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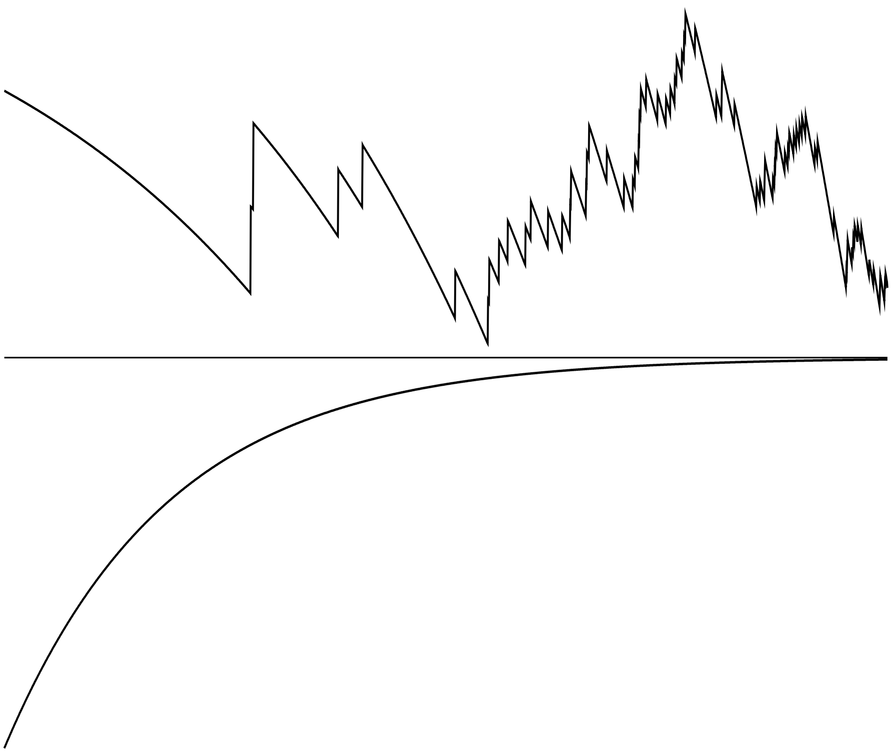
Stochastic Analysis in Discrete and Continuous Settings

With Normal Martingales



Preface

This monograph is an introduction to some aspects of stochastic analysis in the framework of normal martingales, in both discrete and continuous time. The text is mostly self-contained, except for Section 5.7 that requires some background in geometry, and should be accessible to graduate students and researchers having already received a basic training in probability. Prerequisites are mostly limited to a knowledge of measure theory and probability, namely σ-algebras, expectations, and conditional expectations. A short introduction to stochastic calculus for continuous and jump processes is given in Chapter 2 using normal martingales, whose predictable quadratic variation is the Lebesgue measure.

There already exists several books devoted to stochastic analysis for continuous diffusion processes on Gaussian and Wiener spaces, cf. e.g. [53], [65], [67], [76], [87], [88], [96], [132], [138], [147], [150], [151]. The particular feature of this text is to simultaneously consider continuous processes and jump processes in the unified framework of normal martingales.

These notes have grown from several versions of graduate courses given in the Master in Imaging and Computation at the University of La Rochelle and in the Master of Mathematics and Applications at the University of Poitiers, as well as from lectures presented at the universities of Ankara, Greifswald, Marne la Vallée, Tunis, and Wuhan, at the invitations of G. Wallet, M. Arnaudon, H. Körezlioğlu, U. Franz, A. Sulem, H. Ouerdiane, and L.M. Wu, respectively. The text has also benefited from constructive remarks from several colleagues and former students, including D. David, A. Joulin, Y.T. Ma, C. Pintoux, and A. Réveillac. I thank in particular J.C. Breton for numerous suggestions and corrections.

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Contents

Introduction

Stochastic analysis can be viewed as a branch of infinite-dimensional analysis that stems from a combined use of analytic and probabilistic tools, and is developed in interaction with stochastic processes. In recent decades it has turned into a powerful approach to the treatment of numerous theoretical and applied problems ranging from existence and regularity criteria for probability densities and solutions of partial differential equations by the Malliavin calculus, to functional and deviation inequalities, mathematical finance, and anticipative extensions of stochastic calculus.

The basic tools of stochastic analysis consist in a gradient and a divergence operator which are linked by an integration by parts formula. Such gradient operators can be defined by finite differences or by infinitesimal shifts of the paths of a given stochastic process. Whenever possible, the divergence operator is connected to the stochastic integral with respect to that same underlying process. In this way, deep connections can be established between the algebraic and geometric aspects of differentiation and integration by parts on the one hand, and their probabilistic counterpart on the other hand. Note that the term "stochastic analysis" is also used with somewhat different significations especially in engineering or applied probability; here we refer to stochastic analysis from a functional analytic point of view.

Let us turn to the contents of this monograph. Chapter 1 starts with an elementary exposition in a discrete setting in which most of the basic tools of stochastic analysis can be introduced. The simple setting of the discrete case still captures many important properties of the continuous-time case and provides a simple model for its understanding. It also yields non trivial results such as concentration and deviation inequalities, and logarithmic Sobolev inequalities for Bernoulli measures, as well as hedging formulas for contingent claims in discrete time financial models. In addition, the results obtained in the discrete case are directly suitable for computer implementation. We start by introducing discrete time versions of the gradient and divergence operators, of chaos expansions, and of the predictable representation property. We write the discrete time structure equation satisfied by a sequence $(X_n)_{n\in\mathbb{N}}$ of independent Bernoulli random variables defined on the probability space $\Omega = \{-1,1\}^{\mathbb{N}}$, we construct the associated discrete multiple stochastic integrals and prove the chaos representation property for discrete time random walks with independent increments. A gradient operator D acting by finite differences is introduced in connection with the multiple stochastic integrals, and used to state a Clark predictable representation formula. The divergence operator δ, defined as the adjoint of D, turns out to be an extension of the discrete-time stochastic integral, and is used to express the generator of the Ornstein-Uhlenbeck process. The properties of the associated Ornstein-Uhlenbeck process and semi-group are investigated, with applications to covariance identities and deviation inequalities under Bernoulli measures. Covariance identities are stated both from the Clark representation formula and using Ornstein-Uhlenbeck semigroups. Logarithmic Sobolev inequalities are also derived in this framework, with additional applications to deviation inequalities. Finally we prove an Itô type change of variable formula in discrete time and apply it, along with the Clark formula, to option pricing and hedging in the Cox-Ross-Rubinstein discrete-time financial model.

In Chapter 2 we turn to the continuous time case and present an elementary account of continuous time normal martingales. This includes the construction of associated multiple stochastic integrals $I_n(f_n)$ of symmetric deterministic functions f_n of n variables with respect to a normal martingale, and the derivation of structure equations determined by a predictable process $(\phi_t)_{t\in\mathbb{R}_+}$. In case $(\phi_t)_{t\in\mathbb{R}_+}$ is a deterministic function, this family of martingales includes Brownian motion (when ϕ vanishes identically) and the compensated Poisson process (when ϕ is a deterministic constant), which will be considered separately. A basic construction of stochastic integrals and calculus is presented in the framework of normal martingales, with a proof of the Itô formula. In this chapter, the construction of Brownian motion is done via a series of Gaussian random variables and its pathwise properties will not be particularly discussed, as our focus is more on connections with functional analysis. Similarly, the notions of local martingales and semimartingales are not within the scope of this introduction.

Chapter 3 contains a presentation of the continuous time gradient and divergence in an abstract setting. We identify some minimal assumptions to be satisfied by these operators in order to connect them later on to stochastic integration with respect to a given normal martingale. The links between the Clark formula, the predictable representation property and the relation between Skorohod and Itô integrals, as well as covariance identities, are discussed at this level of generality. This general setting gives rise to applications such as the determination of the predictable representation of random variables, and a proof of logarithmic Sobolev inequalities for normal martingales. Generic examples of operators satisfying the hypotheses of Chapter 2 can be constructed by addition of a process with vanishing adapted projection to the gradient operator. Concrete examples of such gradient and divergence operators will be described in the sequel (Chapters 4, 5, 6, and 7), in particular in the Wiener and Poisson cases.

Chapter 4 introduces a first example of a pair of gradient and divergence operators satisfying the hypotheses of Chapter 3, based on the notion of multiple stochastic integral $I_n(f_n)$ of a symmetric function f_n on $\mathbb{R}_+^n$ with respect to a normal martingale. Here the gradient operator D is defined by lowering the degree of multiple stochastic integrals (i.e. as an annihilation operator), while its adjoint δ is defined by raising that degree (i.e. as a creation operator). We give particular attention to the class of normal martingales which can be used to expand any square-integrable random variable into a series of multiple stochastic integrals. This property, called the chaos representation property, is stronger than the predictable representation property and plays a key role in the representation of functionals as stochastic integrals. Note that here the words "chaos" and "chaotic" are not taken in the sense of dynamical systems theory and rather refer to the notion of chaos introduced by N. Wiener [152]. We also present an application to deviation and concentration inequalities in the case of deterministic structure equations. The family of normal martingales having the chaos representation property, includes Brownian motion and the compensated Poisson process, which will be dealt with separately cases in the following sections.

The general results developed in Chapter 3 are detailed in Chapter 5 in the particular case of Brownian motion on the Wiener space. Here the gradient operator has the derivation property and the multiple stochastic integrals can be expressed using Hermite polynomials, cf. Section 5.1. We state the expression of the Ornstein-Uhlenbeck semi-group and the associated covariance identities and Gaussian deviation inequalities obtained. A differential calculus is presented for time changes on Brownian motion, and more generally for random transformations on the Wiener space, with application to Brownian motion on Riemannian path space in Section 5.7.

In Chapter 6 we introduce the main tools of stochastic analysis under Poisson measures on the space of configurations of a metric space X. We review the connection between Poisson multiple stochastic integrals and Charlier polynomials, gradient and divergence operators, and the Ornstein-Uhlenbeck semi-group. In this setting the annihilation operator defined on multiple Poisson stochastic integrals is a difference operator that can be used to formulate the Clark predictable representation formula. It also turns out that the integration by parts formula can be used to characterize Poisson measure. We also derive some deviation and concentration results for random vectors and infinitely divisible random variables.

In Chapter 7 we study a class of local gradient operators on the Poisson space that can also be used to characterize the Poisson measure. Unlike the finite difference gradients considered in Chapter 6, these operators do satisfy the chain rule of derivation. In the case of the standard Poisson process on the real line, they provide another instance of an integration by parts setting that fits into the general framework of Chapter 3. In particular this operator can be used in a Clark predictable representation formula and it is closely connected to the stochastic integral with respect to the compensated Poisson process via its associated divergence operator. The chain rule of derivation, which is not satisfied by the difference operators considered in Chapter 6, turns out to be necessary in a number of application such as deviation inequalities, chaos expansions, or sensitivity analysis.

Chapter 8 is devoted to applications in mathematical finance. We use normal martingales to extend the classical Black-Scholes theory and to construct complete market models with jumps. The results of previous chapters are applied to the pricing and hedging of contingent claims in complete markets driven by normal martingales. Normal martingales play only a modest role in the modeling of financial markets. Nevertheless, in addition to Brownian and Poisson models, they provide examples of complete markets with jumps.

To close this introduction we turn to some informal remarks on the Clark formula and predictable representation in connection with classical tools of finite dimensional analysis. This simple example shows how analytic arguments and stochastic calculus can be used in stochastic analysis. The classical "fundamental theorem of calculus" can be written using entire series as

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \alpha_n x^n \\ &= \alpha_0 + \sum_{n=1}^{\infty} n\alpha_n \int_0^x y^{n-1} dy \\ &= f(0) + \int_0^x f'(y) dy, \end{aligned}$$

and commonly relies on the identity

$$x^n = n \int_0^x y^{n-1} dy, \qquad x \in \mathbb{R}_+. \tag{0.1}$$

Replacing the monomial x^n with the Hermite polynomial $H_n(x,t)$ with parameter $t > 0$, we do obtain an analog of (0.1) as

$$\frac{\partial}{\partial x} H_n(x,t) = n H_{n-1}(x,t),$$

however the argument contained in (0.1) is no longer valid since $H_{2n}(0,t) \neq 0$, $n \geq 1$. The question of whether there exists a simple analog of (0.1) for the Hermite polynomials can be positively answered using stochastic calculus with respect to Brownian motion $(B_t)_{t\in\mathbb{R}_+}$ which provides a way to write $H_n(B_t,t)$ as a stochastic integral of $nH_{n-1}(B_t,t)$, i.e.

$$H_n(B_t,t) = n \int_0^t H_{n-1}(B_s,s) dB_s. \tag{0.2}$$

Consequently $H_n(B_t,t)$ can be written as an n-fold iterated stochastic integral with respect to Brownian motion $(B_t)_{t\in\mathbb{R}_+}$, which is denoted by $I_n(1_{[0,t]^n})$.

This allows us to write down the following expansion of a function f depending on the parameter t into a series of Hermite polynomials, as follows:

$$\begin{aligned} f(B_t, t) &= \sum_{n=0}^{\infty} \beta_n H_n(B_t, t) \\ &= \beta_0 + \sum_{n=1}^{\infty} n\beta_n \int_0^t H_{n-1}(B_s, s) dB_s, \end{aligned}$$

$\beta_n \in \mathbb{R}_+$, $n \in \mathbb{N}$. Using the relation $H_n'(x,t) = nH_{n-1}(x,t)$, this series can be written as

$$f(B_t, t) = \mathbb{E}[f(B_t, t)] + \int_0^t \mathbb{E}\left[\frac{\partial f}{\partial x}(B_t, t) \Big| \mathcal{F}_s\right] dB_s, \tag{0.3}$$

since, by the martingale property of (0.2), $H_{n-1}(B_s, s)$ coincides with the conditional expectation $\mathbb{E}[H_{n-1}(B_t, t) \mid \mathcal{F}_s]$, $s < t$, where $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ is the filtration generated by $(B_t)_{t\in\mathbb{R}_+}$.

It turns out that the above argument can be extended to general functionals of the Brownian path $(B_t)_{t\in\mathbb{R}_+}$ to prove that the square integrable functionals of $(B_t)_{t\in\mathbb{R}_+}$ have the following expansion in series of multiple stochastic integrals $I_n(f_n)$ of symmetric functions $f_n \in L^2(\mathbb{R}_+^n)$:

$$\begin{aligned} F &= \mathbb{E}[F] + \sum_{n=1}^{\infty} I_n(f_n) \\ &= \mathbb{E}[F] + \sum_{n=1}^{\infty} n \int_0^{\infty} I_{n-1}(f_n(*, t) 1_{\{* \le t\}}) dB_t. \end{aligned}$$

Using again stochastic calculus in a way similar to the above argument will show that this relation can be written under the form

$$F = \mathbb{E}[F] + \int_0^{\infty} \mathbb{E}[D_t F \mid \mathcal{F}_t] dB_t, \tag{0.4}$$

where D is a gradient acting on Brownian functionals and $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ is the filtration generated by $(B_t)_{t\in\mathbb{R}_+}$. Relation (0.4) is a generalization of (0.3) to arbitrary dimensions which does not require the use of Hermite polynomials, and can be adapted to other processes such as the compensated Poisson process, and more generally to the larger class of normal martingales.

Classical Taylor expansions for functions of one or several variables can also be interpreted in a stochastic analysis framework, in relation to the explicit determination of chaos expansions of random functionals. Consider for instance the classical formula

$$a_n = \frac{\partial^n f}{\partial x^n}(x)_{|x=0}$$

for the coefficients in the entire series

$$f(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}.$$

In the general setting of normal martingales having the chaos representation property, one can similarly compute the function f_n in the development of

$$F = \sum_{n=0}^{\infty} \frac{1}{n!} I_n(f_n)$$

as

$$f_n(t_1, \ldots, t_n) = \mathbb{E}[D_{t_1} \cdots D_{t_n} F], \quad a.e. \ t_1, \ldots, t_n \in \mathbb{R}_+, \tag{0.5}$$

cf. [70], [142]. This identity holds in particular for Brownian motion and the compensated Poisson process. However, the probabilistic interpretation of $D_t F$ can be difficult to find except in the Wiener and Poisson cases, i.e. in the case of deterministic structure equations.

Our aim in the next chapters will be in particular to investigate to which extent these techniques remain valid in the general framework of normal martingales and other processes with jumps.

Chapter 1
The Discrete Time Case

In this chapter we introduce the tools of stochastic analysis in the simple framework of discrete time random walks. Our presentation relies on the use of finite difference gradient and divergence operators which are defined along with single and multiple stochastic integrals. The main applications of stochastic analysis to be considered in the following chapters, including functional inequalities and mathematical finance, are discussed in this elementary setting. Some technical difficulties involving measurability and integrability conditions, that are typical of the continuous-time case, are absent in the discrete time case.

1.1 Normal Martingales

Consider a sequence $(Y_k)_{k\in\mathbb{N}}$ of (not necessarily independent) random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\mathbb{N} = \{0, 1, 2, \ldots\}$ denotes the set of nonnegative integers. Let $(\mathcal{F}_n)_{n\geq -1}$ denote the filtration generated by $(Y_n)_{n\in\mathbb{N}}$, i.e.

$$\mathcal{F}_{-1} = \{\emptyset, \Omega\},$$

and

$$\mathcal{F}_n = \sigma(Y_0, \ldots, Y_n), \qquad n \geq 0.$$

Recall that a random variable F is said to be $\mathcal{F}_n$-measurable if it can be written as a function

$$F = f_n(Y_0, \ldots, Y_n)$$

of $Y_0, \ldots, Y_n$, where $f_n : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}$ is Borel measurable.

Assumption 1.1.1. *We make the following assumptions on the sequence* $(Y_n)_{n\in\mathbb{N}}$*:*

a) it is conditionally centered:

$$\mathbb{E}[Y_n \mid \mathcal{F}_{n-1}] = 0, \qquad n \geq 0, \tag{1.1.1}$$

b) its conditional quadratic variation satisfies:

$$\mathbb{E}\left[Y_n^2 \mid \mathcal{F}_{n-1}\right] = 1, \qquad n \geq 0.$$

Condition (1.1.1) implies that the process $(Y_0 + \cdots + Y_n)_{n\geq 0}$ is an $\mathcal{F}_n$-martingale, cf. Section 9.4 in the Appendix. More precisely, the sequence $(Y_n)_{n\in\mathbb{N}}$ and the process $(Y_0 + \cdots + Y_n)_{n\geq 0}$ can be viewed respectively as a (correlated) noise and as a normal martingale in discrete time.

1.2 Stochastic Integrals

In this section we construct the discrete stochastic integral of predictable square-summable processes with respect to a discrete-time normal martingale.

Definition 1.2.1. *Let $(u_k)_{k\in\mathbb{N}}$ be a uniformly bounded sequence of random variables with finite support in $\mathbb{N}$, i.e. there exists $N \geq 0$ such that $u_k = 0$ for all $k \geq N$. The stochastic integral $J(u)$ of $(u_n)_{n\in\mathbb{N}}$ is defined as*

$$J(u) = \sum_{k=0}^{\infty} u_k Y_k.$$

The next proposition states a version of the Itô isometry in discrete time. A sequence $(u_n)_{n\in\mathbb{N}}$ of random variables is said to be $\mathcal{F}_n$-predictable if u_n is $\mathcal{F}_{n-1}$-measurable for all $n \in \mathbb{N}$, in particular u_0 is constant in this case.

Proposition 1.2.2. *The stochastic integral operator $J(u)$ extends to square-integrable predictable processes $(u_n)_{n\in\mathbb{N}} \in L^2(\Omega \times \mathbb{N})$ via the (conditional) isometry formula*

$$\mathbb{E}\left[|J(\mathbf{1}_{[n,\infty)}u)|^2| \mid \mathcal{F}_{n-1}\right] = \mathbb{E}\left[\|\mathbf{1}_{[n,\infty)}u\|^2_{\ell^2(\mathbb{N})} \mid \mathcal{F}_{n-1}\right], \qquad n \in \mathbb{N}. \quad (1.2.1)$$

Proof. Let $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ be bounded predictable processes with finite support in $\mathbb{N}$. The product $u_k Y_k v_l$, $0 \leq k < l$, is $\mathcal{F}_{l-1}$-measurable, and $u_k Y_l v_l$ is $\mathcal{F}_{k-1}$-measurable, $0 \leq l < k$. Hence

$$\mathbb{E}\left[\sum_{k=n}^{\infty} u_k Y_k \sum_{l=n}^{\infty} v_l Y_l \middle| \mathcal{F}_{n-1}\right] = \mathbb{E}\left[\sum_{k,l=n}^{\infty} u_k Y_k v_l Y_l \middle| \mathcal{F}_{n-1}\right]$$

$$= \mathbb{E}\left[\sum_{k=n}^{\infty} u_k v_k Y_k^2 + \sum_{n\leq k<l} u_k Y_k v_l Y_l + \sum_{n\leq l<k} u_k Y_k v_l Y_l \middle| \mathcal{F}_{n-1}\right]$$

$$= \sum_{k=n}^{\infty} \mathbb{E}\left[\mathbb{E}\left[u_k v_k Y_k^2 \mid \mathcal{F}_{k-1}\right] \mid \mathcal{F}_{n-1}\right] + \sum_{n\leq k<l} \mathbb{E}\left[\mathbb{E}\left[u_k Y_k v_l Y_l \mid \mathcal{F}_{l-1}\right] \mid \mathcal{F}_{n-1}\right]$$

$$
\begin{aligned}
&+ \sum_{n\le l<k} \mathbb{E}\left[\mathbb{E}\left[u_k Y_k v_l Y_l \mid \mathcal{F}_{k-1}\right] \mid \mathcal{F}_{n-1}\right] \\
&= \sum_{k=0}^{\infty} \mathbb{E}\left[u_k v_k \mathbb{E}\left[Y_k^2 \mid \mathcal{F}_{k-1}\right] \mid \mathcal{F}_{n-1}\right] + 2 \sum_{n\le k<l} \mathbb{E}\left[u_k Y_k v_l \mathbb{E}\left[Y_l \mid \mathcal{F}_{l-1}\right] \mid \mathcal{F}_{n-1}\right] \\
&= \sum_{k=n}^{\infty} \mathbb{E}\left[u_k v_k \mid \mathcal{F}_{n-1}\right] \\
&= \mathbb{E}\left[\sum_{k=n}^{\infty} u_k v_k \middle| \mathcal{F}_{n-1}\right].
\end{aligned}
$$

This proves the isometry property (1.2.1) for J. The extension to $L^2(\Omega\times\mathbb{N})$ is proved using the following Cauchy sequence argument. Consider a sequence of bounded predictable processes with finite support converging to u in $L^2(\Omega\times\mathbb{N})$, for example the sequence $(u^n)_{n\in\mathbb{N}}$ defined as

$$
u^n = (u_k^n)_{k\in\mathbb{N}} = \left(u_k \mathbf{1}_{\{0\le k\le n\}} \mathbf{1}_{\{|u_k|\le n\}}\right)_{k\in\mathbb{N}}, \qquad n\in\mathbb{N}.
$$

Then the sequence $(J(u^n))_{n\in\mathbb{N}}$ is Cauchy and converges in $L^2(\Omega)$, hence we may define

$$
J(u) := \lim_{k\to\infty} J(u^k).
$$

From the isometry property (1.2.1) applied with $n=0$, the limit is clearly independent of the choice of the approximating sequence $(u^k)_{k\in\mathbb{N}}$ as using Fatou's lemma we have

$$
\begin{aligned}
\mathbb{E}\left[\left(\lim_{n\to\infty} J(u^n) - \lim_{n\to\infty} J(v^n)\right)^2\right] &= \mathbb{E}\left[\lim_{n\to\infty} (J(u^n)-J(v^n))^2\right] \\
&\le \liminf_{n\to\infty} \mathbb{E}\left[(J(u^n)-J(v^n))^2\right] \\
&= \liminf_{n\to\infty} \mathbb{E}\left[\|u^n - v^n\|_{\ell^2(\mathbb{N})}^2\right] \\
&= 0.
\end{aligned}
$$

□

Note that by polarization, (1.2.1) can also be written as

$$
\mathbb{E}\left[J(\mathbf{1}_{[n,\infty)}u)J(\mathbf{1}_{[n,\infty)}v)|\mathcal{F}_{n-1}\right] = \mathbb{E}\left[\langle \mathbf{1}_{[n,\infty)}u, \mathbf{1}_{[n,\infty)}v\rangle_{\ell^2(\mathbb{N})} \mid \mathcal{F}_{n-1}\right], \quad n\in\mathbb{N},
$$

and that for $n=0$ we get

$$
\mathbb{E}\left[J(u)J(v)\right] = \mathbb{E}\left[\langle u, v\rangle_{\ell^2(\mathbb{N})}\right], \tag{1.2.2}
$$

and

$$
\mathbb{E}\left[|J(u)|^2\right] = \mathbb{E}\left[\|u\|_{\ell^2(\mathbb{N})}^2\right], \tag{1.2.3}
$$

for all square-integrable predictable processes $u = (u_k)_{k\in\mathbb{N}}$ and $v = (v_k)_{k\in\mathbb{N}}$.

Proposition 1.2.3. *Let $(u_k)_{k\in\mathbb{N}} \in L^2(\Omega \times \mathbb{N})$ be a predictable square-integrable process. We have*

$$\mathbb{E}\left[J(u) \mid \mathcal{F}_k\right] = J(u\mathbf{1}_{[0,k]}), \quad k \in \mathbb{N}.$$

Proof. In case $(u_k)_{k\in\mathbb{N}}$ has finite support in $\mathbb{N}$ it suffices to note that

$$\begin{aligned}
\mathbb{E}\left[J(u) \mid \mathcal{F}_k\right] &= \mathbb{E}\left[\sum_{i=0}^{k} u_i Y_i \Big| \mathcal{F}_k\right] + \sum_{i=k+1}^{\infty} \mathbb{E}\left[u_i Y_i \mid \mathcal{F}_k\right] \\
&= \sum_{i=0}^{k} u_i Y_i + \sum_{i=k+1}^{\infty} \mathbb{E}\left[\mathbb{E}\left[u_i Y_i \mid \mathcal{F}_{i-1}\right] \mid \mathcal{F}_k\right] \\
&= \sum_{i=0}^{k} u_i Y_i + \sum_{i=k+1}^{\infty} \mathbb{E}\left[u_i \mathbb{E}\left[Y_i \mid \mathcal{F}_{i-1}\right] \mid \mathcal{F}_k\right] \\
&= \sum_{i=0}^{k} u_i Y_i \\
&= J\big(u\mathbf{1}_{[0,k]}\big).
\end{aligned}$$

The formula extends to the general case by linearity and density, using the continuity of the conditional expectation on L^2 and the sequence $(u^n)_{n\in\mathbb{N}}$ defined as $u^n = (u^n_k)_{k\in\mathbb{N}} = \big(u_k \mathbf{1}_{\{0\le k\le n\}}\big)_{k\in\mathbb{N}}$, $n \in \mathbb{N}$, i.e.

$$\begin{aligned}
\mathbb{E}\left[\Big(J\big(u\mathbf{1}_{[0,k]}\big) - \mathbb{E}\left[J(u) \mid \mathcal{F}_k\right]\Big)^2\right] &= \lim_{n\to\infty} \mathbb{E}\left[\Big(J\big(u^n\mathbf{1}_{[0,k]}\big) - \mathbb{E}\left[J(u) \mid \mathcal{F}_k\right]\Big)^2\right] \\
&= \lim_{n\to\infty} \mathbb{E}\left[\left(\mathbb{E}\left[J(u^n) - J(u) \mid \mathcal{F}_k\right]\right)^2\right] \\
&\le \lim_{n\to\infty} \mathbb{E}\left[\mathbb{E}\left[\left(J(u^n) - J(u)\right)^2 \Big| \mathcal{F}_k\right]\right] \\
&= \lim_{n\to\infty} \mathbb{E}\left[\left(J(u^n) - J(u)\right)^2\right] \\
&= 0,
\end{aligned}$$

by (1.2.3). □

Corollary 1.2.4. *The indefinite stochastic integral $\big(J\big(u\mathbf{1}_{[0,k]}\big)\big)_{k\in\mathbb{N}}$ is a discrete time martingale with respect to $(\mathcal{F}_n)_{n\ge -1}$.*

Proof. We have

$$\begin{aligned}
\mathbb{E}\left[J\big(u\mathbf{1}_{[0,k+1]}\big) \mid \mathcal{F}_k\right] &= \mathbb{E}\left[\mathbb{E}\left[J\big(u\mathbf{1}_{[0,k+1]}\big) \mid \mathcal{F}_{k+1} \mid \mathcal{F}_k\right]\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[J(u) \mid \mathcal{F}_{k+1} \mid \mathcal{F}_k\right]\right] \\
&= \mathbb{E}\left[J(u) \mid \mathcal{F}_k\right]
\end{aligned}$$

$$= J\big(u\mathbf{1}_{[0,k]}\big).$$

□

1.3 Multiple Stochastic Integrals

The role of multiple stochastic integrals in the orthogonal expansion of a random variable is similar to that of polynomials in the series expansion of a function of a real variable. In some situations, multiple stochastic integrals can be expressed using polynomials, such as in the symmetric case $p_n = q_n = 1/2$, $n \in \mathbb{N}$, in which the Krawtchouk polynomials are used, see Relation (1.5.2) below.

Definition 1.3.1. *Let $\ell^2(\mathbb{N})^{\circ n}$ denote the subspace of $\ell^2(\mathbb{N})^{\otimes n} = \ell^2(\mathbb{N}^n)$ made of functions f_n that are symmetric in n variables, i.e. such that for every permutation σ of $\{1, \ldots, n\}$,*

$$f_n(k_{\sigma(1)}, \ldots, k_{\sigma(n)}) = f_n(k_1, \ldots, k_n), \quad k_1, \ldots, k_n \in \mathbb{N}.$$

Given $f_1 \in \ell^2(\mathbb{N})$ we let

$$J_1(f_1) = J(f_1) = \sum_{k=0}^{\infty} f_1(k) Y_k.$$

As a convention we identify $\ell^2(\mathbb{N}^0)$ to $\mathbb{R}$ and let $J_0(f_0) = f_0$, $f_0 \in \mathbb{R}$. Let

$$\Delta_n = \{(k_1, \ldots, k_n) \in \mathbb{N}^n \ : \ k_i \neq k_j, \ 1 \leq i < j \leq n\}, \qquad n \geq 1.$$

The following proposition gives the definition of multiple stochastic integrals by iterated stochastic integration of predictable processes in the sense of Proposition 1.2.2.

Proposition 1.3.2. *The multiple stochastic integral $J_n(f_n)$ of $f_n \in \ell^2(\mathbb{N})^{\circ n}$, $n \geq 1$, is defined as*

$$J_n(f_n) = \sum_{(i_1, \ldots, i_n) \in \Delta_n} f_n(i_1, \ldots, i_n) Y_{i_1} \cdots Y_{i_n}.$$

It satisfies the recurrence relation

$$J_n(f_n) = n \sum_{k=1}^{\infty} Y_k J_{n-1}(f_n(*, k) \mathbf{1}_{[0,k-1]^{n-1}}(*)) \tag{1.3.1}$$

and the isometry formula

$$\mathbb{E}\left[J_n(f_n)J_m(g_m)\right] = \begin{cases} n!\langle \mathbf{1}_{\Delta_n} f_n, g_m\rangle_{\ell^2(\mathbb{N})^{\otimes n}} & \textit{if } n = m, \\ 0 & \textit{if } n \neq m. \end{cases} \tag{1.3.2}$$

Proof. Note that we have

$$\begin{aligned} J_n(f_n) &= n! \sum_{0\le i_1<\cdots<i_n} f_n(i_1,\ldots,i_n)Y_{i_1}\cdots Y_{i_n} \\ &= n! \sum_{i_n=0}^{\infty} \sum_{0\le i_{n-1}<i_n} \cdots \sum_{0\le i_1<i_2} f_n(i_1,\ldots,i_n)Y_{i_1}\cdots Y_{i_n}. \end{aligned} \tag{1.3.3}$$

Note that since $0 \le i_1 < i_2 < \cdots < i_n$ and $0 \le j_1 < j_2 < \cdots < j_n$ we have

$$\mathbb{E}\left[Y_{i_1}\cdots Y_{i_n}Y_{j_1}\cdots Y_{j_n}\right] = \mathbf{1}_{\{i_1=j_1,\ldots,i_n=j_n\}}.$$

Hence

$$\begin{aligned} &\mathbb{E}\left[J_n(f_n)J_n(g_n)\right] \\ &= (n!)^2\mathbb{E}\left[\sum_{0\le i_1<\cdots<i_n} f_n(i_1,\ldots,i_n)Y_{i_1}\cdots Y_{i_n} \sum_{0\le j_1<\cdots<j_n} g_n(j_1,\ldots,j_n)Y_{j_1}\cdots Y_{j_n}\right] \\ &= (n!)^2 \sum_{0\le i_1<\cdots<i_n,\ 0\le j_1<\cdots<j_n} f_n(i_1,\ldots,i_n)g_n(j_1,\ldots,j_n)\mathbb{E}\left[Y_{i_1}\cdots Y_{i_n}Y_{j_1}\cdots Y_{j_n}\right] \\ &= (n!)^2 \sum_{0\le i_1<\cdots<i_n} f_n(i_1,\ldots,i_n)g_n(i_1,\ldots,i_n) \\ &= n! \sum_{(i_1,\ldots,i_n)\in\Delta_n} f_n(i_1,\ldots,i_n)g_n(i_1,\ldots,i_n) \\ &= n!\langle \mathbf{1}_{\Delta_n} f_n, g_m\rangle_{\ell^2(\mathbb{N})^{\otimes n}}. \end{aligned}$$

When $n < m$ and $(i_1,\ldots,i_n) \in \Delta_n$ and $(j_1,\ldots,j_m) \in \Delta_m$ are two sets of indices, there necessarily exists $k \in \{1,\ldots,m\}$ such that $j_k \notin \{i_1,\ldots,i_n\}$, hence

$$\mathbb{E}\left[Y_{i_1}\cdots Y_{i_n}Y_{j_1}\cdots Y_{j_m}\right] = 0,$$

and this implies the orthogonality of $J_n(f_n)$ and $J_m(g_m)$. The recurrence relation (1.3.1) is a direct consequence of (1.3.3). The isometry property (1.3.2) of J_n also follows by induction from (1.2.1) and the recurrence relation. □

If $f_n \in \ell^2(\mathbb{N}^n)$ is not symmetric we let $J_n(f_n) = J_n(\tilde{f}_n)$, where $\tilde{f}_n$ is the symmetrization of f_n, defined as

$$\tilde{f}_n(i_1,\ldots,i_n) = \frac{1}{n!}\sum_{\sigma\in\Sigma_n} f(i_{\sigma(1)},\ldots,i_{\sigma_n}), \qquad i_1,\ldots,i_n \in \mathbb{N}^n,$$

and Σ_n is the set of all permutations of $\{1, \ldots, n\}$.

In particular, if $(k_1, \ldots, k_n) \in \Delta_n$, the symmetrization of $\mathbf{1}_{\{(k_1,\ldots,k_n)\}}$ in n variables is given by

$$\tilde{\mathbf{1}}_{\{(k_1,\ldots,k_n)\}}(i_1, \ldots, i_n) = \frac{1}{n!}\mathbf{1}_{\{\{i_1,\ldots,i_n\}=\{k_1,\ldots,k_n\}\}}, \quad i_1, \ldots, i_n \in \mathbb{N},$$

and

$$J_n(\tilde{\mathbf{1}}_{\{(k_1,\ldots,k_n)\}}) = Y_{k_1} \cdots Y_{k_n}.$$

Lemma 1.3.3. *For all $n \geq 1$ we have*

$$\mathbb{E}\left[J_n(f_n) \mid \mathcal{F}_k\right] = J_n(f_n \mathbf{1}_{[0,k]^n}),$$

$k \in \mathbb{N}$, $f_n \in \ell^2(\mathbb{N})^{\circ n}$.

Proof. This lemma can be proved in two ways, either as a consequence of Proposition 1.2.3 and Proposition 1.3.2 or via the following direct argument, noting that for all $m = 0, \ldots, n$ and $g_m \in \ell^2(\mathbb{N})^{\circ m}$ we have:

$$\begin{aligned}
&\mathbb{E}\left[(J_n(f_n) - J_n(f_n \mathbf{1}_{[0,k]^n}))J_m(g_m \mathbf{1}_{[0,k]^m})\right] \\
&= \mathbf{1}_{\{n=m\}} n! \langle f_n(1 - \mathbf{1}_{[0,k]^n}), g_m \mathbf{1}_{[0,k]^m} \rangle_{\ell^2(\mathbb{N}^n)} \\
&= 0,
\end{aligned}$$

hence $J_n(f_n \mathbf{1}_{[0,k]^n}) \in L^2(\Omega, \mathcal{F}_k)$, and $J_n(f_n) - J_n(f_n \mathbf{1}_{[0,k]^n})$ is orthogonal to $L^2(\Omega, \mathcal{F}_k)$. □

In other terms we have

$$\mathbb{E}\left[J_n(f_n)\right] = 0, \qquad f_n \in \ell^2(\mathbb{N})^{\circ n}, \qquad n \geq 1,$$

the process $(J_n(f_n \mathbf{1}_{[0,k]^n}))_{k \in \mathbb{N}}$ is a discrete-time martingale, and $J_n(f_n)$ is $\mathcal{F}_k$-measurable if and only if

$$f_n \mathbf{1}_{[0,k]^n} = f_n, \qquad 0 \leq k \leq n.$$

1.4 Structure Equations

Assume now that the sequence $(Y_n)_{n \in \mathbb{N}}$ satisfies the discrete structure equation:

$$Y_n^2 = 1 + \varphi_n Y_n, \qquad n \in \mathbb{N}, \tag{1.4.1}$$

where $(\varphi_n)_{n \in \mathbb{N}}$ is an $\mathcal{F}_n$-predictable process. Condition (1.1.1) implies that

$$\mathbb{E}\left[Y_n^2 \mid \mathcal{F}_{n-1}\right] = 1, \qquad n \in \mathbb{N},$$

hence the hypotheses of the preceding sections are satisfied. Since (1.4.1) is a second order equation, there exists an $\mathcal{F}_n$-adapted process $(X_n)_{n\in\mathbb{N}}$ of Bernoulli $\{-1,1\}$-valued random variables such that

$$Y_n = \frac{\varphi_n}{2} + X_n\sqrt{1+\left(\frac{\varphi_n}{2}\right)^2}, \qquad n \in \mathbb{N}. \tag{1.4.2}$$

Consider the conditional probabilities

$$p_n = \mathbb{P}(X_n = 1 \mid \mathcal{F}_{n-1}) \quad \text{and} \quad q_n = \mathbb{P}(X_n = -1 \mid \mathcal{F}_{n-1}), \qquad n \in \mathbb{N}. \tag{1.4.3}$$

From the relation $\mathbb{E}[Y_n \mid \mathcal{F}_{n-1}] = 0$, rewritten as

$$p_n\left(\frac{\varphi_n}{2} + \sqrt{1+\left(\frac{\varphi_n}{2}\right)^2}\right) + q_n\left(\frac{\varphi_n}{2} - \sqrt{1+\left(\frac{\varphi_n}{2}\right)^2}\right) = 0, \qquad n \in \mathbb{N},$$

we get

$$p_n = \frac{1}{2}\left(1 - \frac{\varphi_n}{\sqrt{4+\varphi_n^2}}\right), \qquad q_n = \frac{1}{2}\left(1 + \frac{\varphi_n}{\sqrt{4+\varphi_n^2}}\right), \tag{1.4.4}$$

and

$$\varphi_n = \sqrt{\frac{q_n}{p_n}} - \sqrt{\frac{p_n}{q_n}} = \frac{q_n - p_n}{\sqrt{p_n q_n}}, \qquad n \in \mathbb{N},$$

hence

$$Y_n = \mathbf{1}_{\{X_n=1\}}\sqrt{\frac{q_n}{p_n}} - \mathbf{1}_{\{X_n=-1\}}\sqrt{\frac{p_n}{q_n}}, \qquad n \in \mathbb{N}. \tag{1.4.5}$$

Letting

$$Z_n = \frac{X_n+1}{2} \in \{0,1\}, \qquad n \in \mathbb{N},$$

we also have the relations

$$Y_n = \frac{q_n - p_n + X_n}{2\sqrt{p_n q_n}} = \frac{Z_n - p_n}{\sqrt{p_n q_n}}, \quad n \in \mathbb{N}, \tag{1.4.6}$$

which yield

$$\mathcal{F}_n = \sigma(X_0,\ldots,X_n) = \sigma(Z_0,\ldots,Z_n), \qquad n \in \mathbb{N}.$$

Remark 1.4.1. *In particular, one can take $\Omega = \{-1,1\}^{\mathbb{N}}$ and construct the Bernoulli process $(X_n)_{n\in\mathbb{N}}$ as the sequence of canonical projections on $\Omega = \{-1,1\}^{\mathbb{N}}$ under a countable product $\mathbb{P}$ of Bernoulli measures on $\{-1,1\}$. In this case the sequence $(X_n)_{n\in\mathbb{N}}$ can be viewed as the dyadic expansion of $X(\omega) \in [0,1]$ defined as:*

$$X(\omega) = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} X_n(\omega).$$

In the symmetric case $p_k = q_k = 1/2$, $k \in \mathbb{N}$, the image measure of $\mathbb{P}$ by the mapping $\omega \longmapsto X(\omega)$ is the Lebesgue measure on $[0,1]$, see [143] for the non-symmetric case.

1.5 Chaos Representation

From now on we assume that the sequence $(p_k)_{k\in\mathbb{N}}$ defined in (1.4.3) is deterministic, which implies that the random variables $(X_n)_{n\in\mathbb{N}}$ are independent. Precisely, X_n will be constructed as the canonical projection $X_n : \Omega \to \{-1,1\}$ on $\Omega = \{-1,1\}^{\mathbb{N}}$ under the measure $\mathbb{P}$ given on cylinder sets by

$$\mathbb{P}(\{\epsilon_0,\ldots,\epsilon_n\} \times \{-1,1\}^{\mathbb{N}}) = \prod_{k=0}^{n} p_k^{(1+\varepsilon_k)/2} q_k^{(1-\varepsilon_k)/2},$$

$\{\epsilon_0,\ldots,\epsilon_n\} \in \{-1,1\}^{n+1}$. The sequence $(Y_k)_{k\in\mathbb{N}}$ can be constructed as a family of independent random variables given by

$$Y_n = \frac{\varphi_n}{2} + X_n\sqrt{1+\left(\frac{\varphi_n}{2}\right)^2}, \qquad n \in \mathbb{N},$$

where the sequence $(\varphi_n)_{n\in\mathbb{N}}$ is deterministic. In this case, all spaces $L^r(\Omega,\mathcal{F}_n)$, $r \geq 1$, have finite dimension 2^{n+1}, with basis

$$\left\{ \mathbf{1}_{\{Y_0=\epsilon_0,\ldots,Y_n=\epsilon_n\}} \ : \ (\epsilon_0,\ldots,\epsilon_n) \in \prod_{k=0}^{n} \left\{ \sqrt{\frac{q_k}{p_k}}, -\sqrt{\frac{p_k}{q_k}} \right\} \right\}$$
$$= \left\{ \mathbf{1}_{\{X_0=\epsilon_0,\ldots,X_n=\epsilon_n\}} \ : \ (\epsilon_0,\ldots,\epsilon_n) \in \prod_{k=0}^{n} \{-1,1\} \right\}.$$

An orthogonal basis of $L^r(\Omega,\mathcal{F}_n)$ is given by

$$\left\{ Y_{k_1}\cdots Y_{k_l} = J_l(\tilde{\mathbf{1}}_{\{(k_1,\ldots,k_l)\}}) \ : \ 0 \leq k_1 < \cdots < k_l \leq n, \ l = 0,\ldots,n+1 \right\}.$$

Let

$$S_n = \sum_{k=0}^{n} \frac{1+X_k}{2} \tag{1.5.1}$$

$$= \sum_{k=0}^{n} Z_k, \qquad n \in \mathbb{N},$$

denote the random walk associated to $(X_k)_{k\in\mathbb{N}}$.

Remark. *In the special case $p_k = p$ for all $k \in \mathbb{N}$, we have*

$$J_n(\mathbf{1}_{[0,N]}^{\circ n}) = K_n(S_N; N+1, p) \tag{1.5.2}$$

coincides with the Krawtchouk polynomial $K_n(\cdot; N+1, p)$ of order n and parameter $(N+1, p)$, evaluated at S_N, cf. Proposition 4 of [119].

Let now $\mathcal{H}_0 = \mathbb{R}$ and let $\mathcal{H}_n$ denote the subspace of $L^2(\Omega)$ made of integrals of order $n \geq 1$, and called chaos of order n:

$$\mathcal{H}_n = \{J_n(f_n) \ : \ f_n \in \ell^2(\mathbb{N})^{\circ n}\}.$$

The space of $\mathcal{F}_n$-measurable random variables is denoted by $L^0(\Omega, \mathcal{F}_n)$.

Lemma 1.5.1. *For all $n \in \mathbb{N}$ we have*

$$L^0(\Omega, \mathcal{F}_n) = (\mathcal{H}_0 \oplus \cdots \oplus \mathcal{H}_{n+1}) \bigcap L^0(\Omega, \mathcal{F}_n). \tag{1.5.3}$$

Proof. It suffices to note that $\mathcal{H}_l \cap L^0(\Omega, \mathcal{F}_n)$ has dimension $\binom{n+1}{l}$, $1 \leq l \leq n+1$. More precisely it is generated by the orthonormal basis

$$\left\{Y_{k_1} \cdots Y_{k_l} = J_l(\tilde{\mathbf{1}}_{\{(k_1,\ldots,k_l)\}}) \ : \ 0 \leq k_1 < \cdots < k_l \leq n\right\},$$

since any element F of $\mathcal{H}_l \cap L^0(\Omega, \mathcal{F}_n)$ can be written as $F = J_l(f_l \mathbf{1}_{[0,n]^l})$. Hence $L^0(\Omega, \mathcal{F}_n)$ and $(\mathcal{H}_0 \oplus \cdots \oplus \mathcal{H}_{n+1}) \bigcap L^0(\Omega, \mathcal{F}_n)$ have same dimension $2^{n+1} = \displaystyle\sum_{k=0}^{n+1} \binom{n+1}{k}$, and this implies (1.5.3) since

$$L^0(\Omega, \mathcal{F}_n) \supset (\mathcal{H}_0 \oplus \cdots \oplus \mathcal{H}_{n+1}) \bigcap L^0(\Omega, \mathcal{F}_n).$$

□

As a consequence of Lemma 1.5.1 we have

$$L^0(\Omega, \mathcal{F}_n) \subset \mathcal{H}_0 \oplus \cdots \oplus \mathcal{H}_{n+1}.$$

Alternatively, Lemma 1.5.1 can be proved by noting that

$$J_n(f_n \mathbf{1}_{[0,N]^n}) = 0, \qquad n > N+1, \quad f_n \in \ell^2(\mathbb{N})^{\circ n},$$

and as a consequence, any $F \in L^0(\Omega, \mathcal{F}_N)$ can be expressed as

$$F = \mathbb{E}[F] + \sum_{n=1}^{N+1} J_n(f_n \mathbf{1}_{[0,N]^n}).$$

Definition 1.5.2. *Let $\mathcal{S}$ denote the linear space spanned by multiple stochastic integrals, i.e.*

$$\begin{aligned}\mathcal{S} &= \mathrm{Vect}\left\{\bigcup_{n=0}^{\infty}\mathcal{H}_n\right\} \qquad (1.5.4)\\ &= \left\{\sum_{k=0}^{n} J_k(f_k) \ : \ f_k \in \ell^2(\mathbb{N})^{\circ k}, \ k=0,\ldots,n, \ n\in\mathbb{N}\right\}.\end{aligned}$$

The completion of $\mathcal{S}$ in $L^2(\Omega)$ is denoted by the direct sum

$$\bigoplus_{n=0}^{\infty}\mathcal{H}_n.$$

The next result is the chaos representation property for Bernoulli processes, which is analogous to the Walsh decomposition, cf. [82]. Here this property is obtained under the assumption that the sequence $(X_n)_{n\in\mathbb{N}}$ is made of independent random variables since $(p_k)_{k\in\mathbb{N}}$ is deterministic, which corresponds to the setting of Proposition 4 in [40]. See [40] and Proposition 5 therein for other instances of the chaos representation property without this independence assumption.

Proposition 1.5.3. *We have the identity*

$$L^2(\Omega) = \bigoplus_{n=0}^{\infty}\mathcal{H}_n.$$

Proof. It suffices to show that $\mathcal{S}$ is dense in $L^2(\Omega)$. Let F be a bounded random variable. Relation (1.5.3) of Lemma 1.5.1 shows that $\mathbb{E}[F \mid \mathcal{F}_n] \in \mathcal{S}$. The martingale convergence theorem, cf. e.g. Theorem 27.1 in [71], implies that $(\mathbb{E}[F \mid \mathcal{F}_n])_{n\in\mathbb{N}}$ converges to F a.s., hence every bounded F is the $L^2(\Omega)$-limit of a sequence in $\mathcal{S}$. If $F \in L^2(\Omega)$ is not bounded, F is the limit in $L^2(\Omega)$ of the sequence $(\mathbf{1}_{\{|F|\leq n\}}F)_{n\in\mathbb{N}}$ of bounded random variables. □

As a consequence of Proposition 1.5.3, any $F \in L^2(\Omega,\mathbb{P})$ has a unique decomposition

$$F = \mathbb{E}[F] + \sum_{n=1}^{\infty} J_n(f_n), \qquad f_n \in \ell^2(\mathbb{N})^{\circ n}, \ n\in\mathbb{N},$$

as a series of multiple stochastic integrals. Note also that the statement of Lemma 1.5.1 is sufficient for the chaos representation property to hold.

1.6 Gradient Operator

We start by defining the operator D on the space $\mathcal{S}$ of finite sums of multiple stochastic integrals, which is dense in in $L^2(\Omega)$ by Proposition 1.5.3.

Definition 1.6.1. *We densely define the linear gradient operator*

$$D : \mathcal{S} \longrightarrow L^2(\Omega \times \mathbb{N})$$

by

$$D_k J_n(f_n) = n J_{n-1}(f_n(*, k)\mathbf{1}_{\Delta_n}(*, k)),$$

$k \in \mathbb{N}$, $f_n \in \ell^2(\mathbb{N})^{\circ n}$, $n \in \mathbb{N}$.

Note that for all $k_1, \ldots, k_{n-1}, k \in \mathbb{N}$, we have

$$\mathbf{1}_{\Delta_n}(k_1, \ldots, k_{n-1}, k) = \mathbf{1}_{\{k \notin (k_1, \ldots, k_{n-1})\}} \mathbf{1}_{\Delta_{n-1}}(k_1, \ldots, k_{n-1}),$$

hence we can write

$$D_k J_n(f_n) = n J_{n-1}(f_n(*, k)\mathbf{1}_{\{k \notin *\}}), \quad k \in \mathbb{N},$$

where in the above relation, "$*$" denotes the first $k-1$ variables $(k_1, \ldots, k_{n-1})$ of $f_n(k_1, \ldots, k_{n-1}, k)$. We also have $D_k F = 0$ whenever $F \in \mathcal{S}$ is $\mathcal{F}_{k-1}$-measurable.

On the other hand, D_k is a continuous operator on the chaos $\mathcal{H}_n$ since

$$\begin{aligned} \|D_k J_n(f_n)\|^2_{L^2(\Omega)} &= n^2 \|J_{n-1}(f_n(*, k))\|^2_{L^2(\Omega)} \qquad (1.6.1) \\ &= n n! \|f_n(*, k)\|^2_{\ell^2(\mathbb{N}^{\otimes(n-1)})}, \qquad f_n \in \ell^2(\mathbb{N}^{\otimes n}), \quad k \in \mathbb{N}. \end{aligned}$$

The following result gives the probabilistic interpretation of D_k as a finite difference operator. Given

$$\omega = (\omega_0, \omega_1, \ldots) \in \{-1, 1\}^{\mathbb{N}},$$

let

$$\omega^k_+ = (\omega_0, \omega_1, \ldots, \omega_{k-1}, +1, \omega_{k+1}, \ldots)$$

and

$$\omega^k_- = (\omega_0, \omega_1, \ldots, \omega_{k-1}, -1, \omega_{k+1}, \ldots).$$

Proposition 1.6.2. *We have for any $F \in \mathcal{S}$:*

$$D_k F(\omega) = \sqrt{p_k q_k}(F(\omega^k_+) - F(\omega^k_-)), \quad k \in \mathbb{N}. \qquad (1.6.2)$$

Proof. We start by proving the above statement for an $\mathcal{F}_n$-measurable $F \in \mathcal{S}$. Since $L^0(\Omega, \mathcal{F}_n)$ is finite dimensional it suffices to consider

$$F = Y_{k_1} \cdots Y_{k_l} = f(X_0, \ldots, X_{k_l}),$$

with from (1.4.6):

$$f(x_0, \ldots, x_{k_l}) = \frac{1}{2^l} \prod_{i=1}^{l} \frac{q_{k_i} - p_{k_i} + x_{k_i}}{\sqrt{p_{k_i} q_{k_i}}}.$$

First we note that from (1.5.3) we have for $(k_1, \ldots, k_n) \in \Delta_n$:

$$\begin{aligned}
&D_k (Y_{k_1} \cdots Y_{k_n}) = D_k J_n(\tilde{\mathbf{1}}_{\{(k_1,\ldots,k_n)\}}) \\
&= n J_{n-1}(\tilde{\mathbf{1}}_{\{(k_1,\ldots,k_n)\}}(*, k)) \\
&= \frac{1}{(n-1)!} \sum_{i=1}^{n} \mathbf{1}_{\{k_i\}}(k) \sum_{(i_1,\ldots,i_{n-1}) \in \Delta_{n-1}} \tilde{\mathbf{1}}_{\{\{i_1,\ldots,i_{n-1}\}=\{k_1,\ldots,k_{i-1},k_{i+1},\ldots,k_n\}\}} \\
&= \sum_{i=1}^{n} \mathbf{1}_{\{k_i\}}(k) J_{n-1}(\tilde{\mathbf{1}}_{\{(k_1,\ldots,k_{i-1},k_{i+1},\ldots,k_n)\}}) \\
&= \mathbf{1}_{\{k_1,\ldots,k_n\}}(k) \prod_{\substack{i=1 \\ k_i \neq k}}^{n} Y_{k_i}. \qquad (1.6.3)
\end{aligned}$$

If $k \notin \{k_1, \ldots, k_l\}$ we clearly have $F(\omega_+^k) = F(\omega_-^k) = F(\omega)$, hence

$$\sqrt{p_k q_k}(F(\omega_+^k) - F(\omega_-^k)) = 0 = D_k F(\omega).$$

On the other hand if $k \in \{k_1, \ldots, k_l\}$ we have

$$F(\omega_+^k) = \sqrt{\frac{q_k}{p_k}} \prod_{\substack{i=1 \\ k_i \neq k}}^{l} \frac{q_{k_i} - p_{k_i} + \omega_{k_i}}{2\sqrt{p_{k_i} q_{k_i}}},$$

$$F(\omega_-^k) = -\sqrt{\frac{p_k}{q_k}} \prod_{\substack{i=1 \\ k_i \neq k}}^{l} \frac{q_{k_i} - p_{k_i} + \omega_{k_i}}{2\sqrt{p_{k_i} q_{k_i}}},$$

hence from (1.6.3) we get

$$\begin{aligned}
\sqrt{p_k q_k}(F(\omega_+^k) - F(\omega_-^k)) &= \frac{1}{2^{l-1}} \prod_{\substack{i=1 \\ k_i \neq k}}^{l} \frac{q_{k_i} - p_{k_i} + \omega_{k_i}}{\sqrt{p_{k_i} q_{k_i}}} \\
&= \prod_{\substack{i=1 \\ k_i \neq k}}^{l} Y_{k_i}(\omega) \\
&= D_k (Y_{k_1} \cdots Y_{k_l})(\omega)
\end{aligned}$$

$$= D_k F(\omega).$$

In the general case, $J_l(f_l)$ is the L^2-limit of the sequence $\mathbb{E}[J_l(f_l) \mid \mathcal{F}_n] = J_l(f_l \mathbf{1}_{[0,n]^l})$ as n goes to infinity, and since from (1.6.1) the operator D_k is continuous on all chaoses $\mathcal{H}_n$, $n \geq 1$, we have

$$\begin{aligned} D_k F &= \lim_{n\to\infty} D_k \mathbb{E}[F \mid \mathcal{F}_n] \\ &= \sqrt{p_k q_k} \lim_{n\to\infty} (\mathbb{E}[F \mid \mathcal{F}_n](\omega_+^k) - \mathbb{E}[F \mid \mathcal{F}_n](\omega_-^k)) \\ &= \sqrt{p_k q_k}(F(\omega_+^k) - F(\omega_-^k)), \quad k \in \mathbb{N}. \end{aligned}$$

□

The next property follows immediately from Proposition 1.6.2 .

Corollary 1.6.3. *A random variable $F : \Omega \to \mathbb{R}$ is $\mathcal{F}_n$-measurable if and only if*

$$D_k F = 0$$

for all $k > n$.

If F has the form $F = f(X_0, \ldots, X_n)$, we may also write

$$D_k F = \sqrt{p_k q_k}(F_k^+ - F_k^-), \qquad k \in \mathbb{N},$$

with

$$F_k^+ = f(X_0, \ldots, X_{k-1}, +1, X_{k+1}, \ldots, X_n),$$

and

$$F_k^- = f(X_0, \ldots, X_{k-1}, -1, X_{k+1}, \ldots, X_n).$$

The gradient D can also be expressed as

$$D_k F(S_\cdot) = \sqrt{p_k q_k}\Big(F\big(S_\cdot + \mathbf{1}_{\{X_k=-1\}}\mathbf{1}_{\{k\leq \cdot\}}\big) - F\big(S_\cdot - \mathbf{1}_{\{X_k=1\}}\mathbf{1}_{\{k\leq \cdot\}}\big)\Big),$$

where $F(S_\cdot)$ is an informal notation for the random variable F estimated on a given path of $(S_n)_{n\in\mathbb{N}}$ defined in (1.5.1) and $S_\cdot + \mathbf{1}_{\{X_k=\mp 1\}}\mathbf{1}_{\{k\leq \cdot\}}$ denotes the path of $(S_n)_{n\in\mathbb{N}}$ perturbed by forcing X_k to be equal to ± 1.

We will also use the gradient ∇_k defined as

$$\begin{aligned} \nabla_k F = X_k \,(& f(X_0, \ldots, X_{k-1}, -1, X_{k+1}, \ldots, X_n) \\ & - f(X_0, \ldots, X_{k-1}, 1, X_{k+1}, \ldots, X_n)), \end{aligned}$$

$k \in \mathbb{N}$, with the relation

$$D_k = -X_k \sqrt{p_k q_k} \nabla_k, \qquad k \in \mathbb{N},$$

hence $\nabla_k F$ coincides with $D_k F$ after squaring and multiplication by $p_k q_k$. From now on, D_k denotes the finite difference operator which is extended to any $F : \Omega \to \mathbb{R}$ using Relation (1.6.2).

The L^2 domain of D, denoted $\mathrm{Dom}\,(D)$, is naturally defined as the space of functionals $F \in L^2(\Omega)$ such that

$$\mathbb{E}\left[\|DF\|^2_{\ell^2(\mathbb{N})}\right] < \infty,$$

or equivalently by (1.6.1),

$$\sum_{n=1}^{\infty} n n! \|f_n\|^2_{\ell^2(\mathbb{N}^n)} < \infty,$$

if $F = \displaystyle\sum_{n=0}^{\infty} J_n(f_n)$.

The following is the product rule for the operator D.

Proposition 1.6.4. *Let $F, G : \Omega \to \mathbb{R}$. We have*

$$D_k(FG) = F D_k G + G D_k F - \frac{X_k}{\sqrt{p_k q_k}} D_k F D_k G, \qquad k \in \mathbb{N}.$$

Proof. Let $F_+^k(\omega) = F(\omega_+^k)$, $F_-^k(\omega) = F(\omega_-^k)$, $k \geq 0$. We have

$$\begin{aligned}
D_k(FG) &= \sqrt{p_k q_k}(F_+^k G_+^k - F_-^k G_-^k) \\
&= \mathbf{1}_{\{X_k=-1\}} \sqrt{p_k q_k} \left(F(G_+^k - G) + G(F_+^k - F) + (F_+^k - F)(G_+^k - G)\right) \\
&\quad + \mathbf{1}_{\{X_k=1\}} \sqrt{p_k q_k} \left(F(G - G_-^k) + G(F - F_-^k) - (F - F_-^k)(G - G_-^k)\right) \\
&= \mathbf{1}_{\{X_k=-1\}} \left(F D_k G + G D_k F + \frac{1}{\sqrt{p_k q_k}} D_k F D_k G\right) \\
&\quad + \mathbf{1}_{\{X_k=1\}} \left(F D_k G + G D_k F - \frac{1}{\sqrt{p_k q_k}} D_k F D_k G\right).
\end{aligned}$$

□

1.7 Clark Formula and Predictable Representation

In this section we prove a predictable representation formula for the functionals of $(S_n)_{n\geq 0}$ defined in (1.5.1).

Proposition 1.7.1. *For all $F \in \mathcal{S}$ we have*

$$F = \mathbb{E}[F] + \sum_{k=0}^{\infty} \mathbb{E}[D_k F \mid \mathcal{F}_{k-1}] Y_k \qquad (1.7.1)$$
$$= \mathbb{E}[F] + \sum_{k=0}^{\infty} Y_k D_k \mathbb{E}[F \mid \mathcal{F}_k].$$

Proof. The formula is obviously true for $F = J_0(f_0)$. Given $n \geq 1$, as a consequence of Proposition 1.3.2 above and Lemma 1.3.3 we have:

$$\begin{aligned} J_n(f_n) &= n \sum_{k=0}^{\infty} J_{n-1}(f_n(*, k) \mathbf{1}_{[0,k-1]^{n-1}}(*)) Y_k \\ &= n \sum_{k=0}^{\infty} J_{n-1}(f_n(*, k) \mathbf{1}_{\Delta_n}(*, k) \mathbf{1}_{[0,k-1]^{n-1}}(*)) Y_k \\ &= n \sum_{k=0}^{\infty} \mathbb{E}[J_{n-1}(f_n(*, k) \mathbf{1}_{\Delta_n}(*, k)) \mid \mathcal{F}_{k-1}] Y_k \\ &= \sum_{k=0}^{\infty} \mathbb{E}[D_k J_n(f_n) \mid \mathcal{F}_{k-1}] Y_k, \end{aligned}$$

which yields (1.7.1) for $F = J_n(f_n)$, since $\mathbb{E}[J_n(f_n)] = 0$. By linearity the formula is established for $F \in \mathcal{S}$.

For the second identity we use the relation

$$\mathbb{E}[D_k F \mid \mathcal{F}_{k-1}] = D_k \mathbb{E}[F \mid \mathcal{F}_k]$$

which clearly holds since $D_k F$ is independent of X_k, $k \in \mathbb{N}$. □

Although the operator D is unbounded we have the following result, which states the boundedness of the operator that maps a random variable to the unique process involved in its predictable representation.

Lemma 1.7.2. *The operator*

$$\begin{aligned} L^2(\Omega) &\longrightarrow L^2(\Omega \times \mathbb{N}) \\ F &\longmapsto (\mathbb{E}[D_k F \mid \mathcal{F}_{k-1}])_{k \in \mathbb{N}} \end{aligned}$$

is bounded with norm equal to one.

Proof. Let $F \in \mathcal{S}$. From Relation (1.7.1) and the isometry formula (1.2.2) for the stochastic integral operator J we get

$$\begin{aligned} \|\mathbb{E}[D_\cdot F \mid \mathcal{F}_{\cdot-1}]\|^2_{L^2(\Omega \times \mathbb{N})} &= \|F - \mathbb{E}[F]\|^2_{L^2(\Omega)} \qquad (1.7.2) \\ &\leq \|F - \mathbb{E}[F]\|^2_{L^2(\Omega)} + (\mathbb{E}[F])^2 \\ &= \|F\|^2_{L^2(\Omega)}, \end{aligned}$$

with equality in case $F = J_1(f_1)$. □

As a consequence of Lemma 1.7.2 we have the following corollary.

Corollary 1.7.3. *The Clark formula of Proposition 1.7.1 extends to any* $F \in L^2(\Omega)$.

Proof. Since $F \longmapsto \mathbb{E}[D_{\cdot}F \mid \mathcal{F}_{\cdot-1}]$ is bounded from Lemma 1.7.2, the Clark formula extends to $F \in L^2(\Omega)$ by a standard Cauchy sequence argument. □

Let us give a first elementary application of the above construction to the proof of a Poincaré inequality on Bernoulli space. Using (1.2.3) we have

$$\begin{aligned}
\mathrm{Var}\,(F) &= \mathbb{E}\left[|F - \mathbb{E}[F]|^2\right] \\
&= \mathbb{E}\left[\left(\sum_{k=0}^{\infty} \mathbb{E}[D_k F \mid \mathcal{F}_{k-1}] Y_k\right)^2\right] \\
&= \mathbb{E}\left[\sum_{k=0}^{\infty} (\mathbb{E}[D_k F \mid \mathcal{F}_{k-1}])^2\right] \\
&\leq \mathbb{E}\left[\sum_{k=0}^{\infty} \mathbb{E}\left[|D_k F|^2 \mid \mathcal{F}_{k-1}\right]\right] \\
&= \mathbb{E}\left[\sum_{k=0}^{\infty} |D_k F|^2\right],
\end{aligned}$$

hence

$$\mathrm{Var}\,(F) \leq \|DF\|^2_{L^2(\Omega\times\mathbb{N})}.$$

More generally the Clark formula implies the following.

