Laplace transform identities for the volume of stopping sets based on Poisson point processes

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Abstract

We derive Laplace transform identities for the volume content of random stopping sets based on Poisson point processes. Our results are based on anticipating Girsanov identities for Poisson point processes under a cyclic vanishing condition for a finite difference gradient. This approach does not require classical assumptions based on set-indexed martingales and the (partial) ordering of index sets. The examples treated focus on stopping sets in finite volume, and include the random missed volume of Poisson convex hulls.

Key words: Poisson point processes; stopping sets; gamma-type distributions; Girsanov identities; anticipating stochastic calculus.
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1 Introduction

Gamma-type results for the area of random domains constructed from a finite number of "typical" Poisson distributed points, and more generally known as complementary theorems, have been obtained in [7], cf. e.g. Theorem 10.4.8 in [14].

Stopping sets are random sets that carry over the notion of stopping time to set-indexed processes, cf. Definition 2.27 in [8], based on stochastic calculus for set-indexed martingales, cf. e.g. [6]. Gamma-type results for the probability law of the volume content of random sets have been obtained in the framework of stopping sets in [15], via Laplace transforms, using the martingale property of set-indexed stochastic exponentials, see [16] for the strong Markov property for point processes, cf. also [2] for extensions to Poisson processes of k-flats in $\mathbb{R}^d$.

The above mentioned approaches make use of changes of measures by modifying the intensity of the underlying Poisson point process, cf. also § 6 of [4]. In this paper we further develop and extend the change of measure approach to the derivation of the probability distribution of random sets, based on anticipating Girsanov identities under a measure with density, cf. Proposition 4.1 and Corollary 5.2. Instead of relying on set-indexed adaptedness, we use a cyclic vanishing condition of quasi-nilpotence type for the finite difference gradient of stochastic processes. This approach does not require any (partial) ordering of index sets, in the spirit of anticipating stochastic calculus on the Poisson and Wiener spaces.

As a consequence of Girsanov identities we derive Laplace transform identities for the volume of stopping sets in finite volume, cf. Proposition 5.4 and Corollaries 5.5-5.6 below. This approach also recovers classical gamma-type identities [9], [15], for the Laplace transform of the volume of stopping sets, cf. Corollary 5.3.

This paper is organized as follows. In Section 2 and 3 we state the definitions and preliminary results needed on stopping sets. In Section 4 we state an extension of the Girsanov identities of [10] to measures under a density. In Section 5 we derive formulas for the conditional Laplace transform of the random volume content of stopping sets. Examples in finite volume are given, including the convex hull of a Poisson point process. Section 6 contains the technical proof of the anticipating Girsanov identities of Section 4.

2 Poisson point processes

We work with a Poisson point process having a sigma-finite diffuse intensity measure σ on a sigma-compact metric space X with Borel sigma-algebra $\mathcal{B}(X)$. The underlying probability space

$$\Omega^X = \{\omega := (x_i)_{i=1,\ldots,N} \subset X,\ x_i \neq x_j,\ \forall i \neq j,\ N \in \mathbb{N} \cup \{\infty\}\}$$

is the space of configurations whose elements $\omega \in \Omega^X$ are at most countable and locally finite subsets of X, which are identified with the Radon point measure

$$\omega = \sum_{x \in \omega} \delta_x,$$

where δ_x denotes the Dirac measure at $x \in X$ and $\omega(K) \in \mathbb{N} \cup \{\infty\}$ represents the cardinality of $K \cap \omega$.

Given K in the collection $\mathcal{K}(X)$ of compact subsets of X we let

$$\mathcal{F}_K = \sigma(\omega(U)\ :\ U \subset K,\ \sigma(U) < \infty)$$

denote the sigma-algebra generated by $\omega \mapsto \omega(U)$, $U \subset K$, $\sigma(U) < \infty$.

Letting $\mathcal{F} = \bigvee_{K \in \mathcal{K}(X)} \mathcal{F}_K$, the space $(\Omega^X, \mathcal{F})$ is endowed with the probability π_σ on X such that for all compact disjoint subsets $K_1, \ldots, K_n$ of X, $n \geq 1$, the mapping $\omega \mapsto (\omega(K_1), \ldots, \omega(K_n))$ is a vector of independent Poisson distributed random variables on $\mathbb{N}$ with respective parameters $\sigma(K_1), \ldots, \sigma(K_n)$.

We will use the finite difference operator D_x defined as

$$D_x F(\omega) = F(\omega \cup \{x\}) - F(\omega), \qquad x \in X,$$

and the iterated difference operator $D_{\mathfrak{s}_k}$ defined by

$$D_{\mathfrak{s}_k} F = D_{s_1} \cdots D_{s_k} F,$$

where $\mathfrak{s}_k = (s_1, \ldots, s_k) \in X^k$, $k \geq 1$, and $D_\emptyset F = F$. Recall that the standard Poisson process $(N_t)_{t \in \mathbb{R}_+}$ on $X = \mathbb{R}_+$ is defined by $N_t(\omega) = \omega([0, t])$, $t \in \mathbb{R}_+$. In particular we have the relation

$$D_{\mathfrak{s}_k} F(\omega) = \sum_{\eta \subset \{s_1,\ldots,s_k\}} (-1)^{k-|\eta|} F(\omega \cup \eta),$$

where the above summation is taken over all (possibly empty) subsets η of Θ.

3 Stopping sets

We recall the definition of *stopping set*, cf. [15] and Definition 2.27 page 335 of [8].

Definition 3.1 *A random compact set* $A(\omega)$ *is called a* stopping set *if*

$$\{\omega \ : \ A(\omega) \subset K\} \in \mathcal{F}_K \quad \textit{for all } K \in \mathcal{K}(X). \tag{3.1}$$

When $X = \mathbb{R}_+$ and $d = 1$, the interval $[0, \tau]$ is a stopping set when τ is a stopping time in the usual sense with respect to the forward filtration generated by $(N_t)_{t \in \mathbb{R}_+}$. In particular, any interval $[0, T_n]$, where T_n is the n-th Poisson jump time is a stopping set; in finite volume with $X = [0, T]$ we can also consider any interval $[0, T_n \wedge T]$ as well as the interval $[T_{N_T}, T]$ where T_{N_T} is the last Poisson jump time before time T, with $T_{N_T} = 0$ if $N_T = 0$ (note that the process $\mathbf{1}_{[0, T_{N_T})}(t)$ is predictable with respect to the *backward* Poisson process filtration generated by $(N_t)_{t \in \mathbb{R}_+}$).

When $X = \mathbb{R}^d$ with $d \geq 1$, examples of compact stopping sets include, in infinite volume (see [4], [5] for other examples),

- the minimal closed ball centered in the origin and containing exactly $n \geq 1$ points,
- the Poisson-Voronoi flower, which is the union of balls centered at the vertices of the Voronoi polygon containing the point 0 and exactly two other process points,
- the closed complement of the convex hull of a Poisson point process inside a convex subset of $\mathbb{R}^d$.

