*_{A\eta^*}$, and to determine of a mapping $\phi : \mathbf{R} \times \mathbf{R}_+ \to \mathbf{R} \times \mathbf{R}_+$ such that

$$(A\eta \cdot B^*\eta^*) \circ \phi(k,t) = (\eta \cdot \eta^*)(k,t), \qquad (k,t) \in \mathbf{R} \times \mathbf{R}_+.$$

This procedure is trivial for the pairs $(e^{\beta N_k}, e^{-\beta N^*_k})$, $\beta \in \mathbf{R}$, $k = 1,2,3$, described in the introduction, for which the associated transformations are the identity and time and space shifts, respectively. In this paper we investigate the role played by N_4, and we show that it is associated to time dilations and a creation operator.

3 Lie algebra generators

Consider $(N_t)_{t\in\mathbf{R}_+}$ a standard Poisson process with forward/backward generators

$$L\eta(k,t) := \frac{\partial\eta}{\partial t}(k,t) + \eta(k+1,t) - \eta(k,t), \tag{7}$$

and

$$L^*\eta^*(k,t) := -\frac{\partial\eta^*}{\partial t}(k,t) + \eta^*(k-1,t) - \eta^*(k,t). \tag{8}$$

In other terms we have

$$Hf(k,t) = f(k+1,t) - f(k,t) \qquad \text{and} \qquad H^*g(k,t) = g(k-1,t) - g(k,t),$$

which are mutually adjoint with respect to the counting measure

$$\lambda(dl) = \sum_{k\in\mathbb{Z}} \delta_k(dl),$$

where δ_x denotes the Dirac measure at $x \in \mathbf{R}$. For example the standard Poisson process is obtained with

$$\eta(k,t) = 1 \quad \text{and} \quad \eta^*(k,t) = 1_{\{k \geq 0\}} \frac{t^k}{k!} e^{-t}.$$

Recall that the Charlier polynomials $(C_n(k,t))_{n\in\mathbf{N}}$ of parameter $t \in \mathbf{R}$, defined through their generating function

$$\psi_\lambda(k) = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} C_n(k,t) = e^{-\lambda t}(1+\lambda)^k, \quad \lambda \in (-1,1), \quad k,t \in \mathbf{R},$$

or by

$$C_0(k,t) = 1, \qquad C_1(k,t) = k - t,$$

and the recurrence relation

$$C_{n+1}(k,t) = (k-n-t)C_n(k,t) - ntC_{n-1}(k,t), \tag{9}$$

are solutions of $L\eta(k,t) = 0$, $n \in \mathbf{N}$, with the initial condition $\eta(k,0) = \prod_{i=1}^{n}(k-i)$.

Definition 1. *Let the creation operator P be defined as*

$$P\eta(k,t) = k\eta(k-1,t) - t\eta(k,t).$$

The Charlier polynomials satisfy the relation

$$PC_n(k,t) = kC_n(k-1,t) - tC_n(k,t) = C_{n+1}(k,t), \qquad n \in \mathbf{N}.$$

We have the commutation relation

$$[P,L] = 0, \tag{10}$$

which is easily proved by direct calculation or as a consequence of the more general result (Proposition 10) stated in the appendix.

Definition 2. *Let $\mathcal{G}$ be the Lie algebra with commutator $[\cdot,\cdot]$ spanned by all first-order differential operators with smooth coefficients, of the form*

$$N = \alpha(k,t)I + \beta(k,t)\frac{\partial}{\partial t} + \gamma(k,t)\frac{\partial}{\partial k}, \tag{11}$$

where I denotes the identity, and verifying the stability property

$$L\eta = 0 \quad \Rightarrow \quad LN\eta = 0.$$

Similarly, let $\mathcal{G}^*$ denote the Lie algebra spanned by all first order differential operators N^* of the form (11) and satisfying

$$L^*\eta^* = 0 \quad \Rightarrow \quad L^*N^*\eta^* = 0.$$

Definition 3. *Let R be defined by $R\eta(k,t) = \eta(-k,-t)$, $k \in \mathbf{R}$, $t \in [u,v]$.*

Note that we have

$$H^* = RHR, \qquad L^* = RLR.$$

In the same way, any element A of the symmetry group G can be associated to an element $A^* := RAR$ of G^*, and the mapping $N \mapsto N^*$ is an isomorphism from $\mathcal{G}$ onto $\mathcal{G}^*$, and from G onto G^*.

Next, we state representation results for the elements of $\mathcal{G}$ and $\mathcal{G}^*$.

Proposition 2. *On the solution space*

$$\mathrm{Ker}\,(L) = \left\{\eta \ : \ \frac{\partial \eta}{\partial t}(k,t) + \eta(k+1,t) - \eta(k,t) = 0\right\}$$

of L, any element of $\mathcal{G}$ can be written as

$$N = \alpha_1 I + \alpha_2 \frac{\partial}{\partial t} + \alpha_3 \frac{\partial}{\partial k} + \alpha_4 PS, \tag{12}$$

for some $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbf{R}$.

Proof. We start by showing that any element N of $\mathcal{G}$ can be represented as

$$N = (\alpha + (k-t)\lambda)I + (\xi + \lambda t)\frac{\partial}{\partial t} + \gamma \frac{\partial}{\partial k}, \tag{13}$$

for some $a, \xi, \lambda, \gamma \in \mathbf{R}$. Indeed, a necessary and sufficient condition for an element N of the form (11) to belong to $\mathcal{G}$ is the existence of a function $\lambda(k,t)$ such that

