Stratified approximations for the pricing of options on averages

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January 25, 2018

Abstract

We propose to use stratified approximations based on the gamma and log-normal distributions for the pricing of options on average such as Asian options and bond prices in the Dothan model. We show that this approach improves on standard numerical approximation methods, and is not subject to the instabilities encountered with closed form integral expressions.

Key words: Asian options, bond pricing, Dothan model, stratified sampling, numerical approximation.

Mathematics Subject Classification (2010): 60J65, 60H30, 60J60, 91B24.

1 Introduction

Options on average are generally difficult to price due to the lack of simple closed form expressions. In [9], Asian options have been priced numerically by Monte Carlo estimates combined with variance reduction based on control variates, and PDE pricing arguments have been developed in e.g. [10], page 91, [18], or § 7.5.3 of [19]. Pricing based on the probability density of the averaged geometric Brownian motion has been considered in e.g. [12], or in [5] by the use of Laguerre series. The time Laplace transform of Asian option prices has been computed in [6], and this expression can be used for pricing by numerical inversion of the Laplace transform, see also [3].

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Other numerical approaches to the pricing of Asian options include [20] which relies on approximations of the average price probability based on the Lognormal distribution. Namely, Asian options on the time integral

$$\Lambda_T := \int_0^T S_t dt$$

of geometric Brownian motion

$$S_t = e^{\sigma B_t + (r - \sigma^2/2)t}, \qquad t \in [0, T], \tag{1.1}$$

have been priced in [20], [11] by approximating Λ_T by a lognormal random variable, as

$$e^{-rT} E\left[\left(\frac{1}{T}\int_0^T S_t dt - K\right)^+\right] \simeq e^{-rT}\left(\frac{1}{T} e^{\hat{\mu} + \hat{\sigma}^2/2} \Phi(d_1) - K\Phi(d_2)\right), \tag{1.2}$$

where

$$d_1 = \frac{\log(E[\Lambda_T]/(KT))}{\hat{\sigma}\sqrt{T}} + \hat{\sigma}\frac{\sqrt{T}}{2} = \frac{\hat{\mu}T + \hat{\sigma}^2 T - \log(KT)}{\hat{\sigma}\sqrt{T}}$$

and

$$d_2 = d_1 - \hat{\sigma}\sqrt{T} = \frac{\log(E[\Lambda_T]/(KT))}{\hat{\sigma}\sqrt{T}} - \hat{\sigma}\frac{\sqrt{T}}{2},$$

and $\hat{\mu}$, $\hat{\sigma}$ are estimated as

$$\hat{\sigma}^2 = \frac{1}{T}\log\left(\frac{E[\Lambda_T^2]}{(E[\Lambda_T])^2}\right) \tag{1.3}$$

and

$$\hat{\mu} = \frac{1}{T}\log E[\Lambda_T] - \frac{1}{2}\hat{\sigma}^2, \tag{1.4}$$

based on the first two moments of the lognormal distribution, cf. (3.5) below.

With respect to the above approaches, this paper is a contribution to the pricing of options on average from a numerical point of view, by providing an alternative to existing closed form integral expressions which suffer from numerical instabilities.

More precisely, we propose a more accurate approximation by applying *stratified sampling* to the computation of (1.2), via the conditioning

$$E\left[\left(\frac{1}{T}\int_0^T S_t dt - K\right)^+\right] = \int_0^\infty E\left[\left(\frac{1}{T}\int_0^T S_t dt - K\right)^+ \Big| S_T = z\right] d\mathbb{P}(S_T = z).$$

Stratified sampling usually acts as a variance reduction method in Monte Carlo simulations, and in the present setting it also improves numerical precision as seen in the graphs of Figures 2 and 3 below. The conditional expectation

$$E\left[\left(\frac{1}{T}\int_0^T S_t dt - K\right)^+ \Big| S_T = z\right]$$

can be computed in a closed integral form from Lemma 2.1 below, however this expression relies on a triple oscillating integral and it lacks sufficient stability for numerical computation, especially in small time T, see Figure 1. For this reason we will replace the integral form by the approximations (3.3) and (3.7) which are based on the gamma and lognormal distributions, cf. [7] and [16] for a related gamma approximation in phylogenetics.

In addition we apply the above method to the stratified computation

$$E\left[\exp\left(-\int_0^T S_t dt\right)\right] = \int_0^\infty E\left[\exp\left(-\int_0^T S_t dt\right) \Big| S_T = z\right] d\mathbb{P}(S_T = z)$$

of bond prices in the Dothan model, cf. [4], [14], [15]. Again, the conditional Laplace transform

$$E\left[\exp\left(-\int_0^T S_t dt\right) \Big| S_T = z\right]$$

can be computed in a closed integral form using Bessel functions, cf. Proposition 4.1 below, however this expression fails for small values of T. For this reason we will estimate the integral by a gamma approximation, cf. (4.4) below. Note that the lognormal approximation is ineffective here since the Laplace transform of the lognormal distribution is not available in closed form.

We proceed as follows. In Section 2 we recall known results on the conditional distribution of Λ_T. In Section 3 we propose stratified approximations of option prices using the gamma and lognormal distributions, with an application to bond pricing in the Dothan model in Section 4. Section 5 contains the computations on conditional mean and variance needed for the approximations.

2 Conditional calculus

In this section we recall and state some facts on the conditional distribution and moments of Λ_T given S_T.

Rewriting (1.1) as the solution of

$$dS_t = (1-p)\frac{\sigma^2}{2}S_t dt + \sigma S_t dB_t,$$

with $p = 1 - 2r/\sigma^2$, the joint probability density of

$$\left(\int_0^T S_t dt,\ B_T - p\sigma T/2\right)$$

can be written as

$$\begin{aligned}
\mathbb{P}&\left(\int_0^T e^{\sigma B_s - p\sigma^2 s/2} ds \in dx, B_T - p\sigma T/2 \in dy\right) \\
&= \frac{\sigma}{2} e^{-p\sigma y/2 - p^2\sigma^2 T/8} \exp\left(-2\frac{1+e^{\sigma y}}{\sigma^2 x}\right) \theta\left(\frac{4e^{\sigma y/2}}{\sigma^2 x}, \frac{\sigma^2 T}{4}\right) \frac{dx}{x} dy \qquad (2.1) \\
&= e^{-p\sigma y/2 - p^2\sigma^2 T/8} \mathbb{P}\left(\int_0^T e^{\sigma B_s} ds \in dx, B_T \in dy\right),
\end{aligned}$$

$y \in \mathbb{R}$, $x, T > 0$, where

$$\theta(v, \tau) = \frac{v e^{\pi^2/(2\tau)}}{\sqrt{2\pi^3 \tau}} \int_0^\infty e^{-\xi^2/(2\tau)} e^{-v\cosh\xi} \sinh(\xi) \sin(\pi\xi/\tau)\, d\xi, \qquad v, \tau > 0, \qquad (2.2)$$

and $e^{-p\sigma B_T/2 - p^2\sigma^2 T/8}$ is the density of the Girsanov shift

$$B_T \mapsto B_T + p\sigma T/2,$$

cf. [22], Proposition 2, and also [13].

Note that the function $\theta(v, \tau)$ in (2.2) is difficult to evaluate numerically due to the oscillating behavior of its integrand, in fact we have

$$\int_0^\infty e^{-\xi^2/(2t)} \sinh(\xi)(\cosh(\xi))^n \sin(\pi\xi/t)\, d\xi = 0$$

for all $n \geq 0$, cf. [2], [8], [17] for several attempts at the numerical computation of the function $\theta(v, \tau)$.