The latter example is a stopping set because the addition of a point within the convex hull will not modify its shape, in other words whether a compact K contains $A(\omega)$ is equivalent to whether K can contain all edges of the convex hull, and this can be decided based on the sole knowledge of the positions of configuration points contained in K.

A stopping set $A(\omega)$ will be said to be *non-increasing* if

$$A(\omega \cup \{x\}) \subset A(\omega), \qquad \omega \in \Omega^X, \quad x \in X, \tag{3.2}$$

which implies in particular

$$D_x \mathbf{1}_A(y) \leq 0, \qquad x, y \in X.$$

A stopping set $A(\omega)$ will be said to be *stable* if

$$x \in A(\omega) \Longrightarrow x \in A(\omega \cup \{x\}), \qquad \omega \in \Omega^X, \quad x \in X, \tag{3.3}$$

i.e. $D_x \mathbf{1}_A(x) \geq 0$, for all $x \in X$. In particular, for $A(\omega)$ a stable and non-increasing stopping set we have

$$D_x \mathbf{1}_A(x) = 0, \qquad x \in X.$$

The above monotonicity and stability conditions are not restrictive in practice because they are satisfied by common examples of stopping sets:

- The closed complement $A(\omega)$ of the convex hull of a Poisson point process inside a convex subset of $\mathbb{R}^d$ is a stable and non-increasing stopping set. The stability follows from the fact that the addition of a point $x \in A(\omega)$ to ω creates a new vertex in the convex hull of $\omega \cup \{x\}$. On the other hand, $A(\omega)$ is non-increasing because the addition of any configuration point can only make the convex hull larger.

- The Poisson-Voronoi flower is also a stable stopping set, which is non-increasing because each disk is defined by three points while only one of them is displaced by the addition of a new configuration point and the modified disk can only have a smaller radius.

The minimal closed ball centered in the origin and containing exactly $n \geq 1$ points is also a stable and non-increasing stopping set. The stability property depends on the openness or closedness of the stopping set. For example the closed complement of the convex hull is stable, while the open complement is not stable according to (3.3).

The following lemma will be needed for the proof of Proposition 4.2 below.

Lemma 3.2 *Let $A(\omega)$ be a non-increasing stopping set. Then for any $\mathcal{F}_A$-measurable random variable $F(\omega)$ we have*

$$D_x F(\omega) = 0, \qquad x \in A^c(\omega), \quad \omega \in \Omega^X.$$

Proof. Consider $B \in \mathcal{F}$ such that

$$B \cap \{\omega \ : \ A(\omega) \subset K\} \in \mathcal{F}_K,$$

hence

$$D_x(\mathbf{1}_B(\omega)\mathbf{1}_{\{A(\omega)\subset K\}}) = 0, \qquad x \in K^c,$$

for all $K \in \mathcal{K}(X)$ and $\omega \in \Omega^X$. Let now $\omega \in \Omega^X$ and $x \in A^c(\omega)$. There exists $K \in \mathcal{K}(X)$ such that

$$x \in K^c \subset A^c(\omega)$$

and in particular $A(\omega \cup \{x\}) \subset A(\omega) \subset K$ since $A(\omega)$ is non-increasing, hence

$$\begin{aligned} D_x \mathbf{1}_B(\omega) &= \mathbf{1}_B(\omega \cup \{x\}) - \mathbf{1}_B(\omega) \\ &= \mathbf{1}_B(\omega \cup \{x\})\mathbf{1}_{\{A(\omega\cup\{x\})\subset K\}} - \mathbf{1}_B(\omega)\mathbf{1}_{\{A(\omega)\subset K\}} \\ &= D_x \mathbf{1}_{B\cap\{A\subset K\}}(\omega) = 0, \end{aligned}$$

and we extend the statement from $B \in \mathcal{F}_A$ to any $\mathcal{F}_A$-measurable $F(\omega)$ by a monotone class argument. □

In particular, Lemma 3.2 shows that

$$D_x \mathbf{1}_{A(\omega)}(y) = 0, \qquad y \in X, \quad x \in A^c(\omega), \quad \omega \in \Omega^X, \tag{3.4}$$

by taking $F = \mathbf{1}_{A(\omega)}(y) \in \mathcal{F}_A$ for $y \in X$.

4 Girsanov identities

Proposition 4.1 is a Girsanov identity for random, non-adapted shifts of a Poisson point process which extends Proposition 2.1 of [10] by including a density F. Recall that the adapted Girsanov identity for a Poisson point process on $X = \mathbb{R}_+$ can be stated as

$$E\left[F \exp\left(-\int_0^T u_t \sigma(dt)\right) \prod_{t\in\omega\cap[0,T]} (1 + u_t)\right] = E[F],$$

provided that

$$E\left[F\exp\left(\int_0^T u_t\sigma(dt)\right)\prod_{t\in\omega\cap[0,T]}(1+u_t)\right]<\infty,$$

for $(u_t)_{t\in\mathbb{R}_+}$ an adapted process such that $u_t > -1$, $t \in \mathbb{R}_+$ and F an independent non-negative random variable which is measurable with respect to sigma-algebra generated by the future increments $(N_t - N_s)_{T\leq s\leq t}$ of the Poisson process $(N_t)_{t\in\mathbb{R}_+}$ after time T.

Proposition 4.1 *Consider* $\phi : \Omega^X \times X \to \mathbb{R}_+$ *a non-negative process and* $F(\omega)$ *a non-negative random variable such that*

$$D_{\Theta_0}F(\omega)D_{\Theta_1}\phi(\omega,x_1)\cdots D_{\Theta_k}\phi(\omega,x_k)=0, \qquad \sigma^{\otimes k}(dx_1,\ldots,dx_k)-a.e., \tag{4.1}$$

for all $\omega \in \Omega^X$, $k \geq 1$, *and all families* $\Theta_1,\ldots,\Theta_k$ *of (possibly empty) subsets of* $\{x_1,\ldots,x_k\}$ *with union* $\Theta_0\cup\Theta_1\cup\cdots\cup\Theta_k=\{x_1,\ldots,x_k\}$. *Then under the condition*

$$E\left[F(\omega)\exp\left(\int_X \phi(\omega,x)\sigma(dx)\right)\prod_{x\in\omega}(1+\phi(\omega,x))\right]<\infty,$$

we have the Girsanov identity

$$E[F(\omega)]=E\left[F(\omega)\exp\left(-\int_X \phi(\omega,x)\sigma(dx)\right)\prod_{x\in\omega}(1+\phi(\omega,x))\right].$$

The proof of Proposition 4.1 is given in the appendix Section 6.