Corollary 1.7.4. *Let $a \in \mathbb{N}$ and $F \in L^2(\Omega)$. We have*

$$F = \mathbb{E}[F \mid \mathcal{F}_a] + \sum_{k=a+1}^{\infty} \mathbb{E}[D_k F \mid \mathcal{F}_{k-1}] Y_k, \tag{1.7.3}$$

and

$$\mathbb{E}\left[F^2\right] = \mathbb{E}\left[(\mathbb{E}[F \mid \mathcal{F}_a])^2\right] + \mathbb{E}\left[\sum_{k=a+1}^{\infty} (\mathbb{E}[D_k F \mid \mathcal{F}_{k-1}])^2\right]. \tag{1.7.4}$$

Proof. From Proposition 1.2.3 and the Clark formula (1.7.1) of Proposition 1.7.1 we have

$$\mathbb{E}[F \mid \mathcal{F}_a] = \mathbb{E}[F] + \sum_{k=0}^{a} \mathbb{E}[D_k F \mid \mathcal{F}_{k-1}] Y_k,$$

which implies (1.7.3). Relation (1.7.4) is an immediate consequence of (1.7.3) and the isometry property of J. □

As an application of the Clark formula of Corollary 1.7.4 we obtain the following predictable representation property for discrete-time martingales.

Proposition 1.7.5. *Let $(M_n)_{n\in\mathbb{N}}$ be a martingale in $L^2(\Omega)$ with respect to $(\mathcal{F}_n)_{n\in\mathbb{N}}$. There exists a predictable process $(u_k)_{k\in\mathbb{N}}$ locally in $L^2(\Omega\times\mathbb{N})$, (i.e. $u(\cdot)\mathbf{1}_{[0,N]}(\cdot)\in L^2(\Omega\times\mathbb{N})$ for all $N>0$) such that*

$$M_n = M_{-1} + \sum_{k=0}^{n} u_k Y_k, \qquad n \in \mathbb{N}. \tag{1.7.5}$$

Proof. Let $k \geq 1$. From Corollaries 1.6.3 and 1.7.4 we have:

$$\begin{aligned} M_k &= \mathbb{E}\left[M_k \mid \mathcal{F}_{k-1}\right] + \mathbb{E}\left[D_k M_k \mid \mathcal{F}_{k-1}\right] Y_k \\ &= M_{k-1} + \mathbb{E}\left[D_k M_k \mid \mathcal{F}_{k-1}\right] Y_k, \end{aligned}$$

hence it suffices to let

$$u_k = \mathbb{E}\left[D_k M_k \mid \mathcal{F}_{k-1}\right], \qquad k \geq 0,$$

to obtain

$$M_n = M_{-1} + \sum_{k=0}^{n} M_k - M_{k-1} = M_{-1} + \sum_{k=0}^{n} u_k Y_k.$$

□

1.8 Divergence Operator

The divergence operator δ is introduced as the adjoint of D. Let $\mathcal{U} \subset L^2(\Omega\times\mathbb{N})$ be the space of processes defined as

$$\mathcal{U} = \left\{ \sum_{k=0}^{n} J_k(f_{k+1}(*,\cdot)), \quad f_{k+1} \in \ell^2(\mathbb{N})^{\circ k} \otimes \ell^2(\mathbb{N}), \quad k = 0,\ldots,n, \ n \in \mathbb{N} \right\}.$$

We refer to Section 9.7 in the appendix for the definition of the tensor product $\ell^2(\mathbb{N})^{\circ k} \otimes \ell^2(\mathbb{N})$, $k \geq 0$.

Definition 1.8.1. *Let $\delta : \mathcal{U} \to L^2(\Omega)$ be the linear mapping defined on $\mathcal{U}$ as*

$$\delta(u) = \delta(J_n(f_{n+1}(*,\cdot))) = J_{n+1}(\tilde{f}_{n+1}), \quad f_{n+1} \in \ell^2(\mathbb{N})^{\circ n} \otimes \ell^2(\mathbb{N}),$$

for $(u_k)_{k\in\mathbb{N}}$ of the form

$$u_k = J_n(f_{n+1}(*, k)), \qquad k \in \mathbb{N},$$

where $\tilde{f}_{n+1}$ *denotes the symmetrization of* f_{n+1} *in* $n+1$ *variables, i.e.*

$$\tilde{f}_{n+1}(k_1, \ldots, k_{n+1}) = \frac{1}{n+1} \sum_{i=1}^{n+1} f_{n+1}(k_1, \ldots, k_{k-1}, k_{k+1}, \ldots, k_{n+1}, k_i).$$

From Proposition 1.5.3, $\mathcal{S}$ is dense in $L^2(\Omega)$, hence $\mathcal{U}$ is dense in $L^2(\Omega \times \mathbb{N})$.

Proposition 1.8.2. *The operator* δ *is adjoint to* D*:*

$$\mathbb{E}\left[\langle DF, u\rangle_{\ell^2(\mathbb{N})}\right] = \mathbb{E}\left[F\delta(u)\right], \quad F \in \mathcal{S}, \ u \in \mathcal{U}.$$

Proof. We consider $F = J_n(f_n)$ and $u_k = J_m(g_{m+1}(*, k))$, $k \in \mathbb{N}$, where $f_n \in \ell^2(\mathbb{N})^{\circ n}$ and $g_{m+1} \in \ell^2(\mathbb{N})^{\circ m} \otimes \ell^2(\mathbb{N})$. We have

$$\begin{aligned}
&\mathbb{E}\left[\langle D.J_n(f_n), J_m(g_{m+1}(*, \cdot))\rangle_{\ell^2(\mathbb{N})}\right] \\
&= n\mathbb{E}\left[\langle J_{n-1}(f_n(*, \cdot)), J_m(g_m(*, \cdot))\rangle_{\ell^2(\mathbb{N})}\right] \\
&= n\mathbb{E}\left[\langle J_{n-1}(f_n(*, \cdot)\mathbf{1}_{\Delta_n}(*, \cdot)), J_m(g_m(*, \cdot))\rangle_{\ell^2(\mathbb{N})}\right] \\
&= n\mathbf{1}_{\{n-1=m\}} \sum_{k=0}^{\infty} \mathbb{E}\left[J_{n-1}(f_n(*, k)\mathbf{1}_{\Delta_n}(*, k))J_m(g_{m+1}(*, k))\right] \\
&= n!\mathbf{1}_{\{n-1=m\}} \sum_{k=0}^{\infty} \langle \mathbf{1}_{\Delta_n}(*, k) f_n(*, k), g_{m+1}(*, k)\rangle_{\ell^2(\mathbb{N}^{n-1})} \\
&= n!\mathbf{1}_{\{n=m+1\}} \langle \mathbf{1}_{\Delta_n} f_n, g_{m+1}\rangle_{\ell^2(\mathbb{N}^n)} \\
&= n!\mathbf{1}_{\{n=m+1\}} \langle \mathbf{1}_{\Delta_n} f_n, \tilde{g}_{m+1}\rangle_{\ell^2(\mathbb{N}^n)} \\
&= \mathbb{E}\left[J_n(f_n)J_m(\tilde{g}_{m+1})\right] \\
&= \mathbb{E}\left[F\delta(u)\right].
\end{aligned}$$

□

The next proposition shows that δ coincides with the stochastic integral operator J on the square-summable predictable processes.

Proposition 1.8.3. *The operator* δ *can be extended to* $u \in L^2(\Omega \times \mathbb{N})$ *with*

$$\delta(u) = \sum_{k=0}^{\infty} u_k Y_k - \sum_{k=0}^{\infty} D_k u_k - \delta(\varphi D u), \tag{1.8.1}$$

provided that all series converges in $L^2(\Omega)$*, where* $(\varphi_k)_{k \in \mathbb{N}}$ *appears in the structure equation* (1.4.1)*. We also have for all* $u \in \mathcal{U}$*:*

$$\mathbb{E}[|\delta(u)|^2] = \mathbb{E}[\|u\|^2_{\ell^2(\mathbb{N})}] + \mathbb{E}\left[\sum_{\substack{k,l=0 \\ k \neq l}}^{\infty} D_k u_l D_l u_k - \sum_{k=0}^{\infty} (D_k u_k)^2\right]. \tag{1.8.2}$$

Proof. Using the expression (1.3.3) of $u_k = J_n(f_{n+1}(*,k))$ we have

$$\begin{aligned}
\delta(u) &= J_{n+1}(\tilde{f}_{n+1}) \\
&= \sum_{(i_1,\ldots,i_{n+1})\in\Delta_{n+1}} \tilde{f}_{n+1}(i_1,\ldots,i_{n+1})Y_{i_1}\cdots Y_{i_{n+1}} \\
&= \sum_{k=0}^{\infty} \sum_{(i_1,\ldots,i_n)\in\Delta_n} \tilde{f}_{n+1}(i_1,\ldots,i_n,k)Y_{i_1}\cdots Y_{i_n}Y_k \\
&\quad -n\sum_{k=0}^{\infty} \sum_{(i_1,\ldots,i_{n-1})\in\Delta_{n-1}} \tilde{f}_{n+1}(i_1,\ldots,i_{n-1},k,k)Y_{i_1}\cdots Y_{i_{n-1}}|Y_k|^2 \\
&= \sum_{k=0}^{\infty} u_k Y_k - \sum_{k=0}^{\infty} D_k u_k |Y_k|^2 \\
&= \sum_{k=0}^{\infty} u_k Y_k - \sum_{k=0}^{\infty} D_k u_k - \sum_{k=0}^{\infty} \varphi_k D_k u_k Y_k.
\end{aligned}$$

Next, we note the commutation relation[1]

$$\begin{aligned}
D_k\delta(u) &= D_k\left(\sum_{l=0}^{\infty} u_l Y_l - \sum_{l=0}^{\infty} |Y_l|^2 D_l u_l\right) \\
&= \sum_{l=0}^{\infty}\left(Y_l D_k u_l + u_l D_k Y_l - \frac{X_k}{\sqrt{p_k q_k}} D_k u_l D_k Y_l\right) \\
&\quad -\sum_{l=0}^{\infty}\left(|Y_l|^2 D_k D_l u_l + D_l u_l D_k |Y_l|^2 - \frac{X_k}{\sqrt{p_k q_k}} D_k |Y_l|^2 D_k D_l u_l\right) \\
&= \delta(D_k u) + u_k D_k Y_k - \frac{X_k}{\sqrt{p_k q_k}} D_k u_k D_k Y_k - D_k u_k D_k |Y_k|^2 \\
&= \delta(D_k u) + u_k - \left(\frac{X_k}{\sqrt{p_k q_k}} + 2Y_k D_k Y_k - \frac{X_k}{\sqrt{p_k q_k}} D_k Y_k D_k Y_k\right) D_k u_k \\
&= \delta(D_k u) + u_k - 2Y_k D_k u_k.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\delta(\mathbf{1}_{\{k\}} D_k u_k) &= \sum_{l=0}^{\infty} Y_l \mathbf{1}_{\{k\}}(l) D_k u_k - \sum_{l=0}^{\infty} |Y_l|^2 D_l(\mathbf{1}_{\{k\}}(l) D_k u_k) \\
&= Y_k D_k u_k - |Y_k|^2 D_k D_k u_k \\
&= Y_k D_k u_k,
\end{aligned}$$

[1] See A. Mantei, Masterarbeit "Stochastisches Kalkül in diskreter Zeit", Satz 6.7, 2015.

hence

$$
\begin{aligned}
\|\delta(u)\|^2_{L^2(\Omega)} &= \mathbb{E}[\langle u, D\delta(u)\rangle_{\ell^2(\mathbb{N})}] \\
&= \mathbb{E}\left[\sum_{k=0}^{\infty} u_k(u_k + \delta(D_k u) - 2Y_k D_k u_k)\right] \\
&= \mathbb{E}[\|u\|^2_{\ell^2(\mathbb{N})}] + \mathbb{E}\left[\sum_{k,l=0}^{\infty} D_k u_l D_l u_k\right] - 2\mathbb{E}\left[\sum_{k=0}^{\infty} u_k Y_k D_k u_k\right] \\
&= \mathbb{E}[\|u\|^2_{\ell^2(\mathbb{N})}] + \mathbb{E}\left[\sum_{k,l=0}^{\infty} D_k u_l D_l u_k - 2\sum_{k=0}^{\infty}(D_k u_k)^2\right],
\end{aligned}
$$

where we used the equality

$$
\begin{aligned}
&\mathbb{E}\left[u_k Y_k D_k u_k\right] \\
&= \mathbb{E}\left[p_k \mathbf{1}_{\{X_k=1\}} u_k(\omega_+^k) Y_k(\omega_+^k) D_k u_k + q_k \mathbf{1}_{\{X_k=-1\}} u_k(\omega_-^k) Y_k(\omega_-^k) D_k u_k\right] \\
&= \sqrt{p_k q_k}\,\mathbb{E}\left[(\mathbf{1}_{\{X_k=1\}} u_k(\omega_+^k) - \mathbf{1}_{\{X_k=-1\}} u_k(\omega_-^k)) D_k u_k\right] \\
&= \mathbb{E}\left[(D_k u_k)^2\right], \qquad k \in \mathbb{N}.
\end{aligned}
$$

□

In the symmetric case $p_k = q_k = 1/2$ we have $\varphi_k = 0$, $k \in \mathbb{N}$, and

$$
\delta(u) = \sum_{k=0}^{\infty} u_k Y_k - \sum_{k=0}^{\infty} D_k u_k.
$$

The last two terms in the right hand side of (1.8.1) vanish when $(u_k)_{k\in\mathbb{N}}$ is predictable, and in this case the Skorohod isometry (1.8.2) becomes the Itô isometry as in the next proposition.

Corollary 1.8.5. *If $(u_k)_{k\in\mathbb{N}}$ satisfies $D_k u_k = 0$, i.e. u_k does not depend on X_k, $k \in \mathbb{N}$, then $\delta(u)$ coincides with the (discrete time) stochastic integral*

$$
\delta(u) = \sum_{k=0}^{\infty} Y_k u_k, \tag{1.8.3}
$$

provided the series converges in $L^2(\Omega)$. If moreover $(u_k)_{k\in\mathbb{N}}$ is predictable and square-summable we have the isometry

$$
\mathbb{E}\left[\delta(u)^2\right] = \mathbb{E}\left[\|u\|^2_{\ell^2(\mathbb{N})}\right], \tag{1.8.4}
$$

and $\delta(u)$ coincides with $J(u)$ on the space of predictable square-summable processes.

1.9 Ornstein-Uhlenbeck Semi-Group and Process

The Ornstein-Uhlenbeck operator L is defined as $L = \delta D$, i.e. L satisfies

$$LJ_n(f_n) = nJ_n(f_n), \qquad f_n \in \ell^2(\mathbb{N})^{\circ n}.$$

Proposition 1.9.1. *For any $F \in \mathcal{S}$ we have*

$$LF = \delta DF = \sum_{k=0}^{\infty} Y_k(D_kF) = \sum_{k=0}^{\infty} \sqrt{p_k q_k} Y_k(F_k^+ - F_k^-),$$

Proof. Note that $D_k D_k F = 0$, $k \in \mathbb{N}$, and use Relation (1.8.1) of Proposition 1.8.3. □

Note that L can be expressed in other forms, for example

$$LF = \sum_{k=0}^{\infty} \Delta_k F,$$

where

$$\begin{aligned}\Delta_k F &= (\mathbf{1}_{\{X_k=1\}} q_k (F(\omega) - F(\omega_-^k)) - \mathbf{1}_{\{X_k=-1\}} p_k (F(\omega_+^k) - F(\omega))) \\ &= F - (\mathbf{1}_{\{X_k=1\}} q_k F(\omega_-^k) + \mathbf{1}_{\{X_k=-1\}} p_k F(\omega_+^k)) \\ &= F - \mathbb{E}\left[F \mid \mathcal{F}_k^c\right], \quad k \in \mathbb{N},\end{aligned}$$

and $\mathcal{F}_k^c$ is the σ-algebra generated by

$$\{X_l \ : \ l \neq k, \ l \in \mathbb{N}\}.$$

Let now $(P_t)_{t\in\mathbb{R}_+} = (\mathrm{e}^{tL})_{t\in\mathbb{R}_+}$ denote the semi-group associated to L and defined as

$$P_t F = \sum_{n=0}^{\infty} \mathrm{e}^{-nt} J_n(f_n), \qquad t \in \mathbb{R}_+,$$

on $F = \displaystyle\sum_{n=0}^{\infty} J_n(f_n) \in L^2(\Omega)$. The next result shows that $(P_t)_{t\in\mathbb{R}_+}$ admits an integral representation by a probability kernel. Let $q_t^N : \Omega \times \Omega \to \mathbb{R}_+$ be defined by

$$q_t^N(\tilde{\omega}, \omega) = \prod_{i=0}^{N} (1 + \mathrm{e}^{-t} Y_i(\omega) Y_i(\tilde{\omega})), \qquad \omega, \tilde{\omega} \in \Omega, \quad t \in \mathbb{R}_+.$$

Lemma 1.9.2. *Let the probability kernel $Q_t(\tilde{\omega}, d\omega)$ be defined by*

$$\mathbb{E}\left[\frac{dQ_t(\tilde{\omega}, \cdot)}{d\mathbb{P}} \Big| \mathcal{F}_N\right](\omega) = q_t^N(\tilde{\omega}, \omega), \qquad N \geq 1, \quad t \in \mathbb{R}_+.$$

For $F \in L^2(\Omega, \mathcal{F}_N)$ *we have*

$$P_t F(\tilde{\omega}) = \int_\Omega F(\omega) Q_t(\tilde{\omega}, d\omega), \qquad \tilde{\omega} \in \Omega, \quad n \geq N. \tag{1.9.1}$$

Proof. Since $L^2(\Omega, \mathcal{F}_N)$ has finite dimension 2^{N+1}, it suffices to consider functionals of the form $F = Y_{k_1} \cdots Y_{k_n}$ with $0 \leq k_1 < \cdots < k_n \leq N$. By Relation (1.4.5) we have for $\omega \in \Omega$, $k \in \mathbb{N}$:

$$\begin{aligned}
&\mathbb{E}\left[Y_k(\cdot)(1 + e^{-t} Y_k(\cdot) Y_k(\omega))\right] \\
&= p_k \sqrt{\frac{q_k}{p_k}} \left(1 + e^{-t} \sqrt{\frac{q_k}{p_k}} Y_k(\omega)\right) - q_k \sqrt{\frac{p_k}{q_k}} \left(1 - e^{-t} \sqrt{\frac{p_k}{q_k}} Y_k(\omega)\right) \\
&= e^{-t} Y_k(\omega),
\end{aligned}$$

which implies, by independence of the sequence $(X_k)_{k \in \mathbb{N}}$,

$$\begin{aligned}
\mathbb{E}\left[Y_{k_1} \cdots Y_{k_n} q_t^N(\omega, \cdot)\right] &= \mathbb{E}\left[Y_{k_1} \cdots Y_{k_n} \prod_{i=1}^N (1 + e^{-t} Y_{k_i}(\omega) Y_{k_i}(\cdot))\right] \\
&= \prod_{i=1}^N \mathbb{E}\left[Y_{k_i}(\cdot)(1 + e^{-t} Y_{k_i}(\omega) Y_{k_i}(\cdot))\right] \\
&= e^{-nt} Y_{k_1}(\omega) \cdots Y_{k_n}(\omega) \\
&= e^{-nt} J_n(\tilde{\mathbf{1}}_{\{(k_1,\ldots,k_n)\}})(\omega) \\
&= P_t J_n(\tilde{\mathbf{1}}_{\{(k_1,\ldots,k_n)\}})(\omega) \\
&= P_t(Y_{k_1} \cdots Y_{k_n})(\omega).
\end{aligned}$$

□

Consider the Ω-valued stationary process

$$(X(t))_{t \in \mathbb{R}_+} = ((X_k(t))_{k \in \mathbb{N}})_{t \in \mathbb{R}_+}$$

with independent components and distribution given by

$$\mathbb{P}(X_k(t) = 1 \mid X_k(0) = 1) = p_k + e^{-t} q_k, \tag{1.9.2}$$

$$\mathbb{P}(X_k(t) = -1 \mid X_k(0) = 1) = q_k - e^{-t} q_k, \tag{1.9.3}$$

$$\mathbb{P}(X_k(t) = 1 \mid X_k(0) = -1) = p_k - e^{-t} p_k, \tag{1.9.4}$$

$$\mathbb{P}(X_k(t) = -1 \mid X_k(0) = -1) = q_k + e^{-t} p_k, \tag{1.9.5}$$

$k \in \mathbb{N}$, $t \in \mathbb{R}_+$.

Proposition 1.9.3. *The process* $(X(t))_{t\in\mathbb{R}_+} = ((X_k(t))_{k\in\mathbb{N}})_{t\in\mathbb{R}_+}$ *is the Ornstein-Uhlenbeck process associated to* $(P_t)_{t\in\mathbb{R}_+}$*, i.e. we have*

$$P_tF = \mathbb{E}\left[F(X(t)) \mid X(0)\right], \qquad t \in \mathbb{R}_+, \tag{1.9.6}$$

for F *bounded and* $\mathcal{F}_n$*-measurable on* Ω*,* $n \in \mathbb{N}$*.*

Proof. By construction of $(X(t))_{t\in\mathbb{R}_+}$ in Relations (1.9.2)-(1.9.5) we have

$$\mathbb{P}(X_k(t) = 1 \mid X_k(0)) = p_k\left(1 + \mathrm{e}^{-t}Y_k(0)\sqrt{\frac{q_k}{p_k}}\right),$$

$$\mathbb{P}(X_k(t) = -1 \mid X_k(0)) = q_k\left(1 - \mathrm{e}^{-t}Y_k(0)\sqrt{\frac{p_k}{q_k}}\right),$$

where $Y_k(0)$ is defined by (1.4.6), i.e.

$$Y_k(0) = \frac{q_k - p_k + X_k(0)}{2\sqrt{p_kq_k}}, \qquad k \in \mathbb{N},$$

thus

$$d\mathbb{P}(X_k(t)(\tilde{\omega}) = \epsilon \mid X(0))(\omega) = \left(1 + \mathrm{e}^{-t}Y_k(\omega)Y_k(\tilde{\omega})\right) d\mathbb{P}(X_k(\tilde{\omega}) = \epsilon),$$

$\varepsilon = \pm 1$. Since the components of $(X_k(t))_{k\in\mathbb{N}}$ are independent, this shows that the law of $(X_0(t), \ldots, X_n(t))$ conditionally to $X(0)$ has the density $q_t^n(\tilde{\omega}, \cdot)$ with respect to $\mathbb{P}$:

$$\begin{aligned} &d\mathbb{P}(X_0(t)(\tilde{\omega}) = \epsilon_0, \ldots, X_n(t)(\tilde{\omega}) = \epsilon_n \mid X(0))(\tilde{\omega}) \\ &\qquad = q_t^n(\tilde{\omega}, \omega)d\mathbb{P}(X_0(\tilde{\omega}) = \epsilon_0, \ldots, X_n(\tilde{\omega}) = \epsilon_n). \end{aligned}$$

Consequently we have

$$\mathbb{E}\left[F(X(t)) \mid X(0) = \tilde{\omega}\right] = \int_\Omega F(\omega)q_t^N(\tilde{\omega}, \omega)\mathbb{P}(d\omega), \tag{1.9.7}$$

hence from (1.9.1), Relation (1.9.6) holds for $F \in L^2(\Omega, \mathcal{F}_N)$, $N \geq 0$. □

The independent components $X_k(t)$, $k \in \mathbb{N}$, can be constructed from the data of $X_k(0) = \epsilon$ and an independent exponential random variable τ_k via the following procedure. If $\tau_k > t$, let $X_k(t) = X_k(0) = \epsilon$, otherwise if $\tau_k < t$, take $X_k(t)$ to be an independent copy of $X_k(0)$. This procedure is illustrated in the following equalities:

$$\begin{aligned} \mathbb{P}(X_k(t) = 1 \mid X_k(0) = 1) &= \mathbb{E}\left[\mathbf{1}_{\{\tau_k > t\}}\right] + \mathbb{E}\left[\mathbf{1}_{\{\tau_k < t\}}\mathbf{1}_{\{X_k = 1\}}\right] \\ &= \mathrm{e}^{-t} + p_k(1 - \mathrm{e}^{-t}), \end{aligned} \tag{1.9.8}$$

$$\mathbb{P}(X_k(t) = -1 \mid X_k(0) = 1) = \mathbb{E}\left[\mathbf{1}_{\{\tau_k < t\}}\mathbf{1}_{\{X_k = -1\}}\right] \\ = q_k(1 - \mathrm{e}^{-t}), \tag{1.9.9}$$

$$\mathbb{P}(X_k(t) = -1 \mid X_k(0) = -1) = \mathbb{E}\left[\mathbf{1}_{\{\tau_k > t\}}\right] + \mathbb{E}\left[\mathbf{1}_{\{\tau_k < t\}}\mathbf{1}_{\{X_k = -1\}}\right] \\ = \mathrm{e}^{-t} + q_k(1 - \mathrm{e}^{-t}), \tag{1.9.10}$$

$$\mathbb{P}(X_k(t) = 1 \mid X_k(0) = -1) = \mathbb{E}\left[\mathbf{1}_{\{\tau_k < t\}}\mathbf{1}_{\{X_k = 1\}}\right] \\ = p_k(1 - \mathrm{e}^{-t}). \tag{1.9.11}$$

The operator $L^2(\Omega \times \mathbb{N}) \to L^2(\Omega \times \mathbb{N})$ which maps $(u_k)_{k\in\mathbb{N}}$ to $(P_t u_k)_{k\in\mathbb{N}}$ is also denoted by P_t. As a consequence of the representation of P_t given in Lemma 1.9.2 we obtain the following bound.

Lemma 1.9.4. *For $F \in \mathrm{Dom}\,(D)$ we have*

$$\|P_t u\|_{L^\infty(\Omega,\ell^2(\mathbb{N}))} \leq \|u\|_{L^\infty(\Omega,\ell^2(\mathbb{N}))}, \qquad t \in \mathbb{R}_+, \quad u \in L^2(\Omega \times \mathbb{N}).$$

Proof. As a consequence of the representation formula (1.9.7) we have $\mathbb{P}(d\tilde{\omega})$-a.s.:

$$\begin{aligned} \|P_t u\|^2_{\ell^2(\mathbb{N})}(\tilde{\omega}) &= \sum_{k=0}^\infty |P_t u_k(\tilde{\omega})|^2 \\ &= \sum_{k=0}^\infty \left(\int_\Omega u_k(\omega) Q_t(\tilde{\omega}, d\omega)\right)^2 \\ &\leq \sum_{k=0}^\infty \int_\Omega |u_k(\omega)|^2 Q_t(\tilde{\omega}, d\omega) \\ &= \int_\Omega \|u\|^2_{\ell^2(\mathbb{N})}(\omega) Q_t(\tilde{\omega}, d\omega) \\ &\leq \|u\|^2_{L^\infty(\Omega,\ell^2(\mathbb{N}))}. \end{aligned}$$

□

1.10 Covariance Identities

In this section we state the covariance identities which will be used for the proof of deviation inequalities in the next section. The covariance $\mathrm{Cov}\,(F, G)$ of $F, G \in L^2(\Omega)$ is defined as

$$\begin{aligned} \mathrm{Cov}\,(F, G) &= \mathbb{E}\,[(F - \mathbb{E}\,[F])(G - \mathbb{E}\,[G])] \\ &= \mathbb{E}\,[FG] - \mathbb{E}\,[F]\,\mathbb{E}\,[G]\,. \end{aligned}$$

Proposition 1.10.1. *For all $F, G \in L^2(\Omega)$ such that $\mathbb{E}\left[\|DF\|^2_{\ell^2(\mathbb{N})}\right] < \infty$ we have*

$$\mathrm{Cov}\,(F,G) = \mathbb{E}\left[\sum_{k=0}^{\infty} \mathbb{E}\,[D_k G \mid \mathcal{F}_{k-1}]\, D_k F\right]. \qquad (1.10.1)$$

Proof. This identity is a consequence of the Clark formula (1.7.1):

$$\begin{aligned}
\mathrm{Cov}\,(F,G) &= \mathbb{E}\,[(F-\mathbb{E}\,[F])(G-\mathbb{E}\,[G])] \\
&= \mathbb{E}\left[\left(\sum_{k=0}^{\infty} \mathbb{E}\,[D_k F \mid \mathcal{F}_{k-1}]\, Y_k\right)\left(\sum_{l=0}^{\infty} \mathbb{E}\,[D_l G \mid \mathcal{F}_{l-1}]\, Y_l\right)\right] \\
&= \mathbb{E}\left[\sum_{k=0}^{\infty} \mathbb{E}\,[D_k F \mid \mathcal{F}_{k-1}]\, \mathbb{E}\,[D_k G \mid \mathcal{F}_{k-1}]\right] \\
&= \sum_{k=0}^{\infty} \mathbb{E}\,[\mathbb{E}\,[\mathbb{E}\,[D_k G \mid \mathcal{F}_{k-1}]\, D_k F \mid \mathcal{F}_{k-1}]] \\
&= \mathbb{E}\left[\sum_{k=0}^{\infty} \mathbb{E}\,[D_k G \mid \mathcal{F}_{k-1}]\, D_k F\right],
\end{aligned}$$

and of its extension to $G \in L^2(\Omega)$ in Corollary 1.7.3. □

A covariance identity can also be obtained using the semi-group $(P_t)_{t\in\mathbb{R}_+}$.

Proposition 1.10.2. *For any $F, G \in L^2(\Omega)$ such that*

$$\mathbb{E}\left[\|DF\|^2_{\ell^2(\mathbb{N})}\right] < \infty \qquad \textit{and} \qquad \mathbb{E}\left[\|DG\|^2_{\ell^2(\mathbb{N})}\right] < \infty,$$

we have

$$\mathrm{Cov}\,(F,G) = \mathbb{E}\left[\sum_{k=0}^{\infty} \int_0^{\infty} \mathrm{e}^{-t} (D_k F) P_t D_k G dt\right]. \qquad (1.10.2)$$

Proof. Consider $F = J_n(f_n)$ and $G = J_m(g_m)$. We have

$$\begin{aligned}
&\mathrm{Cov}\,(J_n(f_n), J_m(g_m)) = \mathbb{E}\,[J_n(f_n) J_m(g_m)] \\
&= \mathbf{1}_{\{n=m\}} n! \langle f_n, g_n \mathbf{1}_{\Delta_n}\rangle_{\ell^2(\mathbb{N}^n)} \\
&= \mathbf{1}_{\{n=m\}} n! n \int_0^{\infty} \mathrm{e}^{-nt} dt \langle f_n, g_n \mathbf{1}_{\Delta_n}\rangle_{\ell^2(\mathbb{N}^n)} \\
&= \mathbf{1}_{\{n-1=m-1\}} n! n \int_0^{\infty} \mathrm{e}^{-t} \sum_{k=0}^{\infty} \langle f_n(*,k), \mathrm{e}^{-(n-1)t} g_n(*,k) \mathbf{1}_{\Delta_n}(*,k)\rangle_{\ell^2(\mathbb{N}^{n-1})} dt \\
&= nm \mathbb{E}\left[\int_0^{\infty} \mathrm{e}^{-t} \sum_{k=0}^{\infty} J_{n-1}(f_n(*,k)\mathbf{1}_{\Delta_n}(*,k)) \mathrm{e}^{-(m-1)t} J_{m-1}(g_m(*,k)\mathbf{1}_{\Delta_m}(*,k)) dt\right] \\
&= nm \mathbb{E}\left[\int_0^{\infty} \mathrm{e}^{-t} \sum_{k=0}^{\infty} J_{n-1}(f_n(*,k)\mathbf{1}_{\Delta_n}(*,k)) P_t J_{m-1}(g_m(*,k)\mathbf{1}_{\Delta_m}(*,k)) dt\right]
\end{aligned}$$

$$= \mathbb{E}\left[\int_0^\infty e^{-t} \sum_{k=0}^\infty D_k J_n(f_n) P_t D_k J_m(g_m) dt\right].$$

□

By the relations (1.9.8)-(1.9.11) the covariance identity (1.10.2) shows that

$$\begin{aligned}
\mathrm{Cov}\,(F,G) &= \mathbb{E}\left[\sum_{k=0}^\infty \int_0^\infty e^{-t} D_k F P_t D_k G dt\right] \\
&= \mathbb{E}\left[\int_0^1 \sum_{k=0}^\infty D_k F P_{(-\log\alpha)} D_k G d\alpha\right] \\
&= \int_0^1 \int_{\Omega\times\Omega} \sum_{k=0}^\infty D_k F(\omega) D_k G((\omega_i \mathbf{1}_{\{\tau_i<-\log\alpha\}} + \omega_i' \mathbf{1}_{\{\tau_i<-\log\alpha\}})_{i\in\mathbb{N}}) d\alpha \mathbb{P}(d\omega)\mathbb{P}(d\omega') \\
&= \int_0^1 \int_{\Omega\times\Omega} \sum_{k=0}^\infty D_k F(\omega) D_k G((\omega_i \mathbf{1}_{\{\xi_i<\alpha\}} + \omega_i' \mathbf{1}_{\{\xi_i>\alpha\}})_{i\in\mathbb{N}}) \mathbb{P}(d\omega)\mathbb{P}(d\omega') d\alpha,
\end{aligned} \tag{1.10.3}$$

where $(\xi_i)_{i\in\mathbb{N}}$ is a family of independent identically distributed (i.i.d.) random variables, uniformly distributed on $[0,1]$. Note that the marginals of $(X_k, X_k \mathbf{1}_{\{\xi_k<\alpha\}} + X_k' \mathbf{1}_{\{\xi_i>\alpha\}})$ are identical when X_k' is an independent copy of X_k. Letting

$$\phi_\alpha(s,t) = \mathbb{E}\left[e^{isX_k} e^{it(X_k+\mathbf{1}_{\{\xi_k<\alpha\}})+it(X_k'+\mathbf{1}_{\{\xi_k>\alpha\}})}\right],$$

we have the relation

$$\begin{aligned}
\mathrm{Cov}\,(e^{isX_k}, e^{itX_k}) &= \phi_1(s,t) - \phi_0(s,t) \\
&= \int_0^1 \frac{d\phi_\alpha}{d\alpha}(s,t) d\alpha.
\end{aligned}$$

Next we prove an iterated version of the covariance identity in discrete time, which is an analog of a result proved in [58] for the Wiener and Poisson processes.

Theorem 1.10.3. *Let $n \in \mathbb{N}$ and $F, G \in L^2(\Omega)$. We have*

$$\begin{aligned}
&\mathrm{Cov}\,(F,G) \\
&= \sum_{d=1}^n (-1)^{d+1} \mathbb{E}\left[\sum_{\{1\le k_1<\cdots<k_d\}} (D_{k_d}\cdots D_{k_1}F)(D_{k_d}\cdots D_{k_1}G)\right] \\
&\quad + (-1)^n \mathbb{E}\left[\sum_{\{1\le k_1<\cdots<k_{n+1}\}} (D_{k_{n+1}}\cdots D_{k_1}F)\mathbb{E}\left[D_{k_{n+1}}\cdots D_{k_1}G \mid \mathcal{F}_{k_{n+1}-1}\right]\right].
\end{aligned} \tag{1.10.4}$$

Proof. Take $F = G$. For $n = 0$, (1.10.4) is a consequence of the Clark formula. Let $n \geq 1$. Applying Lemma 1.7.4 to $D_{k_n} \cdots D_{k_1} F$ with $a = k_n$ and $b = k_{n+1}$, and summing on $(k_1, \ldots, k_n) \in \Delta_n$, we obtain

$$\mathbb{E}\left[\sum_{\{1 \leq k_1 < \cdots < k_n\}} (\mathbb{E}[D_{k_n} \cdots D_{k_1} F \mid \mathcal{F}_{k_n - 1}])^2\right]$$
$$= \mathbb{E}\left[\sum_{\{1 \leq k_1 < \cdots < k_n\}} | D_{k_n} \cdots D_{k_1} F|^2\right]$$
$$-\mathbb{E}\left[\sum_{\{1 \leq k_1 < \cdots < k_{n+1}\}} \left(\mathbb{E}\left[D_{k_{n+1}} \cdots D_{k_1} F \mid \mathcal{F}_{k_{n+1}-1}\right]\right)^2\right],$$

which concludes the proof by induction and polarization. □

As a consequence of Theorem 1.10.3, letting $F = G$ we get the variance inequality

$$\sum_{k=1}^{2n} \frac{(-1)^{k+1}}{k!} \mathbb{E}\left[\|D^k F\|^2_{\ell^2(\Delta_k)}\right] \leq \operatorname{Var}(F) \leq \sum_{k=1}^{2n-1} \frac{(-1)^{k+1}}{k!} \mathbb{E}\left[\|D^k F\|^2_{\ell^2(\Delta_k)}\right],$$

since

$$\mathbb{E}\left[\sum_{\{1 \leq k_1 < \cdots < k_{n+1}\}} (D_{k_{n+1}} \cdots D_{k_1} F) \mathbb{E}\left[D_{k_{n+1}} \cdots D_{k_1} F \mid \mathcal{F}_{k_{n+1}-1}\right]\right]$$
$$= \mathbb{E}\Bigg[\sum_{\{1 \leq k_1 < \cdots < k_{n+1}\}}$$
$$\mathbb{E}\left[(D_{k_{n+1}} \cdots D_{k_1} F) \mathbb{E}\left[D_{k_{n+1}} \cdots D_{k_1} F \mid \mathcal{F}_{k_{n+1}-1}\right] \mid \mathcal{F}_{k_{n+1}-1}\right]\Bigg]$$
$$= \mathbb{E}\left[\sum_{\{1 \leq k_1 < \cdots < k_{n+1}\}} (\mathbb{E}\left[D_{k_{n+1}} \cdots D_{k_1} F \mid \mathcal{F}_{k_{n+1}-1}\right])^2\right]$$
$$\geq 0,$$

see Relation (2.15) in [58] in continuous time. In a similar way, another iterated covariance identity can be obtained from Proposition 1.10.2.

Corollary 1.10.4. *Let $n \in \mathbb{N}$ and $F, G \in L^2(\Omega, \mathcal{F}_N)$. We have*

$$\operatorname{Cov}(F, G) = \sum_{d=1}^{n} (-1)^{d+1} \mathbb{E}\left[\sum_{\{1 \leq k_1 < \cdots < k_d \leq N\}} (D_{k_d} \cdots D_{k_1} F)(D_{k_d} \cdots D_{k_1} G)\right]$$

$$+(-1)^n \int_{\Omega\times\Omega} \sum_{\{1\le k_1<\cdots<k_{n+1}\le N\}} D_{k_{n+1}}\cdots D_{k_1}F(\omega)D_{k_{n+1}}\cdots D_{k_1}G(\omega')$$
$$q_t^N(\omega,\omega')\mathbb{P}(d\omega)\mathbb{P}(d\omega'). \qquad (1.10.5)$$

Using the tensorization property

$$\begin{aligned}\mathrm{Var}\,(FG) &= \mathbb{E}\left[F^2\right]\mathrm{Var}\,(G) + (\mathbb{E}\,[G])^2\mathrm{Var}\,(F)\\ &\le \mathbb{E}\left[F^2\mathrm{Var}\,(G)\right] + \mathbb{E}\left[G^2\mathrm{Var}\,(F)\right]\end{aligned}$$

of the variance for independent random variable F, G, most of the identities in this section can be obtained by tensorization of elementary one dimensional covariance identities.

The following lemma is an elementary consequence of the covariance identity proved in Proposition 1.10.1.

Lemma 1.10.5. *Let $F, G \in L^2(\Omega)$ such that*

$$\mathbb{E}\,[D_kF|\mathcal{F}_{k-1}]\cdot\mathbb{E}\,[D_kG|\mathcal{F}_{k-1}] \ge 0, \qquad k\in\mathbb{N}.$$

Then F and G are non-negatively correlated:

$$\mathrm{Cov}\,(F,G)\ge 0.$$

According to the next definition, a non-decreasing functional F satisfies $D_kF\ge 0$ for all $k\in\mathbb{N}$.

Definition 1.10.6. *A random variable $F:\Omega\to\mathbb{R}$ is said to be non-decreasing if for all $\omega_1,\omega_2\in\Omega$ we have*

$$\omega_1(k)\le\omega_2(k), \qquad k\in\mathbb{N}, \qquad \Rightarrow\ F(\omega_1)\le F(\omega_2).$$

The following result is then immediate from Proposition 1.6.2 and Lemma 1.10.5, and shows that the FKG inequality holds on Ω. It can also be obtained from Proposition 1.10.2.

Proposition 1.10.7. *If $F, G\in L^2(\Omega)$ are non-decreasing then F and G are non-negatively correlated:*

$$\mathrm{Cov}\,(F,G)\ge 0.$$

Note however that the assumptions of Lemma 1.10.5 are actually weaker as they do not require F and G to be non-decreasing.

1.11 Deviation Inequalities

In this section, which is based on [61], we recover a deviation inequality of [20] in the case of Bernoulli measures, using covariance representations instead of the logarithmic Sobolev inequalities to be presented in Section 1.12. The method relies on a bound on the Laplace transform $L(t) = \mathbb{E}\left[e^{tF}\right]$ obtained via a differential inequality and Chebychev's inequality.

Proposition 1.11.1. *Let $F \in L^1(\Omega)$ be such that $|F_k^+ - F_k^-| \leq K$, $k \in \mathbb{N}$, for some $K \geq 0$, and $\|DF\|_{L^\infty(\Omega,\ell^2(\mathbb{N}))} < \infty$. Then*

$$\begin{aligned}\mathbb{P}(F - \mathbb{E}[F] \geq x) &\leq \exp\left(-\frac{\|DF\|^2_{L^\infty(\Omega,\ell^2(\mathbb{N}))}}{K^2} g\left(\frac{xK}{\|DF\|^2_{L^\infty(\Omega,\ell^2(\mathbb{N}))}}\right)\right) \\ &\leq \exp\left(-\frac{x}{2K}\log\left(1 + \frac{xK}{\|DF\|^2_{L^\infty(\Omega,\ell^2(\mathbb{N}))}}\right)\right),\end{aligned}$$

with $g(u) = (1+u)\log(1+u) - u$, $u \geq 0$.

Proof. Although D_k does not satisfy a derivation rule for products, from Proposition 1.6.4 we have

$$\begin{aligned} D_k e^F &= \mathbf{1}_{\{X_k=1\}}\sqrt{p_k q_k}(e^F - e^{F_k^-}) + \mathbf{1}_{\{X_k=-1\}}\sqrt{p_k q_k}(e^{F_k^+} - e^F) \\ &= \mathbf{1}_{\{X_k=1\}}\sqrt{p_k q_k}e^F(1 - e^{-\frac{1}{\sqrt{p_k q_k}}D_k F}) + \mathbf{1}_{\{X_k=-1\}}\sqrt{p_k q_k}e^F(e^{\frac{1}{\sqrt{p_k q_k}}D_k F} - 1) \\ &= -X_k\sqrt{p_k q_k}e^F(e^{-\frac{X_k}{\sqrt{p_k q_k}}D_k F} - 1), \end{aligned}$$

hence

$$D_k e^F = X_k\sqrt{p_k q_k}e^F(1 - e^{-\frac{X_k}{\sqrt{p_k q_k}}D_k F}), \tag{1.11.1}$$

and since the function $x \longmapsto (e^x - 1)/x$ is positive and increasing on $\mathbb{R}$ we have:

$$\begin{aligned}\frac{e^{-sF}D_k e^{sF}}{D_k F} &= -\frac{X_k\sqrt{p_k q_k}}{D_k F}\left(e^{-s\frac{X_k}{\sqrt{p_k q_k}}D_k F} - 1\right) \\ &\leq \frac{e^{sK} - 1}{K},\end{aligned}$$

or in other terms:

$$\begin{aligned}\frac{e^{-sF}D_k e^{sF}}{D_k F} &= \mathbf{1}_{\{X_k=1\}}\frac{e^{s(F_k^- - F_k^+)} - 1}{F_k^- - F_k^+} + \mathbf{1}_{\{X_k=-1\}}\frac{e^{s(F_k^+ - F_k^-)} - 1}{F_k^+ - F_k^-} \\ &\leq \frac{e^{sK} - 1}{K}.\end{aligned}$$

We first assume that F is a bounded random variable with $\mathbb{E}[F] = 0$. From Proposition 1.10.2 applied to F and e^{sF}, noting that since F is bounded,

$$\mathbb{E}\left[\|De^{sF}\|_{\ell^2(\mathbb{N})}^2\right] \leq C_K \mathbb{E}\left[e^{2sF}\right] \|DF\|_{L^\infty(\Omega,\ell^2(\mathbb{N}))}^2 < \infty,$$

for some $C_K > 0$, we have

$$\begin{aligned}
\mathbb{E}\left[F\mathrm{e}^{sF}\right] &= \mathrm{Cov}\left(F, \mathrm{e}^{sF}\right) \\
&= \mathbb{E}\left[\int_0^\infty \mathrm{e}^{-v} \sum_{k=0}^\infty D_k \mathrm{e}^{sF} P_v D_k F dv\right] \\
&\leq \left\|\frac{\mathrm{e}^{-sF} D\mathrm{e}^{sF}}{DF}\right\|_\infty \mathbb{E}\left[\mathrm{e}^{sF} \int_0^\infty \mathrm{e}^{-v} \|DFP_v DF\|_{\ell^1(\mathbb{N})} dv\right] \\
&\leq \frac{\mathrm{e}^{sK}-1}{K} \mathbb{E}\left[\mathrm{e}^{sF} \|DF\|_{\ell^2(\mathbb{N})} \int_0^\infty \mathrm{e}^{-v} \|P_v DF\|_{\ell^2(\mathbb{N})} dv\right] \\
&\leq \frac{\mathrm{e}^{sK}-1}{K} \mathbb{E}\left[\mathrm{e}^{sF}\right] \|DF\|_{L^\infty(\Omega,\ell^2(\mathbb{N}))}^2 \int_0^\infty \mathrm{e}^{-v} dv \\
&\leq \frac{\mathrm{e}^{sK}-1}{K} \mathbb{E}\left[\mathrm{e}^{sF}\right] \|DF\|_{L^\infty(\Omega,\ell^2(\mathbb{N}))}^2,
\end{aligned}$$

where we also applied Lemma 1.9.4 to $u = DF$.

In the general case, letting $L(s) = \mathbb{E}\left[\mathrm{e}^{s(F-\mathbb{E}[F])}\right]$, we have

$$\begin{aligned}
\log(\mathbb{E}\left[\mathrm{e}^{t(F-\mathbb{E}[F])}\right]) &= \int_0^t \frac{L'(s)}{L(s)} ds \\
&\leq \int_0^t \frac{\mathbb{E}\left[(F-\mathbb{E}[F])\mathrm{e}^{s(F-\mathbb{E}[F])}\right]}{\mathbb{E}\left[\mathrm{e}^{s(F-\mathbb{E}[F])}\right]} ds \\
&\leq \frac{1}{K} \|DF\|_{L^\infty(\Omega,\ell^2(\mathbb{N}))}^2 \int_0^t (\mathrm{e}^{sK}-1) ds \\
&= \frac{1}{K^2} (\mathrm{e}^{tK} - tK - 1) \|DF\|_{L^\infty(\Omega,\ell^2(\mathbb{N}))}^2,
\end{aligned}$$

$t \in \mathbb{R}_+$. We have for all $x \geq 0$ and $t \in \mathbb{R}_+$:

$$\begin{aligned}
\mathbb{P}(F - \mathbb{E}[F] \geq x) &\leq \mathrm{e}^{-tx} \mathbb{E}\left[\mathrm{e}^{t(F-\mathbb{E}[F])}\right] \\
&\leq \exp\left(\frac{1}{K^2}(\mathrm{e}^{tK} - tK - 1)\|DF\|_{L^\infty(\Omega,\ell^2(\mathbb{N}))}^2 - tx\right),
\end{aligned}$$

The minimum in $t \in \mathbb{R}_+$ in the above expression is attained with

$$t = \frac{1}{K} \log \left(1 + \frac{xK}{\|DF\|^2_{L^\infty(\Omega,\ell^2(\mathbb{N}))}} \right),$$

hence

$$\begin{aligned}
&\mathbb{P}(F - \mathbb{E}[F] \geq x) \\
&\leq \exp\left(-\frac{1}{K} \left(x + \frac{\|DF\|^2_{L^\infty(\Omega,\ell^2(\mathbb{N}))}}{K} \right) \log \left(1 + \frac{Kx}{\|DF\|^2_{L^\infty(\Omega,\ell^2(\mathbb{N}))}} \right) - \frac{x}{K} \right) \\
&\leq \exp\left(-\frac{x}{2K} \log\left(1 + xK \|DF\|^{-2}_{L^\infty(\Omega,\ell^2(\mathbb{N}))} \right) \right),
\end{aligned}$$

where we used the inequality

$$\frac{u}{2}\log(1+u) \leq (1+u)\log(1+u) - u, \qquad u \in \mathbb{R}_+.$$

If $K = 0$, the above proof is still valid by replacing all terms by their limits as $K \to 0$. Finally if F is not bounded the conclusion holds for $F_n = \max(-n, \min(F, n))$, $n \geq 1$, and $(F_n)_{n\in\mathbb{N}}$, $(DF_n)_{n\in\mathbb{N}}$, converge respectively almost surely and in $L^2(\Omega\times\mathbb{N})$ to F and DF, with $\|DF_n\|^2_{L^\infty(\Omega,L^2(\mathbb{N}))} \leq \|DF\|^2_{L^\infty(\Omega,L^2(\mathbb{N}))}$. □

In case $p_k = p$ for all $k \in \mathbb{N}$, the conditions

$$|D_k F| \leq \beta, \quad k \in \mathbb{N}, \quad \text{and} \quad \|DF\|^2_{L^\infty(\Omega,\ell^2(\mathbb{N}))} \leq \alpha^2,$$

give

$$\begin{aligned}
\mathbb{P}(F - \mathbb{E}[F] \geq x) &\leq \exp\left(-\frac{\alpha^2 pq}{\beta^2} g\left(\frac{x\beta}{\alpha^2\sqrt{pq}} \right) \right) \\
&\leq \exp\left(-\frac{x\sqrt{pq}}{2\beta} \log\left(1 + \frac{x\beta}{\alpha^2\sqrt{pq}} \right) \right),
\end{aligned}$$

which is Relation (13) in [20]. In particular if F is $\mathcal{F}_N$-measurable, then

$$\begin{aligned}
\mathbb{P}(F - \mathbb{E}[F] \geq x) &\leq \exp\left(-N g\left(\frac{x\sqrt{pq}}{\beta N} \right) \right) \\
&\leq \exp\left(-\frac{x\sqrt{pq}}{\beta} \left(\log\left(1 + \frac{x\sqrt{pq}}{\beta N} \right) - 1 \right) \right).
\end{aligned}$$

Finally we show a Gaussian concentration inequality for functionals of $(S_n)_{n\in\mathbb{N}}$, using the covariance identity (1.10.1). We refer to [18], [19], [63], [79], for other versions of this inequality.

Proposition 1.11.2. *Let $F \in L^1(\Omega)$ be such that*

$$\left\| \sum_{k=0}^{\infty} \frac{1}{2(p_k \wedge q_k)} |D_k F| \|D_k F\|_\infty \right\|_\infty \leq K^2.$$

Then

$$\mathbb{P}(F - \mathbb{E}[F] \geq x) \leq \exp\left(-\frac{x^2}{2K^2}\right), \qquad x \geq 0. \tag{1.11.2}$$

Proof. Again, we assume that F is a bounded random variable with $\mathbb{E}[F] = 0$. Using the inequality

$$|\mathrm{e}^{tx} - \mathrm{e}^{ty}| \leq \frac{t}{2}|x - y|(\mathrm{e}^{tx} + \mathrm{e}^{ty}), \qquad x, y \in \mathbb{R}, \tag{1.11.3}$$

we have, replacing the lack of chain rule of derivation for D_k by an upper bound,

$$\begin{aligned}
|D_k \mathrm{e}^{tF}| &= \sqrt{p_k q_k} |\mathrm{e}^{tF_k^+} - \mathrm{e}^{tF_k^-}| \\
&\leq \frac{1}{2}\sqrt{p_k q_k}\, t |F_k^+ - F_k^-| (\mathrm{e}^{tF_k^+} + \mathrm{e}^{tF_k^-}) \\
&= \frac{1}{2} t |D_k F| (\mathrm{e}^{tF_k^+} + \mathrm{e}^{tF_k^-}) \\
&\leq \frac{t}{2(p_k \wedge q_k)} |D_k F| \mathbb{E}\left[\mathrm{e}^{tF} \mid X_i,\ i \neq k\right] \qquad (1.11.4) \\
&= \frac{1}{2(p_k \wedge q_k)} t \mathbb{E}\left[\mathrm{e}^{tF} |D_k F| \mid X_i,\ i \neq k\right],
\end{aligned}$$

where in (1.11.4) the inequality is due to the absence of chain rule of derivation for the operator D_k. Now, Proposition 1.10.1 yields

$$\begin{aligned}
\mathbb{E}\left[F \mathrm{e}^{tF}\right] &= \mathrm{Cov}\,(F, \mathrm{e}^{sF}) \\
&= \sum_{k=0}^{\infty} \mathbb{E}\left[\mathbb{E}[D_k F \mid \mathcal{F}_{k-1}]\, D_k \mathrm{e}^{tF}\right] \\
&\leq \sum_{k=0}^{\infty} \|D_k F\|_\infty \mathbb{E}\left[|D_k \mathrm{e}^{tF}|\right] \\
&\leq \frac{t}{2} \sum_{k=0}^{\infty} \frac{1}{p_k \wedge q_k} \|D_k F\|_\infty \mathbb{E}\left[\mathbb{E}\left[\mathrm{e}^{tF} |D_k F| \mid X_i,\ i \neq k\right]\right] \\
&= \frac{t}{2} \mathbb{E}\left[\mathrm{e}^{tF} \sum_{k=0}^{\infty} \frac{1}{p_k \wedge q_k} \|D_k F\|_\infty |D_k F|\right] \\
&\leq \frac{t}{2} \mathbb{E}\left[\mathrm{e}^{tF}\right] \left\| \sum_{k=0}^{\infty} \frac{1}{p_k \wedge q_k} |D_k F| \|D_k F\|_\infty \right\|_\infty.
\end{aligned}$$

This shows that

$$\log(\mathbb{E}\left[e^{t(F-\mathbb{E}[F])}\right]) = \int_0^t \frac{\mathbb{E}\left[(F-\mathbb{E}[F])e^{s(F-\mathbb{E}[F])}\right]}{\mathbb{E}\left[e^{s(F-\mathbb{E}[F])}\right]} ds$$
$$\leq K^2 \int_0^t s ds$$
$$= \frac{t^2}{2} K^2,$$

hence

$$e^x \mathbb{P}(F-\mathbb{E}[F] \geq x) \leq \mathbb{E}\left[e^{t(F-\mathbb{E}[F])}\right]$$
$$\leq e^{t^2K^2/2}, \qquad t \in \mathbb{R}_+,$$

and

$$\mathbb{P}(F-\mathbb{E}[F] \geq x) \leq e^{\frac{t^2}{2}K^2 - tx}, \quad t \in \mathbb{R}_+.$$

The best inequality is obtained for $t = x/K^2$.

Finally if F is not bounded the conclusion holds for $F_n = \max(-n, \min(F, n))$, $n \geq 0$, and $(F_n)_{n\in\mathbb{N}}$, $(DF_n)_{n\in\mathbb{N}}$, converge respectively to F and DF in $L^2(\Omega)$, resp. $L^2(\Omega \times \mathbb{N})$, with $\|DF_n\|^2_{L^\infty(\Omega,\ell^2(\mathbb{N}))} \leq \|DF\|^2_{L^\infty(\Omega,\ell^2(\mathbb{N}))}$. □

In case $p_k = p$, $k \in \mathbb{N}$, we obtain

$$\mathbb{P}(F-\mathbb{E}[F] \geq x) \leq \exp\left(-\frac{px^2}{\|DF\|^2_{\ell^2(\mathbb{N},L^\infty(\Omega))}}\right).$$

Proposition 1.11.3. *We have* $\mathbb{E}\left[e^{\alpha|F|}\right] < \infty$ *for all* $\alpha > 0$, *and* $\mathbb{E}\left[e^{\alpha F^2}\right] < \infty$ *for all* $\alpha < 1/(2K^2)$.

Proof. Let $\lambda < c/e$. The bound (1.11.2) implies

$$\mathbb{E}\left[e^{\alpha|F|}\right] = \int_0^\infty \mathbb{P}(e^{\alpha|F|} \geq t) dt$$
$$= \int_{-\infty}^\infty \mathbb{P}(\alpha|F| \geq y) e^y dy$$
$$\leq 1 + \int_0^\infty \mathbb{P}(\alpha|F| \geq y) e^y dy$$
$$\leq 1 + \int_0^\infty \exp\left(-\frac{(\mathbb{E}[|F|] + y/\alpha)^2}{2K^2}\right) e^y dy$$
$$< \infty,$$

for all $\alpha > 0$. On the other hand we have

$$\mathbb{E}\left[e^{\alpha F^2}\right] = \int_0^\infty \mathbb{P}(e^{\alpha F^2} \geq t) dt$$
$$= \int_{-\infty}^\infty \mathbb{P}(\alpha F^2 \geq y) e^y dy$$

$$\leq 1 + \int_0^\infty \mathbb{P}(|F| \geq (y/\alpha)^{1/2}) e^y dy$$
$$\leq 1 + \int_0^\infty \exp\left(-\frac{(\mathbb{E}[|F|] + (y/\alpha)^{1/2})^2}{2K^2} \right) e^y dy$$
$$< \infty,$$

provided $2K^2\alpha < 1$. □

1.12 Logarithmic Sobolev Inequalities

The logarithmic Sobolev inequalities on Gaussian space provide an infinite dimensional analog of Sobolev inequalities, cf. e.g. [81]. On Riemannian path space [24] and on Poisson space [7], [155], martingale methods have been successfully applied to the proof of logarithmic Sobolev inequalities. Here, discrete time martingale methods are used along with the Clark predictable representation formula (1.7.1) as in [48], to provide a proof of logarithmic Sobolev inequalities for Bernoulli measures. Here we are only concerned with modified logarithmic Sobolev inequalities, and we refer to [131], Theorem 2.2.8 and references therein, for the standard version of the logarithmic Sobolev inequality on the hypercube under Bernoulli measures.

The entropy of a random variable $F > 0$ is defined by

$$\mathrm{Ent}\,[F] = \mathbb{E}[F \log F] - \mathbb{E}[F] \log \mathbb{E}[F],$$

for sufficiently integrable F.

Lemma 1.12.1. *The entropy has the tensorization property, i.e. if F, G are sufficiently integrable independent random variables we have*

$$\mathrm{Ent}\,[FG] = \mathbb{E}[F\mathrm{Ent}\,[G]] + \mathbb{E}[G\mathrm{Ent}\,[F]]. \tag{1.12.1}$$

Proof. We have

$$\begin{aligned}
\mathrm{Ent}\,[FG] &= \mathbb{E}[FG\log(FG)] - \mathbb{E}[FG]\log\mathbb{E}[FG] \\
&= \mathbb{E}[FG(\log F + \log G)] - \mathbb{E}[F]\,\mathbb{E}[G]\,(\log\mathbb{E}[F] + \log\mathbb{E}[G]) \\
&= \mathbb{E}[G]\,\mathbb{E}[F\log F] + \mathbb{E}[F]\,\mathbb{E}[G\log G)] - \mathbb{E}[F]\,\mathbb{E}[G]\,(\log\mathbb{E}[F] + \log\mathbb{E}[G]) \\
&= \mathbb{E}[F\mathrm{Ent}\,[G]] + \mathbb{E}[G\mathrm{Ent}\,[F]].
\end{aligned}$$

□

In the next proposition we recover the modified logarithmic Sobolev inequality of [20] using the Clark representation formula in discrete time.

Theorem 1.12.2. *Let* $F \in \mathrm{Dom}\,(D)$ *with* $F > \eta$ *a.s. for some* $\eta > 0$. *We have*

$$\mathrm{Ent}\,[F] \leq \mathbb{E}\left[\frac{1}{F}\|DF\|^2_{\ell^2(\mathbb{N})}\right]. \tag{1.12.2}$$

Proof. Assume that F is $\mathcal{F}_N$-measurable and let $M_n = \mathbb{E}[F \mid \mathcal{F}_n]$, $0 \leq n \leq N$. Using Corollary 1.6.3 and the Clark formula (1.7.1) we have

$$M_n = M_{-1} + \sum_{k=0}^{n} u_k Y_k, \qquad 0 \leq n \leq N,$$

with $u_k = \mathbb{E}[D_k F \mid \mathcal{F}_{k-1}]$, $0 \leq k \leq n \leq N$, and $M_{-1} = \mathbb{E}[F]$. Letting $f(x) = x \log x$ and using the bound

$$\begin{aligned} f(x+y) - f(x) &= y \log x + (x+y) \log\left(1 + \frac{y}{x}\right) \\ &\leq y(1 + \log x) + \frac{y^2}{x}, \end{aligned}$$

we have:

$$\begin{aligned} \mathrm{Ent}\,[F] &= \mathbb{E}[f(M_N)] - \mathbb{E}[f(M_{-1})] \\ &= \mathbb{E}\left[\sum_{k=0}^{N} f(M_k) - f(M_{k-1})\right] \\ &= \mathbb{E}\left[\sum_{k=0}^{N} f\left(M_{k-1} + Y_k u_k\right) - f(M_{k-1})\right] \\ &\leq \mathbb{E}\left[\sum_{k=0}^{N} Y_k u_k (1 + \log M_{k-1}) + \frac{Y_k^2 u_k^2}{M_{k-1}}\right] \\ &= \mathbb{E}\left[\sum_{k=0}^{N} \frac{1}{\mathbb{E}[F \mid \mathcal{F}_{k-1}]} (\mathbb{E}[D_k F \mid \mathcal{F}_{k-1}])^2\right] \\ &\leq \mathbb{E}\left[\sum_{k=0}^{N} \mathbb{E}\left[\frac{1}{F}|D_k F|^2 \mid \mathcal{F}_{k-1}\right]\right] \\ &= \mathbb{E}\left[\frac{1}{F}\sum_{k=0}^{N} |D_k F|^2\right]. \end{aligned}$$

where we used the Jensen inequality (9.3.1) and the convexity of $(u, v) \longmapsto v^2/u$ on $(0, \infty) \times \mathbb{R}$, or the Schwarz inequality applied to

$$1/\sqrt{F} \qquad \text{and} \qquad (D_k F/\sqrt{F})_{k \in \mathbb{N}},$$

as in the Wiener and Poisson cases [24] and [7]. This inequality is extended by density to $F \in \mathrm{Dom}\,(D)$. □

By a one-variable argument, letting $\mathrm{d}f = f(1) - f(-1)$, we have

$$\begin{aligned}
\mathrm{Ent}\,[f] &= pf(1)\log f(1) + qf(-1)\log f(-1) - \mathbb{E}[f]\log \mathbb{E}[f] \\
&= p(\mathbb{E}[f] + q\mathrm{d}f)\log(\mathbb{E}[f] + q\mathrm{d}f) \\
&\quad + q(\mathbb{E}[f] - p\mathrm{d}f)\log(\mathbb{E}[f] - p\mathrm{d}f) - (pf(1) + qf(-1))\log \mathbb{E}[f] \\
&= p\mathbb{E}[f]\log\left(1 + q\frac{\mathrm{d}f}{\mathbb{E}[f]}\right) + pq\mathrm{d}f\log f(1) \\
&\quad + q\mathbb{E}[f]\log\left(1 - p\frac{\mathrm{d}f}{\mathbb{E}[f]}\right) - qp\mathrm{d}f\log f(-1) \\
&\leq pq\mathrm{d}f\log f(1) - pq\mathrm{d}f\log f(-1) \\
&= pq\mathbb{E}[\mathrm{d}f\mathrm{d}\log f],
\end{aligned}$$

which, by tensorization, recovers the following L^1 inequality of [49], [31], and proved in [155] in the Poisson case. In the next proposition we state and prove this inequality in the multidimensional case, using the Clark representation formula, similarly to Theorem 1.12.2.

