Pricing CIR Asian options by conditional moment matching

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Abstract

Asian yield options are priced in the CIR model using conditional moment matching for the gamma distribution. This method is fast and simple to implement, and it shows a high degree of accuracy without being subject to the numerical instabilities that can be encountered with more sophisticated approaches.

Keywords: CIR model; Asian options; Asian caps, conditional moment matching; stratified approximation.

We consider Asian call options priced as

$$\mathrm{AO}^c(K,T) := \mathbb{E}\left[e^{-\Lambda_T}\left(\frac{\Lambda_T}{T}-K\right)^+\right]$$

where

$$\Lambda_T := \int_0^T S_t dt,$$

and $(S_t)_{t\in\mathbb{R}_+}$ is the Cox-Ingersoll-Ross (CIR) solution of the stochastic differential equation

$$dS_t = (a - bS_t)dt + \sigma\sqrt{S_t}dW_t. \tag{0.1}$$

Unconditional gamma approximation

Using moment matching, $\mathrm{AO}^c(K,T)$ can be estimated as

$$\mathrm{AO}^c(K,T) := \mathbb{E}\left[e^{-\Lambda_T}\left(\frac{\Lambda_T}{T}-K\right)^+\right] \tag{0.2}$$
$$\approx \frac{\theta_T}{(1+\theta_T)^{\nu_T+1}}\left(\nu_T Q\left(1+\nu_T, KT+\frac{KT}{\theta_T}\right)-\left(KT+\frac{KT}{\theta_T}\right)Q\left(\nu_T, KT+\frac{KT}{\theta_T}\right)\right)$$

under the unconditional gamma approximation, where

$$\theta_T := \frac{\mathrm{Var}[\Lambda_T]}{\mathbb{E}[\Lambda_T]} \quad \text{and} \quad \nu_T := \frac{\mathbb{E}[\Lambda_T]}{\theta_T} = \frac{(\mathbb{E}[\Lambda_T])^2}{\mathrm{Var}[\Lambda_T]},$$

and

$$\mathbb{E}[\Lambda_T] = S_0\frac{1-e^{-bT}}{b} + a\frac{e^{-bT}+bT-1}{b^2},$$
$$\mathrm{Var}[\Lambda_T] = \sigma^2 S_0\frac{1-2bTe^{-bT}-e^{-2bT}}{b^3} + \sigma^2 a\frac{5-2bT-e^{-2bT}-4(bT+1)e^{-bT}}{2b^4}.$$

Conditional gamma approximation

Under the conditional gamma approximation we find

$$\begin{aligned}\mathrm{AO}^c(K,T) &= \mathbb{E}\left[e^{-\Lambda_T}\left(\frac{1}{T}\int_0^T S_t dt - K\right)^+\right] \qquad (0.3)\\ &\approx \frac{1}{T}\int_0^\infty \frac{\theta_T(y)}{(1+\theta_T(y))^{\nu_T(y)+1}}\left(\nu_T(y)Q\left(1+\nu_T(y), KT+\frac{KT}{\theta_T(y)}\right)\right.\\ &\qquad \left. -\left(KT+\frac{KT}{\theta_T(y)}\right)Q\left(\nu_T(y), KT+\frac{KT}{\theta_T(y)}\right)\right) f_{S_T}(y)dy,\end{aligned}$$

see § 4.4 of [4], where

$$\theta_T(y) := \frac{\mathrm{Var}[\Lambda_T \mid S_T=y]}{\mathbb{E}[\Lambda_T \mid S_T=y]} \quad \text{and} \quad \nu_T(y) := \frac{\mathbb{E}[\Lambda_T \mid S_T=y]}{\theta_T(y)} = \frac{(\mathbb{E}[\Lambda_T \mid S_T=y])^2}{\mathrm{Var}[\Lambda_T \mid S_T=y]},$$