We show in Proposition 4.2 below that Condition (4.1) of Proposition 4.1 is satisfied by the indicator functions of stopping sets. Given $A(\omega)$ a stopping set we define the stopped sigma-algebra

$$\mathcal{F}_A=\sigma(B\in\mathcal{F}\ :\ B\cap\{\omega\ :\ A(\omega)\subset K\}\in\mathcal{F}_K,\ K\in\mathcal{K}(X)), \tag{4.2}$$

cf. e.g. Definition 1 of [15].

Next we show that the indicator function of a stable and non-increasing stopping set $A(\omega)$ satisfies Condition (4.1) of Proposition 4.1, cf. also Proposition 3.3 of [3] for a particular situation.

Proposition 4.2 *For any* stable *and* non-increasing *stopping set* $A(\omega)$, *Condition* (4.1) *is satisfied by* $\phi(\omega, x) := \mathbf{1}_{A^c(\omega)}(x)$ *and any* $\mathcal{F}_A$*-measurable random variable* $F(\omega)$.

Proof. Let $x_1, \ldots, x_k \in X$. We consider the following cases.
(i) $\{x_1, \ldots, x_k\} \cap A^c(\omega) \neq \emptyset$. First, if there exists $i \in \{1, \ldots, k\}$ such that $x_i \in A^c(\omega)$, then $x_i \in A^c(\omega \cup \eta)$ for any $\eta \subset \{x_1, \ldots, x_k\}$ because $A(\omega)$ is non-increasing, and this shows $D_{x_i}\mathbf{1}_{A(\omega\cup\eta)}(x) = 0$ for all $x \in X$ by (3.4), hence $D_\Theta \mathbf{1}_{A(\omega)}(x) = D_\Theta \mathbf{1}_{A^c(\omega)}(x) = 0$ whenever $\{x_i\} \subset \Theta \subset \{x_1, \ldots, x_k\}$. This shows that

$$D_{\Theta_1}\mathbf{1}_{A^c(\omega)}(x_1) \cdots D_{\Theta_k}\mathbf{1}_{A^c(\omega)}(x_k) = 0 \tag{4.3}$$

provided that $\Theta_1 \cup \cdots \cup \Theta_k \neq \emptyset$. If $\Theta_1 \cup \cdots \cup \Theta_k = \emptyset$, then $\Theta_0 = \{x_1, \ldots, x_k\}$ and we can assume again that $\{x_1, \ldots, x_k\} \subset A^c(\omega)$, since otherwise we would have

$$D_{\Theta_0}F(\omega)D_{\Theta_1}\mathbf{1}_{A^c(\omega)}(x_1) \cdots D_{\Theta_k}\mathbf{1}_{A^c(\omega)}(x_k) = (D_{\Theta_0}F(\omega))\mathbf{1}_{A^c(\omega)}(x_1) \cdots \mathbf{1}_{A^c(\omega)}(x_k) = 0.$$

Under the condition $\{x_1, \ldots, x_k\} \subset A^c(\omega)$ we have $D_{x_i}F(\omega\cup\eta) = 0$ for all $i = 1, \ldots, k$ by Lemma 3.2 since $A^c(\omega \cup \eta) \supset A^c(\omega)$ for any $\eta \subset \{x_1, \ldots, x_k\}$, and this shows that $D_{\Theta_0}F(\omega) = 0$ due to the relation

$$D_\Theta F(\omega) = D_{\Theta\setminus\{x_i\}}D_{x_i}F(\omega) = \sum_{\eta\subset\Theta\setminus\{x_i\}} (-1)^{|\Theta|-|\eta|} D_{x_i}F(\omega \cup \eta) = 0,$$

where the above summation is taken over all (possibly empty) subsets η of $\Theta \setminus \{x_i\}$.
(ii) $\{x_1, \ldots, x_k\} \cap A^c(\omega) = \emptyset$. Next, if $\{x_1, \ldots, x_k\} \subset A(\omega)$ then it follows from Lemma 4.3 below that there exists $x_e \in \{x_1, \ldots, x_k\}$ such that $x_e \in A(\omega\cup\{x_1, \ldots, x_k\})$. Hence since $A(\omega)$ is non-increasing we have

$$\mathbf{1}_{A(\omega\cup\eta)}(x_e) = \mathbf{1}_{A(\omega)}(x_e) = 1$$

for all $\eta \subset \{x_1, \ldots, x_k\}$, and

$$D_\Theta \mathbf{1}_{A(\omega)}(x_e) = 0,$$

for all non-empty $\Theta \subset \{x_1, \ldots, x_k\}$, by the relation

$$\begin{aligned} D_\Theta \mathbf{1}_{A(\omega)}(x_e) &= \sum_{\eta\subset\Theta} (-1)^{|\Theta|+1-|\eta|} \mathbf{1}_{A(\omega\cup\eta)}(x_e) \\ &= \mathbf{1}_{A(\omega)}(x_e) \sum_{\eta\subset\Theta} (-1)^{|\Theta|+1-|\eta|} \end{aligned}$$

$$\begin{aligned} &= \mathbf{1}_{A(\omega)}(x_e)(1-1)^{|\Theta|+1} \\ &= 0, \end{aligned} \tag{4.4}$$

where the summation above is taken over all (possibly empty) subsets η of Θ. As a consequence, a factor in the product

$$D_{\Theta_1}\mathbf{1}_{A^c(\omega)}(x_1)\cdots D_{\Theta_k}\mathbf{1}_{A^c(\omega)}(x_k) = \prod_{l=1}^{k} D_{\Theta_l}\mathbf{1}_{A^c(\omega)}(x_l)$$

has to vanish when $\Theta_1 \cup \cdots \cup \Theta_k \neq \emptyset$. In case $\Theta_1 \cup \cdots \cup \Theta_k = \emptyset$ we can show as in (i) above that $D_{\Theta_0}F(\omega) = 0$, which concludes the proof. $\square$

The next lemma has been used in the proof of Proposition 4.2.

Lemma 4.3 *Let $A(\omega)$ be a stable and non-increasing stopping set. For any $\omega \in \Omega^X$, and $x_1, \ldots, x_k \in A(\omega)$, there exists $i \in \{1, \ldots, k\}$ such that $x_i \in A(\omega \cup \{x_1, \ldots, x_k\})$.*

Proof. Assume that $\{x_1, \ldots, x_k\} \subset A^c(\omega \cup \{x_1, \ldots, x_k\})$. We will show that

$$A^c(\omega \cup \{x_1, \ldots, x_k\}) = A^c\left(\omega \cup \cup_{i=j}^{k}\{x_i\}\right), \tag{4.5}$$

by induction on $j = 1, \ldots, k+1$, with the convention $\cup_{i=k+1}^{k}\{x_i\} = \emptyset$. This leads to $A^c(\omega \cup \{x_1, \ldots, x_k\}) = A^c(\omega)$ for $j = k+1$, and to $x_j \in A^c(\omega)$, $j = 1, \ldots, k$, which is a contradiction since we assumed that $\{x_1, \ldots, x_k\} \subset A(\omega)$.