$$[N, L] = \lambda(k,t)L, \tag{14}$$

on $\{\eta \ : \ L\eta = 0\}$. Now, Relation (14) reads

$$\begin{aligned}
& NL\eta(k,t) - LN\eta(k,t) \\
&= \frac{\partial \gamma}{\partial t}(k,t)\frac{\partial \eta}{\partial k}(k,t) + \gamma(k,t)\frac{\partial^2 \eta}{\partial t \partial k}(k,t) + \eta(k,t)\frac{\partial \alpha}{\partial t}(k,t) + \alpha(k,t)\frac{\partial \eta}{\partial t}(k,t) \\
&\quad + \frac{\partial \beta}{\partial t}(k,t)\frac{\partial \eta}{\partial t}(k,t) + \beta(k,t)\frac{\partial^2 \eta}{\partial t^2}(k,t) \\
&\quad + \gamma(k+1,t)\frac{\partial \eta}{\partial k}(k+1,t) + \alpha(k+1,t)\eta(k+1,t) + \beta(k+1,t)\frac{\partial \eta}{\partial t}(k+1,t) \\
&\quad - \gamma(k,t)\frac{\partial \eta}{\partial k}(k,t) - \alpha(k,t)\eta(k,t) - \beta(k,t)\frac{\partial \eta}{\partial t}(k,t) \\
&\quad - \left(\gamma(k,t)\frac{\partial^2 \eta}{\partial k \partial t}(k,t) + \gamma(k,t)\frac{\partial \eta}{\partial k}(k+1,t) - \gamma(k,t)\frac{\partial \eta}{\partial k}(k,t)\right. \\
&\quad + \alpha(k,t)\frac{\partial \eta}{\partial t}(k,t) + \alpha(k,t)\eta(k+1,t) - \alpha(k,t)\eta(k,t) \\
&\quad \left. + \beta(k,t)\frac{\partial^2 \eta}{\partial t^2}(k,t) + \beta(k,t)\frac{\partial \eta}{\partial t}(k+1,t) - \beta(k,t)\frac{\partial \eta}{\partial t}(k,t)\right) \\
&= \lambda(k,t)\left(\frac{\partial \eta}{\partial t}(k,t) + \eta(k+1,t) - \eta(k,t)\right) \\
&= \lambda(k,t)L\eta(k,t),
\end{aligned}$$

from which we deduce

$$\begin{aligned}
& \frac{\partial \gamma}{\partial t}(k,t) = 0, \\
& \gamma(k+1,t) - \gamma(k,t) = 0, \\
& \frac{\partial \alpha}{\partial t}(k,t) = -\lambda(k,t), \\
& \alpha(k+1,t) - \alpha(k,t) = \lambda(k,t), \\
& \frac{\partial \beta}{\partial t}(k,t) = \lambda(k,t),
\end{aligned}$$

$$\beta(k+1,t) - \beta(k,t) = 0.$$

It follows that $\lambda(k+1,t) = \lambda(k,t)$ and $\alpha(k,t) = a(t) - k\lambda(t)$, hence

$$a'(t) - k\lambda'(t) = -\lambda(t),$$

which implies $\lambda'(t) = 0$ and $a'(t) = -\lambda(t)$, i.e. $a(t) = a - \lambda t$, for some $a, \lambda \in \mathbf{R}$. On the other hand we have

$$\alpha(k,t) = a + (k-t)\lambda, \qquad \beta(k,t) = \xi + \lambda t, \qquad \gamma(k,t) = \gamma,$$

for some $\xi, \gamma \in \mathbf{R}$, which proves (13). Finally we have

$$\begin{aligned}
N\eta(k,t) &= \alpha_3 \frac{\partial \eta}{\partial k}(k,t) + (\alpha_1 + (k-t)\alpha_4)\eta(k,t) + (\alpha_2 + \alpha_4 t)\frac{\partial \eta}{\partial t}(k,t) \\
&= \alpha_3 \frac{\partial \eta}{\partial k}(k,t) + (\alpha_1 + (k-t)\alpha_4)\eta(k,t) + \alpha_4 t(\eta(k,t) - \eta(k+1,t)) + \alpha_2 \frac{\partial \eta}{\partial t}(k,t) \\
&= \alpha_3 \frac{\partial \eta}{\partial k}(k,t) + \alpha_1 \eta(k,t) + \alpha_4 (k\eta(k,t) - t\eta(k+1,t)) + \alpha_2 \frac{\partial \eta}{\partial t}(k,t) \\
&= \alpha_3 \frac{\partial \eta}{\partial k}(k,t) + \alpha_1 \eta(k,t) + \alpha_4 PS\eta(k,t) + \alpha_2 \frac{\partial \eta}{\partial t}(k,t).
\end{aligned}$$

□

The first three generators N_1, N_2, N_3, resp. N_1^*, N_2^*, N_3^*, of $\mathcal{G}$, resp. $\mathcal{G}^*$ are given by

$$N_1 = N_1^* = I, \quad N_2 = -N_2^* = \frac{\partial}{\partial t}, \quad \text{and} \quad N_3 = -N_3^* = \frac{\partial}{\partial k},$$

and we let $N_4 = PS$, i.e. (12) is written as

$$N = \alpha_1 N_1 + \alpha_2 N_2 + \alpha_3 N_3 + \alpha_4 N_4.$$

We have the commutation table

$[\cdot,\cdot]$	N_1	N_2	N_3	N_4
N_1	0	0	0	0
N_2	0	0	0	$-N_1$
N_3	0	0	0	N_1
N_4	0	$-N_1$	N_1	0

The operators P^* and S^* satisfy

$$P^*\eta^*(k,t) = -k\eta^*(k+1,t) + t\eta^*(k,t), \quad \text{and} \quad S^*\eta^*(k,t) = \eta^*(k-1,t),$$

with $S^* = e^{N_3^*}$, and the relations

$$[P^*, S^*] = -I, \qquad [P^*, L^*] = 0,$$

We may also let

$$C_n^*(k,t) := RC_n(k,t) = C_n(-k,-t), \qquad n \in \mathbf{N}, \quad k,t \in \mathbf{R},$$

and in this case

$$P^* C_n^*(k,t) = kC_n^*(k+1,t) - tC_n^*(k,t) = C_{n+1}^*(k,t), \qquad n \in \mathbf{N}.$$

Similarly to the above, we have the following proposition, with $N_4 = P^*S^*$.

Proposition 3. *On the solution space*

$$\mathrm{Ker}\,(L^*) = \left\{ \eta^* \ : \ -\frac{\partial \eta^*}{\partial t}(k,t) + \eta^*(k-1,t) - \eta^*(k,t) = 0 \right\}$$

of L^*, any element of $\mathcal{G}^*$ can be written as

$$N^* = \alpha_1 N_1^* + \alpha_2 N_2^* + \alpha_3 N_3^* + \alpha_4 N_4^*,$$

for some α_1, α_2, α_3, $\alpha_4 \in \mathbf{R}$.