The next lemma, which will be used in Section 4, follows from (2.1) combined with the lognormal distribution

$$d\mathbb{P}(e^{\sigma B_T - p\sigma^2 T/2} = y) = \frac{1}{y\sqrt{2\pi\sigma^2 T}} e^{-(p\sigma^2 T/2 + \log y)^2/(2\sigma^2 T)}.$$

Note that the conditional law of Λ_T given $S_T = z$ does not depend on the parameter $p \in \mathbb{R}$.

Lemma 2.1 *For all $z, T > 0$ we have*

$$\mathbb{P}\left(\Lambda_T \in dx \;\middle|\; S_T = z\right) = \sigma\sqrt{\frac{\pi T}{2}} \exp\left(\frac{(\log z)^2}{2\sigma^2 T} - 2\frac{1+z}{\sigma^2 x}\right) \theta\left(\frac{4\sqrt{z}}{\sigma^2 x}, \frac{\sigma^2 T}{4}\right) \frac{dx}{x},$$

$x > 0$. (2.3)

Next, we define the functions

$$a_T(z) := \frac{1}{\sigma^2 p(z)} \left(\Phi\left(\frac{\log z}{\sqrt{\sigma^2 T}} + \frac{1}{2}\sqrt{\sigma^2 T}\right) - \Phi\left(\frac{\log z}{\sqrt{\sigma^2 T}} - \frac{1}{2}\sqrt{\sigma^2 T}\right)\right), \tag{2.4}$$

and

$$b_T(z) = \frac{1}{\sigma^2 q(z)} \left(\Phi\left(\frac{\log z}{\sqrt{\sigma^2 T}} + \sqrt{\sigma^2 T}\right) - \Phi\left(\frac{\log z}{\sqrt{\sigma^2 T}} - \sqrt{\sigma^2 T}\right)\right),$$

where

$$p(z) = \frac{1}{\sqrt{2\pi\sigma^2 T}} e^{-\left(\sigma^2 T/2 + \log z\right)^2/(2\sigma^2 T)}, \quad \text{and} \quad q(z) = \frac{1}{\sqrt{2\pi\sigma^2 T}} e^{-\left(\sigma^2 T + \log z\right)^2/(2\sigma^2 T)},$$

$z > 0$, and

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2} dy, \qquad x \in \mathbb{R},$$

is the standard Gaussian cumulative distribution function.

In Proposition 2.2 which is proved in the Appendix Section 5 we derive the closed form expressions of $E[\Lambda_T \mid S_T = z]$ and $\mathrm{Var}[\Lambda_T \mid S_T = z]$.

Proposition 2.2 *We have*

$$E[\Lambda_T \mid S_T = z] = a_T(z) = \frac{1}{\sigma^2 p(z)} \left(\Phi\left(\frac{\log z}{\sqrt{\sigma^2 T}} + \frac{1}{2}\sqrt{\sigma^2 T} \right) - \Phi\left(\frac{\log z}{\sqrt{\sigma^2 T}} - \frac{1}{2}\sqrt{\sigma^2 T} \right) \right),$$

and

$$E[(\Lambda_T)^2 \mid S_T = z] = \frac{2}{\sigma^2}(b_T(z) - (1+z)a_T(z)), \qquad z > 0. \tag{2.5}$$

As T tends to zero we have the small time asymptotics

$$\begin{aligned} E[\Lambda_T \mid S_T = z] &= \frac{\sqrt{T}}{\sigma} e^{(\sqrt{\sigma^2 T}/2 + (\log z)/\sqrt{\sigma^2 T})^2/2} \int_{(\log z)/\sqrt{\sigma^2 T} - \sqrt{\sigma^2 T}/2}^{(\log z)/\sqrt{\sigma^2 T} + \sqrt{\sigma^2 T}/2} e^{-y^2/2} dy \\ &= \frac{T}{2} \int_{-1}^{1} e^{\sigma^2 T/8 + (\log z)/2 - y^2 \sigma^2 T/8 - (y \log z)/2} dy \\ &\simeq \frac{T\sqrt{z}}{2} \int_{-1}^{1} e^{-(y \log z)/2} dy + o(T) \\ &= T \frac{z-1}{\log z} + o(T), \end{aligned} \tag{2.6}$$

and $E[(\Lambda_T)^2 \mid S_T = z] = o(T)$. Finally we note the scaling relation

$$\Lambda_T = \int_0^T e^{\sigma B_t - p\sigma^2 t/2} dt \simeq \int_0^T e^{B_{\sigma^2 t} - p\sigma^2 t/2} dt = \frac{1}{\sigma^2} \int_0^{\sigma^2 T} e^{B_t - pt/2} dt, \quad T > 0. \tag{2.7}$$

3 Stratified Asian option pricing

Gamma approximation for Asian options

We use the gamma probability density function

$$f(x) = \frac{1}{\theta^\nu \Gamma(\nu)} x^{\nu-1} e^{-x/\theta}, \qquad x > 0, \tag{3.1}$$

with mean and variance

$$E[X] = \nu\theta, \qquad \mathrm{Var}[X] = \nu\theta^2,$$

where the shape parameter $\nu > 0$ and the scale parameter $\theta > 0$ can be estimated from the first two moments of X as

$$\theta = \frac{\mathrm{Var}[X]}{E[X]}, \qquad \nu = \frac{(E[X])^2}{\mathrm{Var}[X]} = \frac{E[X]}{\theta}. \tag{3.2}$$

In the next proposition, as a consequence of (3.2) and Proposition 2.2 we fit the conditional distribution of Λ_T given $S_T = z$ to a gamma distribution using its first two moments.

Proposition 3.1 *For any $z > 0$, the gamma random variable with scale parameter*

$$\theta(z) := \frac{2}{\sigma^2}\left(\frac{b_T(z)}{a_T(z)} - 1 - z\right) - a_T(z),$$

and shape parameter $\nu(z) := a_T(z)/\theta(z)$ has same first and second moments as Λ_T given $S_T = z$.

Based on Proposition 3.1 we will use the approximation

$$d\mathbb{P}\left(\Lambda_T = x \,\middle|\, S_T = z\right) \simeq \frac{e^{-x/\theta(z)}}{\theta(z)} \frac{(x/\theta(z))^{-1+\nu(z)}}{\Gamma\left(\nu(z)\right)} dx \tag{3.3}$$

$x > 0$, under which the conditional Asian option price is approximated as

$$\begin{aligned}
& e^{-rT} E\left[\left(\frac{1}{T}\int_0^T S_t dt - K\right)^+ \,\middle|\, S_T = z\right] = \frac{e^{-rT}}{T} E\left[(\Lambda_T - KT)^+ \,\middle|\, S_T = z\right] \\
&= \frac{e^{-rT}}{T} \int_{KT}^\infty (x - KT)^+ d\mathbb{P}\left(\Lambda_T = x \,\middle|\, S_T = z\right) \\
&\simeq \frac{e^{-rT}}{T\Gamma(\nu(z))} \int_{KT}^\infty (x - KT) e^{-x/\theta(z)} \frac{x^{-1+\nu(z)}}{\theta^{\nu(z)}(z)} dx \\
&= \frac{e^{-rT}}{T\Gamma(\nu(z))} \int_{KT}^\infty e^{-x/\theta(z)} (x/\theta(z))^{\nu(z)} dx - \frac{K}{\Gamma(\nu(z))} e^{-rT} \int_{KT}^\infty e^{-x/\theta(z)} \frac{x^{-1+\nu(z)}}{\theta^{\nu(z)}(z)} dx \\
&= \frac{e^{-rT}}{T} \frac{\theta(z)}{\Gamma(\nu(z))} \int_{KT/\theta(z)}^\infty e^{-x} x^{\nu(z)} dx - \frac{K}{\Gamma(\nu(z))} e^{-rT} \int_{KT/\theta(z)}^\infty e^{-x} x^{-1+\nu(z)} dx \\
&= \frac{e^{-rT}}{T} \theta(z) \frac{\Gamma(1 + \nu(z), KT/\theta(z))}{\Gamma(\nu(z))} - K e^{-rT} \frac{\Gamma(\nu(z), KT/\theta(z))}{\Gamma(\nu(z))},
\end{aligned}$$