Relation (4.5) clearly holds for $j = 1$, and we suppose that it holds for some $j \in \{1, \ldots, k\}$. By assumption we have $x_j \in A^c(\omega \cup \{x_1, \ldots, x_k\})$, which implies

$$x_j \in A^c(\omega \cup \{x_1, \ldots, x_k\}) = A^c(\omega \cup \cup_{i=j}^{k}\{x_i\}),$$

hence

$$x_j \in A^c(\omega \cup \cup_{i=j+1}^{k}\{x_i\})$$

by the stability condition (3.3). Consequently, by (3.4) or Lemma 3.2 we have

$$A^c(\omega \cup \{x_{j+1}, \ldots, x_k\}) = A^c(\omega \cup \cup_{i=j}^{k}\{x_i\})$$

since $A(\omega)$ is a stable and non-increasing stopping set. $\square$

5 Laplace transforms of stopping sets

In the following consequence of Propositions 4.1 and 4.2 we start by recovering the conditional moment generating function of $\sum_{x\in\omega} g(x)\mathbf{1}_{A^c(\omega)}(x)$ given $\mathcal{F}_A$, for g in the space $\mathcal{C}_c(X)$ of continuous functions with compact support in X. Given that this moment generating function characterizes the point process distribution, we recover the intuitive fact that given $\mathcal{F}_A$, the restriction of ω to $A^c(\omega)$ is a Poisson point process with intensity $\mathbf{1}_{A^c(\omega)}(x)\sigma(dx)$ when $A(\omega)$ is a stopping set.

Proposition 5.1 *For any* stable *and* non-increasing *stopping set $A(\omega)$ we have*

$$E\left[\exp\left(\sum_{x\in\omega\cap A^c(\omega)} g(x)\right)\Big|\mathcal{F}_A\right] = E\left[\exp\left(\int_{A^c(\omega)}(e^{g(x)}-1)\sigma(dx)\right)\Big|\mathcal{F}_A\right],$$

for all non-negative $g \in \mathcal{C}_c(X)$.

Proof. Taking $f(x) = e^{g(x)} - 1$, $x \in X$, by Propositions 4.1 and 4.2 we have

$$E\left[F(\omega)\exp\left(-\int_{A_c(\omega)} f(x)\sigma(dx)\right)\prod_{x\in\omega\cap A^c(\omega)}(1+f(x))\right] = E[F(\omega)] \qquad (5.1)$$

for any $\mathcal{F}_A$-measurable bounded random variable $F(\omega)$. Next, letting

$$F = Ge^{\int_{A^c(\omega)} f(x)\sigma(dx)} = Ge^{\int_X f(x)\sigma(dx) - \int_{A(\omega)} f(x)\sigma(dx)},$$

where G is a $\mathcal{F}_A$-measurable bounded random variable, we get

$$E\left[G\prod_{x\in\omega\cap A^c(\omega)}(1+f(x))\right] = E\left[G\exp\left(-\int_{A^c(\omega)} f(x)\sigma(dx)\right)\right].$$

□

As a consequence of Propositions 4.1 and 4.2 with $\phi := z\mathbf{1}_{A^c(\omega)}$, $z > 0$, we also obtain the following corollary.

Corollary 5.2 *Consider $A(\omega)$ a* stable *and* non-increasing *stopping set and $F(\omega)$ a non-negative $\mathcal{F}_A$-measurable random variable with*

$$E\left[F(\omega)e^{z\sigma(A^c)}(1+z)^{\omega(A^c)}\right] < \infty, \qquad (5.2)$$

for some $z > 0$. We have the Girsanov identity

$$E[F(\omega)] = E\left[F(\omega)e^{-z\sigma(A^c)}(1+z)^{\omega(A^c)}\right], \qquad z > 0. \qquad (5.3)$$

The gamma Laplace transform $E[e^{-zT_n}] = (1+z)^{-n}$ of the n-th Poisson jump time T_n can be recovered as a straightforward application of Girsanov identities to the stopping set $A = [0, T_n]$ with respect to the standard Poisson process filtration.

The following consequence of (5.3) on the conditional Laplace transform is consistent with the gamma-type results Theorem 2 of [9] and Theorem 2 of [15].

Corollary 5.3 *Let $A(\omega)$ be a* stable *and* non-increasing *stopping set. We have the conditional Laplace transform*

$$E[e^{-z\sigma(A)} \mid \omega(A) = n] = \frac{1}{(1+z)^n} \frac{\mathbb{P}_z(\{\omega(A) = n\})}{\mathbb{P}(\{\omega(A) = n\})}, \qquad z > 0, \quad n \in \mathbb{N}, \tag{5.4}$$

where $\mathbb{P}_z$ denotes the Poisson point process distribution with intensity $z\sigma(dx)$.

Proof. Taking $F = \mathbf{1}_{\{\omega(A)=n\}} e^{-z\sigma(A)} \in \mathcal{F}_A$, by (5.3) we get

$$\begin{aligned} E[e^{-z\sigma(A)} \mathbf{1}_{\{\omega(A)=n\}}] &= E[e^{-z(\sigma(X)-\sigma(A^c))} \mathbf{1}_{\{\omega(A)=n\}}] \\ &= e^{-z\sigma(X)} E[(1+z)^{\omega(A^c)} \mathbf{1}_{\{\omega(A)=n\}}] \\ &= \frac{1}{(1+z)^n} e^{-z\sigma(X)} E[(1+z)^{\omega(X)} \mathbf{1}_{\{\omega(A)=n\}}] \\ &= \frac{1}{(1+z)^n} \mathbb{P}_z(\{\omega(A) = n\}). \end{aligned}$$

□

When $\mathbb{P}_z(\{\omega(A) = n\})$ does not depend on $z > 0$ as assumed in [15], Corollary 5.3 recovers the gamma Laplace transform

$$E[e^{-z\sigma(A)} \mid \omega(A) = n] = \frac{1}{(1+z)^n}, \qquad z > 0,$$

conditionally to the number $n \geq 0$ of points in A.

Stopping sets in finite volume

In case $\sigma(X) < \infty$, taking $F = e^{-z\sigma(A)}$ in Corollary 5.2 we can derive the Laplace transform

$$E[e^{-z\sigma(A)}] = e^{-z\sigma(X)} E[(1+z)^{\omega(A^c)}], \quad z > 0,$$

of the random variable $\sigma(A(\omega))$ where $A(\omega)$ is a stable and non-increasing stopping set. More generally, Relation (5.3) in Corollary 5.2 shows that

$$E[f(\sigma(A^c))\mathbf{1}_{\{\omega(A)=n\}}] = E[f(\sigma(A^c))e^{-z\sigma(A^c)}\mathbf{1}_{\{\omega(A)=n\}}(1+z)^{\omega(A^c)}], \tag{5.5}$$

for all $z > 0$. In the next proposition we provide a more explicit form for (5.5) by denoting $\sigma_n^c(x_1, \ldots, x_n)$ the volume content of the complement $A^c(\omega)$ in X when $A \cap \omega = \{x_1, \ldots, x_n\}$ has $n \in \mathbb{N}$ points.