and

$$\mathbb{E}[\Lambda_T \mid S_T=y] = -\frac{\sigma^2}{b^2} + \frac{1}{b(e^{bT}-1)^2}\left(\frac{\sigma^2 T}{2}(e^{2bT}-1) + (S_0+y)(e^{2bT}-2bTe^{bT}-1)\right.$$
$$\left. + \sqrt{yS_0e^{bT}}\left(e^{bT}(bT-2)+bT+2\right)\frac{I_{2a/\sigma^2}\left(\frac{2b\sqrt{yS_0}}{\sigma^2\sinh(bT/2)}\right)+I_{2a/\sigma^2-2}\left(\frac{2b\sqrt{yS_0}}{\sigma^2\sinh(bT/2)}\right)}{I_{2a/\sigma^2-1}\left(\frac{2b\sqrt{yS_0}}{\sigma^2\sinh(bT/2)}\right)}\right).$$

$$
\begin{aligned}
\mathrm{Var}[\Lambda_T \mid S_T = y] &= -2\frac{\sigma^4}{b^4} + \sigma^2\frac{(e^{bT}(2bT+1)-1)}{b^2(e^{bT}-1)}\left(\frac{\sigma^2}{b^2} + \mathbb{E}[\Lambda_T \mid S_T = y]\right) \\
&+ \frac{1}{b(e^{bT}-1)^2}\Bigg(-\sigma^4 T^2\frac{e^{2bT}}{b} - 2\frac{\sigma^2 T}{b}(S_0+y)\left(e^{2bT} - e^{bT}(bT+1)\right) \\
&- \frac{\sigma^2 T}{2b}\sqrt{yS_0e^{bT}}\left(e^{bT}(3bT-4)+bT+4\right)\frac{I_{2a/\sigma^2}\left(\frac{4b\sqrt{yS_0e^{-bT}}}{\sigma^2(1-e^{-bT})}\right) + I_{2a/\sigma^2-2}\left(\frac{4b\sqrt{yS_0e^{-bT}}}{\sigma^2(1-e^{-bT})}\right)}{I_{2a/\sigma^2-1}\left(\frac{2b\sqrt{yS_0}}{\sigma^2\sinh(bT/2)}\right)} \\
&+ \frac{yS_0e^{bT}}{b(1-e^{-bT})^2}\left(e^{bT}(bT-2)+bT+2\right)^2\left(2 + \frac{I_{2a/\sigma^2-3}+I_{2a/\sigma^2+1}}{I_{2a/\sigma^2-1}} - \frac{(I_{2a/\sigma^2-2}+I_{2a/\sigma^2})^2}{(I_{2a/\sigma^2-1})^2}\right)\Bigg),
\end{aligned}
$$

with $I_z := I_z\left(\frac{2b\sqrt{yS_0}}{\sigma^2\sinh(bT/2)}\right)$, and

$$
I_\lambda(z) := \left(\frac{z}{2}\right)^\lambda \sum_{k=0}^\infty \frac{(z^2/4)^k}{k!\Gamma(\lambda+k+1)}, \qquad z, \lambda \in \mathbb{R},
$$

is the modified Bessel function of the first kind, $z \in \mathbb{R}$. In (0.3) above, f_{S_T} is the non-central chi-square probability density function

$$
f_{S_T}(y) := \frac{2b}{\sigma^2(1-e^{-bT})}\exp\left(-\frac{2b(S_0+ye^{bT})}{\sigma^2(e^{bT}-1)}\right)\left(\frac{ye^{bT}}{S_0}\right)^{a/\sigma^2-1/2} I_{2a/\sigma^2-1}\left(\frac{2b\sqrt{yS_0}}{\sigma^2\sinh(bT/2)}\right), \tag{0.4}
$$

$y > 0$, and

$$
\Gamma(\lambda) := \int_0^\infty x^{\lambda-1}e^x dx
$$

denotes the gamma function.

Numerical results

In Table 1 we compare our results to the joint density (JD) method of [2], [3], with the parameters of [1].