where

$$\Gamma(\nu, y) = \int_y^\infty t^{\nu-1} e^{-t} dt, \qquad y > 0,$$

is the upper incomplete gamma function. Hence we find

$$\begin{aligned}
& e^{-rT} E\left[\left(\frac{1}{T}\int_0^T S_t dt - K\right)^+\right] \qquad (3.4) \\
&\simeq \frac{e^{-rT}}{T} \int_0^\infty \frac{\left(\theta(z)\Gamma(1 + \nu(z), KT/\theta(z)) - KT\Gamma(\nu(z), KT/\theta(z))\right)}{\Gamma(\nu(z))} d\mathbb{P}(S_T = z),
\end{aligned}$$

with

$$d\mathbb{P}(S_T = z) = \frac{1}{\sigma\sqrt{2\pi T}} e^{-(p\sigma^2 T/2 + \log z)^2/(2\sigma^2 T)} \frac{dz}{z}.$$

The change of variable $z = e^{\sigma y - p\sigma^2 T/2}$ can be applied in order to replace $d\mathbb{P}(S_T = z)$ with the Gaussian density $e^{-y^2/2}dy/\sqrt{2\pi}$ in the integral (3.4).

Figure 1 compares the integral density expression (2.3) for Λ_T with the gamma and lognormal density approximations (3.3) and (3.7) based on the first two moments.

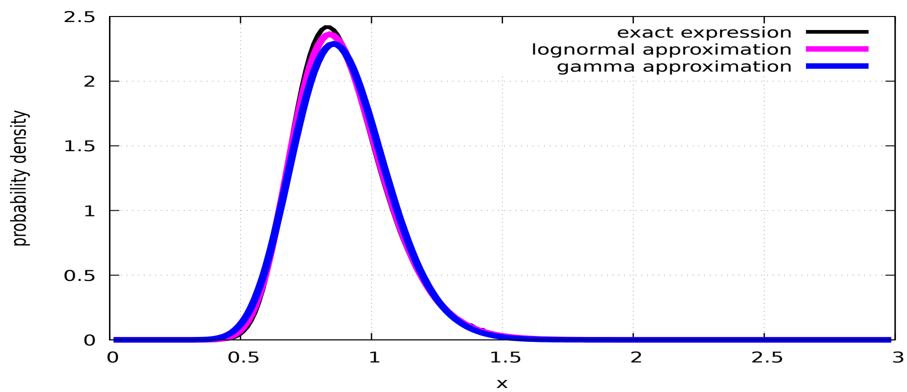

Figure 1: Lognormal and gamma density approximations.

Lognormal approximation for Asian options

The lognormal distribution with mean $-p\sigma^2 T/2$ and variance $\sigma^2 T$ has the probability density function

$$g(x) = \frac{1}{\sigma x\sqrt{2\pi T}} e^{-(p\sigma^2 T/2 + \log x)^2/(2\sigma^2 T)},$$

where $x > 0$, $\mu \in \mathbb{R}$, $\sigma > 0$, and moments

$$E[X] = e^{(1-p)\sigma^2 T/2} \quad \text{and} \quad E[X^2] = e^{(2-p)\sigma^2 T}, \tag{3.5}$$

i.e.

$$p = 1 - \frac{2}{\sigma^2 T} \log E[X] \quad \text{and} \quad \sigma^2 = \frac{1}{T} \log\left(\frac{E[X^2]}{(E[X])^2}\right). \tag{3.6}$$

In the next proposition, as a consequence of (3.6) and Proposition 2.2 we fit the conditional distribution of Λ_T given $S_T = z$ to a lognormal distribution using its first two moments.

Proposition 3.2 *Given $z > 0$, letting*

$$\sigma^2(z) = \frac{1}{T}\log\left(\frac{2}{\sigma^2 a_T(z)}\left(\frac{b_T(z)}{a_T(z)} - 1 - z\right)\right) \quad \textit{and} \quad p(z) := 1 - \frac{2}{T\sigma^2(z)}\log a_T(z),$$

the lognormal random variable with parameter $(-p(z)\sigma^2(z)T/2, \sigma^2(z)T)$ has same first and second moments as Λ_T given $S_T = z$.

Based on Proposition 3.2 we will approximate the law of Λ_T given $S_T = z$ as

$$d\mathbb{P}\left(\Lambda_T = x \,\middle|\, S_T = z\right) \simeq \frac{1}{\sigma(z)\sqrt{2\pi T}} e^{-(p(z)\sigma^2(z)T/2 + \log x)^2/(2\sigma^2(z)T)}\frac{dx}{x}, \tag{3.7}$$

$x > 0$. In particular we have

$$\begin{aligned} e^{-rT}E\left[\left(\frac{1}{T}\int_0^T S_t dt - K\right)^+\right] &= \frac{1}{T}e^{-rT}E\left[\left(\int_0^T S_t dt - KT\right)^+\right] \\ &\simeq \frac{e^{-rT}}{T}\int_0^\infty \left(e^{-p(z)\sigma^2(z)T/2 + \sigma^2(z)T/2}\Phi(d_1(z)) - KT\Phi(d_2(z))\right) d\mathbb{P}(S_T = z), \end{aligned} \tag{3.8}$$

where

$$d_1(z) = \frac{\log(E[\Lambda_T \mid S_T = z]/(KT))}{\sigma(z)\sqrt{T}} + \sigma(z)\frac{\sqrt{T}}{2} = \frac{1}{2\sigma(z)\sqrt{T}}\log\left(\frac{2(b_T(z) - (1+z)a_T(z))}{\sigma^2 K^2 T^2}\right),$$

and

$$d_2(z) = d_1(z) - \sigma(z)\sqrt{T} = \frac{1}{2\sigma(z)\sqrt{T}}\log\left(\frac{2(b_T(z) - (1+z)a_T(z))}{\sigma^2 K^2 T^2}\right) - \sigma(z)\sqrt{T}.$$

Figure 2 compares the Asian option prices obtained from (3.4) (stratified gamma approximation), (3.8) (stratified lognormal approximation), and the standard lognormal approximation (1.2) with the Monte Carlo method. Significant discrepancies in the approximations can be observed for large values of time to maturity, and the stratified approximations appear to perform better than the standard lognormal approximation. A (non-stratified) gamma approximation similar to (1.2) is also included for reference.

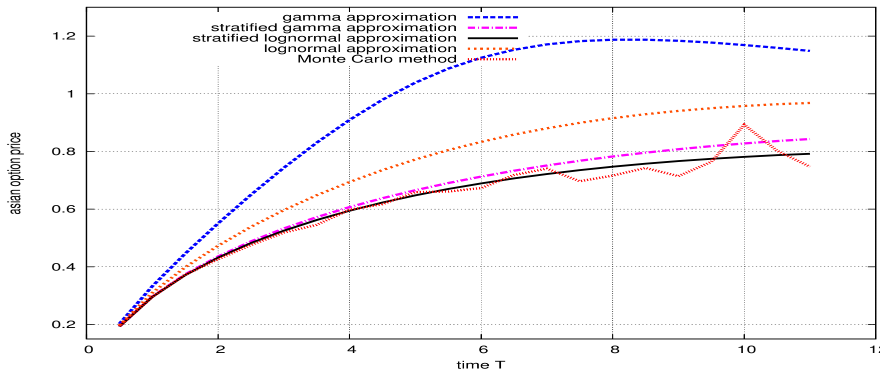

Figure 2: Asian option prices with $\sigma = 1$, $r = 0.05$, $K/S_0 = 1.1$, $S_0 = 1.5$.