Proposition 5.4 *Assume that $\sigma(X) < \infty$ and consider a stable and non-increasing stopping set $A(\omega)$. We have*

$$E[f(\sigma(A^c(\omega)))\mathbf{1}_{\{\omega(A)=n\}}] = \frac{e^{-\sigma(X)}}{n!}\int_{X^n} e^{\sigma_n^c(x_1,\ldots,x_n)} f(\sigma_n^c(x_1, \ldots, x_n))\mu_n(dx_1, \ldots, dx_n), \tag{5.6}$$

$n \geq 1$, for f bounded and measurable on $\mathbb{R}$, where

$$\mu_n(dx_1, \ldots, dx_n) := \mathbf{1}_{\{A(\{x_1,\ldots,x_n\})\supset\{x_1,\ldots,x_n\}\}}\sigma(dx_1)\cdots\sigma(dx_n), \qquad n \geq 1.$$

Proof. By (5.5) we have, conditioning on the number k of points in $A^c(\omega)$,

$$\begin{aligned}
& E[f(\sigma(A^c))e^{-z\sigma(A^c)}\mathbf{1}_{\{\omega(A)=n\}}(1+z)^{\omega(A^c)}] \\
= \; & \frac{e^{-\sigma(X)}}{n!}\sum_{k=0}^{\infty}\frac{(1+z)^k}{k!} \\
& \int_{X^n}(\sigma_n^c(x_1, \ldots, x_n))^k e^{-z\sigma_n^c(x_1,\ldots,x_n)} f(\sigma_n^c(x_1, \ldots, x_n))\mu_n(dx_1, \ldots, dx_n) \\
= \; & \frac{e^{-\sigma(X)}}{n!}\sum_{k=0}^{\infty}\sum_{l=0}^{k}\frac{1}{(k-l)!}\frac{z^l}{l!} \\
& \int_{X^n}(\sigma_n^c(x_1, \ldots, x_n))^k e^{-z\sigma_n^c(x_1,\ldots,x_n)} f(\sigma_n^c(x_1, \ldots, x_n))\mu_n(dx_1, \ldots, dx_n) \\
= \; & \frac{e^{-\sigma(X)}}{n!}\sum_{m=0}^{\infty}\frac{1}{m!}\sum_{l=0}^{\infty}\frac{z^l}{l!} \\
& \int_{X^n}(\sigma_n^c(x_1, \ldots, x_n))^{m+l} e^{-z\sigma_n^c(x_1,\ldots,x_n)} f(\sigma_n^c(x_1, \ldots, x_n))\mu_n(dx_1, \ldots, dx_n) \\
= \; & \frac{e^{-\sigma(X)}}{n!}\sum_{m=0}^{\infty}\frac{1}{m!}\int_{X^n}(\sigma_n^c(x_1, \ldots, x_n))^m f(\sigma_n^c(x_1, \ldots, x_n))\mu_n(dx_1, \ldots, dx_n),
\end{aligned}$$

which yields (5.6) by (5.5). □

By (5.6) we have

$$\begin{aligned} E[e^{-z\sigma(A)}\mathbf{1}_{\{\omega(A)=n\}}] &= \frac{e^{-(1+z)\sigma(X)}}{n!}\int_{X^n} e^{(1+z)\sigma_n^c(x_1,\ldots,x_n)}\mu_n(dx_1,\ldots,dx_n) \\ &= \frac{1}{n!}\int_0^{\sigma(X)} e^{-(1+z)x}\nu_n(dx), \end{aligned} \tag{5.7}$$

where $\nu_n(dx)$ is the image measure on $[0,\sigma(X)]$ of $\mu_n(dx_1,\ldots,dx_n)$ by

$$(x_1,\ldots,x_n)\mapsto \sigma(X)-\sigma_n^c(x_1,\ldots,x_n),$$

with $\nu_1(dx)=\nu_2(dx)=\delta_{\sigma(X)}(dx)$. Hence

$$\begin{aligned} \mathbb{P}(\{\omega(A)=n\}) &= \frac{e^{-\sigma(X)}}{n!}\int_{X^n} e^{\sigma_n^c(x_1,\ldots,x_n)}\mu_n(dx_1,\ldots,dx_n) \\ &= \frac{1}{n!}\int_0^{\sigma(X)} e^{-x}\nu_n(dx), \end{aligned} \tag{5.8}$$

and the probability distribution of the random variable $\sigma(A^c)$ on $\{\omega(A)=n\}$ is given by

$$\frac{1}{n!}e^{-x}\nu_n(dx), \qquad n\geq 1.$$

Consequently we have the following corollary of Proposition 5.4 which, in comparison with Corollary 5.5, provides an expression for the ratio $\mathbb{P}_z(\{\omega(A)=n\})/\mathbb{P}(\{\omega(A)=n\})$, $z>0$.

Corollary 5.5 *Assume that* $\sigma(X)<\infty$. *For any stable and non-increasing stopping set* $A(\omega)$ *we have*

$$E[e^{-z\sigma(A)}\mid\omega(A)=n]=\frac{\int_0^{\sigma(X)} e^{-(1+z)x}\nu_n(dx)}{\int_0^{\sigma(X)} e^{-x}\nu_n(dx)}, \qquad z\in\mathbb{R}_+, \quad n\in\mathbb{N}. \tag{5.9}$$

Proof. By Relation (5.4) in Corollary 5.3 and Relations (5.7)-(5.8) we have

$$\begin{aligned} E[e^{-z\sigma(A)}\mid\omega(A)=n] &= \frac{1}{(1+z)^n}\frac{\mathbb{P}_z(\{\omega(A)=n\})}{\mathbb{P}(\{\omega(A)=n\})} \\ &= \frac{E[e^{-z\sigma(A)}\mathbf{1}_{\{\omega(A)=n\}}]}{\mathbb{P}(\{\omega(A)=n\})}=\frac{\int_0^{\sigma(X)} e^{-(1+z)x}\nu_n(dx)}{\int_0^{\sigma(X)} e^{-x}\nu_n(dx)}. \end{aligned}$$

□

The above analysis also yields the Laplace transform of $\sigma(A)$.

Corollary 5.6 *Assume that $\sigma(X) < \infty$. For any stable and non-increasing stopping set $A(\omega)$ we have the Laplace transform*

$$E[e^{-z\sigma(A)}] = e^{-\sigma(X)} + e^{-(1+z)\sigma(X)} \sum_{n=1}^{\infty} \frac{1}{n!} \int_{X^n} e^{(1+z)\sigma_n^c(x_1,\ldots,x_n)} \mu_n(dx_1,\ldots,dx_n), \quad (5.10)$$

$z > 0$.