Theorem 1.12.3. *Let $F > 0$ be $\mathcal{F}_N$-measurable. We have*

$$\mathrm{Ent}\,[F] \leq \mathbb{E}\left[\sum_{k=0}^{N} D_k F D_k \log F\right]. \tag{1.12.3}$$

Proof. Let $f(x) = x\log x$ and

$$\Psi(x,y) = (x+y)\log(x+y) - x\log x - (1+\log x)y, \quad x,\ x+y > 0.$$

From the relation

$$\begin{aligned}
Y_k u_k &= Y_k \mathbb{E}[D_k F \mid \mathcal{F}_{k-1}] \\
&= q_k \mathbf{1}_{\{X_k=1\}} \mathbb{E}\left[(F_k^+ - F_k^-) \mid \mathcal{F}_{k-1}\right] + p_k \mathbf{1}_{\{X_k=-1\}} \mathbb{E}\left[(F_k^- - F_k^+) \mid \mathcal{F}_{k-1}\right] \\
&= \mathbf{1}_{\{X_k=1\}} \mathbb{E}\left[(F_k^+ - F_k^-)\mathbf{1}_{\{X_k=-1\}} \mid \mathcal{F}_{k-1}\right] \\
&\quad + \mathbf{1}_{\{X_k=-1\}} \mathbb{E}\left[(F_k^- - F_k^+)\mathbf{1}_{\{X_k=1\}} \mid \mathcal{F}_{k-1}\right],
\end{aligned}$$

we have, using the convexity of Ψ:

$$\begin{aligned}
\mathrm{Ent}\,[F] &= \mathbb{E}\left[\sum_{k=0}^{N} f(M_{k-1} + Y_k u_k) - f(M_{k-1})\right] \\
&= \mathbb{E}\left[\sum_{k=0}^{N} \Psi(M_{k-1}, Y_k u_k) + Y_k u_k (1 + \log M_{k-1})\right]
\end{aligned}$$

$$
\begin{aligned}
&= \mathbb{E}\left[\sum_{k=0}^{N} \Psi(M_{k-1}, Y_k u_k)\right] \\
&= \mathbb{E}\left[\sum_{k=0}^{N} p_k \Psi\left(\mathbb{E}\left[F \mid \mathcal{F}_{k-1}\right], \mathbb{E}\left[(F_k^+ - F_k^-)\mathbf{1}_{\{X_k=-1\}} \mid \mathcal{F}_{k-1}\right]\right)\right. \\
&\quad \left. + q_k \Psi\left(\mathbb{E}\left[F \mid \mathcal{F}_{k-1}\right], \mathbb{E}\left[(F_k^- - F_k^+)\mathbf{1}_{\{X_k=1\}} \mid \mathcal{F}_{k-1}\right]\right)\right] \\
&\leq \mathbb{E}\left[\sum_{k=0}^{N} \mathbb{E}\left[p_k \Psi\left(F, (F_k^+ - F_k^-)\mathbf{1}_{\{X_k=-1\}}\right) + q_k \Psi\left(F, (F_k^- - F_k^+)\mathbf{1}_{\{X_k=1\}}\right) \Big| \mathcal{F}_{k-1}\right]\right] \\
&= \mathbb{E}\left[\sum_{k=0}^{N} p_k \mathbf{1}_{\{X_k=-1\}} \Psi\left(F_k^-, F_k^+ - F_k^-\right) + q_k \mathbf{1}_{\{X_k=1\}} \Psi\left(F_k^+, F_k^- - F_k^+\right)\right] \\
&= \mathbb{E}\left[\sum_{k=0}^{N} p_k q_k \Psi(F_k^-, F_k^+ - F_k^-) + p_k q_k \Psi(F_k^+, F_k^- - F_k^+)\right] \\
&= \mathbb{E}\left[\sum_{k=0}^{N} p_k q_k (\log F_k^+ - \log F_k^-)(F_k^+ - F_k^-)\right] \\
&= \mathbb{E}\left[\sum_{k=0}^{N} D_k F D_k \log F\right].
\end{aligned}
$$

□

The application of Theorem 1.12.3 to e^F gives the following inequality for $F > 0$, $\mathcal{F}_N$-measurable:

$$
\begin{aligned}
\operatorname{Ent}\left[\mathrm{e}^F\right] &\leq \mathbb{E}\left[\sum_{k=0}^{N} D_k F D_k \mathrm{e}^F\right] \\
&= \mathbb{E}\left[\sum_{k=0}^{N} p_k q_k \Psi(\mathrm{e}^{F_k^-}, \mathrm{e}^{F_k^+} - \mathrm{e}^{F_k^-}) + p_k q_k \Psi(\mathrm{e}^{F_k^+}, \mathrm{e}^{F_k^-} - \mathrm{e}^{F_k^+})\right] \\
&= \mathbb{E}\left[\sum_{k=0}^{N} p_k q_k \mathrm{e}^{F_k^-}\left((F_k^+ - F_k^-)\mathrm{e}^{F_k^+ - F_k^-} - \mathrm{e}^{F_k^+ - F_k^-} + 1\right)\right. \\
&\quad \left. + p_k q_k \mathrm{e}^{F_k^+}\left((F_k^- - F_k^+)\mathrm{e}^{F_k^- - F_k^+} - \mathrm{e}^{F_k^- - F_k^+} + 1\right)\right] \\
&= \mathbb{E}\left[\sum_{k=0}^{N} p_k \mathbf{1}_{\{X_k=-1\}} \mathrm{e}^{F_k^-}\left((F_k^+ - F_k^-)\mathrm{e}^{F_k^+ - F_k^-} - \mathrm{e}^{F_k^+ - F_k^-} + 1\right)\right. \\
&\quad \left. + q_k \mathbf{1}_{\{X_k=1\}} \mathrm{e}^{F_k^+}\left((F_k^- - F_k^+)\mathrm{e}^{F_k^- - F_k^+} - \mathrm{e}^{F_k^- - F_k^+} + 1\right)\right] \\
&= \mathbb{E}\left[\mathrm{e}^F \sum_{k=0}^{N} \sqrt{p_k q_k} |Y_k| (\nabla_k F \mathrm{e}^{\nabla_k F} - \mathrm{e}^{\nabla_k F} + 1)\right]. \qquad (1.12.4)
\end{aligned}
$$

This implies

$$\operatorname{Ent}\left[e^F\right] \le \mathbb{E}\left[e^F \sum_{k=0}^{N} (\nabla_k F e^{\nabla_k F} - e^{\nabla_k F} + 1)\right]. \tag{1.12.5}$$

As noted in [31], Relation (1.12.3) and the Poisson limit theorem yield the L^1 inequality of [155]. More precisely, letting $M_n = (n + X_1 + \cdots + X_n)/2$, $F = \varphi(M_n)$ and $p_k = \lambda/n$, $k \in \mathbb{N}$, , $n \ge 1$, $\lambda > 0$, we have, from Proposition 1.6.2,

$$\begin{aligned}
&\sum_{k=0}^{n} D_k F D_k \log F \\
&= \frac{\lambda}{n}\left(1 - \frac{\lambda}{n}\right)(n - M_n)(\varphi(M_n + 1) - \varphi(M_n)) \log(\varphi(M_n + 1) - \varphi(M_n)) \\
&\quad + \frac{\lambda}{n}\left(1 - \frac{\lambda}{n}\right) M_n(\varphi(M_n) - \varphi(M_n - 1)) \log(\varphi(M_n) - \varphi(M_n - 1)),
\end{aligned}$$

in the limit as n goes to infinity we obtain

$$\operatorname{Ent}\left[\varphi(U)\right] \le \lambda \mathbb{E}\left[(\varphi(U+1) - \varphi(U))(\log \varphi(U+1) - \log \varphi(U))\right],$$

where U is a Poisson random variable with parameter λ. In one variable we have, still letting $\mathrm{d}f = f(1) - f(-1)$,

$$\begin{aligned}
\operatorname{Ent}\left[e^f\right] &\le pq \mathbb{E}\left[\mathrm{d}e^f \,\mathrm{d}\log e^f\right] \\
&= pq(e^{f(1)} - e^{f(-1)})(f(1) - f(-1)) \\
&= pq e^{f(-1)}((f(1) - f(-1))e^{f(1)-f(-1)} - e^{f(1)-f(-1)} + 1) \\
&\quad + pq e^{f(1)}((f(-1) - f(1))e^{f(-1)-f(1)} - e^{f(-1)-f(1)} + 1) \\
&\le q e^{f(-1)}((f(1) - f(-1))e^{f(1)-f(-1)} - e^{f(1)-f(-1)} + 1) \\
&\quad + p e^{f(1)}((f(-1) - f(1))e^{f(-1)-f(1)} - e^{f(-1)-f(1)} + 1) \\
&= \mathbb{E}\left[e^f(\nabla f e^{\nabla f} - e^{\nabla f} + 1)\right],
\end{aligned}$$

where ∇_k is the gradient operator defined in (1.6.4). This last inequality is not comparable to the optimal constant inequality

$$\operatorname{Ent}\left[e^F\right] \le \mathbb{E}\left[e^F \sum_{k=0}^{N} p_k q_k (|\nabla_k F| e^{|\nabla_k F|} - e^{|\nabla_k F|} + 1)\right], \tag{1.12.6}$$

of [20] since when $F_k^+ - F_k^- \ge 0$ the right-hand side of (1.12.6) grows as $F_k^+ e^{2F_k^+}$, instead of $F_k^+ e^{F_k^+}$ in (1.12.5). In fact we can prove the following inequality which improves (1.12.2), (1.12.3) and (1.12.6).

Theorem 1.12.4. *Let F be $\mathcal{F}_N$-measurable. We have*

$$\text{Ent}\left[\mathrm{e}^F\right] \le \mathbb{E}\left[\mathrm{e}^F \sum_{k=0}^N p_k q_k (\nabla_k F \mathrm{e}^{\nabla_k F} - \mathrm{e}^{\nabla_k F} + 1)\right]. \qquad (1.12.7)$$

Clearly, (1.12.7) is better than (1.12.6), (1.12.4) and (1.12.3). It also improves (1.12.2) from the bound

$$x\mathrm{e}^x - \mathrm{e}^x + 1 \le (\mathrm{e}^x - 1)^2, \quad x \in \mathbb{R},$$

which implies

$$\mathrm{e}^F(\nabla F \mathrm{e}^{\nabla F} - \mathrm{e}^{\nabla F} + 1) \le \mathrm{e}^F(\mathrm{e}^{\nabla F} - 1)^2 = \mathrm{e}^{-F}|\nabla \mathrm{e}^F|^2.$$

By the tensorization property (1.12.1), the proof of (1.12.7) reduces to the following one dimensional lemma.

Lemma 1.12.5. *For any* $0 \le p \le 1$, $t \in \mathbb{R}$, $a \in \mathbb{R}$, $q = 1 - p$,

$$\begin{aligned} &pt\mathrm{e}^t + qa\mathrm{e}^a - \left(p\mathrm{e}^t + q\mathrm{e}^a\right)\log\left(p\mathrm{e}^t + q\mathrm{e}^a\right) \\ &\quad \le pq\left(q\mathrm{e}^a\left((t-a)\mathrm{e}^{t-a} - \mathrm{e}^{t-a} + 1\right) + p\mathrm{e}^t\left((a-t)\mathrm{e}^{a-t} - \mathrm{e}^{a-t} + 1\right)\right). \end{aligned}$$

Proof. Set

$$\begin{aligned} g(t) = {} & pq\left(q\mathrm{e}^a\left((t-a)\mathrm{e}^{t-a} - \mathrm{e}^{t-a} + 1\right) + p\mathrm{e}^t\left((a-t)\mathrm{e}^{a-t} - \mathrm{e}^{a-t} + 1\right)\right) \\ & - pt\mathrm{e}^t - qa\mathrm{e}^a + \left(p\mathrm{e}^t + q\mathrm{e}^a\right)\log\left(p\mathrm{e}^t + q\mathrm{e}^a\right). \end{aligned}$$

Then

$$g'(t) = pq\left(q\mathrm{e}^a(t-a)\mathrm{e}^{t-a} + p\mathrm{e}^t\left(-\mathrm{e}^{a-t} + 1\right)\right) - pt\mathrm{e}^t + p\mathrm{e}^t\log(p\mathrm{e}^t + q\mathrm{e}^a)$$

and $g''(t) = p\mathrm{e}^t h(t)$, where

$$h(t) = -a - 2pt - p + 2pa + p^2 t - p^2 a + \log(p\mathrm{e}^t + q\mathrm{e}^a) + \frac{p\mathrm{e}^t}{p\mathrm{e}^t + q\mathrm{e}^a}.$$

Now,

$$\begin{aligned} h'(t) &= -2p + p^2 + \frac{2p\mathrm{e}^t}{p\mathrm{e}^t + q\mathrm{e}^a} - \frac{p^2\mathrm{e}^{2t}}{(p\mathrm{e}^t + q\mathrm{e}^a)^2} \\ &= \frac{pq^2(\mathrm{e}^t - \mathrm{e}^a)(p\mathrm{e}^t + (q+1)\mathrm{e}^a)}{(p\mathrm{e}^t + q\mathrm{e}^a)^2}, \end{aligned}$$

which implies that $h'(a) = 0$, $h'(t) < 0$ for any $t < a$ and $h'(t) > 0$ for any $t > a$. Hence, for any $t \neq a$, $h(t) > h(a) = 0$, and so $g''(t) \ge 0$ for any $t \in \mathbb{R}$ and $g''(t) = 0$ if and only if $t = a$. Therefore, g' is strictly increasing. Finally, since $t = a$ is the unique root of $g' = 0$, we have that $g(t) \ge g(a) = 0$ for all $t \in \mathbb{R}$. □

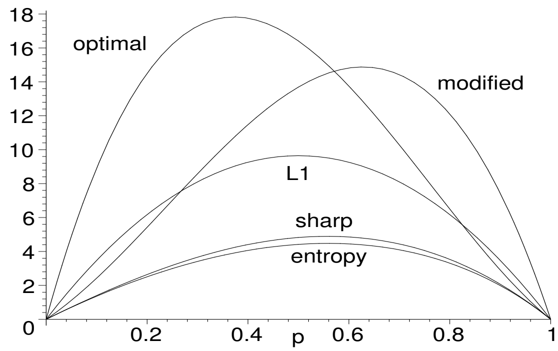

Fig. 1.1: Graph of bounds on the entropy as a function of $p \in [0, 1]$.

This inequality improves (1.12.2), (1.12.3), and (1.12.6), as illustrated in one dimension in Figure 1.12, where the entropy is represented as a function of $p \in [0, 1]$ with $f(1) = 1$ and $f(-1) = 3.5$:

The inequality (1.12.7) is a discrete analog of the sharp inequality on Poisson space of [155]. In the symmetric case $p_k = q_k = 1/2$, $k \in \mathbb{N}$, we have

$$\begin{aligned}
\mathrm{Ent}\left[e^F\right] &\le \mathbb{E}\left[e^F \sum_{k=0}^{N} p_k q_k (\nabla_k F e^{\nabla_k F} - \nabla_k F + 1)\right] \\
&= \frac{1}{8}\mathbb{E}\left[\sum_{k=0}^{N} e^{F_k^-}((F_k^+ - F_k^-)e^{F_k^+ - F_k^-} - e^{F_k^+ - F_k^-} + 1)\right. \\
&\qquad \left. + e^{F_k^+}((F_k^- - F_k^+)e^{F_k^- - F_k^+} - e^{F_k^- - F_k^+} + 1)\right] \\
&= \frac{1}{8}\mathbb{E}\left[\sum_{k=0}^{N} (e^{F_k^+} - e^{F_k^-})(F_k^+ - F_k^-)\right] \\
&= \frac{1}{2}\mathbb{E}\left[\sum_{k=0}^{N} D_k F D_k e^F\right],
\end{aligned}$$

which improves on (1.12.3).

Similarly the sharp inequality of [155] can be recovered by taking $F = \varphi(M_n)$ in

$$\begin{aligned}
&\mathrm{Ent}\left[e^F\right] \le \mathbb{E}\left[e^F \sum_{k=0}^{N} p_k q_k (\nabla_k F e^{\nabla_k F} - \nabla_k F + 1)\right] \\
&= \frac{\lambda}{n}\left(1 - \frac{\lambda}{n}\right)\mathbb{E}\left[M_n e^{\varphi(M_n)}\right.
\end{aligned}$$

$$\times\left((\varphi(M_n)-\varphi(M_n-1))\mathrm{e}^{\varphi(M_n)-\varphi(M_n-1)}-\mathrm{e}^{\varphi(M_n)-\varphi(M_n-1)}+1)\right]$$
$$+\frac{\lambda}{n}\left(1-\frac{\lambda}{n}\right)\mathbb{E}\left[(n-M_n)\mathrm{e}^{\varphi(M_n)}\right.$$
$$\left.\times((\varphi(M_n+1)-\varphi(M_n))\mathrm{e}^{\varphi(M_n+1)-\varphi(M_n)}-\mathrm{e}^{\varphi(M_n+1)-\varphi(M_n)}+1)\right],$$

which, in the limit as n goes to infinity, yields

$$\mathrm{Ent}\left[\mathrm{e}^{\varphi(U)}\right]\leq\lambda\mathbb{E}\left[\mathrm{e}^{\varphi(U)}((\varphi(U+1)-\varphi(U))\mathrm{e}^{\varphi(U+1)-\varphi(U)}-\mathrm{e}^{\varphi(U+1)-\varphi(U)}+1)\right],$$

where U is a Poisson random variable with parameter λ.

1.13 Change of Variable Formula

In this section we state a discrete-time analog of Itô's change of variable formula which will be useful for the predictable representation of random variables and for option hedging.

Proposition 1.13.1. *Let $(M_n)_{n\in\mathbb{N}}$ be a square-integrable martingale and $f:\mathbb{R}\times\mathbb{N}\to\mathbb{R}$. We have*

$$f(M_n,n)$$
$$=f(M_{-1},-1)+\sum_{k=0}^{n}D_kf(M_k,k)Y_k+\sum_{k=0}^{n}\mathbb{E}\left[f(M_k,k)-f(M_{k-1},k-1)\mid\mathcal{F}_{k-1}\right]. \quad (1.13.1)$$

Proof. By Proposition 1.7.5 there exists square-integrable process $(u_k)_{k\in\mathbb{N}}$ such that

$$M_n=M_{-1}+\sum_{k=0}^{n}u_kY_k,\qquad n\in\mathbb{N}.$$

We write

$$f(M_n,n)-f(M_{-1},-1)=\sum_{k=0}^{n}(f(M_k,k)-f(M_{k-1},k-1))$$
$$=\sum_{k=0}^{n}(f(M_k,k)-f(M_{k-1},k)+f(M_{k-1},k)-f(M_{k-1},k-1))$$
$$=\sum_{k=0}^{n}\sqrt{\frac{p_k}{q_k}}\left(f\left(M_{k-1}+u_k\sqrt{\frac{q_k}{p_k}},k\right)-f(M_{k-1},k)\right)Y_k$$
$$+\frac{p_k}{q_k}\mathbf{1}_{\{X_k=-1\}}\left(f\left(M_{k-1}+u_k\sqrt{\frac{q_k}{p_k}},k\right)-f(M_{k-1},k)\right)$$

$$+\mathbf{1}_{\{X_k=-1\}}\left(f\left(M_{k-1}-u_k\sqrt{\frac{p_k}{q_k}},k\right)-f(M_{k-1},k)\right)$$
$$+\sum_{k=0}^{n}(f(M_{k-1},k)-f(M_{k-1},k-1))$$
$$=\sum_{k=0}^{n}\sqrt{\frac{p_k}{q_k}}\left(f\left(M_{k-1}+u_k\sqrt{\frac{q_k}{p_k}},k\right)-f(M_{k-1},k)\right)Y_k$$
$$+\sum_{k=0}^{n}\frac{1}{q_k}\mathbf{1}_{\{X_k=-1\}}\mathbb{E}\left[f(M_k,k)-f(M_{k-1},k)\mid\mathcal{F}_{k-1}\right]$$
$$+\sum_{k=0}^{n}(f(M_{k-1},k)-f(M_{k-1},k-1)).$$

Similarly we have

$$f(M_n,n)$$
$$=f(M_{-1},-1)-\sum_{k=0}^{n}\sqrt{\frac{q_k}{p_k}}\left(f\left(M_{k-1}-u_k\sqrt{\frac{p_k}{q_k}},k\right)-f(M_{k-1},k)\right)Y_k$$
$$+\sum_{k=0}^{n}\frac{1}{p_k}\mathbf{1}_{\{X_k=1\}}\mathbb{E}\left[f(M_k,k)-f(M_{k-1},k)\mid\mathcal{F}_{k-1}\right]$$
$$+\sum_{k=0}^{n}(f(M_{k-1},k)-f(M_{k-1},k-1)).$$

Multiplying each increment in the above formulas respectively by q_k and p_k and summing on k we get

$$f(M_n,n)=f(M_{-1},-1)+\sum_{k=0}^{n}(f(M_k,k)-f(M_{k-1},k-1))$$
$$=f(M_{-1},-1)+\sum_{k=0}^{n}q_k(f(M_k,k)-f(M_{k-1},k-1))$$
$$+\sum_{k=0}^{n}p_k(f(M_k,k)-f(M_{k-1},k-1))$$
$$=f(M_{-1},-1)+\sum_{k=0}^{n}\sqrt{p_kq_k}\left(f\left(M_{k-1}+u_k\sqrt{\frac{q_k}{p_k}},k\right)-f(M_{k-1},k)\right)Y_k$$
$$-\sum_{k=0}^{n}\sqrt{p_kq_k}\left(f\left(M_{k-1}-u_k\sqrt{\frac{p_k}{q_k}},k\right)-f(M_{k-1},k)\right)Y_k$$
$$+\sum_{k=0}^{n}\mathbb{E}\left[f(M_k,k)\mid\mathcal{F}_{k-1}\right]-f(M_{k-1},k)$$

$$
\begin{aligned}
&+\sum_{k=0}^{n}(f(M_{k-1},k)-f(M_{k-1},k-1))\\
&= f(M_{-1},-1)\\
&\quad+\sum_{k=0}^{n}\sqrt{p_kq_k}\left(f\left(M_{k-1}+u_k\sqrt{\frac{q_k}{p_k}},k\right)-f\left(M_{k-1}-u_k\sqrt{\frac{p_k}{q_k}},k\right)\right)Y_k\\
&\quad+\sum_{k=0}^{n}\mathbb{E}\left[f(M_k,k)\mid\mathcal{F}_{k-1}\right]-f(M_{k-1},k-1).
\end{aligned}
$$

□

Note that in (1.13.1) we have

$$
D_kf(M_k,k)=\sqrt{p_kq_k}\left(f\left(M_{k-1}+u_k\sqrt{\frac{q_k}{p_k}},k\right)-f\left(M_{k-1}-u_k\sqrt{\frac{p_k}{q_k}},k\right)\right),
$$

$k\in\mathbb{N}$.

On the other hand, the term

$$
\mathbb{E}\left[f(M_k,k)-f(M_{k-1},k-1)\mid\mathcal{F}_{k-1}\right]
$$

is analog to the generator part in the continuous time Itô formula, and can be written as

$$
p_kf\left(M_{k-1}+u_k\sqrt{\frac{q_k}{p_k}},k\right)+q_kf\left(M_{k-1}-u_k\sqrt{\frac{p_k}{q_k}},k\right)-f\left(M_{k-1},k-1\right).
$$

When $p_n=q_n=1/2$, $n\in\mathbb{N}$, we have

$$
\begin{aligned}
f(M_n,n)=f(M_{-1},-1)&+\sum_{k=0}^{n}\frac{f\left(M_{k-1}+u_k,k\right)-f\left(M_{k-1}-u_k,k\right)}{2}Y_k\\
&+\sum_{k=0}^{n}\frac{f\left(M_{k-1}+u_k,k\right)+f\left(M_{k-1}-u_k,k\right)-2f\left(M_{k-1},k-1\right)}{2}.
\end{aligned}
$$

The above proposition also provides an explicit version of the Doob decomposition for supermartingales. Naturally if $(f(M_n,n))_{n\in\mathbb{N}}$ is a martingale we have

$$
\begin{aligned}
&f(M_n,n)=f(M_{-1},-1)\\
&\quad+\sum_{k=0}^{n}\sqrt{p_kq_k}\left(f\left(M_{k-1}+u_k\sqrt{\frac{q_k}{p_k}},k\right)-f\left(M_{k-1}-u_k\sqrt{\frac{p_k}{q_k}},k\right)\right)Y_k\\
&=f(M_{-1},-1)+\sum_{k=0}^{n}D_kf(M_k,k)Y_k.
\end{aligned}
$$

In this case the Clark formula, the martingale representation formula Proposition 1.7.5 and the change of variable formula all coincide, and we have in particular

$$\begin{aligned} D_k f(M_k, k) &= \mathbb{E}\left[D_k f(M_n, n) \mid \mathcal{F}_{k-1}\right] \\ &= \mathbb{E}\left[D_k f(M_k, k) \mid \mathcal{F}_{k-1}\right], \qquad k \in \mathbb{N}. \end{aligned}$$

If F is an $\mathcal{F}_N$-measurable random variable and f is a function such that

$$\mathbb{E}\left[F \mid \mathcal{F}_n\right] = f(M_n, n), \qquad -1 \le n \le N,$$

we have $F = f(M_N, N)$, $\mathbb{E}[F]] = f(M_{-1}, -1)$ and

$$\begin{aligned} F &= \mathbb{E}[F] + \sum_{k=0}^{n} \mathbb{E}\left[D_k f(M_N, N) \mid \mathcal{F}_{k-1}\right] Y_k \\ &= \mathbb{E}[F] + \sum_{k=0}^{n} D_k f(M_k, k) Y_k \\ &= \mathbb{E}[F] + \sum_{k=0}^{n} D_k \mathbb{E}\left[f(M_N, N) \mid \mathcal{F}_k\right] Y_k. \end{aligned}$$

Such a function f exists if $(M_n)_{n\in\mathbb{N}}$ is Markov and $F = h(M_N)$. In this case, consider the semi-group $(P_{k,n})_{0\le k<n\le N}$ associated to $(M_n)_{n\in\mathbb{N}}$ and defined by

$$(P_{k,n}h)(x) = \mathbb{E}\left[h(M_n) \mid M_k = x\right].$$

Letting $f(x, n) = (P_{n,N}h)(x)$ we can write

$$\begin{aligned} F &= \mathbb{E}[F] + \sum_{k=0}^{n} \mathbb{E}\left[D_k h(M_N) \mid \mathcal{F}_{k-1}\right] Y_k \\ &= \mathbb{E}[F] + \sum_{k=0}^{n} D_k (P_{k,N} h(M_k)) Y_k. \end{aligned}$$

1.14 Option Hedging

In this section we give a presentation of the Black-Scholes formula in discrete time, or in the Cox-Ross-Rubinstein model, see e.g. [47], [78], [129], or §15-1 of [153], as an application of the Clark formula.

In order to be consistent with the notation of the previous sections we choose to use the time scale $\mathbb{N}$, hence the index 0 is that of the first random value of any stochastic process, while the index -1 corresponds to its deterministic initial value.

Let $(A_k)_{k\in\mathbb{N}}$ be a riskless asset with initial value A_{-1}, and defined by

$$A_n = A_{-1}\prod_{k=0}^{n}(1+r_k), \qquad n\in\mathbb{N},$$

where $(r_k)_{k\in\mathbb{N}}$, is a sequence of deterministic numbers such that $r_k > -1$, $k\in\mathbb{N}$. Consider a stock price with initial value S_{-1}, given in discrete time as

$$S_n = \begin{cases} (1+b_n)S_{n-1} \text{ if } X_n = 1, \\ \\ (1+a_n)S_{n-1} \text{ if } X_n = -1, \quad n\in\mathbb{N}, \end{cases}$$

where $(a_k)_{k\in\mathbb{N}}$ and $(b_k)_{k\in\mathbb{N}}$ are sequences of deterministic numbers such that

$$-1 < a_k < r_k < b_k, \qquad k\in\mathbb{N}.$$

We have

$$S_n = S_{-1}\prod_{k=0}^{n}\sqrt{(1+b_k)(1+a_k)}\left(\frac{1+b_k}{1+a_k}\right)^{X_k/2}, \quad n\in\mathbb{N}.$$

Consider now the discounted stock price given as

$$\begin{aligned}\tilde{S}_n &= S_n\prod_{k=0}^{n}(1+r_k)^{-1} \\ &= S_{-1}\prod_{k=0}^{n}\left(\frac{1}{1+r_k}\sqrt{(1+b_k)(1+a_k)}\left(\frac{1+b_k}{1+a_k}\right)^{X_k/2}\right), \quad n\in\mathbb{N}.\end{aligned}$$

If $-1 < a_k < r_k < b_k$, $k\in\mathbb{N}$, then $(\tilde{S}_n)_{n\in\mathbb{N}}$ is a martingale with respect to $(\mathcal{F}_n)_{n\geq -1}$ under the probability $\mathbb{P}^*$ given by

$$p_k = \frac{r_k - a_k}{b_k - a_k}, \quad q_k = \frac{b_k - r_k}{b_k - a_k}, \quad k\in\mathbb{N}.$$

In other terms, under $\mathbb{P}^*$ we have

$$\mathbb{E}^*\left[S_{n+1} \mid \mathcal{F}_n\right] = (1+r_{n+1})S_n, \qquad n\geq -1,$$

where $\mathbb{E}^*$ denotes the expectation under $\mathbb{P}^*$. Recall that under this probability measure there is absence of arbitrage and the market is complete. From the change of variable formula Proposition 1.13.1 or from the Clark formula (1.7.1) we have the martingale representation

$$\tilde{S}_n = S_{-1} + \sum_{k=0}^{n} Y_k D_k \tilde{S}_k = S_{-1} + \sum_{k=0}^{n} \tilde{S}_{k-1}\sqrt{p_k q_k}\frac{b_k - a_k}{1+r_k} Y_k.$$

Definition 1.14.1. *A portfolio strategy is represented by a pair of predictable processes* $(\eta_k)_{k\in\mathbb{N}}$ *and* $(\zeta_k)_{k\in\mathbb{N}}$ *where* η_k*, resp.* ζ_k *represents the numbers of units invested over the time period* $(k, k+1]$ *in the asset* S_k*, resp.* A_k*, with* $k \geq 0$.

The value at time $k \geq -1$ of the portfolio $(\eta_k, \zeta_k)_{0\leq k\leq N}$ is defined as

$$V_k = \zeta_{k+1} A_k + \eta_{k+1} S_k, \qquad k \geq -1, \tag{1.14.1}$$

and its discounted value is defined as

$$\tilde{V}_n = V_n \prod_{k=0}^{n} (1+r_k)^{-1}, \qquad n \geq -1. \tag{1.14.2}$$

Definition 1.14.2. *A portfolio* $(\eta_k, \zeta_k)_{k\in\mathbb{N}}$ *is said to be self-financing if*

$$A_k(\zeta_{k+1} - \zeta_k) + S_k(\eta_{k+1} - \eta_k) = 0, \qquad k \geq 0.$$

Note that the self-financing condition implies

$$V_k = \zeta_k A_k + \eta_k S_k, \qquad k \geq 0.$$

Our goal is to hedge an arbitrary claim on Ω, i.e. given an $\mathcal{F}_N$-measurable random variable F we search for a portfolio $(\eta_k, \zeta_k)_{0\leq k\leq n}$ such that the equality

$$F = V_N = \zeta_N A_N + \eta_N S_N \tag{1.14.3}$$

holds at time $N \in \mathbb{N}$.

Proposition 1.14.3. *Assume that the portfolio* $(\eta_k, \zeta_k)_{0\leq k\leq N}$ *is self-financing. Then we have the decomposition*

$$V_n = V_{-1} \prod_{k=0}^{n} (1+r_k) + \sum_{i=0}^{n} \eta_i S_{i-1} \sqrt{p_i q_i}(b_i - a_i) Y_i \prod_{k=i+1}^{n} (1+r_k). \tag{1.14.4}$$

Proof. Under the self-financing assumption we have

$$\begin{aligned}
V_i - V_{i-1} &= \zeta_i (A_i - A_{i-1}) + \eta_i (S_i - S_{i-1}) \\
&= r_i \zeta_i A_{i-1} + (a_i \mathbf{1}_{\{X_i=-1\}} + b_i \mathbf{1}_{\{X_i=1\}}) \eta_i S_{i-1} \\
&= \eta_i S_{i-1} (a_i \mathbf{1}_{\{X_i=-1\}} + b_i \mathbf{1}_{\{X_i=1\}} - r_i) + r_i V_{i-1} \\
&= \eta_i S_{i-1} ((a_i - r_i) \mathbf{1}_{\{X_i=-1\}} + (b_i - r_i) \mathbf{1}_{\{X_i=1\}}) + r_i V_{i-1} \\
&= (b_i - a_i) \eta_i S_{i-1} (-p_i \mathbf{1}_{\{X_i=-1\}} + q_i \mathbf{1}_{\{X_i=1\}}) + r_i V_{i-1}
\end{aligned}$$

$$= \eta_i S_{i-1} \sqrt{p_i q_i}(b_i - a_i) Y_i + r_i V_{i-1}, \qquad i \in \mathbb{N},$$

by Relation (1.4.5), hence for the discounted portfolio we get:

$$\begin{aligned}
\tilde{V}_i - \tilde{V}_{i-1} &= \prod_{k=1}^{i} (1 + r_k)^{-1} V_i - \prod_{k=1}^{i-1} (1 + r_k)^{-1} V_{i-1} \\
&= \prod_{k=1}^{i} (1 + r_k)^{-1} (V_i - V_{i-1} - r_i V_{i-1}) \\
&= \eta_i S_{i-1} \sqrt{p_i q_i}(b_i - a_i) Y_i \prod_{k=1}^{i} (1 + r_k)^{-1}, \qquad i \in \mathbb{N},
\end{aligned}$$

which successively yields (1.14.4). □

As a consequence of (1.14.4) and (1.14.2) we immediately obtain

$$\tilde{V}_n = \tilde{V}_{-1} + \sum_{i=0}^{n} \eta_i S_{i-1} \sqrt{p_i q_i}(b_i - a_i) Y_i \prod_{k=0}^{i} (1 + r_k)^{-1}, \tag{1.14.5}$$

$n \geq -1$. The next proposition provides a solution to the hedging problem under the constraint (1.14.3).

Proposition 1.14.4. *Given $F \in L^2(\Omega, \mathcal{F}_N)$, let*

$$\eta_n = \frac{1}{S_{n-1} \sqrt{p_n q_n}(b_n - a_n)} \mathbb{E}^* \left[D_n F \mid \mathcal{F}_{n-1} \right] \prod_{k=n+1}^{N} (1 + r_k)^{-1}, \tag{1.14.6}$$

$0 \leq n \leq N$, and

$$\zeta_n = A_n^{-1} \left(\prod_{k=n+1}^{N} (1 + r_k)^{-1} \mathbb{E}^* \left[F \mid \mathcal{F}_n \right] - \eta_n S_n \right), \tag{1.14.7}$$

$0 \leq n \leq N$. Then the portfolio $(\eta_k, \zeta_k)_{0 \leq k \leq n}$ is self financing and satisfies

$$\zeta_n A_n + \eta_n S_n = \prod_{k=n+1}^{N} (1 + r_k)^{-1} \mathbb{E}^* \left[F \mid \mathcal{F}_n \right],$$

$0 \leq n \leq N$, in particular we have $V_N = F$, hence $(\eta_k, \zeta_k)_{0 \leq k \leq N}$ is a hedging strategy leading to F.

Proof. Let $(\eta_k)_{-1 \leq k \leq N}$ be defined by (1.14.6) and $\eta_{-1} = 0$, and consider the process $(\zeta_n)_{0 \leq n \leq N}$ defined by

$$\zeta_{-1} = \frac{\mathbb{E}^* [F]}{S_{-1}} \prod_{k=0}^{N} (1 + r_k)^{-1} \quad \text{and} \quad \zeta_{k+1} = \zeta_k - \frac{(\eta_{k+1} - \eta_k) S_k}{A_k},$$

$k = -1, \ldots, N-1$. Then $(\eta_k, \zeta_k)_{-1 \le k \le N}$ satisfies the self-financing condition

$$A_k(\zeta_{k+1} - \zeta_k) + S_k(\eta_{k+1} - \eta_k) = 0, \qquad -1 \le k \le N-1.$$

Let now

$$V_{-1} = \mathbb{E}^* [F] \prod_{k=0}^{N} (1 + r_k)^{-1}, \quad \text{and} \quad V_n = \zeta_n A_n + \eta_n S_n, \qquad 0 \le n \le N,$$

and

$$\tilde{V}_n = V_n \prod_{k=0}^{n} (1 + r_k)^{-1}, \qquad -1 \le n \le N.$$

Since $(\eta_k, \zeta_k)_{-1 \le k \le N}$ is self-financing, Relation (1.14.5) shows that

$$\tilde{V}_n = \tilde{V}_{-1} + \sum_{i=0}^{n} Y_i \eta_i S_{i-1} \sqrt{p_i q_i} (b_i - a_i) \prod_{k=1}^{i} (1 + r_k)^{-1}, \tag{1.14.8}$$

$-1 \le n \le N$. On the other hand, from the Clark formula (1.7.1) and the definition of $(\eta_k)_{-1 \le k \le N}$ we have

$$\begin{aligned}
&\mathbb{E}^* [F \mid \mathcal{F}_n] \prod_{k=0}^{N} (1 + r_k)^{-1} \\
&= \mathbb{E}^* \left[\mathbb{E}^* [F] \prod_{k=0}^{N} (1 + r_k)^{-1} + \sum_{i=0}^{N} Y_i \mathbb{E}^* [D_i F \mid \mathcal{F}_{i-1}] \prod_{k=0}^{N} (1 + r_k)^{-1} \middle| \mathcal{F}_n \right] \\
&= \mathbb{E}^* [F] \prod_{k=0}^{N} (1 + r_k)^{-1} + \sum_{i=0}^{n} Y_i \mathbb{E}^* [D_i F \mid \mathcal{F}_{i-1}] \prod_{k=0}^{N} (1 + r_k)^{-1} \\
&= \mathbb{E}^* [F] \prod_{k=0}^{N} (1 + r_k)^{-1} + \sum_{i=0}^{n} Y_i \eta_i S_{i-1} \sqrt{p_i q_i} (b_i - a_i) \prod_{k=1}^{i} (1 + r_k)^{-1} \\
&= \tilde{V}_n
\end{aligned}$$

from (1.14.8). Hence

$$\tilde{V}_n = \mathbb{E}^* [F \mid \mathcal{F}_n] \prod_{k=0}^{N} (1 + r_k)^{-1}, \qquad -1 \le n \le N,$$

and

$$V_n = \mathbb{E}^* [F \mid \mathcal{F}_n] \prod_{k=n+1}^{N} (1 + r_k)^{-1}, \qquad -1 \leq n \leq N.$$

In particular we have $V_N = F$. To conclude the proof we note that from the relation $V_n = \zeta_n A_n + \eta_n S_n$, $0 \leq n \leq N$, the process $(\zeta_n)_{0 \leq n \leq N}$ coincides with $(\zeta_n)_{0 \leq n \leq N}$ defined by (1.14.7). □

Note that we also have

$$\zeta_{n+1} A_n + \eta_{n+1} S_n = \mathbb{E}^* [F \mid \mathcal{F}_n] \prod_{k=n+1}^{N} (1 + r_k)^{-1}, \qquad -1 \leq n \leq N.$$

The above proposition shows that there always exists a hedging strategy starting from

$$\tilde{V}_{-1} = \mathbb{E}^* [F] \prod_{k=0}^{N} (1 + r_k)^{-1}.$$

Conversely, if there exists a hedging strategy leading to

$$\tilde{V}_N = F \prod_{k=0}^{N} (1 + r_k)^{-1},$$

then $(\tilde{V}_n)_{-1 \leq n \leq N}$ is necessarily a martingale with initial value

$$\tilde{V}_{-1} = \mathbb{E}^* \left[\tilde{V}_N \right] = \mathbb{E}^* [F] \prod_{k=0}^{N} (1 + r_k)^{-1}.$$

When $F = h(\tilde{S}_N)$, we have $\mathbb{E}^* \left[h(\tilde{S}_N) \mid \mathcal{F}_k \right] = f(\tilde{S}_k, k)$ with

$$f(x, k) = \mathbb{E}^* \left[h \left(x \prod_{i=k+1}^{n} \frac{\sqrt{(1 + b_k)(1 + a_k)}}{1 + r_k} \left(\frac{1 + b_k}{1 + a_k} \right)^{X_k/2} \right) \right].$$

The hedging strategy is given by

$$\begin{aligned} \eta_k &= \frac{1}{S_{k-1} \sqrt{p_k q_k} (b_k - a_k)} D_k f(\tilde{S}_k, k) \prod_{i=k+1}^{N} (1 + r_i)^{-1} \\ &= \frac{\prod_{i=k+1}^{N} (1 + r_i)^{-1}}{S_{k-1} (b_k - a_k)} \left(f \left(\tilde{S}_{k-1} \frac{1 + b_k}{1 + r_k}, k \right) - f \left(\tilde{S}_{k-1} \frac{1 + a_k}{1 + r_k}, k \right) \right), \end{aligned}$$

$k \geq -1$. Note that η_k is non-negative (i.e. there is no short-selling) when f is an increasing function, e.g. in the case of European options we have $f(x) = (x - K)^+$.

1.15 Notes and References

This chapter is a revision of [117] with some additions, and is mainly based on [61] and [119]. It is included for the sake of consistency and for the role it plays as an introduction to the next chapters. Other approaches to discrete-time stochastic analysis include [55], [56], [50], [82] and [92]; see [9] for an approach based on quantum probability. Deviation inequalities and logarithmic Sobolev inequalities are treated in [20], [48], [61]. We also refer to [6], [18], [19], [63], [79], for other versions of logarithmic Sobolev inequalities in discrete settings. See [78], §15-1 of [153], and [129], for other derivations of the Black-Scholes formula in the discrete time Cox-Ross-Rubinstein model.

Chapter 2
Continuous-Time Normal Martingales

This chapter is concerned with the basics of stochastic calculus in continuous time. In continuation of Chapter 1 we keep considering the point of view of normal martingales and structure equations, which provides a unified treatment of stochastic integration and calculus that applies to both continuous and discontinuous processes. In particular we cover the construction of single and multiple stochastic integrals with respect to normal martingales and we discuss other classical topics such as quadratic variations and the Itô formula.

2.1 Normal Martingales

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space equipped with a right-continuous filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$, i.e. an increasing family of sub σ-algebras of $\mathcal{F}$ such that

$$\mathcal{F}_t = \bigcap_{s>t} \mathcal{F}_s, \qquad t \in \mathbb{R}_+.$$

We refer to Section 9.5 in the Appendix for recalls on martingales in continuous time.

Definition 2.1.1. *A square-integrable martingale $(M_t)_{t\in\mathbb{R}_+}$ such that*

$$\mathbb{E}\left[(M_t - M_s)^2 | \mathcal{F}_s\right] = t - s, \qquad 0 \le s < t, \tag{2.1.1}$$

is called a normal martingale.

Every square-integrable process $(M_t)_{t\in\mathbb{R}_+}$ with centered independent increments and generating the filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ satisfies

$$\mathbb{E}\left[(M_t - M_s)^2 | \mathcal{F}_s\right] = \mathbb{E}\left[(M_t - M_s)^2\right], \qquad 0 \le s \le t,$$

hence the following remark.

Remark 2.1.2. *A square-integrable process $(M_t)_{t\in\mathbb{R}_+}$ with centered independent increments is a normal martingale if and only if*

$$\mathbb{E}\left[(M_t - M_s)^2\right] = t - s, \qquad 0 \le s \le t.$$

In our presentation of stochastic integration we will restrict ourselves to normal martingales. As will be seen in the next sections, this family contains Brownian motion and the standard Poisson process as particular cases.

Remark 2.1.3. *A martingale $(M_t)_{t\in\mathbb{R}_+}$ is normal if and only if $(M_t^2 - t)_{t\in\mathbb{R}_+}$ is a martingale, i.e.*

$$\mathbb{E}\left[M_t^2 - t|\mathcal{F}_s\right] = M_s^2 - s, \qquad 0 \le s < t.$$

Proof. This follows from the equalities

$$\begin{aligned}
&\mathbb{E}\left[(M_t - M_s)^2|\mathcal{F}_s\right] - (t-s)\\
&= \mathbb{E}\left[M_t^2 - M_s^2 - 2(M_t - M_s)M_s|\mathcal{F}_s\right] - (t-s)\\
&= \mathbb{E}\left[M_t^2 - M_s^2|\mathcal{F}_s\right] - 2M_s\mathbb{E}\left[M_t - M_s|\mathcal{F}_s\right] - (t-s)\\
&= \mathbb{E}\left[M_t^2|\mathcal{F}_s\right] - t - (\mathbb{E}\left[M_s^2|\mathcal{F}_s\right] - s).
\end{aligned}$$

□

Throughout the remainder of this chapter, $(M_t)_{t\in\mathbb{R}_+}$ will be a normal martingale and $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ will be the right-continuous filtration generated by $(M_t)_{t\in\mathbb{R}_+}$, i.e.

$$\mathcal{F}_t = \sigma(M_s \ : \ 0 \le s \le t), \qquad t \in \mathbb{R}_+.$$

2.2 Brownian Motion

In this section and the next one we present Brownian motion and the compensated Poisson process as the fundamental examples of normal martingales. Stochastic processes, as sequences of random variables can be naturally constructed in an infinite dimensional setting. Similarly to Remark 1.4.1 where an infinite product of discrete Bernoulli measures is mapped to the Lebesgue measure, one can map the uniform measure on "infinite dimensional spheres" to a Gaussian measure, cf. e.g. [89], [51], and references therein. More precisely, the surface of the n-dimensional sphere with radius r is

$$s_n(r) = \frac{n\pi^{n/2}}{\Gamma(\frac{n}{2}+1)} r^{n-1} \simeq \pi^{n/2}\sqrt{2n\pi}\mathrm{e}^{n/2}\left(\frac{n}{2}\right)^{-n/2} r^{n-1},$$

with $s_1(r) = 2$, where the equivalence is given by Stirling's formula as n goes to infinity. The set of points on the sphere $S_n(\sqrt{n})$ whose first coordinate x_1 lies between a and $a + da$ has the measure

$$\begin{aligned}
\sigma_n(\{(x_1,\ldots,x_n) \in S_n(\sqrt{n}) \ : \ a \leq x_1 \leq a+da\}) \\
&= \frac{s_{n-1}(\sqrt{n-a^2})}{s_n(\sqrt{n})}\frac{da}{\sqrt{1-a^2/n}} \\
&\simeq \frac{1}{\sqrt{2\pi}}\left(\sqrt{1-\frac{a^2}{n}}\right)^{n-2} da \\
&\to \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-a^2/2}da, \qquad [n\to\infty].
\end{aligned}$$

When a point is chosen uniformly on the sphere, its components have an approximately Gaussian distribution as n becomes large. Namely, the uniform measure $\sigma_n(dx)$ on $S_n(\sqrt{n})$ converges weakly as n goes to infinity to the infinite dimensional Gaussian measure

$$\gamma_{\mathbb{N}}(dx) = \bigotimes_{k=0}^{\infty} \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-x_k^2/2}dx_k, \tag{2.2.1}$$

on $(\mathbb{R}^{\mathbb{N}}, \mathcal{B}_{\mathbb{R}}^{\otimes\mathbb{N}})$, cf. e.g. Ex. 5, page 66 of [54], which gives a numerical model of Gaussian space in the sense of [87], §I-3. Since the n-dimensional sphere with radius r has curvature $(n-1)/r^2$, $S_n(\sqrt{n})$ has curvature $1-1/n$, and can be viewed as an infinite dimensional space with unit curvature when n is large. We refer to [30] and to Chapter 5 of [52] for approaches to this phenomenon using non standard analysis and white noise analysis respectively.

Thus our starting point is now a family $(\xi_n)_{n\in\mathbb{N}}$ of independent standard Gaussian random variables under $\gamma_{\mathbb{N}}$, constructed as the canonical projections from $(\mathbb{R}^{\mathbb{N}}, \mathcal{B}_{\mathbb{R}^{\mathbb{N}}}, \gamma_{\mathbb{N}})$ into $\mathbb{R}$. The measure $\gamma_{\mathbb{N}}$ is characterized by its Fourier transform

$$\begin{aligned}
\alpha \longmapsto \mathbb{E}\left[\exp\left(i\langle\xi,\alpha\rangle_{\ell^2(\mathbb{N})}\right)\right] &= \mathbb{E}\left[\exp\left(i\sum_{n=0}^{\infty}\xi_n\alpha_n\right)\right] \\
&= \prod_{n=0}^{\infty}\mathrm{e}^{-\alpha_n^2/2} \\
&= \mathrm{e}^{-\frac{1}{2}\|\alpha\|^2_{\ell^2(\mathbb{N})}}, \quad \alpha\in\ell^2(\mathbb{N}),
\end{aligned}$$

i.e. $\langle\xi,\alpha\rangle_{\ell^2(\mathbb{N})}$ is a centered Gaussian random variable with variance $\|\alpha\|^2_{\ell^2(\mathbb{N})}$. Let $(e_n)_{n\in\mathbb{N}}$ denote an orthonormal basis $(e_n)_{n\in\mathbb{N}}$ of $L^2(\mathbb{R}_+)$.

Definition 2.2.1. *Given $u\in L^2(\mathbb{R}_+)$ with decomposition*

$$u = \sum_{n=0}^{\infty}\langle u, e_n\rangle e_n,$$

we let $J_1 : L^2(\mathbb{R}_+) \longrightarrow L^2(\mathbb{R}^{\mathbb{N}}, \gamma^{\mathbb{N}})$ *be defined as*

$$J_1(u) = \sum_{n=0}^{\infty} \xi_n \langle u, e_n \rangle. \tag{2.2.2}$$

We have the isometry property

$$\begin{aligned} \mathbb{E}\left[|J_1(u)|^2\right] &= \sum_{k=0}^{\infty} |\langle u, e_n \rangle|^2 \mathbb{E}\left[\xi_n^2\right] \\ &= \sum_{k=0}^{\infty} |\langle u, e_n \rangle|^2 \\ &= \|u\|^2_{L^2(\mathbb{R}_+)}, \end{aligned} \tag{2.2.3}$$

and the characteristic function of $J_1(u)$ is given by

$$\begin{aligned} \mathbb{E}\left[\mathrm{e}^{iJ_1(u)}\right] &= \prod_{n=0}^{\infty} \mathbb{E}\left[\mathrm{e}^{i\xi_n \langle u, e_n \rangle}\right] \\ &= \prod_{n=0}^{\infty} \mathrm{e}^{-\frac{1}{2}\langle u, e_n \rangle^2_{L^2(\mathbb{R}_+)}} \\ &= \exp\left(-\frac{1}{2}\|u\|^2_{L^2(\mathbb{R}_+)}\right), \end{aligned}$$

hence $J_1(u)$ is a centered Gaussian random variable with variance $\|u\|^2_{L^2(\mathbb{R}_+)}$, cf. Section 9.2. Next is a constructive definition of Brownian motion, using the mapping J_1 and the decomposition

$$\mathbf{1}_{[0,t]} = \sum_{n=0}^{\infty} e_n \int_0^t e_n(s) ds.$$

Definition 2.2.2. *For all* $t \in \mathbb{R}_+$, *let*

$$B_t = J_1(\mathbf{1}_{[0,t]}) = \sum_{n=0}^{\infty} \xi_n \int_0^t e_n(s) ds. \tag{2.2.4}$$

Clearly, $B_t - B_s = J_1(\mathbf{1}_{[s,t]})$ is a Gaussian centered random variable with variance:

$$\mathbb{E}[(B_t - B_s)^2] = \mathbb{E}[|J_1(\mathbf{1}_{[s,t]})|^2] = \|\mathbf{1}_{[s,t]}\|^2_{L^2(\mathbb{R}_+)} = t - s. \tag{2.2.5}$$

Moreover, the isometry formula (2.2.3) shows that if $u_1, \ldots, u_n$ are orthogonal in $L^2(\mathbb{R}_+)$ then $J_1(u_1), \ldots, J_1(u_n)$ are also mutually orthogonal in $L^2(\Omega)$, hence from Corollary 16.1 of [71], see Proposition 9.2.1 in the Appendix, we get the following.

Proposition 2.2.3. *Let $u_1, \ldots, u_n$ be an orthogonal family in $L^2(\mathbb{R}_+)$, i.e.*

$$\langle u_i, u_j \rangle_{L^2(\mathbb{R}_+)} = 0, \qquad 1 \leq i \neq j \leq n.$$

Then $(J_1(u_1), \ldots, J_1(u_n))$ is a vector of independent Gaussian centered random variables with respective variances $\|u_1\|^2_{L^2(\mathbb{R}_+)}, \ldots, \|u_1\|^2_{L^2(\mathbb{R}_+)}$.

As a consequence of Proposition 2.2.3, $(B_t)_{t\in\mathbb{R}_+}$ has centered independent increments hence it is a martingale from Proposition 9.5.2 in the Appendix.

Moreover, from (2.2.5) and Remark 2.1.2 we deduce the following proposition.

Proposition 2.2.4. *The Brownian motion $(B_t)_{t\in\mathbb{R}_+}$ is a* normal martingale.

2.3 Compensated Poisson Martingale

The compensated Poisson process will provide a second example of a normal martingale. As mentioned at the beginning of Section 2, Gaussian distributions arise from uniform measures on infinite-dimensional spheres. They also can be constructed from the central limit theorem which states that if $(Y_1^n, \ldots, Y_n^n)$ is a sequence of independent, identically distributed centered random variables with variance σ^2/n,

$$Y_1^n + \cdots + Y_n^n, \qquad n \geq 1,$$

converges in distribution to a centered Gaussian random variable with variance σ^2.

In a discrete setting we let

$$S_n = Z_1^n + \cdots + Z_n^n, \qquad n \geq 1,$$

where $Z_1^n, \ldots, Z_n^n \in \{0, 1\}$ is a family of independent Bernoulli random variables with same parameter λ/n, i.e.

$$\mathbb{P}(Z_k^n = 1) = \frac{\lambda}{n}, \qquad 1 \leq k \leq n.$$

Then S_n has a binomial distribution with parameter $(n, \lambda/n)$:

$$\mathbb{P}(Z_1^n + \cdots + Z_n^n = k) = \binom{n}{k} \left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^{n-k}$$

which converges to $\frac{\lambda^k}{k!}\mathrm{e}^{-\lambda}$ as n goes to infinity, i.e.

$$Z_1^n + \cdots + Z_n^n$$

converges in distribution to a Poisson random variable with intensity $\lambda > 0$. This defines a probability measure π_λ on $\mathbb{Z}$ as

$$\pi_\lambda(\{k\}) = p_\lambda(k) := \mathbf{1}_{\{k\geq 0\}}\frac{\lambda^k}{k!}\mathrm{e}^{-\lambda}, \qquad k \in \mathbb{N}, \tag{2.3.1}$$

with the convolution property

$$\pi_\lambda \star \pi_\mu = \pi_{\lambda+\mu}.$$

Let now $(\tau_n)_{n\geq 1}$ denote a sequence of independent and identically exponentially distributed random variables, with parameter $\lambda > 0$, i.e.

$$\mathbb{E}[f(\tau_1,\ldots,\tau_n)] = \lambda^n \int_0^\infty \cdots \int_0^\infty \mathrm{e}^{-\lambda(s_1+\cdots+s_n)} f(s_1,\ldots,s_n) ds_1 \cdots ds_n, \tag{2.3.2}$$

for all sufficiently integrable $f : \mathbb{R}_+^n \to \mathbb{R}$. Let also

$$T_n = \tau_1 + \cdots + \tau_n, \qquad n \geq 1.$$

Next we consider the canonical point process associated to $(T_k)_{k\geq 1}$.

Definition 2.3.1. *The point process $(N_t)_{t\in\mathbb{R}_+}$ defined by*

$$N_t = \sum_{k=1}^\infty \mathbf{1}_{[T_k,\infty)}(t), \qquad t \in \mathbb{R}_+ \tag{2.3.3}$$

is called the standard Poisson point process with intensity $\lambda > 0$.

Relation (2.3.2) can be rewritten as

$$\mathbb{E}[f(T_1,\ldots,T_n)] = \lambda^n \int_0^\infty \cdots \int_0^\infty \mathrm{e}^{-\lambda t_n} f(t_1,\ldots,t_n)\mathbf{1}_{\{t_1<\cdots<t_n\}} dt_1 \cdots dt_n, \tag{2.3.4}$$

hence the law of $(T_1,\ldots,T_n)$ has density

$$(t_1,\ldots,t_n) \longmapsto \lambda^n \mathrm{e}^{-\lambda t_n}\mathbf{1}_{\{0\leq t_1<\cdots<t_n\}}$$

on $\mathbb{R}_+^n$.

From this we may directly show that N_t has a Poisson distribution with parameter λt:

$$\mathbb{P}(N_t = k) = \mathbb{P}(T_k \leq t < T_{k+1})$$

$$
\begin{aligned}
&= \lambda^{k+1} \int_0^\infty e^{-\lambda t_{k+1}} \int_0^{t_{k+1}} \cdots \int_0^{t_2} \mathbf{1}_{\{t_k < t \le t_{k+1}\}} dt_1 \cdots dt_{k+1} \\
&= \lambda^{k+1} \int_t^\infty e^{-\lambda t_{k+1}} \int_0^t \int_0^{t_k} \cdots \int_0^{t_2} dt_1 \cdots dt_{k+1} \\
&= \frac{t^k}{k!} \lambda^{k+1} \int_t^\infty e^{-\lambda t_{k+1}} dt_{k+1} \\
&= \frac{(\lambda t)^k}{k!} e^{-\lambda t}, \qquad t > 0.
\end{aligned}
$$

Proposition 2.3.2. *The law of T_n has the density $t \longmapsto \lambda^n e^{-\lambda t} \frac{t^{n-1}}{(n-1)!}$ on $\mathbb{R}_+$, $n \ge 1$.*

Proof. We have

$$
\mathbb{P}(T_1 > t) = \mathbb{P}(N_t = 0) = e^{-\lambda t}, \qquad t \in \mathbb{R}_+,
$$

and by induction, assuming that

$$
\mathbb{P}(T_{n-1} > t) = \lambda \int_t^\infty e^{-\lambda s} \frac{(\lambda s)^{n-2}}{(n-2)!} ds, \qquad n \ge 2,
$$

we obtain

$$
\begin{aligned}
\mathbb{P}(T_n > t) &= \mathbb{P}(T_n > t \ge T_{n-1}) + \mathbb{P}(T_{n-1} > t) \\
&= \mathbb{P}(N_t = n-1) + \mathbb{P}(T_{n-1} > t) \\
&= e^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!} + \lambda \int_t^\infty e^{-\lambda s} \frac{(\lambda s)^{n-2}}{(n-2)!} ds \\
&= \lambda \int_t^\infty e^{-\lambda s} \frac{(\lambda s)^{n-1}}{(n-1)!} ds, \qquad t \in \mathbb{R}_+.
\end{aligned}
$$

□

For any bounded measurable functions $g : \mathbb{R}_+ \to \mathbb{R}$ and $f : \mathbb{R}_+^n \to \mathbb{R}$ we also have

$$
\begin{aligned}
&\mathbb{E}[g(T_{n+1})\mathbb{E}[f(T_1, \ldots, T_n)|T_{n+1}]] = \mathbb{E}[g(T_{n+1}) f(T_1, \ldots, T_n)] \\
&= \lambda^{n+1} \int_0^\infty g(t_{n+1}) e^{-\lambda t_{n+1}} \int_0^{t_{n+1}} \cdots \int_0^{t_2} f(t_1, \ldots, t_n) dt_1 \cdots dt_{n+1} \\
&= \int_0^\infty \cdots \int_0^\infty \frac{n!}{t_{n+1}^n} f(t_1, \ldots, t_n) \mathbf{1}_{\{t_1 < \cdots < t_{n+1}\}} dt_1 \cdots dt_n g(t_{n+1}) d\mathbb{P}(T_{n+1} = t_{n+1}) \\
&= \int_0^\infty \mathbb{E}[f(T_1, \ldots, T_n)|T_{n+1} = t_{n+1}] g(t_{n+1}) d\mathbb{P}(T_{n+1} = t_{n+1}),
\end{aligned}
$$

where $d\mathbb{P}(T_{n+1} = t_{n+1})$ denotes $\mathbb{P}(T_{n+1} \in dt_{n+1})$. As a consequence, we have the next proposition.

Proposition 2.3.3. *The conditional density of* $(T_1, \ldots, T_n)$ *given that* $T_{n+1} = T$ *is*

$$(t_1, \ldots, t_n) \longmapsto \frac{n!}{T^n} \mathbf{1}_{\{0 \leq t_1 < \cdots < t_n \leq T\}}.$$

Moreover we have

$$\begin{aligned}
&\mathbb{E}\left[f\left(\frac{T_1}{T_{n+1}}, \ldots, \frac{T_n}{T_{n+1}} \right) g(T_{n+1}) \right] \\
&= \lambda^{n+1} \int_0^\infty \mathrm{e}^{-\lambda t_{n+1}} \int_0^{t_{n+1}} \cdots \int_0^{t_2} f\left(\frac{t_1}{t_{n+1}}, \ldots, \frac{t_n}{t_{n+1}} \right) g(t_{n+1}) dt_1 \cdots dt_{n+1} \\
&= \lambda^{n+1} \int_0^\infty (t_{n+1})^n g(t_{n+1}) \mathrm{e}^{-\lambda t_{n+1}} \int_0^1 \int_0^{s_n} \cdots \int_0^{s_2} f(s_1, \ldots, s_n) ds_1 \cdots ds_n dt_{n+1} \\
&= n! \int_0^1 \int_0^{s_n} \cdots \int_0^{s_2} f(s_1, \ldots, s_n) ds_1 \cdots ds_n \int_0^\infty g(t_{n+1}) d\mathbb{P}(T_{n+1} = t_{n+1}) \\
&= \mathbb{E}\left[f\left(\frac{T_1}{T_{n+1}}, \ldots, \frac{T_n}{T_{n+1}} \right) \right] \mathbb{E}[g(T_{n+1})],
\end{aligned}$$

hence $\left(\frac{T_1}{T_{n+1}}, \ldots, \frac{T_n}{T_{n+1}} \right)$ is independent of T_{n+1} and has density

$$(s_1, \ldots, s_n) \longmapsto n! \mathbf{1}_{\{0 \leq s_1 < \cdots < s_n \leq 1\}}$$

on $[0,1]^n$, cf. e.g. [28]. As will be seen in Proposition 2.5.10 below, the random sum

$$\sum_{k=1}^{N_a} f(T_k)$$

used in the next proposition can be interpreted as the Stieltjes integral

$$\int_0^a f(t) dN_t$$

of f with respect to $(N_t)_{t \in \mathbb{R}_+}$.

Proposition 2.3.4. *Let* $a > 0$ *and let* $f : [0, a] \to \mathbb{R}$ *be measurable. We have*

$$\mathbb{E}\left[\exp\left(i \sum_{k=1}^{N_a} f(T_k) \right) \right] = \exp\left(\lambda \int_0^a (\mathrm{e}^{if(t)} - 1) dt \right).$$

Proof. Let g_n, $n \geq 1$, be defined as

$$g_n(t_1, \ldots, t_n) = \sum_{k=1}^n \mathbf{1}_{\{t_{k-1} < a < t_k\}} \mathrm{e}^{if(t_1) + \cdots + if(t_{k-1})} + \mathbf{1}_{\{t_n < a\}} \mathrm{e}^{if(t_1) + \cdots + if(t_n)},$$

with $t_0 = 0$, so that

$$\exp\left(i\sum_{k=1}^{n\wedge N_a} f(T_k)\right) = g_n(T_1,\ldots,T_n).$$

Then we have

$$\begin{aligned}
&\mathbb{E}\left[\exp\left(i\sum_{k=1}^{n\wedge N_a} f(T_k)\right)\right] = \mathbb{E}[g_n(T_1,\ldots,T_n)] \\
&= \lambda^n\sum_{k=1}^{n}\int_0^\infty \mathrm{e}^{-\lambda t_n}\int_0^{t_n}\cdots\int_0^{t_2}\mathbf{1}_{\{t_{k-1}<a<t_k\}}\mathrm{e}^{if(t_1)+\cdots+if(t_{k-1})}dt_1\cdots dt_n \\
&\quad+\lambda^n\int_0^a \mathrm{e}^{-\lambda t_n}\int_0^{t_n}\cdots\int_0^{t_2}\mathrm{e}^{if(t_1)+\cdots+if(t_n)}dt_1\cdots dt_n \\
&= \lambda^n\sum_{k=1}^{n}\int_a^\infty \mathrm{e}^{-\lambda t_n}\frac{(t_n-a)^{n-k}}{(n-k-1)!}dt_n \\
&\quad\times\int_0^a\int_0^{t_{k-1}}\cdots\int_0^{t_2}\mathrm{e}^{if(t_1)+\cdots+if(t_{k-1})}dt_1\cdots dt_{k-1} \\
&\quad+\lambda^n\int_0^a \mathrm{e}^{-\lambda t_n}\int_0^{t_n}\cdots\int_0^{t_2}\mathrm{e}^{if(t_1)+\cdots+if(t_n)}dt_1\cdots dt_n \\
&= \mathrm{e}^{-\lambda a}\sum_{k=1}^{n}\lambda^k\int_0^\infty \mathrm{e}^{-\lambda t}\frac{(\lambda t)^{n-k}}{(n-k-1)!}dt \\
&\quad\times\int_0^a\int_0^{t_{k-1}}\cdots\int_0^{t_2}\mathrm{e}^{if(t_1)+\cdots+if(t_{k-1})}dt_1\cdots dt_{k-1} \\
&\quad+\lambda^n\int_0^a \mathrm{e}^{-\lambda t_n}\int_0^{t_n}\cdots\int_0^{t_2}\mathrm{e}^{if(t_1)+\cdots+if(t_n)}dt_1\cdots dt_n \\
&= \mathrm{e}^{-\lambda a}\sum_{k=1}^{n}\lambda^k\int_0^a\int_0^{t_{k-1}}\cdots\int_0^{t_2}\mathrm{e}^{if(t_1)+\cdots+if(t_{k-1})}dt_1\cdots dt_{k-1} \\
&\quad+\mathrm{e}^{-\lambda a}\sum_{k=n}^{\infty}\lambda^k\int_0^a\frac{(a-t_n)^{k-n}}{(k-n)!}\int_0^{t_n}\cdots\int_0^{t_2}\mathrm{e}^{if(t_1)+\cdots+if(t_n)}dt_1\cdots dt_n \\
&= \mathrm{e}^{-\lambda a}\sum_{k=1}^{n}\lambda^k\int_0^a\int_0^{t_{k-1}}\cdots\int_0^{t_2}\mathrm{e}^{if(t_1)+\cdots+if(t_{k-1})}dt_1\cdots dt_{k-1} \\
&\quad+\mathrm{e}^{-\lambda a}\sum_{k=n}^{\infty}\lambda^k\int_0^a\int_{t_n}^a\int_{t_n}^{t_k}\cdots\int_{t_n}^{t_{n+2}}dt_{n+1}\cdots dt_k \\
&\quad\times\int_0^{t_{n+1}}\cdots\int_0^{t_2}\mathrm{e}^{if(t_1)+\cdots+if(t_n)}dt_1\cdots dt_n \\
&= \mathrm{e}^{-\lambda a}\sum_{k=0}^{\infty}\lambda^k\int_0^a\int_0^{t_k}\cdots\int_0^{t_2}\mathrm{e}^{if(t_1)+\cdots+if(t_{k\wedge n})}dt_1\cdots dt_k \\
&= \mathrm{e}^{-\lambda a}\sum_{k=0}^{\infty}\frac{\lambda^k}{k!}\int_0^a\cdots\int_0^a\mathrm{e}^{if(t_1)+\cdots+if(t_{k\wedge n})}dt_1\cdots dt_k.
\end{aligned}$$

Hence as n goes to infinity,

$$\mathbb{E}\left[\exp\left(i\sum_{k=1}^{N_a} f(T_k)\right)\right] = \lim_{n\to\infty}\mathbb{E}\left[\exp\left(i\sum_{k=1}^{n\wedge N_a} f(T_k)\right)\right]$$
$$= \lim_{n\to\infty} \mathrm{e}^{-\lambda a}\sum_{k=0}^{\infty}\frac{\lambda^k}{k!}\int_0^a\cdots\int_0^a \mathrm{e}^{if(t_1)+\cdots+if(t_{k\wedge n})}dt_1\cdots dt_k$$
$$= \mathrm{e}^{-\lambda a}\sum_{k=0}^{\infty}\frac{\lambda^k}{k!}\int_0^a\cdots\int_0^a \mathrm{e}^{if(t_1)+\cdots+if(t_k)}dt_1\cdots dt_k$$
$$= \exp\left(\lambda\int_0^a(\mathrm{e}^{if(t)}-1)dt\right).$$

□

The next corollary states the standard definition of the Poisson process.