Figure 3 is consistent with the numerical result of [11], page 486.

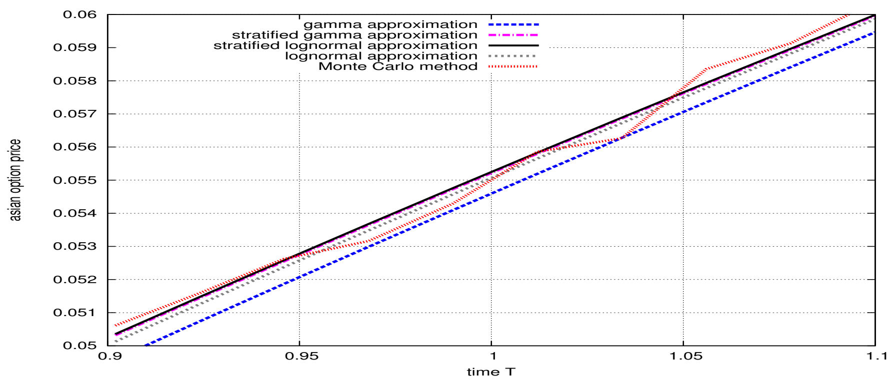

Figure 3: Asian option prices with $\sigma = 0.3$, $r = 0.05$, $K/S_0 = 1.1$, $S_0 = 1.5$.

4 Stratified bond pricing in the Dothan model

The following proposition, deduced from Lemma 2.1, provides a closed form integral expression for the conditional Laplace transform of Λ_T. In the next proposition we use the modified Bessel function of the second kind

$$K_\zeta(z) = \frac{z^\zeta}{2^{\zeta+1}} \int_0^\infty \exp\left(-u - \frac{z^2}{4u}\right) \frac{du}{u^{\zeta+1}}, \qquad \zeta \in \mathbb{R}, \quad z \in \mathbb{C}, \tag{4.1}$$

cf. e.g. [21] page 183, provided the real part $\mathcal{R}(z^2)$ of $z^2 \in \mathbb{C}$ is positive.

Proposition 4.1 *For all $\lambda, z > 0$ we have*

$$E\left[\exp\left(-\lambda \Lambda_T\right) \;\middle|\; S_T = z\right] \tag{4.2}$$

$$= \frac{4e^{-\sigma^2 T/8}}{\pi^{3/2}\sigma^2 p(z)} \sqrt{\frac{\lambda}{T}} \int_0^\infty e^{2(\pi^2-\xi^2)/(\sigma^2 T)} \sin\left(\frac{4\pi\xi}{\sigma^2 T}\right) \sinh(\xi) \frac{K_1\left(\sqrt{8\lambda}\sqrt{1+2\sqrt{z}\cosh\xi + z}/\sigma\right)}{\sqrt{1+2\sqrt{z}\cosh\xi+z}} d\xi.$$

Proof. By the scaling relation

$$E\left[\exp(-\lambda\Lambda_T) \;\Big|\; S_T = z\right] = E\left[\exp\left(-\frac{\lambda}{\sigma^2}\int_0^{\sigma^2 T} e^{B_t - pt/2} dt\right) \;\Big|\; S_T = z\right], \quad z > 0,$$

that follows from (2.7), it suffices to do the proof for $\sigma = 1$. By the Fubini theorem we have

$$\int_0^\infty e^{-u\lambda} \exp\left(-2\frac{1+z}{u}\right) \theta\left(\frac{4\sqrt{z}}{u}, \frac{T}{4}\right) \frac{du}{u} \tag{4.3}$$
$$= \frac{4e^{2\pi^2/T}\sqrt{z}}{\sqrt{\pi^3 T/2}} \int_0^\infty e^{-2\xi^2/T} \sin\left(\frac{4\pi\xi}{T}\right) \sinh(\xi) \int_0^\infty \exp\left(-\lambda u - 2\frac{1+2\sqrt{z}\cosh\xi + z}{u}\right) \frac{du}{u^2} d\xi,$$

since the above integrand belongs to $L^1(\mathbb{R}_+^2)$ as it is bounded by

$$(\xi, u) \mapsto e^{-2\xi^2/T} \sinh(\xi) \exp\left(-\lambda u - 2\frac{1+z}{u}\right).$$

Next, we have

$$\int_0^\infty \exp\left(-\lambda u - 2\frac{1+2\sqrt{z}\cosh\xi+z}{u}\right)\frac{du}{u^2} = \sqrt{2\lambda}\frac{K_1\left(\sqrt{8\lambda}\sqrt{1+2\sqrt{z}\cosh\xi+z}\right)}{\sqrt{1+2\sqrt{z}\cosh\xi+z}},$$

where we used the identity (4.1). Hence we find

$$E\left[\exp(-\lambda\Lambda_T) \;\Big|\; S_T = z\right] = \int_0^\infty e^{-u\lambda}\mathbb{P}\left(\Lambda_T \in du \;\Big|\, S_T = z\right)$$
$$= \sqrt{\frac{\pi T}{2}} e^{(\log z)^2/(2T)} \int_0^\infty e^{-u\lambda} \exp\left(-2\frac{1+z}{u}\right)\theta\left(\frac{4\sqrt{z}}{u}, T/4\right)\frac{du}{u} dz$$
$$= \frac{4\sqrt{2\lambda z}}{\pi} e^{(\log z)^2/(2T) + 2\pi^2/T} \int_0^\infty e^{-2\xi^2/T} \sin\left(\frac{4\pi\xi}{T}\right) \sinh(\xi) \frac{K_1\left(\sqrt{8\lambda}\sqrt{1+2\sqrt{z}\cosh\xi+z}\right)}{\sqrt{1+2\sqrt{z}\cosh\xi+z}} d\xi.$$

□

Under the Gamma approximation of Proposition 3.1 we approximate the conditional bond price on the short rate S_t as

$$E\left[\exp\left(-\lambda\int_0^T S_t dt\right) \;\Big|\, S_T = z\right] \;\simeq\; (1+\lambda\theta(z))^{-\nu(z)},$$

hence by stratification we have

$$E\left[\exp\left(-\lambda\int_0^T S_t dt\right)\right] \simeq \int_0^\infty (1+\lambda\theta(z))^{-\nu(z)}\, d\mathbb{P}(S_T = z). \qquad (4.4)$$

Figures 4 and 5 show that the pricing formula based on the integral expression (4.2) fails for small values of $T > 0$ when $\sigma = 0.3$ and $\sigma = 0.5$ while the stratified gamma approximation (4.4) is more stable and matches the Monte Carlo simulations. For small values of T, we may also use the asymptotics (2.6) of $E[\Lambda_T \mid S_T = z]$ to derive the small time approximation

$$E\left[\exp\left(-\lambda\int_0^T S_t dt\right)\ \Big| S_T = z\right] \simeq 1 - \lambda E[\Lambda_T \mid S_T = z] \simeq 1 + \frac{\lambda T(1-z)}{\log z} + o(T).$$

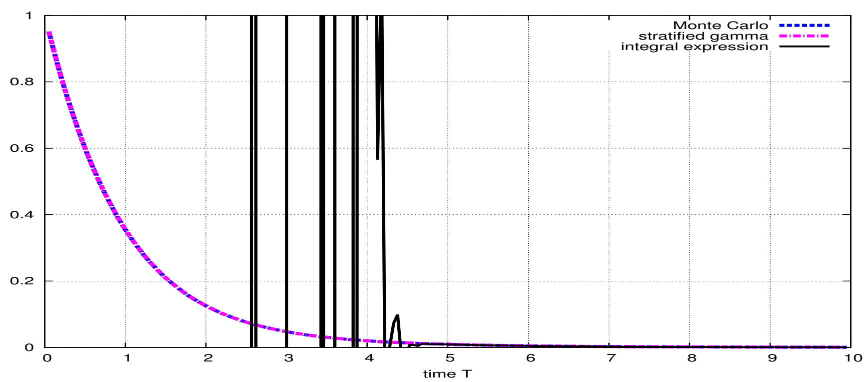

Figure 4: Approximations of Dothan bond prices with $\sigma = 0.3$.