Proof. By Proposition 5.4 we have

$$\begin{aligned} & E[f(\sigma(A^c))] = E[f(\sigma(A^c))e^{-z\sigma(A^c)}(1+z)^{\omega(A^c)}] \\ = \;& e^{-\sigma(X)} f(0) + e^{-\sigma(X)} \sum_{n=1}^{\infty} \frac{1}{n!} \int_{X^n} e^{\sigma_n^c(x_1,\ldots,x_n)} f(\sigma_n^c(x_1,\ldots,x_n)) \mu_n(dx_1,\ldots,dx_n). \end{aligned}$$

□

Next we consider some examples of applications for Corollaries 5.5 and 5.6.

Convex hulls of Poisson point processes

The closed complement $A(\omega)$ of the (open) convex hull $A^c(\omega)$ of a Poisson point process in a convex domain X of finite volume in $\mathbb{R}^d$ is a stable and non-increasing stopping set, cf. Section 3 and Figure 1 for an illustration.

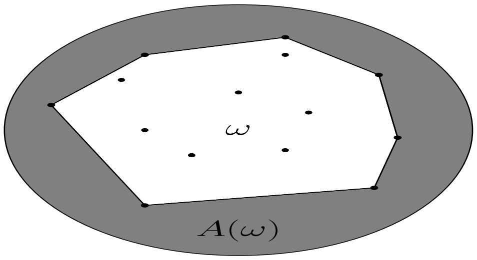

Figure 1: Convex hull of a Poisson point process.

When $n = 1, 2$ we clearly have $\mathbb{P}(\{\omega(A) = n\}) = \mathbb{P}(\{\omega(X) = n\})$ and $\sigma_1^c(x_1) = 0$ and $\sigma_2^c(x_1, x_2) = 0$. When $n = 3$, $\sigma_3^c(x_1, x_2, x_3)$ is given when $X \subset \mathbb{R}^2$ by Heron's formula

$$\begin{aligned} & \sigma_3(x_1, x_2, x_3) \\ = \;& \frac{1}{4}\sqrt{(|x_1 - x_2| + |x_1 - x_3| + |x_2 - x_3|)(-|x_1 - x_2| + |x_1 - x_3| + |x_2 - x_3|)} \end{aligned}$$

$$\times \sqrt{(|x_1 - x_2| - |x_1 - x_3| + |x_2 - x_3|)(|x_1 - x_2| + |x_1 - x_3| - |x_2 - x_3|)},$$

$x_1, x_2, x_3 \in X$, which can also be used to compute $\sigma_n^c(x_1, \ldots, x_n)$ for any convex polytope by triangulation. In higher dimensions, Heron's formula can be replaced with simplex volumes that can be computed with help of the Cayley-Menger determinants, and the Laplace transform of $\sigma(A)$ can be computed from (5.10). By Corollaries 5.5 and 5.6 and the expression of $\sigma_n^c(x_1, \ldots, x_n)$ we can compute the conditional and unconditional Laplace transforms $E[e^{-z\sigma(A)} \mid \omega(A) = n]$ and $E[e^{-z\sigma(A)}]$, $z > 0$, $n \in \mathbb{N}$.

Last Poisson jump time

When $X = [0, T]$, the last Poisson jump time T_{N_T} before time T, with $T_{N_T} = 0$ if $N_T = 0$, is not a stopping time for the forward filtration of the Poisson process, however T_{N_T} can be seen as the first jump time of the time reversed Poisson process $(N_{T-t})_{t \in [0,T]}$ and the process

$$u(t, \omega) = z \mathbf{1}_{[T_{N_T}, T]}(t) = z(1 - \mathbf{1}_{[0, T_{N_T})}(t)), \qquad t \in \mathbb{R}_+,$$

is backward predictable on $[0, T]$. In this case, by (5.1) with $A = [T_{N_T}, T]$ and $\sigma(dt) = dt$, $F = f(T_{N_T})$ and $\phi(t) = \mathbf{1}_{[0,T]}(t)$, we simply have $\mu_1(dt_1) = dt_1$ and $\mu_n(dt_1, \ldots, dt_n) = 0$, $n \geq 2$, hence by (5.10) we find

$$\begin{aligned} & E[f(\sigma(A^c)) e^{-z\sigma(A^c)} (1+z)^{\omega(A^c)}] \\ &= f(0) \mathbb{P}(\{N_T = 0\}) + E\left[f(T_{N_T}) e^{-zT_{N_T}} (1+z)^{N_T - 1} \mathbf{1}_{\{N_T \geq 1\}} \right] \\ &= f(0) e^{-\sigma(T)} + e^{-T} \int_X e^{\sigma_1^c(t)} f(\sigma_1^c(t)) \mu_1(dt) \\ &= e^{-T} f(0) + \int_0^T e^{-(T-t)} f(t) dt. \end{aligned}$$

Hence, taking $f(x) = e^{xz} \mathbf{1}_{(0,\infty)}(x)$, Relation (5.9) recovers the Laplace transform

$$E[e^{-z(T - T_{N_T})} \mid N_T \geq 1] = \frac{1}{1+z} \left(\frac{e^T - e^{-zT}}{e^T - 1} \right), \qquad z > -1,$$

of the truncated exponential distribution on $[0, T]$.

Annuli in finite volume

In the case where X is a ball centered at 0 in $\mathbb{R}^d$ we can consider the stable and non-increasing stopping set $A(\omega) = B_m(\omega) \cap X$ where $B_m(\omega)$ is the smallest closed ball centered at the origin and containing $m \geq 1$ process points in ω. Here we have

$$\sigma_n^c(x_1, \ldots, x_n) = \mathbf{1}_{\{n \geq m\}}(\sigma(X) - v_d(\max(|x_1|, \ldots, |x_n|))), \qquad n \in \mathbb{N},$$

where $v_d(r)$ is the volume of the d-dimensional ball with radius r. In case $d = 1$, $X = [0, T]$ and $A = [0, T_m \wedge T]$, we have

$$\sigma_n^c(x_1, \ldots, x_n) = \mathbf{1}_{\{n \geq m\}}(T - \max(x_1, \ldots, x_n)), \qquad n \in \mathbb{N}.$$