Corollary 2.3.5. *The counting process $(N_t)_{t\in\mathbb{R}_+}$ defined by (2.3.3) has independent increments which are distributed according to the Poisson law, i.e. for all $0 \le t_0 \le t_1 < \cdots < t_n$,*

$$(N_{t_1} - N_{t_0}, \ldots, N_{t_n} - N_{t_{n-1}})$$

is a vector of independent Poisson random variables with respective parameters

$$(\lambda(t_1 - t_0), \ldots, \lambda(t_n - t_{n-1})).$$

Proof. Letting

$$f = \sum_{k=1}^{n}\alpha_k \mathbf{1}_{(t_{k-1},t_k]},$$

from Proposition 2.3.4 we get

$$\mathbb{E}\left[\exp\left(i\sum_{k=1}^{n}\alpha_k(N_{t_k} - N_{t_{k-1}})\right)\right] = \prod_{k=1}^{n}\mathrm{e}^{\lambda(t_k-t_{k-1})(\mathrm{e}^{i\alpha_k}-1)}$$
$$= \exp\left(\lambda\sum_{k=1}^{n}(t_k - t_{k-1})(\mathrm{e}^{i\alpha_k}-1)\right)$$
$$= \prod_{k=1}^{n}\mathbb{E}\left[\mathrm{e}^{i\alpha_k(N_{t_k}-N_{t_{k-1}})}\right],$$

for all $0 = t_0 \le t_1 < \cdots < t_n$, hence the components of the vector

$$(N_{t_1} - N_{t_0}, \ldots, N_{t_n} - N_{t_{n-1}})$$

are independent Poisson random variables with parameters

$$(\lambda(t_n - t_{n-1}), \ldots, \lambda(t_1 - t_0)).$$

□

In particular, for all $n \in \mathbb{Z}$ and $t \in \mathbb{R}_+$, we have

$$\mathbb{P}(N_t = n) = p_n(t) = \mathrm{e}^{-\lambda t} \frac{(\lambda t)^n}{n!},$$

i.e. $p_{n-1} : \mathbb{R}_+ \to \mathbb{R}_+$, $n \geq 1$, is the density function of T_n, and

$$p'_n(t) = p_{n-1}(t) - p_n(t), \qquad n \in \mathbb{Z}, \ t \in \mathbb{R}_+.$$

Relation (2.3.4) above can be extended as a statement in conditional expectation using the independent increment property of the Poisson process. Let

$$\Delta_n = \{(t_1, \ldots, t_n) \in \mathbb{R}_+^n \ : \ 0 \leq t_1 < \cdots < t_n\}, \qquad n \geq 1.$$

Proposition 2.3.6. *For any* $f \in L^1(\Delta_n, \mathrm{e}^{-s_n} ds_1 \cdots ds_n)$ *we have*

$$\begin{aligned} &\mathbb{E}[f(T_1, \ldots, T_n) | \mathcal{F}_t] \qquad (2.3.5)\\ &= \int_t^\infty \mathrm{e}^{-(s_n - t)} \int_t^{s_n} \cdots \int_t^{s_{N_t+2}} f(T_1, \ldots, T_{N_t}, s_{N_t+1}, \ldots, s_n) ds_{N_t+1} \cdots ds_n. \end{aligned}$$

Proof. Apply Relation (2.3.4) using the fact that for fixed $t > 0$, $(N_s - N_t)_{s \geq t}$ is a standard Poisson process independent of $\mathcal{F}_t$. □

In particular we have

$$\mathbb{E}[f(T_n) | \mathcal{F}_t] = 1_{\{N_t \geq n\}} f(T_n) + \int_t^\infty p_{n-1-N_t}(x - t) f(x) dx, \qquad (2.3.6)$$

and taking $t = 0$ in (2.3.5) recovers Relation (2.3.4).

Proposition 2.3.7. *Given that* $\{N_T \geq n\}$*, the (unordered) jump times* $\{T_1, \ldots, T_n\}$ *of the Poisson process* $(N_t)_{t \in \mathbb{R}_+}$ *are independent uniformly distributed random variables on* $[0, T]^n$*.*

Proof. For all $n \geq 1$ and $f_n \in \mathcal{C}_c([0, T]^n)$ a *symmetric* function in n variables, we have

$$\begin{aligned} &\mathbb{E}\left[f_n(T_1, \ldots, T_n) \mathbf{1}_{\{N_T \geq n\}}\right] = \lambda^n \int_0^T \mathrm{e}^{-\lambda t_n} \int_0^{t_n} \cdots \int_0^{t_2} f_n(t_1, \ldots, t_n) dt_1 \cdots dt_n \\ &= \mathrm{e}^{-\lambda T} \lambda^n \sum_{k=n}^\infty \lambda^{k-n} \int_0^T \frac{(T - t_n)^{k-n}}{(k-n)!} \int_0^{t_n} \cdots \int_0^{t_2} f_n(t_1, \ldots, t_n) dt_1 \cdots dt_n \\ &= \mathrm{e}^{-\lambda T} \sum_{k=n}^\infty \lambda^k \int_{t_n}^T \int_{t_n}^{t_k} \cdots \int_{t_n}^{t_{n+2}} dt_{n+1} \cdots dt_k \\ &\quad \times \int_0^{t_{n+1}} \cdots \int_0^{t_2} f_n(t_1, \ldots, t_n) dt_1 \cdots dt_n \end{aligned}$$

$$
\begin{aligned}
&= e^{-\lambda T} \sum_{k=n}^{\infty} \lambda^k \int_0^T \int_0^{t_k} \cdots \int_0^{t_2} f_n(t_1, \ldots, t_n) dt_1 \cdots dt_k \\
&= e^{-\lambda T} \sum_{k=n}^{\infty} \frac{\lambda^k}{k!} \int_0^T \cdots \int_0^T f_n(t_1, \ldots, t_n) dt_1 \cdots dt_k \qquad (2.3.7) \\
&= \sum_{k=n}^{\infty} \frac{1}{T^k} \mathbb{P}(N_T = k) \int_0^T \cdots \int_0^T f_n(t_1, \ldots, t_n) dt_1 \cdots dt_k \\
&= \frac{1}{T^n} \int_0^T \cdots \int_0^T f_n(t_1, \ldots, t_n) dt_1 \cdots dt_n \sum_{k=n}^{\infty} \mathbb{P}(N_T = k) \\
&= \frac{1}{T^n} \mathbb{P}(N_T \geq n) \int_0^T \cdots \int_0^T f_n(t_1, \ldots, t_n) dt_1 \cdots dt_n \\
&= \mathbb{E}\left[f_n(T_1, \ldots, T_n) | N_T \geq n\right] \mathbb{P}(N_T \geq n),
\end{aligned}
$$

where we used the symmetry of f_n in (2.3.7), hence we have

$$
\mathbb{E}\left[f_n(T_1, \ldots, T_n) | N_T \geq n\right] = \frac{1}{T^n} \int_0^T \cdots \int_0^T f_n(t_1, \ldots, t_n) dt_1 \cdots dt_n.
$$

□

As a consequence of Corollary 2.3.5, $(M_t)_{t \in \mathbb{R}_+} = \lambda^{-1/2}(N_t - \lambda t)_{t \in \mathbb{R}_+}$ has centered independent increments, hence it is a martingale. Moreover a direct variance computation under the Poisson law shows that we have

$$
\begin{aligned}
\mathbb{E}[(M_t - M_s)^2 | \mathcal{F}_s] &= \mathbb{E}[(M_t - M_s)^2] \\
&= t - s,
\end{aligned}
$$

hence from Remark 2.1.2 we get the next proposition.

Proposition 2.3.8. *The normalized compensated Poisson process*

$$
\lambda^{-1/2}(N_t - \lambda t)_{t \in \mathbb{R}_+}
$$

is a normal martingale.

2.4 Compound Poisson Martingale

In this section, $(Y_k)_{k \geq 1}$ denotes an *i.i.d.* sequence of random variables with probability distribution $\nu(dy)$ on $\mathbb{R}$.

Definition 2.4.1. *The process*

$$
X_t = \sum_{k=1}^{N_t} Y_k, \qquad t \in \mathbb{R}_+,
$$

is called a compound Poisson process.

For simplicity we only consider real-valued compound processes, however this notion is easily extended to the $\mathbb{R}^d$-valued case, see Relation (6.3.2) in Section 6.3 below.

The compound Poisson processes provide other examples of normal martingales.

Proposition 2.4.2. *For any* $t \in [0, T]$ *we have*

$$\mathbb{E}\left[\exp\left(i\alpha(X_T - X_t)\right)\right] = \exp\left(\lambda(T-t)\int_{-\infty}^{\infty}(\mathrm{e}^{iy\alpha} - 1)\nu(dy)\right),$$

$\alpha \in \mathbb{R}$.

Proof. Since N_t has a Poisson distribution with parameter $t > 0$ and is independent of $(Y_k)_{k\geq 1}$, for all $\alpha \in \mathbb{R}$ we have by conditioning:

$$\begin{aligned}
\mathbb{E}\left[\exp\left(i\alpha(X_T - X_t)\right)\right] &= \mathbb{E}\left[\mathbb{E}\left[\exp\left(i\alpha\sum_{k=N_t}^{N_T} Y_k\right)\Big| N_T - N_t\right]\right] \\
&= \mathrm{e}^{-\lambda(T-t)}\sum_{n=0}^{\infty}\frac{\lambda^n}{n!}(T-t)^n\mathbb{E}\left[\exp\left(i\alpha\sum_{k=1}^{n} Y_k\right)\Big| N_T - N_t = n\right] \\
&= \mathrm{e}^{-\lambda(T-t)}\sum_{n=0}^{\infty}\frac{\lambda^n}{n!}(T-t)^n\mathbb{E}\left[\exp\left(i\alpha\sum_{k=1}^{n} Y_k\right)\right] \\
&= \mathrm{e}^{-\lambda(T-t)}\sum_{n=0}^{\infty}\frac{\lambda^n}{n!}(T-t)^n\left(\mathbb{E}\left[\exp\left(i\alpha Y_1\right)\right]\right)^n \\
&= \mathrm{e}^{-\lambda(T-t)}\sum_{n=0}^{\infty}\frac{1}{n!}\left(\lambda(T-t)\int_{-\infty}^{\infty}\mathrm{e}^{i\alpha y}\nu(dy)\right)^n \\
&= \exp\left(\lambda(T-t)\int_{-\infty}^{\infty}(\mathrm{e}^{i\alpha y} - 1)\nu(dy)\right),
\end{aligned}$$

since $\nu(dy)$ is the probability distribution of Y_1. □

In particular we have

$$\begin{aligned}
\mathbb{E}[X_t] &= \mathbb{E}\left[\mathbb{E}\left[\sum_{k=1}^{N_t} Y_k \Big| N_t\right]\right] \\
&= \mathrm{e}^{-\lambda t}\sum_{n=0}^{\infty}\frac{\lambda^n t^n}{n!}\mathbb{E}\left[\sum_{k=1}^{n} Y_k \Big| N_t = n\right] \\
&= \mathrm{e}^{-\lambda t}\sum_{n=0}^{\infty}\frac{\lambda^n t^n}{n!}\mathbb{E}\left[\sum_{k=1}^{n} Y_k\right]
\end{aligned}$$

$$= \mathrm{e}^{-\lambda t}\mathbb{E}[Y_1]\sum_{n=1}^{\infty}\frac{\lambda^n t^n}{(n-1)!}$$
$$= \lambda t\mathbb{E}[Y_1],$$

and

$$\begin{aligned}
\mathrm{Var}\,[X_t] &= \mathbb{E}\left[\left(\sum_{k=1}^{N_t} Y_k - \mathbb{E}[X_t]\right)^2\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[\left(\sum_{k=1}^{N_t} Y_k - \mathbb{E}[X_t]\right)^2 \Big| N_t\right]\right] \\
&= \mathrm{e}^{-\lambda t}\sum_{n=0}^{\infty}\frac{\lambda^n t^n}{n!}\mathbb{E}\left[\mathbb{E}\left[\left(\sum_{k=1}^{n} Y_k - \lambda t\mathbb{E}[Y_1]\right)^2 \Big| N_t = n\right]\right] \\
&= \mathrm{e}^{-\lambda t}\sum_{n=0}^{\infty}\frac{\lambda^n t^n}{n!}\mathbb{E}\left[\left(\sum_{k=1}^{n} Y_k - \lambda t\mathbb{E}[Y_1]\right)^2\right] \\
&= \mathrm{e}^{-\lambda t}\sum_{n=0}^{\infty}\frac{\lambda^n t^n}{n!}\mathbb{E}\left[2\sum_{1\le k<l\le n} Y_k Y_l + \sum_{k=1}^{n}|Y_k|^2 - 2\lambda t\mathbb{E}[Y_1]\sum_{k=1}^{n} Y_k + \lambda^2 t^2(\mathbb{E}[Y_1])^2\right] \\
&= \mathrm{e}^{-\lambda t}\sum_{n=0}^{\infty}\frac{\lambda^n t^n}{n!}(n(n-1)(\mathbb{E}[Y_1])^2 + n\mathbb{E}[|Y_1|^2] - 2n\lambda t(\mathbb{E}[Y_1])^2 + \lambda^2 t^2(\mathbb{E}[Y_1])^2) \\
&= \mathrm{e}^{-\lambda t}(\mathbb{E}[Y_1])^2\sum_{n=2}^{\infty}\frac{\lambda^n t^n}{(n-2)!} + \mathrm{e}^{-\lambda t}\mathbb{E}[|Y_1|^2]\sum_{n=1}^{\infty}\frac{\lambda^n t^n}{(n-1)!} \\
&\quad -2\mathrm{e}^{-\lambda t}\lambda t(\mathbb{E}[Y_1])^2\sum_{n=1}^{\infty}\frac{\lambda^n t^n}{(n-1)!} + \lambda^2 t^2(\mathbb{E}[Y_1])^2) \\
&= \lambda t\mathbb{E}[|Y_1|^2].
\end{aligned}$$

Both relations can be recovered from the characteristic function of X_t, as

$$\begin{aligned}
\mathbb{E}[X_t] &= -i\frac{d}{d\alpha}\mathbb{E}[\mathrm{e}^{i\alpha X_t}]_{|\alpha=0} \\
&= \lambda t\int_{-\infty}^{\infty} y\mu(dy) \\
&= \lambda t\mathbb{E}[Y_1],
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{Var}\,[X_t] &= \mathbb{E}[|X_t|^2] - (\mathbb{E}[X_t])^2 \\
&= -\frac{d^2}{d\alpha^2}\mathbb{E}[\mathrm{e}^{i\alpha X_t}]_{|\alpha=0} - \lambda^2 t^2(\mathbb{E}[Y_1])^2
\end{aligned}$$

$$= \lambda t \int_{-\infty}^{\infty} |y|^2 \mu(dy)$$
$$= \lambda t \mathbb{E}[|Y_1|^2].$$

We can show as in Corollary 2.3.5 that $(X_t)_{t\in\mathbb{R}_+}$ has independent increments, in particular for all $0 \leq t_0 \leq t_1 \cdots \leq t_n$ and $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ we have

$$\begin{aligned}\mathbb{E}\left[\prod_{k=1}^n \mathrm{e}^{i\alpha_k(X_{t_k}-X_{t_{k-1}})}\right] &= \exp\left(\lambda \sum_{k=1}^n (t_k - t_{k-1}) \int_{-\infty}^{\infty} (\mathrm{e}^{i\alpha_k y} - 1)\nu(dy)\right) \\ &= \prod_{k=1}^n \exp\left(\lambda(t_k - t_{k-1}) \int_{-\infty}^{\infty} (\mathrm{e}^{i\alpha_k y} - 1)\nu(dy)\right) \\ &= \prod_{k=1}^n \mathbb{E}\left[\mathrm{e}^{i\alpha_k(X_{t_k}-X_{t_{k-1}})}\right].\end{aligned}$$

Consider the compensated and rescaled process $(M_t)_{t\in\mathbb{R}_+}$ defined by

$$M_t := \frac{X_t - \lambda t \mathbb{E}[Y_1]}{\sqrt{\lambda \mathbb{E}[Y_1^2]}}, \qquad t \in \mathbb{R}_+.$$

We note that

$$\mathbb{E}[(M_t - M_s)^2 | \mathcal{F}_s] = \mathbb{E}[(M_t - M_s)^2] = t - s, \qquad 0 \leq s \leq t.$$

hence the following proposition.

Proposition 2.4.3. *The compensated and rescaled process $(M_t)_{t\in\mathbb{R}_+}$ is a* normal martingale*.*

Compound Poisson processes belong to the more general family of Lévy processes, see Section 6.1 of Chapter 6.

2.5 Stochastic Integrals

In this section we construct the Itô stochastic integral of square-integrable adapted processes with respect to normal martingales. Recall that the filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ is generated by $(M_t)_{t\in\mathbb{R}_+}$, i.e.

$$\mathcal{F}_t = \sigma(M_s \ : \ 0 \leq s \leq t), \qquad t \in \mathbb{R}_+.$$

A process $(X_t)_{t\in\mathbb{R}_+}$ is said to be $\mathcal{F}_t$-adapted if X_t is $\mathcal{F}_t$-measurable for all $t \in \mathbb{R}_+$.

Definition 2.5.1. *Let $L^p_{ad}(\Omega \times \mathbb{R}_+)$, $p \in [1, \infty]$, denote the space of $\mathcal{F}_t$-adapted processes in $L^p(\Omega \times \mathbb{R}_+)$.*

Stochastic integrals will be first constructed as integrals of simple predictable processes.

Definition 2.5.2. *Let $\mathcal{S}$ be a space of random variables dense in $L^2(\Omega,\mathcal{F},\mathbb{P})$. Consider the following spaces of simple processes:*

i) let $\mathcal{U}$ denote the space of simple processes of the form

$$u_t = \sum_{i=1}^n F_i \mathbf{1}_{(t_{i-1}^n, t_i^n]}(t), \qquad t \in \mathbb{R}_+,$$

$F_1,\ldots,F_n \in \mathcal{S}$, $t_0^n := 0 \leq t_1^n < \cdots < t_n^n$, $n \geq 1$.

ii) let $\mathcal{P}$ denote the subspace of $\mathcal{U}$ made of simple predictable processes $(u_t)_{t\in\mathbb{R}_+}$ of the form

$$u_t = \sum_{i=1}^n F_i \mathbf{1}_{(t_{i-1}^n, t_i^n]}(t), \qquad t \in \mathbb{R}_+, \tag{2.5.1}$$

where F_i is $\mathcal{F}_{t_{i-1}^n}$-measurable, $i = 1,\ldots,n$.

One easily checks that the set $\mathcal{P}$ of simple predictable processes forms a linear space. Part (ii) of the next proposition also follows from Lemma 1.1 of [66], pages 22 and 46.

Proposition 2.5.3. *Let $p \geq 1$.*

i) The space $\mathcal{U}$ of simple processes is dense in $L^p(\Omega\times\mathbb{R}_+)$.

ii) The space $\mathcal{P}$ of simple predictable processes is dense in $L^p_{ad}(\Omega\times\mathbb{R}_+)$.

Proof. We will prove both (i) and (ii) by the same argument, starting with the case $p = 2$. Let $(u_t)_{t\in[0,T]} \in L^2(\Omega\times[0,T])$ and consider the sequence $(u^n)_{n\in\mathbb{N}}$ of simple processes defined as

$$u_t^n = \sum_{i=1}^n \mathbf{1}_{(t_{i-1}^n, t_i^n]} \frac{1}{t_{i-1}^n - t_{i-2}^n} \int_{t_{i-2}^n}^{t_{i-1}^n} u_s ds, \quad t \in \mathbb{R}_+, \quad n \geq 1, \tag{2.5.2}$$

where $0 = t_{-1}^n < t_0^n < t_1^n < \cdots < t_{n-1}^n < t_n^n = T$ is a subdivision of $[0,T]$. Clearly, u^n belongs to $\mathcal{U}$ and in addition it is predictable and belongs to $\mathcal{P}$ when $(u_t)_{t\in[0,T]}$ is adapted. We have

$$\begin{aligned}&\|u-u^n\|^2_{L^2(\Omega\times\mathbb{R}_+)}\\ &= \mathbb{E}\left[\int_0^\infty \left(u_s - \sum_{i=1}^n \mathbf{1}_{(t_{i-1}^n, t_i^n]}(s) \frac{1}{t_{i-1}^n - t_{i-2}^n} \int_{t_{i-2}^n}^{t_{i-1}^n} u_\tau d\tau\right)^2 ds\right]\end{aligned}$$

$$
\begin{aligned}
&= \mathbb{E}\left[\int_0^\infty \left(\sum_{i=1}^n \mathbf{1}_{(t_{i-1}^n, t_i^n]}(s)\left(u_s - \frac{1}{t_{i-1}^n - t_{i-2}^n}\int_{t_{i-2}^n}^{t_{i-1}^n} u_\tau d\tau\right)\right)^2 ds\right] \\
&= \mathbb{E}\left[\int_0^\infty \left(\sum_{i=1}^n \mathbf{1}_{(t_{i-1}^n, t_i^n]}(s)\frac{1}{t_{i-1}^n - t_{i-2}^n}\int_{t_{i-2}^n}^{t_{i-1}^n} (u_s - u_\tau) d\tau\right)^2 ds\right] \\
&= \mathbb{E}\left[\int_0^\infty \sum_{i=1}^n \mathbf{1}_{(t_{i-1}^n, t_i^n]}(s)\left(\int_{t_{i-2}^n}^{t_{i-1}^n} \frac{u_s - u_\tau}{t_{i-1}^n - t_{i-2}^n} d\tau\right)^2 ds\right] \\
&\leq \mathbb{E}\left[\int_0^\infty \sum_{i=1}^n \mathbf{1}_{(t_{i-1}^n, t_i^n]}(s)\frac{1}{t_{i-1}^n - t_{i-2}^n}\int_{t_{i-2}^n}^{t_{i-1}^n} (u_s - u_\tau)^2 d\tau ds\right] \\
&= \sum_{i=1}^n (t_i^n - t_{i-1}^n)\int_{t_{i-1}^n}^{t_i^n}\int_{t_{i-2}^n}^{t_{i-1}^n} \mathbb{E}\left[\frac{(u_s - u_\tau)^2}{(t_i^n - t_{i-1}^n)(t_{i-1}^n - t_{i-2}^n)}\right] d\tau ds,
\end{aligned}
$$

which tends to 0 provided that $(u_t)_{t\in[0,T]}$ is (uniformly) continuous in $L^2(\Omega)$ on $[0,T]$. If $(u_t)_{t\in[0,T]} \in L^2(\Omega \times [0,T])$ is uniformly bounded but not continuous in $L^2(\Omega)$, then it can be approximated in $L^2(\Omega \times [0,T])$ by continuous processes in $L^2(\Omega)$ using convolution by a smooth approximation of unity:

$$
\begin{aligned}
u_t^\varepsilon(\omega) &= \frac{1}{\varepsilon}\int_{-\infty}^\infty u_s(\omega)\varphi\left(\frac{t-s}{\varepsilon}\right) ds \\
&= \frac{1}{\varepsilon}\int_{-\infty}^t u_s(\omega)\varphi\left(\frac{t-s}{\varepsilon}\right) ds,
\end{aligned}
$$

as $\varepsilon > 0$ tends to 0, where $\varphi \in \mathcal{C}_c^\infty(\mathbb{R})$ is supported on $[0,1]$ and such that $\int_{-\infty}^\infty \varphi(x)dx = 1$. Moreover, the process u^ε is adapted when $(u_t)_{t\in[0,T]}$ is adapted.

The conclusion follows by approximation of $(u_t)_{t\in[0,T]}$ in $L^p(\Omega \times [0,T])$ by uniformly bounded (adapted) processes, $p \geq 1$. □

Note that an argument similar to the above shows that the simple processes

$$
\sum_{i=1}^n u_{t_{i-1}^n} \mathbf{1}_{(t_{i-1}^n, t_i^n]}
$$

also converges to $(u_t)_{t\in[0,T]}$ in $L^2(\Omega \times [0,T])$ when $(u_t)_{t\in[0,T]}$ is continuous on $[0,T]$, a.e. uniformly on $[0,T] \times \Omega$, since

$$
\mathbb{E}\left[\int_0^\infty \left(u_s - \sum_{i=1}^n u_{t_{i-1}^n}\mathbf{1}_{(t_{i-1}^n, t_i^n]}(s)\right)^2 ds\right]
$$

$$= \mathbb{E}\left[\int_0^\infty \left(\sum_{i=1}^n \mathbf{1}_{(t_{i-1}^n, t_i^n]}(s)\left(u_s - u_{t_{i-1}^n}\right)\right)^2 ds\right]$$
$$= \mathbb{E}\left[\int_0^\infty \sum_{i=1}^n \mathbf{1}_{(t_{i-1}^n, t_i^n]}(s)\left(u_s - u_{t_{i-1}^n}\right)^2 ds\right]$$
$$= \sum_{i=1}^n (t_i^n - t_{i-1}^n)\int_{t_{i-1}^n}^{t_i^n} \mathbb{E}\left[\frac{(u_s - u_{t_{i-1}^n})^2}{(t_i^n - t_{i-1}^n)}\right] ds.$$

The stochastic integral of a simple predictable processes $(u_t)_{t\in\mathbb{R}_+}$ of the form (2.5.1) with respect to the normal martingale $(M_t)_{t\in\mathbb{R}_+}$ is defined as

$$\int_0^\infty u_t dM_t := \sum_{i=1}^n F_i(M_{t_i} - M_{t_{i-1}}). \tag{2.5.3}$$

Proposition 2.5.4. *The definition* (2.5.3) *of the stochastic integral with respect to the normal martingale* $(M_t)_{t\in\mathbb{R}_+}$ *on simple predictable processes extends to* $u \in L^2_{ad}(\Omega \times \mathbb{R}_+)$ *via the (conditional) isometry formula*

$$\mathbb{E}\left[\int_0^\infty \mathbf{1}_{[s,\infty)} u_t dM_t \int_0^\infty \mathbf{1}_{[s,\infty)} v_t dM_t \Big| \mathcal{F}_s\right] = \mathbb{E}\left[\int_s^\infty u_t v_t dt \Big| \mathcal{F}_s\right], \qquad s \in \mathbb{R}_+. \tag{2.5.4}$$

Proof. We start by showing that the isometry (2.5.4) holds for the simple predictable process $u = \sum_{i=1}^n G_i \mathbf{1}_{(t_{i-1}, t_i]}$, with $s = t_0 < t_1 < \cdots t_n$:

$$\mathbb{E}\left[\left(\int_0^\infty u_t dM_t\right)^2 \Big| \mathcal{F}_s\right] = \mathbb{E}\left[\left(\sum_{i=1}^n G_i(M_{t_i} - M_{t_{i-1}})\right)^2 \Big| \mathcal{F}_s\right]$$
$$= \mathbb{E}\left[\sum_{i=1}^n |G_i|^2 (M_{t_i} - M_{t_{i-1}})^2 \Big| \mathcal{F}_s\right]$$
$$+ 2\mathbb{E}\left[\sum_{1\le i<j\le n} G_i G_j (M_{t_i} - M_{t_{i-1}})(M_{t_j} - M_{t_{j-1}}) \Big| \mathcal{F}_s\right]$$
$$= \sum_{i=1}^n \mathbb{E}\left[\mathbb{E}\left[|G_i|^2 (M_{t_i} - M_{t_{i-1}})^2 | \mathcal{F}_{t_{i-1}}\right] | \mathcal{F}_s\right]$$
$$+ 2\sum_{1\le i<j\le n} \mathbb{E}\left[\mathbb{E}\left[G_i G_j (M_{t_i} - M_{t_{i-1}})(M_{t_j} - M_{t_{j-1}}) | \mathcal{F}_{t_{j-1}}\right] | \mathcal{F}_s\right]$$
$$= \sum_{i=1}^n \mathbb{E}\left[|G_i|^2 \mathbb{E}\left[(M_{t_i} - M_{t_{i-1}})^2 | \mathcal{F}_{t_{i-1}}\right] | \mathcal{F}_s\right]$$
$$+ 2\sum_{1\le i<j\le n} \mathbb{E}[G_i G_j (M_{t_i} - M_{t_{i-1}}) \mathbb{E}[(M_{t_j} - M_{t_{j-1}}) | \mathcal{F}_{t_{j-1}}] | \mathcal{F}_s]$$

$$= \mathbb{E}\left[\sum_{i=1}^{n} |G_i|^2 (t_i - t_{i-1}) \Big| \mathcal{F}_s\right]$$
$$= \mathbb{E}\left[\int_0^\infty |u_t|^2 dt \Big| \mathcal{F}_s\right]$$
$$= \mathbb{E}\left[\|u\|^2_{L^2(\mathbb{R}_+)} \Big| \mathcal{F}_s\right].$$

As in the discrete case, cf. Proposition 1.2.2, the stochastic integral operator extends to $L^2_{ad}(\Omega\times\mathbb{R}_+)$ by density and a Cauchy sequence argument, applying the isometry (2.5.4) with $s = 0$, i.e.

$$\mathbb{E}\left[\int_0^\infty u_t dM_t \int_0^\infty v_t dM_t\right] = \mathbb{E}\left[\int_0^\infty u_t v_t dt\right]. \tag{2.5.5}$$

Similarly, the limit is clearly independent of the choice of the approximating sequence $(u^k)_{k\in\mathbb{N}}$ as we have

$$\mathbb{E}\left[\left(\lim_{n\to\infty}\int_0^\infty u_t^n dM_t - \lim_{n\to\infty}\int_0^\infty v_t^n dM_t\right)^2\right]$$
$$= \mathbb{E}\left[\lim_{n\to\infty}\left(\int_0^\infty (u_t^n - v_t^n) dM_t\right)^2\right]$$
$$\leq \liminf_{n\to\infty} \mathbb{E}\left[\left(\int_0^\infty (u_t^n - v_t^n) dM_t\right)^2\right]$$
$$= \lim_{n\to\infty} \mathbb{E}\left[\|u^n - v^n\|^2_{L^2(\mathbb{R}_+)}\right]$$
$$= 0.$$

□

In particular, $\int_0^\infty f(t)dM_t$ is defined for all deterministic functions $f \in L^2(\mathbb{R}_+)$, with the isometry formula

$$\mathbb{E}\left[\left(\int_0^\infty f(t)dM_t\right)^2\right] = \int_0^\infty |f(t)|^2 dt.$$

The Itô integral with respect to the normal martingale $(M_t)_{t\in\mathbb{R}_+}$ has the following locality property.

Proposition 2.5.5. *Let $A \in \mathcal{F}$ and $u \in L^2_{ad}(\Omega\times\mathbb{R}_+)$ such that*

$$u_s(\omega) = 0, \qquad \mathbf{1}_A(\omega)ds \times \mathbb{P}(d\omega) - a.e.$$

Then we have

$$\int_0^\infty u_s dM_s = 0,$$

$\mathbb{P}(d\omega)$-a.e. on A.

Proof. Consider the sequence $(u_s^n)_{s\in\mathbb{R}_+}$, $n \in \mathbb{N}$, of simple predictable processes defined as

$$u^n = \sum_{i=1}^{n} \mathbf{1}_{(t_{i-1}^n, t_i^n]} \frac{1}{t_{i-1}^n - t_{i-2}^n} \int_{t_{i-2}^n}^{t_{i-1}^n} u_s ds, \qquad n \in \mathbb{N},$$

in the proof Proposition 2.5.3, which converges to u in $L^2(\Omega \times \mathbb{R}_+)$. Clearly, u_s^n vanishes on A, $dsd\mathbb{P}$-a.e., hence $\int_0^T u_s^n dM_s = 0$, $\mathbb{P}(d\omega)$-a.e. on A for all $n \geq 0$. We get the desired result by taking the limit as n goes to infinity. □

The Itô integral of $u \in L_{ad}^2(\Omega \times \mathbb{R}_+)$ on the interval (a, b) is defined by

$$\int_a^b u_s dM_s := \int_0^\infty \mathbf{1}_{[a,b]}(s) u_s dM_s,$$

with the Chasles relation

$$\int_a^c u_s dM_s = \int_a^b u_s dM_s + \int_b^c u_s dM_s, \qquad 0 \leq a \leq b \leq c. \tag{2.5.6}$$

Proposition 2.5.6. *For all $T > 0$, the indefinite integral process*

$$\left(\int_0^t u_s dM_s \right)_{t\in[0,T]}$$

has a measurable version in $L_{ad}^2(\Omega \times [0,T])$.

Proof. Let $(u^n)_{n\in\mathbb{N}}$ be a sequence of simple predictable processes converging to u in $L^2(\Omega \times [0,T])$. We have

$$\begin{aligned} \mathbb{E}\left[\int_0^T \left(\int_0^t u_s dM_s - \int_0^t u_s^n dM_s \right)^2 dt \right] &= \mathbb{E}\left[\int_0^T \int_0^t |u_s - u_s^n|^2 ds dt \right] \\ &\leq T \times \mathbb{E}\left[\int_0^T |u_s - u_s^n|^2 ds \right] \\ &< \infty. \end{aligned}$$

Hence

$$\lim_{n\to\infty} \int_0^t u_s^n dM_s = \int_0^t u_s dM_s, \qquad t \in [0,T],$$

in $L^2(\Omega \times [0,T])$, and the convergence holds $dt \times \mathbb{P}(d\omega)$-a.e. for a certain subsequence of $(u^n)_{n\in\mathbb{N}}$. □

As a consequence, if a process $(u_t)_{t\in\mathbb{R}_+}$ is locally in $L_{ad}^2(\Omega \times \mathbb{R}_+)$:

$$\mathbb{E}\left[\int_0^T |u_s|^2 ds \right] < \infty, \qquad T > 0,$$

then the indefinite integral process $\left(\int_0^t u_s dM_s\right)_{t\in\mathbb{R}_+}$ has a version which also belongs locally to $L^2_{ad}(\Omega\times\mathbb{R}_+)$. Note also that the sequence

$$\sum_{i=1}^n \mathbf{1}_{[t_{i-1},t_i]} \int_0^{t_{i-1}} u_s dM_s, \qquad n\geq 1,$$

converges to $\left(\int_0^t u_s dM_s\right)_{t\in[0,T]}$ in $L^2_{ad}(\Omega\times[0,T])$ as the mesh of the partition $0=t_0\leq t_1<\cdots<t_n=T$ goes to zero.

As a consequence of Proposition 2.5.7 below with $\mathcal{F}_0=\{\emptyset,\Omega\}$ we find in particular that the Itô integral is a centered random variable:

$$\mathbb{E}\left[\int_0^\infty u_s dM_s\right]=0, \qquad u\in L^2_{ad}(\Omega\times\mathbb{R}_+). \tag{2.5.7}$$

Proposition 2.5.7. *For any $u\in L^2_{ad}(\Omega\times\mathbb{R}_+)$ we have*

$$\mathbb{E}\left[\int_0^\infty u_s dM_s\Big|\mathcal{F}_t\right]=\int_0^t u_s dM_s, \qquad t\in\mathbb{R}_+.$$

In particular, $\int_0^t u_s dM_s$ is $\mathcal{F}_t$-measurable, $t\in\mathbb{R}_+$.

Proof. Let $u\in\mathcal{U}$ of the form $u=G\mathbf{1}_{(a,b]}$, where G is bounded and $\mathcal{F}_a$-measurable.

i) If $0\leq a\leq t$ we have

$$\begin{aligned}\mathbb{E}\left[\int_0^\infty u_s dM_s\Big|\mathcal{F}_t\right]&=\mathbb{E}\left[G(M_b-M_a)|\mathcal{F}_t\right]\\&=G\mathbb{E}\left[(M_b-M_a)|\mathcal{F}_t\right]\\&=G\mathbb{E}\left[(M_b-M_t)|\mathcal{F}_t\right]+G\mathbb{E}\left[(M_t-M_a)|\mathcal{F}_t\right]\\&=G(M_t-M_a)\\&=\int_0^\infty \mathbf{1}_{[0,t]}(s)u_s dM_s.\end{aligned}$$

ii) If $0\leq t\leq a$ we have for all bounded $\mathcal{F}_t$-measurable random variable F:

$$\mathbb{E}\left[F\int_0^\infty u_s dM_s\right]=\mathbb{E}\left[FG(M_b-M_a)\right]=0,$$

hence

$$\begin{aligned}\mathbb{E}\left[\int_0^\infty u_s dM_s\Big|\mathcal{F}_t\right]&=\mathbb{E}\left[G(M_b-M_a)|\mathcal{F}_t\right]\\&=0\\&=\int_0^\infty \mathbf{1}_{[0,t]}(s)u_s dM_s.\end{aligned}$$

This statement is extended by linearity and density, since by Fatou's lemma and continuity of the conditional expectation on L^2 we have:

$$\begin{aligned}
&\mathbb{E}\left[\left(\int_0^t u_s dM_s - \mathbb{E}\left[\int_0^\infty u_s dM_s \Big| \mathcal{F}_t\right]\right)^2\right] \\
&= \mathbb{E}\left[\lim_{n\to\infty}\left(\int_0^t u_s^n dM_s - \mathbb{E}\left[\int_0^\infty u_s dM_s \Big| \mathcal{F}_t\right]\right)^2\right] \\
&\leq \liminf_{n\to\infty} \mathbb{E}\left[\left(\int_0^t u_s^n dM_s - \mathbb{E}\left[\int_0^\infty u_s dM_s \Big| \mathcal{F}_t\right]\right)^2\right] \\
&= \liminf_{n\to\infty} \mathbb{E}\left[\left(\mathbb{E}\left[\int_0^\infty u_s^n dM_s - \int_0^\infty u_s dM_s \Big| \mathcal{F}_t\right]\right)^2\right] \\
&\leq \liminf_{n\to\infty} \mathbb{E}\left[\mathbb{E}\left[\left(\int_0^\infty u_s^n dM_s - \int_0^\infty u_s dM_s\right)^2 \Big| \mathcal{F}_t\right]\right] \\
&= \liminf_{n\to\infty} \mathbb{E}\left[\left(\int_0^\infty (u_s^n - u_s) dM_s\right)^2\right] \\
&= \liminf_{n\to\infty} \mathbb{E}\left[\int_0^\infty |u_s^n - u_s|^2 ds\right] \\
&= 0,
\end{aligned}$$

as in the proof of Proposition 1.2.3. □

The following is an immediate corollary of Proposition 2.5.7.

Corollary 2.5.8. *The indefinite stochastic integral* $\left(\int_0^t u_s dM_s\right)_{t\in\mathbb{R}_+}$ *of* $u \in L^2_{ad}(\Omega \times \mathbb{R}_+)$ *is a martingale, i.e.:*

$$\mathbb{E}\left[\int_0^t u_\tau dM_\tau \Big| \mathcal{F}_s\right] = \int_0^s u_\tau dM_\tau, \qquad 0 \leq s \leq t.$$

Recall that since the Poisson martingale $(M_t)_{t\in\mathbb{R}_+} = (N_t - t)_{t\in\mathbb{R}_+}$ is a normal martingale, the stochastic integral

$$\int_0^T u_t dM_t$$

is defined in Itô sense as an $L^2(\Omega)$-limit of stochastic integrals of simple adapted processes.

In the particular case of Brownian motion, i.e. when

$$(M_t)_{t\in\mathbb{R}_+} = (B_t)_{t\in\mathbb{R}_+},$$

we find from (2.2.2), (2.2.4) and (2.5.3) that for all deterministic $f \in L^2(\mathbb{R}_+)$, the random variable $J_1(f)$ coincides with the single stochastic integral with respect to $(B_t)_{t\in\mathbb{R}_+}$, i.e. we have

$$J_1(f) = \int_0^\infty f(t)dB_t.$$

In particular, it follows from page 62 and the Itô isometry (2.5.4) that $\int_0^\infty f(t)dB_t$ is a centered Gaussian random variable with variance

$$\int_0^\infty f^2(t)dt.$$

The next result is an integration by parts for Brownian motion.

Remark 2.5.9. *In case $(M_t)_{t\in\mathbb{R}_+} = (B_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion, we have*

$$\int_0^\infty f(t)dB_t = -\int_0^\infty f'(t)B_t dt,$$

provided that $f \in L^2(\mathbb{R}_+)$ is $\mathcal{C}^1$ on $\mathbb{R}_+$ and such that $\lim_{t\to\infty} t|f(t)|^2 = 0$.

Proof. First, we note that we have

$$f(t)\int_0^t f(s)ds \le \sqrt{t}f(t)\sqrt{\int_0^t |f(s)|^2 ds} \le \|f\|_{L^2(\mathbb{R}_+)}\sqrt{t}f(t), \quad t\in\mathbb{R}_+,$$

hence

$$\lim_{t\to\infty} f(t)\int_0^t f(s)ds = 0,$$

since $\lim_{t\to\infty} t|f(t)|^2 = 0$. Next, we have

$$\begin{aligned}
&\left\|\int_0^\infty f(t)dB_t + \int_0^\infty f'(t)B_t dt\right\|^2_{L^2(\Omega)} \\
&= \left\|\int_0^\infty f(t)dB_t\right\|^2_{L^2(\Omega)} + \left\|\int_0^\infty f'(t)B_t dt\right\|^2_{L^2(\Omega)} \\
&\quad +2\left\langle \int_0^\infty f(t)dB_t, \int_0^\infty f'(t)B_t dt\right\rangle_{L^2(\Omega)} \\
&= \int_0^\infty |f(t)|^2 dt + \mathbb{E}\left[\int_0^\infty\int_0^\infty f'(t)f'(s)B_tB_s ds dt\right] \\
&\quad +2\left\langle \int_0^\infty f(s)dB_s, \int_0^\infty f'(t)\int_0^t dB_s dt\right\rangle_{L^2(\Omega)} \\
&= \int_0^\infty |f(t)|^2 dt + \int_0^\infty\int_0^\infty f'(t)f'(s)\mathbb{E}[B_tB_s]ds dt + 2\int_0^\infty f'(t)\int_0^t f(s)ds dt \\
&= \int_0^\infty |f(t)|^2 dt + \int_0^\infty\int_0^\infty f'(t)f'(s)(t\wedge s)ds dt + 2\int_0^\infty f'(t)\int_0^t f(s)ds dt \\
&= \int_0^\infty |f(t)|^2 dt + 2\int_0^\infty f'(t)\int_0^t sf'(s)ds dt - 2\int_0^\infty |f(t)|^2 dt \\
&= \int_0^\infty |f(t)|^2 dt - 2\int_0^\infty tf'(t)f(t)dt - 2\int_0^\infty |f(t)|^2 dt
\end{aligned}$$

$$= -\lim_{t\to\infty} tf^2(t)$$
$$= 0.$$

□

Since the Itô integral is defined in the $L^2(\Omega)$ sense on adapted processes in $L^2(\Omega\times\mathbb{R}_+, d\mathbb{P}\otimes dt)$ the integrals of the adapted and predictable version of a given square-integrable process coincide, as follows from Proposition 2.5.3-*ii*), see also [33], page 199.

In particular, for the compensated Poisson martingale $(M_t)_{t\in\mathbb{R}_+} = (N_t - \lambda t)_{t\in\mathbb{R}_+}$, the integrals

$$\int_0^\infty u_t dM_t \qquad \text{and} \qquad \int_0^\infty u_{t^-} dM_t$$

coincide in $L^2(\Omega)$ whenever $t \longmapsto u_t$ has left and right limits, $\mathbb{P}$-a.s., since the set of discontinuities of a function having left and right limits in every point is always countable, cf. e.g. [54], page 5.

In the Poisson case, in addition to its definition in L^2 sense, the Itô integral also admits a pathwise interpretation as the Stieltjes integral of $(u_{t^-})_{t>0}$ under uniform continuity conditions, as shown in the next proposition, in which

$$(M_t)_{t\in\mathbb{R}_+} = \lambda^{-1/2}(N_t - \lambda t)_{t\in\mathbb{R}_+}$$

is the compensated Poisson martingale with intensity $\lambda > 0$.

Proposition 2.5.10. *Let $(u_t)_{t\in[0,T]} \in L^2(\Omega\times[0,T])$ be an adapted process with a càdlàg version (i.e. continuous on the right with left limits), $(\bar{u}_t)_{t\in[0,T]}$ such that*

$$\lim_{\varepsilon\to 0} \sup_{t\in[\varepsilon,T]} |\bar{u}_{t^-} - \bar{u}_{t-\varepsilon}|^2 = 0, \tag{2.5.8}$$

in $L^4(\Omega)$. Then we have

$$\int_0^T u_t dM_t = \lambda^{-1/2}\int_0^T \bar{u}_{t^-}(\omega(dt) - \lambda dt),$$

$\mathbb{P}(d\omega)$-almost surely, where $\omega(dt)$ denotes the random measure

$$\omega(dt) = \sum_{k=1}^\infty \delta_{T_k}(dt).$$

Proof. Let

$$A_\varepsilon^u = \sup_{t\in[\varepsilon,T]} |\bar{u}_{t^-} - \bar{u}_{t-\varepsilon}|^2, \qquad 0<\varepsilon<T,$$

and $\bar{u}_t^- = \bar{u}_{t^-}$, $t\in\mathbb{R}_+$. Given a subdivision

$$\pi = \{0 = t_0 < t_1 < \cdots < t_n = T\}$$

of $[0,T]$, with $|\pi| = \max_{i=1,\ldots,n} |t_i - t_{i-1}|$, we have

$$\begin{aligned}
&\left\| \bar{u}_{\cdot}^{-} \mathbf{1}_{[0,T]}(\cdot) - \sum_{i=1}^{n} u_{t_{i-1}}^{n} \mathbf{1}_{(t_{i-1},t_i]}(\cdot) \right\|_{L^2(\Omega\times[0,T])}^2 \\
&\qquad = \left\| \sum_{i=1}^{n} (\bar{u}^{-} - u_{t_{i-1}}^{n}) \mathbf{1}_{(t_{i-1},t_i]} \right\|_{L^2(\Omega\times[0,T])}^2 \leq T \|A_{|\pi|}^{u}\|_{L^2(\Omega)}^2,
\end{aligned}$$

hence from (2.5.8), $\displaystyle\sum_{i=1}^{n} u_{t_i}^{n} \mathbf{1}_{(t_{i-1},t_i]}$ defined by (2.5.2) converges to $\bar{u}^{-}$ in $L^2(\Omega \times [0,T])$, and

$$\sum_{i=1}^{n} u_{t_i}^{n} (M_{t_i} - M_{t_{i-1}})$$

converges to $\int_0^\infty \bar{u}_{t^-} dM_t$ as $|\pi|$ goes to zero. On the other hand we have

$$\begin{aligned}
&\left\| \lambda^{-1/2} \int_0^T \bar{u}_{t^-} (\omega(dt) - \lambda dt) - \sum_{i=1}^{n} u_{t_{i-1}}^{n} (M_{t_i} - M_{t_{i-1}}) \right\|_{L^2(\Omega)} \\
&= \lambda^{-1/2} \left\| \sum_{k=1}^{N_T} \bar{u}_{T_k^-} - \int_0^T u_s ds + \sum_{i=1}^{N_T} u_{t_{i-1}}^{n} (t_i - t_{i-1}) - \sum_{k=1}^{\infty} \sum_{i=1}^{n} u_{t_{i-1}}^{n} \mathbf{1}_{[t_{i-1},t_i]}(T_k) \right\|_{L^2(\Omega)} \\
&\leq \lambda^{-1/2} \left\| \sum_{k=1}^{N_T} \bar{u}_{T_k^-} - \sum_{k=1}^{\infty} \sum_{i=1}^{n} u_{t_{i-1}}^{n} \mathbf{1}_{[t_{i-1},t_i]}(T_k) \right\|_{L^2(\Omega)} \\
&\quad + \lambda^{-1/2} \left\| \int_0^T u_s ds - \sum_{i=1}^{n} u_{t_{i-1}}^{n} (t_i - t_{i-1}) \right\|_{L^2(\Omega)} \\
&= \lambda^{-1/2} \left\| \sum_{k=1}^{N_T} \sum_{i=1}^{n} \mathbf{1}_{[t_{i-1},t_i]}(T_k) (\bar{u}_{T_k^-} - u_{t_{i-1}}^{n}) \right\|_{L^2(\Omega)} \\
&\quad + \lambda^{-1/2} \left\| \int_0^T \sum_{i=1}^{n} \mathbf{1}_{[t_{i-1},t_i]} (u_s - u_{t_{i-1}}^{n}) ds \right\|_{L^2(\Omega)} \\
&\leq \lambda^{-1/2} \|A_\varepsilon^u\|_{L^4(\Omega)}^2 \|N_T\|_{L^4(\Omega)}^2 + \lambda^{-1/2} T^{1/2} \|A_\varepsilon^u\|_{L^2(\Omega)},
\end{aligned}$$

hence as $|\pi|$ goes to zero,

$$\int_0^T u_t dM_t = \lim_{|\pi|\to 0} \sum_{i=1}^{n} u_{t_{i-1}} (M_{t_i} - M_{t_{i-1}})$$

$$= \lambda^{-1/2} \int_0^T \bar{u}_{t^-} (\omega(dt) - \lambda dt),$$

where the limit is taken in $L^2(\Omega)$. □

Proposition 2.5.10 admits the following simplified corollary.

Corollary 2.5.11. *Let $T > 0$ and let $(u_t)_{t\in[0,T]} \in L^2(\Omega \times [0,T])$ be an adapted process with a uniformly càdlàg version $(\bar{u}_t)_{t\in[0,T]} \in L^2(\Omega \times [0,T])$ and such that $u \in L^4(\Omega, L^\infty([0,T]))$, i.e*

$$\sup_{t\in[0,T]} |u_t| \in L^4(\Omega). \qquad (2.5.9)$$

Then we have, $\mathbb{P}(d\omega)$-almost surely,

$$\int_0^T u_t dM_t = \lambda^{-1/2} \int_0^T \bar{u}_{t^-} (\omega(dt) - \lambda dt), \qquad T > 0.$$

Proof. It suffices to check that Condition (2.5.8) holds under the hypothesis (2.5.9). □

Concerning the compound Poisson process

$$X_t = \sum_{k=1}^{N_t} Y_k, \qquad t \in \mathbb{R}_+,$$

of Section 2.4, under similar conditions we get

$$\int_0^T u_t dM_t = (\lambda \mathrm{Var}\,[Y_1])^{-1/2} \int_0^T u_{t^-} (Y_{N_t} \omega(dt) - \lambda \mathbb{E}[Y_1] dt),$$

where

$$M_t := \frac{X_t - \lambda t \mathbb{E}[Y_1]}{\sqrt{\lambda \mathrm{Var}\,[Y_1]}}, \qquad t \in \mathbb{R}_+.$$

2.6 Predictable Representation Property

Definition 2.6.1. *We say that the martingale $(M_t)_{t\in\mathbb{R}_+}$ has the predictable representation property if*

$$\left\{ c + \int_0^\infty u_t dM_t \ : \ c \in \mathbb{R}, \ u \in \mathcal{P} \right\}$$

is dense in $L^2(\Omega)$.

By the Itô isometry (2.5.4), the predictable representation property of Definition 2.6.1 is equivalent to stating that any $F \in L^2(\Omega)$ can be written as

$$F = E[F] + \int_0^\infty u_t dM_t$$

where $u \in L^2_{ad}(\Omega \times \mathbb{R}_+)$ is a certain adapted square-integrable process.

The next proposition is the continuous-time analog of Proposition 1.7.5.

Proposition 2.6.2. *The martingale $(M_t)_{t\in\mathbb{R}_+}$ has the predictable representation property if and only if any square-integrable martingale $(X_t)_{t\in\mathbb{R}_+}$ with respect to $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ can be represented as*

$$X_t = X_0 + \int_0^t u_s dM_s, \qquad t \in \mathbb{R}_+, \tag{2.6.1}$$

where $(u_t)_{t\in\mathbb{R}_+} \in L^2_{ad}(\Omega \times \mathbb{R}_+)$ is an adapted process such that

$$u\mathbf{1}_{[0,T]} \in L^2(\Omega \times \mathbb{R}_+)$$

for all $T > 0$.

Proof. Assume that for any square-integrable martingale $(X_t)_{t\in\mathbb{R}_+}$ a representation of the form (2.6.1) exists. Given $F \in L^2(\Omega)$, letting

$$X_t = \mathbb{E}[F \mid \mathcal{F}_t], \qquad t \in \mathbb{R}_+,$$

defines a square-integrable martingale $(X_t)_{t\in\mathbb{R}_+}$. If F is in $L^2(\Omega)$, Proposition 9.4.1 in the Appendix shows that $(X_n)_{n\in\mathbb{N}}$ converges to F in $L^2(\Omega)$. On the other hand, X_n can be represented from (2.6.1) as

$$X_n = \mathbb{E}[F] + \int_0^n u_s^n dM_s, \qquad n \geq 1,$$

where for all $n \geq 1$ the process u^n can be approximated by a sequence of elements of $\mathcal{P}$ by Proposition 2.5.3. Hence

$$\left\{ c + \int_0^\infty u_t dM_t \ : \ c \in \mathbb{R}, \ u \in \mathcal{P} \right\}$$

is dense in $L^2(\Omega)$.

Conversely, assume that the predictable representation property of Definition 2.6.1 holds and let $(X_t)_{t\in\mathbb{R}_+}$ be an L^2 martingale. Then for all $n \geq 1$ there exists a sequence $(u_t^{n,k})_{t\in[0,n]}$ in $\mathcal{P}$ such that the limit

$$X_n = X_0 + \lim_{k\to\infty} \int_0^n u_t^{n,k} dM_t$$

exists in $L^2(\Omega)$. By the Itô isometry (2.5.4), the sequence $(u_t^{n,k})_{t\in[0,n]}$ is Cauchy in $L^2(\Omega \times [0,n])$ and by completeness of L^p spaces it converges to a process $(u_t^n)_{t\in[0,n]} \in L^2(\Omega \times [0,n])$ such that

$$X_n = X_0 + \int_0^n u_s^n dM_s, \qquad n \geq 1.$$

Then from Proposition 2.5.7 we have, for all $n \in \mathbb{N}$ and $t \in [n, n+1)$:

$$\begin{aligned}
X_t &= \mathbb{E}[X_{n+1} | \mathcal{F}_t] \\
&= \mathbb{E}\left[X_0 + \int_0^{n+1} u_s^{n+1} dM_s \Big| \mathcal{F}_t \right] \\
&= \mathbb{E}\left[X_0 + \int_0^n u_s^{n+1} dM_s + \int_n^{n+1} u_s^{n+1} dM_s \Big| \mathcal{F}_t \right] \\
&= \mathbb{E}\left[X_0 + \int_0^n u_s^{n+1} dM_s \Big| \mathcal{F}_n \right] + \mathbb{E}\left[\int_n^{n+1} u_s^{n+1} dM_s \Big| \mathcal{F}_t \right] \\
&= \mathbb{E}\left[X_0 + \int_0^{n+1} u_s^{n+1} dM_s \Big| \mathcal{F}_n \right] + \mathbb{E}\left[\int_n^{n+1} u_s^{n+1} dM_s \Big| \mathcal{F}_t \right] \\
&= \mathbb{E}\left[X_{n+1} \Big| \mathcal{F}_n \right] + \mathbb{E}\left[\int_n^{n+1} u_s^{n+1} dM_s \Big| \mathcal{F}_t \right] \\
&= X_n + \mathbb{E}\left[\int_n^{n+1} u_s^{n+1} dM_s \Big| \mathcal{F}_t \right] \\
&= X_n + \int_n^t u_s^{n+1} dM_s.
\end{aligned}$$

Letting the process $(u_s)_{s \in \mathbb{R}_+}$ be defined by

$$u_s = u_s^{n+1}, \qquad n \leq s < n+1, \quad n \in \mathbb{N},$$

we obtain

$$X_t = X_0 + \int_0^t u_s dM_s, \quad t \in \mathbb{R}_+,$$

which is (2.6.1). □

In the sequel we will show that Brownian motion and the compensated Poisson process have the predictable representation property. This is however not true of compound Poisson processes in general, see (2.10.6) below.

2.7 Multiple Stochastic Integrals

Multiple stochastic integrals have originally been defined with respect to Brownian motion, cf. [68], and with respect to Lévy processes, cf. [69]. In this section we construct the multiple stochastic integral with respect to a normal martingale $(M_t)_{t \in \mathbb{R}_+}$. Let $L^2(\mathbb{R}_+)^{\circ n}$ denote the subspace of $L^2(\mathbb{R}_+)^{\otimes n} = L^2(\mathbb{R}_+^n)$, made of symmetric functions f_n in n variables (see Section 9.7 in the Appendix for a review of tensor products). The multiple stochastic integral of a symmetric function $f_n \in L^2(\mathbb{R}_+)^{\circ n}$ is defined as an iterated integral. First we let

$$I_1(f) = \int_0^\infty f(t) dM_t, \qquad f \in L^2(\mathbb{R}_+).$$

As a convention we identify $L^2(\mathbb{R}_+)^{\circ 0}$ to $\mathbb{R}$ and let

$$I_0(f_0) = f_0, \qquad f_0 \in L^2(\mathbb{R}_+)^{\circ 0} \simeq \mathbb{R}.$$

Proposition 2.7.1. *The multiple stochastic integral $I_n(f_n)$ of $f_n \in L^2(\mathbb{R}_+)^{\circ n}$, defined by induction as*

$$I_n(f_n) = n \int_0^\infty I_{n-1}(f_n(*, t)\mathbf{1}_{[0,t]^{n-1}}(*)) dM_t, \quad n \geq 1, \tag{2.7.1}$$

satisfies the isometry formula

$$\mathbb{E}[I_n(f_n) I_m(g_m)] = n! \mathbf{1}_{\{n=m\}} \langle f_m, g_m \rangle_{L^2(\mathbb{R}_+^n)},$$

$f_n \in L^2(\mathbb{R}_+)^{\circ n}$, $f_m \in L^2(\mathbb{R}_+)^{\circ m}$, $n, m \in \mathbb{N}$.

Proof. Note that the process

$$t \longmapsto I_{n-1}(f_n(*, t)\mathbf{1}_{[0,t]^{n-1}}(*))$$

is $\mathcal{F}_t$-adapted, cf. Proposition 2.5.7, hence the iterated stochastic integral in (2.7.1) is well-defined by Proposition 2.5.4. If $n = m \geq 1$ we have from (2.7.1) and the Itô isometry (2.5.5):

$$\mathbb{E}[|I_n(f_n)|^2] = n^2 \int_0^\infty \mathbb{E}[|I_{n-1}(f_n(*, t)\mathbf{1}_{[0,t]^{n-1}}(*))|^2] dt,$$

with the convention $I_0(f_0) = f_0 \in \mathbb{R}$. By induction on $n \geq 1$ this yields:

$$\begin{aligned} \mathbb{E}[I_n(f_n)^2] &= n!^2 \int_0^\infty \int_0^{t_n} \cdots \int_0^{t_2} |f_n(t_1, \ldots, t_n)|^2 dt_1 \cdots dt_n \\ &= n! \|f_n\|^2_{L^2(\mathbb{R}_+^n)}. \end{aligned}$$

On the other hand, from (2.5.7) we have

$$\mathbb{E}[I_1(f_1) I_0(g_0)] = I_0(g_0) \mathbb{E}[I_1(f_1)] = 0.$$

By induction on the rank $n \geq 1$ of $I_n(f_n)$, assuming that

$$\mathbb{E}[I_n(f_n) I_k(f_k)] = 0, \qquad 0 \leq k < n,$$

$f_k \in L^2(\mathbb{R}_+)^{\circ k}$, $0 \leq k \leq n$, we have for all $0 \leq k \leq n$:

$$\mathbb{E}[I_{n+1}(f_n) I_k(f_k)]$$

$$= k(n+1)\int_0^\infty \mathbb{E}[I_n(f_{n+1}(*,t)\mathbf{1}_{[0,t]^n}(*))I_{k-1}(f_k(*,t)\mathbf{1}_{[0,t]^{k-1}}(*))]dt$$
$$= 0,$$

hence for all $n \geq 1$ we have

$$\mathbb{E}[I_n(f_n)I_k(g_k)] = 0, \qquad 0 \leq k \leq n-1.$$

□

In particular we have $\mathbb{E}[I_n(f_n)] = 0$ for all $n \geq 1$.

We also have

$$I_n(f_n) = n! \int_0^\infty \int_0^{t_n} \cdots \int_0^{t_2} f_n(t_1, \ldots, t_n) dM_{t_1} \cdots dM_{t_n}. \tag{2.7.2}$$

On the other hand, the symmetric tensor product $u \circ f_n$ satisfies

$$u \circ f_n(t_1, \ldots, t_{n+1}) = \frac{1}{n+1} \sum_{i=1}^{n+1} u(t_i) f_n(t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_{n+1}), \tag{2.7.3}$$

$u \in L^2(\mathbb{R}_+)$, $f_n \in L^2(\mathbb{R}_+)^{\circ n}$, hence

$$\begin{aligned} I_{n+1}(u \circ f_n) = n \int_0^\infty I_n(f_n(*,s) \circ u(\cdot)\mathbf{1}_{[0,s]^n}(*,\cdot))dM_s \\ + \int_0^\infty u(s) I_n(f_n \mathbf{1}_{[0,s]^n}) dM_s. \end{aligned} \tag{2.7.4}$$

Lemma 2.7.2. *For all $f_n \in L^2(\mathbb{R}_+)^{\circ n}$, $n \geq 1$, we have*

$$\mathbb{E}[I_n(f_n) \mid \mathcal{F}_t] = I_n(f_n \mathbf{1}_{[0,t]^n}), \qquad t \in \mathbb{R}_+.$$

Proof. Since the indefinite Itô integral is a martingale from (2.7.2) and Proposition 2.5.7 we have

$$\begin{aligned} \mathbb{E}[I_n(f_n) \mid \mathcal{F}_t] &= n!\mathbb{E}\left[\int_0^\infty \int_0^{t_n} \cdots \int_0^{t_2} f_n(t_1, \ldots, t_n) dM_{t_1} \cdots dM_{t_n} \Big| \mathcal{F}_t\right] \\ &= n! \int_0^t \int_0^{t_n} \cdots \int_0^{t_2} f_n(t_1, \ldots, t_n) dM_{t_1} \cdots dM_{t_n} \\ &= I_n(f_n \mathbf{1}_{[0,t]^n}). \end{aligned}$$

□

As a consequence of Lemma 2.7.2, $I_n(f_n)$ is $\mathcal{F}_t$-measurable if and only if

$$f_n = f_n \mathbf{1}_{[0,t]^n},$$

i.e. $f_n = 0$ over $\mathbb{R}^n \setminus [0,t]^n$.

2.8 Chaos Representation Property

Let now

$$\mathcal{S} = \left\{ f_0 + \sum_{k=1}^{n} I_k(f_k) \ : \ f_0 \in \mathbb{R}, \ f_k \in L^2(\mathbb{R}_+)^{\circ k} \cap L^4(\mathbb{R}_+^k), \ k = 1, 2, \ldots, n, \ n \in \mathbb{N} \right\}.$$

Let also $\mathcal{H}_0 = \mathbb{R}$ and

$$\mathcal{H}_n = \{ I_n(f_n) \ : \ f_n \in L^2(\mathbb{R}_+)^{\circ n} \}, \quad n \geq 1.$$

We have

$$\mathcal{S} \subset \mathrm{Vect} \left\{ \bigcup_{n=0}^{\infty} \mathcal{H}_n \right\},$$

where Vect $\{U\}$ denotes the smallest linear space generated by U.

The following is the definition of the Fock space over $L^2(\mathbb{R}_+)$.

Definition 2.8.1. *The closure of $\mathcal{S}$ in $L^2(\Omega)$ is denoted by the direct sum*

$$\bigoplus_{n=0}^{\infty} \mathcal{H}_n.$$

The chaos representation property states that every $F \in L^2(\Omega, \mathcal{F}, \mathbb{P})$ admits a decomposition

$$F = f_0 + \sum_{n=1}^{\infty} I_n(f_n),$$

where $f_0 = \mathbb{E}[F]$ and $f_n \in L^2(\mathbb{R}_+)^{\circ n}$, $n \geq 1$.

It is equivalent to stating that $\mathcal{S}$ is dense in $L^2(\Omega)$, and can also be formulated as in the next definition.

Definition 2.8.2. *The martingale $(M_t)_{t \in \mathbb{R}_+}$ has the chaos representation property if*

$$L^2(\Omega, \mathcal{F}, \mathbb{P}) = \bigoplus_{n=0}^{\infty} \mathcal{H}_n.$$

In case $(M_t)_{t \in \mathbb{R}_+}$ has the chaos representation property, the multiple stochastic integrals I_n provide an isometric isomorphism between $L^2(\Omega)$ and the Fock space $\Phi(L^2(\mathbb{R}_+))$ defined as the direct sum

$$\Phi(L^2(\mathbb{R}_+)) = \mathbb{R} \oplus \bigoplus_{n=1}^{\infty} L^2(\mathbb{R}_+)^{\circ n}.$$

February 23, 2018 (with corrections).

Definition 2.8.3. *The number operator L is defined on $\mathcal{S}$ by $L = \delta D$.*

The operator L satisfies

$$LI_n(f_n) = nI_n(f_n), \qquad f_n \in L^2(\mathbb{R}_+)^{\circ n}, \quad n \in \mathbb{N}.$$

We will note later, cf. Proposition 5.1.5 and Proposition 6.3.2, that Brownian motion and the compensated Poisson martingale have the chaos representation property.

Moreover, the chaos representation property of Definition 2.6.1 implies the predictable representation property, as will be shown in Proposition 4.2.4 below.

2.9 Quadratic Variation

Next, we introduce the quadratic variation of a normal martingale.