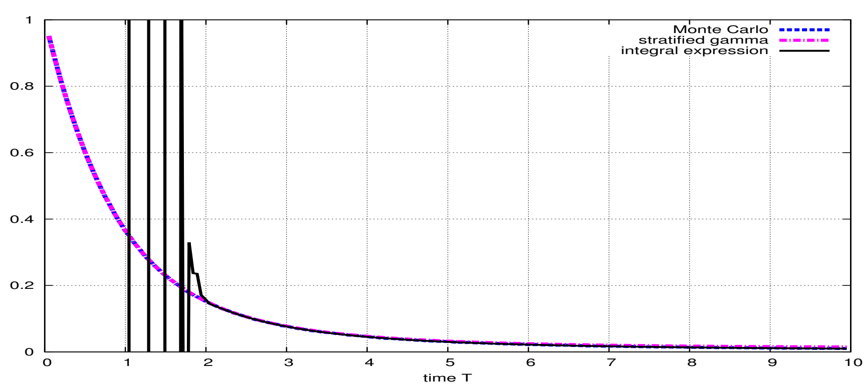

Figure 5: Approximations of Dothan bond prices with $\sigma = 0.5$.

On the other hand, Figure 6 for $\sigma = 1$ shows that the gamma approximation (4.4) becomes less accurate for large values of $\sigma^2 T$.

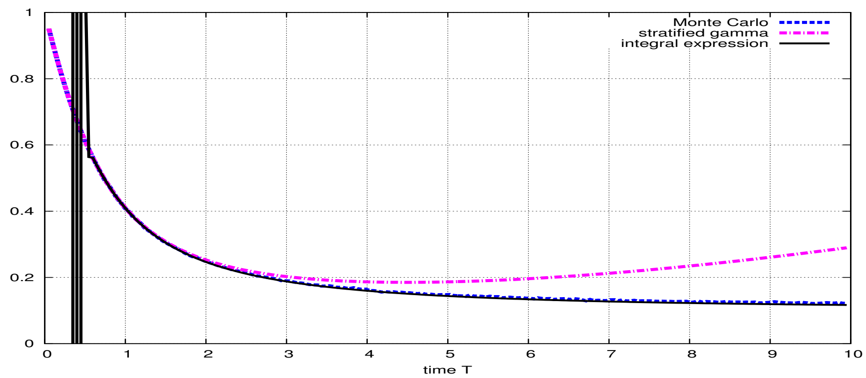

Figure 6: Approximations of Dothan bond prices with $\sigma = 1$.

We are not proposing a bond price approximation based on the lognormal distribution since its Laplace transform does not admit a closed form expression. Note however that the Laplace transform of the lognormal distribution can be approximated [1].

5 Appendix - conditional mean and variance

We now prove Proposition 2.2 which contains the closed form expressions of the conditional mean and variance, $E[\Lambda_T \mid S_T = z]$ and $\mathrm{Var}[\Lambda_T \mid S_T = z]$.

Proof of Proposition 2.2. By scaling it suffices to do the proof for $\sigma = 1$. Under conditioning we write

$$S_t = e^{\sigma(B_t - tB_T/T) + t(\log z)/T}, \qquad t \in [0, T],$$

hence

$$\begin{aligned} E[\Lambda_T \mid S_T = z] &= \int_0^T E[S_t \mid S_T = z] dt \\ &= \int_0^T e^{t(\log z)/T + t(T-t)/(2T)} dt \\ &= \int_0^T e^{t((\log z)/T + 1/2) - t^2/(2T)} dt \\ &= e^{(T/2 + \log z)^2/(2T)} \int_0^T e^{-(t - T/2 - \log z)^2/(2T)} dt \end{aligned}$$

$$
\begin{aligned}
&= e^{(T/2+\log z)^2/(2T)} \int_{-T/2-\log z}^{T/2-\log z} e^{-x^2/(2T)} dx \\
&= \sqrt{T} e^{(T/2+\log z)^2/(2T)} \int_{-\frac{\log z}{\sqrt{T}}-\sqrt{T}/2}^{-\frac{\log z}{\sqrt{T}}+\sqrt{T}/2} e^{-y^2/2} dy \\
&= \sqrt{2\pi T} e^{(T/2+\log z)^2/(2T)} \left(\Phi\left(-\frac{\log z}{\sqrt{T}} + \frac{1}{2}\sqrt{T} \right) - \Phi\left(-\frac{\log z}{\sqrt{T}} - \frac{1}{2}\sqrt{T} \right) \right),
\end{aligned}
$$

from which we conclude by (2.7). Next, we have

$$
\begin{aligned}
E[(\Lambda_T)^2 \mid S_T = z] &= E\left[\left(\int_0^T e^{B_t - tB_T/T + t(\log z)/T} dt \right)^2 \right] \\
&= 2 \int_0^T \int_0^t e^{(s+t)(\log z)/T} E[e^{B_s - sB_T/T + B_t - tB_T/T}] ds dt \\
&= 2 \int_0^T \int_0^t e^{((s+t)(\log z)/T + \frac{1}{2T}(t(T-t)+s(T-s)+2s(T-t)))} ds dt, \\
&= 2 e^{(3T/2+\log z)^2/(2T)} \int_0^T e^{-t} \int_0^t e^{-\frac{1}{2T}(3T/2-s-t+\log z)^2} ds dt \\
&= 2 e^{(3T/2+\log z)^2/(2T)} \int_0^T e^{-t} \int_{-3T/2+t-\log z}^{-3T/2+2t-\log z} e^{-x^2/(2T)} dx dt \\
&= 2 e^{(3T/2+\log z)^2/(2T)} \sqrt{T} \int_0^T e^{-t} \int_{-\frac{\log z}{\sqrt{T}} - \frac{3}{2}\sqrt{T} + t\sqrt{\frac{1}{T}}}^{-\frac{\log z}{\sqrt{T}} - \frac{3}{2}\sqrt{T} + 2t\sqrt{\frac{1}{T}}} e^{-y^2/2} dy dt \\
&= 2\sqrt{2\pi T} e^{(3T/2+\log z)^2/(2T)} \int_0^T e^{-t} \Phi\left(-\frac{\log z}{\sqrt{T}} - \frac{3}{2}\sqrt{T} + 2t\sqrt{\frac{1}{T}} \right) dt \\
&\quad -2\sqrt{2\pi T} e^{(3T/2+\log z)^2/(2T)} \int_0^T e^{-t} \Phi\left(-\frac{\log z}{\sqrt{T}} - \frac{3}{2}\sqrt{T} + t\sqrt{\frac{1}{T}} \right) dt.
\end{aligned}
$$