4 Probabilistic interpretation of G

Consider again two respective (λ-a.e. strictly positive) solutions $\eta(k,t)$, $\eta^*(k,t)$ of the finite difference partial differential equations

$$L\eta(k,t) = 0 \quad \text{and} \quad L^*\eta^*(k,t) = 0.$$

Using (5) and (6), the forward and backward generators of $(Z_\eta(t))_{t\in\mathrm{R}_+}$ and $(Z^*_{\eta^*}(t))_{t\in\mathrm{R}_+}$ can be computed as follows:

$$\mathcal{L}_\eta f(k) = \frac{\eta(k+1,t)}{\eta(k,t)}(f(k+1) - f(k)),$$

and

$$\mathcal{L}^*_{\eta^*} f(k) = \frac{\eta^*(k-1,t)}{\eta^*(k,t)}(f(k-1) - f(k)).$$

In particular $(Z_\eta(t))_{t\in\mathrm{R}_+}$ and $(Z^*_{\eta^*}(t))_{t\in\mathrm{R}_+}$ are point processes with respective forward and backward intensities

$$\frac{\eta(Z_\eta(t^-)+1,t)}{\eta(Z_\eta(t^-),t)} \qquad \text{and} \qquad \frac{\eta^*(Z^*_{\eta^*}(t^+)-1,t)}{\eta^*(Z^*_{\eta^*}(t^+),t)}.$$

We will denote by $(\mathcal{F}_t)_{t\in[u,v]}$, resp. $(\mathcal{F}^*_t)_{t\in[u,v]}$ the forward, resp. backward filtration generated by the transformations $(Z_\eta(t))_{t\in[u,v]}$, resp. $(Z^*_{\eta^*}(t))_{t\in[u,v]}$. The generators $(N_1, -N_1^*) = (I, -I)$, $(N_2, -N_2^*) = \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right)$, and $(N_3, -N_3^*) = \left(\frac{\partial}{\partial k}, \frac{\partial}{\partial k}\right)$, are respectively associated to:

- the identity:

$$e^{\alpha N_1}\eta(k,s)e^{-\alpha N_1^*}\eta^*(k,s) = \eta(k,s)\eta^*(k,s),$$

- time translations:

$$e^{\alpha N_2}\eta(k,s)e^{-\alpha N_2^*}\eta^*(k,s) = \eta(k,s+\alpha)\eta^*(k,s+\alpha),$$

  corresponding to $(Z_\eta(s))_{s\in[u,v]} \mapsto (Z_\eta(s+t))_{s\in[u+t,v+t]}$,
- space translations:

$$e^{\alpha N_3}\eta(k,s)e^{-\alpha N_3^*}\eta^*(k,s) = \eta(k+\alpha,s)\eta^*(k+\alpha,s),$$

  corresponding to $(Z_\eta(s))_{s\in[u,v]} \mapsto (k + Z_\eta(s))_{s\in[u,v]}$,

and similarly for $(Z^*_{\eta^*}(s))_{s\in[u,v]}$.

We now focus on the role of the operators $N_4 = PS$ and $N_4^* = P^*S^*$, and show that they are linked to pathwise transformations of $(Z_\eta(t))_{t\in\mathrm{R}_+}$ and $(Z^*_{\eta^*}(t))_{t\in\mathrm{R}_+}$ by time dilations. We identify the one-parameter semigroup of solutions of $L\eta = 0$ generated by $N_4 = PS$ and we show that it corresponds to time dilations.

Proposition 4. *For all $\eta \in \mathcal{S}$ and all $k \in \mathbf{R}$, $t \in \mathbf{R}_+$, $\beta \in \mathbf{R}$ we have*

$$e^{\beta N_4}\eta(k,t) = \exp(k\beta - (e^\beta - 1)t)\eta(k, e^\beta t), \tag{15}$$

and

$$e^{\beta N_4^*}\eta^*(k,t) = \exp(-k\beta + (e^\beta - 1)t)\eta^*(k, e^\beta t). \tag{16}$$

Proof. For convenience of notation we make the change of variable $\beta = \log(1+\alpha)$, $\alpha > -1$. Let

$$\eta_\alpha(k,t) = (1+\alpha)^k e^{-\alpha t}\eta(k,(1+\alpha)t), \qquad k \in \mathbf{R}, \quad t \in \mathbf{R}_+, \quad \eta \in \mathcal{S}.$$

We have

$$\begin{aligned}
\frac{\partial \eta_\alpha}{\partial \alpha}(k,t) &= k(1+\alpha)^{k-1}e^{-\alpha t}\eta(k,(1+\alpha)t) \\
&\quad -t(1+\alpha)^k e^{-\alpha t}\eta(k,(1+\alpha)t) + t(1+\alpha)^k e^{-\alpha t}\frac{\partial \eta}{\partial t}(k,(1+\alpha)t) \\
&= k(1+\alpha)^{k-1}e^{-\alpha t}\eta(k,(1+\alpha)t) - t(1+\alpha)^k e^{-\alpha t}\eta(k,(1+\alpha)t) \\
&\quad +t(1+\alpha)^k e^{-\alpha t}(\eta(k,(1+\alpha)t) - \eta(k+1,(1+\alpha)t)) \\
&= k(1+\alpha)^{k-1}e^{-\alpha t}\eta(k,(1+\alpha)t) - t(1+\alpha)^k e^{-\alpha t}\eta(k+1,(1+\alpha)t) \\
&= (1+\alpha)^{-1}\left(kS\eta_\alpha(k-1,t) - tS\eta_\alpha(k,t)\right) \\
&= (1+\alpha)^{-1}PS\eta_\alpha(k,t),
\end{aligned}$$

which shows that η_α is solution in $\mathcal{S}$ of

$$\begin{cases} \dfrac{\partial \eta_\alpha}{\partial \alpha} = \dfrac{1}{1+\alpha}N_4\eta_\alpha, & \alpha > -1, \\ \eta_0 = \eta, \end{cases}$$

hence $\eta_\alpha = \exp(\log(1+\alpha)N_4)\eta$, $\alpha > -1$. Relation (16) can be obtained by direct transfer using the mapping R, or from the solution of the equation