Definition 2.9.1. *The quadratic variation of the martingale $(M_t)_{t\in\mathbb{R}_+}$ is the process $([M,M]_t)_{t\in\mathbb{R}_+}$ defined as*

$$[M,M]_t = M_t^2 - 2\int_0^t M_s dM_s, \qquad t \in \mathbb{R}_+. \tag{2.9.1}$$

Note that we have

$$[M,M]_t - [M,M]_s = (M_t - M_s)^2 - 2\int_s^t (M_\tau - M_s) dM_\tau, \quad 0 < s < t, \tag{2.9.2}$$

since

$$M_s(M_t - M_s) = \int_s^t M_s dM_\tau, \qquad 0 \le s \le t,$$

as an immediate consequence of the definition 2.5.3 of the stochastic integral. Let now

$$\pi^n = \{0 = t_0^n < t_1^n < \cdots < t_{n-1}^n < t_n^n = t\}$$

denote a family of subdivision of $[0,t]$, such that $|\pi^n| := \max_{i=1,\ldots,n} |t_i^n - t_{i-1}^n|$ converges to 0 as n goes to infinity.

Proposition 2.9.2. *Let $(M_t)_{t\in\mathbb{R}_+}$ be a normal martingale. We have*

$$[M,M]_t = \lim_{n\to\infty} \sum_{i=1}^n (M_{t_i^n} - M_{t_{i-1}^n})^2, \quad t \ge 0,$$

where the limit exists in $L^2(\Omega)$ and is independent of the sequence $(\pi^n)_{n\in\mathbb{N}}$ of subdivisions chosen.

Proof. We have

$$[M,M]_{t_i^n} - [M,M]_{t_{i-1}^n} = M_{t_i^n}^2 - M_{t_{i-1}^n}^2 - 2\int_{t_{i-1}^n}^{t_i^n} M_s dM_s$$
$$= (M_{t_i^n} - M_{t_{i-1}^n})^2 + 2\int_{t_{i-1}^n}^{t_i^n} (M_{t_{i-1}^n} - M_s) dM_s,$$

hence

$$\begin{aligned}
&\mathbb{E}\left[\left([M,M]_t - \sum_{i=1}^n (M_{t_i^n} - M_{t_{i-1}^n})^2\right)^2\right] \\
&= \mathbb{E}\left[\left(\sum_{i=1}^n [M,M]_{t_i^n} - [M,M]_{t_{i-1}^n} - (M_{t_i^n} - M_{t_{i-1}^n})^2\right)^2\right] \\
&= 4\mathbb{E}\left[\left(\int_0^t \sum_{i=1}^n \mathbf{1}_{(t_{i-1}^n, t_i^n]}(s)(M_s - M_{t_{i-1}^n}) dM_s\right)^2\right] \\
&= 4\mathbb{E}\left[\int_{t_{i-1}^n}^{t_i^n} \sum_{i=1}^n (M_s - M_{t_{i-1}^n})^2 ds\right] \\
&= 4\sum_{i=1}^n \int_{t_{i-1}^n}^{t_i^n} (s - t_{i-1}^n) ds \\
&\leq 4t|\pi^n|,
\end{aligned}$$

which tends to 0 as n tends to infinity. □

Proposition 2.9.3. *The quadratic variation of Brownian motion* $(B_t)_{t\in\mathbb{R}_+}$ *is*

$$[B,B]_t = t, \quad t \in \mathbb{R}_+.$$

Proof. (cf. e.g. [125], Theorem I-28). For every subdivision $\{0 = t_0^n < \cdots < t_n^n = t\}$ we have

$$\begin{aligned}
&\mathbb{E}\left[\left(t - \sum_{i=1}^n (B_{t_i^n} - B_{t_{i-1}^n})^2\right)^2\right] \\
&= \mathbb{E}\left[\left(\sum_{i=1}^n (B_{t_i^n} - B_{t_{i-1}^n})^2 - (t_i^n - t_{i-1}^n)\right)^2\right] \\
&= \sum_{i=1}^n (t_i^n - t_{i-1}^n)^2 \mathbb{E}\left[\left(\frac{(B_{t_i^n} - B_{t_{i-1}^n})^2}{t_i^n - t_{i-1}^n} - 1\right)^2\right] \\
&= \mathbb{E}[(Z^2-1)^2] \sum_{i=0}^n (t_i^n - t_{i-1}^n)^2
\end{aligned}$$

$$\leq |\pi^n| \mathbb{E}[(Z^2-1)^2] \sum_{i=0}^{n} (t_i^n - t_{i-1}^n)$$
$$= t|\pi^n| \mathbb{E}[(Z^2-1)^2],$$

where Z is a standard Gaussian random variable. □

Concerning the Poisson process, a simple analysis of the paths of $(N_t)_{t\in\mathbb{R}_+}$ shows that the quadratic variation of the compensated Poisson process $(M_t)_{t\in\mathbb{R}_+} = (N_t - t)_{t\in\mathbb{R}_+}$ is

$$[M, M]_t = N_t, \qquad t \in \mathbb{R}_+.$$

Similarly for the compensated compound Poisson martingale $(M_t)_{t\in\mathbb{R}_+}$ defined as

$$M_t := \frac{X_t - \lambda t \mathbb{E}[Y_1]}{\sqrt{\lambda \mathbb{E}[Y_1^2]}}, \qquad t \in \mathbb{R}_+,$$

where

$$X_t = \sum_{k=1}^{N_t} Y_k, \qquad t \in \mathbb{R}_+,$$

we have

$$[M, M]_t = \frac{1}{\lambda \mathbb{E}[Y_1^2]} \sum_{k=1}^{N_t} |Y_k|^2, \qquad t \in \mathbb{R}_+.$$

Definition 2.9.4. *The angle bracket $\langle M, M\rangle_t$ is defined as the unique increasing process such that*

$$M_t^2 - \langle M, M\rangle_t, \qquad t \in \mathbb{R}_+,$$

is a martingale.

As a consequence of Remark 2.1.3 we have

$$\langle M, M\rangle_t = t, \qquad t \in \mathbb{R}_+,$$

for all normal martingales.

2.10 Structure Equations

We refer to [39] for the following definition.

Definition 2.10.1. *An equation of the form*

$$[M, M]_t = t + \int_0^t \phi_s dM_s, \quad t \in \mathbb{R}_+, \tag{2.10.1}$$

where $(\phi_t)_{t\in\mathbb{R}_+} \in L^2_{ad}(\Omega\times\mathbb{R}_+)$ *is a square-integrable adapted process, is called a structure equation.*

In the sequel we will always consider a right-continuous version of $(\phi_t)_{t\in\mathbb{R}_+}$.

From (2.9.2) we get that for any normal martingale satisfying the structure equation (2.10.1) we have

$$(M_t-M_s)^2 = 2\int_s^t (M_\tau - M_\tau)dM_\tau + \int_s^t \phi_\tau dM_\tau + t - s, \qquad 0 \le s \le t. \quad (2.10.2)$$

Moreover,

$$[M,M]_t - \langle M,M\rangle_t, \qquad t\in\mathbb{R}_+,$$

is also a martingale as a consequence of Remark 2.1.3 and Corollary 2.5.8, since by Definition 2.9.1 we have

$$\begin{aligned}[M,M]_t - \langle M,M\rangle_t &= [M,M]_t - t \qquad (2.10.3)\\ &= M_t^2 - t - 2\int_0^t M_s dM_s, \qquad t\in\mathbb{R}_+.\end{aligned}$$

As a consequence we have the following proposition.

Proposition 2.10.2. *Assume that* $(M_t)_{t\in\mathbb{R}_+}$ *is a normal martingale in* L^4 *having the predictable representation property. Then* $(M_t)_{t\in\mathbb{R}_+}$ *satisfies the structure equation* (2.10.1)*, i.e. there exists a square-integrable adapted process* $(\phi_t)_{t\in\mathbb{R}_+}$ *such that*

$$[M,M]_t = t + \int_0^t \phi_s dM_s, \qquad t\in\mathbb{R}_+.$$

Proof. Since $([M,M]_t - t)_{t\in\mathbb{R}_+}$ is a martingale by (2.10.3) and $(M_t)_{t\in\mathbb{R}_+}$ has the chaos representation property, Proposition 2.6.2 shows the existence of a square-integrable adapted process $(\phi_t)_{t\in\mathbb{R}_+}$ such that

$$[M,M]_t - t = \int_0^t \phi_s dM_s, \qquad t\in\mathbb{R}_+.$$

□

In general, letting

$$i_t = \mathbf{1}_{\{\phi_t=0\}} \quad \text{and} \quad j_t = 1 - i_t = \mathbf{1}_{\{\phi_t\neq 0\}}, \qquad t\in\mathbb{R}_+, \qquad (2.10.4)$$

the continuous part of $(M_t)_{t\in\mathbb{R}_+}$ is given by $dM_t^c = i_t dM_t$ and the eventual jump of $(M_t)_{t\in\mathbb{R}_+}$ at time $t\in\mathbb{R}_+$ is given as $\Delta M_t = \phi_t$ on $\{\Delta M_t \neq 0\}$, $t\in\mathbb{R}_+$, see [39], page 70.

In particular,

a) Brownian motion $(B_t)_{t\in\mathbb{R}_+}$ satisfies the structure equation (2.10.1) with $\phi_t = 0$, since the quadratic variation of $(B_t)_{t\in\mathbb{R}_+}$ is $[B,B]_t = t$, $t \in \mathbb{R}_+$. In (2.10.7) we have $\Delta B_t = \pm\sqrt{\Delta t}$ with equal probabilities 1/2.

b) The compensated Poisson martingale $(M_t)_{t\in\mathbb{R}_+} = \lambda(N^\lambda_t - t/\lambda^2)_{t\in\mathbb{R}_+}$, where $(N^\lambda_t)_{t\in\mathbb{R}_+}$ is a standard Poisson process with intensity $1/\lambda^2$, satisfies the structure equation (2.10.1) with $\phi_t = \lambda \in \mathbb{R}$, $t \in \mathbb{R}_+$, since

$$[M,M]_t = \lambda^2 N^\lambda_t = t + \lambda M_t, \qquad t \in \mathbb{R}_+.$$

In this case, $\Delta M_t \in \{0,\lambda\}$ in (2.10.7), with respective probabilities $1-\lambda^{-2}\Delta t$ and $\lambda^{-2}\Delta t$.

c) If $(\phi_t)_{t\in\mathbb{R}_+}$ is deterministic, then $(M_t)_{t\in\mathbb{R}_+}$ can be represented as

$$dM_t = i_t dB_t + \phi_t(dN_t - \lambda_t dt), \qquad t \in \mathbb{R}_+, \qquad M_0 = 0, \tag{2.10.5}$$

with $\lambda_t = j_t/\phi_t^2$, $t \in \mathbb{R}_+$, where $(B_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion, and $(N_t)_{t\in\mathbb{R}_+}$ a Poisson process independent of $(B_t)_{t\in\mathbb{R}_+}$, with intensity $\nu_t = \int_0^t \lambda_s ds$, $t \in \mathbb{R}_+$, cf. [39].

d) The Azéma martingales correspond to $\phi_t = \beta M_t$, $\beta \in [-2,0)$, and provide other examples of processes having the chaos representation property, and dependent increments, cf. [39].

e) Not all normal martingales satisfy a structure equation and have the predictable representation property. For instance, for the compound Poisson process

$$X_t = \sum_{k=1}^{N_t} Y_k, \qquad t \in \mathbb{R}_+,$$

and the compensated compound Poisson martingale

$$M_t = \frac{X_t - \lambda t\mathbb{E}[Y_1]}{\sqrt{\lambda\,\mathbb{E}[Y_1^2]}}, \qquad t \in \mathbb{R}_+, \tag{2.10.6}$$

of Section 2.4, we have

$$\begin{aligned}
[M,M]_t &= (\lambda\,\mathbb{E}[Y_1^2])^{-1}\sum_{k=1}^{N_t}|Y_k|^2 \\
&= (\lambda\,\mathbb{E}[Y_1^2])^{-1}\int_0^t |Y_{1+N_{s-}}|^2 dN_s \\
&= (\lambda\,\mathbb{E}[Y_1^2])^{-1}\int_0^t \frac{|Y_{1+N_{s-}}|^2}{Y_{N_s}} d(X_s - \lambda\mathbb{E}[Y_1]ds)
\end{aligned}$$

$$+\mathbb{E}[Y_1](\mathbb{E}[Y_1^2])^{-1}\int_0^t \frac{|Y_{1+N_{s-}}|^2}{Y_{N_s}}ds$$
$$= \frac{1}{\sqrt{\lambda\,\mathbb{E}[Y_1^2]}}\int_0^t \frac{|Y_{1+N_{s-}}|^2}{Y_{N_s}}dM_s$$
$$+\mathbb{E}[Y_1](\mathbb{E}[Y_1^2])^{-1/2}\int_0^t \frac{|Y_{1+N_{s-}}|^2}{Y_{N_s}}ds,$$

$t \in \mathbb{R}_+$, hence $(M_t)_{t\in\mathbb{R}_+}$ does not satisfy a structure equation and as a consequence of Proposition 2.6.2 it does not have the predictable representation property, and it does not satisfy (2.10.1), unless Y_1 is a constant.

Another way to verify this fact is to consider for example the sum

$$M_t = N_t^1 - t + \alpha(N_t^2 - t)$$

where $|\alpha| \neq 1$ and (N_t^1), (N_t^2) are independent standard Poisson processes. In this case,

$$(M_T)^2 - 2\int_0^T M_{s-}dM_s = N_T^1 + |\alpha|^2 N_T^2,$$

can clearly not be represented as a stochastic integral with respect to $(M_t)_{t\in\mathbb{R}_+}$ when $|\alpha| \neq 1$.

The structure equation (2.10.1) can be informally written as

$$(\Delta M_t)^2 = \Delta t + \phi_t \Delta M_t,$$

with solution

$$\Delta M_t = \frac{\phi_t}{2} \pm \sqrt{\left(\frac{\phi_t}{2}\right)^2 + \Delta t}, \tag{2.10.7}$$

which is a continuous-time analog of Relation (1.4.2). By the martingale property of $(M_t)_{t\in\mathbb{R}_+}$ we have the equation $\mathbb{E}[\Delta M_t] = 0$ which yields the respective probabilities

$$\frac{1}{2} \mp \frac{\phi_t}{2\sqrt{\phi_t^2 + 4\Delta t}},$$

compare with (1.4.2) and (1.4.4). This provides a procedure to simulate sample paths of a normal martingale.

Figure 2.1 presents a simulation of the paths of an Azéma martingale, in which case we have

$$\Delta M_t = \frac{\beta M_{t-}}{2} \pm \sqrt{\left(\frac{\beta M_{t-}}{2}\right)^2 + \Delta t},$$

with probabilities

$$\frac{1}{2} \mp \frac{\beta M_{t^-}}{2\sqrt{(\beta M_{t^-})^2 + 4\Delta t}},$$

for some $\beta \in \mathbb{R}$.

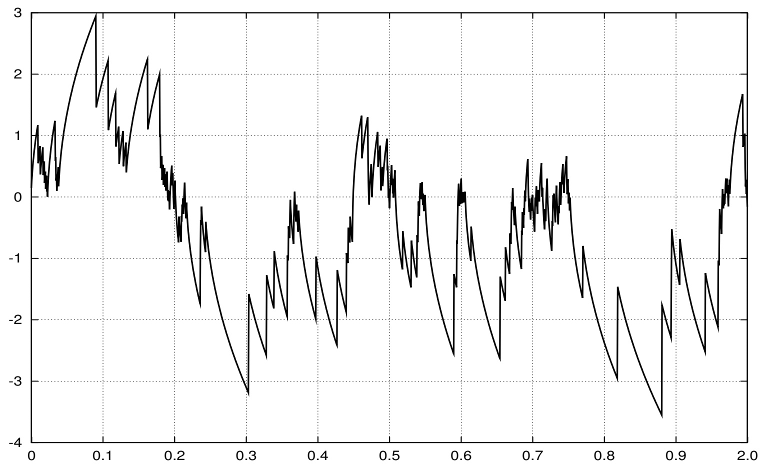

Fig. 2.1: Sample path of an Azéma martingale with $\beta = -0.5$.

Informally, the above calculations of quadratic variations can be obtained by applying the rules

$$\begin{cases} |dB_t|^2 = dt, \\ |dN_t|^2 = dN_t, \\ |dX_t|^2 = |Y_{N_t}|^2 dN_t, \\ |dt|^2 = 0. \end{cases}$$

2.11 Product Formula for Stochastic Integrals

In this section we present a multiplication formula in $L^2(\Omega)$ for stochastic integrals with respect to normal martingales. For this we need to be able to control their L^4 norms as in the next proposition.

Assumption 2.11.1. *Assume that for some constant $K > 0$,*

$$\mathbb{E}\left[\int_a^b \phi_s^2 ds \Big| \mathcal{F}_a\right] \leq K^2(b-a), \qquad \mathbb{P}-a.s., \quad 0 \leq a \leq b. \tag{2.11.1}$$

Note that Condition (2.11.1) holds whenever $(\phi_t)_{t\in[0,T]} \in L^\infty_{ad}(\Omega \times [0,T])$. This condition is satisfied in all cases of interest here, since for Azéma martingales we have $(\phi_t)_{t\in[0,T]} = (\beta M_t)_{t\in[0,T]}$, $\beta \in (-2,0)$, and $\sup_{t\in[0,T]} |M_t| \leq (-2/\beta)^{1/2} < \infty$ (see [39], page 83). In addition, Brownian motion and the compensated Poisson martingales satisfy this hypothesis. Assumption 2.11.1 leads to the next proposition.

Proposition 2.11.2. *Under the Assumption 2.11.1 we have*

$$\begin{aligned}&\left\|\int_0^\infty u_s dM_s\right\|_{L^4(\Omega)} \\ &\leq C(\|u\|_{L^4(\Omega\times\mathbb{R}_+)} + \|u\|_{L^4(\Omega,L^2(\mathbb{R}_+))} + \|u\|^{1/2}_{L^4(\Omega,L^2(\mathbb{R}_+))}\|u\|^{1/2}_{L^4(\Omega\times\mathbb{R}_+)}),\end{aligned} \tag{2.11.2}$$

for all $u \in L^4_{ad}(\Omega\times\mathbb{R}_+) \cap L^4(\Omega, L^2(\mathbb{R}_+))$, where $C > 0$ is a constant.

Proof. Let $u \in \mathcal{P}$ a simple predictable process of the form

$$u = \sum_{i=1}^n G_i \mathbf{1}_{(t_{i-1},t_i]}, \tag{2.11.3}$$

for a given a subdivision $\pi = \{0 = t_0 < t_1 < \cdots < t_n\}$ with

$$|\pi| := \max_{i=1,\ldots,n} |t_i - t_{i-1}| \leq 1.$$

We have, using Relation (2.1.1),

$$\begin{aligned}\mathbb{E}\left[\left(\int_0^\infty u_s dM_s\right)^4\right] &= \mathbb{E}\left[\left(\sum_{i=1}^n G_i(M_{t_i} - M_{t_{i-1}})\right)^4\right] \\ &= \mathbb{E}\left[\sum_{i=1}^n G_i^4 (M_{t_i} - M_{t_{i-1}})^4\right] \\ &\quad +2\mathbb{E}\left[\sum_{1\leq i<j\leq n} G_i^2 G_j^2 (M_{t_i} - M_{t_{i-1}})^2 (M_{t_j} - M_{t_{j-1}})^2\right] \\ &= \mathbb{E}\left[\sum_{i=1}^n G_i^4 (M_{t_i} - M_{t_{i-1}})^4\right] \\ &\quad +2\mathbb{E}\left[\sum_{1\leq i<j\leq n} G_i^2 G_j^2 (M_{t_i} - M_{t_{i-1}})^2 (t_j - t_{j-1})\right]\end{aligned}$$

$$
\begin{aligned}
&\le \mathbb{E}\left[\sum_{i=1}^n G_i^4 (M_{t_i}-M_{t_{i-1}})^4\right]\\
&\quad +2\mathbb{E}\left[\sum_{i=1}^n G_i^2 (M_{t_i}-M_{t_{i-1}})^2 \int_{t_i}^\infty |u_s|^2 ds\right]\\
&\le \mathbb{E}\left[\sum_{i=1}^n G_i^4 (M_{t_i}-M_{t_{i-1}})^4\right]\\
&\quad +2\left(\mathbb{E}\left[\left(\int_0^\infty |u_s|^2 ds\right)^2\right]\right)^{1/2}\left(\mathbb{E}\left[\left(\sum_{i=1}^n G_i^2 (M_{t_i}-M_{t_{i-1}})^2\right)^2\right]\right)^{1/2}.
\end{aligned}
$$

We will deal successively with the three terms in the above expression. From Relation (2.10.2) we have

$$
\begin{aligned}
&\mathbb{E}\left[\left(\sum_{i=1}^n G_i^2 (M_{t_i}-M_{t_{i-1}})^2\right)^2\right] \qquad (2.11.4)\\
&= \mathbb{E}\left[\left(\sum_{i=1}^n 2G_i^2 \int_{t_{i-1}}^{t_i} (M_\tau - M_{t_{i-1}}) dM_\tau + G_i^2 \int_{t_{i-1}}^{t_i} \phi_\tau dM_\tau + G_i^2 (t_{i-1}-t_i)\right)^2\right]\\
&\le 3\mathbb{E}\left[\sum_{i=1}^n 4G_i^4 \left(\int_{t_{i-1}}^{t_i} (M_\tau - M_{t_{i-1}}) dM_\tau\right)^2 + G_i^4 \left(\int_{t_{i-1}}^{t_i} \phi_\tau d\tau\right)^2 + G_i^4 (t_{i-1}-t_i)^2\right]\\
&\le 3\mathbb{E}\left[\sum_{i=1}^n 4G_i^4 \int_{t_{i-1}}^{t_i} (M_\tau - M_{t_{i-1}})^2 d\tau + G_i^4 (t_i - t_{i-1}) \int_{t_{i-1}}^{t_i} |\phi_\tau|^2 d\tau + G_i^4 (t_{i-1}-t_i)^2\right]\\
&\le 3\mathbb{E}\left[\sum_{i=1}^n 4G_i^4 \int_{t_{i-1}}^{t_i} (\tau - t_{i-1}) d\tau + G_i^4 (t_i - t_{i-1}) \int_{t_{i-1}}^{t_i} |\phi_\tau|^2 d\tau + G_i^4 (t_{i-1}-t_i)^2\right]\\
&\le 3\mathbb{E}\left[\sum_{i=1}^n G_i^4 \left(4\int_{t_{i-1}}^{t_i} (\tau - t_{i-1}) d\tau + (K^2+1)(t_i - t_{i-1})^2\right)\right]\\
&\le 3(5+K^2)|\pi| \|u\|_{L^4(\Omega\times\mathbb{R}_+)}^4.
\end{aligned}
$$

Next from Relation (2.10.2) we have

$$
\begin{aligned}
(M_t - M_s)^4 &= (t-s)\left(2\int_s^t (M_\tau - M_s) dM_\tau + \int_s^t \phi_\tau dM_\tau + t - s\right)\\
&\quad +2\int_s^t (M_\tau - M_s) dM_\tau \left(2\int_s^t (M_\tau - M_s) dM_\tau + \int_s^t \phi_\tau dM_\tau + t - s\right)\\
&\quad +\int_s^t \phi_\tau dM_\tau \left(2\int_s^t (M_\tau - M_s) dM_\tau + \int_s^t \phi_\tau dM_\tau + t - s\right),
\end{aligned}
$$

hence from Proposition 2.5.7, Relations (2.1.1) and (2.5.4), and the conditional Itô isometry (2.5.4) we have

$$\begin{aligned}
\mathbb{E}[(M_t - M_s)^4|\mathcal{F}_s] &= (t-s)^2 + 4\mathbb{E}\left[\int_s^t (M_\tau - M_s)^2 d\tau \middle| \mathcal{F}_s\right] \\
&\quad +4\mathbb{E}\left[\int_s^t \phi_\tau (M_\tau - M_s) d\tau \middle| \mathcal{F}_s\right] + \mathbb{E}\left[\int_s^t \phi_\tau^2 d\tau \middle| \mathcal{F}_s\right] \\
&= (t-s)^2 + 4\int_s^t (\tau - s)^2 d\tau \\
&\quad +4\mathbb{E}\left[\int_s^t \phi_\tau (M_\tau - M_s) d\tau \middle| \mathcal{F}_s\right] + \mathbb{E}\left[\int_s^t \phi_\tau^2 d\tau \middle| \mathcal{F}_s\right] \\
&\leq (t-s)^2 + 4\int_s^t (\tau - s)^2 d\tau + \mathbb{E}\left[\int_s^t \phi_\tau^2 d\tau \middle| \mathcal{F}_s\right] \\
&\quad +4\left(\mathbb{E}\left[\int_s^t \phi_\tau^2 d\tau \middle| \mathcal{F}_s\right]\right)^{1/2} \left(\mathbb{E}\left[\int_s^t (M_\tau - M_s)^2 d\tau \middle| \mathcal{F}_s\right]\right)^{1/2} \\
&\leq (t-s)^2 + 4\int_s^t (\tau - s)^2 d\tau + \mathbb{E}\left[\int_s^t \phi_\tau^2 d\tau \middle| \mathcal{F}_s\right] \\
&\quad +4\left(\mathbb{E}\left[\int_s^t \phi_\tau^2 d\tau \middle| \mathcal{F}_s\right]\right)^{1/2} \left(\int_s^t (\tau - s)^2 d\tau\right)^{1/2} \\
&\leq (t-s)^2 + \frac{4}{3}(t-s)^3 + \frac{4}{\sqrt{3}}|t-s|^{3/2}\left(\mathbb{E}\left[\int_s^t \phi_\tau^2 d\tau \middle| \mathcal{F}_s\right]\right)^{1/2} \\
&\quad +\mathbb{E}\left[\int_s^t \phi_\tau^2 d\tau \middle| \mathcal{F}_s\right] \\
&\leq (t-s)\left(K^2 + t - s + \frac{4K}{\sqrt{3}}(t-s) + \frac{4}{3}(t-s)^2\right),
\end{aligned}$$

which yields

$$\begin{aligned}
\mathbb{E}\left[\sum_{i=1}^n G_i^4 (M_{t_i} - M_{t_{i-1}})^4\right] &= \sum_{i=1}^n \mathbb{E}[G_i^4 \mathbb{E}[(M_{t_i} - M_{t_{i-1}})^4 | \mathcal{F}_{t_{i-1}}]] \\
&\leq \left(K^2 + |\pi| + \frac{4K}{\sqrt{3}}|\pi| + \frac{4}{3}|\pi|^2\right) \mathbb{E}\left[\sum_{i=1}^n G_i^4 (t_i - t_{i-1})\right] \qquad (2.11.5) \\
&\leq \tilde{K}^2 \|u\|^4_{L^4(\Omega\times\mathbb{R}_+)},
\end{aligned}$$

for some constant $\tilde{K} > 0$. Finally, (2.11.4) and (2.11.5) lead to (2.11.2). □

Next we state a multiplication formula for stochastic integrals with respect to a normal martingale.

Proposition 2.11.3. *Under the Assumption 2.11.1, for all* $u \in L^4_{ad}(\Omega \times \mathbb{R}_+) \cap L^4(\Omega, L^2(\mathbb{R}_+))$ *we have* $\int_0^\infty u_s dM_s \in L^4(\Omega)$ *and*

$$\left(\int_0^\infty u_s dM_s\right)^2 = 2\int_0^\infty u_s \int_0^s u_\tau dM_\tau dM_s + \int_0^\infty |u_s|^2 \phi_s dM_s + \int_0^\infty |u_s|^2 ds. \tag{2.11.6}$$

Proof. For simple predictable processes $(u_s)_{s\in\mathbb{R}_+}$ of the form (2.11.3), formula (2.11.6) follows from (2.10.2). It is extended to $u \in L^4_{ad}(\Omega \times \mathbb{R}_+) \cap L^4(\Omega, L^2(\mathbb{R}_+))$ using (2.11.2) and the Itô isometry (2.5.4) which shows that

$$\begin{aligned}
\mathbb{E}\left[\left(\int_0^\infty |u_s|^2 \phi_s dM_s\right)^2\right] &= \mathbb{E}\left[\sum_{i=1}^n |G_i|^4 \int_{t_{i-1}}^{t_i} |\phi_s|^2 ds\right] \\
&\leq K\mathbb{E}\left[\sum_{i=1}^n |G_i|^4 (t_i - t_{i-1})\right] \\
&= K\|u\|^4_{L^4(\Omega\times\mathbb{R}_+)},
\end{aligned}$$

and

$$\begin{aligned}
&\mathbb{E}\left[\left(\int_0^\infty u_s \int_0^s u_\tau dM_\tau dM_s\right)^2\right] = \mathbb{E}\left[\int_0^\infty |u_s|^2 \left(\int_0^s u_\tau dM_\tau\right)^2 ds\right] \\
&\leq \int_0^\infty \mathbb{E}\left[|u_s|^4\right]^{1/2} ds \mathbb{E}\left[\left(\int_0^s u_\tau dM_\tau\right)^4\right]^{1/2} \\
&\leq C\|u\|^2_{L^4(\Omega\times\mathbb{R}_+)}(\|u\|^2_{L^4(\Omega\times\mathbb{R}_+)} + \|u\|^2_{L^4(\Omega,L^2(\mathbb{R}_+))}\|u\|^2_{L^4(\Omega\times\mathbb{R}_+)}),
\end{aligned}$$

for some constant $C > 0$. □

The proof of Proposition 2.11.3 can easily be modified to show that we have

$$\begin{aligned}
&\left(\int_0^t u_s dM_s + \int_0^t a_s ds\right)\left(\int_0^t v_s dM_s + \int_0^t b_s ds\right) \\
&\quad = \int_0^t u_s \int_0^s b_\tau d\tau dM_s + \int_0^t b_s \int_0^s u_\tau dM_\tau ds \\
&\qquad + \int_0^t v_s \int_0^s a_\tau d\tau dM_s + \int_0^t a_s \int_0^s v_\tau dM_\tau ds \\
&\qquad + \int_0^t a_s \int_0^s b_\tau d\tau ds + \int_0^t b_s \int_0^s a_\tau d\tau ds \\
&\qquad + \int_0^t u_s \int_0^s v_\tau dM_\tau dM_s + \int_0^t v_s \int_0^s u_\tau dM_\tau dM_s \\
&\qquad + \int_0^t \phi_s u_s v_s dM_s + \int_0^t u_s v_s ds,
\end{aligned} \tag{2.11.7}$$

where all terms belong to $L^4(\Omega)$, $t \in \mathbb{R}_+$, for all $u, v, a, b \in L^4_{ad}(\Omega \times \mathbb{R}_+) \cap L^4(\Omega, L^2(\mathbb{R}_+))$.

As a corollary we have the following multiplication formula for multiple stochastic integrals.

Corollary 2.11.4. *Let $u \in L^\infty(\mathbb{R}_+)$ and $v \in L^4(\mathbb{R}_+)$. Then under Assumption 2.11.1 we have for all $n \geq 1$ and $t \in \mathbb{R}_+$:*

$$\begin{aligned}
&I_1(u\mathbf{1}_{[0,t]})I_n(\mathbf{1}_{[0,t]^n}v^{\otimes n})\\
&= \int_0^t u_s I_n(\mathbf{1}_{[0,s]^n}v^{\otimes n})dM_s + n\int_0^t v_s I_{n-1}(\mathbf{1}_{[0,s]^{n-1}}v^{\otimes(n-1)})\int_0^s u_\tau dM_\tau dM_s\\
&+n\int_0^t \phi_s u_s v_s I_{n-1}(\mathbf{1}_{[0,s]^{n-1}}v^{\otimes(n-1)})dM_s + n\int_0^t u_s v_s I_{n-1}(\mathbf{1}_{[0,s]^{n-1}}v^{\otimes(n-1)})ds.
\end{aligned}$$

Proof. Applying Proposition 2.11.3 to u_s and $v_s I_{n-1}(\mathbf{1}_{[0,s]^{n-1}}v^{\otimes(n-1)})$, we have

$$\begin{aligned}
I_1(u\mathbf{1}_{[0,t]})I_n(\mathbf{1}_{[0,t]^n}v^{\otimes n}) &= n\int_0^t u_s dM_s \int_0^t v_s I_{n-1}(\mathbf{1}_{[0,s]^{n-1}}v^{\otimes(n-1)})dM_s\\
&= n\int_0^t u_s \int_0^s v_\tau I_{n-1}(\mathbf{1}_{[0,\tau]^{n-1}}v^{\otimes(n-1)})dM_\tau dM_s\\
&\quad +n\int_0^t v_s I_{n-1}(\mathbf{1}_{[0,s]^{n-1}}v^{\otimes(n-1)})\int_0^s u_\tau dM_\tau dM_s\\
&\quad +n\int_0^t \phi_s u_s v_s I_{n-1}(\mathbf{1}_{[0,s]^{n-1}}v^{\otimes(n-1)})dM_s\\
&\quad +n\int_0^t u_s v_s I_{n-1}(\mathbf{1}_{[0,s]^{n-1}}v^{\otimes(n-1)})ds\\
&= \int_0^t u_s I_n(\mathbf{1}_{[0,s]^n}v^{\otimes n})dM_s + n\int_0^t v_s I_{n-1}(\mathbf{1}_{[0,s]^{n-1}}v^{\otimes(n-1)})\int_0^s u_\tau dM_\tau dM_s\\
&\quad +n\int_0^t \phi_s u_s v_s I_{n-1}(\mathbf{1}_{[0,s]^{n-1}}v^{\otimes(n-1)})dM_s\\
&\quad +n\int_0^t u_s v_s I_{n-1}(\mathbf{1}_{[0,s]^{n-1}}v^{\otimes(n-1)})ds.
\end{aligned}$$

□

2.12 Itô Formula

Consider a normal martingale $(M_t)_{t\in\mathbb{R}_+}$ satisfying the structure equation

$$d[M,M]_t = dt + \phi_t dM_t.$$

Such an equation is satisfied in particular if $(M_t)_{t\in\mathbb{R}_+}$ has the predictable representation property, cf. Proposition 2.10.2.

The following is a statement of Itô's formula for normal martingales, cf. [39], page 70, for a proof using formal semimartingales. The proof presented here is in the L^2 sense, and when $\phi_s = 0$ the fractions in (2.12.2) should be replaced with their limits as $\phi_s = 0$ tends to 0 (i.e. with their corresponding first and second derivatives).

Proposition 2.12.1. *Assume that* $\phi \in L^\infty_{ad}([0,T] \times \Omega)$. *Let* $(X_t)_{t\in[0,T]}$ *be a process given by*

$$X_t = X_0 + \int_0^t u_s dM_s + \int_0^t v_s ds, \qquad t \in [0,T], \tag{2.12.1}$$

where $(u_s)_{s\in[0,T]}$ *and* $(v_s)_{s\in[0,T]}$ *are adapted processes in* $L^2_{ad}(\Omega\times[0,T])$. *Then for all* $f \in \mathcal{C}^{1,2}(\mathbb{R}_+ \times \mathbb{R})$ *we have*

$$\begin{aligned} f(t,X_t) - f(0,X_0) &= \int_0^t \frac{f(s,X_{s^-}+\phi_s u_s) - f(s,X_{s^-})}{\phi_s} dM_s \\ &\quad + \int_0^t \frac{f(s,X_s+\phi_s u_s) - f(s,X_s) - \phi_s u_s \frac{\partial f}{\partial x}(s,X_s)}{\phi_s^2} ds \\ &\quad + \int_0^t v_s \frac{\partial f}{\partial x}(s,X_s) ds + \int_0^t \frac{\partial f}{\partial s}(s,X_s) ds, \qquad t \in [0,T]. \end{aligned} \tag{2.12.2}$$

Proof. We prove the formula in the case where f does not depend on time and $v_s = 0$, $s \in \mathbb{R}_+$, since this generalization can be done using standard calculus arguments. Assume now that $u \in \mathcal{U}$ is a simple predictable process of the form $u = G\mathbf{1}_{[a,b]}$, with $G \in L^\infty(\Omega, \mathcal{F}_a)$. Clearly the formula (2.12.2) holds when $f = c$ is a constant function. By induction on $n \geq 1$, we assume that it holds when applied to $f(x) = x^n$, i.e.

$$X_t^n = X_0^n + \int_0^t L_s^n ds + \int_0^t U_s^n dM_s,$$

with

$$U_s^n = \frac{(X_{s^-} + \phi_s u_s)^n - (X_{s^-})^n}{\phi_s},$$

and

$$L_s^n = \frac{(X_s + \phi_s u_s)^n - (X_s)^n - n\phi_s u_s (X_s)^{n-1}}{\phi_s^2}, \quad s \in \mathbb{R}_+.$$

From Proposition 2.11.3 we have

$$\begin{aligned} X_t^{n+1} - X_0^n &= (X_t - X_0)(X_t^n - X_0^n) + X_0(X_t^n - X_0^n) + X_0^n(X_t - X_0) \\ &= \int_0^t u_s \int_0^s L_\tau^n d\tau dM_s + \int_0^t u_s \int_0^s U_\tau^n dM_\tau dM_s \\ &\quad + \int_0^t L_s^n \int_0^s u_\tau dM_\tau ds + \int_0^t U_s^n \int_0^s u_\tau dM_\tau dM_s \\ &\quad + \int_0^t \phi_s u_s U_s^n dM_s + \int_0^t u_s U_s^n ds \\ &\quad + X_0 \int_0^t L_s^n ds + X_0 \int_0^t U_s^n dM_s, + X_0^n \int_0^t u_s dM_s \\ &= \int_0^t X_s U_s^n dM_s + \int_0^t X_s L_s^n ds + \int_0^t X_s^n u_s dM_s \end{aligned}$$

$$+\int_0^t u_s\phi_s U_s^n dM_s + \int_0^t u_s U_s^n ds$$
$$= \int_0^t L_s^{n+1} ds + \int_0^t U_s^{n+1} dM_s,$$

since

$$U_s^{n+1} = u_s(X_{s^-})^n + X_{s^-}U_s^n + u_s\phi_s U_s^n = u_s X_{s^-}^n + X_{s^-}U_s^n + u_s\phi_s U_s^n,$$

and

$$L_s^{n+1} = U_s^n u_s + X_{s^-}L_s^n, \qquad s \in \mathbb{R}_+, \quad n \in \mathbb{N}.$$

This proves the formula for f polynomial and $u \in \mathcal{P}$. For all $n \geq 1$, let

$$\tau_n = \inf\{s \in \mathbb{R}_+ \ : \ |X_s| > n\}.$$

Let $f \in \mathcal{C}_b^2(\mathbb{R})$ and let $(f_m)_{m\in\mathbb{N}}$ be a sequence of polynomials converging uniformly to f on $[-n, n]$. Since $\mathbf{1}_{\{s\leq\tau_n\}} = 1$, a.s., $0 \leq s \leq t$, on $\{t \leq \tau_n\}$, from the locality property of the stochastic integral (Proposition 2.5.5) we have for all $m \in \mathbb{N}$:

$$\begin{aligned}
\mathbf{1}_{\{t\leq\tau_n\}}(f_m(X_t) - f_m(X_0)) &= \mathbf{1}_{\{t\leq\tau_n\}} \int_0^t \frac{f_m(X_{s^-} + \phi_s u_s) - f_m(X_{s^-})}{\phi_s} dM_s \\
&+ \mathbf{1}_{\{t\leq\tau_n\}} \int_0^t \frac{f_m(X_{s^-} + \phi_s u_s) - f_m(X_{s^-}) - \phi_s u_s f_m'(X_{s^-})}{\phi_s^2} ds \\
&= \mathbf{1}_{\{t\leq\tau_n\}} \int_0^t \mathbf{1}_{\{s\leq\tau_n\}} \frac{f_m(X_{s^-} + \phi_s u_s) - f_m(X_{s^-})}{\phi_s} dM_s \\
&+ \mathbf{1}_{\{t\leq\tau_n\}} \int_0^t \mathbf{1}_{\{s\leq\tau_n\}} \frac{f_m(X_{s^-} + \phi_s u_s) - f_m(X_{s^-}) - \phi_s u_s f_m'(X_{s^-})}{\phi_s^2} ds.
\end{aligned}$$

Letting m go to infinity we get

$$\begin{aligned}
&\mathbf{1}_{\{t\leq\tau_n\}}(f(X_t) - f(X_0)) \\
&= \mathbf{1}_{\{t\leq\tau_n\}} \int_0^t \mathbf{1}_{\{s\leq\tau_n\}} \frac{f(X_{s^-} + \phi_s u_s) - f(X_{s^-})}{\phi_s} dM_s \\
&\quad + \mathbf{1}_{\{t\leq\tau_n\}} \int_0^t \mathbf{1}_{\{s\leq\tau_n\}} \frac{f(X_{s^-} + \phi_s u_s) - f(X_{s^-}) - \phi_s u_s f'(X_{s^-})}{\phi_s^2} ds,
\end{aligned}$$

and it remains to let n go to infinity, which proves the formula for $f \in \mathcal{C}_b^2(\mathbb{R})$ by locality of the Itô integral, see e.g. Proposition 2.5.5. The formula is then extended to $u \in L_{ad}^2([0, T] \times \Omega)$, by density of $\mathcal{U}$ in $L_{ad}^2([0, T] \times \Omega)$, and finally to all $f \in \mathcal{C}^2(\mathbb{R})$, again by locality of the Itô integral. □

Note that if $\phi_s = 0$, the terms

$$\frac{f(X_{s^-} + \phi_s u_s) - f(X_{s^-})}{\phi_s}$$

and

$$\frac{f(X_s+\phi_s u_s)-f(X_s)-\phi_s u_s f'(X_s)}{\phi_s^2}$$

have to be replaced by their respective limits $u_s f'(X_{s^-})$ and $\frac{1}{2}u_s^2 f''(X_{s^-})$ as $\phi_s \to 0$. This gives

$$\begin{aligned}
f(t,X_t)-f(0,X_0) &= \int_0^t i_s u_s \frac{\partial f}{\partial x}(s,X_s)dM_s + \frac{1}{2}\int_0^t i_s u_s^2 \frac{\partial^2 f}{\partial x^2}(s,X_s)ds \\
&+ \int_0^t j_s \frac{f(s,X_{s^-}+\phi_s u_s)-f(s,X_{s^-})}{\phi_s} dM_s \\
&+ \int_0^t j_s \frac{f(s,X_s+\phi_s u_s)-f(s,X_s)-\phi_s u_s \frac{\partial f}{\partial x}(s,X_s)}{\phi_s^2} ds \\
&+ \int_0^t v_s \frac{\partial f}{\partial x}(s,X_s)ds + \int_0^t \frac{\partial f}{\partial s}(s,X_s)ds, \qquad t\in[0,T],
\end{aligned}$$

where the processes $(i_t)_{t\in\mathbb{R}_+}$ and $(j_t)_{t\in\mathbb{R}_+}$ have been defined in (2.10.4).

Examples

i) In the case of Brownian motion $(B_t)_{t\in\mathbb{R}_+}$ we have $\phi_s = 0$, $s\in\mathbb{R}_+$, hence the Itô formula reads

$$f(B_t) = f(0) + \int_0^t f'(B_s)dB_s + \frac{1}{2}\int_0^t f''(B_s)ds.$$

ii) For the compensated Poisson process $(N_t - t)_{t\in\mathbb{R}_+}$ we have $\phi_s = 1$, $s\in\mathbb{R}_+$, hence

$$\begin{aligned}
f(N_t - t) &= f(0) + \int_0^t (f(1+N_{s^-}-s)-f(N_{s^-}-s))d(N_s-s) \\
&\quad + \int_0^t (f(1+N_s-s)-f(N_s-s)-f'(N_s-s))ds.
\end{aligned}$$

In the Poisson case this formula can actually be recovered by elementary calculus, as follows:

$$\begin{aligned}
f(N_t-t) &= f(0) + f(N_t-t) - f(N_t - T_{N_t}) \\
&\quad + \sum_{k=1}^{N_t} f(k-T_k) - f(k-1-T_{k-1}) \\
&= f(0) + \sum_{k=1}^{N_t} f(k-T_k) - f(k-1-T_k) \\
&\quad - \int_{T_{N_t}}^t f'(N_t-s)ds - \sum_{k=1}^{N_t}\int_{T_{k-1}}^{T_k} f'(k-1-s)ds
\end{aligned}$$

$$= f(0) + \int_0^t (f(1+N_{s^-}-s) - f(N_{s^-}-s))dN_s - \int_0^t f'(N_s - s)ds$$
$$= f(0) + \int_0^t (f(1+N_{s^-}-s) - f(N_{s^-}-s))(dN_s - ds)$$
$$+ \int_0^t (f(1+N_s-s) - f(N_s-s) - f'(N_s-s))ds.$$

iii) More generally, in case $(\phi_t)_{t\in\mathbb{R}_+}$ is deterministic we have

$$M_t = \int_0^t \mathbf{1}_{\{\phi_s=0\}} dB_s + \int_0^t \mathbf{1}_{\{\phi_s\neq 0\}} \phi_s \left(dN_s - \frac{ds}{\phi_s^2} \right), \qquad t \in \mathbb{R}_+,$$

and

$$f(M_t) = f(0) + \int_0^t \mathbf{1}_{\{\phi_s=0\}} f'(M_s) dB_s + \frac{1}{2}\int_0^t \mathbf{1}_{\{\phi_s=0\}} f''(M_s) dB_s$$
$$+ \int_0^t \mathbf{1}_{\{\phi_s\neq 0\}} (f(M_{s^-}+\phi_s) - f(M_{s^-})) \left(dN_s - \frac{ds}{\phi_s^2} \right)$$
$$+ \int_0^t \mathbf{1}_{\{\phi_s\neq 0\}} (f(M_{s^-}+\phi_s) - f(M_{s^-}) - \phi_s f'(M_s)) \frac{ds}{\phi_s^2}.$$

iv) For the compound Poisson martingale

$$M_t := \frac{X_t - \lambda t \mathbb{E}[Y_1]}{\sqrt{\lambda \mathrm{Var}\,[Y_1]}}, \qquad t \in \mathbb{R}_+,$$

of Section 2.4 we have

$$f(M_t) = f(0) + \int_0^t (f(Y_{N_s} + M_{s^-}) - f(M_{s^-})) d(N_s - \lambda s) \tag{2.12.3}$$
$$+ \lambda \int_0^t \left(f(Y_{N_s} + M_s) - f(M_s) - \frac{\mathbb{E}[Y_1]}{\sqrt{\lambda \mathrm{Var}\,[Y_1]}} f'(M_s) \right) ds.$$

However, as noted above the compound Poisson martingale (2.10.6) does not have the predictable representation property. Thus it does not satisfy the hypotheses of this section and the above formula is actually distinct from (2.12.1) since here the stochastic integral is not with respect to $(M_t)_{t\in\mathbb{R}_+}$ itself. Note also that the pathwise stochastic integral (2.12.3) is not in the sense of Proposition 2.5.10 since the integrand $(f(Y_{N_t} + M_{t^-}) - f(M_{t^-}))$ is not a left limit due to the presence of Y_{N_t}.

The change of variable formula can be extended to the multidimensional case as in Proposition 2 of [39]. Here we use the convention $0/0 = 0$.

Proposition 2.12.2. *Let $X = (X_t)_{t\in\mathbb{R}_+}$ be a $\mathbb{R}^n$-valued process given by*

$$dX_t = R_t dt + K_t dM_t, \quad X_0 > 0,$$

where $(R_t)_{t\in\mathbb{R}_+}$ *and* $(K_t)_{t\in\mathbb{R}_+}$ *are predictable square-integrable* $\mathbb{R}^n$*-valued processes. For any* $f \in \mathcal{C}_b^2(\mathbb{R}_+ \times \mathbb{R}^n;\mathbb{R})$ *we have*

$$f(t, X_t) = f(0, X_0) + \int_0^t L_s f(s, X_s) dM_s + \int_0^t U_s f(s, X_s) ds + \int_0^t \frac{\partial f}{\partial s}(s, X_s) ds, \tag{2.12.4}$$

where

$$L_s f(s, X_s) = i_s \langle K_s, \nabla f(s, X_s)\rangle + \frac{j_s}{\phi_s}(f(s, X_{s^-} + \phi_s K_{s^-}) - f(s, X_{s^-})), \tag{2.12.5}$$

and

$$U_s f(s, X_s) = R_s \nabla f(s, X_s) + \alpha_s^2 \left(\frac{1}{2} i_s \langle \text{Hess } f(s, X_s), K_s \otimes K_s\rangle \right.$$
$$\left. + \frac{j_s}{\phi_s^2}(f(s, X_{s^-} + \phi_s K_{s^-}) - f(s, X_{s^-}) - \phi_s \langle K_s, \nabla f(s, X_s)\rangle) \right).$$

2.13 Exponential Vectors

The exponential vectors are a stochastic analog of the generating function for polynomials. In this section they are presented as the solutions of linear stochastic differential equations with respect to a normal martingale $(M_t)_{t\in\mathbb{R}_+}$ having the predictable representation property, and satisfying a structure equation of the form (2.10.1).

In the next proposition, the solution (2.13.2) of Equation 2.13.1 can be derived using the Itô formula, and the uniqueness can be proved using the Itô isometry and classical arguments.

Proposition 2.13.1. *For any* $u \in L^2_{ad}(\Omega \times \mathbb{R}_+)$*, the equation*

$$Z_t = 1 + \int_0^t Z_{s^-} u_s dM_s, \quad t \in [0, T] \tag{2.13.1}$$

has a unique solution $(\xi_t(u))_{t\in\mathbb{R}_+}$ *given by*

$$\xi_t(u) = \exp\left(\int_0^t u_s dM_s - \frac{1}{2}\int_0^t u_s^2 \mathbf{1}_{\{\phi_s = 0\}} ds \right) \prod_{s\in J_M^t} (1 + u_s \phi_s) \mathrm{e}^{-u_s\phi_s}, \tag{2.13.2}$$

where J_M^t *denotes the set of jump times of* $(M_s)_{s\in[0,t]}$, $t \in [0, T]$.

In the pure jump case, from Proposition 2.5.10, $(Z_s)_{s\in\mathbb{R}_+}$ satisfies the pathwise stochastic differential equation with jumps

$$dG_t = u(t) G_{t^-}(dN_t - dt), \quad G_0 = 1, \tag{2.13.3}$$

which can be directly solved on each interval $(T_{k-1}, T_k]$, $k \geq 1$, to get

$$G_t = \mathrm{e}^{-\int_0^t u(s)ds} \prod_{k=1}^{N_t} (1 + u(T_k)), \qquad t \in \mathbb{R}_+.$$

Proposition 2.13.2. *Given $u \in L^\infty([0,T])$, $\xi_T(u)$ can be represented as*

$$\xi_T(u) = \sum_{n=0}^{\infty} \frac{1}{n!} I_n(u^{\otimes n} \mathbf{1}_{[0,T]^n}). \tag{2.13.4}$$

Proof. Letting

$$Z_t^n = 1 + \sum_{k=1}^{n} \frac{1}{k!} I_k(u^{\otimes k} \mathbf{1}_{[0,t]^k}),$$

we have

$$\begin{aligned} 1 + \int_0^t u_\tau Z_\tau^n dM_\tau &= 1 + \sum_{k=1}^{n+1} \frac{1}{k!} I_k(u^{\otimes k} \mathbf{1}_{[0,t]^k}) \\ &= Z_t^{n+1}, \end{aligned}$$

which yields

$$Z_t = 1 + \int_0^t u_\tau Z_\tau dM_\tau, \qquad t \in \mathbb{R}_+,$$

as n goes to infinity, where integrals and sums have been interchanged in the L^2 sense. Hence $\xi_t(u)$ and Z_t coincide since they solve the same equation (2.13.1). □

In particular, letting T go to infinity in (2.13.2) and (2.13.4) yields the identity

$$\xi(u) = \exp\left(\int_0^\infty u_s dM_s - \frac{1}{2} \int_0^\infty u_s^2 \mathbf{1}_{\{\phi_s = 0\}} ds \right) \prod_{\Delta N_s \neq 0} (1 + u_s \phi_s) \mathrm{e}^{-u_s \phi_s}. \tag{2.13.5}$$

Definition 2.13.3. *Let $\mathcal{E}$ denote the linear space generated by exponential vectors of the form $\xi(u)$, where $u \in L^\infty([0,T])$.*

Under the chaos representation property of Definition 2.8.2 the space $\mathcal{E}$ is dense in $L^2(\Omega)$, and from the following lemma, $\mathcal{E}$ is an algebra for the pointwise multiplication of random variables when $(\phi_t)_{t \in [0,T]}$ is a deterministic function.

Lemma 2.13.4. *For any $u, v \in L^\infty(\mathbb{R}_+)$, we have the relation*

$$\xi(u)\xi(v) = \exp(\langle u, v \rangle_{L^2(\mathbb{R}_+)}) \xi(u + v + \phi uv). \tag{2.13.6}$$

Proof. From Proposition 2.11.3 we have for $u, v \in L^\infty(\mathbb{R}_+)$:

$$d(\xi_t(u)\xi_t(v))$$

$$
\begin{aligned}
&= u_t\xi_{t^-}(u)\xi_{t^-}(v)dM_t + v_t\xi_{t^-}(v)\xi_{t^-}(u)dM_t + v_tu_t\xi_{t^-}(v)\xi_{t^-}(u)d[M,M]_t \\
&= u_t\xi_{t^-}(u)\xi_{t^-}(v)dM_t + v_t\xi_{t^-}(v)\xi_{t^-}(u)dM_t + v_tu_t\xi_t(v)\xi_t(u)dt \\
&\quad +\phi_tu_tv_t\xi_{t^-}(v)\xi_{t^-}(u)dM_t \\
&= v_tu_t\xi_t(v)\xi_t(u)dt + \xi_{t^-}(v)\xi_{t^-}(u)(u_t+v_t+\phi_tu_tv_t)dM_t.
\end{aligned}
$$

hence

$$
d(\mathrm{e}^{-\int_0^t u_sv_sds}\xi_t(u)\xi_t(v)) = \mathrm{e}^{-\int_0^t u_sv_sds}\xi_{t^-}(v)\xi_{t^-}(u)(u_t+v_t+\phi_tu_tv_t)dM_t,
$$

which shows that

$$
\exp(-\langle u,v\rangle_{L^2([0,T])})\xi(u)\xi(v) = \xi(u+v+\phi uv).
$$

Relation (2.13.6) then follows by comparison with (2.13.1). □

2.14 Vector-Valued Case

In this section we consider multiple stochastic integrals of vector-valued functions with respect to a d-dimensional normal martingale $(M_t)_{0\le t\le T}$ with independent components $M_t = (M_t^{(1)},\ldots,M_t^{(d)})$.

Let $g_n \in L^2([0,T]^n)$ and let $(e_1,\ldots,e_d)$ denote the canonical basis of $\mathbb{R}^d$. We define the n-th iterated integral of $g_ne_{i_1}\otimes\ldots\otimes e_{i_n}$, with $1\le i_1,\ldots,i_n\le d$, as

$$
I_n(g_ne_{i_1}\otimes\ldots\otimes e_{i_n}) = n!\int_0^T\int_0^{t_n}\cdots\int_0^{t_2} g_n(t_1,\ldots,t_n)dM_{t_1}^{(i_1)}\ldots dM_{t_n}^{(i_n)}. \tag{2.14.1}
$$

For $g_n\in L^2([0,T]^n)$, $h_m\in L^2([0,T]^m)$, with $1\le i_1,\ldots,i_n\le d$ and $1\le j_1,\ldots,j_m\le d$, we have

$$
\begin{aligned}
&\mathbb{E}\left[I_n(g_ne_{i_1}\otimes\ldots\otimes e_{i_n})I_m(h_me_{j_1}\otimes\ldots\otimes e_{j_m})\right] \\
&= \begin{cases} n!\langle g_n,h_m\rangle_{L^2([0,T]^n)} & \text{if} \quad n=m \quad \text{and} \quad i_l=j_l,\ 1\le l\le n, \\ \\ 0 & \text{otherwise.}\end{cases}
\end{aligned}
$$

Given $f_n = (f_n^{(i_1,\ldots,i_n)})_{1\le i_1,\ldots,i_n\le d}\in L^2([0,T],\mathbb{R}^d)^{\otimes n}$, we define the n-th iterated integral of f_n by

$$
I_n(f_n) = n!\sum_{i_1,\ldots,i_n=1}^d\int_0^T\int_0^{t_n}\cdots\int_0^{t_2} f_n^{(i_1,\ldots,i_n)}(t_1,\ldots,t_n)dM_{t_1}^{(i_1)}\ldots dM_{t_n}^{(i_n)}
$$

$$= \sum_{i_1,\ldots,i_n=1}^{d} I_n(f_n^{(i_1,\ldots,i_n)} e_{i_1} \otimes \ldots \otimes e_{i_n}).$$

Let Σ_n denote the set of all permutations of $\{1,\ldots,n\}$. We have

$$I_n(\tilde{f}_n) = I_n(f_n),$$

where $\tilde{f}_n$ is the symmetrization of f_n in $L^2([0,T],\mathbb{R}^d)^{\circ n}$, i.e for $1 \le i_1,\ldots,i_n \le d$, we have

$$\tilde{f}_n^{(i_1,\ldots,i_n)}(t_1,\ldots,t_n) = \frac{1}{n!} \sum_{\sigma \in \Sigma_n} f_n^{(i_{\sigma(1)},\ldots,i_{\sigma(n)})}(t_{\sigma(1)},\ldots,t_{\sigma(n)}).$$

Given

$$f_n = (f_n^{(i_1,\ldots,i_n)})_{1 \le i_1,\ldots,i_n \le d} \in L^2([0,T],\mathbb{R}^d)^{\otimes n}$$

and

$$g_m = (g_m^{(j_1,\ldots,j_m)})_{1 \le j_1,\cdots,j_m \le d} \in L^2([0,T],\mathbb{R}^d)^{\otimes m}$$

we have

$$\mathbb{E}\left[I_n(f_n) I_m(g_m)\right] = \begin{cases} \displaystyle\sum_{\substack{i_1,\cdots,i_n=1 \\ j_1,\cdots,j_m=1}}^{d} n! \langle f_n^{(i_1,\ldots,i_n)}, g_m^{(j_1,\ldots,j_m)} \rangle_{L^2([0,T]^n)} & \text{if } i_l = j_l,\ 1 \le l \le n, \\ 0 & \text{otherwise.} \end{cases}$$

Finally we consider the multidimensional Poisson and mixed Brownian-Poisson cases.

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbb{R}_+}, P)$ be a probability space with the filtration $(\mathcal{F}_t)_{t \in \mathbb{R}_+}$ generated by a d-dimensional Poisson process $N_t = (N_t^{(1)},\ldots,N_t^{(p)})$, $0 \le t \le T$, and independent components, with deterministic intensity

$$\left(\int_0^t \lambda_s^{(1)} ds, \ldots, \int_0^t \lambda_s^{(d)} ds \right).$$

The iterated stochastic integral of a symmetric function

$$f_n = (f_n^{(i_1,\ldots,i_n)})_{1 \le i_1,\ldots,i_n \le d} \in L^2([0,T],\mathbb{R}^d)^{\circ n},$$

where $f_n^{(i_1,\ldots,i_n)} \in L^2([0,T]^n)$, is defined by

$$I_n(f_n) := n! \sum_{i_1,\ldots,i_n=1}^{d} \int_0^T \int_0^{t_n} \cdots \int_0^{t_2}$$

$$f_n^{(i_1,\ldots,i_n)}(t_1,\ldots,t_n)(dN_{t_1}^{(i_1)}-\lambda_{t_1}^{(i_1)}dt_1)\ldots(dN_{t_n}^{(i_n)}-\lambda_{t_n}^{(i_n)}dt_n).$$

In the mixed Brownian-Poisson case let $(B_t)_{0\le t\le T}$ with $B_t=(B_t^{(1)},\ldots,B_t^{(d)})$ be a d-dimensional Brownian motion with independent components, let $(N_t)_{0\le t\le T}$ with $N_t=(N_t^{(1)},\ldots,N_t^{(p)})$ be a p-dimensional Poisson process with independent components and intensity

$$\left(\int_0^t \lambda_s^{(1)}ds,\ldots,\int_0^t \lambda_s^{(d)}ds\right).$$

We work on a filtered probability space $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\in[0,T]},P)$, where $\mathcal{F}_t$ is the filtration generated by $(B_t)_{t\in\mathbb{R}_+}$ and $(N_t)_{t\in\mathbb{R}_+}$, which are assumed to be independent. We denote by $M=(M^{(1)},\ldots,M^{(p)})$ the compensated Poisson process given by

$$dM^{(l)}=dN_t^{(l)}-\lambda_t^{(l)}dt,\qquad t\in[0,T],\qquad l=1,\ldots,p.$$

Let

$$(X_t^{(1)},\ldots,X_t^{(d)},X_t^{(d+1)},\ldots,X_t^{(d+p)})=(B_t^{(1)},\ldots,B_t^{(d)},M_t^{(1)},\ldots,M_t^{(p)}).$$

The iterated stochastic integral of a symmetric function

$$f_n=(f_n^{(i_1,\ldots,i_n)})_{1\le i_1,\ldots,i_n\le d+p}\in L^2([0,T],\mathbb{R}^{(d+p)})^{\otimes n},$$

where $f_n^{(i_1,\ldots,i_n)}\in L^2([0,T]^n)$, is given by

$$I_n(f_n):=n!\sum_{i_1,\ldots,i_n=1}^{d+p}\int_0^T\int_0^{t_n}\cdots\int_0^{t_2}f_n^{(i_1,\ldots,i_n)}(t_1,\ldots,t_n)dX_{t_1}^{(i_1)}\ldots dX_{t_n}^{(i_n)}.$$

The chaos representation property holds in the multidimensional case, i.e. for any $F\in L^2(\Omega)$ there exists a unique sequence $(f_n)_{n\in\mathbb{N}}$ of symmetric deterministic functions

$$f_n=(f_n^{(i_1,\ldots,i_n)})_{i_1,\ldots,i_n\in\{1,\ldots,d\}}\in L^2([0,T],\mathbb{R}^d)^{\circ n}$$

such that

$$F=\sum_{n=0}^{\infty}I_n(f_n).$$

2.15 Notes and References

Our presentation of stochastic calculus is restricted to normal martingales, which are well fitted to the construction of multiple stochastic integrals and chaos expansions in the L^2 sense. We refer to e.g. [125] for the the standard approach to stochastic calculus using local martingales and semimartingales. The systematic study of normal martingales started in [39], and d-dimensional normal martingales have been considered in [10], [11]. Several books and surveys exist on multiple stochastic integrals, see [41], [60], [77], [86]. A number of properties of Brownian motion, such as almost sure path continuity, have been omitted here and can be found in [126]. See [28], page 317, and [26], [157] for the absence of predictable representation property of compound Poisson processes. The Itô change of variable formula Proposition 2.12.1 can be found in Proposition 2 in [39] with a different formulation and proof. We refer to [125], Theorem 36, page 77, for results on exponential vectors. Lemma 2.13.4 is a version of Yor's formula [156], cf. also Theorem 37 of [123], page 79, for martingales with deterministic bracket $\langle M, M\rangle_t$. Proposition 2.3.6 can be proved using the independence of increments of the Poisson process and arguments of [91]. The presentation of multidimensional Poisson and mixed Brownian-Poisson integrals is based on [73].

Chapter 3
Gradient and Divergence Operators

In this chapter we construct an abstract framework for stochastic analysis in continuous time with respect to a normal martingale $(M_t)_{t\in\mathbb{R}_+}$, using the construction of stochastic calculus presented in Section 2. In particular we identify some minimal properties that should be satisfied in order to connect a gradient and a divergence operator to stochastic integration, and to apply them to the predictable representation of random variables. Some applications, such as logarithmic Sobolev and deviation inequalities, are formulated in this general setting. In the next chapters we will examine concrete examples of operators that can be included in this framework, in particular when $(M_t)_{t\in\mathbb{R}_+}$ is a Brownian motion or a compensated Poisson process.

3.1 Definition and Closability

In this chapter, $(M_t)_{t\in\mathbb{R}_+}$ denotes a normal martingale as considered in Chapter 2. We let $\mathcal{S}$, $\mathcal{U}$, and $\mathcal{P}$ denote the spaces of random variables, simple processes and simple predictable processes introduced in Definition 2.5.2, and we note that $\mathcal{S}$ is dense in $L^2(\Omega)$ by Definition 2.5.2 and $\mathcal{U}$, $\mathcal{P}$ are dense in $L^2(\Omega\times\mathbb{R}_+)$ and $L^2_{ad}(\Omega\times\mathbb{R}_+)$ respectively from Proposition 2.5.3.

Let now

$$D : L^2(\Omega, d\mathbb{P}) \to L^2(\Omega\times\mathbb{R}_+, d\mathbb{P}\times dt)$$

and

$$\delta : L^2(\Omega\times\mathbb{R}_+, d\mathbb{P}\times dt) \to L^2(\Omega, d\mathbb{P})$$

be linear operators defined respectively on $\mathcal{S}$ and $\mathcal{U}$. In particular, $(D_tF)_{t\in\mathbb{R}_+}$ is a stochastic process for $F\in\mathcal{S}$, and we assume that the following duality relation holds.

Assumption 3.1.1. *(Duality relation) The operators D and δ satisfy the relation*

$$\mathbb{E}[\langle DF, u\rangle_{L^2(\mathbb{R}_+)}] = \mathbb{E}[F\delta(u)], \qquad F \in \mathcal{S},\ u \in \mathcal{U}. \tag{3.1.1}$$

Note that $D\mathbf{1} = 0$ is equivalent to $\mathbb{E}[\delta(u)] = 0$, for all $u \in \mathcal{U}$. In the next proposition we use the notion of closability for operators in normed linear spaces, whose definition is recalled in Section 9.8 of the Appendix. The next proposition is actually a general result on the closability of the adjoint of a densely defined operator.