By integration by parts we have

$$
\begin{aligned}
\int_0^T e^{-t} \Phi\left(-\frac{\log z}{\sqrt{T}} - \frac{3}{2}\sqrt{T} + 2t\sqrt{\frac{1}{T}} \right) dt &= \sqrt{\frac{2}{\pi T}} \int_0^T e^{-t-(3T/2-2t+\log z)^2/(2T)} dt \\
&\quad + \Phi\left(-\frac{\log z}{\sqrt{T}} - \frac{3}{2}\sqrt{T} \right) - e^{-T} \Phi\left(-\frac{\log z}{\sqrt{T}} + \frac{1}{2}\sqrt{T} \right) \\
&= \sqrt{\frac{2}{\pi T}} \int_0^T e^{-(2t-T-\log z)^2/(2T) - (3T/2+\log z)^2/(2T) + (T+\log z)^2/(2T)} dt \\
&\quad + \Phi\left(-\frac{\log z}{\sqrt{T}} - \frac{3}{2}\sqrt{T} \right) - e^{-T} \Phi\left(-\frac{\log z}{\sqrt{T}} + \frac{1}{2}\sqrt{T} \right)
\end{aligned}
$$

$$
\begin{aligned}
&= -e^{-T}\Phi\left(-\frac{\log z}{\sqrt{T}}+\frac{1}{2}\sqrt{T}\right)+\Phi\left(-\frac{\log z}{\sqrt{T}}-\frac{3}{2}\sqrt{T}\right)\\
&\quad +\frac{e^{-5T/8}}{\sqrt{z}}\left(\Phi\left(-\frac{\log z}{\sqrt{T}}+\sqrt{T}\right)-\Phi\left(-\frac{\log z}{\sqrt{T}}-\sqrt{T}\right)\right),
\end{aligned}
$$

and similarly,

$$
\begin{aligned}
&\int_0^T e^{-t}\Phi\left(-\frac{\log z}{\sqrt{T}}-\frac{3}{2}\sqrt{T}+t\sqrt{\frac{1}{T}}\right)dt\\
&= -e^{-T}\Phi\left(-\frac{\log z}{\sqrt{T}}-\frac{1}{2}\sqrt{T}\right)+\Phi\left(-\frac{\log z}{\sqrt{T}}-\frac{3}{2}\sqrt{T}\right)+\sqrt{\frac{1}{2\pi T}}\int_0^T e^{-t-(3T/2-t+\log z)^2/(2T)}dt\\
&= -e^{-T}\Phi\left(-\frac{\log z}{\sqrt{T}}-\frac{1}{2}\sqrt{T}\right)+\Phi\left(-\frac{\log z}{\sqrt{T}}-\frac{3}{2}\sqrt{T}\right)\\
&\quad +\frac{e^{-T}}{z}\left(\Phi\left(-\frac{\log z}{\sqrt{T}}+\frac{1}{2}\sqrt{T}\right)-\Phi\left(-\frac{\log z}{\sqrt{T}}-\frac{1}{2}\sqrt{T}\right)\right).
\end{aligned}
$$

Consequently we have

$$
\begin{aligned}
&E[(\Lambda_T)^2] = 2\sqrt{2\pi T}\left(\Phi\left(\frac{\log z}{\sqrt{T}}+\sqrt{T}\right)-\Phi\left(\frac{\log z}{\sqrt{T}}-\sqrt{T}\right)\right)e^{(3T/2+\log z)^2/(2T)-5T/8-(\log z)/2}\\
&-2\sqrt{2\pi T}\left(\Phi\left(\frac{\log z}{\sqrt{T}}+\frac{1}{2}\sqrt{T}\right)-\Phi\left(\frac{\log z}{\sqrt{T}}-\frac{1}{2}\sqrt{T}\right)\right)\left(e^{\frac{1}{2T}(3T/2+\log z)^2-T}+e^{(T/2+\log z)^2/(2T)}\right)\\
&= 2\sqrt{2\pi T}\left(\Phi\left(\frac{\log z}{\sqrt{T}}+\sqrt{T}\right)-\Phi\left(\frac{\log z}{\sqrt{T}}-\sqrt{T}\right)\right)e^{(T+\log z)^2/(2T)}\\
&\quad -2\sqrt{2\pi T}(1+z)e^{(T/2+\log z)^2/(2T)}\left(\Phi\left(\frac{\log z}{\sqrt{T}}+\frac{1}{2}\sqrt{T}\right)-\Phi\left(\frac{\log z}{\sqrt{T}}-\frac{1}{2}\sqrt{T}\right)\right),
\end{aligned}
$$

which yields (2.5) by (2.7). □

In the next proposition, for reference we also compute the unconditional mean and variance of Λ_T, which have been used in (1.2), cf. also (7) and (8) page 480 of [11]. Note that closed-form expressions are available for the moments of Λ_T of all orders, cf. Corollary 2 page 33 of [22] and the references given in Postscript #3 page 54 therein.

Proposition 5.1 *We have*

$$
E[\Lambda_T] = \frac{e^{rT}-1}{r},
$$

and

$$
E[(\Lambda_T)^2] = 2\frac{re^{(2r+\sigma^2)T}-(2r+\sigma^2)e^{rT}+(r+\sigma^2)}{r(r+\sigma^2)(2r+\sigma^2)}.
$$

Proof. For the second moment we have

$$\begin{aligned}
E[(\Lambda_T)^2] &= \int_0^T \int_0^T e^{-p\sigma^2 a/2 - p\sigma^2 b/2} E[e^{\sigma B_a} e^{\sigma B_b}] db da \\
&= 2\int_0^T \int_0^a e^{-p\sigma^2 a/2 - p\sigma^2 b/2} e^{\sigma^2(a+b)/2} e^{b\sigma^2} db da \\
&= 2\int_0^T e^{-(p-1)\sigma^2 a/2} \int_0^a e^{-(p-3)\sigma^2 b/2} db da \\
&= \frac{4}{(p-3)\sigma^2} \int_0^T e^{-(p-1)\sigma^2 a/2} (1 - e^{-(p-3)\sigma^2 a/2}) da \\
&= \frac{4}{(p-3)\sigma^2} \int_0^T e^{-(p-1)\sigma^2 a/2} da - \frac{4}{(p-3)\sigma^2} \int_0^T e^{-(p-1)\sigma^2 a/2} e^{-(p-3)\sigma^2 a/2} da \\
&= \frac{8}{(p-3)(p-1)\sigma^4} (1 - e^{-(p-1)\sigma^2 T/2}) - \frac{4}{(p-3)\sigma^2} \int_0^T e^{-(2p-4)\sigma^2 a/2} da \\
&= \frac{8}{(p-3)(p-1)\sigma^4} (1 - e^{-(p-1)\sigma^2 T/2}) - \frac{4}{(p-3)(p-2)\sigma^4} (1 - e^{-(p-2)\sigma^2 T}) \\
&= 2\frac{re^{(2r+\sigma^2)T} - (2r+\sigma^2)e^{rT} + (r+\sigma^2)}{r(r+\sigma^2)(2r+\sigma^2)},
\end{aligned}$$

since $r - \sigma^2/2 = -p\sigma^2/2$. □

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6 Addendum - error bounds in the Wasserstein distance