$$\begin{cases} \dfrac{\partial \eta^*_\alpha}{\partial \alpha} = \dfrac{1}{1+\alpha}N_4^*\eta^*_\alpha, & \alpha > -1, \\ \eta^*_0 = \eta^*, \end{cases} \tag{17}$$

which is given by

$$\eta^*_\alpha(k,t) = \exp(\log(1+\alpha)N_4^*)\eta^*(k,t) = (1+\alpha)^{-k}e^{\alpha t}\eta^*(k,(1+\alpha)t),$$

since we have

$$\begin{aligned}
\frac{\partial \eta^*_\alpha}{\partial \alpha}(k,t) &= -k(1+\alpha)^{-k-1}e^{\alpha t}\eta^*(k,(1+\alpha)t) \\
&\quad +t(1+\alpha)^{-k}e^{\alpha t}\eta^*(k,(1+\alpha)t) + t(1+\alpha)^{-k}e^{-\alpha t}\frac{\partial \eta^*}{\partial t}(k,(1+\alpha)t) \\
&= -k(1+\alpha)^{-k-1}e^{-\alpha t}\eta^*(k,(1+\alpha)t) + t(1+\alpha)^{-k}e^{\alpha t}\eta^*(k,(1+\alpha)t) \\
&\quad +t(1+\alpha)^{-k}e^{\alpha t}(\eta^*(k-1,(1+\alpha)t) - \eta^*(k,(1+\alpha)t)) \\
&= -(1+\alpha)^{-1}\left(k(1+\alpha)^{-k}e^{-\alpha t}\eta(k,(1+\alpha)t) - t(1+\alpha)^{-(k-1)}e^{-\alpha t}\eta(k-1,(1+\alpha)t)\right) \\
&= (1+\alpha)^{-1}P^*S^*\eta^*_\alpha(k,t).
\end{aligned}$$

□

The following proposition is now obvious and shows that $(Z_{\eta_\alpha}(s))_{s\in[u,v]}$, $(Z^*_{\eta^*_\alpha}(s))_{s\in[u,v]}$ have same laws as $(Z_\eta((1+\alpha)s))_{s\in[u,v]}$ and $(Z^*_{\eta^*}((1+\alpha)s))_{s\in[u,v]}$.

Proposition 5. *We have*

$$e^{\beta N_4}\eta(k,t)e^{-\beta N_4^*}\eta^*(k,t) = (\eta^*\eta)(k, e^\beta t), \qquad k \in \mathbf{R}, \quad t \in \mathbf{R}_+.$$

In other terms, $(Z_{\eta_\alpha}(s))_{s\in[u,v]}$ and $(Z^*_{\eta^*_\alpha}(s))_{s\in[u,v]}$ have the forward and backward intensities

$$\frac{\eta_\alpha(Z_{\eta_\alpha}(t^-)+1,t)}{\eta_\alpha(Z_{\eta_\alpha}(t^-),t)} = (1+\alpha)\frac{\eta(Z_{\eta_\alpha}(t^-)+1,t)}{\eta(Z_{\eta_\alpha}(t^-),t)},$$

and

$$\frac{\eta^*_\alpha(Z^*_{\eta^*_\alpha}(t^+)-1,t)}{\eta^*_\alpha(Z^*_{\eta^*_\alpha}(t^+),t)} = (1+\alpha)\frac{\eta(Z^*_{\eta^*_\alpha}(t^+)-1,t)}{\eta(Z^*_{\eta^*_\alpha}(t^+),t)},$$

i.e. $(Z_{\eta_\alpha}(\tau^{-1}(t)))_{t\in\mathrm{R}_+}$ is a standard Poisson process, where the time change $\tau(\cdot)$ is defined by

$$\tau(t) = \int_0^t \frac{\eta_\alpha(Z_{\eta_\alpha}(s)+1,s)}{\eta_\alpha(Z_{\eta_\alpha}(s),s)}ds = (1+\alpha)\int_0^t \frac{\eta(Z_{\eta_\alpha}(s)+1,s)}{\eta(Z_{\eta_\alpha}(s),s)}ds, \qquad t \in \mathbf{R}_+.$$

5 Absolute continuity

The aim of this section is to study the change of measure generated by the transformations

$$(Z_\eta(t))_{t\in[u,v]} \mapsto (Z_\zeta(t))_{t\in[u,v]}$$

and

$$(Z^*_{\eta^*}(t))_{t\in[u,v]} \mapsto (Z^*_{\zeta^*}(t))_{t\in[u,v]},$$

given η, ζ, resp. η^*, ζ^*, two a.e. strictly positive solutions of $\{L\eta = 0\}$, resp. $\{L^*\eta^* = 0\}$.

Clearly, the (unconditional) density of $Z_\zeta(t)$ with respect to $Z_\eta(t)$ is

$$\Psi(Z_\eta(t),t) = \frac{\zeta(Z_\eta(t),t)}{\eta(Z_\eta(t),t)}\frac{\zeta^*(Z_\eta(t),t)}{\eta^*(Z_\eta(t),t)}, \qquad t \in [u,v],$$

as follows from

$$\begin{aligned} E[f(Z_\zeta(t))] &= \int f(k)\zeta(k,t)\zeta^*(k,t)\lambda(dk) \\ &= \int f(k)\Psi(k,t)\eta(k,t)\eta^*(k,t)\lambda(dk) \\ &= E[f(Z_\eta(t))\Psi(Z_\eta(t),t)]. \end{aligned}$$

Similarly the density of $Z^*_{\zeta^*}(t)$ with respect to $Z^*_{\eta^*}(t)$ is

$$\Psi(Z^*_\eta(t),t) = \frac{\zeta(Z^*_{\eta^*}(t),t)}{\eta(Z^*_{\eta^*}(t),t)}\frac{\zeta^*(Z^*_{\eta^*}(t),t)}{\eta^*(Z^*_{\eta^*}(t),t)}, \qquad t \in [u,v].$$

We now turn to conditional densities.