For $n = 1, \ldots, m$ we have $\mathbb{P}(\{\omega(A) = n\}) = \mathbb{P}(N_T = n)$ and $\sigma(X) = T$, while for $n \geq m + 1$ we have $\mathbb{P}(\{\omega(A) = n\}) = \mathbb{P}(T_n \leq T)$, and by Proposition 5.6,

$$\begin{aligned} & E[f(\sigma(A)) e^{-z\sigma(A^c)} (1+z)^{\omega(A^c)} \mathbf{1}_{\{\omega(A)=n\}}] \\ &= \frac{1}{n!} \int_{X^n} e^{\sigma(X) - \sigma_n^c(x_1, \ldots, x_n)} f(\sigma(X) - \sigma_n^c(x_1, \ldots, x_n)) dx_1 \cdots dx_n \\ &= \frac{1}{n!} \int_0^T \cdots \int_0^T e^{-\max(t_1, \ldots, t_n)} f(\max(t_1, \ldots, t_n)) dt_1 \cdots dt_n \\ &= \frac{1}{(n-1)!} \int_0^T e^{-t} f(t) t^{n-1} dt, \end{aligned}$$

and (5.9) becomes the Laplace transform

$$E[e^{-zT_m} \mid T_m < T] = \frac{1}{(1+z)^m} \frac{\mathbb{P}_z(N_T \geq m)}{\mathbb{P}(N_T \geq m)} = \frac{1 - e^{-(1+z)T} \sum_{k=0}^{m-1} ((1+z)T)^k / k!}{(1+z)^m (1 - e^{-T} \sum_{k=0}^{m-1} T^k / k!)}$$

of the truncated gamma distribution on $[0, T]$.

6 Appendix

In this section we give the proof of Proposition 4.1 which extends the argument of Proposition 2.1 in [10] to take into account a density $F(\omega)$. We will use the multiple Poisson stochastic integral

$$I_n(f_n)(\omega) := \int_{\Delta_n} f_n(x_1, \ldots, x_n)(\omega(dx_1) - \sigma(dx_1)) \cdots (\omega(dx_n) - \sigma(dx_n)),$$

where f_n is a symmetric function of n variables in the space $L^2_\sigma(X^n)$ of functions on X^n which are square-integrable with respect to $\sigma^{\otimes n}$, and

$$\Delta_n = \{(x_1, \ldots, x_n) \in X^n \ : \ x_i \neq x_j, \ \forall i \neq j\}.$$

The proof of Proposition 4.1 will be based on the following Lemmas 6.1 and 6.2. In the sequel we let $u^{\otimes n}(x_1, \ldots, x_n) := u(x_1) \cdots u(x_n)$, $x_1, \ldots, x_n \in X$, for $u \in L^2_\sigma(X)$.

Lemma 6.1 *Let $A(\omega)$ be a random set and $F(\omega)$ a bounded random variable. We have*

$$E\left[F(\omega)I_n(\mathbf{1}_{A(\omega)}^{\otimes n})\right] = E\left[\int_{X^n} D_{x_1}\cdots D_{x_n}\left(F(\omega)\prod_{p=1}^n \mathbf{1}_{A(\omega)}(x_p)\right)\sigma(dx_1)\cdots\sigma(dx_n)\right].$$

Proof. For all (possibly random) disjoint subsets $A_1(\omega),\ldots,A_n(\omega)$ of X with finite measure, denoting by $\mathbf{1}_{A_1^{k_1}(\omega)}\circ\cdots\circ\mathbf{1}_{A_n^{k_n}(\omega)}$ the symmetrization in $k_1+\cdots+k_n$ variables of the function $\mathbf{1}_{A_1^{k_1}(\omega)}\otimes\cdots\otimes\mathbf{1}_{A_n^{k_n}(\omega)}$, we have the relation

$$I_N(\mathbf{1}_{A_1^{k_1}(\omega)}\circ\cdots\circ\mathbf{1}_{A_n^{k_n}(\omega)}) = \prod_{i=1}^n C_{k_i}(\omega(A_i),\sigma(A_i)) \tag{6.1}$$

between the multiple Poisson integrals and the Charlier polynomial defined as

$$C_n(x,\lambda) = \sum_{k=0}^n x^k \sum_{l=0}^n \binom{n}{l}(-\lambda)^{n-l}s(l,k), \qquad x,\lambda\in\mathbb{R}.$$

cf. § 4.3.3 of [13], where

$$s(k,l) = \frac{1}{l!}\sum_{i=0}^l (-1)^i\binom{l}{i}(l-i)^k$$

is the Stirling number of the first kind, cf. page 824 of [1], i.e. $(-1)^{k-l}s(k,l)$ is the number of permutations of k elements which contain exactly l permutation cycles, $n\in\mathbb{N}$. By the moment identity

$$E\left[F(\omega)\left(\omega(A)\right)^k\right] = E\left[\int_{X^j}\varepsilon_{\mathfrak{s}_j}^+\left(F(\omega)\mathbf{1}_{A(\omega)}(x_1)\cdots\mathbf{1}_{A(\omega)}(x_j)\right)\sigma(dx_1)\cdots\sigma(dx_j)\right] \tag{6.2}$$

cf. Proposition 3.1 of [11] or Theorem 1 of [12], where $\varepsilon_{\mathfrak{s}_k}^+$ is the addition operator defined on any random variable $F:\Omega^X\longrightarrow\mathbb{R}$ by

$$\varepsilon_{\mathfrak{s}_k}^+F(\omega) = F(\omega\cup\{s_1,\ldots,s_k\}), \qquad \omega\in\Omega^X, \quad s_1,\ldots,s_k\in X,$$

and using the Stirling inversion formula

$$\sum_{k=l}^n S(n,k)s(k,l) = \sum_{k=0}^n S(n,k)s(k,l) = \mathbf{1}_{\{n=l\}}, \qquad n,l\geq 0, \tag{6.3}$$

cf. e.g. page 825 of [1], we have

$$E\left[F(\omega)I_n(\mathbf{1}_{A(\omega)}^{\otimes n})\right] = E\left[F(\omega)C_n(\omega(A),\sigma(A))\right]$$

$$
\begin{aligned}
&= \sum_{l=0}^{n}\sum_{k=0}^{n} E\left[F(\omega)\,(\omega(A))^k \binom{n}{l}(-\sigma(A))^{n-l}s(l,k)\right]\\
&= \sum_{l=0}^{n}\binom{n}{l}s(l,k)\sum_{k=0}^{n}\\
&\quad \times\sum_{j=0}^{k}S(k,j)E\left[\int_{X^j}\varepsilon^+_{\mathfrak{s}_j}\left(F(\omega)(-\sigma(A))^{n-l}\mathbf{1}_{A(\omega)}(x_1)\cdots\mathbf{1}_{A(\omega)}(x_j)\right)\sigma(dx_1)\cdots\sigma(dx_j)\right]\\
&= \sum_{l=0}^{n}(-1)^{n-l}\binom{n}{l}\sum_{j=0}^{n}\sum_{k=0}^{n}s(l,k)S(k,j)\\
&\quad \times E\left[\int_{X^{n-l+j}}\varepsilon^+_{\mathfrak{s}_j}\left(F(\omega)\mathbf{1}_{A(\omega)}(x_1)\cdots\mathbf{1}_{A(\omega)}(x_{n-l+j})\right)\sigma(dx_1)\cdots\sigma(dx_{n-l+j})\right]\\
&= \sum_{l=0}^{n}(-1)^{n-l}\binom{n}{l}E\left[\int_{X^n}\varepsilon^+_{\mathfrak{s}_l}(F(\omega)\mathbf{1}_{A(\omega)}(x_1)\cdots\mathbf{1}_{A(\omega)}(x_n))\sigma(dx_1)\cdots\sigma(dx_n)\right]\\
&= E\left[\int_{X^n}D_{x_1}\cdots D_{x_n}\left(F(\omega)\prod_{p=1}^{n}\mathbf{1}_{A(\omega)}(x_p)\right)\sigma(dx_1)\cdots\sigma(dx_n)\right].
\end{aligned}
$$

□

The next lemma is in the proof of Proposition 4.1.