This section deals with approximation error bounds based on the Malliavin calculus and the Stein method. In particular we apply recent results on the Malliavin calculus and the Stein method, cf. [NP09], to study the error generated by the gamma and log-normal approximations using Wasserstein type distance estimates between probability measures. Recall that letting

$$I_1(f) = \int_0^T f(t) dB_t$$

denote the first order integral of $f \in L^2([0,T])$ with respect to Brownian motion, the Malliavin gradient is the operator D_t defined as

$$D_t F = \sum_{k=1}^n f_k(t) \frac{\partial g}{\partial x_k}(I_1(f_1), \ldots, I_1(f_n)), \qquad t \in [0,T],$$

where the random variable F has the form $F = g(I_1(f_1), \ldots, I_1(f_n))$, the function g is in the space $\mathcal{C}^1([0,T]^n)$ of continuously differentiable functions on $[0,T]^n$, and $f_1, \ldots, f_n \in L^2([0,T])$, $n \geq 1$, cf. e.g. [Üst95] and references therein. We denote by $(\mathcal{F}_t)_{t\in[0,T]}$ the filtration generated by the Brownian motion $(B_t)_{t\in[0,T]}$ built on the Wiener space W as the coordinate process $B_t(\omega) = \omega(t)$, $\omega \in W$. Recall also that the Ornstein-Uhlenbeck operator L can be defined via its semigroup $(P_t)_{t\in\mathrm{R}} = (e^{tL})_{t\in\mathrm{R}_+}$ by the Mehler formula

$$e^{-tL} F(\omega) = \tilde{E}[F(e^{-t}\omega + \sqrt{1-e^{-2t}}\tilde{\omega})], \qquad t > 0, \tag{6.1}$$

cf. § I.2 page 15 of [Üst95], where $\tilde{\omega}$ denotes an independent copy of $\omega \in W$. We will consider the Wasserstein type distance

$$d(X,Y) := \sup_{h\in\mathcal{H}} |\mathrm{E}[h(X)] - \mathrm{E}[h(Y)]|, \tag{6.2}$$

between the laws of random variables X, Y, where

$$\mathcal{H} := \{h \in \mathcal{C}_b^2(\mathbb{R}) : \ \max\{\|h\|_\infty, \|h'\|_\infty, \|h''\|_\infty\} \leq 1\}.$$

In the sequel we let $\langle \cdot, \cdot \rangle$ denote the inner product defined by

$$\langle f, g \rangle := \int_0^T f(t) g(t) dt, \qquad f, g \in L^2([0,T]).$$

Gamma error bounds

Letting $\Gamma_{\hat{\theta},\hat{\nu}}$ denote the gamma distribution with parameters $(\hat{\nu},\hat{\theta})$ given from (3.2). We have $E[\Lambda_T] = E[\Gamma_{\hat{\theta},\hat{\nu}}]$ and the bounds

$$\begin{aligned} d(\Lambda_T, \Gamma_{\hat{\theta},\hat{\nu}}) &= d(\Lambda_T - E[\Lambda_T], \Gamma_{\hat{\theta},\hat{\nu}} - E[\Gamma_{\hat{\theta},\hat{\nu}}]) \\ &\leq K\sqrt{E[(2\hat{\theta}\Lambda_T - \langle D\Lambda_T, D(-L)^{-1}(\Lambda_T - E[\Lambda_T])\rangle)^2]}, \end{aligned}$$

for $K > 0$ a constant, cf. [NP09], Theorem 3.11, and

$$\begin{aligned} d(\Lambda_T, \Gamma_{\hat{\theta},\hat{\nu}}) &= d(\Lambda_T - E[\Lambda_T], \Gamma_{\hat{\theta},\hat{\nu}} - E[\Gamma_{\hat{\theta},\hat{\nu}}]) \\ &\leq d(\Lambda_T, \Gamma_{\hat{\theta},\hat{\nu}}) \leq K\sqrt{E[(2\hat{\theta}\Lambda_T - \langle D.\Lambda_T, E[D.\Lambda_T \mid \mathcal{F}.]\rangle)^2]}, \end{aligned} \qquad (6.3)$$

cf. [PT13], Corollary 3.4, after rescaling with respect to the parameter $\hat{\theta}$.

Lognormal error bounds

On the other hand, applying Theorem 1 and § 4.4 of [KT12] with $b(x) = E[\mathcal{L}_{p\sigma^2 T/2,\sigma^2}] - x$ we have $E[b(Y)] = E[b(\Lambda_T)] = E[\Lambda_T] - E[\mathcal{L}_{-\hat{p}\hat{\sigma}^2 T/2,\hat{\sigma}^2}] = 0$ and

$$d(\Lambda_T, \mathcal{L}_{-\hat{p}\hat{\sigma}^2 T/2,\hat{\sigma}^2 T}) \leq K\sqrt{E[(a_T(\Lambda_T) - \langle D\Lambda_T, D(-L)^{-1}(\Lambda_T - E[\Lambda_T])\rangle)^2]}, \qquad (6.4)$$

where $\mathcal{L}_{-p\sigma^2 T/2,\sigma^2}$ is a lognormal random variable with mean $-\hat{p}\hat{\sigma}^2 T/2$ and variance $\hat{\sigma}^2 T$ given by (1.3)-(1.4) and

$$a_T(z) = \frac{1}{p(z)}\left(\Phi\left(\frac{(\log z) - \delta}{\sigma}\right) - \Phi\left(\frac{(\log z) - \delta}{\sigma} - \sigma\right)\right),$$

is defined in (2.4). Replacing the use of the Ornstein-Uhlenbeck covariance representation in the derivation of (6.4) with the Clark-Ocone covariance representation as in (6.3) and [PT13] yields the bound

$$d(\Lambda_T, \mathcal{L}_{-p\sigma^2 T/2,\sigma^2} \leq K\sqrt{E[(a_T(\Lambda_T) - \langle D.\Lambda_T, E[D.\Lambda_T \mid \mathcal{F}.]\rangle)^2]}. \qquad (6.5)$$

Next, we show how the terms

$$\langle D\Lambda_T, D(-L)^{-1}(\Lambda_T - E[\Lambda_T])\rangle \quad \text{and} \quad \langle D.\Lambda_T, E[D.\Lambda_T \mid \mathcal{F}.]\rangle$$

appearing in (6.3)-(6.5) can be computed in the unconditional case.

Proposition 6.1 *We have*

$$\langle D.\Lambda_T, E[D.\Lambda_T \mid \mathcal{F}.]\rangle = \int_0^T \int_0^T e^{\sigma B_s - p\sigma^2(s+t)/2} \int_0^{s\wedge t} e^{\sigma B_u + \sigma^2(T-u)/2} du ds dt.$$

Proof. For any $f \in L^2([0,T])$ we have

$$D_t e^{I_1(f)} = f(t) e^{I_1(f)},$$

and

$$\begin{aligned} E[D_t e^{I_1(g)} \mid \mathcal{F}_t] &= g(t) E[e^{I_1(g)} \mid \mathcal{F}_t] \\ &= g(t) e^{\int_0^T g^2(s)ds/2} E[e^{I_1(g) - \int_0^T g^2(s)ds/2} \mid \mathcal{F}_t] \\ &= g(t) e^{\int_0^T g^2(s)ds/2} e^{\int_0^t g(s)dB_s - \int_0^t g^2(s)ds/2} \\ &= g(t) e^{\int_0^t g(s)dB_s + \int_t^T g^2(s)ds/2}, \end{aligned}$$

hence

$$\langle D.e^{I_1(f)}, E[D.e^{I_1(g)} \mid \mathcal{F}.]\rangle = e^{I_1(f)} \int_0^T f(u)g(u) e^{\int_0^u g(s)dB_s + \int_u^T g^2(s)ds/2} du,$$

and

$$\langle D.e^{\sigma B_s}, E[D.e^{\sigma B_t} \mid \mathcal{F}.]\rangle = e^{\sigma B_s} \int_0^{s\wedge t} e^{\sigma B_u + \sigma^2(T-u)/2} du.$$

This yields

$$\langle D.e^{\sigma B_s - p\sigma^2 s/2}, E[D.e^{\sigma B_t - p\sigma^2 t/2} \mid \mathcal{F}.]\rangle = e^{\sigma B_s - p\sigma^2(s+t)/2} \int_0^{s\wedge t} e^{\sigma B_u + \sigma^2(T-u)/2} du,$$

and

$$\langle D.\Lambda_T, E[D.\Lambda_T \mid \mathcal{F}.]\rangle = \int_0^T \int_0^T \langle D.e^{\sigma B_s - p\sigma^2 s/2}, E[D.e^{\sigma B_t - p\sigma^2 t/2} \mid \mathcal{F}.]\rangle ds dt.$$

□

As for (6.3) and (6.4) we have the following result. In order to compute the term $(-L)^{-1}(\Lambda_T - E[\Lambda_T])$ we will use the representation formula

$$(-L)^{-1}F(\omega) = \int_0^\infty e^{-tL} F dt = \int_0^1 \tilde{E}[F(a\omega + \sqrt{1-a^2}\tilde{\omega})] da, \qquad (6.6)$$

for $F \in L^2(\Omega)$ with $E[F] = 0$, that follows from the Mehler formula (6.1), cf. also Lemma 3.8 of [Vie09].