Strike	Type	Maturity T = 0.1 JD[3]	Stratified (0.3)	Gamma (0.2)	Maturity T = 0.5 JD[3]	Stratified (0.3)	Gamma (0.2)
0.08	AO^c	0.0199	0.0199	0.0199	0.0201	0.0201	0.0201
0.12	AO^c	0.0002	0.0002	0.0002	0.0018	0.0018	0.0018
		T = 1			T = 2		
Strike	Type	JD[3]	Stratified (0.3)	Gamma (0.2)	JD[3]	Stratified (0.3)	Gamma (0.2)
0.08	AO^c	0.0193	0.0193	0.0194	0.0170	0.0170	0.0171
0.12	AO^c	0.0023	0.0023	0.0023	0.0019	0.0019	0.0018
		T = 5			T = 10		
Strike	Type	JD[3]	Stratified (0.3)	Gamma (0.2)	JD[3]	Stratified (0.3)	Gamma (0.2)
0.08	AO^c	0.0118	0.0118	0.0118	0.0069	0.0069	0.0069
0.12	AO^c	0.0006	0.0006	0.0006	0.0001	0.0001	0.0001

Table 1: Asian prices, $S_0 = 0.1$, $a = 0.15$, $b = 1.5$, $\sigma = 0.2$.

Figure 1 presents the evolution of prices according to maturity times. All three methods show consistent numerical results.

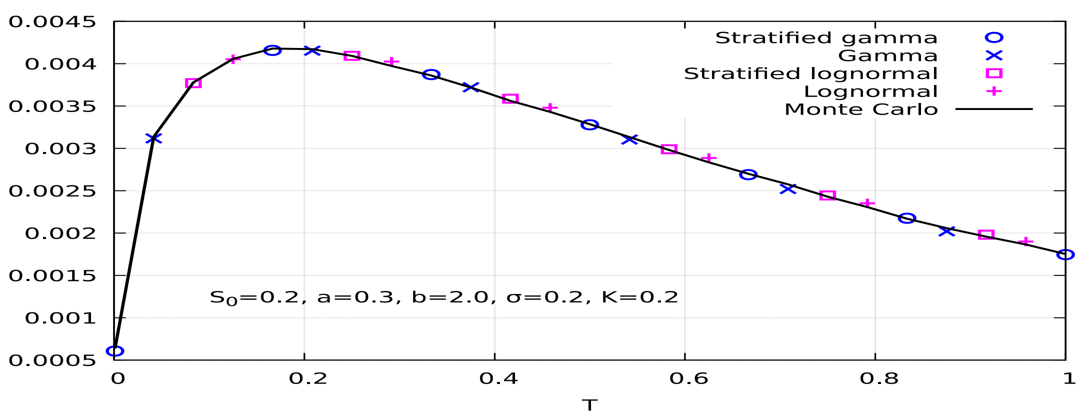

Figure 1: Regular Asian (A) cap prices for $T \in [0, 1]$.

Table 2 presents a sample of computation times for comparison of the different methods, cf. [4] for details.

Parameters						Time			
S_0	a	b	σ	T	K	Stratified (0.3)	Gamma (0.2)	Monte Carlo	JD[3]
2.1	0.0	-0.05	0.72	1.0	2.0	1.32e-02	2.60e-5	144.46	8.62

Table 2: Computation times in seconds.*

Call-put parity

The relations

$$E\left[e^{\eta\Lambda_T}\right] = e^{-S_0\psi(\eta)-a\phi(\eta)}, \tag{0.5}$$

where $\bar{b} := \sqrt{b^2 - 2\eta\sigma^2}$ and

$$\psi(\eta) := \frac{2\eta(e^{-\bar{b}T}-1)}{\bar{b}+b+e^{-\bar{b}T}(\bar{b}-b)} \qquad \phi(\eta) := \frac{1}{\sigma^2}(\bar{b}-b)T + \frac{2}{\sigma^2}\log\frac{\bar{b}+b+e^{-\bar{b}T}(\bar{b}-b)}{2\bar{b}},$$