Lemma 6.2 *Assume that* $\phi : \Omega^X \times X \to \mathbb{R}_+$ *is a non-negative process and* $F(\omega)$ *is a non-negative random variable satisfying Condition* (4.1)*. Then for all bounded non-negative random processes* $\phi : X \times \Omega^X \to \mathbb{R}_+$ *with compact support we have*

$$
E\left[F(\omega)I_n(\phi^{\otimes n})\right] = 0.
$$

Proof. (*i*) We start with a random set $A(\omega)$ and a non-negative random variable $F(\omega)$ that satisfy the condition

$$
D_{\Theta_0}F(\omega)D_{\Theta_1}\mathbf{1}_{A(\omega)}(x_1)\cdots D_{\Theta_k}\mathbf{1}_{A(\omega)}(x_k) = 0, \qquad \sigma^{\otimes k}(dx_1,\ldots,dx_k)-a.e., \quad \omega\in\Omega^X,
$$

for all $k \geq 1$, whenever $\Theta_0 \cup \Theta_1 \cup \cdots \cup \Theta_k = \{x_1,\ldots,x_k\}$, $\omega \in \Omega^X$. By Lemma 6.1 we have

$$
E\left[F(\omega)I_n(\mathbf{1}_{A(\omega)}^{\otimes n})\right] = E\left[\int_{X^n}D_{s_1}\cdots D_{s_n}\left(F(\omega)\prod_{p=1}^{n}\mathbf{1}_{A(\omega)}(s_p)\right)\sigma(ds_1)\cdots\sigma(ds_n)\right]. \tag{6.4}
$$

Next we have

$$
D_{x_1}\cdots D_{x_k}(F(\omega)\mathbf{1}_{A(\omega)}(x_1)\cdots\mathbf{1}_{A(\omega)}(x_k))
$$

$$= \sum_{\Theta_1 \cup \cdots \cup \Theta_k = \{1,\ldots,k\}} D_{\Theta_1}(F(\omega)\mathbf{1}_{A(\omega)}(x_1)) D_{\Theta_2}\mathbf{1}_{A(\omega)}(x_2) \cdots D_{\Theta_k}\mathbf{1}_{A(\omega)}(x_k),$$

where the above sum runs over all (possibly empty) subsets $\Theta_1, \ldots, \Theta_k$ of $\{1, \ldots, k\}$ such that $\Theta_1 \cup \cdots \cup \Theta_k = \{1, \ldots, k\}$. In addition,

$$D_{\Theta_1}(F(\omega)\mathbf{1}_{A(\omega)}(x_1)) = \sum_{\eta_1 \cup \eta_2 = \Theta_1} D_{\eta_1}F(\omega) D_{\eta_2}\mathbf{1}_{A(\omega)}(x_k),$$

where the sum runs over all sets η_1, η_2 such that $\eta_1 \cup \eta_2 = \Theta_1$, hence

$$\begin{aligned} & D_{\Theta_1}(F(\omega)\mathbf{1}_{A(\omega)}(x_1)) \cdots D_{\Theta_k}\mathbf{1}_{A(\omega)}(x_k) \\ &= \sum_{\eta_1 \cup \eta_2 = \Theta_1} D_{\eta_1}F(\omega) D_{\eta_2}\mathbf{1}_{A(\omega)}(x_1) D_{\Theta_2}\mathbf{1}_{A(\omega)}(x_2) \cdots D_{\Theta_k}\mathbf{1}_{A(\omega)}(x_k) \\ &= 0, \end{aligned}$$

$\sigma^{\otimes k}(dx_1, \ldots, dx_k) - a.e.$, $\omega \in \Omega^X$, for all $k \geq 1$, which yields

$$E\left[F(\omega) I_n(\mathbf{1}_{A(\omega)}^{\otimes n})\right] = 0 \tag{6.5}$$

by (6.4).

(ii) Assuming without loss of generality that ϕ takes values in $[0, 1]$ we consider the step process approximation

$$0 \leq \phi_m(\omega, t) := \sum_{k=0}^{2^m-1} \frac{k}{2^m} \mathbf{1}_{\{k/2^m \leq \phi(\omega,t) < (k+1)/2^m\}} = \sum_{k=1}^{2^m} \frac{k}{2^m} \mathbf{1}_{B_k(\omega)}(t) \leq \phi(\omega, t),$$

$t \in X$, $m \geq 1$, where

$$B_k(\omega) = \{t \ : \ k/2^m \leq \phi(\omega, t) < (k+1)/2^m\}, \qquad k = 0, 1, \ldots, 2^m - 1.$$

By the polarization identity

$$h_1 \circ \cdots \circ h_n = \frac{1}{n!} \sum_{k=1}^{k=n} (-1)^{n-k} \sum_{l_1 < \cdots < l_k} (h_{l_1} + \cdots + h_{l_k})^{\circ n}$$

we can extend (6.5) to $\phi_m^{\otimes n}$ as $E\left[F(\omega) I_n(\phi_m^{\otimes n}(\omega, \cdot))\right] = 0$ for all $m \geq 1$, and the extension to the general case follows by dominated convergence as m goes to infinity.

$\square$

Proof of Proposition 4.1. As in the proof of Proposition 2.1 of [10] we apply Lemma 6.2 to a step function approximation $\phi_m(\omega, t)$ of $\phi(\omega, t)$ and deduce by Fubini's theorem that

$$\begin{aligned} & E\left[F(\omega)e^{-\int_K \phi_m(\omega,x)\sigma(dx)} \prod_{x\in K\cap\omega}(1+\phi_m(\omega,x))\right] \\ &\quad = \; E[F(\omega)] + E\left[F(\omega)\sum_{n=1}^{\infty}\frac{1}{n!}I_n(\mathbf{1}_{K^n}(\cdot)\phi_m^{\otimes n}(\omega,\cdot))\right] \\ &\quad = \; E[F(\omega)] + \sum_{n=1}^{\infty}\frac{1}{n!}E\left[F(\omega)I_n(\mathbf{1}_{K^n}(\cdot)\phi_m^{\otimes n}(\omega,\cdot))\right] = E[F(\omega)], \end{aligned}$$

and we complete the proof by the same steps as in [10] □

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