Proposition 6. *The density of $(Z_\zeta(u))_{s\le u\le t}$ with respect to $(Z_\eta(u))_{s\le u\le t}$ given $\mathcal{F}_s$ is*

$$\Lambda_{s,t}(Z_\zeta(s), Z_\eta(t)) = \frac{\zeta(Z_\eta(t),t)}{\eta(Z_\eta(t),t)}\frac{\eta(Z_\zeta(s),s)}{\zeta(Z_\zeta(s),s)}.$$

Proof. By (5), the process $(Z_\eta(s))_{s\in[u,v]}$ has the forward transition semigroup

$$p_\eta(t,k,u,dl) = \frac{\eta(l,u)}{\eta(k,t)} h(t,k,u,dl),$$

where $h(s,l,t,dk)$ denotes the kernel of $e^{(t-s)H}$, i.e.

$$e^{(t-s)H} f(l) = \int f(k) h(s,l,t,dk), \qquad l \in \mathbf{R}, \quad 0 \le s < t.$$

By the Markov property of $(Z_\eta(t))_{t\in[u,v]}$, it suffices to check that the finite-dimensional distributions satisfy:

$$\begin{aligned}
&E[f(Z_\eta(t_1),\dots,Z_\eta(t_n))\Lambda_{s,t}(Z_\zeta(s),Z_\eta(t))|Z_\eta(s)=k]\\
&= \int\cdots\int f(k_1,\dots,k_n) h(s,k,t_1,dk_1)\cdots h(t_n,k_n,t,dl)\Lambda_{s,t}(k,l)\frac{\eta(l,t)}{\eta(k,s)}\\
&= \int\cdots\int f(k_1,\dots,k_n) h(s,k,t_1,dk_1)\cdots h(t_n,k_n,t,dl)\frac{\zeta(l,t)}{\zeta(k,s)}\\
&= E[f(Z_\zeta(t_1),\dots,Z_\zeta(t_n))|Z_\zeta(s)=k],
\end{aligned}$$

for all $s \le t_1 < \cdots < t_n \le t$, $f \in \mathcal{C}_b(\mathbf{R})$. □

Note that the unconditional density $\Psi(Z_\eta(t),t)$ can be recovered from the conditional density $\Lambda_{s,t}(Z_\zeta(s),Z_\eta(t))$, as follows:

$$\begin{aligned}
E[f(Z_\zeta(t))] &= E[E[f(Z_\zeta(t)) \mid Z_\zeta(s)]]\\
&= E[E[f(Z_\eta(t))\Lambda_{s,t}(Z_\zeta(s),Z_\eta(t)) \mid Z_\zeta(s)]]\\
&= E[f(Z_\eta(t))\Lambda_{s,t}(Z_\zeta(s),Z_\eta(t))]\\
&= E\left[f(Z_\eta(t))\int \Lambda_{s,t}(l,Z_\eta(t))dP(Z_\zeta(s)=l \mid Z_\eta(t))\right], \qquad f \in \mathcal{C}_b^\infty(\mathbf{R}),
\end{aligned}$$

which implies

$$\Psi(k,t) = \int \Lambda_{s,t}(l,k)dP(Z_\zeta(s)=l \mid Z_\eta(t)=k),$$

$k \in \mathbf{R}$, $0 \le s \le t$. More explicitly, using the duality

$$h(s,l,t,dk)\lambda(dl) = h^*(s,dl,t,k)\lambda(dk) \tag{18}$$

between H and H^* we have

$$\begin{aligned}
\Psi(k,t) &= \frac{\zeta(k,t)}{\eta(k,t)}\frac{\zeta^*(k,t)}{\eta^*(k,t)}\\
&= \int \zeta(k,t)\zeta^*(l,s)\frac{h^*(s,dl,t,k)}{\eta^*(k,t)\eta(k,t)}\\
&= \int \frac{\zeta(k,t)}{\zeta(l,s)}\frac{h(s,l,t,dk)}{\eta^*(k,t)\eta(k,t)\lambda(dk)}\zeta(l,s)\zeta^*(l,s)\lambda(dl)\\
&= \int \frac{\zeta(k,t)}{\zeta(l,s)}\frac{\eta(l,s)}{\eta(k,t)}\frac{h(s,l,t,dk)}{\eta^*(k,t)\eta(k,t)\lambda(dk)}\frac{\eta(k,t)}{\eta(l,s)}\zeta(l,s)\zeta^*(l,s)\lambda(dl)\\
&= \int \Lambda_{s,t}(l,k)\frac{p_\eta(s,l,t,dk)}{\eta^*(k,t)\eta(k,t)\lambda(dk)}dP(Z_\zeta(s)=l)\\
&= \int \Lambda_{s,t}(l,k)dP(Z_\zeta(s)=l \mid Z_\eta(t)=k).
\end{aligned}$$

The analogous time reversed statement on conditional densities is as follows.

Proposition 7. *The density of* $(Z^*_{\zeta^*}(u))_{s\le u\le t}$ *with respect to* $(Z^*_{\eta^*}(u))_{s\le u\le t}$ *given* $\mathcal{F}^*_t$ *is*

$$\Lambda^*_{s,t}(k,l)=\frac{\zeta^*(Z^*_{\zeta^*}(s),s)}{\eta^*(Z^*_{\eta^*}(t),t)}\frac{\eta^*(Z^*_{\zeta^*}(s),s)}{\zeta^*(Z^*_{\eta^*}(t),t)}.$$

Proof. By (6), the process $(Z^*_{\eta}(s))_{s\in[u,v]}$ has the backward transition semigroup

$$p^*_{\eta^*}(s,dj,t,k)=\frac{\eta^*(j,s)}{\eta^*(k,t)}h(s,dj,t,k),$$

where $h^*(s,dl,t,k)$ denotes the kernels of $e^{(t-s)H^*}$, i.e.

$$e^{(t-s)H^*}f(k)=\int f(k)h^*(s,dl,t,k),\qquad k\in\mathbf{R},\quad 0\le s<t.$$