allow us to estimate the regular Asian floor price as

$$\begin{aligned}
\mathrm{AO}^f(K,T) &= \mathbb{E}\left[e^{-\Lambda_T}\left(K-\frac{\Lambda_T}{T}\right)^+\right] \\
&= \mathbb{E}\left[e^{-\Lambda_T}\left(\frac{\Lambda_T}{T}-K\right)^+\right] - \left(\frac{1}{T}\,\mathbb{E}\left[\Lambda_T e^{-\Lambda_T}\right] - K\,\mathbb{E}\left[e^{-\Lambda_T}\right]\right) \\
&= \mathrm{AO}^c(K,T) + K\,\mathbb{E}\left[e^{-\Lambda_T}\right] - \frac{1}{T}\,\mathbb{E}\left[\Lambda_T e^{-\Lambda_T}\right],
\end{aligned}$$

from

$$\begin{aligned}
&\mathbb{E}\left[\Lambda_T e^{-\Lambda_T}\right] = -(S_0\psi'(-1) + a\phi'(-1))e^{-S_0\psi(-1)-a\phi(-1)} \\
&= \; -S_0\left(\frac{2\bar{b}(e^{-\bar{b}T}-1) - 2\sigma^2 T e^{-\bar{b}T}}{\bar{b}(\bar{b}+b+e^{-\bar{b}T}(\bar{b}-b))} - \frac{2\sigma^2(e^{-\bar{b}T}-1)(1-e^{-\bar{b}T}(\bar{b}-b)T+e^{-\bar{b}T})}{\bar{b}(\bar{b}+b+e^{-\bar{b}T}(\bar{b}-b))^2}\right) E\left[e^{-\Lambda_T}\right] \\
&\quad +\frac{a}{\bar{b}}\left(T + 2\frac{1-e^{-\bar{b}T}(\bar{b}-b)T+e^{-\bar{b}T}}{\bar{b}+b+e^{-\bar{b}T}(\bar{b}-b)} - \frac{2}{\bar{b}}\right) E\left[e^{-\Lambda_T}\right],
\end{aligned}$$

with $\bar{b}' = -\sigma^2/\bar{b}$ and $\eta = -1$, since

$$\begin{aligned}
\psi'(\eta) &= \frac{2(e^{-\bar{b}T}-1)}{\bar{b}+b+e^{-\bar{b}T}(\bar{b}-b)} + \frac{2\eta T\sigma^2 e^{-\bar{b}T}}{\bar{b}(\bar{b}+b+e^{-\bar{b}T}(\bar{b}-b))} \\
&\quad -\frac{2\eta(e^{-\bar{b}T}-1)}{(\bar{b}+b+e^{-\bar{b}T}(\bar{b}-b))^2}(-\sigma^2/\bar{b} + (\sigma^2 T/\bar{b})e^{-\bar{b}T}(\bar{b}-b) - e^{-\bar{b}T}(\sigma^2/\bar{b})) \\
&= \frac{2\bar{b}(e^{-\bar{b}T}-1) + 2\eta T\sigma^2 e^{-\bar{b}T}}{\bar{b}(\bar{b}+b+e^{-\bar{b}T}(\bar{b}-b))} + \frac{2\eta\sigma^2(e^{-\bar{b}T}-1)(1-Te^{-\bar{b}T}(\bar{b}-b)+e^{-\bar{b}T})}{\bar{b}(\bar{b}+b+e^{-\bar{b}T}(\bar{b}-b))^2},
\end{aligned}$$

and

$$\begin{aligned}
\phi'(\eta) &= -\frac{T}{\bar{b}} + \frac{2(-\sigma^2/\bar{b} + (\sigma^2 T/\bar{b})e^{-\bar{b}T}(\bar{b}-b) + e^{-\bar{b}T}(-\sigma^2/\bar{b}))}{\sigma^2(\bar{b}+b+e^{-\bar{b}T}(\bar{b}-b))} + \frac{2(2\sigma^2/\bar{b})}{\sigma^2(2\bar{b})} \\
&= -\frac{T}{\bar{b}} - 2\frac{1-e^{-\bar{b}T}(\bar{b}-b)T+e^{-\bar{b}T}}{\bar{b}(\bar{b}+b+e^{-\bar{b}T}(\bar{b}-b))} + \frac{2}{\bar{b}^2}.
\end{aligned}$$

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