Proposition 3.1.2. *The duality assumption 3.1.1 implies that D and δ are closable.*

Proof. If $(F_n)_{n\in\mathbb{N}}$ converges to 0 in $L^2(\Omega)$ and $(DF_n)_{n\in\mathbb{N}}$ converges to $U \in L^2(\Omega \times \mathbb{R}_+)$, the relation

$$\mathbb{E}[\langle DF_n, u\rangle_{L^2(\mathbb{R}_+)}] = \mathbb{E}[F_n\delta(u)], \qquad u \in \mathcal{U},$$

implies

$$\begin{aligned}
|\langle U, u\rangle_{L^2(\Omega\times\mathbb{R}_+)}| &= |\,\mathbb{E}[\langle U, u\rangle_{L^2(\mathbb{R}_+)}]| \\
&\leq |\,\mathbb{E}[\langle DF_n, u\rangle_{L^2(\mathbb{R}_+)}] - \mathbb{E}[\langle U, u\rangle_{L^2(\mathbb{R}_+)}]| + |\,\mathbb{E}[\langle DF_n, u\rangle_{L^2(\mathbb{R}_+)}| \\
&= |\,\mathbb{E}[\langle DF_n - U, u\rangle_{L^2(\mathbb{R}_+)}]| + |\,\mathbb{E}[F_n\delta(u)]| \\
&\leq \|DF_n - U\|_{L^2(\Omega\times\mathbb{R}_+)}\|u\|_{L^2(\Omega\times\mathbb{R}_+)} + \|F_n\|_{L^2(\Omega)}\|\delta(u)\|_{L^2(\Omega)},
\end{aligned}$$

hence as n goes to infinity we get $\mathbb{E}[\langle U, u\rangle_{L^2(\mathbb{R}_+)}] = 0$, $u \in \mathcal{U}$, i.e. $U = 0$ since $\mathcal{U}$ is dense in $L^2(\Omega \times \mathbb{R}_+)$. The proof of closability of δ is similar: if $(u_n)_{n\in\mathbb{N}}$ converges to 0 in $L^2(\Omega \times \mathbb{R}_+)$ and $(\delta(u_n))_{n\in\mathbb{N}}$ converges to $F \in L^2(\Omega)$, we have for all $G \in \mathcal{S}$:

$$\begin{aligned}
|\,\mathbb{E}[FG]| &\leq |\,\mathbb{E}[\langle DG, u_n\rangle_{L^2(\mathbb{R}_+)}] - \mathbb{E}[FG]| + |\,\mathbb{E}[\langle DG, u_n\rangle_{L^2(\mathbb{R}_+)}]| \\
&= |\,\mathbb{E}[G(\delta(u_n) - F)]| + |\,\mathbb{E}[\langle DG, u_n\rangle_{L^2(\Omega\times\mathbb{R}_+)}]| \\
&\leq \|\delta(u_n) - F\|_{L^2(\Omega)}\|G\|_{L^2(\Omega)} + \|u_n\|_{L^2(\Omega\times\mathbb{R}_+)}\|DG\|_{L^2(\Omega\times\mathbb{R}_+)},
\end{aligned}$$

hence $\mathbb{E}[FG] = 0$, $G \in \mathcal{S}$, i.e. $F = 0$ since $\mathcal{S}$ is dense in $L^2(\Omega)$. □

From the above proposition these operators are respectively extended to their closed domains Dom (D) and Dom (δ), and for simplicity their extensions will remain denoted by D and δ.

3.2 Clark Formula and Predictable Representation

In this section we study the connection between D, δ, and the predictable representation of random variables using stochastic integrals.

Assumption 3.2.1. *(Clark formula). Every $F \in \mathcal{S}$ can be represented as*

$$F = \mathbb{E}[F] + \int_0^\infty \mathbb{E}[D_t F | \mathcal{F}_t] dM_t. \tag{3.2.1}$$

This assumption is connected to the predictable representation property for the martingale $(M_t)_{t\in\mathbb{R}_+}$, cf. Proposition 3.2.8 and Proposition 3.2.6 below.

Definition 3.2.2. *Given $k \geq 1$, let $\mathbb{D}_{2,k}([a,\infty))$, $a > 0$, denote the completion of $\mathcal{S}$ under the norm*

$$\|F\|_{\mathbb{D}_{2,k}([a,\infty))} = \|F\|_{L^2(\Omega)} + \sum_{i=1}^{k} \sqrt{\mathbb{E}\left[\int_a^\infty |D_t^i F|^2 dt\right]},$$

where $D_t^i = D_t \cdots D_t$ denotes the i-th iterated power of D_t, $i \geq 1$.

In other words, for any $F \in \mathbb{D}_{2,k}([a,\infty))$, the process $(D_t F)_{t\in[a,\infty)}$ exists in $L^2(\Omega \times [a,\infty))$. Clearly we have $\mathrm{Dom}\,(D) = \mathbb{D}_{2,1}([0,\infty))$. Under the Clark formula Assumption 3.2.1, a representation result for $F \in \mathbb{D}_{2,1}([a,\infty))$ can be stated as a consequence of the Clark formula:

Proposition 3.2.3. *For all $t \in \mathbb{R}_+$ and $F \in \mathbb{D}_{2,1}([t,\infty))$ we have*

$$\mathbb{E}[F|\mathcal{F}_t] = \mathbb{E}[F] + \int_0^t \mathbb{E}[D_s F|\mathcal{F}_s] dM_s, \tag{3.2.2}$$

and

$$F = \mathbb{E}[F|\mathcal{F}_t] + \int_t^\infty \mathbb{E}[D_s F|\mathcal{F}_s] dM_s, \qquad t \in \mathbb{R}_+. \tag{3.2.3}$$

Proof. This is a direct consequence of (3.2.1) and Proposition 2.5.7. □

By uniqueness of the predictable representation of $F \in L^2(\Omega)$, an expression of the form

$$F = c + \int_0^\infty u_t dM_t$$

where $c \in \mathbb{R}$ and $(u_t)_{t\in\mathbb{R}_+}$ is adapted and square-integrable, implies

$$u_t = \mathbb{E}[D_t F|\mathcal{F}_t], \qquad dt \times d\mathbb{P} - a.e.$$

The covariance identity proved in the next lemma is a consequence of Proposition 3.2.3 and the Itô isometry (2.5.5).

Lemma 3.2.4. *For all $t \in \mathbb{R}_+$ and $F \in \mathbb{D}_{2,1}([t,\infty))$ we have*

$$\mathbb{E}[(\mathbb{E}[F|\mathcal{F}_t])^2] = (\mathbb{E}[F])^2 + \mathbb{E}\left[\int_0^t (\mathbb{E}[D_s F|\mathcal{F}_s])^2 ds\right] \tag{3.2.4}$$

$$= \mathbb{E}[F^2] - \mathbb{E}\left[\int_t^\infty (\mathbb{E}[D_s F|\mathcal{F}_s])^2 ds\right]. \tag{3.2.5}$$

Proof. From the Itô isometry (2.5.4) and Relation (3.2.2) we have

$$\begin{aligned}\mathbb{E}[(\mathbb{E}[F|\mathcal{F}_t])^2] &= \mathbb{E}\left[\left(\mathbb{E}[F] + \int_0^t \mathbb{E}[D_s F|\mathcal{F}_s] dM_s\right)^2\right] \\ &= (\mathbb{E}[F])^2 + \mathbb{E}\left[\left(\int_0^t \mathbb{E}[D_s F|\mathcal{F}_s] dM_s\right)^2\right] \\ &= (\mathbb{E}[F])^2 + \mathbb{E}\left[\int_0^t (\mathbb{E}[D_s F|\mathcal{F}_s])^2 ds\right], \qquad t \in \mathbb{R}_+,\end{aligned}$$

which shows (3.2.4). Next, concerning (3.2.5) we have

$$\begin{aligned}\mathbb{E}[F^2] &= \mathbb{E}\left[\left(\mathbb{E}[F|\mathcal{F}_t] + \int_t^\infty \mathbb{E}[D_s F|\mathcal{F}_s] dM_s\right)^2\right] \\ &= \mathbb{E}\left[(\mathbb{E}[F|\mathcal{F}_t])^2\right] + \mathbb{E}\left[\mathbb{E}[F|\mathcal{F}_t] \int_t^\infty \mathbb{E}[D_s F|\mathcal{F}_s] dM_s\right] \\ &\quad + \mathbb{E}\left[\left(\int_t^\infty \mathbb{E}[D_s F|\mathcal{F}_s] dM_s\right)^2\right] \\ &= \mathbb{E}\left[(\mathbb{E}[F|\mathcal{F}_t])^2\right] + \mathbb{E}\left[\int_t^\infty \mathbb{E}[F|\mathcal{F}_t]\, \mathbb{E}[D_s F|\mathcal{F}_s] dM_s\right] \\ &\quad + \mathbb{E}\left[\int_t^\infty (\mathbb{E}[D_s F|\mathcal{F}_s])^2\, ds\right] \\ &= \mathbb{E}\left[(\mathbb{E}[F|\mathcal{F}_t])^2\right] + \mathbb{E}\left[\int_t^\infty (\mathbb{E}[D_s F|\mathcal{F}_s])^2\, ds\right], \quad t \in \mathbb{R}_+,\end{aligned}$$

since from (2.5.7) the Itô stochastic integral has expectation 0, which shows (3.2.5). □

The next remark applies in general to any mapping sending a random variable to the process involved in its predictable representation with respect to a normal martingale.

Lemma 3.2.5. *The operator*

$$F \longmapsto (\mathbb{E}[D_t F|\mathcal{F}_t])_{t\in\mathbb{R}_+}$$

defined on $\mathcal{S}$ *extends to a continuous operator from* $L^2(\Omega)$ *into* $L^2(\Omega \times \mathbb{R}_+)$.

Proof. This follows from the bound

$$\begin{aligned}\|\, \mathbb{E}[D_\cdot F|\mathcal{F}_\cdot]\|^2_{L^2(\Omega\times\mathbb{R}_+)} &= \|F\|^2_{L^2(\Omega)} - (\mathbb{E}[F])^2 \\ &\leq \|F\|^2_{L^2(\Omega)},\end{aligned}$$

which is consequence of Relation (3.2.5) with $t = 0$, or of (3.2.4) with $t = +\infty$. □

As a consequence of Lemma 3.2.5, the Clark formula can be extended in Proposition 3.2.6 below as in the discrete case, cf. Proposition 1.7.2.

Proposition 3.2.6. *The Clark formula*

$$F = \mathbb{E}[F] + \int_0^\infty \mathbb{E}[D_t F|\mathcal{F}_t]dM_t$$

can be extended to all F in $L^2(\Omega)$.

Similarly, the results of Proposition 3.2.3 and Lemma 3.2.4 also extend to $F \in L^2(\Omega)$.

The Clark representation formula naturally implies a Poincaré type inequality.

Proposition 3.2.7. *Under Assumption 3.2.1, for all $F \in \mathrm{Dom}\,(D)$ we have*

$$\mathrm{Var}\,(F) \leq \|DF\|^2_{L^2(\Omega\times\mathbb{R}_+)}.$$

Proof. From Lemma 3.2.4 we have

$$\begin{aligned}
\mathrm{Var}\,(F) &= \mathbb{E}[|F - \mathbb{E}[F]|^2] \\
&= \mathbb{E}\left[\left(\int_0^\infty \mathbb{E}[D_t F|\mathcal{F}_t]dM_t\right)^2\right] \\
&= \mathbb{E}\left[\int_0^\infty (\mathbb{E}[D_t F|\mathcal{F}_t])^2\, dt\right] \\
&\leq \mathbb{E}\left[\int_0^\infty \mathbb{E}[|D_t F|^2|\mathcal{F}_t]dt\right] \\
&= \int_0^\infty \mathbb{E}\left[\mathbb{E}[|D_t F|^2|\mathcal{F}_t]\right] dt \\
&\leq \int_0^\infty \mathbb{E}\left[|D_t F|^2\right] dt \\
&\leq \mathbb{E}\left[\int_0^\infty |D_t F|^2 dt\right],
\end{aligned}$$

hence the conclusion. □

Since the space $\mathcal{S}$ is dense in $L^2(\Omega)$ by Definition 2.5.2, the Clark formula Assumption 3.2.1 implies the predictable representation property of Definition 2.6.1 for $(M_t)_{t\in\mathbb{R}_+}$ as a consequence of the next corollary.

Corollary 3.2.8. *Under the Clark formula Assumption 3.2.1 the martingale $(M_t)_{t\in\mathbb{R}_+}$ has the predictable representation property.*

Proof. Definition 2.6.1 is satisfied because $\mathcal{S}$ is dense in $L^2(\Omega)$ and the process $(\mathbb{E}[D_t F \mid \mathcal{F}_t])_{t\in\mathbb{R}_+}$ in (3.2.1) can be approximated by a sequence in $\mathcal{P}$ from Proposition 2.5.3.

Alternatively, one may use Proposition 2.6.2 and proceed as follows. Consider a square-integrable martingale $(X_t)_{t\in\mathbb{R}_+}$ with respect to $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ and let

$$u_s := \mathbb{E}[D_s X_{n+1}|\mathcal{F}_n], \qquad s \in [n, n+1).$$

Then $(u_t)_{t\in\mathbb{R}_+}$ is an adapted process such that $u\mathbf{1}_{[0,T]} \in L^2(\Omega\times\mathbb{R}_+)$ for all $T > 0$, and the Clark formula Assumption 3.2.1 and Proposition 3.2.6 imply

$$\begin{aligned}
X_t &= \mathbb{E}[X_{n+1} \mid \mathcal{F}_t] \\
&= \mathbb{E}\left[X_0 + \int_0^{n+1} \mathbb{E}[D_s X_{n+1} \mid \mathcal{F}_s] dM_s \Big| \mathcal{F}_t\right] \\
&= X_0 + \int_0^t \mathbb{E}\left[D_s X_{n+1} \mid \mathcal{F}_s\right] dM_s \\
&= X_0 + \int_0^n \mathbb{E}\left[D_s X_{n+1} \mid \mathcal{F}_s\right] dM_s + \int_n^t \mathbb{E}\left[D_s X_{n+1} \mid \mathcal{F}_s\right] dM_s \\
&= X_n + \int_n^t \mathbb{E}[D_s X_{n+1} \mid \mathcal{F}_s] dM_s \\
&= X_n + \int_n^t u_s dM_s, \qquad n \leq t \leq n+1, \quad n \in \mathbb{N},
\end{aligned}$$

where we used the Chasles relation (2.5.6), hence

$$X_t = X_0 + \int_0^t u_s dM_s, \qquad t \in \mathbb{R}_+, \tag{3.2.6}$$

hence from Proposition 2.6.2, $(M_t)_{t\in\mathbb{R}_+}$ has the predictable representation property. □

In particular, the Clark formula Assumption 3.2.1 and Relation (3.2.3) of Proposition 3.2.3 imply the following proposition.

Proposition 3.2.9. *For any $\mathcal{F}_T$-measurable $F \in L^2(\Omega)$ we have*

$$\mathbb{E}[D_t F | \mathcal{F}_T] = 0, \qquad 0 \leq T \leq t. \tag{3.2.7}$$

Proof. From from Relation (3.2.3) we have $F = \mathbb{E}[F|\mathcal{F}_T]$ if and only if

$$\int_T^\infty \mathbb{E}[D_t F | \mathcal{F}_t] dM_t = 0,$$

which implies $\mathbb{E}[D_t F|\mathcal{F}_t] = 0$, $t \geq T$, by the Itô isometry (2.5.4), hence (3.2.7) holds as

$$\mathbb{E}[D_t F | \mathcal{F}_T] = \mathbb{E}[\mathbb{E}[D_t F | \mathcal{F}_t] | \mathcal{F}_T] = 0, \qquad t \geq T,$$

by the tower property of conditional expectations stated in Section 9.3. □

The next assumption is a stability property for the gradient operator D.

Assumption 3.2.10. *(Stability property) For all $\mathcal{F}_T$-measurable $F \in \mathcal{S}$, $D_t F$ is $\mathcal{F}_T$-measurable for all $t \geq T$.*

Proposition 3.2.11. *Let $T > 0$. Under the stability Assumption 3.2.10, for any $\mathcal{F}_T$-measurable random variable $F \in L^2(\Omega)$ we have $F \in \mathbb{D}_{[T,\infty)}$ and*

$$D_t F = 0, \qquad t \geq T.$$

Proof. Since F is $\mathcal{F}_T$-measurable, $D_t F$ is $\mathcal{F}_T$-measurable, $t \geq T$, by the stability Assumption 3.2.10, and from Proposition 3.2.9 we have

$$D_t F = \mathbb{E}[D_t F | \mathcal{F}_T] = 0, \qquad 0 \leq T \leq t.$$

□

3.3 Divergence and Stochastic Integrals

In this section we are interested in the connection between the operator δ and the stochastic integral with respect to $(M_t)_{t\in\mathbb{R}_+}$.

Proposition 3.3.1. *Under the duality Assumption 3.1.1 and the Clark formula Assumption 3.2.1, the operator δ applied to any square-integrable adapted process $(u_t)_{t\in\mathbb{R}_+} \in L^2_{ad}(\Omega\times\mathbb{R}_+)$ coincides with the stochastic integral*

$$\delta(u) = \int_0^\infty u_t dM_t, \qquad u \in L^2_{ad}(\Omega\times\mathbb{R}_+), \tag{3.3.1}$$

of $(u_t)_{t\in\mathbb{R}_+}$ with respect to $(M_t)_{t\in\mathbb{R}_+}$, and the domain Dom (δ) *of δ contains $L^2_{ad}(\Omega\times\mathbb{R}_+)$.*

Proof. Let $u \in \mathcal{P}$ be a simple $\mathcal{F}_t$-predictable process as in Definition 2.5.2. From the duality Assumption 3.1.1 and the fact (2.5.7) that

$$\mathbb{E}\left[\int_0^\infty u_t dM_t\right] = 0,$$

we have:

$$\begin{aligned}
\mathbb{E}\left[F\int_0^\infty u_t dM_t\right] &= \mathbb{E}[F]\,\mathbb{E}\left[\int_0^\infty u_t dM_t\right] + \mathbb{E}\left[(F-\mathbb{E}[F])\int_0^\infty u_t dM_t\right] \\
&= \mathbb{E}\left[(F-\mathbb{E}[F])\int_0^\infty u_t dM_t\right] \\
&= \mathbb{E}\left[\int_0^\infty \mathbb{E}[D_tF|\mathcal{F}_t]dM_t \int_0^\infty u_t dM_t\right] \\
&= \mathbb{E}\left[\int_0^\infty u_t\, \mathbb{E}[D_tF|\mathcal{F}_t]dt\right] \\
&= \mathbb{E}\left[\int_0^\infty \mathbb{E}[u_tD_tF|\mathcal{F}_t]dt\right] \\
&= \int_0^\infty \mathbb{E}\left[\mathbb{E}[u_tD_tF|\mathcal{F}_t]\right]dt \\
&= \int_0^\infty \mathbb{E}\left[u_tD_tF\right]dt \\
&= \mathbb{E}\left[\int_0^\infty u_tD_tF dt\right] \\
&= \mathbb{E}[\langle DF, u\rangle_{L^2(\mathbb{R}_+)}] \\
&= \mathbb{E}[F\delta(u)],
\end{aligned}$$

for all $F \in \mathcal{S}$, hence the deneness of $\mathcal{S}$ in $L^2(\Omega)$ we have

$$\delta(u) = \int_0^\infty u_t dM_t$$

for all $\mathcal{F}_t$-predictable $u \in \mathcal{P}$. In the general case, from Proposition 2.5.3 we approximate $u \in L^2_{ad}(\Omega \times \mathbb{R}_+)$ by a sequence $(u^n)_{n\in\mathbb{N}} \subset \mathcal{P}$ of simple $\mathcal{F}_t$-predictable processes converging to u in $L^2(\Omega \times \mathbb{R}_+)$ and use the Itô isometry (2.5.4). □

As a consequence of the proof of Proposition 3.3.1 we have the isometry

$$\|\delta(u)\|_{L^2(\Omega)} = \|u\|_{L^2(\Omega\times\mathbb{R}_+)}, \qquad u \in L^2_{ad}(\Omega \times \mathbb{R}_+). \tag{3.3.2}$$

We also have the following partial converse to Proposition 3.3.1.

Proposition 3.3.2. *Assume that*

i) $(M_t)_{t\in\mathbb{R}_+}$ has the predictable representation property, and

ii) the operator δ coincides with the stochastic integral with respect to $(M_t)_{t\in\mathbb{R}_+}$ on the space $L^2_{ad}(\Omega \times \mathbb{R}_+)$ of square-integrable adapted processes.

Then the Clark formula Assumption 3.2.1 holds for the adjoint D of δ.

Proof. For all $F \in \mathrm{Dom}\,(D)$ and square-integrable adapted process u we have:

$$\begin{aligned}
\mathbb{E}[(F - \mathbb{E}[F])\delta(u)] &= \mathbb{E}[F\delta(u)] \\
&= \mathbb{E}[\langle DF, u\rangle_{L^2(\mathbb{R}_+)}] \\
&= \mathbb{E}\left[\int_0^\infty u_t\, \mathbb{E}[D_tF|\mathcal{F}_t]dt\right] \\
&= \mathbb{E}\left[\int_0^\infty u_t dM_t \int_0^\infty \mathbb{E}[D_tF|\mathcal{F}_t]dM_t\right] \\
&= \mathbb{E}\left[\delta(u)\int_0^\infty \mathbb{E}[D_tF|\mathcal{F}_t]dM_t\right],
\end{aligned}$$

hence

$$F - \mathbb{E}[F] = \int_0^\infty \mathbb{E}[D_tF|\mathcal{F}_t]dM_t,$$

since by (ii) we have

$$\left\{\delta(u) \ : \ u \in L^2_{ad}(\Omega \times \mathbb{R}_+)\right\} = \left\{\int_0^\infty u_t dM_t \ : \ u \in L^2_{ad}(\Omega \times \mathbb{R}_+)\right\},$$

which is dense in $\{F \in L^2(\Omega) \ : \ \mathbb{E}[F] = 0\}$ by (i) and Definition 2.6.1. □

3.4 Covariance Identities

Covariance identities will be useful in the proof of concentration and deviation inequalities. The Clark formula and the Itô isometry imply the following covariance identity, which uses the L^2 extension of the Clark formula, cf. Proposition 3.2.6.

Proposition 3.4.1. *For any $F, G \in L^2(\Omega)$ we have*

$$\mathrm{Cov}\,(F,G) = \mathbb{E}\left[\int_0^\infty \mathbb{E}[D_tF|\mathcal{F}_t]\,\mathbb{E}[D_tG|\mathcal{F}_t]dt\right]. \qquad (3.4.1)$$

Proof. We have

$$\begin{aligned}\mathrm{Cov}\,(F,G) &= \mathbb{E}[(F-\mathbb{E}[F])(G-\mathbb{E}[G])]\\ &= \mathbb{E}\left[\int_0^\infty \mathbb{E}[D_tF|\mathcal{F}_t]dM_t \int_0^\infty \mathbb{E}[D_tG|\mathcal{F}_t]dM_t\right]\\ &= \mathbb{E}\left[\int_0^\infty \mathbb{E}[D_tF|\mathcal{F}_t]\,\mathbb{E}[D_tG|\mathcal{F}_t]dt\right].\end{aligned}$$

□

The identity (3.4.1) can be rewritten as

$$\begin{aligned}\mathrm{Cov}\,(F,G) &= \mathbb{E}\left[\int_0^\infty \mathbb{E}[D_tF|\mathcal{F}_t]\,\mathbb{E}[D_tG|\mathcal{F}_t]dt\right]\\ &= \mathbb{E}\left[\int_0^\infty \mathbb{E}[\mathbb{E}[D_tF|\mathcal{F}_t]D_tG|\mathcal{F}_t]dt\right]\\ &= \mathbb{E}\left[\int_0^\infty \mathbb{E}[D_tF|\mathcal{F}_t]D_tG dt\right],\end{aligned}$$

provided that $G \in \mathrm{Dom}\,(D)$.

As is well known, if X is a real random variable and f, g are monotone functions then $f(X)$ and $g(X)$ are non-negatively correlated. Lemma 3.4.2, which is an immediate consequence of (3.4.1), provides an analog of this result for normal martingales, replacing the ordinary derivative with the adapted process $(\mathbb{E}[D_tF|\mathcal{F}_t])_{t\in[0,1]}$.

Lemma 3.4.2. *Let $F, G \in L^2(\Omega)$ such that*

$$\mathbb{E}[D_tF|\mathcal{F}_t]\cdot\mathbb{E}[D_tG|\mathcal{F}_t] \geq 0, \qquad dt \times d\mathbb{P} - a.e.$$

Then F and G are non-negatively correlated:

$$\mathrm{Cov}\,(F,G) \geq 0.$$

If $G \in \mathrm{Dom}\,(D)$, resp. $F, G \in \mathrm{Dom}\,(D)$, the above condition can be replaced by

$$\mathbb{E}[D_tF|\mathcal{F}_t] \geq 0 \quad \text{and} \quad D_tG \geq 0, \qquad dt \times d\mathbb{P} - a.e.,$$

resp.

$$D_tF \geq 0 \quad \text{and} \quad D_tG \geq 0, \qquad dt \times d\mathbb{P} - a.e..$$

Iterated versions of Lemma 3.2.4 can also be proved. Let

$$\Delta_n = \{(t_1,\dots,t_n) \in \mathbb{R}_+^n \ :\ 0 \leq t_1 < \cdots < t_n\},$$

and assume further that

Assumption 3.4.3. *(Domain condition) For all $F \in \mathcal{S}$ we have*

$$D_{t_n} \cdots D_{t_1} F \in \mathbb{D}_{2,1}([t_n, \infty)), \qquad a.e.\ (t_1, \ldots, t_n) \in \Delta_n.$$

We denote by $\mathbb{D}_{2,k}(\Delta_k)$ the L^2 domain of D^k, i.e. the completion of $\mathcal{S}$ under the norm

$$\|F\|^2_{\mathbb{D}_{2,k}(\Delta_k)} = \mathbb{E}\left[F^2\right] + \mathbb{E}\left[\int_{\Delta_k} |D_{t_k} \cdots D_{t_1} F|^2 dt_1 \cdots dt_k\right].$$

Note the inclusion $\mathbb{D}_{2,k}(\Delta_k) \subset \mathbb{D}_{2,1}(\Delta_k)$, $k \geq 1$.

Next we prove an extension of the covariance identity of [58], with a shortened proof.

Theorem 3.4.4. *Let $n \in \mathbb{N}$ and $F, G \in \bigcap_{k=1}^{n+1} \mathbb{D}_{2,k}(\Delta_k)$. We have*

$$\begin{aligned} \mathrm{Cov}\,(F,G) = {} & \sum_{k=1}^{n} (-1)^{k+1}\, \mathbb{E}\left[\int_{\Delta_k} (D_{t_k} \cdots D_{t_1} F)(D_{t_k} \cdots D_{t_1} G) dt_1 \cdots dt_k\right] \\ & + (-1)^n\, \mathbb{E}\left[\int_{\Delta_{n+1}} D_{t_{n+1}} \cdots D_{t_1} F\, \mathbb{E}\left[D_{t_{n+1}} \cdots D_{t_1} G | \mathcal{F}_{t_{n+1}}\right] dt_1 \cdots dt_{n+1}\right]. \end{aligned} \tag{3.4.2}$$

Proof. By polarization we may take $F = G$. For $n = 0$, (3.4.2) is a consequence of the Clark formula. Let $n \geq 1$. Applying Lemma 3.2.4 to $D_{t_n} \cdots D_{t_1} F$ with $t = t_n$ and $ds = dt_{n+1}$, and integrating on $(t_1, \ldots, t_n) \in \Delta_n$ we obtain

$$\begin{aligned} & \mathbb{E}\left[\int_{\Delta_n} \left(\mathbb{E}[D_{t_n} \cdots D_{t_1} F | \mathcal{F}_{t_n}]\right)^2 dt_1 \cdots dt_n\right] \\ & \quad = \mathbb{E}\left[\int_{\Delta_n} |D_{t_n} \cdots D_{t_1} F|^2 dt_1 \cdots dt_n\right] \\ & \qquad - \mathbb{E}\left[\int_{\Delta_{n+1}} \left(\mathbb{E}\left[D_{t_{n+1}} \cdots D_{t_1} F | \mathcal{F}_{t_{n+1}}\right]\right)^2 dt_1 \cdots dt_{n+1}\right], \end{aligned}$$

which concludes the proof by induction. □

The variance inequality

$$\sum_{k=1}^{2n} (-1)^{k+1} \|D^k F\|^2_{L^2(\Delta_k)} \leq \mathrm{Var}\,(F) \leq \sum_{k=1}^{2n-1} (-1)^{k+1} \|D^k F\|^2_{L^2(\Delta_k)},$$

for $F \in \bigcap_{k=1}^{2n} \mathbb{D}_{2,k}(\Delta_k)$, is a consequence of Theorem 3.4.4, and extends (2.15) in [58]. It also recovers the Poincaré inequality Proposition 3.2.7 when $n = 1$.

3.5 Logarithmic Sobolev Inequalities

The logarithmic Sobolev inequalities on Gaussian space provide an infinite dimensional analog of Sobolev inequalities, cf. e.g. [81]. In this section logarithmic Sobolev inequalities for normal martingales are proved as an application of the Itô and Clark formulas. Recall that the entropy of a sufficiently integrable random variable $F > 0$ is defined by

$$\mathrm{Ent}\,[F] = \mathbb{E}[F \log F] - \mathbb{E}[F] \log \mathbb{E}[F].$$

Proposition 3.5.1. *Let $F \in \mathrm{Dom}\,(D)$ be lower bounded with $F > \eta$ a.s. for some $\eta > 0$. We have*

$$\mathrm{Ent}\,[F] \leq \frac{1}{2}\,\mathbb{E}\left[\frac{1}{F}\int_0^\infty (2 - \mathbf{1}_{\{\phi_t = 0\}})|D_t F|^2 dt\right]. \tag{3.5.1}$$

Proof. Let us assume that F is bounded and $\mathcal{F}_T$-measurable, and let

$$X_t = \mathbb{E}[F \mid \mathcal{F}_t] = X_0 + \int_0^t u_s dM_s, \qquad t \in \mathbb{R}_+,$$

with $u_s = \mathbb{E}[D_s F \mid \mathcal{F}_s]$, $s \in \mathbb{R}_+$. The change of variable formula Proposition 2.12.1 applied to $f(x) = x \log x$ shows that, since $X_T = F$,

$$\begin{aligned} F \log F - \mathbb{E}[F] \log \mathbb{E}[F] &= f(X_T) - f(X_0) \\ &= \int_0^T \frac{f(X_{t^-} + \phi_t u_t) - f(X_{t^-})}{\phi_t} dM_t + \int_0^T i_t u_t f'(X_{t^-}) dM_t \\ &\quad + \int_0^T \frac{j_t}{\phi_t^2} \Psi(X_{t^-}, \phi_t u_t) dt + \frac{1}{2}\int_0^T i_t \frac{u_t^2}{X_t} dt, \end{aligned}$$

with the convention $0/0 = 0$, and

$$\Psi(u, v) = (u + v)\log(u + v) - u \log u - v(1 + \log u), \qquad u, u + v > 0.$$

Using the inequality

$$\Psi(u, v) \leq v^2/u, \qquad u > 0, \quad u + v > 0,$$

we obtain

$$\begin{aligned} \mathrm{Ent}\,[F] &= \mathbb{E}\left[\int_0^T \frac{j_t}{\phi_t^2}\Psi(X_t, \phi_t u_t) dt + \frac{1}{2}\int_0^T i_t \frac{u_t^2}{X_t} dt\right] \\ &\leq \frac{1}{2}\,\mathbb{E}\left[\int_0^T (2 - i_t)\frac{u_t^2}{X_t} dt\right] \qquad (3.5.2) \\ &\leq \frac{1}{2}\,\mathbb{E}\left[\int_0^T \mathbb{E}\left[(2 - i_t)\frac{|D_t F|^2}{F}\Big|\mathcal{F}_t\right] dt\right] \qquad (3.5.3) \end{aligned}$$

$$= \frac{1}{2} \int_0^T \mathbb{E}\left[\mathbb{E}\left[(2 - i_t) \frac{|D_t F|^2}{F} \Big| \mathcal{F}_t \right] \right] dt$$
$$= \frac{1}{2} \int_0^T \mathbb{E}\left[(2 - i_t) \frac{|D_t F|^2}{F} \right] dt$$
$$= \frac{1}{2} \mathbb{E}\left[\frac{1}{F} \int_0^T (2 - i_t) |D_t F|^2 dt \right].$$

where from (3.5.2) to (3.5.3) we applied Jensen's inequality (9.3.1) to the convex function $(u, v) \to v^2/u$ on $\mathbb{R} \times (0, \infty)$. Finally we apply the above to the approximating sequence $F_n = F \wedge n$, $n \in \mathbb{N}$, and let n go to infinity. $\square$

If $\phi_t = 0$, i.e. $i_t = 1$, $t \in \mathbb{R}_+$, then $(M_t)_{t \in \mathbb{R}_+}$ is a Brownian motion and we obtain the classical modified Sobolev inequality

$$\operatorname{Ent}[F] \leq \frac{1}{2} \mathbb{E}\left[\frac{1}{F} \|DF\|^2_{L^2([0,T])} \right]. \tag{3.5.4}$$

If $\phi_t = 1$, $t \in \mathbb{R}_+$ then $i_t = 0$, $t \in \mathbb{R}_+$, $(M_t)_{t \in \mathbb{R}_+}$ is a standard compensated Poisson process and we obtain the modified Sobolev inequality

$$\operatorname{Ent}[F] \leq \mathbb{E}\left[\frac{1}{F} \|DF\|^2_{L^2([0,T])} \right]. \tag{3.5.5}$$

More generally, the logarithmic Sobolev inequality (3.5.4) can be proved to hold as

$$\operatorname{Ent}[F^2] \leq 2\, \mathbb{E}\left[\|DF\|^2_{L^2([0,T])} \right]$$

for any gradient operator D satisfying both the derivation rule Assumption 3.5.2 below and the Clark formula Assumption 3.2.1, see Proposition 5.7.5 in Chapter 5.7 for Riemannian Brownian motion, and also (7.2.5) in Chapter 7 for another example on the Poisson space.

Assumption 3.5.2. *(Derivation rule) For all $F, G \in \mathcal{S}$ we have*

$$D_t(FG) = F D_t G + G D_t F, \qquad t \in \mathbb{R}_+. \tag{3.5.6}$$

Note that by polynomial approximation, Relation (3.5.6) extends as

$$D_t f(F) = f'(F) D_t F, \qquad t \in \mathbb{R}_+, \tag{3.5.7}$$

for $f \in \mathcal{C}_b^1(\mathbb{R})$.

3.6 Deviation Inequalities

In this section we assume that D is a gradient operator satisfying both the Clark formula Assumption 3.2.1 and the derivation rule Assumption 3.5.2. Examples of such operators will be provided in the Wiener and Poisson cases in Chapters 5 and 7.

Under the derivation rule Assumption 3.5.2 we get the following deviation bound.

Proposition 3.6.1. *Let* $F \in \mathrm{Dom}\,(D)$. *If* $\|DF\|_{L^2(\mathbb{R}_+,L^\infty(\Omega))} \leq C$ *for some* $C > 0$, *then*

$$\mathbb{P}(F - \mathbb{E}[F] \geq x) \leq \exp\left(-\frac{x^2}{2C\|DF\|_{L^\infty(\Omega,L^2(\mathbb{R}_+))}}\right), \qquad x \geq 0. \qquad (3.6.1)$$

In particular we have

$$\mathbb{E}[\mathrm{e}^{\lambda F^2}] < \infty, \qquad \lambda < \frac{1}{2C\|DF\|_{L^\infty(\Omega,L^2(\mathbb{R}_+))}}. \qquad (3.6.2)$$

Proof. We first consider a bounded random variable $F \in \mathrm{Dom}\,(D)$. The general case follows by approximating $F \in \mathrm{Dom}\,(D)$ by the sequence

$$(\max(-n, \min(F, n)))_{n\geq 1}.$$

Let

$$\eta_F(t) = \mathbb{E}[D_t F \mid \mathcal{F}_t], \qquad t \in [0, T].$$

Since F is bounded, the derivation rule (3.5.7) shows that

$$D_t \mathrm{e}^{sF} = s\mathrm{e}^{sF} D_t F, \qquad s, t \in \mathbb{R}_+,$$

hence assuming first that $\mathbb{E}[F] = 0$ and letting

$$L(s) = \mathbb{E}[\exp(sF)], \qquad s \in \mathbb{R}_+,$$

by Proposition 3.4.1 we get

$$\begin{aligned}
L'(s) &= \mathbb{E}[F\mathrm{e}^{sF}] \\
&= \mathbb{E}\left[\int_0^T D_u \mathrm{e}^{sF} \cdot \eta_F(u) du\right] \\
&= s\,\mathbb{E}\left[\mathrm{e}^{sF} \int_0^T D_u F \cdot \eta_F(u) du\right] \\
&\leq s\,\mathbb{E}\left[\mathrm{e}^{sF} \|DF\|_{L^2(\mathbb{R}_+)}\ \|\eta_F\|_{L^2(\mathbb{R}_+)}\right] \\
&\leq s\,\mathbb{E}\left[\mathrm{e}^{sF}\right] \|\eta_F\|_{L^\infty(\Omega,L^2(\mathbb{R}_+))} \|DF\|_{L^\infty(\Omega,L^2(\mathbb{R}_+))}
\end{aligned}$$

$$
\begin{aligned}
&\leq sC\,\mathbb{E}\left[e^{sF}\right]\|DF\|_{L^\infty(\Omega,L^2(\mathbb{R}_+))}\\
&\leq sCL(s)\|DF\|_{L^\infty(\Omega,L^2(\mathbb{R}_+))},
\end{aligned}
$$

where

$$
\begin{aligned}
\|\eta_F\|_{L^\infty(\Omega,L^2(\mathbb{R}_+))} &= \|\,\mathbb{E}[D_\cdot F \mid \mathcal{F}_\cdot]\|_{L^\infty(\Omega,L^2(\mathbb{R}_+))}\\
&= \left\|\int_0^\infty |\,\mathbb{E}[D_tF \mid \mathcal{F}_\cdot]|^2 dt\right\|_{L^\infty(\Omega)}\\
&\leq \left\|\int_0^\infty \mathbb{E}[|D_tF|^2 \mid \mathcal{F}_\cdot] dt\right\|_{L^\infty(\Omega)}\\
&\leq \int_0^\infty \left\||D_tF|^2 \mid \mathcal{F}_\cdot]\right\|_{L^\infty(\Omega)} dt\\
&\leq \|DF\|_{L^2(\mathbb{R}_+,L^\infty(\Omega))}\\
&\leq C.
\end{aligned}
$$

In the general case, letting

$$
L(s) = \mathbb{E}[\exp(s(F-\mathbb{E}[F]))], \qquad s \in \mathbb{R}_+,
$$

we obtain:

$$
\begin{aligned}
\log\left(\mathbb{E}\left[\exp\left(t(F-\mathbb{E}[F])\right)\right]\right) &= \int_0^t \frac{L'(s)}{L(s)} ds\\
&\leq \int_0^t \frac{\mathbb{E}\left[(F-\mathbb{E}[F])\exp\left(s(F-\mathbb{E}[F])\right)\right]}{\mathbb{E}\left[\exp\left(s(F-\mathbb{E}[F])\right)\right]} ds\\
&= \frac{1}{2}t^2C\|DF\|_{L^2(\mathbb{R}_+,L^\infty(\Omega))}, \qquad t \in \mathbb{R}_+.
\end{aligned}
$$

By Chebyshev's inequality we conclude that, for all $x \in \mathbb{R}_+$ and $t \in [0,T]$,

$$
\begin{aligned}
\mathbb{P}(F-\mathbb{E}[F] \geq x) &\leq e^{-tx}\,\mathbb{E}\left[\exp\left(t(F-\mathbb{E}[F])\right)\right]\\
&\leq \exp\left(\frac{1}{2}t^2C\|DF\|_{L^2(\mathbb{R}_+,L^\infty(\Omega))} - tx\right),
\end{aligned}
$$

which yields (3.6.1) after minimization in $t \in [0,T]$. The proof of (3.6.2) is completed as in Proposition 1.11.3. □

3.7 Markovian Representation

This subsection presents a predictable representation method that can be used to compute $\mathbb{E}[D_tF|\mathcal{F}_t]$, based on the Itô formula and the Markov property, cf. Section 9.6 in the appendix. It can applied to Delta hedging in mathematical finance, cf. Proposition 8.2.2 in Chapter 8, and [124]. Let $(X_t)_{t\in[0,T]}$ be a

$\mathbb{R}^n$-valued Markov (not necessarily time homogeneous) process defined on Ω, generating a filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ and satisfying a change of variable formula of the form

$$f(X_t) = f(X_0) + \int_0^t L_s f(X_s) dM_s + \int_0^t U_s f(X_s) ds, \quad t \in [0,T], \qquad (3.7.1)$$

where L_s, U_s are operators defined on $f \in \mathcal{C}^2(\mathbb{R}^n)$. Let the (non homogeneous) semi-group $(P_{s,t})_{0\le s\le t\le T}$ associated to $(X_t)_{t\in[0,T]}$ be defined on $\mathcal{C}_b^2(\mathbb{R}^n)$ functions by

$$\begin{aligned} P_{s,t} f(X_s) &= \mathbb{E}[f(X_t) \mid X_s] \\ &= \mathbb{E}[f(X_t) \mid \mathcal{F}_s], \quad 0 \le s \le t \le T, \end{aligned}$$

with

$$P_{s,t} \circ P_{t,u} = P_{s,u}, \qquad 0 \le s \le t \le u \le T.$$

Proposition 3.7.1. *For any $f \in \mathcal{C}_b^2(\mathbb{R}^n)$, the process $(P_{t,T} f(X_t))_{t\in[0,T]}$ is an $\mathcal{F}_t$-martingale.*

Proof. By the tower property of conditional expectations, cf. Section 9.3, we have

$$\begin{aligned} \mathbb{E}[P_{t,T} f(X_t) \mid \mathcal{F}_s] &= \mathbb{E}[\mathbb{E}[f(X_T) \mid \mathcal{F}_t] \mid \mathcal{F}_s] \\ &= \mathbb{E}[f(X_T) \mid \mathcal{F}_s] \\ &= P_{s,T} f(X_s), \end{aligned}$$

$0 \le s \le t \le T$. □

Next we use above the framework with application to the Clark formula. When $(\phi_t)_{t\in[0,T]}$ is random the probabilistic interpretation, of D is unknown in general, nevertheless it is possible to explicitly compute the predictable representation of $f(X_T)$ using (3.7.1) and the Markov property.

Lemma 3.7.2. *Let $f \in \mathcal{C}_b^2(\mathbb{R}^n)$. We have*

$$\mathbb{E}[D_t f(X_T) \mid \mathcal{F}_t] = (L_t(P_{t,T} f))(X_t), \quad t \in [0,T]. \qquad (3.7.2)$$

Proof. We apply the change of variable formula (3.7.1) to $t \longmapsto P_{t,T} f(X_t) = \mathbb{E}[f(X_T) \mid \mathcal{F}_t]$, since $P_{t,T} f$ is $\mathcal{C}^2$. Using the fact that the finite variation term vanishes since $(P_{t,T} f(X_t))_{t\in[0,T]}$ is a martingale, (see e.g. Corollary 1, p. 64 of [123]), we obtain:

$$P_{t,T} f(X_t) = P_{0,T} f(X_0) + \int_0^t (L_s(P_{s,T} f))(X_s) dM_s, \quad t \in [0,T],$$

with $P_{0,T} f(X_0) = \mathbb{E}[f(X_T)]$. Letting $t = T$, we obtain (3.7.2) by uniqueness of the representation (4.2.2) applied to $F = f(X_T)$. □

In practice we can use Proposition 3.2.6 to extend $(\mathbb{E}[D_t f(X_T) \mid \mathcal{F}_t])_{t\in[0,T]}$ to a less regular function $f : \mathbb{R}^n \to \mathbb{R}$.

As an example, if ϕ_t is written as $\phi_t = \varphi(t, M_t)$, and

$$dS_t = \sigma(t, S_t)dM_t + \mu(t, S_t)dt,$$

we can apply Proposition 2.12.2, with $(X_t)_{t\in[0,T]} = ((S_t, M_t))_{t\in[0,T]}$ and

$$\begin{aligned} L_t f(S_t, M_t) &= i_t\sigma(t, S_t)\partial_1 f(S_t, M_t) + i_t\partial_2 f(S_t, M_t) \\ &+ \frac{j_t}{\varphi(t, M_t)}(f(S_t + \varphi(t, M_t)\sigma(t, S_t), M_t + \varphi(t, M_t)) - f(S_t, M_t)), \end{aligned}$$

where $j_t = \mathbf{1}_{\{\phi_t\neq 0\}}$, $t \in \mathbb{R}_+$, since the eventual jump of $(M_t)_{t\in[0,T]}$ at time t is $\varphi(t, M_t)$. Here, ∂_1, resp. ∂_2, denotes the partial derivative with respect to the first, resp. second, variable. Hence

$$\begin{aligned} \mathbb{E}[D_t f(S_T, M_T) \mid \mathcal{F}_t] &= i_t\sigma(t, S_t)(\partial_1 P_{t,T} f)(S_t, M_t) + i_t(\partial_2 P_{t,T} f)(S_t, M_t) \\ &+ \frac{j_t}{\varphi(t, M_t)}(P_{t,T} f)(S_t + \varphi(t, M_t)\sigma(t, S_t), M_t + \varphi(t, M_t)) \\ &- \frac{j_t}{\varphi(t, M_t)}(P_{t,T} f)(S_t, M_t). \end{aligned}$$

When $(\phi_t)_{t\in\mathbb{R}_+}$ and $\sigma(t, x) = \sigma_t$, are deterministic functions of time and $\mu(t, x) = 0$, $t \in \mathbb{R}_+$, the semi-group $P_{t,T}$ can be explicitly computed as follows.

In this case, from (2.10.5), the martingale $(M_t)_{t\in\mathbb{R}_+}$ can be represented as

$$dM_t = i_t dB_t + \phi_t(dN_t - \lambda_t dt), \qquad t \in \mathbb{R}_+, \qquad M_0 = 0,$$

with $\lambda_t = j_t/\phi_t^2$, $t \in \mathbb{R}_+$, where $(N_t)_{t\in\mathbb{R}_+}$ is an independent Poisson process with intensity λ_t, $t \in \mathbb{R}_+$. Let

$$\Gamma_t(T) = \int_t^T \mathbf{1}_{\{\phi_s=0\}}\sigma_s^2 ds, \qquad 0 \le t \le T,$$

denote the variance of $\int_t^T i_s\sigma_s dB_s = \int_t^T \mathbf{1}_{\{\phi_s=0\}}\sigma_s dB_s$, $0 \le t \le T$, and let

$$\Gamma_t(T) = \int_t^T \lambda_s ds, \qquad 0 \le t \le T,$$

denote the intensity parameter of the Poisson random variable $N_T - N_t$.

Proposition 3.7.3. *We have for* $f \in \mathcal{C}_b(\mathbb{R})$

$$P_{t,T}f(x) = \frac{1}{\sqrt{2\pi}} \sum_{k=0}^{\infty} \frac{e^{-\Gamma_t(T)}}{k!} \int_{-\infty}^{\infty} e^{-t_0^2/2} \int_{[t,T]^k} \lambda_{t_1} \cdots \lambda_{t_k}$$
$$f\left(x e^{-\frac{\Gamma_t(T)}{2} + \sqrt{\Gamma_t(T)} t_0 - \int_t^T \phi_s \lambda_s \sigma_s ds} \prod_{i=1}^{k} (1 + \sigma_{t_i} \phi_{t_i}) \right) dt_1 \cdots dt_k dt_0.$$

Proof. We have $P_{t,T}f(x) = \mathbb{E}[f(S_T)|S_t = x] = \mathbb{E}[f(S_{t,T}^x)]$, and

$$P_{t,T}f(x) = \exp(-\Gamma_t(T)) \sum_{k=0}^{\infty} \frac{(\Gamma_t(T))^k}{k!} \mathbb{E}\left[f(S_{t,T}^x) \Big| N_T - N_t = k \right]$$

$k \in \mathbb{N}$. It can be shown (see e.g. Proposition 6.1.8 below) that the time changed process $\left(N_{\Gamma_t^{-1}(s)} - N_t\right)_{s \in \mathbb{R}_+}$ is a standard Poisson process with jump times $(\tilde{T}_k)_{k \geq 1} = (\Gamma_t(T_{k+N_t}))_{k \geq 1}$. Hence from Proposition 2.3.7, conditionally to $\{N_T - N_t = k\}$, the jump times $(\tilde{T}_1, \ldots, \tilde{T}_k)$ have the law

$$\frac{k!}{(T-t)^k} \mathbf{1}_{\{0 < t_1 < \cdots < t_k < T-t\}} dt_1 \cdots dt_k.$$

over $[0, T-t]^k$. Consequently, conditionally to $\{N_T - N_t = k\}$, the k first jump times $(T_1, \ldots, T_k)$ of $(N_s)_{s \in [t,T]}$ have the distribution

$$\frac{k!}{(\Gamma_t(T))^k} \mathbf{1}_{\{t < t_1 < \cdots < t_k < T\}} \lambda_{t_1} \cdots \lambda_{t_k} dt_1 \cdots dt_k.$$

We then use the identity in law between $S_{t,T}^x$ and

$$x X_{t,T} \exp\left(- \int_t^T \phi_s \lambda_s (1 + \phi_s \psi_s) \sigma_s ds \right) \prod_{k=1+N_t}^{N_T} (1 + \sigma_{T_k} \phi_{T_k}),$$

where $X_{t,T}$ has same distribution as

$$\exp\left(W \sqrt{\Gamma_t(T)} - \Gamma_t(T)/2 \right),$$

and W a standard Gaussian random variable, independent of $(N_t)_{t \in [0,T]}$, which holds because $(B_t)_{t \in [0,T]}$ is a standard Brownian motion, independent of $(N_t)_{t \in [0,T]}$. □

3.8 Notes and References

Several examples of gradient operators satisfying the hypotheses of this chapter will be provided in Chapters 4, 5, 6, and 7, on the Wiener and Poisson space and also on Riemannian path space. The Itô formula has been used for the proof of logarithmic Sobolev inequalities in [5], [7], [155] for the Poisson process, and in [24] on Riemannian path space, and Proposition 3.5.1 can be found in [115]. The probabilistic interpretations of D as a derivation operator and as a finite difference operator has been studied in [120] and will be presented in more detail in the sequel. The extension of the Clark formula presented in Proposition 3.2.6 is related to the approach of [93] of [146]. The covariance identity (3.4.1) can be found in Proposition 2.1 of [61]. See also [8] for a unified presentation of the Malliavin calculus based on the Fock space.

Chapter 4
Annihilation and creation operators

In this chapter we present a first example of a pair of gradient and divergence operators satisfying the duality Assumption 3.1.1, the Clark formula Assumption 3.2.1 and the stability Assumption 3.2.10 of Section 3.1. This construction is based on annihilation and creation operators acting on multiple stochastic integrals with respect to a normal martingale. In the following chapters we will implement several constructions of such operators, respectively when the normal martingale $(M_t)_{t\in\mathbb{R}_+}$ is a Brownian motion or a compensated Poisson process. Other examples of operators satisfying the above assumptions will be built in the sequel by addition of a process with vanishing adapted projection to the gradient D, such as in Section 7.7 on the Poisson space.

4.1 Duality Relation

The annihilation and creation operators on multiple stochastic integrals provide a first concrete example of operators D, δ satisfying the hypotheses of Chapter 2. Let the spaces $\mathcal{S}$ and $\mathcal{U}$ of Section 3.1 be taken equal to

$$\mathcal{S} = \left\{ \sum_{k=0}^{n} I_k(f_k) \ : \ f_k \in L^4(\mathbb{R}_+)^{\circ k}, \ k = 0, \ldots, n, \ n \in \mathbb{N} \right\}, \qquad (4.1.1)$$

and

$$\mathcal{U} = \left\{ \sum_{i=1}^{n} \mathbf{1}_{[t_i, t_{i-1})} F_i \ : \ F_i \in \mathcal{S}, \ 0 = t_0 \leq t_1 < \cdots < t_n, \ n \geq 1 \right\},$$

which is contained in

$$\tilde{\mathcal{U}} := \left\{ \sum_{k=0}^{n} I_k(g_k(*, \cdot)) \ : \ g_k \in L^2(\mathbb{R}_+)^{\circ k} \otimes L^2(\mathbb{R}_+), \ k = 0, \ldots, n, \ n \in \mathbb{N} \right\},$$

where the symmetric tensor product $\circ$ is defined in (9.7.1), cf. Section 9.7 of the Appendix.

In the following statements, the definition of the operators D and δ are stated on multiple stochastic integrals (random variables and processes), whose linear combinations span $\mathcal{S}$ and $\mathcal{U}$.

Definition 4.1.1. *Let*

$$D : \mathcal{S} \longrightarrow L^2(\Omega \times \mathbb{R}_+)$$

be the linear operator defined by

$$D_t I_n(f_n) = n I_{n-1}(f_n(*, t)), \qquad d\mathbb{P} \times dt - a.e., \quad f_n \in L^2(\mathbb{R}_+)^{\circ n}.$$

Due to its role as a lowering operator on the degree of multiple stochastic integrals, the operator D is called an *annihilation* operator in the sequel, in reference to the use of Fock space expansions (see Definition 2.8.1) in quantum field theory.

Definition 4.1.2. *Let*

$$\delta : \tilde{\mathcal{U}} \longrightarrow L^2(\Omega)$$

be the linear operator defined by

$$\delta(I_n(f_{n+1}(*, \cdot))) = I_{n+1}(\tilde{f}_{n+1}), \quad f_{n+1} \in L^2(\mathbb{R}_+)^{\circ n} \otimes L^2(\mathbb{R}_+), \tag{4.1.2}$$

where $\tilde{f}_{n+1}$ is the symmetrization of f_{n+1} in $n+1$ variables defined as:

$$\tilde{f}_{n+1}(t_1, \ldots, t_{n+1}) = \frac{1}{n+1} \sum_{k=1}^{n+1} f_{n+1}(t_1, \ldots, t_{k-1}, t_{k+1}, \ldots, t_{n+1}, t_k).$$

In particular we have

$$f \circ g_n(t_1, \ldots, t_{n+1}) = \frac{1}{n+1} \sum_{k=1}^{n+1} f(t_k) g_n(t_1, \ldots, t_{k-1}, t_{k+1}, \ldots, t_{n+1}), \tag{4.1.3}$$

i.e. $f \circ g_n$ is the symmetrization of $f \otimes g_n$ in $n+1$ variables, cf. Section 9.7.

Similarly to the above, the operator δ is usually referred to as a *creation* operator, due to the fact that it raises the degree of multiple stochastic integrals. The operator δ is also called the Skorohod integral.

Note that

$$\delta(f) = I_1(f) = \int_0^\infty f(t) dM_t, \qquad f \in L^2(\mathbb{R}_+),$$

and, in particular, from (2.7.4) we have

$$\delta(uI_n(f_n)) = n\int_0^\infty I_n(f_n(*,s)\circ u.\mathbf{1}_{[0,s]^n}(*,\cdot))dM_s + \int_0^\infty u_s I_n(f_n \mathbf{1}_{[0,s]^n})dM_s,$$

$u \in L^2(\mathbb{R}_+)$, $g_n \in L^2(\mathbb{R}_+)^{\circ n}$, where as a convention "$*$" denotes the $n-1$ first variables and "$\cdot$" denotes the last integration variable in I_n. In the next proposition we show that D and δ satisfy the duality Assumption 3.1.1.

Proposition 4.1.3. *The operators D and δ satisfy the duality relation*

$$\mathbb{E}[F\delta(u)] = \mathbb{E}[\langle DF, u\rangle_{L^2(\mathbb{R}_+)}], \qquad F \in \mathcal{S},\ u \in \mathcal{U}. \tag{4.1.4}$$

Proof. As in Proposition 1.8.2, we consider $F = I_n(f_n)$ and $u_t = I_m(g_{m+1}(*,t))$, $t \in \mathbb{R}_+$, $f_n \in L^2(\mathbb{R}_+)^{\circ n}$, $g_{m+1} \in L^2(\mathbb{R}_+)^{\circ m} \otimes L^2(\mathbb{R}_+)$. We have

$$\begin{aligned}
&\mathbb{E}[F\delta(u)] = \mathbb{E}[I_{m+1}(\tilde{g}_{m+1})I_n(f_n)]\\
&= n!\mathbf{1}_{\{n=m+1\}}\langle f_n, \tilde{g}_n\rangle_{L^2(\mathbb{R}_+^n)}\\
&= n!\mathbf{1}_{\{n=m+1\}}\langle f_n, g_n\rangle_{L^2(\mathbb{R}_+^n)}\\
&= n!\mathbf{1}_{\{n-1=m\}}\int_0^\infty\cdots\int_0^\infty f_n(s_1,\ldots,s_{n-1},t)g_n(s_1,\ldots,s_{n-1},t)ds_1\cdots ds_{n-1}dt\\
&= n\mathbf{1}_{\{n-1=m\}}\int_0^\infty \mathbb{E}[I_{n-1}(f_n(*,t))I_{n-1}(g_n(*,t))]dt\\
&= \mathbb{E}[\langle D.I_n(f_n), I_m(g_{m+1}(*,\cdot))\rangle_{L^2(\mathbb{R}_+)}]\\
&= \mathbb{E}[\langle DF, u\rangle_{L^2(\mathbb{R}_+)}].
\end{aligned}$$

□

In Proposition 4.2.3 below we will show that the Clark formula Assumption 3.2.1 is also satisfied by D.

Proposition 4.1.4. *For any $u \in \tilde{\mathcal{U}}$ we have*

$$D_t\delta(u) = u_t + \delta(D_t u), \qquad t \in \mathbb{R}_+.$$

Proof. Letting $u_t = f(t)I_n(g_n)$, $t \in \mathbb{R}_+$, $f \in L^2(\mathbb{R}_+)$, $g_n \in L^2(\mathbb{R}_+)^{\circ n}$, we have, by (4.1.3),

$$\begin{aligned}
D_t\delta(u) &= D_t\delta(fI_n(g_n))\\
&= D_t I_{n+1}(\widetilde{f\otimes g_n})\\
&= D_t I_{n+1}(f\circ g_n)\\
&= (n+1)I_{n+1}((f\circ g_n)(*,t))\\
&= f(t)I_n(g_n) + nI_n(f\circ g_n(*,t))\\
&= f(t)I_n(g_n) + n\delta(I_{n-1}(f\circ g_n(*,t)))\\
&= f(t)I_n(g_n) + \delta(D_t I_n(f\circ g_n(*,t)))\\
&= u_t + \delta(D_t u).
\end{aligned}$$

□

Remark 4.1.5. *By construction, the operator D satisfies the stability Assumption 3.2.10 of Chapter 2, thus the conclusion of Proposition 3.2.11 is valid for D, i.e. we have $D_s F = 0$, $s > t$, for any $\mathcal{F}_t$-measurable $F \in \mathcal{S}$, $t \in \mathbb{R}_+$.*

4.2 Annihilation Operator

From now on we will assume that $\mathcal{S}$ is dense in $L^2(\Omega)$, which is equivalent to saying that $(M_t)_{t\in\mathbb{R}_+}$ has the chaos representation property according to Definition 2.8.2. As a consequence of Proposition 3.1.2 and Proposition 4.1.3 we have the following.

Proposition 4.2.1. *The operators D and δ are closable in the sense of Section 9.8 on $L^2(\Omega)$ and $L^2(\Omega \times \mathbb{R}_+)$ respectively.*

It also follows from the density of $\mathcal{S}$ in $L^2(\Omega)$ that $\mathcal{U}$ is dense in $L^2(\Omega \times \mathbb{R}_+)$.

Proposition 4.2.2. *The domain* $\mathrm{Dom}\,(D) = \mathbb{D}([0,\infty))$ *of D consists in the space of square-integrable random variables with chaos expansion*

$$F = \mathbb{E}[F] + \sum_{k=1}^{\infty} I_k(f_k), \tag{4.2.1}$$

such that the series

$$\sum_{k=1}^{n} k I_{k-1}(f_k(*,\cdot))$$

converges in $L^2(\Omega \times \mathbb{R}_+)$ as n goes to infinity.

Given $F \in \mathrm{Dom}\,(D)$ with the expansion (4.2.1) we have

$$\mathbb{E}\left[\|DF\|^2_{L^2(\mathbb{R}_+)}\right] = \sum_{k=1}^{\infty} k k! \|f_k\|^2_{L^2(\mathbb{R}_+^k)} < \infty,$$

and

$$D_t F = f_1(t) + \sum_{k=1}^{\infty} k I_{k-1}(f_k(*,t)), \qquad dt d\mathbb{P} - a.e.,$$

by Definition 4.1.1. In particular, we have

$$D_t I_k(f_k) = k I_{k-1}(f_k(*,t)), \qquad dt d\mathbb{P} - a.e.$$

and the exponential vector $\xi(u)$, of (2.13.4) belongs to $\mathrm{Dom}\,(D)$ for all $u \in L^2(\mathbb{R}_+)$, with

$$D_s \xi_t(u) = \mathbf{1}_{[0,t]}(s) u(s) \xi_t(u), \qquad s,t \in [0,T].$$

The following Proposition 4.2.3 shows that the Clark formula Assumption 3.2.1 is satisfied by D. Its proof parallels the classical argument

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \alpha_n x^n \\ &= \alpha_0 + \sum_{n=1}^{\infty} n\alpha_n \int_0^x y^{n-1} dy \\ &= f(0) + \int_0^x f'(y) dy, \end{aligned}$$

described in the introduction for functions of one variable, using the identity

$$x^n = n \int_0^x y^{n-1} dy.$$

It shows in particular that the operator D defined in this chapter satisfies the Clark formula Assumption 3.2.1 on predictable representation.

Proposition 4.2.3. *Every $F \in \mathcal{S}$ can be represented as*

$$F = \mathbb{E}[F] + \int_0^\infty \mathbb{E}[D_t F \mid \mathcal{F}_t] dM_t. \tag{4.2.2}$$

Proof. By linearity, in order to prove the statement for $F \in \mathcal{S}$, it suffices to consider $F = I_n(f_n)$. By the definitions of $I_n(f_n)$ and $D_t I_n(f_n)$ and using Lemma 2.7.2 we have, since $\mathbb{E}[I_n(f_n)] = 0$,

$$\begin{aligned} I_n(f_n) &= n \int_0^\infty I_{n-1}(f_n(*, t) \mathbf{1}_{[0,t]^{n-1}}(*)) dM_t \\ &= n \int_0^\infty \mathbb{E}[I_{n-1}(f_n(*, t)) \mid \mathcal{F}_t] dM_t \\ &= \int_0^\infty \mathbb{E}[D_t I_n(f_n) \mid \mathcal{F}_t] dM_t. \end{aligned}$$

□

As in the abstract framework of Chapter 3, the Clark formula (4.2.2) extends to $\mathrm{Dom}\,(D)$ from the closability of D as in Proposition 3.2.3, and to $L^2(\Omega)$ by continuity of $F \longmapsto \mathbb{E}[D_. F \mid \mathcal{F}_.]$, cf. Proposition 3.2.6.

Since $\mathcal{S}$ defined by (4.1.1) is assumed to be dense in $L^2(\Omega)$, Corollary 3.2.8 and Proposition 4.2.3 show that $(M_t)_{t\in\mathbb{R}_+}$ has the chaos representation property as in Definition 2.8.2.

More generally, the following proposition follows from the fact that the denseness of $\mathcal{S}$ is equivalent to the chaos representation property.

Proposition 4.2.4. *If $(M_t)_{t\in\mathbb{R}_+}$ has the chaos representation property then it has the predictable representation property.*

By iteration, the Clark formula (4.2.2) yields the following proposition.

Proposition 4.2.5. *For all* $F \in \cap_{n\geq 1}\mathrm{Dom}\,(D^n)$ *we have*

$$F = \mathbb{E}[F] + \sum_{n=1}^{\infty} I_n(f_n),$$

where

$$f_n(t_1,\ldots,t_n) = \frac{1}{n!}\,\mathbb{E}[D_{t_1}\cdots D_{t_n}F],$$

$dt_1\cdots dt_n d\mathbb{P}$*-a.e.,* $n \geq 1$.

Proof. It suffices to note that

$$D_{t_1}\cdots D_{t_n}F = n! f_n(t_1,\ldots,t_n) + \sum_{k=n+1}^{\infty} \frac{k!}{(k-n)!} I_{k-n}(f_k(*,t_1,\ldots,t_n)),$$

and to use the fact that

$$\mathbb{E}[I_{k-n}(f_k(*,t_1,\ldots,t_n))] = 0, \qquad dt_1\cdots dt_n - a.e., \quad k > n \geq 1,$$

that follows from Proposition 2.7.1 or Lemma 2.7.2. □

The above result is analogous to the following expression of Taylor's formula:

$$f(x) = f(0) + \sum_{n=1}^{\infty} \frac{x^n}{n!}\frac{\partial^n f}{\partial x^n}(0),$$

with the following correspondence:

calculus on $\mathbb{R}$	stochastic analysis
$f(x)$	F
$f(0)$	$\mathbb{E}[F]$
$\frac{\partial^n}{\partial x^n}$	D^n
$\frac{\partial^n f}{\partial x^n}(0)$	$\mathbb{E}[D^n F]$

The gradient operator D can be extended to the multidimensional case using the vector-valued multiple stochastic integral (2.14.1).

Definition 4.2.6. *Let* $l \in \{1,\ldots,d\}$*. We define the operator*

$$D^{(l)} : \mathrm{Dom}\,(D^{(l)}) \subset L^2(\Omega) \to L^2(\Omega \times [0,T])$$

which maps any $F \in \mathrm{Dom}\,(D^{(l)})$ *with decomposition*

$$F = \sum_{n=0}^{\infty} I_n(f_n),$$

to the process $(D_t^{(l)} F)_{t\in[0,T]}$ *given by*

$$\begin{aligned} D_t^{(l)} F = & \sum_{n=1}^{\infty} \sum_{h=1}^{n} \sum_{i_1,\ldots,i_n=1}^{d} \mathbf{1}_{\{i_h=l\}} \\ & I_{n-1}(f_n^{i_1,\ldots,i_n}(t_1,\ldots,t_{l-1},t,t_{l+1}\ldots,t_n) e_{i_1} \otimes \ldots \otimes e_{i_{h-1}} \otimes e_{i_{h+1}} \ldots \otimes e_{i_n}) \\ = & \sum_{n=1}^{\infty} n I_{n-1}(f_n^l(*,t)), \qquad d\mathbb{P} \times dt - a.e. \end{aligned} \tag{4.2.3}$$

with

$$f_n^l = (f_n^{i_1,\ldots,i_{n-1},l} e_{i_1} \otimes \ldots \otimes e_{i_{n-1}})_{1\le i_1,\ldots,i_{n-1}\le d}.$$

The domain of $D^{(l)}$ is given by

$$\mathrm{Dom}\,(D^{(l)}) = \left\{ F = \sum_{n=0}^{\infty} \sum_{i_1,\ldots,i_n=1}^{d} I_n(f_n^{i_1,\ldots,i_n} e_{i_1} \otimes \ldots \otimes e_{i_n}) \ : \right.$$
$$\left. \sum_{n=1}^{\infty} n n! \sum_{i_1,\ldots,i_n=1}^{d} \|f_n^{i_1,\ldots,i_n}\|^2_{L^2([0,T]^n)} < \infty \right\}.$$

The Clark formula extends to the multidimensional setting of Section 2.14 as the next proposition.