It suffices to check that the finite-dimensional distributions satisfy:

$$\begin{aligned}
&E[f(Z^*_{\eta^*}(t_1),\ldots,Z^*_{\eta^*}(t_n))\Lambda^*_{s,t}(Z^*_{\zeta^*}(s),Z^*_{\eta^*}(t))|Z^*_{\eta^*}(t)=l]\\
&=\int\cdots\int f(k_1,\ldots,k_n)h^*(s,dk,t_1,k_1)\cdots h^*(t_n,dk_n,t,l)\Lambda^*_{s,t}(k,l)\frac{\eta^*(k,s)}{\eta^*(l,t)}\\
&=\int\cdots\int f(k_1,\ldots,k_n)h^*(s,dk,t_1,k_1)\cdots h^*(t_n,dk_n,t,k)\frac{\zeta^*(k,s)}{\zeta^*(l,t)}\\
&=E[f(Z^*_{\zeta^*}(t_1),\ldots,Z^*_{\zeta^*}(t_n))|Z^*_{\zeta^*}(t)=l],
\end{aligned}$$

for all $s\le t_1<\cdots<t_n\le t$. $\square$

Similarly to the above, the unconditional density $\Psi(Z^*_{\eta^*}(s),s)$ can be recovered from the conditional density $\Lambda^*_{s,t}(Z^*_{\eta^*}(s),Z^*_{\eta^*}(t))$:

$$\begin{aligned}
E[f(Z^*_{\zeta^*}(s))]&=E[E[f(Z^*_{\zeta^*}(s))\mid Z^*_{\zeta^*}(t)]]\\
&=E[E[f(Z^*_{\eta^*}(s))\Lambda^*_{s,t}(Z^*_{\zeta^*}(s),Z^*_{\eta^*}(t))\mid Z^*_{\eta^*}(t)]]\\
&=E[f(Z^*_{\eta^*}(s))\Lambda^*_{s,t}(Z^*_{\zeta^*}(t),Z^*_{\eta^*}(s))]\\
&=E[E[f(Z^*_{\eta^*}(s))\Lambda^*_{s,t}(Z^*_{\zeta^*}(t),Z^*_{\eta^*}(s))\mid Z^*_{\eta^*}(t)]]\\
&=E\left[f(Z^*_{\eta^*}(s))\int\Lambda^*_{s,t}(Z^*_{\eta^*}(s),k)dP(Z^*_{\zeta^*}(t)=k\mid Z^*_{\eta^*}(s))\right],
\end{aligned}$$

hence

$$\Psi(l,s)=\int\Lambda^*_{s,t}(l,k)dP(Z^*_{\zeta^*}(t)=k\mid Z^*_{\eta^*}(s)=l),$$

$k\in\mathbf{R}$, $0\le s\le t$. Again, the above calculation can be confirmed as follows using (18):

$$\begin{aligned}
\Psi(l,s)&=\frac{\zeta(l,s)}{\eta(l,s)}\frac{\zeta^*(l,s)}{\eta^*(l,s)}\\
&=\int\zeta(k,t)\zeta^*(l,s)\frac{h(s,l,t,dk)}{\eta^*(l,s)\eta(l,s)}\\
&=\int\frac{\zeta^*(l,s)}{\zeta^*(k,t)}\frac{h^*(s,dl,t,k)}{\eta^*(l,s)\eta(l,s)\lambda(dl)}\zeta(k,t)\zeta^*(k,t)\lambda(dk)\\
&=\int\frac{\zeta^*(l,s)}{\zeta^*(k,t)}\frac{\eta^*(k,t)}{\eta^*(l,s)}\frac{h^*(s,dl,t,k)}{\eta^*(l,s)\eta(l,s)\lambda(dl)}\frac{\eta^*(l,s)}{\eta^*(k,t)}\zeta(l,s)\zeta^*(l,s)\lambda(dk)\\
&=\int\Lambda^*_{s,t}(l,k)\frac{p^*_{\eta^*}(s,dl,t,k)}{\eta^*(l,s)\eta(l,s)\lambda(dl)}dP(Z^*_{\zeta^*}(s)=l)
\end{aligned}$$

$$= \int \Lambda^*_{s,t}(l,k) dP(Z^*_{\zeta^*}(t) = k \mid Z^*_{\eta^*}(s) = l).$$

The density processes $\dfrac{\zeta(Z_\eta(t),t)}{\eta(Z_\eta(t),t)}$ and $\dfrac{\zeta^*(Z^*_\eta(t),t)}{\eta^*(Z^*_\eta(t),t)}$ are forward/backward martingales since by construction,

$$\mathcal{L}_\eta \left(\frac{\zeta}{\eta} \right)(k,t) = 0,$$

and

$$\mathcal{L}^*_{\eta^*} \left(\frac{\zeta^*}{\eta^*} \right)(k,t) = 0.$$

This is the case in particular for

$$\frac{e^{\beta N_4}\eta}{\eta}(Z_\eta(t),t) = \exp(\beta Z_\eta(t) - (e^\beta - 1)t) \frac{\eta(Z_\eta(t), e^\beta t)}{\eta(Z_\eta(t),t)},$$

and

$$\frac{e^{\beta N^*_4}\eta^*}{\eta^*}(Z^*_{\eta^*}(t),t) = \exp(-\beta Z^*_{\eta^*}(t) + (e^\beta - 1)t) \frac{\eta^*(Z^*_{\eta^*}(t), e^\beta t)}{\eta^*(Z^*_{\eta^*}(t),t)}.$$

In the framework of the previous section the Girsanov densities are given by the classical expression

$$\Psi(k,t) = \frac{\eta_\alpha(k,t)\eta^*_\alpha(k,t)}{\eta(k,t)\eta^*(k,t)} = \frac{\eta(k,(1+\alpha)t)\eta^*(k,(1+\alpha)t)}{\eta(k,t)\eta^*(k,t)},$$

and

$$\begin{aligned}\Lambda_{s,t}(Z_{\eta_\alpha}(s), Z_\eta(t)) &= \frac{\eta_\alpha(Z_\eta(t),t)}{\eta(Z_\eta(t),t)} \frac{\eta(Z_{\eta_\alpha}(s),s)}{\eta_\alpha(Z_{\eta_\alpha}(s),s)} \\ &= (1+\alpha)^{Z_\eta(t) - Z_{\eta_\alpha}(s)} e^{-\alpha(t-s)} \frac{\eta(Z_\eta(t),(1+\alpha)t)\eta(Z_{\eta_\alpha}(s),s)}{\eta(Z_\eta(t),t)\eta(Z_{\eta_\alpha}(s),(1+\alpha)s)},\end{aligned}$$