Proposition 6.2 *We have*

$$\langle D.\Lambda_T, D.(-L)^{-1}(\Lambda_T - E[\Lambda_T])\rangle = \int_0^T \int_0^T \frac{s\wedge t}{t} e^{-p(s+t)\sigma^2/2} e^{\sigma B_s + \sigma^2 t/2}$$
$$\times \left(1 - e^{\sigma B_t - \sigma^2 t/2} + \sqrt{\frac{2\pi}{t}} B_t e^{B_t^2/(2t)} \left(\Phi\left(B_t/\sqrt{t}\right) - \Phi\left(B_t/\sqrt{t} - \sigma\sqrt{t}\right)\right)\right) dsdt.$$

Proof. By (6.6) we have, denoting by $\tilde{I}_1(g)$ the stochastic integral of $g \in L^2([0,T])$ with respect to $\tilde{\omega}$,

$$\begin{aligned}
(-L)^{-1}(e^{I_1(g)} - E[e^{I_1(g)}]) &= \tilde{E}\left[\int_0^1 e^{aI_1(g)+\sqrt{1-a^2}\tilde{I}_1(g)} da\right] \\
&= \int_0^1 e^{aI_1(g)+(1-a^2)\eta^2/2} da \\
&= e^{\eta^2/2}\int_0^1 e^{aI_1(g)-a^2\eta^2/2} da \\
&= e^{\eta^2/2} e^{(I_1(g))^2/(2\eta^2)} \int_0^1 e^{-(I_1(g)/\eta - a\eta)^2/2} da \\
&= \frac{1}{\eta} e^{\eta^2/2} e^{(I_1(g))^2/(2\eta^2)} \int_0^\eta e^{-(a - I_1(g)/\eta)^2/2} da \\
&= \frac{1}{\eta} e^{\eta^2/2} e^{(I_1(g))^2/(2\eta^2)} \int_{-I_1(g)/\eta}^{\eta - I_1(g)/\eta} e^{-a^2/2} da \\
&= \frac{\sqrt{2\pi}}{\eta} e^{\eta^2/2} e^{(I_1(g))^2/(2\eta^2)} \left(\Phi\left(I_1(g)/\eta\right) - \Phi\left(I_1(g)/\eta - \eta\right)\right),
\end{aligned}$$

with $\eta^2 = \int_0^T g^2(s)ds$, hence

$$\begin{aligned}
D_t(-L)^{-1}(e^{I_1(g)} - E[e^{I_1(g)}]) &= \frac{\sqrt{2\pi}}{\eta} e^{\eta^2/2} D_t\left(e^{(I_1(g))^2/(2\eta^2)}\left(\Phi\left(I_1(g)/\eta\right) - \Phi\left(I_1(g)/\eta - \eta\right)\right)\right) \\
&= \frac{\sqrt{2\pi}}{\eta} e^{\eta^2/2} e^{(I_1(g))^2/(2\eta^2)} D_t\left(\Phi\left(I_1(g)/\eta\right) - \Phi\left(I_1(g)/\eta - \eta\right)\right) \\
&\quad + \frac{\sqrt{2\pi}}{\eta} e^{\eta^2/2}\left(\Phi\left(I_1(g)/\eta\right) - \Phi\left(I_1(g)/\eta - \eta\right)\right) D_t e^{(I_1(g))^2/(2\eta^2)} \\
&= \frac{1}{\eta^2} g(t) e^{\eta^2/2} e^{(I_1(g))^2/(2\eta^2)} \left(e^{-(I_1(g)/\eta)^2/2} - e^{-(I_1(g)/\eta - \eta)^2/2}\right) \\
&\quad + \frac{\sqrt{2\pi}}{\eta^3} e^{\eta^2/2} e^{(I_1(g))^2/(2\eta^2)} \left(\Phi\left(I_1(g)/\eta\right) - \Phi\left(I_1(g)/\eta - \eta\right)\right) g(t) I_1(g) \\
&= \frac{1}{\eta^2} g(t)\left(e^{\eta^2/2} - e^{I_1(g)}\right) + \frac{\sqrt{2\pi}}{\eta^3} e^{\eta^2/2} g(t) I_1(g) e^{(I_1(g))^2/(2\eta^2)} \left(\Phi\left(I_1(g)/\eta\right) - \Phi\left(I_1(g)/\eta - \eta\right)\right),
\end{aligned}$$

hence, for any $f \in L^2([0,T])$,

$$\begin{aligned}
&\langle De^{I_1(f)}, D(-L)^{-1}(e^{I_1(g)} - E[e^{I_1(g)}])\rangle \\
&= \frac{1}{\eta^2} e^{I_1(f)+\eta^2/2} \langle f, g\rangle \left(1 - e^{I_1(g)-\eta^2/2} + \sqrt{2\pi}\frac{I_1(g)}{\eta} e^{(I_1(g))^2/(2\eta^2)} \left(\Phi\left(I_1(g)/\eta\right) - \Phi\left(I_1(g)/\eta - \eta\right)\right)\right),
\end{aligned}$$

and

$$\begin{aligned}
&\langle De^{\sigma B_s}, D(-L)^{-1}(e^{\sigma B_t} - E[e^{\sigma B_t}])\rangle \\
&= \frac{s \wedge t}{t} e^{\sigma B_s} \left(e^{\sigma^2 t/2} - e^{\sigma B_t} + \sqrt{2\pi/t} B_t e^{B_t^2/(2t)+\sigma^2 t/2} \left(\Phi\left(B_t/\sqrt{t}\right) - \Phi\left(B_t/\sqrt{t} - \sigma\sqrt{t}\right)\right)\right),
\end{aligned}$$

which yields

$$\begin{aligned}
&\langle De^{\sigma B_s - p\sigma^2 s/2}, D(-L)^{-1}(e^{\sigma B_t - p\sigma^2 t/2} - E[e^{\sigma B_t - p\sigma^2 t/2}])\rangle \qquad (6.7)\\
&= \frac{s \wedge t}{t} e^{-p(s+t)\sigma^2/2} e^{\sigma B_s + \sigma^2 t/2} \\
&\quad \times \left(1 - e^{\sigma B_t - \sigma^2 t/2} + \sqrt{\frac{2\pi}{t}} B_t e^{B_t^2/(2t)} \left(\Phi\left(B_t/\sqrt{t}\right) - \Phi\left(B_t/\sqrt{t} - \sigma\sqrt{t}\right)\right)\right).
\end{aligned}$$

□

Note also that the above bounds can also be computed under conditioning given $S_T = z$, by writing

$$S_t = z^{t/T} e^{\sigma U_t} = z^{t/T} e^{\sigma(B_t - \frac{t}{T} B_T)},$$

where

$$U_t := B_t - \frac{t}{T} B_T, \qquad t \in [0,T],$$

is a standard Brownian bridge with $U_0 = U_T = 0$.

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