Proposition 4.2.7. *Let* $F \in \bigcap_{l=1}^{d} \mathrm{Dom}\,(D^{(l)})$*. We have*

$$F = \mathbb{E}[F] + \sum_{l=1}^{d} \int_0^T \mathbb{E}[D_t^{(l)} F \mid \mathcal{F}_t] dM_t^{(l)}. \tag{4.2.4}$$

In the multidimensional Poisson case we define the operator $D^{N^{(l)}}$ as in (4.2.3) and we have the following Clark formula:

$$F = \mathbb{E}[F] + \sum_{l=1}^{d} \int_0^{\infty} |\lambda_t^{(l)}|^{-1/2}\, \mathbb{E}[D_t^{N^{(l)}} F \mid \mathcal{F}_t](dN_t^{(l)} - \lambda_t^{(l)} dt),$$

for $F \in \bigcap_{l=1}^{d} \mathrm{Dom}\,(D^{N^{(l)}})$. In the mixed Poisson-Brownian setting of Section 2.14, the operators $D^{X^{(l)}}$ are also defined as in (4.2.3) and we have the

Clark formula

$$F = \mathbb{E}[F] + \sum_{l=1}^{d} \int_0^T \mathbb{E}[D_t^{(l)} F \mid \mathcal{F}_t] dB_t^{(l)} + \sum_{l=1}^{p} \int_0^T \mathbb{E}[D_t^{N^{(l)}} F \mid \mathcal{F}_t] dM_t^{(l)},$$

for $F \in \bigcap_{l=1}^{d+p} \mathrm{Dom}\,(D^{X^{(l)}})$.

4.3 Creation Operator

The domain $\mathrm{Dom}\,(\delta)$ of δ is the space of processes $(u_t)_{t\in\mathbb{R}_+} \in L^2(\Omega \times \mathbb{R}_+)$ with

$$u_t = \sum_{n=0}^{\infty} I_n(f_{n+1}(*, t)),$$

and such that

$$\mathbb{E}[|\delta(u)|^2] = \sum_{n=1}^{\infty} (n+1)! \|\tilde{f}_n\|^2_{L^2(\mathbb{R}_+^{n+1})} < \infty.$$

We will sometimes use the notation

$$\int_a^b u_s \delta M_s := \delta(\mathbf{1}_{[a,b]} u), \tag{4.3.1}$$

to denote the Skorohod integral of $u \in \mathrm{Dom}\,(\delta)$ on the interval $[a, b]$, $0 \le a \le b \le \infty$. The creation operator δ satisfies the following Itô-Skorohod type isometry, also called an energy identity for the Skorohod integral.

Proposition 4.3.1. *Let $u \in \mathrm{Dom}\,(\delta)$ such that $u_t \in \mathrm{Dom}\,(D)$, dt-a.e., and $(D_s u_t)_{s,t\in\mathbb{R}_+} \in L^2(\Omega \times \mathbb{R}_+^2)$. We have*

$$\mathbb{E}[|\delta(u)|^2] = \mathbb{E}\left[\|u\|^2_{L^2(\mathbb{R}_+)}\right] + \mathbb{E}\left[\int_0^\infty \int_0^\infty D_s u_t D_t u_s ds dt\right], \tag{4.3.2}$$

Proof. By polarization, orthogonality and density it suffices to choose $u = g I_n(f^{\otimes n})$, $f, g \in L^2(\mathbb{R}_+)$, and to note that by the Definition 4.1.2 of δ we have

$$\begin{aligned}
&\mathbb{E}[|\delta(u)|^2] = \mathbb{E}[|\delta(g I_n(f^{\otimes n}))|^2] \\
&= \mathbb{E}[|I_{n+1}(f^{\otimes n} \circ g)|^2] \\
&= \frac{1}{(n+1)^2} \mathbb{E}\left[\left(\sum_{i=0}^{n} I_{n+1}(f^{\otimes i} \otimes g \otimes f^{\otimes (n-i)})\right)^2\right]
\end{aligned}$$

$$= \frac{1}{(n+1)^2}\Big((n+1)!(n+1)\|f\|^{2n}_{L^2(\mathbb{R}_+)}\|g\|^2_{L^2(\mathbb{R}_+)}$$
$$+n(n+1)(n+1)!\|f\|^{2n-2}_{L^2(\mathbb{R}_+)}\langle f,g\rangle^2_{L^2(\mathbb{R}_+)}\Big)$$
$$= n!\|f\|^{2n}_{L^2(\mathbb{R}_+)}\|g\|^2_{L^2(\mathbb{R}_+)} + (n-1)!n^2\|f\|^{2n-2}_{L^2(\mathbb{R}_+)}\langle f,g\rangle^2_{L^2(\mathbb{R}_+)}$$
$$= \mathbb{E}\left[\|u\|^2_{L^2(\mathbb{R}_+)}\right] + \langle\langle g, DI_n(f^{\otimes n})\rangle_{L^2(\mathbb{R}_+)}, \langle g, DI_n(f^{\otimes n})\rangle_{L^2(\mathbb{R}_+)}\rangle_{L^2(\Omega)}$$
$$= \mathbb{E}\left[\|u\|^2_{L^2(\mathbb{R}_+)}\right] + \mathbb{E}\left[\int_0^\infty\int_0^\infty D_s u_t D_t u_s ds dt\right].$$

□

By polarization, if u and v satisfy the conditions of Proposition 4.3.1 we also have

$$\langle\delta(u),\delta(v)\rangle_{L^2(\Omega)} = \langle u,v\rangle_{L^2(\Omega\times\mathbb{R}_+)} + \int_0^\infty\int_0^\infty \langle D_s u_t, D_t v_s\rangle_{L^2(\Omega)} ds dt.$$

The proof of (4.3.2) does not depend on the particular type of normal martingale we are considering, and it can be rewritten as a Weitzenböck type identity, cf. [137] and Section 7.6 for details, i.e.:

$$\|\delta(u)\|^2_{L^2(\Omega)} + \frac{1}{2}\int_0^\infty\int_0^\infty \|D_s u_t - D_t u_s\|^2_{L^2(\Omega)} ds dt = \|u\|^2_{L^2(\Omega\times\mathbb{R}_+)} + \|Du\|^2_{L^2(\Omega\times\mathbb{R}_+^2)}. \tag{4.3.3}$$

For Riemannian Brownian motion the study of identities such as (4.3.3) can be developed via intrinsic differential operators on Riemannian path space, cf. [29].

Definition 4.3.2. *Let $\mathbb{L}_{p,1}$ denote the space of random processes $(u_t)_{t\in\mathbb{R}_+}$ such that $u_t \in \mathrm{Dom}\,(D)$, dt-a.e., and*

$$\|u\|^p_{p,1} := \mathbb{E}\left[\|u\|^p_{L^2(\mathbb{R}_+)}\right] + \mathbb{E}\left[\int_0^\infty\int_0^\infty |D_s u_t|^p ds dt\right] < \infty.$$

The next result is a direct consequence of Proposition 4.3.1 and Definition 4.3.2 for $p = 2$.

Proposition 4.3.3. *We have $\mathbb{L}_{2,1} \subset \mathrm{Dom}\,(\delta)$.*

As a consequence of Proposition 3.3.1, Proposition 4.1.3 and Proposition 4.2.3, the operator δ coincides with the Itô integral with respect to $(M_t)_{t\in\mathbb{R}_+}$ on the square-integrable adapted processes, as stated in the next proposition.

Proposition 4.3.4. *Let $(u_t)_{t\in\mathbb{R}_+} \in L^2_{ad}(\Omega\times\mathbb{R}_+)$ be a square-integrable adapted process. We have*

$$\delta(u) = \int_0^\infty u_t dM_t.$$

Proof. This result can also be recovered from the definition (4.1.2) of δ via multiple stochastic integrals. Since the adaptedness of $(u_t)_{t\in\mathbb{R}_+} = (I_{n-1}(f_n(*,t)))_{t\in\mathbb{R}_+}$ implies

$$f_n(*,t) = f_n(*,t)\mathbf{1}_{[0,t]^{n-1}}(*), \qquad t \in \mathbb{R}_+,$$

by Lemma 2.7.2, we have

$$\begin{aligned}\delta(I_{n-1}(f_n(*,\cdot))) &= I_n(\tilde{f}_n)\\ &= n\int_0^\infty I_{n-1}(\tilde{f}_n(*,t)\mathbf{1}_{[0,t]^{n-1}}(*))dM_t\\ &= \int_0^\infty I_{n-1}(f_n(*,t)\mathbf{1}_{[0,t]^{n-1}}(*))dM_t\\ &= \int_0^\infty I_{n-1}(f_n(*,t))dM_t, \quad n \geq 1.\end{aligned}$$

□

Note that when $(u_t)_{t\in\mathbb{R}_+} \in L^2_{ad}(\Omega\times\mathbb{R}_+)$ is a square-integrable adapted process, then Relation (4.3.2) becomes the Itô isometry as a consequence of Proposition 4.3.4, i.e. we have

$$\begin{aligned}\|\delta(u)\|_{L^2(\Omega)} &= \left\|\int_0^\infty u_t dM_t\right\|_{L^2(\Omega)} \\ &= \|u\|_{L^2(\Omega\times\mathbb{R}_+)}, \qquad u \in L^2_{ad}(\Omega\times\mathbb{R}_+),\end{aligned} \tag{4.3.4}$$

as follows from Remark 4.1.5 since $D_t u_s = 0$, $0 \leq s \leq t$, cf. also Relation (3.3.2) of Proposition 3.3.1.

The following proposition is a Fubini type property for the exchange of Skorohod and Itô stochastic integrals with respect to normal martingales.

Lemma 4.3.5. *Let $u, v \in L^4_{ad}(\Omega\times\mathbb{R}_+)$. For all $t > 0$ we have*

$$\int_0^t u_s \int_s^t v_r dM_r \delta M_s = \int_0^t \int_0^r u_s v_r \delta M_s dM_r, \tag{4.3.5}$$

where the indefinite Skorohod integral is defined in (4.3.1).

Proof. First, note that

$$\int_0^r u_s v_r \delta M_s = \delta\left(u_\cdot \mathbf{1}_{\{\cdot<r\}} v_r\right)$$

is $\mathcal{F}_r$-measurable, $r \in \mathbb{R}_+$, hence the stochastic integral in the right hand side of (4.3.5) exists in the Itô sense by Proposition 2.5.4. On the other hand, by the duality relation (4.1.4) between D and δ and using the Itô-Skorohod isometry (4.3.4), we have

$$
\begin{aligned}
&\mathbb{E}\left[I_n(f^{\otimes n})\delta\left(\mathbf{1}_{[0,t]}(\cdot)u_\cdot\int_\cdot^t v_r dM_r\right)\right]\\
&= \mathbb{E}\left[\int_0^t u_s \int_s^t v_r dM_r D_s I_n(f^{\otimes(n-1)})ds\right]\\
&= n\int_0^t f(s)\,\mathbb{E}\left[\delta(f(\cdot)I_{n-2}(f^{\otimes(n-2)}))\delta(\mathbf{1}_{[s,t]}u_s v_\cdot)\right]ds\\
&= n\int_0^t f(s)\,\mathbb{E}\left[I_{n-2}(f^{\otimes(n-2)})u_s\int_s^t f(r)v_r dr\right]ds\\
&= n\,\mathbb{E}\left[I_{n-2}(f^{\otimes(n-2)})\int_0^t f(s)u_s\int_s^t f(r)v_r dr ds\right]\\
&= n\,\mathbb{E}\left[I_{n-2}(f^{\otimes(n-2)})\int_0^t f(r)v_r\int_0^r f(s)u_s ds dr\right]\\
&= n\,\mathbb{E}\left[\int_0^t \delta(u_\cdot\mathbf{1}_{\{\cdot<r\}}v_r)f(r)dr I_{n-1}(f^{\otimes(n-1)})\right]\\
&= \mathbb{E}\left[\int_0^t \delta(u_\cdot\mathbf{1}_{\{\cdot<r\}}v_r)D_r I_n(f^{\otimes n})dr\right]\\
&= \mathbb{E}\left[I_n(f^{\otimes n})\int_0^t \delta(u_\cdot\mathbf{1}_{\{\cdot<r\}}v_r)\delta M_r\right]\\
&= \mathbb{E}\left[I_n(f^{\otimes n})\int_0^t \delta(u_\cdot\mathbf{1}_{\{\cdot<r\}}v_r)dM_r\right],
\end{aligned}
$$

for $n \geq 1$ and $f \in L^2(\mathbb{R}_+)$, since the processes u and v are adapted. Hence by density of $\mathcal{S}$ in $L^2(\Omega)$ we get

$$
\delta\left(\mathbf{1}_{[0,t]}(\cdot)u_\cdot\int_\cdot^t v_r dM_r\right) = \int_0^t \delta\left(u_\cdot\mathbf{1}_{\{\cdot<r\}}v_r\right)dM_r,
$$

which implies (4.3.5) by (4.3.1). □

As a consequence of Proposition 2.11.3 we have the following divergence formula. The hypothesis of the next proposition is satisfied in particular when $\phi \in L^\infty_{ad}([0,T]\times\Omega)$.

Proposition 4.3.6. *Suppose that Assumption 2.11.1 holds, i.e.*

$$
\mathbb{E}\left[\int_a^b \phi_s^2 ds\Big|\mathcal{F}_a\right] \leq K^2(b-a), \qquad \mathbb{P}-a.s., \quad 0\leq a\leq b.
$$

Then for all $T>0$ and $u\in L^\infty([0,T])$ we have

$$
I_1(u)F = \delta(uF) + \langle u, DF\rangle_{L^2(\mathbb{R}_+)} + \delta(u\phi DF), \qquad F\in\mathcal{S}. \tag{4.3.6}
$$

Proof. We prove this result by induction for all $F = I_n(\mathbf{1}_{[0,t]^n}v^{\otimes n})$, $n\in\mathbb{N}$. The formula clearly holds for $n=0$. From Corollary 2.11.4 we have

$$
\begin{aligned}
& I_1(u\mathbf{1}_{[0,t]})I_n(\mathbf{1}_{[0,t]^n}v^{\otimes n}) \\
& = \int_0^t u_s I_n(\mathbf{1}_{[0,s]^n}v^{\otimes n})dM_s + n\int_0^t v_s I_{n-1}(\mathbf{1}_{[0,s]^{n-1}}v^{\otimes(n-1)})\int_0^s u_\tau dM_\tau dM_s \\
& \quad +n\int_0^t \phi_s u_s v_s I_{n-1}(\mathbf{1}_{[0,s]^{n-1}}v^{\otimes(n-1)})dM_s \\
& \quad +n\int_0^t u_s v_s I_{n-1}(\mathbf{1}_{[0,s]^{n-1}}v^{\otimes(n-1)})ds.
\end{aligned}
$$

Applying the induction hypothesis and Relation (4.1.2) we get

$$
\begin{aligned}
& I_{n-1}(\mathbf{1}_{[0,s]^{n-1}}v^{\otimes(n-1)})\int_0^s u_\tau dM_\tau = \delta(\mathbf{1}_{[0,s]}uI_{n-1}(\mathbf{1}_{[0,s]^{n-1}}v^{\otimes(n-1)})) \\
& \quad +(n-1)\int_0^s u_\tau v_\tau d\tau I_{n-2}(\mathbf{1}_{[0,s]^{n-2}}v^{\otimes(n-2)}) \\
& \quad +(n-1)\delta(\mathbf{1}_{[0,t]}\phi uvI_{n-2}(\mathbf{1}_{[0,s]^{n-2}}v^{\otimes(n-2)})) \\
& = I_n(\mathbf{1}_{[0,s]^n}v^{\otimes(n-1)}\circ u) + (n-1)\int_0^s u_\tau v_\tau d\tau I_{n-2}(\mathbf{1}_{[0,s]^{n-2}}v^{\otimes(n-2)}) \\
& \quad +(n-1)\delta(\mathbf{1}_{[0,s]}\phi uvI_{n-2}(\mathbf{1}_{[0,s]^{n-1}}v^{\otimes(n-2)})),
\end{aligned}
$$

hence

$$
\begin{aligned}
& I_1(u\mathbf{1}_{[0,t]})I_n(\mathbf{1}_{[0,t]^n}v^{\otimes n}) = \int_0^t u_s I_n(\mathbf{1}_{[0,s]^n}v^{\otimes n})dM_s \\
& \quad +n(n-1)\int_0^t v_s\int_0^s u_\tau v_\tau d\tau I_{n-2}(\mathbf{1}_{[0,s]^{n-2}}v^{\otimes(n-2)})dM_s \\
& \quad +n\int_0^t v_s I_n(\mathbf{1}_{[0,s]^n}v^{\otimes(n-1)}\circ u)dM_s \\
& \quad +n(n-1)\int_0^t v_s\delta(\mathbf{1}_{[0,s]}\phi uvI_{n-2}(\mathbf{1}_{[0,s]^{n-1}}v^{\otimes(n-2)}))dM_s \\
& \quad +n\int_0^t \phi_s u_s v_s I_{n-1}(\mathbf{1}_{[0,s]^{n-1}}v^{\otimes(n-1)})dM_s \\
& \quad +n\int_0^t u_s v_s I_{n-1}(\mathbf{1}_{[0,s]^{n-1}}v^{\otimes(n-1)})ds \\
& = I_{n+1}(\mathbf{1}_{[0,t]^{n+1}}v^{\otimes n}\circ u) \\
& \quad +n(n-1)\int_0^t v_s\delta(\mathbf{1}_{[0,s]}\phi uvI_{n-2}(\mathbf{1}_{[0,s]^{n-1}}v^{\otimes(n-2)}))dM_s \\
& \quad +n\int_0^t \phi_s u_s v_s I_{n-1}(\mathbf{1}_{[0,s]^{n-1}}v^{\otimes(n-1)})dM_s \\
& \quad +n(n-1)\int_0^t v_s\int_0^s \phi_\tau v_\tau d\tau I_{n-2}(\mathbf{1}_{[0,s]^{n-2}}v^{\otimes(n-2)})dM_s \\
& \quad +n\int_0^t u_s v_s I_{n-1}(\mathbf{1}_{[0,s]^{n-1}}v^{\otimes(n-1)})ds \\
& = I_{n+1}(\mathbf{1}_{[0,t]^{n+1}}v^{\otimes n}\circ u) \\
& \quad +n(n-1)\int_0^t v_s\delta(\mathbf{1}_{[0,s]}\phi uvI_{n-2}(\mathbf{1}_{[0,s]^{n-1}}v^{\otimes(n-2)}))dM_s \\
& \quad +n\int_0^t \phi_s u_s v_s I_{n-1}(\mathbf{1}_{[0,s]^{n-1}}v^{\otimes(n-1)})dM_s
\end{aligned}
$$

$$
\begin{aligned}
&+n \int_0^t u_s v_s ds I_{n-1}(\mathbf{1}_{[0,t]^{n-1}} v^{\otimes(n-1)}) ds \\
&= \delta(u \mathbf{1}_{[0,t]} I_n(\mathbf{1}_{[0,t]^n} v^{\otimes n})) + \delta(u\phi \mathbf{1}_{[0,t]} D I_n(\mathbf{1}_{[0,t]^n} v^{\otimes n})) \\
&+\langle u \mathbf{1}_{[0,t]}, D I_n(\mathbf{1}_{[0,t]^n} v^{\otimes n})\rangle_{L^2(\mathbb{R}_+)},
\end{aligned}
$$

where in the final equality we used the relations

$$
\begin{aligned}
I_{n-1}(\mathbf{1}_{[0,t]^{n-1}} v^{\otimes(n-1)}) \int_0^t u_s v_s ds &= \int_0^t \phi_s u_s v_s I_{n-1}(\mathbf{1}_{[0,s]^{n-1}} v^{\otimes(n-1)}) dM_s \\
&+(n-1) \int_0^t v_s \int_0^s u_\tau v_\tau d\tau I_{n-2}(\mathbf{1}_{[0,s]^{n-2}} v^{\otimes(n-2)}) dM_s,
\end{aligned}
$$

cf. (2.11.7), and

$$
\begin{aligned}
&\delta(\phi u v \mathbf{1}_{[0,t]} I_{n-1}(\mathbf{1}_{[0,t]^{n-1}} v^{\otimes(n-1)})) \\
&= (n-1)\delta\left(\phi u v \mathbf{1}_{[0,t]} \int_0^t v_s I_{n-2}(\mathbf{1}_{[0,s]^{n-2}} v^{\otimes(n-2)}) dM_s\right) \\
&= (n-1)\delta\left(\phi.u.v.\mathbf{1}_{[0,t]}(\cdot) \int_\cdot^t v_s I_{n-2}(\mathbf{1}_{[0,s]^{n-2}} v^{\otimes(n-2)}) dM_s\right) \\
&\quad +(n-1)\delta\left(\phi.u.v.\mathbf{1}_{[0,t]}(\cdot) \int_0^\cdot v_s I_{n-2}(\mathbf{1}_{[0,s]^{n-2}} v^{\otimes(n-2)}) dM_s\right) \\
&= (n-1) \int_0^t \phi.u.v.\mathbf{1}_{[0,t]}(\cdot) \int_\cdot^t v_r I_{n-2}(\mathbf{1}_{[0,r]^{n-2}} v^{\otimes(n-2)}) dM_r \delta M_s \\
&\quad +\delta\left(\phi.u.v.\mathbf{1}_{[0,t]}(\cdot) I_{n-1}(\mathbf{1}_{[0,\cdot]^{n-1}} v^{\otimes(n-1)})\right) \\
&= (n-1) \int_0^t v_s \delta(\mathbf{1}_{[0,s]}\phi.u.v.I_{n-2}(\mathbf{1}_{[0,s]^{n-2}} v^{\otimes(n-2)})) dM_s \\
&\quad + \int_0^t \phi_s u_s v_s I_{n-1}(\mathbf{1}_{[0,s]^{n-1}} v^{\otimes(n-1)}) dM_s,
\end{aligned}
$$

that follows from Lemma 4.3.5. □

4.4 Ornstein-Uhlenbeck Semi-Group

As in the discrete case, a covariance identity can be obtained from the Clark formula in Section 3.4. In this section we focus on covariance identities obtained from the Ornstein-Uhlenbeck (O.-U.) semi-group $(P_t)_{t\in\mathbb{R}_+}$ defined as

$$
P_t F = \sum_{n=0}^{\infty} \mathrm{e}^{-nt} I_n(f_n), \tag{4.4.1}
$$

with $F = \sum_{n=0}^{\infty} I_n(f_n)$, i.e. $P_t = \mathrm{e}^{-tL}$ with $L = \delta D$.

Proposition 4.4.1. *Let $F, G \in \mathrm{Dom}\,(D)$. We have the covariance identity*

$$\text{Cov}(F,G) = \mathbb{E}\left[\int_0^\infty \int_0^\infty e^{-s} D_u F P_s D_u G du ds\right]. \tag{4.4.2}$$

Proof. It suffices to prove this identity for $F = I_n(f_n)$ and $G = I_n(g_n)$ as

$$\begin{aligned}
\text{Cov}(F,G) &= \mathbb{E}[I_n(f_n)I_n(g_n)] \\
&= n!\langle f_n, g_n\rangle_{L^2(\mathbb{R}_+^n)} \\
&= \frac{1}{n}\,\mathbb{E}\left[\int_0^\infty D_u F D_u G du\right] \\
&= \mathbb{E}\left[\int_0^\infty \int_0^\infty D_u F e^{-ns} D_u G du ds\right] \\
&= \mathbb{E}\left[\int_0^\infty e^{-s} \int_0^\infty D_u F P_s D_u G du ds\right].
\end{aligned}$$

□

Let L^{-1} denote the inverse of the number operator $L = \delta D$, cf. Definition 2.8.3, defined on

$$\{F \in L^2(\Omega) \ : \ \mathbb{E}[F] = 0\}$$

as

$$L^{-1}F = \sum_{n=1}^\infty \frac{1}{n} I_n(f_n)$$

provided F is written as

$$F = \sum_{n=1}^\infty I_n(f_n).$$

Note that using the identity

$$L^{-1} = \int_0^\infty e^{-tL} dt = \int_0^\infty P_t dt,$$

and the commutation relation $DP_t = e^{-t} P_t D$, Relation (4.4.2) can also be obtained from a general semi-group argument:

$$\begin{aligned}
\text{Cov}(F,G) &= \mathbb{E}[LL^{-1}(F - \mathbb{E}[F])G] \\
&= \mathbb{E}[\langle DL^{-1}(F - \mathbb{E}[F]), DG\rangle_{L^2(X,\sigma)}] \\
&= \mathbb{E}\left[\int_0^\infty \langle DP_t(F - \mathbb{E}[F]), DG\rangle_{L^2(X,\sigma)} dt\right] \\
&= \mathbb{E}\left[\int_0^\infty e^{-t}\langle P_t D(F - \mathbb{E}[F]), DG\rangle_{L^2(X,\sigma)} dt\right] \\
&= \mathbb{E}\left[\int_0^\infty e^{-t}\langle P_t DF, DG\rangle_{L^2(X,\sigma)} dt\right].
\end{aligned}$$

Relation (4.4.2) implies the covariance inequality

$$|\text{Cov}(F,G)| \leq \left|\mathbb{E}\left[\|DF\|_{L^2(\mathbb{R}_+)} \int_0^\infty e^{-s} \|P_s DG\|_{L^2(\mathbb{R}_+)} ds\right]\right|$$

$$\leq \|DG\|_{L^\infty(\Omega,L^2(\mathbb{R}_+))} \mathbb{E}\left[\|DF\|_{L^2(\mathbb{R}_+)}\right]$$
$$\leq \|DG\|_{L^\infty(\Omega,L^2(\mathbb{R}_+))} \|DF\|_{L^\infty(\Omega,L^2(\mathbb{R}_+))},$$

$F, G \in \mathrm{Dom}(D)$, provided P_s satisfies the following continuity property.

Assumption 4.4.2. *(Continuity property) For all* $F \in \mathrm{Dom}(D)$ *we have*

$$\|P_t DF\|_{L^\infty(\Omega,L^2(\mathbb{R}_+))} \leq \|DF\|_{L^\infty(\Omega,L^2(\mathbb{R}_+))}, \qquad t \in \mathbb{R}_+. \tag{4.4.3}$$

This property is satisfied in particular when $(\phi_t)_{t\in\mathbb{R}_+}$ is deterministic, cf. Section 4.7.

4.5 Deterministic Structure Equations

When the process $(\phi_t)_{t\in\mathbb{R}_+}$ is a deterministic function, Corollary 2.11.4 or Proposition 4.3.6 can be rewritten as a multiplication formula for multiple stochastic integrals, without requiring a smoothness condition on $(\phi_t)_{t\in\mathbb{R}_+}$.

Proposition 4.5.1. *Assume that* $\phi \in L^\infty(\mathbb{R}_+)$ *is a bounded, deterministic, function. Then we have*

$$I_1(u)I_n(v^{\otimes n}) = I_{n+1}(v^{\otimes n} \circ u) + nI_n((\phi uv) \circ v^{\otimes(n-1)}) \\ +n\langle u, v\rangle_{L^2(\mathbb{R}_+)} I_{n-1}(v^{\otimes(n-1)}), \tag{4.5.1}$$

for all $u \in L^\infty(\mathbb{R}_+) \cap L^2(\mathbb{R}_+)$, $v \in L^2(\mathbb{R}_+)$.

From the above proposition we obtain in particular that for every $n \geq 1$ there exists a polynomial $Q_n(x)$ such that

$$I_n(v^{\otimes n}) = Q_n(I_1(v)), \qquad n \geq 1, \tag{4.5.2}$$

see [120] for details when $(\phi_s)_{s\in\mathbb{R}_+}$ is a random process. As seen in Chapters 5 and 6, the Hermite and Charlier polynomials are respectively used to represent multiple stochastic integrals with respect to Brownian motion and the compensated Poisson process.

On the other hand, if $s_1 < \cdots < s_n$ and $n = n_1 + \cdots + n_d$, we have

$$\mathbf{1}_{[t_0,t_1]}^{\otimes n_1} \circ \cdots \circ \mathbf{1}_{[t_{d-1},t_d]}^{\otimes n_d}(s_1, \ldots, s_n) = \frac{n_1! \cdots n_d!}{n!} \mathbf{1}_{[t_0,t_1]^{n_1}\times\cdots\times[t_{d-1},t_d]^{n_d}}(s_1, \ldots, s_n),$$

hence if $0 \leq t_0 < \cdots < t_d$,

$$I_n(\mathbf{1}_{[t_0,t_1]}^{\otimes n_1} \circ \cdots \circ \mathbf{1}_{[t_{d-1},t_d]}^{\otimes n_d})$$
$$= n! \int_0^\infty \int_0^{s_n} \cdots \int_0^{s_2} \mathbf{1}_{[t_0,t_1]}^{\otimes n_1} \circ \cdots \circ \mathbf{1}_{[t_{d-1},t_d]}^{\otimes n_d}(s_1, \ldots, s_n) dM_{s_1} \cdots dM_{s_n}$$

$$
\begin{aligned}
&= n_1! \cdots n_d! \int_0^\infty \int_0^{s_n} \cdots \int_0^{s_2} \mathbf{1}_{[t_0,t_1]^{n_1}\times\cdots\times[t_{d-1},t_d]^{n_d}}(s_1,\ldots,s_n) dM_{s_1}\cdots dM_{s_n} \\
&= \prod_{k=1}^d \left(n_k! \int_0^\infty \int_0^{s_{n_k}} \cdots \int_0^{s_2} \mathbf{1}^{\otimes n_k}_{[t_{k-1},t_k]}(s_1,\ldots,s_{n_k}) dM_{s_1}\cdots dM_{s_{n_k}} \right) \\
&= \prod_{k=1}^d I_{n_k}(\mathbf{1}^{\otimes n_k}_{[t_{k-1},t_k]}).
\end{aligned} \tag{4.5.3}
$$

The following is a product rule for the operator D.

Proposition 4.5.2. *Assume that $\phi \in L^\infty(\mathbb{R}_+)$ is a bounded, deterministic function. We have*

$$
D_t(FG) = F D_t G + G D_t F + \phi_t D_t F D_t G, \tag{4.5.4}
$$

$t \in \mathbb{R}_+$, $F, G \in \mathcal{S}$.

Proof. We first notice that for $F = I_1(u)$ and $G = I_n(f_n)$, this formula is a consequence of the multiplication formula Proposition 4.5.1 since

$$
\begin{aligned}
&D_t(I_1(u)I_n(f_n)) \\
&= D_t \left(I_{n+1}(f_n \circ u) + n \int_0^\infty u_s I_{n-1}(f_n(\cdot,s) ds + n I_n(f_n \circ (\phi u)) \right) \\
&= I_n(f_n) D_t I_1(u) + n I_n(f_n(\cdot,t) \circ u) + n(n-1) \int_0^\infty u_s I_{n-2}(f_n(\cdot,t,s)) ds \\
&\quad + n(n-1) I_n(f_n(\cdot,t) \circ (\phi u)) + \phi_t D_t I_1(u) D_t I_n(f_n) \\
&= I_n(f_n) D_t I_1(u) + I_1(u) D_t I_n(f_n) + \phi_t D_t I_1(u) D_t I_n(f_n), \quad t \in \mathbb{R}_+.
\end{aligned}
$$

Next, we prove by induction on $k \in \mathbb{N}$ that

$$
\begin{aligned}
D_t(I_n(f_n)(I_1(u))^k) = (I_1(u))^k D_t I_n(f_n) + I_n(f_n) D_t (I_1(u))^k \\
+ \phi_t D_t (I_1(u))^k D_t I_n(f_n),
\end{aligned}
$$

for all $n \in \mathbb{N}$. Clearly this formula holds for $k = 0$. From Proposition 4.5.1 we have

$$
I_n(f_n) I_1(u) \in \mathcal{H}_{n-1} \oplus \mathcal{H}_n \oplus \mathcal{H}_{n+1}, \qquad n \geq 1,
$$

hence by the induction hypothesis applied at the rank k we have

$$
\begin{aligned}
&D_t(I_n(f_n)(I_1(u))^{k+1}) \\
&= (I_1(u))^k D_t(I_n(f_n) I_1(u)) + I_n(f_n) I_1(u) D_t(I_1(u))^k \\
&\quad + \phi_t D_t (I_1(u))^k D_t(I_n(f_n) I_1(u)) \\
&= (I_1(u))^{k+1} D_t I_n(f_n) + I_n(f_n) I_1(u) D_t (I_1(u))^k + I_n(f_n)(I_1(u))^k D_t I_1(u) \\
&\quad + \phi_t I_n(f_n) D_t I_1(u) D_t (I_1(u))^k + \phi_t I_1(u) D_t (I_1(u))^k D_t I_n(f_n) \\
&\quad + \phi_t (I_1(u))^k D_t I_1(u) D_t I_n(f_n) + \phi_t^2 D_t I_1(u) D_t (I_1(u))^k D_t I_n(f_n) \\
&= (I_1(u))^{k+1} D_t I_n(f_n) + I_n(f_n) D_t (I_1(u))^{k+1} + \phi_t D_t (I_1(u))^{k+1} D_t I_n(f_n).
\end{aligned}
$$

Consequently, (4.5.4) holds for any polynomial in single stochastic integrals, hence from Relation (4.5.2) it holds for any F and G of the form $F = I_n(u^{\otimes n})$, $G = I_n(v^{\otimes n})$. The extension of $F, G \in \mathcal{S}$ is obtained by an approximation argument in $L^2(\Omega)$ from Proposition (2.11.2). □

In case $\phi_t = 0$, $t \in \mathbb{R}_+$, in order for the product relation (4.5.4) of Proposition 4.5.2 to be satisfied it suffices that D_t be a derivation operator. On the other hand, if $\phi_t \neq 0$, $t \in \mathbb{R}_+$, Relation (4.5.4) is satisfied by any finite difference operator of the form

$$F \longmapsto \frac{1}{\phi_t}(F_t^\phi - F).$$

By induction on $r \geq 1$ we obtain the following generalization of Relation (4.5.4).

Corollary 4.5.3. *For all $F, G \in \mathcal{S}$ we have*

$$D_{t_1} \cdots D_{t_r}(FG) = \sum_{p=0}^{r} \sum_{q=r-p}^{r} \sum_{\{k_1<\cdots<k_p\}\cup\{l_1<\cdots<l_q\}=\{1,\ldots,r\}} D_{t_{k_1}} \cdots D_{t_{k_p}} F D_{t_{l_1}} \cdots D_{t_{l_q}} G \prod_{i\in\{k_1,\ldots,k_p\}\cap\{l_1,\ldots,l_q\}} \phi(t_i), \tag{4.5.5}$$

$t_1, \ldots t_r \in \mathbb{R}_+$.

From Proposition 4.5.2, Proposition 4.3.6 can be extended to random $u \in \mathcal{U}$ as in the next result.

Proposition 4.5.4. *Let $T \in \mathbb{R}_+$ and assume that $\phi \in L^\infty([0,T])$ is a locally bounded deterministic function. Then for all $u \in \mathcal{U}$ and $F \in \mathcal{S}$ we have*

$$\delta(u)F = \delta(uF) + \langle DF, u\rangle_{L^2(\mathbb{R}_+)} + \delta(\phi u DF). \tag{4.5.6}$$

Proof. The proof of this statement follows by duality from Proposition 4.5.2. Letting $u = vG$ we have for $F, G_1, G_2 \in \mathcal{S}$:

$$\begin{aligned}
\mathbb{E}[FG_1\delta(u)] &= \mathbb{E}[G_2\langle v, D(FG_1)\rangle_{L^2(\mathbb{R}_+)}] \\
&= \mathbb{E}[G_2F\langle v, DG_1\rangle_{L^2(\mathbb{R}_+)}] + \mathbb{E}[G_2G_1\langle v, DF\rangle_{L^2(\mathbb{R}_+)}] \\
&\quad + \mathbb{E}[G_2\langle v, \phi DFDG_1\rangle_{L^2(\mathbb{R}_+)}] \\
&= \mathbb{E}[G_1\delta(uF)] + \mathbb{E}[G_1\langle vG_2, DF\rangle_{L^2(\mathbb{R}_+)}] + \mathbb{E}[G_1\delta(G_2v\phi DF)].
\end{aligned}$$

□

If $(\phi_t)_{t\in\mathbb{R}_+}$ is random the probabilistic interpretation of the gradient operator D is unknown, however we have the following conditional product rule.

Proposition 4.5.5. *For $F, G \in \mathcal{S}$ we have*

$$\mathbb{E}[D_t(FG) \mid \mathcal{F}_t] = \mathbb{E}[FD_tG \mid \mathcal{F}_t] + \mathbb{E}[GD_tF \mid \mathcal{F}_t] + \phi_t\, \mathbb{E}[D_tFD_tG \mid \mathcal{F}_t],$$

$F, G \in \mathcal{S}$, $t \in \mathbb{R}_+$.

Proof. We write (4.5.6) for $u \in \mathcal{U}$ adapted and apply the duality between D and δ:

$$\begin{aligned}
&\mathbb{E}[\langle u, D(FG)\rangle] = \mathbb{E}[\delta(u)FG] \\
&= \mathbb{E}[G(\delta(uF) + \langle u, DF\rangle_{L^2(\mathbb{R}_+)} + \delta(u\phi DF)] \\
&= \mathbb{E}[\langle u, FDG\rangle_{L^2(\mathbb{R}_+)} + \langle u, GDF\rangle_{L^2(\mathbb{R}_+)} + \langle u, \phi DFDG\rangle_{L^2(\mathbb{R}_+)}].
\end{aligned}$$

□

With help of (4.5.4) and Proposition 4.2.5, the following multiplication formula can been proved as a generalization of (4.5.1), cf. [111]. For $f_n \in L^2(\mathbb{R}_+)^{\circ n}$ and $g_m \in L^2(\mathbb{R}_+)^{\circ m}$, we define $f_n \otimes_k^l g_m$, $0 \leq l \leq k$, to be the function

$$\begin{aligned}
&(x_{l+1}, \ldots, x_n, y_{k+1}, \ldots, y_m) \longmapsto \\
&\phi(x_{l+1}) \cdots \phi(x_k) \int_{\mathbb{R}_+^l} f_n(x_1, \ldots, x_n) g_m(x_1, \ldots, x_k, y_{k+1}, \ldots, y_m) dx_1 \cdots dx_l
\end{aligned}$$

of $n + m - k - l$ variables. We denote by $f_n \circ_k^l g_m$ the symmetrization in $n + m - k - l$ variables of $f_n \otimes_k^l g_m$, $0 \leq l \leq k$.

Proposition 4.5.6. *We have the chaos expansion*

$$I_n(f_n) I_m(g_m) = \sum_{s=0}^{2(n \wedge m)} I_{n+m-s}(h_{n,m,s}), \tag{4.5.7}$$

if and only if the functions

$$h_{n,m,s} = \sum_{s \leq 2i \leq 2(s \wedge n \wedge m)} i! \binom{n}{i} \binom{m}{i} \binom{i}{s-i} f_n \circ_i^{s-i} g_m$$

belong to $L^2(\mathbb{R}_+)^{\circ n+m-s}$, $0 \leq s \leq 2(n \wedge m)$.

Proof. From Corollary 4.5.3 we have

$$\begin{aligned}
D_{t_1} \cdots D_{t_r}(I_n(f_n) I_m(g_m)) = \sum_{p=0}^{r} \sum_{q=r-p}^{r} & \sum_{\{k_1 < \cdots < k_p\} \cup \{l_1 < \cdots < l_q\} = \{1, \ldots, r\}} \\
& \frac{n!}{(n-p)!} \frac{m!}{(m-q)!} I_{n-p}(f_n(\cdot, t_{k_1}, \ldots, t_{k_p})) I_{m-q}(g_m(\cdot, t_{l_1}, \ldots, t_{l_q})) \\
& \times \prod_{i \in \{k_1, \ldots, k_p\} \cap \{l_1, \ldots, l_q\}} \phi(t_i).
\end{aligned}$$

Define a function $h_{n,m,n+m-r} \in L^2(\mathbb{R}_+)^{\circ r}$ as

$$h_{n,m,n+m-r}(t_1,\ldots,t_r) = \frac{1}{r!}\,\mathbb{E}[D_{t_1}\cdots D_{t_r}(I_n(f_n)I_m(g_m))]$$
$$= \frac{1}{r!}\sum_{p=0}^{r\wedge n}\sum_{q=r-p}^{r}\mathbf{1}_{\{n-p=m-q\}}\frac{n!}{(n-p)!}\frac{m!}{(m-q)!}(n-p)!a_{n,m,p,r}f_n\circ_{q+p-r}^{n-p} g_m(t_1,\ldots,t_r),$$
$$= \frac{1}{r!}\sum_{n-m+r\le 2p\le 2(n\wedge r)}\frac{n!m!}{(n-p)!}a_{n,m,p,r}f_n\circ_{m-r+p}^{n-p} g_m(t_1,\ldots,t_r),$$

where $a_{n,m,p,r}$ is the number of sequences $k_1<\cdots<k_p$ and $l_1<\cdots<l_q$ such that $\{k_1,\ldots,k_p\}\cup\{l_1,\ldots,l_q\}=\{1,\ldots,r\}$, with exactly $m-r+p-(n-p)$ terms in common. This number is

$$a_{n,m,p,r} = \frac{r!}{(r-p)!p!}\frac{p!}{(m-n-r+2p)!(n+r-m-p)!}.$$

Hence

$$h_{n,m,n+m-r}$$
$$= \sum_{n-m+r\le 2p\le 2(n\wedge r)}\frac{n!m!f_n\circ_{m-r+p}^{n-p} g_m}{(r-p)!(m-n-r+2p)!(n+r-m-p)!(n-p)!}$$
$$= \sum_{n+m-r\le 2i\le 2((n+m-r)\wedge n\wedge m)}\frac{n!}{(n-i)!}\frac{m!}{(m-i)!}\frac{1}{(2i-l)!}\frac{1}{(l-i)!}f_n\circ_i^{l-i} g_m$$
$$= \sum_{l\le 2i\le 2(l\wedge n\wedge m)} i!\binom{n}{i}\binom{m}{i}\binom{i}{l-i}f_n\circ_i^{l-i} g_m,$$

with $l=n+m-r$ and $i=p+m-r$. The chaos expansion follows from Proposition 4.2.5, first for f_n, g_m continuous with compact supports. The general case follows by a density argument. □

In the next remark we give a necessary condition for the independence of multiple stochastic integrals.

Remark 4.5.7. *Let $f_n\in L^2(\mathbb{R}_+)^{\circ n}$ and $g_m\in L^2(\mathbb{R}_+)^{\circ m}$ and assume that the $I_n(f_n)$ and $I_m(g_m)$ are independent. Then*

$$\int_{\mathbb{R}_+^{s-i}} f_n(x_1,\ldots,x_n)g_m(x_1,\ldots,x_k,y_{i+1},\ldots,y_m)dx_1\cdots dx_{s-i}=0, \quad (4.5.8)$$

$\phi(x_{s-i+1})\cdots\phi(x_i)dx_{s-i+1}\cdots dx_n dy_{s-i+1}\cdots dx_m$ − a.e., $1\le 2i\le s\le 2(n\wedge m)$.

Proof. If $I_n(f_n)$ and $I_m(g_m)$ are independent, then $I_n(f_n)I_m(g_m)\in L^2(\Omega)$ and

$$(n+m)!\mid f_n\otimes g_m\mid^2_{L^2(\mathbb{R}_+)^{\otimes(n+m)}}=\mid f_n\circ g_m\mid^2_{L^2(\mathbb{R}_+)^{\circ(m+m)}}$$
$$\ge n!m!\mid f_n\mid^2_{L^2(\mathbb{R}_+)^{\otimes n}}\mid g_m\mid^2_{L^2(\mathbb{R}_+)^{\otimes m}}$$

$$
\begin{aligned}
&= E\left[I_n(f_n)^2\right] E\left[I_m(g_m)^2\right] = E\left[(I_n(f_n)I_m(g_m))^2\right] \\
&= \sum_{r=0}^{2(n\wedge m)} (n+m-r)! \mid h_{n,m,r} \mid^2_{L^2(\mathbb{R}_+)^{\otimes(n+m-r)}} \\
&= (n+m)! \mid f_n \otimes g_m \mid^2_{L^2(\mathbb{R}_+)^{\otimes(n+m)}} \\
&\quad + \sum_{r=1}^{2(n\wedge m)} (n+m-r)! \mid h_{n,m,r} \mid^2_{L^2(\mathbb{R}_+)^{\otimes(n+m-r)}},
\end{aligned}
$$

hence $h_{n,m,r} = 0$, $r = 1, \ldots, 2(n \wedge m)$, which implies (4.5.8). □

4.6 Exponential Vectors

We define a linear transformation T_t^ϕ on the space $\mathcal{E}$ spanned by the exponential vectors introduced in Definition 2.13.3.

Definition 4.6.1. *Given $\phi \in L^\infty(\mathbb{R}_+)$, we let*

$$
T_t^\phi \xi(u) = (1 + u_t\phi_t)\xi(u), \qquad t \in \mathbb{R}_+, \quad u \in L^2(\mathbb{R}_+).
$$

The transformation T_t^ϕ is well-defined on $\mathcal{E}$ because $\xi(u_1), \ldots, \xi(u_n)$, are linearly independent if $u_1, \ldots, u_n$ are distinct elements of $L^2(\mathbb{R}_+)$.

Lemma 4.6.2. *The transformation T_t^ϕ is multiplicative, i.e.*

$$
T_t^\phi(FG) = (T_t^\phi F)(T_t^\phi G), \quad F, G \in \mathcal{E}.
$$

Proof. From Lemma 2.13.4 we have

$$
\begin{aligned}
T_t^\phi(\xi(u)\xi(v)) &= \exp(\langle u, v\rangle_{L^2(\mathbb{R}_+)}) T_t^\phi \xi(u + v + \phi uv) \\
&= \exp(\langle u, v\rangle_{L^2(\mathbb{R}_+)})(1 + \phi_t(u_t + v_t + \phi_t u_t v_t))\xi(u + v + \phi uv) \\
&= (1 + \phi_t u_t)(1 + \phi_t v_t)\xi(u)\xi(v) \\
&= T_t^\phi \xi(u) T_t^\phi \xi(v).
\end{aligned}
$$

□

The following proposition provides an interpretation of T_t^ϕ using the construction of exponential vectors as solutions of stochastic differential equations, cf. Proposition 2.13.1.

Proposition 4.6.3. *For all $u \in L^2(\mathbb{R}_+)$, $T_t^\phi \xi_T(u)$ coincides $dt \times d\mathbb{P}$-a.e. with the limit as T goes to infinity of the solution Z_T^t to the equation*

$$
Z_s^t = 1 + \int_0^s Z_{\tau^-}^t u_\tau dM_\tau^t, \qquad s \in \mathbb{R}_+, \tag{4.6.1}
$$

where $(M_s^t)_{s\in\mathbb{R}_+}$ *is defined as*

$$M_s^t = M_s + \phi_t \mathbf{1}_{[t,\infty)}(s), \qquad s \in \mathbb{R}_+.$$

Proof. Clearly by Proposition 2.13.1 we have $Z_s^t = \xi_s(u)$, $s < t$. Next, at time t we have

$$\begin{aligned} Z_t^t &= (1+\phi_t u_t) Z_{t^-}^t \\ &= (1+\phi_t u_t)\xi_{t^-}(u) \\ &= (1+\phi_t u_t)\xi_t(u), \end{aligned}$$

since $\xi_{t^-}(u) = \xi_t(u)$ a.s. for fixed t because $\Delta M_t = 0$, $dt \times d\mathbb{P}$-a.e. Finally, for $s > t$ we have

$$\begin{aligned} Z_s^t &= Z_t^t + \int_t^s Z_{\tau^-}^t u_\tau dM_\tau \\ &= (1+\phi_t u_t)\xi_t(u) + \int_t^s Z_{\tau^-}^t u_\tau dM_\tau, \end{aligned}$$

hence

$$\frac{Z_s^t}{1+\phi_t u_t} = \xi_t(u) + \int_t^s \frac{Z_{\tau^-}^t}{1+\phi_t u_t} u_\tau dM_\tau, \quad s > t,$$

which implies from (2.13.1):

$$\frac{Z_s^t}{1+\phi_t u_t} = \xi_s(u), \quad s > t,$$

and

$$Z_T^t = (1+\phi_t u_t)\xi(u) = T_t^\phi \xi(u),$$

$\mathbb{P}$-a.s., $t \in \mathbb{R}_+$. □

In other words, $T_t^\phi F$, $F \in \mathcal{E}$, can be interpreted as the evaluation of F on the trajectories of $(M_s)_{s\in\mathbb{R}_+}$ perturbed by addition of a jump of height ϕ_t at time t.

In Chapters 5 and 6 we will express the multiple stochastic integrals in terms of polynomials in the Brownian and Poisson cases. Note that such expressions using polynomials are not available in other cases, see e.g. [120] in the case $(\phi_t)_{t\in\mathbb{R}_+}$ is random, in particular for the Azéma martingales.

Finally we turn to the probabilistic interpretation of the gradient D. In case $(\phi_t)_{t\in\mathbb{R}_+}$ is a deterministic function, the probabilistic interpretation of D_t is known and $D_t F$ can be explicitly computed. Define the operator

$$D^B : \mathcal{E} \to L^2(\Omega \times \mathbb{R}_+, d\mathbb{P} \times dt)$$

on the space $\mathcal{E}$ of exponential vectors as

$$\langle D^B F, u\rangle_{L^2(\mathbb{R}_+)} = \frac{d}{d\varepsilon} F\left(M(\cdot) + \varepsilon \int_0^{\cdot} i_s u_s ds\right)_{|\varepsilon=0}, \quad F \in \mathcal{E}, \quad u \in L^2(\mathbb{R}_+). \tag{4.6.2}$$

We have, for $F = \xi(u)$ and $g \in L^2(\mathbb{R}_+)$,

$$\begin{aligned}\langle D^B F, g\rangle_{L^2(\mathbb{R}_+)} &= \frac{d}{d\varepsilon} \exp\left(\varepsilon \int_0^\infty g_s u_s i_s ds\right) \xi(u)_{|\varepsilon=0} \\ &= \int_0^\infty g_s u_s i_s ds \ \xi(u),\end{aligned}$$

hence $D_t^B \xi(u) = i_t u_t \xi(u)$, $t \in \mathbb{R}_+$, where

$$\xi(u) = \exp\left(\int_0^\infty u_s dM_s - \frac{1}{2}\int_0^\infty |u_s|^2 i_s ds\right) \prod_{s\in J_M} (1 + u_s\phi_s) \mathrm{e}^{-u_s\phi_s}, \tag{4.6.3}$$

and J_M denotes the set of jump times of $(M_t)_{t\in\mathbb{R}_+}$. We have the following proposition, which recovers and makes more precise the statement of Proposition (4.5.2). Let again $i_t = \mathbf{1}_{\{\phi_t=0\}}$ and $j_t = 1 - i_t = \mathbf{1}_{\{\phi_t\neq 0\}}$, $t \in \mathbb{R}_+$.

Proposition 4.6.4. *We have*

$$D_t F = D_t^B F + \frac{j_t}{\phi_t}(T_t^\phi F - F), \quad t \in \mathbb{R}_+, \quad F \in \mathcal{E}. \tag{4.6.4}$$

Proof. When $\phi_t = 0$ we have $D_t^B F = i_t u_t \xi(u) = i_t D_t F$, hence

$$\begin{aligned}D_t\xi(u) &= i_t D_t \xi(u) + j_t D_t \xi(u) \\ &= i_t u_t \xi(u) + j_t u_t \xi(u) \\ &= D_t^B \xi(u) + \frac{j_t}{\phi_t}(T_t^\phi \xi(u) - \xi(u)), \qquad t \in \mathbb{R}_+.\end{aligned}$$

Concerning the product rule we have from Lemma 2.13.4:

$$\begin{aligned}D_t(\xi(u)\xi(v)) &= \exp\left(\int_0^\infty u_s v_s ds\right) D_t\xi(u + v + \phi uv) \\ &= \exp\left(\int_0^\infty u_s v_s ds\right) (u_t + v_t + \phi_t u_t v_t)\xi(u + v + \phi uv) \\ &= (u_t + v_t + \phi_t u_t v_t)\xi(u)\xi(v) \\ &= \xi(u) D_t\xi(v) + \xi(v) D_t \xi(u) + \phi_t D_t\xi(u) D_t\xi(v),\end{aligned}$$

$u, v \in L^\infty(\mathbb{R}_+)$, see also Relation (6) of [111]. □

4.7 Deviation Inequalities

In this section we work under the continuity Assumption 4.4.2, which is satisfied when $(\phi_t)_{t\in\mathbb{R}_+}$ is a deterministic function since in this case an Ornstein-Uhlenbeck process $(X_t)_{t\in\mathbb{R}_+}$ can be associated to the semi-group $(P_s)_{s\in\mathbb{R}_+}$. The proof of the next lemma makes forward references to Lemmas 5.3.1 and 6.8.1.

Lemma 4.7.1. *The continuity Assumption 4.4.2 is satisfied if $(\phi_t)_{t\in\mathbb{R}_+}$ is a deterministic function.*

Proof. Let $(M_t)_{t\in\mathbb{R}_+}$ be defined as in (2.10.5) on the product space $\Omega = \Omega_1 \times \Omega_2$ of independent Brownian motion $(B_t)_{t\in\mathbb{R}_+}$ and Poisson process $(N_t)_{t\in\mathbb{R}_+}$. Using the decomposition (2.10.5), i.e.

$$dM_t = i_t dB_t + \phi_t(dN_t - \lambda_t dt), \qquad t \in \mathbb{R}_+,$$

any element

$$G = f(I_1(u_1), \ldots, I_1(u_n))$$

of $\mathcal{S}$ can be constructed as a functional $G : \Omega_1 \times \Omega_2 \to \mathbb{R}$. From Lemma 5.3.1 and Lemma 6.8.1 we have

$$P_t G(\omega) = \int_{\Omega_1\times\Omega_2} G(\mathcal{T}_t^1(\omega_1, \tilde{\omega}_1), \mathcal{T}_t^2(\omega_2, \tilde{\omega}_2)) p_t(\omega_1, \omega_2, d\tilde{\omega}_1, d\tilde{\omega}_2),$$

for some probability kernel p_t and mappings

$$\mathcal{T}_t^1 : \Omega_1 \times \Omega_1 \to \Omega_1, \qquad \mathcal{T}_t^2 : \Omega_1 \times \Omega_1 \to \Omega_1.$$

This implies

$$\begin{aligned}\|P_t DF\|_{L^\infty(\Omega, L^2(\mathbb{R}_+))} &\leq \|P_t \|DF\|_{L^2(\mathbb{R}_+)}\|_{L^\infty(\Omega)} \\ &\leq \|DF\|_{L^\infty(\Omega, L^2(\mathbb{R}_+))}, \qquad t \in \mathbb{R}_+,\end{aligned}$$

for all $F \in \mathrm{Dom}\,(D)$. □

Proposition 4.7.2. *Let $F \in \mathrm{Dom}\,(D)$ be such that $\mathbb{E}[\mathrm{e}^{T|F|}] < \infty$, and $\mathrm{e}^{sF} \in \mathrm{Dom}\,(D)$, $0 < s \leq T$, for some $T > 0$. Let h be the function defined by*

$$h(s) = \left\| \frac{\mathrm{e}^{-sF} D\mathrm{e}^{sF}}{DF} \right\|_\infty \|DF\|^2_{L^\infty(\Omega, L^2(\mathbb{R}_+))}, \qquad s \in [0, T].$$

Then

$$\mathbb{P}(F - \mathbb{E}[F] \geq x) \leq \exp\left(-\int_0^x h^{-1}(s) ds \right), \qquad 0 < x < h(T),$$

where h^{-1} is the inverse of h.

If h is not strictly increasing we may use the left-continuous inverse of h:

$$h^{-1}(x) = \inf\{t > 0 \ : \ h(t) \geq x\}, \qquad 0 < x < h(T^-).$$

Proof. Assume first that $\mathbb{E}[F] = 0$. Since the Ornstein-Uhlenbeck semigroup $(P_t)_{t\in\mathbb{R}_+}$ satisfies the continuity Assumption 4.4.2, then using Proposition 4.4.1 we have

$$\begin{aligned}
&\mathbb{E}[F\mathrm{e}^{sF}] = \mathrm{Cov}\,(F, \mathrm{e}^{sF}) \\
&= \mathbb{E}\left[\int_0^\infty \mathrm{e}^{-v} \int_0^\infty D_u \mathrm{e}^{sF} P_v D_u F du dv\right] \\
&\leq \left\| \frac{\mathrm{e}^{-sF} D\mathrm{e}^{sF}}{DF} \right\|_\infty \mathbb{E}\left[\mathrm{e}^{sF} \int_0^\infty \mathrm{e}^{-v} \int_0^\infty D_u F P_v D_u F dv du\right] \\
&\leq \left\| \frac{\mathrm{e}^{-sF} D\mathrm{e}^{sF}}{DF} \right\|_\infty \mathbb{E}\left[\mathrm{e}^{sF} \int_0^\infty \mathrm{e}^{-v} \|DF\|_{L^2(\mathbb{R}_+)} \|P_v DF\|_{L^2(\mathbb{R}_+)} dv\right] \\
&\leq \left\| \frac{\mathrm{e}^{-sF} D\mathrm{e}^{sF}}{DF} \right\|_\infty \mathbb{E}\left[\mathrm{e}^{sF}\right] \|DF\|_{L^\infty(\Omega, L^2(\mathbb{R}_+))} \left\| \int_0^\infty \mathrm{e}^{-v} P_v \|DF\|_{L^2(\mathbb{R}_+)} dv \right\|_\infty \\
&\leq \left\| \frac{\mathrm{e}^{-sF} D\mathrm{e}^{sF}}{DF} \right\|_\infty \mathbb{E}\left[\mathrm{e}^{sF}\right] \|DF\|_{L^\infty(\Omega, L^2(\mathbb{R}_+))} \int_0^\infty \mathrm{e}^{-v} \|DF\|_{L^\infty(\Omega, L^2(\mathbb{R}_+))} dv \\
&\leq \mathbb{E}\left[\mathrm{e}^{sF}\right] \left\| \frac{\mathrm{e}^{-sF} D\mathrm{e}^{sF}}{DF} \right\|_\infty \|DF\|^2_{L^\infty(\Omega, L^2(\mathbb{R}_+))}. \\
&\leq h(s)\, \mathbb{E}\left[\mathrm{e}^{sF}\right].
\end{aligned}$$

In the general case, letting $L(s) = \mathbb{E}[\mathrm{e}^{s(F-\mathbb{E}[F])}]$, we have

$$\begin{aligned}
\log(\mathbb{E}[\mathrm{e}^{t(F-\mathbb{E}[F])}]) &= \int_0^t \frac{L'(s)}{L(s)} ds \\
&= \int_0^t \frac{\mathbb{E}[(F - \mathbb{E}[F])\mathrm{e}^{s(F-\mathbb{E}[F])}]}{\mathbb{E}[\mathrm{e}^{s(F-\mathbb{E}[F])}]} ds \\
&= \int_0^t h(s) ds,
\end{aligned}$$

$0 \leq t \leq T$. We have for all $x \in \mathbb{R}_+$:

$$\begin{aligned}
\mathrm{e}^{tx} \mathbb{P}(F - \mathbb{E}[F] \geq x) &\leq \mathbb{E}[\mathrm{e}^{t(F-\mathbb{E}[F])}] \\
&\leq \mathrm{e}^{H(t)}, \qquad 0 \leq t \leq T,
\end{aligned}$$

where

$$H(t) = \int_0^t h(s) ds, \qquad 0 \leq t \leq T.$$

For any $0 < t < T$ we have $\frac{d}{dt}(H(t) - tx) = h(t) - x$, hence

$$
\begin{aligned}
\min_{0<t<T}(H(t)-tx) &= -xh^{-1}(x)+H(h^{-1}(x)) \\
&= -xh^{-1}(x)+\int_0^{h^{-1}(x)} h(s)ds \\
&= -xh^{-1}(x)+\int_0^{x} s dh^{-1}(s) \\
&= -\int_0^{x} h^{-1}(s)ds.
\end{aligned}
$$

□

From now on we work with $(\phi_t)_{t\in\mathbb{R}_+}$ a deterministic function, i.e. $(M_t)_{t\in\mathbb{R}_+}$ is written as in (2.10.5) and from Lemma 4.7.1, the continuity Assumption 4.4.2 is satisfied.

This covers the Gaussian case for $\phi_t = 0$, $t \in \mathbb{R}_+$, and also the general Poisson case when ϕ_t is a non-zero constant.

Proposition 4.7.3. *Let $K \geq 0$ and $F \in \mathrm{Dom}\,(D)$ be such that $\phi_t D_t F \leq K$, $dtd\mathbb{P}$-a.e. for some $K \geq 0$ and $\|DF\|_{L^\infty(\Omega,L^2(\mathbb{R}_+))} < \infty$. Then we have*

$$
\begin{aligned}
\mathbb{P}(F-\mathbb{E}[F]\geq x) &\leq \exp\left(-\frac{\|DF\|^2_{L^\infty(\Omega,L^2(\mathbb{R}_+))}}{K^2} g\left(\frac{xK}{\|DF\|^2_{L^\infty(\Omega,L^2(\mathbb{R}_+))}}\right)\right) \\
&\leq \exp\left(-\frac{x}{2K}\log\left(1+\frac{xK}{\|DF\|^2_{L^\infty(\Omega,L^2(\mathbb{R}_+))}}\right)\right), \qquad (4.7.1)
\end{aligned}
$$

$x \geq 0$, with $g(u) = (1+u)\log(1+u) - u$, $u \geq 0$. If $K = 0$ (decreasing functionals) we have

$$
\mathbb{P}(F-\mathbb{E}[F]\geq x) \leq \exp\left(-\frac{x^2}{2\|DF\|^2_{L^\infty(\Omega,L^2(\mathbb{R}_+))}}\right). \qquad (4.7.2)
$$

Proof. We first assume that $F \in \mathrm{Dom}\,(D)$ is a bounded random variable. Let us assume that $\mathbb{E}[F] = 0$. From Proposition 4.5.2 we have as in the proof of Proposition 1.11.1:

$$
\begin{aligned}
0 &\leq \frac{\mathrm{e}^{-sF} D_u \mathrm{e}^{sF}}{D_u F} \\
&= \frac{1}{\phi_u D_u F}\left(\mathrm{e}^{s\phi_u D_u F}-1\right) \\
&\leq \frac{\mathrm{e}^{sK}-1}{K},
\end{aligned}
$$

since the function $x \longmapsto (\mathrm{e}^x - 1)/x$ is positive and increasing on $\mathbb{R}$. Hence in Proposition 4.7.2 we can take

$$h(s) = \left| \frac{e^{sK} - 1}{K} \right| \|DF\|^2_{L^\infty(\Omega, L^2(\mathbb{R}_+))}, \qquad s \in [0, T],$$

and

$$\begin{aligned}
&\min_{0<t<T} (H(t) - tx) = -\int_0^x h^{-1}(s) ds \\
&\leq -\frac{1}{K} \int_0^x \log \left(1 + tK \|DF\|^{-2}_{L^\infty(\Omega, L^2(\mathbb{R}_+))}\right) dt \\
&= -\frac{1}{K} \left((x + \frac{1}{K} \|DF\|^2_{L^\infty(\Omega, L^2(\mathbb{R}_+))}) \log \left(1 + xK \|DF\|^{-2}_{L^\infty(\Omega, L^2(\mathbb{R}_+))}\right) - x \right) \\
&\leq -\frac{x}{2K} \log \left(1 + \frac{xK}{\|DF\|^2_{L^\infty(\Omega, L^2(\mathbb{R}_+))}} \right).
\end{aligned}$$

If $K = 0$, the above proof is still valid by replacing all terms by their limits as $K \to 0$. If $F \in \mathrm{Dom}\,(D)$ is not bounded the conclusion holds for

$$F_n = \max(-n, \min(F, n)) \in \mathrm{Dom}\,(D), \qquad n \geq 1,$$

and $(F_n)_{n\in\mathbb{N}}$, $(DF_n)_{n\in\mathbb{N}}$, converge respectively to F and DF in $L^2(\Omega)$, resp. $L^2(\Omega \times \mathbb{R}_+)$, with $\|DF_n\|^2_{L^\infty(\Omega, L^2(\mathbb{R}_+))} \leq \|DF\|^2_{L^\infty(\Omega, L^2(\mathbb{R}_+))}$. □

By the same argument as in Proposition 1.11.3, the bounds (4.7.1) and (4.7.2) respectively imply

$$\mathbb{E}[e^{\alpha|F| \log_+ |F|}] < \infty$$

for some $\alpha > 0$, and

$$\mathbb{E}[e^{\alpha F^2}] < \infty$$

for all $\alpha < (2\|DF\|^2_{L^\infty(\Omega, L^2(\mathbb{R}_+))})^{-1}$.

Applying Proposition 4.7.3 with the condition $\phi DF \leq \phi K$ for constant $\phi_t = \phi \in \mathbb{R}_+$, $t \in \mathbb{R}_+$, we have the following.