$$\begin{aligned}\Lambda^*_{s,t}(Z^*_{\eta_\alpha}(s), Z^*_\eta(t)) &= \frac{\eta^*_\alpha(Z^*_\eta(t),t)}{\eta^*(Z^*_\eta(t),t)} \frac{\eta^*(Z^*_{\eta^*_\alpha}(s),s)}{\eta^*_\alpha(Z^*_{\eta^*_\alpha}(s),s)} \\ &= (1+\alpha)^{-(Z^*_{\eta^*}(t) - Z^*_{\eta^*_\alpha}(s))} e^{\alpha(t-s)} \frac{\eta^*(Z^*_\eta(t),(1+\alpha)t)\eta^*(Z^*_{\eta^*_\alpha}(s),s)}{\eta^*(Z^*_{\eta^*}(t),t)\eta^*(Z^*_{\eta^*_\alpha}(s),(1+\alpha)s)}.\end{aligned}$$

For example in the case of the forward Poisson process we can take

$$\eta(k,t) = 1 \quad \text{and} \quad \eta^*(k,t) = 1_{\{k \geq 0\}} \frac{t^k}{k!} e^{-t},$$

in this case the Girsanov densities are given by

$$\Psi(Z_\eta(t),t) = (1+\alpha)^{Z_\eta(t)} e^{-\alpha t}, \qquad \Psi(Z^*_\eta(t),t) = (1+\alpha)^{Z^*_\eta(t)} e^{-\alpha t},$$

and

$$\Lambda_{s,t}(Z_{\eta_\alpha}(s), Z_\eta(t)) = 1, \qquad \Lambda^*_{s,t}(Z^*_{\eta^*_\alpha}(s), Z^*_\eta(t)) = (1+\alpha)^{Z^*_{\eta^*}(t) - Z^*_{\eta^*_\alpha}(s)} e^{-\alpha(t-s)}.$$

6 Appendix

In this section we note (using quantum stochastic calculus and normal martingales, see Proposition 10 below) that the commutation property (10) of the creation operator P with the generator L actually holds in both the Poisson and Wiener cases. Let $(M_t)_{t\in\mathbf{R}_+}$ be a martingale with deterministic angle bracket $d\langle M_t, M_t\rangle = dt$. The multiple stochastic integral $I_n(f_n)$ is defined as

$$I_n(f_n) = n! \int_0^\infty \int_0^{t_n} \cdots \int_0^{t_2} f_n(t_1, \ldots, t_n) dM_{t_1} \cdots dM_{t_n},$$

$f_n \in L^2(\mathbf{R}_+)^{\circ n}$, $n \geq 1$, where $L^2(\mathbb{R}_+)^{\circ n}$ is the space of symmetric square integrable functions on $\mathbf{R}_+^n$, with the isometry property

$$E[I_n(f_n)I_m(g_m)] = n!\mathbf{1}_{\{n=m\}}\langle f_n, g_m\rangle_{L^2(\mathbf{R}_+)^{\circ n}}. \tag{19}$$

We assume that $(M_t)_{t\in\mathbf{R}_+}$ has the chaos representation property (CRP), i.e. every $F \in L^2(\Omega, \mathcal{F}, P)$ has a decomposition

$$F = \sum_{n=0}^\infty I_n(f_n).$$

If $(M_t)_{t\in\mathbf{R}_+}$ is in $L^4(\Omega, \mathcal{F}, P)$ then the CRP implies the existence of a square-integrable predictable process $(\phi_t)_{t\in\mathbf{R}_+}$ such that the structure equation

$$d[M_t, M_t] = dt + \phi_t dM_t, \quad t \in \mathbf{R}_+, \tag{20}$$

is satisfied, cf. Proposition 2 of [3]. Recall that $(M_t)_{t\in\mathbf{R}_+}$ is a compensated Poisson process if $\phi_t = 1$, $t \in \mathbf{R}_+$, and is a Brownian motion for $\phi_t = 0$, $t \in \mathbf{R}_+$. The class of normal martingales also includes the Azéma martingales and we have the following change of variable formula, cf. [3].

Proposition 8. *For $\eta \in \mathcal{C}^2(\mathbf{R} \times \mathbf{R}_+)$, let*

$$\nabla_\phi \eta(k,t) := \begin{cases} \dfrac{\eta(k+\phi_t, t) - \eta(k,t)}{\phi_t}, & \phi_t \neq 0, \\[2ex] \dfrac{\partial \eta}{\partial k}(k,t), & \phi_t = 0, \end{cases}$$

and

$$L_\phi \eta(k,t) := \begin{cases} \dfrac{1}{\phi_t^2}\left(\eta(k+\phi_t,t) - \eta(k,t) - \phi_t \dfrac{\partial \eta}{\partial k}(k,t)\right) + \dfrac{\partial \eta}{\partial t}(k,t), & \phi_t \neq 0, \\[2ex] \dfrac{1}{2}\dfrac{\partial^2 \eta}{\partial k^2}(k,t) + \dfrac{\partial \eta}{\partial t}(k,t), & \phi_t = 0, \end{cases} \tag{21}$$

We have

$$\eta(M_t, t) = \eta(M_0, 0) + \int_0^t \nabla_\phi \eta(M_{s^-}, s) dM_s + \int_0^t L_\phi \eta(M_s, s) ds, \tag{22}$$

$\eta \in \mathcal{C}^2(\mathbf{R} \times \mathbf{R}_+)$.

Let

$$D : L^2(\Omega, d\mathbb{P}) \longrightarrow L^2(\Omega \times \mathbf{R}_+, d\mathbb{P} \times dt)$$

denote the (unbounded) closable gradient operator defined as

$$D_t F = \sum_{n=1}^\infty n I_{n-1}(f_n(*, t)), \quad d\mathbb{P} \times dt - a.e.,$$

with $F = \sum_{n=0}^{\infty} I_n(f_n)$. Let also the divergence operator $\delta : L^2(\Omega \times \mathbf{R}_+, d\mathbb{P} \times dt) \longrightarrow L^2(\Omega, d\mathbb{P})$ be defined as

$$\delta(u) = \sum_{n=0}^{\infty} I_{n+1}(\tilde{f}_{n+1}), \quad d\mathbb{P} - a.e.,$$

with $u_t = \sum_{n=0}^{\infty} I_n(f_{n+1}(*, t))$, $t \in \mathbf{R}_+$, and $\tilde{f}_{n+1}$ denotes the symmetrization of f_{n+1} in $n+1$ variables. The stochastic differentials da_t^-, da_t^+ and da_t° can be defined through the identities

$$\int_0^\infty U_t da_t^- F = \int_0^\infty U_t D_t F dt, \quad \int_0^\infty U_t da_t^+ F = \delta(U.F), \quad \int_0^\infty U_t da_t^\circ F = \delta(U.D.F),$$

$F \in \mathcal{S}$, for $(U_t)_{t\in\mathbf{R}_+}$ an adapted operator-valued process satisfying suitable domain conditions. They satisfy the Hudson-Parthasarathy Itô table [5]:

$\cdot$	da_t^-	da_t°	da_t^+
da_t^+	0	0	0
da_t°	0	da_t°	da_t^+
da_t^-	0	da_t^-	dt

(23)

On the other hand, Proposition 18 of [2], which can be interpreted as $dM_t = da_t^- + da_t^+ + \phi_t da_t^\circ$, yields the following representation of the multiplication operator by $\eta(M_t, t)$.

Proposition 9. *The multiplication operator ζ_t^η by $\eta(M_t, t)$ has the decomposition*

$$\zeta_t^\eta = \zeta_0^\eta + \int_0^t \nabla_\phi \eta(M_s, s) da_s^- + \int_0^t \nabla_\phi \eta(M_s, s) da_s^+ + \int_0^t \phi_s \nabla_\phi \eta(M_s, s) da_s^\circ$$
$$+ \int_0^t L_\phi \eta(M_s, s) ds. \tag{24}$$

In other terms we have

$$\eta(M_t, t)F = \eta(M_0, 0)F + \int_0^t \nabla_\phi \eta(M_s, s) D_s F ds + \delta(1_{[0,t]}(\cdot) F \nabla_\phi \eta(M_\cdot, \cdot))$$
$$+\delta(1_{[0,t]}(\cdot) F \phi . \nabla_\phi \eta(M_\cdot, \cdot) D.F) + F \int_0^t L_\phi \eta(M_s, s) ds,$$

for sufficiently regular F.

Wiener case

We have

$$Pf(x, t) = xf(x, t) - tf'(x, t),$$

and $(P^n \mathbf{1}(\cdot, t))_{n\in\mathbf{N}} = (H_n(\cdot, t))_{n\in\mathbf{N}}$ is the family of Hermite polynomials with parameter $t > 0$.

Poisson case

We have

$$Pf(k, t) = kf(k-1, t) - tf(k, t),$$

and $(P^n \mathbf{1}(\cdot, t))_{n\in\mathbf{N}} = (C_n(\cdot, t))_{n\in\mathbf{N}}$ is the family of Charlier polynomials with parameter $t > 0$.

It is known that $I_n(1_{[0,t]}^{\circ n})$ is function of M_t only in the Wiener and Poisson cases, i.e. for constant deterministic ϕ, cf. [10]. More precisely we have the following.

Remark 1. The operator $P : \mathcal{C}(\mathbf{R} \times \mathbf{R}_+) \to \mathcal{C}(\mathbf{R} \times \mathbf{R}_+)$ satisfies the relation

$$[Pf](X_t, t) = a_t^+[f(X_t, t)], \qquad f \in \mathcal{C}(\mathbf{R} \times \mathbf{R}_+), \tag{25}$$

with $X_t = M_t$ in the Wiener case and $X_t = M_t + t$ in the Poisson case, i.e. for constant deterministic ϕ, cf. [10].

For $f = \mathbf{1}$, (25) reads:

$$[P^n \mathbf{1}](X_t, t) = I_n(1_{[0,t]}^{\circ n}).$$

The mapping P can be seen as a finite-dimensional projection of the creation operator a_t^+. Finally we give a proof of the commutation relation (10) in the general context of normal martingales.

Proposition 10. *We have*

$$[L_\phi, P] = 0 \quad \textit{and} \quad [\nabla_\phi, P] = I \quad .$$

Proof. From (24) and the Itô table (23) we get

$$\begin{aligned} d(a_t^+ \zeta_t^\eta) &= a_t^+ d\zeta_t^\eta + \zeta_t^\eta da_t^+ + da_t^+ \cdot d\zeta_t^\eta \\ &= a_t^+ d\zeta_t^\eta + \zeta_t^\eta da_t^+ + \eta(X_t, t) da_t^+ \cdot (da_t^- + da_t^+ + \phi_t da_t^\circ) + L_\phi \eta(X_t, t) da_t^+ \cdot dt \\ &= a_t^+ d\zeta_t^\eta + \zeta_t^\eta da_t^+ \\ &= a_t^+ \nabla_\phi \eta(X_t, t) da_t^+ + a_t^+ (L_\phi \eta(X_t, t)) dt + \eta(X_t, t) da_t^+ \\ &= (P\nabla_\phi \eta(X_t, t) + \eta(X_t, t)) da_t^+ + PL_\phi \eta(X_t, t) dt, \end{aligned}$$

hence

$$d(a_t^+ \zeta_t^\eta)\mathbf{1} = (P\nabla_\phi \eta(X_t, t) + \eta(X_t, t)) dX_t + PL_\phi \eta(X_t, t) dt.$$

On the other hand from the classical Itô formula (22) we have

$$d(P\eta(X_t, t)) = \nabla_\phi P\eta(X_t, t) dX_t + L_\phi P\eta(X_t, t) dt.$$

The identification $d(a_t^+ \zeta_t^f)\mathbf{1} = d(P\eta(X_t, t))$ due to Definition 1 we obtain

$$\nabla_\phi P\eta = \eta + P\nabla_\phi \eta, \quad \text{and} \quad PL_\phi \eta = L_\phi P\eta.$$

□

The relation $[\nabla_\phi, P] = I$ can be viewed as a one-dimensional projection of the canonical commutation relation

$$D_t \delta(u) = \delta(D_t u) + u_t$$

between D and δ, or $[a_t^-, a_t^+] = tI$.

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