Corollary 4.7.4. *Assume that $\phi_t = \phi \in \mathbb{R}_+$, $t \in \mathbb{R}_+$, is constant. Let $F \in \mathrm{Dom}\,(D)$ be such that $DF \leq K$ for some $K \geq 0$ and $\|DF\|_{L^\infty(\Omega, L^2(\mathbb{R}_+))} < \infty$. Then*

$$\begin{aligned}
\mathbb{P}(F - \mathbb{E}[F] \geq x) &\leq \exp \left(-\frac{\|DF\|^2_{L^\infty(\Omega, L^2(\mathbb{R}_+))}}{\phi^2 K^2} g \left(\frac{x\phi K}{\|DF\|^2_{L^\infty(\Omega, L^2(\mathbb{R}_+))}} \right) \right) \\
&\leq \exp \left(-\frac{x}{2\phi K} \log \left(1 + \frac{x\phi K}{\|DF\|^2_{L^\infty(\Omega, L^2(\mathbb{R}_+))}} \right) \right),
\end{aligned}$$

with $g(u) = (1 + u) \log(1 + u) - u$, $u \geq 0$. If $\phi = 0$ (Wiener case) or $K = 0$ (decreasing functionals) we have

$$\mathbb{P}(F - \mathbb{E}[F] \geq x) \leq \exp\left(-\frac{x^2}{2\|DF\|^2_{L^\infty(\Omega,L^2(\mathbb{R}_+))}}\right). \tag{4.7.3}$$

In particular if F is $\mathcal{F}_T$-measurable, then $\|DF\|_{L^\infty(\Omega,L^2(\mathbb{R}_+))} \leq KT$ and

$$\begin{aligned}\mathbb{P}(F - \mathbb{E}[F] \geq x) &\leq \exp\left(-\frac{T}{\phi^2} g\left(\frac{\phi x}{KT}\right)\right)\\ &\leq \exp\left(-\frac{x}{2K}\log\left(1+\frac{\phi x}{KT}\right)\right),\end{aligned}$$

which improves (as in [155]) the inequality

$$\mathbb{P}(F - \mathbb{E}[F] \geq x) \leq \exp\left(-\frac{x}{4\phi K}\log\left(1+\frac{\phi x}{2KT}\right)\right), \tag{4.7.4}$$

which follows from Proposition 6.1 in [7], and relies on modified (not sharp) logarithmic Sobolev inequalities on Poisson space.

4.8 Derivation of Fock Kernels

In this section we introduce some differential operators which will used to construct other instances of operators satisfying Assumptions 3.1.1-3.4.3, namely on the Wiener space in Section 5.8 and on the Poisson space in Section 7.7, by infinitesimal time changes on the paths of the underling process. We let $\mathcal{C}_c^1(\mathbb{R}_+)$ denote the space of continuously differentiable functions with compact support in $\mathbb{R}_+$.

Definition 4.8.1. *We define the linear operator*

$$\nabla^\ominus : \mathcal{S} \to L^2(\Omega \times \mathbb{R}_+)$$

on $\mathcal{S}$ by

$$\nabla_t^\ominus I_n(f^{\otimes n}) = -nI_n((f'\mathbf{1}_{[t,\infty)}) \circ f^{\otimes(n-1)}),$$

$t \in \mathbb{R}_+$, $f \in \mathcal{C}_c^1(\mathbb{R}_+)$, $n \in \mathbb{N}$, and by polarization of this expression.

The operator $\nabla^\ominus$ is unbounded, densely defined, and maps functionals of the n-th chaos $\mathcal{H}_n$ into $\mathcal{H}_n$, $n \geq 1$.

For $h \in L^2(\mathbb{R}_+)$, let $\overset{\circ}{h}$ denote the function defined by

$$\overset{\circ}{h}(t) = \int_0^t h(s)ds, \qquad t \in \mathbb{R}_+.$$

Definition 4.8.2. *We define the linear operator $\nabla^\oplus : \mathcal{U} \to L^2(\Omega)$ by*

$$\nabla^{\oplus}(hI_n(f^{\otimes n})) = nI_n((f\mathring{h})' \circ f^{\otimes(n-1)}),$$

$f, h \in \mathcal{C}_c^1(\mathbb{R}_+)$, and extend it by linearity and polarization.

Next we show that the operators $\nabla^{\oplus}$ and $\nabla^{\ominus}$ are mutually adjoint.

Proposition 4.8.3. *The operators*

$$\nabla^{\ominus} : \mathcal{S} \to L^2(\Omega \times \mathbb{R}_+)$$

and

$$\nabla^{\oplus} : \mathcal{U} \to L^2(\Omega)$$

satisfy the duality relation

$$\mathbb{E}\left[\langle \nabla^{\ominus} F, u\rangle_{L^2(\mathbb{R}_+)}\right] = \mathbb{E}\left[F\nabla^{\oplus}(u)\right], \qquad F \in \mathcal{S}, \quad u \in \mathcal{U}. \tag{4.8.1}$$

Proof. By polarization, we need to prove the following. Letting $F = I_n(f^{\otimes n})$, $u = hI_n(g^{\otimes n})$ and $f, g, h \in \mathcal{C}_c^1(\mathbb{R}_+)$, we have

$$\begin{aligned}
&\mathbb{E}\left[\langle \nabla^{\ominus} F, u)\rangle_{L^2(\mathbb{R}_+)}\right] \\
&= \mathbb{E}\left[\langle \nabla^{\ominus} I_n(f^{\otimes n}), h)\rangle_{L^2(\mathbb{R}_+)} I_n(g^{\otimes n}\right] \\
&= -n\,\mathbb{E}\left[\int_0^\infty I_n((f'\mathbf{1}_{[t,\infty)}) \circ f^{\otimes(n-1)}) I_n(g^{\otimes n}) h(t)dt\right] \\
&= -n^2 \langle I_{n-1}(f^{\otimes(n-1)}), I_{n-1}(g^{\otimes(n-1)})\rangle_{L^2(\mathbb{R}_+)^{\circ(n-1)}} \int_0^\infty h(t) \int_t^\infty f'(s)g(s)dsdt \\
&= -n^2(n-1)! \langle f^{\otimes(n-1)}, g^{\otimes(n-1)}\rangle_{L^2(\mathbb{R}_+)^{\circ(n-1)}} \int_0^\infty f'(t)g(t)\mathring{h}(t)dt \\
&= n^2(n-1)! \langle f^{\otimes(n-1)}, g^{\otimes(n-1)}\rangle_{L^2(\mathbb{R}_+)^{\circ(n-1)}} \int_0^\infty f(t)(\mathring{h}g)'(t)dt \\
&= n\,\mathbb{E}\left[I_n(f^{\otimes n}) I_n((g\mathring{h})' \circ g^{\otimes(n-1)})\right] \\
&= \mathbb{E}\left[I_n(f^{\otimes n}) \nabla^{\oplus}(hI_n(g^{\otimes n}))\right] \\
&= \mathbb{E}\left[F\nabla^{\oplus}(u)\right],
\end{aligned}$$

hence Relation (4.8.1) holds. □

Note that the operators $\nabla^{\ominus}$ and $\nabla^{\oplus}$ are closable from Propositions 3.1.2 and 4.8.3.

4.9 Notes and References

The terminology "Skorohod integral" is adopted in reference to [140]. The first systematic investigation of the relations between the multiple stochastic integrals with respect to normal and the associated annihilation operator and Skorohod integral appeared in [85]. Proposition 4.2.5 is also known as the Stroock formula, cf. [142] and Relations (7.4) and (7.5), pages 26-27 of [70]. We refer to page 216 of [33], and to [72], [144], [145], for other versions of Proposition 4.5.6 in the Poisson case. In [130] a more general result is proved, and yields a decomposition the product $I_n(f_n)I_m(g_m)$ as a sum of $n \wedge m$ integral terms. Those terms are not necessarily linear combinations of multiple stochastic integrals with respect to $(M_t)_{t\in\mathbb{R}_+}$, except when the bracket $d[M,M]_t$ is a linear deterministic combination of dt and dM_t, cf. [120]. Remark 4.5.7 is an extension the necessary condition for independence proved in the Wiener case in [148]. The necessary and sufficient conditions obtained in [111], [113], [145] are true only when f_n and g_m have constant signs. Necessary and sufficient condition for the independence of multiple stochastic integrals with respect to symmetric α-stable random measures with $0 < \alpha < 2$ have been obtained in [128] as a disjoint support condition on f_n and g_m. However, finding a necessary and sufficient condition two given symmetric functions f_n and g_m for the independence of $I_n(f_n)$ and $I_m(g_m)$ in the Poisson case is still an open problem. We refer to [105] for the classical Gaussian deviation inequality (4.7.2) in the case $\phi_t = 0$, $t \in \mathbb{R}_+$, i.e. on Wiener space. The material on multidimensional stochastic integrals is taken from [73]. White noise versions of the annihilation and creation operators, as well as connections with quantum field theory can be found in [53]. The Skorohod isometry Proposition 4.3.1 has also been stated for Brownian motion on Lie groups and on Riemannian manifolds respectively in [45] and [29].

Chapter 5
Analysis on the Wiener Space

In this chapter we consider the particular case where the normal martingale $(M_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion. The general results stated in Chapters 3 and 4 are developed in this particular setting of a continuous martingale. Here, the gradient operator has the derivation property and can be interpreted as a derivative in the directions of Brownian paths, while the multiple stochastic integrals are connected to the Hermite polynomials. The connection is also made between the gradient and divergence operators and other transformations of Brownian motion, e.g. by time changes. We also describe in more detail the specific forms of covariance identities and deviation inequalities that can be obtained on the Wiener space and on Riemannian path space.

5.1 Multiple Wiener Integrals

In this chapter we consider in detail the particular case where $(M_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion, i.e. $(M_t)_{t\in\mathbb{R}_+}$ solves the structure equation (2.10.1) with $\phi_t = 0$, $t \in \mathbb{R}_+$, i.e.

$$[M, M]_t = t, \qquad t \in \mathbb{R}_+.$$

The Hermite polynomials will be used to represent the multiple Wiener integrals.

Definition 5.1.1. *The Hermite polynomial $H_n(x;\sigma^2)$ of degree $n \in \mathbb{N}$ and parameter $\sigma^2 > 0$ is defined with*

$$H_0(x;\sigma^2) = 1, \qquad H_1(x;\sigma^2) = x, \qquad H_2(x;\sigma^2) = x^2 - \sigma^2,$$

and more generally from the recurrence relation

$$H_{n+1}(x;\sigma^2) = xH_n(x;\sigma^2) - n\sigma^2 H_{n-1}(x;\sigma^2), \qquad n \geq 1. \tag{5.1.1}$$

In particular we have

$$H_n(x;0) = x^n, \qquad n \in \mathbb{N}.$$

The generating function of Hermite polynomials is defined as

$$\psi_\lambda(x,\sigma^2) = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} H_n(x;\sigma^2), \quad \lambda \in (-1,1).$$

Proposition 5.1.2. *The following statements hold on the Hermite polynomials:*
i) Generating function:

$$\psi_\lambda(x,\sigma^2) = \mathrm{e}^{\lambda x - \lambda^2\sigma^2/2}, \quad x, \lambda \in \mathbb{R}.$$

ii) Derivation rule:

$$\frac{\partial H_n}{\partial x}(x;\sigma^2) = nH_{n-1}(x;\sigma^2), \tag{5.1.2}$$

iii) Creation rule:

$$H_{n+1}(x;\sigma^2) = \left(x - \sigma^2 \frac{\partial}{\partial x}\right) H_n(x;\sigma^2).$$

Proof. The recurrence relation (5.1.1) shows that the generating function ψ_λ satisfies the differential equation

$$\begin{cases} \dfrac{\partial \psi_\lambda}{\partial \lambda}(x,\sigma) = (x - \lambda\sigma^2)\psi_\lambda(x,\sigma^2), \\ \\ \psi_0(x,\sigma^2) = 1, \end{cases}$$

which proves (i). From the expression of the generating function we deduce (ii), and by rewriting (5.1.1) we obtain (iii). □

Let

$$\phi_d^\sigma(s_1,\ldots,s_d) = \frac{1}{(2\pi)^{d/2}} \mathrm{e}^{-(s_1^2+\cdots+s_d^2)/2}, \qquad (s_1,\ldots,s_d) \in \mathbb{R}^d,$$

denote the standard Gaussian density function with covariance $\sigma^2\mathrm{Id}$ on $\mathbb{R}^n$.

From Relation (5.1.2) we have

$$\frac{\partial}{\partial x}(\phi_1^\sigma(x)H_n(x;\sigma^2)) = \phi_1^\sigma(x)\left(\frac{\partial H_n}{\partial x}(x;\sigma^2) - \frac{x}{\sigma^2}H_n(x;\sigma^2)\right)$$

$$= -\frac{\phi_1^\sigma(x)}{\sigma^2} H_{n+1}(x;\sigma^2),$$

hence by induction, Proposition 5.1.2-(iii) implies

$$\sigma^{2(k_1+\cdots+k_d)} \frac{(-1)^{k_1+\cdots+k_d}}{\phi_d^\sigma(x_1,\ldots x_d)} \partial_{x_1}^{k_1} \cdots \partial_{x_d}^{k_d} \phi_d^\sigma(x_1,\ldots,x_d) = \prod_{i=1}^d H_{k_i}(x_i;\sigma^2). \tag{5.1.3}$$

Let now $I_n(f_n)$ denote the multiple stochastic integral of $f_n \in L^2(\mathbb{R}_+)^{\circ n}$ with respect to $(B_t)_{t\in\mathbb{R}_+}$, as defined in Section 2.7.

Note that here $I_1(f)$ coincides with $J_1(f)$ defined in (2.2.2) for all $f \in L^2(\mathbb{R}_+)$. In particular,

$$I_1(f) = \int_0^\infty f(t)dB_t \simeq \mathcal{N}\left(0, \int_0^\infty |f(t)|^2 dt\right)$$

has a centered Gaussian distribution with variance

$$\|f\|_2^2 := \|f\|_{L^2(\mathbb{R}_+)}^2 = \int_0^\infty |f(t)|^2 dt, \qquad f \in L^2(\mathbb{R}_+).$$

In addition, the multiplication formula (4.5.1) of Proposition 4.5.1 reads

$$I_1(u)I_n(v^{\otimes n}) = I_{n+1}(v^{\otimes n}\circ u) + n\langle u,v\rangle_{L^2(\mathbb{R}_+)} I_{n-1}(v^{\otimes(n-1)}) \tag{5.1.4}$$

for $n \geq 1$, since with $\phi_t = 0$, $t \in \mathbb{R}_+$, and we have in particular

$$I_1(u)I_1(v) = I_2(v\circ u) + \langle u,v\rangle_{L^2(\mathbb{R}_+)}$$

for $n = 1$. More generally, Relation (4.5.7) of Proposition 4.5.6 reads

$$I_n(f_n)I_m(g_m) = \sum_{s=0}^{n\wedge m} \binom{n}{s}\binom{m}{s} I_{n+m-2s}(h_{n,m,2s}),$$

where $h_{n,m,2s}$ is the symmetrization in $n+m-2s$ variables of

$$(x_{s+1},\ldots,x_n,y_{s+1},\ldots,y_m) \longmapsto \int_{\mathbb{R}_+^s} f_n(x_1,\ldots,x_n) g_m(x_1,\ldots,x_i,y_{s+1},\ldots,y_m) dx_1\cdots dx_s.$$

Proposition 5.1.3. *For any orthogonal family* $\{u_1,\ldots,u_d\}$ *in* $L^2(\mathbb{R}_+)$ *we have*

$$I_n(u_1^{\otimes n_1}\circ\cdots\circ u_d^{\otimes n_d}) = \prod_{k=1}^d H_{n_k}(I_1(u_k); \|u_k\|_2^2),$$

where $n = n_1 + \cdots + n_d$.

Proof. We have

$$H_0(I_1(u);\|u\|_2^2) = I_0(u^{\otimes 0}) = 1 \quad \text{and} \quad H_1(I_1(u);\|u\|_2^2) = I_1(u),$$

hence the proof follows by induction on $n \geq 1$, by comparison of the recurrence formula (5.1.1) with the multiplication formula (5.1.4). □

In particular,

$$\begin{aligned} I_n\left(\mathbf{1}_{[0,t]}^{\otimes n}\right) &= n! \int_0^t \int_0^{s_n} \cdots \int_0^{s_2} dB_{s_1} \cdots dB_{s_n} \\ &= H_n(B_t;t), \end{aligned} \tag{5.1.5}$$

and from (4.5.3) we have

$$\begin{aligned} I_n\left(\mathbf{1}_{[t_0,t_1]}^{\otimes n_1} \circ \cdots \circ \mathbf{1}_{[t_{d-1},t_d]}^{\otimes n_d}\right) &= \prod_{k=1}^d I_{n_k}\left(\mathbf{1}_{[t_{k-1},t_k]}^{\otimes n_k}\right) \\ &= \prod_{k=1}^d H_{n_k}(B_{t_k} - B_{t_{k-1}}; t_k - t_{k-1}). \end{aligned}$$

From this we recover the orthonormality properties of the Hermite polynomials with respect to the Gaussian density:

$$\begin{aligned} \int_{-\infty}^{\infty} H_n(x;t) H_m(x;t) \mathrm{e}^{-x^2/(2t)} \frac{dx}{\sqrt{2\pi t}} &= \mathbb{E}[H_n(B_t;t) H_m(B_t;t)] \\ &= \mathbb{E}[I_n(\mathbf{1}_{[0,t]}^{\otimes n}) I_m(\mathbf{1}_{[0,t]}^{\otimes m})] \\ &= \mathbf{1}_{\{n=m\}} n! t^n. \end{aligned}$$

In addition, by Lemma 2.7.2 we have

$$\begin{aligned} H_n(B_t;t) &= I_n\left(\mathbf{1}_{[0,t]}^{\otimes n}\right) \\ &= \mathbb{E}\left[I_n(\mathbf{1}_{[0,T]}^{\otimes n}) \Big| \mathcal{F}_t \right], \qquad t \in \mathbb{R}_+, \end{aligned}$$

is a martingale which, from Itô's formula, can be written as

$$\begin{aligned} & H_n(B_t;t) = I_n(\mathbf{1}_{[0,t]}^{\otimes n}) \\ & = H_n(0;0) + \int_0^t \frac{\partial H_n}{\partial x}(B_s;s) dB_s + \frac{1}{2} \int_0^t \frac{\partial^2 H_n}{\partial x^2}(B_s;s) ds + \int_0^t \frac{\partial H_n}{\partial s}(B_s;s) ds \\ & = n \int_0^t I_{n-1}(\mathbf{1}_{[0,s]}^{\otimes(n-1)}) dB_s \\ & = n \int_0^t H_{n-1}(B_s;s) dB_s \end{aligned}$$

from Proposition 2.12.1. By identification we recover Proposition 5.1.2-(*ii*), i.e.

$$\frac{\partial H_n}{\partial x}(x;s) = n H_{n-1}(x;s), \tag{5.1.6}$$

and the partial differential equation

$$\frac{\partial H_n}{\partial s}(x;s) = -\frac{1}{2}\frac{\partial^2 H_n}{\partial x^2}(x;s),$$

i.e the heat equation with initial condition

$$H_n(x;0) = x^n, \qquad x \in \mathbb{R}, \quad n \in \mathbb{N}.$$

Given $f_n \in L^2(\mathbb{R}_+)^{\otimes n}$ with orthogonal expansion

$$f_n = \sum_{\substack{n_1+\cdots+n_d=n \\ k_1,\ldots,k_d \geq 0}} a^{n_1,\ldots,n_d}_{k_1,\ldots,k_d} e_{k_1}^{\otimes n_1} \circ \cdots \circ e_{k_d}^{\otimes n_d},$$

in an orthonormal basis $(e_n)_{n\in\mathbb{N}}$ of $L^2(\mathbb{R}_+)$, we have

$$I_n(f_n) = \sum_{\substack{n_1+\cdots+n_d=n \\ k_1,\ldots,k_d \geq 0}} a^{n_1,\ldots,n_d}_{k_1,\ldots,k_d} H_{n_1}(I_1(e_{k_1});1)\cdots H_{n_d}(I_1(e_{k_d});1),$$

where the coefficients $a^{n_1,\ldots,n_d}_{k_1,\ldots,k_d}$ are given by

$$\begin{aligned} a^{n_1,\ldots,n_d}_{k_1,\ldots,k_d} &= \frac{1}{n_1!\cdots n_d!}\langle I_n(f_n), I_n(e_{k_1}^{\otimes n_1} \circ \cdots \circ u_{k_d}^{\otimes n_d})\rangle_{L^2(\Omega)} \\ &= \langle f_n, e_{k_1}^{\otimes n_1} \circ \cdots \circ e_{k_d}^{\otimes n_d}\rangle_{L^2(\mathbb{R}_+^n)}. \end{aligned}$$

Proposition 2.13.1 implies the following relation for exponential vectors, that can be recovered independently using the Hermite polynomials.

Proposition 5.1.4. *We have*

$$\xi(u) = \sum_{k=0}^{\infty} \frac{1}{n!} I_n(u^{\otimes n}) = \exp\left(I_1(u) - \frac{1}{2}\|u\|^2_{L^2(\mathbb{R}_+)}\right). \tag{5.1.7}$$

Proof. Relation (5.1.7) follows from Proposition 5.1.2-i) and Proposition 5.1.3 which reads $I_n(u^{\otimes n}) = H_n(I_1(u); \|u\|^2_{L^2(\mathbb{R}_+)})$, $n \geq 1$. □

We refer to Section 2.8 for the notion of chaos representation property.

Proposition 5.1.5. *The Brownian motion $(B_t)_{t\in\mathbb{R}_+}$ has the chaos representation property.*

Proof. Theorem 4.1, p. 134 of [52], shows by a Fourier transform argument that the linear space spanned by the exponential vectors

$$\left\{\exp\left(I_1(u) - \frac{1}{2}\|u\|^2_{L^2(\mathbb{R}_+)}\right) \quad : \quad u \in L^2(\mathbb{R}_+)\right\}$$

is dense in $L^2(\Omega)$. To conclude we note that the exponential vectors belong to the closure of $\mathcal{S}$ in $L^2(\Omega)$. □

From Proposition 5.1.5, any $F \in L^2(\Omega)$ has a chaos decomposition

$$F = \sum_{k=0}^{\infty} I_k(g_k),$$

where

$$\begin{aligned} I_k(g_k) & \qquad (5.1.8) \\ &= \sum_{d=1}^{k} \sum_{k_1+\cdots+k_d=k} \frac{1}{k_1!\cdots k_d!} \mathbb{E}[F I_k(u_1^{\otimes k_1} \circ \cdots \circ u_d^{\otimes k_d})] I_k(u_1^{\otimes k_1} \circ \cdots \circ u_d^{\otimes k_d}) \\ &= \sum_{d=1}^{k} \sum_{k_1+\cdots+k_d=k} \frac{1}{k_1!\cdots k_d!} \mathbb{E}[F I_k(u_1^{\otimes k_1} \circ \cdots \circ u_d^{\otimes k_d})] \prod_{i=1}^{n} H_{k_i}(I_1(u_i); \|u_i\|_2^2), \end{aligned}$$

is a finite sum since for all $m \geq 1$ and $l > k$,

$$\mathbb{E}[I_m(e_l^{\otimes m}) g(I_1(e_1), \ldots, I_1(e_k))] = 0.$$

Lemma 5.1.6. *Assume that F has the form $F = g(I_1(e_1), \ldots, I_1(e_k))$ for some*

$$g \in L^2(\mathbb{R}^k, (2\pi)^{-k/2} \mathrm{e}^{-|x|^2/2} dx),$$

and admits the chaos expansion

$$F = \mathbb{E}[F] + \sum_{n=1}^{\infty} I_n(f_n).$$

Then for all $n \geq 1$ there exists a (multivariate) Hermite polynomial P_n of degree n such that

$$I_n(f_n) = P_n(I_1(e_1), \ldots, I_1(e_k)).$$

Proof. The polynomial P_n is given by (5.1.8) above, which is a finite sum. □

Lemma 5.1.6 can also be recovered from the relation

$$\begin{aligned} f(I_1(e_1), \ldots, I_1(e_d)) & \qquad (5.1.9) \\ = \sum_{n=0}^{\infty} \sum_{\substack{k_1+\cdots+k_d=n \\ k_1 \geq 0, \ldots, k_d \geq 0}} \frac{(-1)^n}{k_1!\cdots k_d!} \langle f, \partial_1^{k_1} \cdots \partial_d^{k_d} \phi_d^1 \rangle_{L^2(\mathbb{R}^d)} I_n(e_1^{\otimes k_1} \circ \cdots \circ e_d^{\otimes k_d}), \end{aligned}$$

which follows from (5.1.6) and (5.1.3).

5.2 Gradient and Divergence Operators

In the Brownian case D has the derivation property, as an application of Proposition 4.5.2 with $\phi_t = 0$, $t \in \mathbb{R}_+$, i.e.

$$D_t(FG) = FD_tG + GD_tF, \qquad F, G \in \mathcal{S}. \tag{5.2.1}$$

More precisely we have the following result.

Proposition 5.2.1. *Let $u_1, \ldots, u_n \in L^2(\mathbb{R}_+)$ and*

$$F = f(I_1(u_1), \ldots, I_1(u_n)),$$

where f is a polynomial or $f \in \mathcal{C}_b^1(\mathbb{R}^n)$. We have

$$D_tF = \sum_{i=1}^n u_i(t) \frac{\partial f}{\partial x_i}(I_1(u_1), \ldots, I_1(u_n)), \qquad t \in \mathbb{R}_+. \tag{5.2.2}$$

Proof. Using the derivation rule (5.1.2), Definition 4.1.1 and Proposition 5.1.3, this statement is obvious when

$$F = I_n(u^{\otimes n}) = H_n(I_1(u); \|u\|_2^2), \qquad u \in L^2(\mathbb{R}_+).$$

By the product rule (5.2.1), the above relation extends to multivariate polynomials f (precisely, to linear combinations of products of Hermite polynomials), and then to $F \in \mathcal{S}$. In the general case we may assume that $u_1, \ldots, u_n \in L^2(\mathbb{R}_+)$ are orthonormal, and that $f \in \mathcal{C}_c^1(\mathbb{R}^n)$. Then from Lemma 5.1.6, we have the chaotic decomposition

$$\begin{aligned} F &= f(I_1(u_1), \ldots, I_1(u_n)) \\ &= \sum_{k=0}^\infty I_k(g_k), \end{aligned}$$

where $I_k(g_k)$ is a polynomial in $I_1(u_1), \ldots, I_1(u_n)$. The sequence

$$F_k := \sum_{l=0}^k I_l(g_l), \quad k \in \mathbb{N}, \qquad k \in \mathbb{N},$$

is a sequence of polynomial functionals contained in $\mathcal{S}$ and converging to F in $L^2(\Omega)$. By tensorization of the finite-dimensional integration by parts

$$\begin{aligned} &\int_{-\infty}^\infty f'(x) H_n(x;1) e^{-x^2/2} \frac{dx}{\sqrt{2\pi}} \\ &= \int_{-\infty}^\infty f(x)(xH_n(x;1) - H_n'(x;1)) e^{-x^2/2} \frac{dx}{\sqrt{2\pi}} \end{aligned}$$

$$= \int_{-\infty}^{\infty} f(x)(xH_n(x;1) - nH_{n-1}(x;1))\mathrm{e}^{-x^2/2}\frac{dx}{\sqrt{2\pi}}$$
$$= \int_{-\infty}^{\infty} f(x)H_{n+1}(x;1)\mathrm{e}^{-x^2/2}\frac{dx}{\sqrt{2\pi}},$$

we get

$$\begin{aligned}
&\mathbb{E}\left[I_k(u_1^{\otimes k_1}\circ\cdots\circ u_n^{\otimes k_n})\frac{\partial f}{\partial x_i}(I_1(u_1),\ldots,I_1(u_n))\right]\\
&\quad= \mathbb{E}[f(I_1(u_1),\ldots,I_1(u_n))I_{k+1}(u_1^{\otimes k_1}\circ\cdots\circ u_n^{\otimes k_n}\circ u_i)]\\
&\quad= \mathbb{E}[I_{k+1}(g_{k+1})I_{k+1}(u_1^{\otimes k_1}\circ\cdots\circ u_n^{\otimes k_n}\circ u_i)]\\
&\quad= \mathbb{E}[\langle DI_{k+1}(g_{k+1}),u_i\rangle_{L^2(\mathbb{R}_+)}I_k(u_1^{\otimes k_1}\circ\cdots\circ u_n^{\otimes k_n})].
\end{aligned}$$

This shows that $\dfrac{\partial f}{\partial x_i}(I_1(u_1),\ldots,I_1(u_n))$ has the chaotic decomposition

$$\frac{\partial f}{\partial x_i}(I_1(u_1),\ldots,I_1(u_n)) = \sum_{l=1}^{\infty}\langle DI_l(g_l),u_i\rangle_{L^2(\mathbb{R}_+)},$$

where the series converges in $L^2(\Omega)$, hence

$$DF_k = \sum_{i=1}^n u_i\sum_{l=1}^k\langle DI_l(g_l),u_i\rangle_{L^2(\mathbb{R}_+)}, \qquad k\in\mathbb{N},$$

converges in $L^2(\Omega\times\mathbb{R}_+)$ to

$$\begin{aligned}
\sum_{i=1}^n u_i\frac{\partial f}{\partial x_i}(I_1(u_1),\ldots,I_1(u_n)) &= \sum_{i=1}^n u_i\sum_{k=1}^{\infty}\langle DI_k(g_k),u_i\rangle_{L^2(\mathbb{R}_+)}\\
&= \sum_{k=1}^{\infty} DI_k(g_k).
\end{aligned}$$

□

In particular, for f polynomial and for $f\in\mathcal{C}_b^1(\mathbb{R}^n)$ we have

$$D_tf(B_{t_1},\ldots B_{t_n}) = \sum_{i=1}^n \mathbf{1}_{[0,t_i]}(t)\frac{\partial f}{\partial x_i}(B_{t_1},\ldots B_{t_n}), \qquad 0\le t_1<\cdots<t_n, \tag{5.2.3}$$

and (5.2.2) can also be written as

$$\begin{aligned}
&\langle DF,h\rangle_{L^2(\mathbb{R}_+)} \qquad (5.2.4)\\
&= \frac{d}{d\varepsilon}f\left(\int_0^\infty u_1(t)(dB(t)+\varepsilon h(t)dt),\ldots,\int_0^\infty u_n(t)(dB(t)+\varepsilon h(t)dt)\right)_{|\varepsilon=0},
\end{aligned}$$

$$= \frac{d}{d\varepsilon} F(\omega + \epsilon h)_{|\varepsilon=0},$$

$h \in L^2(\mathbb{R}_+)$, where the limit exists in $L^2(\Omega)$. We refer to the above identity as the *probabilistic interpretation* of the gradient operator D on the Wiener space.

From Proposition 4.2.3, the operator D satisfies the Clark formula Assumption 3.2.1.

Corollary 5.2.2. *For all $F \in L^2(\Omega)$ we have*

$$F = \mathbb{E}[F] + \int_0^\infty \mathbb{E}[D_t F | \mathcal{F}_t] dB_t.$$

Moreover, since $\phi_t = 0$, $t \in \mathbb{R}_+$, Proposition 4.5.4 becomes a divergence formula as in the next proposition.

Proposition 5.2.3. *For all $u \in \mathcal{U}$ and $F \in \mathcal{S}$ we have*

$$\delta(u)F = \delta(uF) + \langle DF, u \rangle_{L^2(\mathbb{R}_+)}.$$

On the other hand, applying Proposition 4.5.6 yields the following multiplication formula for Wiener multiple stochastic integrals:

$$I_n(f_n)I_m(g_m) = \sum_{k=0}^{n\wedge m} k! \binom{n}{k} \binom{m}{k} I_{n+m-2k}(f_n \circ_k g_m),$$

where $f_n \circ_k g_m$ is the symmetrization in $n+m-2k$ variables of the contraction

$$(t_{k+1}, \ldots, t_n, s_{k+1}, \ldots, s_m) \longmapsto \\ \int_0^\infty \cdots \int_0^\infty f_n(t_1, \ldots t_n) g_m(t_1, \ldots, t_k, s_{k+1}, \ldots, s_m) dt_1 \ldots dt_k,$$

$t_{k+1}, \ldots, t_n, s_{k+1}, \ldots, s_m \in \mathbb{R}_+$.

From Proposition 4.3.4, the Skorohod integral $\delta(u)$ coincides with the Itô integral of $u \in L^2(W; H)$ with respect to Brownian motion, i.e.

$$\delta(u) = \int_0^\infty u_t dB_t,$$

when u is square-integrable and adapted with respect to the Brownian filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$.

We have the following corollary, that completes Proposition 4.2.2 and can be proved using the density property of smooth functions in finite-dimensional Sobolev spaces, cf. e.g. Lemma 1.2 of [95] or [100].

For simplicity we work with a Brownian motion $(B_t)_{t\in[0,1]}$ on $[0,1]$ and we assume that $(e_n)_{n\in\mathbb{N}}$ is the dyadic basis of $L^2([0,1])$ given by

$$e_k = 2^{n/2}\mathbf{1}_{\left[\frac{k-2^n}{2^n},\frac{k+1-2^n}{2^n}\right]}, \qquad 2^n \leq k \leq 2^{n+1}-1, \quad n \in \mathbb{N}. \tag{5.2.5}$$

Corollary 5.2.4. *Given $F \in L^2(\Omega)$ define the σ-algebra*

$$\mathcal{G}_n = \sigma(I_1(e_{2^n}),\ldots,I_1(e_{2^{n+1}-1})),$$

let $F_n = \mathbb{E}[F|\mathcal{G}_n]$, $n \in \mathbb{N}$, and consider f_n a square-integrable function with respect to the standard Gaussian measure on $\mathbb{R}^{2^n}$, such that

$$F_n = f_n(I_1(e_{2^n}),\ldots,I_1(e_{2^{n+1}-1})).$$

Then $F \in \mathrm{Dom}\,(D)$ if and only if f_n belongs for all $n \geq 1$ to the Sobolev space $W^{2,1}(\mathbb{R}^{2^n})$ with respect to the standard Gaussian measure on $\mathbb{R}^{2^n}$, and the sequence

$$D_tF_n := \sum_{i=1}^{2^n} e_{2^n+i-1}(t)\frac{\partial f_n}{\partial x_i}(I_1(e_{2^n}),\ldots,I_1(e_{2^{n+1}-1})), \qquad t \in [0,1],$$

converges in $L^2(\Omega\times[0,1])$. In this case we have

$$DF = \lim_{n\to\infty} DF_n.$$

We close this section by considering the case of a d-dimensional Brownian motion $(B_t)_{0\leq t\leq T} = (B_t^{(1)},\ldots,B_t^{(d)})_{0\leq t\leq T}$, where $(B_t^{(1)})_{t\in\mathbb{R}_+},\ldots,(B_t^{(d)})_{t\in\mathbb{R}_+}$, are independent copies of Brownian motion. In this case the gradient D can be defined with values in $H = L^2(\mathbb{R}_+, X\otimes\mathbb{R}^d)$, where X is a Hilbert space, by

$$D_tF = \sum_{i=1}^{n}\mathbf{1}_{[0,t_i]}(t)\nabla_i f(B_{t_1},\ldots,B_{t_n}), \qquad t \in \mathbb{R}_+,,$$

for F of the form

$$F = f(B_{t_1},\ldots,B_{t_n}), \tag{5.2.6}$$

$f \in \mathcal{C}_b^\infty(\mathbb{R}^n, X)$, $t_1,\ldots,t_n \in \mathbb{R}_+$, $n \geq 1$.

We let $\mathbb{D}_{p,k}(X)$ denote the completion of the space of smooth X-valued random variables under the norm

$$\|u\|_{\mathbb{D}_{p,k}(X)} = \sum_{l=0}^{k}\|D^l u\|_{L^p(W,X\otimes H^{\otimes l})}, \qquad p > 1,$$

where $X\otimes H$ denotes the completed symmetric tensor product of X and H. For all $p,q>1$ such that $p^{-1}+q^{-1}=1$ and $k\geq 1$, the Skorohod integral operator

$$\delta : \mathbb{D}_{p,k}(X \otimes H) \to \mathbb{D}_{q,k-1}(X)$$

adjoint of

$$D : \mathbb{D}_{p,k}(X) \to \mathbb{D}_{q,k-1}(X \otimes H),$$

satisfies

$$E[\langle F, \delta(u)\rangle_X] = E[\langle DF, u\rangle_{X\otimes H}],$$

$F \in \mathbb{D}_{p,k}(X)$, $u \in \mathbb{D}_{q,k}(X \otimes H)$.

Finally we note that the chaos representation property extends to d-dimensional Brownian motion.

Theorem 5.2.5. *For any $F \in L^2(\Omega)$ there exists a unique sequence $(f_n)_{n\in\mathbb{N}}$ of deterministic symmetric functions*

$$f_n = (f_n^{(i_1,\ldots,i_n)})_{i_1,\ldots,i_n\in\{1,\ldots,d\}} \in L^2([0,T], \mathbb{R}^d)^{\circ n}$$

such that

$$F = \mathbb{E}[F] + \sum_{n=1}^{\infty} I_n(f_n).$$

Moreover we have

$$\|F\|^2_{L^2(\Omega)} = \sum_{n=0}^{\infty} \sum_{i_1,\ldots,i_n=1}^{d} n!\|f_n^{(i_1,\ldots,i_n)}\|^2_{L^2([0,T]^n)}.$$

Given $F = f(B_{t_1}, \ldots, B_{t_n}) \in L^2(\Omega)$ where $(t_1, \ldots, t_n) \in [0,T]^n$ and

$$f(x^{1,1}, \ldots, x^{d,1}, \ldots, x^{1,n}, \ldots, x^{d,n})$$

is in $\mathcal{C}_b^\infty(\mathbb{R}^{dn})$, for $l = 1, \ldots, d$ we have:

$$D_t^{(l)} F = \sum_{k=1}^{n} \frac{\partial f}{\partial x^{l,k}}(B_{t_1}, \ldots, B_{t_n}) \mathbf{1}_{[0,t_k]}(t).$$

Similarly the Clark formula of Corollary 5.2.2 extends to the d-dimensional case as

$$F = \mathbb{E}[F] + \int_0^\infty \mathbb{E}[D_t F | \mathcal{F}_t] \cdot dB_t, \qquad (5.2.7)$$

$F \in L^2(\Omega)$.

5.3 Ornstein-Uhlenbeck Semi-Group

Recall the Definition 4.4.1 of the Ornstein-Uhlenbeck semi-group $(P_t)_{t\in\mathbb{R}_+}$ as

$$P_t F = \mathbb{E}[F] + \sum_{n=1}^{\infty} \mathrm{e}^{-nt} I_n(f_n), \qquad t \in \mathbb{R}_+, \tag{5.3.1}$$

for any $F \in L^2(\Omega)$ with the chaos representation

$$F = \mathbb{E}[F] + \sum_{n=1}^{\infty} I_n(f_n).$$

In this section we show that on the Wiener space, P_t admits the integral representation, known as the Mehler formula,

$$P_t F(\omega) = \int_\Omega F(\mathrm{e}^{-t}\omega + \sqrt{1-\mathrm{e}^{-2t}}\tilde{\omega}) d\mathbb{P}(\tilde{\omega}), \qquad \mathbb{P}(d\omega) - a.s., \tag{5.3.2}$$

$F \in L^2(\Omega)$, $t \in \mathbb{R}_+$, cf. e.g. [96], [147], [151]. Precisely we have the following.

Lemma 5.3.1. *Let F of the form*

$$F = f(I_1(u_1), \ldots, I_1(u_n)),$$

where $f \in \mathcal{C}_b(\mathbb{R}^n)$ and $u_1, \ldots, u_n \in L^2(\mathbb{R}_+)$ are mutually orthogonal. For all $t \in \mathbb{R}_+$ we have:

$$P_t F(\omega) = \int_\Omega f(\mathrm{e}^{-t} I_1(u_1)(\omega) + \sqrt{1-\mathrm{e}^{-2t}} I_1(u_1)(\tilde{\omega}), \ldots$$
$$\ldots, \mathrm{e}^{-t} I_1(u_n)(\omega) + \sqrt{1-\mathrm{e}^{-2t}} I_1(u_n)(\tilde{\omega}))\mathbb{P}(d\tilde{\omega}).$$

Proof. Since, by Proposition 5.1.5, the exponential vectors are total in $L^2(\Omega)$ and P_t is continuous on $L^2(\Omega)$, it suffices to consider

$$f_u(x) = \exp\left(x - \frac{1}{2}\|u\|_2^2\right),$$

and to note that by Proposition 5.1.4 we have

$$\begin{aligned} \xi(f_u) &= \exp\left(I_1(u) - \frac{1}{2}\|u\|^2_{L^2(\mathbb{R}_+)}\right) \\ &= \sum_{k=0}^{\infty} \frac{1}{n!} H_n(I_1(u); \|u\|^2_{L^2(\mathbb{R}_+)}) \\ &= \sum_{k=0}^{\infty} \frac{1}{n!} I_n(u^{\otimes n}). \end{aligned}$$

Hence

$$P_t \xi(f_u) = \sum_{k=0}^{\infty} \frac{\mathrm{e}^{-nt}}{n!} I_n(u^{\otimes n})$$

$$= \exp\left(e^{-t}I_1(u) - \frac{1}{2}e^{-2t}\|u\|^2_{L^2(\mathbb{R}_+)}\right),$$

and

$$\begin{aligned}
&\int_\Omega f_u(e^{-t}I_1(u)(\omega) + \sqrt{1-e^{-2t}}I_1(u)(\tilde{\omega}))\mathbb{P}(d\tilde{\omega})\\
&= \int_{-\infty}^{\infty} \exp\left(e^{-t}I_1(u)(\omega) + \sqrt{1-e^{-2t}}y - \frac{1}{2}\|u\|_2^2 - \frac{y^2}{2\|u\|_2^2}\right)\frac{dy}{\sqrt{2\pi}\|u\|_2}\\
&= \int_{-\infty}^{\infty} \exp\left(e^{-t}I_1(u)(\omega) - \frac{1}{2}\|e^{-t}u\|_2^2 - \frac{(y-\sqrt{1-e^{-2t}}\|u\|_2^2)^2}{2\|u\|_2^2}\right)\frac{dy}{\sqrt{2\pi}\|u\|_2}\\
&= \exp\left(e^{-t}I_1(u)(\omega) - \frac{1}{2}\|e^{-t}u\|_2^2\right)\\
&= P_t f_u(I_1(u))(\omega).
\end{aligned}$$

The result is extended by density of the exponential vectors in $L^2(\Omega)$ since Brownian motion has the chaos representation property from Proposition 5.2.5. □

Lemma 5.3.1 implies the following bound which is used in the proof of deviation inequalities in Section 4.7, cf. Lemma 4.7.1.

Lemma 5.3.2. *We have for* $u \in L^2(\Omega \times \mathbb{R}_+)$*:*

$$\|P_t u\|_{L^\infty(\Omega, L^2(\mathbb{R}_+))} \leq \|u\|_{L^\infty(\Omega, L^2(\mathbb{R}_+))}, \qquad t \in \mathbb{R}_+.$$

Proof. Using the integral representation of Lemma 5.3.1 together with Jensen's inequality (9.3.1) we have

$$\begin{aligned}
\|P_s u(\omega)\|^2_{L^2(\mathbb{R}_+)} &= \int_0^\infty |P_s u_t(\omega)|^2 dt\\
&\leq \int_0^\infty P_s|u_t(\omega)|^2 dt\\
&\leq \|u\|^2_{L^\infty(\Omega, L^2(\mathbb{R}_+))}.
\end{aligned}$$

□

5.4 Covariance Identities and Inequalities

In this section we present some covariance identities and inequalities that can be obtained in the particular setting of the Wiener space, in addition to the general results of Section 3.4, 3.5 and 4.4, and to the logarithmic Sobolev inequality (3.5.4).

We consider the order relation introduced in [12] when $\Omega = \mathcal{C}_0(\mathbb{R}_+)$ is the space of continuous functions on $\mathbb{R}_+$ starting at 0.

Definition 5.4.1. *Given $\omega_1, \omega_2 \in \Omega$, we say that $\omega_1 \preceq \omega_2$ if and only if we have*

$$\omega_1(t_2) - \omega_1(t_1) \leq \omega_2(t_2) - \omega_2(t_1), \qquad 0 \leq t_1 \leq t_2.$$

The class of non-decreasing functionals with respect to $\preceq$ is larger than that of non-decreasing functionals with respect to the pointwise order on Ω defined by

$$\omega_1(t) \leq \omega_2(t), \qquad t \in \mathbb{R}_+, \quad \omega_1, \omega_2 \in \Omega.$$

Definition 5.4.2. *A random variable $F : \Omega \to \mathbb{R}$ is said to be non-decreasing if*

$$\omega_1 \preceq \omega_2 \;\Rightarrow\; F(\omega_1) \leq F(\omega_2), \qquad \mathbb{P}(d\omega_1) \otimes \mathbb{P}(d\omega_2) - a.s.$$

The next result is the FKG inequality on the Wiener space. It recovers Theorem 4 of [12] under weaker (i.e. almost-sure) hypotheses.

Theorem 5.4.3. *For any non-decreasing functionals $F, G \in L^2(\Omega)$ we have*

$$\mathrm{Cov}\,(F, G) \geq 0.$$

The proof of this result is a direct consequence of Lemma 3.4.2 and the next lemma.

Lemma 5.4.4. *For every non-decreasing $F \in \mathrm{Dom}\,(D)$ we have*

$$D_t F \geq 0, \qquad dt \times d\mathbb{P} - a.e.$$

Proof. Without loss of generality we state the proof for a Brownian motion on the interval $[0, 1]$. Let

$$H = \left\{ h : [0,1] \to \mathbb{R} \;:\; \int_0^1 |\dot{h}(s)|^2 ds < \infty \right\}$$

denote the Cameron-Martin space, i.e. the space of absolutely continuous functions with square-integrable derivative.

For $n \in \mathbb{N}$, let π_n denotes the orthogonal projection from $L^2([0,1])$ onto the linear space generated by the sequence $(e_k)_{2^n \leq k < 2^{n+1}}$ introduced in (5.2.5). Given $h \in H$, let

$$h_n(t) = \int_0^t [\pi_n \dot{h}](s) ds, \qquad t \in [0,1], \quad n \in \mathbb{N}.$$

Let Λ_n denote the square-integrable and $\mathcal{G}_n$-measurable random variable

$$\Lambda_n = \exp\left(\int_0^1 [\pi_n \dot{h}](s) dB_s - \frac{1}{2} \int_0^1 |[\pi_n \dot{h}](s)|^2 ds \right),$$

where $\mathcal{G}_n$ is defined in Corollary 5.2.4. Letting $F_n = \mathbb{E}[F \mid \mathcal{G}_n]$, $n \in \mathbb{N}$, a suitable change of variable on $\mathbb{R}^n$ with respect to the standard Gaussian density (or an application of the Cameron-Martin theorem cf. e.g. [150]) shows that for all $n \in \mathbb{N}$ and $\mathcal{G}_n$-measurable bounded random variable G_n shows that

$$\begin{aligned}\mathbb{E}[F_n(\cdot + h_n)G_n] &= \mathbb{E}[\Lambda_n F_n G_n(\cdot - h_n)] \\ &= \mathbb{E}[\Lambda_n \mathbb{E}[F|\mathcal{G}_n] G_n(\cdot - h_n)] \\ &= \mathbb{E}[\mathbb{E}[\Lambda_n F G_n(\cdot - h_n)|\mathcal{G}_n]] \\ &= \mathbb{E}[\Lambda_n F G_n(\cdot - h_n)] \\ &= \mathbb{E}[F(\cdot + h_n)G_n],\end{aligned}$$

hence

$$F_n(\omega + h_n) = \mathbb{E}[F(\cdot + h_n)|\mathcal{G}_n](\omega), \qquad \mathbb{P}(d\omega) - a.s.$$

If $\dot{h}$ is non-negative, then $\pi_n \dot{h}$ is non-negative by construction hence $\omega \preceq \omega + h_n$, $\omega \in \Omega$, and we have

$$F(\omega) \leq F(\omega + h_n), \qquad \mathbb{P}(d\omega) - a.s.,$$

since from the Cameron-Martin theorem, $\mathbb{P}(\{\omega + h_n \ : \ \omega \in \Omega\}) = 1$. Hence we have

$$\begin{aligned}&F_n(\omega + h) \\ &= f_n(I_1(e_{2^n}) + \langle e_{2^n}, \dot{h}\rangle_{L^2([0,1])}, \ldots, I_1(e_{2^{n+1}-1}) + \langle e_{2^{n+1}-1}, \dot{h}\rangle_{L^2([0,1])}) \\ &= f_n(I_1(e_{2^n}) + \langle e_{2^n}, \pi_n\dot{h}\rangle_{L^2([0,1])}, \ldots, I_1(e_{2^{n+1}-1}) + \langle e_{2^{n+1}-1}, \pi_n\dot{h}\rangle_{L^2([0,1])}) \\ &= F_n(\omega + h_n) \\ &= \mathbb{E}[F(\cdot + h_n)|\mathcal{G}_n](\omega) \\ &\geq \mathbb{E}[F|\mathcal{G}_n](\omega) \\ &= F_n(\omega), \qquad \mathbb{P}(d\omega) - a.s.,\end{aligned}$$

where $(e_k)_{k\in\mathbb{N}}$ is the dyadic basis defined in (5.2.5). Consequently, for any $\varepsilon_1 \leq \varepsilon_2$ and $h \in H$ such that $\dot{h}$ is non-negative we have

$$F_n(\omega + \varepsilon_1 h) \leq F_n(\omega + \varepsilon_2 h),$$

i.e. the smooth function $\varepsilon \longmapsto F_n(\omega + \varepsilon h)$ is non-decreasing in ε on $[-1, 1]$, $\mathbb{P}(d\omega)$-a.s. As a consequence,

$$\langle DF_n, \dot{h}\rangle_{L^2([0,1])} = \frac{d}{d\varepsilon} F_n(\omega + \epsilon h)_{|\varepsilon=0} \geq 0,$$

for all $h \in H$ such that $\dot{h} \geq 0$, hence $DF_n \geq 0$. Taking the limit of $(DF_n)_{n\in\mathbb{N}}$ as n goes to infinity shows that $DF \geq 0$. □

Next, we extend Lemma 5.4.4 to $F \in L^2(\Omega)$.

Proposition 5.4.5. *For any non-decreasing functional $F \in L^2(\Omega)$ we have*

$$\mathbb{E}[D_t F|\mathcal{F}_t] \geq 0, \qquad dt \times d\mathbb{P} - a.e.$$

Proof. Assume that $F \in L^2(\Omega)$ is non-decreasing. Then $P_{1/n}F$, $n \geq 1$, is non-decreasing from (5.3.2), and belongs to Dom (D) from Relation (5.3.1). From Lemma 5.4.4 we have

$$D_t P_{1/n} F \geq 0, \qquad dt \times d\mathbb{P} - a.e.,$$

hence

$$\mathbb{E}[D_t P_{1/n} F|\mathcal{F}_t] \geq 0, \qquad dt \times d\mathbb{P} - a.e.$$

Taking the limit as n goes to infinity yields $\mathbb{E}[D_t F|\mathcal{F}_t] \geq 0$, $dt \times d\mathbb{P}$-a.e. from Proposition 3.2.6 and the fact that $P_{1/n}F$ converges to F in $L^2(\Omega)$ as n goes to infinity. □

Finally, using the change of variable $\alpha = \mathrm{e}^{-t}$, the covariance identity (4.4.2) can be rewritten with help of Lemma 5.3.1 as

$$\begin{aligned}
&\mathrm{Cov}\,(F,G) \\
&= \int_0^1 \int_\Omega \int_\Omega \langle \nabla f(I_1(u_1),\ldots,I_1(u_n))(\omega), \nabla g(\alpha I_1(u_1)(\omega) + \sqrt{1-\alpha^2} I_1(u_1)(\tilde{\omega}), \\
&\quad \ldots, \alpha I_1(u_n)(\omega) + \sqrt{1-\alpha^2} I_1(u_n)(\tilde{\omega}))\rangle_{\mathbb{R}^n} \mathbb{P}(d\omega)\mathbb{P}(d\tilde{\omega})d\alpha.
\end{aligned}$$

This identity can be recovered using characteristic function: letting

$$\varphi(t) = \mathbb{E}[\mathrm{e}^{itI_1(u)}] = \mathrm{e}^{-t^2\|u\|_2^2/2}$$

and

$$\varphi_\alpha(s,t) := \mathbb{E}[\mathrm{e}^{is\alpha I_1(u)(\omega) + it\sqrt{1-\alpha^2} I_1(u)(\tilde{\omega})}] = (\varphi(s+t))^\alpha (\varphi(s))^{1-\alpha} (\varphi(t))^{1-\alpha},$$

we have

$$\begin{aligned}
&\mathrm{Var}\,[\mathrm{e}^{isI_1(u)}] = \varphi_1(s,t) - \varphi_0(s,t) \\
&= \int_0^1 \frac{\partial \varphi_\alpha}{\partial \alpha}(s,t) d\alpha \\
&= \int_0^1 \frac{\partial}{\partial \alpha}((\varphi(t))^{1-\alpha}(\varphi(t+s))^\alpha (\varphi(s))^{1-\alpha}) d\alpha \\
&= \int_0^1 \log\left(\frac{\varphi(s+t)}{\varphi(s)\varphi(t)}\right) \varphi_\alpha(s,t) d\alpha \\
&= -st\|u\|^2_{L^2(\mathbb{R}_+)} \int_0^1 \varphi_\alpha(s,t) d\alpha \\
&= \int_0^1 \int_\Omega \int_\Omega \langle D\mathrm{e}^{isI_1(u)}(\omega), D\mathrm{e}^{itI_1(u)}(\alpha\omega + \sqrt{1-\alpha^2}\tilde{\omega})\rangle_{L^2(\mathbb{R}_+)} \mathbb{P}(d\omega)\mathbb{P}(d\tilde{\omega}) d\alpha,
\end{aligned}$$

hence

$$\begin{aligned}
&\text{Cov}\,(\mathrm{e}^{isI_1(u)}, \mathrm{e}^{isI_1(v)}) \\
&= \int_0^1 \int_\Omega \int_\Omega \langle D\mathrm{e}^{isI_1(u)}(\omega), D\mathrm{e}^{itI_1(v)}(\alpha\omega + \sqrt{1-\alpha^2}\tilde{\omega}))\rangle_{L^2(\mathbb{R}_+)} \mathbb{P}(d\omega)\mathbb{P}(d\tilde{\omega})d\alpha.
\end{aligned}$$

Since D is a derivation operator from Proposition 5.2.1, the deviation results of Proposition 3.6.1 hold, i.e. for any $F \in \text{Dom}\,(D)$ such that $\|DF\|_{L^2(\mathbb{R}_+, L^\infty(\Omega))} \leq C$ for some $C > 0$ we have

$$\mathbb{P}(F - \mathbb{E}[F] \geq x) \leq \exp\left(-\frac{x^2}{2C\|DF\|_{L^2(\mathbb{R}_+, L^\infty(\Omega))}}\right), \qquad x \geq 0.$$

5.5 Moment identities for Skorohod integrals

In this section we prove a moment identity that extends the Skorohod isometry to arbitrary powers of the Skorohod integral on the Wiener space. As simple consequences of this identity we obtain sufficient conditions for the Gaussianity of the law of the Skorohod integral and a recurrence relation for the moments of second order Wiener integrals.

Here, $(B_t)_{t\in\mathbb{R}_+}$ is a standard $\mathbb{R}^d$-valued Brownian motion on the Wiener space (W, μ) with $W = \mathcal{C}_0(\mathbb{R}_+, \mathbb{R}^d)$.

Each element of $X \otimes H$ is naturally identified to a linear operator from H to X via

$$(a \otimes b)c = a\langle b, c\rangle, \qquad a \otimes b \in X \otimes H, \quad c \in H.$$

For $u \in \mathbb{D}_{2,1}(H)$ we identify $Du = (D_t u_s)_{s,t\in\mathbb{R}_+}$ to the random operator $Du : H \to H$ almost surely defined by

$$(Du)v(s) = \int_0^\infty (D_t u_s) v_t dt, \quad s \in \mathbb{R}_+, \quad v \in L^2(W; H),$$

and define its adjoint $D^* u$ on $H \otimes H$ as

$$(D^* u)v(s) = \int_0^\infty (D_s^\dagger u_t) v_t dt, \qquad s \in \mathbb{R}_+, \quad v \in L^2(W; H),$$

where $D_s^\dagger u_t$ denotes the transpose matrix of $D_s u_t$ in $\mathbb{R}^d \otimes \mathbb{R}^d$.

The Skorohod isometry of Proposition 4.3.1 reads

$$\mathbb{E}[|\delta(u)|^2] = \mathbb{E}[\langle u, u\rangle_H] + \mathbb{E}\left[\text{trace}\,(Du)^2\right], \qquad u \in \mathbb{D}_{2,1}(H), \qquad (5.5.1)$$

with

$$\text{trace}\,(Du)^2 = \langle Du, D^* u\rangle_{H\otimes H}$$

$$= \int_0^\infty \int_0^\infty \langle D_s u_t, D_t^\dagger u_s \rangle_{\mathbb{R}^d \otimes \mathbb{R}^d} ds dt,$$

and the commutation relation

$$D\delta(u) = u + \delta(D^* u), \qquad u \in \mathbb{D}_{2,2}(H). \tag{5.5.2}$$

Next we state a moment identity for Skorohod integrals.

Theorem 5.5.1. *For any $n \geq 1$ and $u \in \mathbb{D}_{n+1,2}(H)$ we have*

$$\mathbb{E}[(\delta(u))^{n+1}] = \sum_{k=1}^{n} \frac{n!}{(n-k)!} \mathbb{E}\Big[(\delta(u))^{n-k} \tag{5.5.3}$$

$$\left(\langle (Du)^{k-1} u, u \rangle_H + \text{trace}\,(Du)^{k+1} + \sum_{i=2}^{k} \frac{1}{i} \langle (Du)^{k-i} u, D\text{trace}\,(Du)^i \rangle_H \right) \Big],$$

where

$$\text{trace}\,(Du)^{k+1}$$
$$= \int_0^\infty \cdots \int_0^\infty \langle D_{t_{k-1}}^\dagger u_{t_k}, D_{t_{k-2}} u_{t_{k-1}} \cdots D_{t_0} u_{t_1} D_{t_k} u_{t_0} \rangle_{\mathbb{R}^d \otimes \mathbb{R}^d} dt_0 \cdots dt_k.$$

For $n = 1$ the above identity coincides with the Skorohod isometry (5.5.1). The proof of Theorem 5.5.1 will be given at the end of this section.

In particular we obtain the following immediate consequence of Theorem 5.5.1. Recall that $\text{trace}\,(Du)^k = 0$, $k \geq 2$, when the process u is adapted with respect to the Brownian filtration.

Corollary 5.5.2. *Let $n \geq 1$ and $u \in \mathbb{D}_{n+1,2}(H)$ such that $\langle u, u \rangle_H$ is deterministic and*

$$\text{trace}\,(Du)^{k+1} + \sum_{i=2}^{k} \frac{1}{i} \langle (Du)^{k-i} u, D\text{trace}\,(Du)^i \rangle_H = 0, \quad a.s., \qquad 1 \leq k \leq n. \tag{5.5.4}$$

Then $\delta(u)$ has the same first $n+1$ moments as the centered Gaussian distribution with variance $\langle u, u \rangle_H$.

Proof. We have

$$\begin{aligned} D_t \langle u, u \rangle &= D_t \int_0^\infty \langle u_s, u_s \rangle ds \\ &= \int_0^\infty \langle u_s, D_t u_s \rangle ds + \int_0^\infty \langle D_t u_s, u_s \rangle ds \\ &= 2 \int_0^\infty \langle D_t^\dagger u_s, u_s \rangle ds \\ &= 2(D^* u) u, \end{aligned}$$

shows that

$$\begin{aligned}\langle (D^{k-1}u)u, u\rangle &= \langle (D^*u)^{k-1}u, u\rangle \\ &= \frac{1}{2}\langle u, (D^*)^{k-2}D\langle u, u\rangle\rangle \\ &= 0,\end{aligned} \tag{5.5.5}$$

$k \geq 2$, when $\langle u, u\rangle$ is deterministic, $u \in \mathbb{D}_{2,1}(H)$. Hence under Condition (5.5.4), Theorem 5.5.1 yields

$$\mathbb{E}[(\delta(u))^{n+1}] = n\langle u, u\rangle_H \, \mathbb{E}\left[(\delta(u))^{n-1}\right],$$

and by induction

$$\mathbb{E}[(\delta(u))^{2m}] = \frac{(2m)!}{2^m m!}\langle u, u\rangle_H^m, \qquad 0 \leq 2m \leq n+1,$$

and $\mathbb{E}[(\delta(u))^{2m+1}] = 0$, $0 \leq 2m \leq n$, while $\mathbb{E}[\delta(u)] = 0$ for all $u \in \mathbb{D}_{2,1}(H)$. □

As a consequence of Corollary 5.5.2 we recover Theorem 2.1-b) of [149], i.e. $\delta(Rh)$ has a centered Gaussian distribution with variance $\langle h, h\rangle_H$ when $u = Rh$, $h \in H$, and R is a random mapping with values in the isometries of H, such that $Rh \in \cap_{p>1}\mathbb{D}_{p,2}(H)$ and $\operatorname{trace}(DRh)^{k+1} = 0$, $k \geq 1$. Note that in [149] the condition $Rh \in \cap_{p>1,k\geq 2}\mathbb{D}_{p,k}(H)$ is assumed instead of $Rh \in \cap_{p>1}\mathbb{D}_{p,2}(H)$.

In the sequel, all scalar products will be simply denoted by $\langle \cdot, \cdot\rangle$. We will need the following lemma.

Lemma 5.5.3. *Let $n \geq 1$ and $u \in \mathbb{D}_{n+1,2}(H)$. Then for all $1 \leq k \leq n$ we have*

$$\begin{aligned}&\mathbb{E}\left[(\delta(u))^{n-k}\langle (Du)^{k-1}u, D\delta(u)\rangle\right] - (n-k)\,\mathbb{E}\left[(\delta(u))^{n-k-1}\langle (Du)^k u, D\delta(u)\rangle\right] \\ &= \mathbb{E}\Bigg[(\delta(u))^{n-k} \\ &\quad \left(\langle (Du)^{k-1}u, u\rangle + \operatorname{trace}(Du)^{k+1} + \sum_{i=2}^{k}\frac{1}{i}\langle (Du)^{k-i}u, D\operatorname{trace}(Du)^i\rangle\right)\Bigg].\end{aligned}$$

Proof. We have $(Du)^{k-1}u \in \mathbb{D}_{(n+1)/k,1}(H)$, $\delta(u) \in \mathbb{D}_{(n+1)/(n-k+1),1}(\mathbb{R})$, and using Relation (5.5.2) we obtain

$$\begin{aligned}&\mathbb{E}\left[(\delta(u))^{n-k}\langle (Du)^{k-1}u, D\delta(u)\rangle\right] \\ &= \mathbb{E}\left[(\delta(u))^{n-k}\langle (Du)^{k-1}u, u + \delta(D^*u)\rangle\right] \\ &= \mathbb{E}\left[(\delta(u))^{n-k}\langle (Du)^{k-1}u, u\rangle\right] + \mathbb{E}\left[(\delta(u))^{n-k}\langle (Du)^{k-1}u, \delta(Du)\rangle\right] \\ &= \mathbb{E}\left[(\delta(u))^{n-k}\langle (Du)^{k-1}u, u\rangle\right] + \mathbb{E}\left[\langle D^*u, D((\delta(u))^{n-k}(Du)^{k-1}u)\rangle\right]\end{aligned}$$

$$
\begin{aligned}
&= \mathbb{E}\left[(\delta(u))^{n-k}\langle (Du)^{k-1}u, u\rangle\right] + \mathbb{E}\left[(\delta(u))^{n-k}\langle D^*u, D((Du)^{k-1}u)\rangle\right] \\
&\quad + \mathbb{E}\left[\langle D^*u, ((Du)^{k-1}u) \otimes D(\delta(u))^{n-k}\rangle\right] \\
&= \mathbb{E}\left[(\delta(u))^{n-k}\left(\langle (Du)^{k-1}u, u\rangle + \langle D^*u, D((Du)^{k-1}u)\rangle\right)\right] \\
&\quad +(n-k)\,\mathbb{E}\left[(\delta(u))^{n-k-1}\langle D^*u, ((Du)^{k-1}u) \otimes D\delta(u)\rangle\right] \\
&= \mathbb{E}\left[(\delta(u))^{n-k}\left(\langle (Du)^{k-1}u, u\rangle + \langle D^*u, D((Du)^{k-1}u)\rangle\right)\right] \\
&\quad +(n-k)\,\mathbb{E}\left[(\delta(u))^{n-k-1}\langle (Du)^{k}u, D\delta(u)\rangle\right].
\end{aligned}
$$

Next,

$$
\begin{aligned}
&\langle D^*u, D((Du)^{k-1}u)\rangle \\
&= \int_0^\infty \cdots \int_0^\infty \langle D^\dagger_{t_{k-1}} u_{t_k}, D_{t_k}(D_{t_{k-2}} u_{t_{k-1}} \cdots D_{t_0} u_{t_1} u_{t_0})\rangle dt_0 \cdots dt_k \\
&= \int_0^\infty \cdots \int_0^\infty \langle D^\dagger_{t_{k-1}} u_{t_k}, D_{t_{k-2}} u_{t_{k-1}} \cdots D_{t_0} u_{t_1} D_{t_k} u_{t_0}\rangle dt_0 \cdots dt_k \\
&\quad + \int_0^\infty \cdots \int_0^\infty \langle D^\dagger_{t_{k-1}} u_{t_k}, D_{t_k}(D_{t_{k-2}} u_{t_{k-1}} \cdots D_{t_0} u_{t_1}) u_{t_0}\rangle dt_0 \cdots dt_k \\
&= \mathrm{trace}\,(Du)^{k+1} + \sum_{i=0}^{k-2} \int_0^\infty \cdots \int_0^\infty \\
&\quad \langle D^\dagger_{t_{k-1}} u_{t_k}, D_{t_k} u_{t_{k+1}} \cdots D_{t_{i+1}} u_{t_{i+2}} (D_{t_i} D_{t_k} u_{t_{i+1}}) D_{t_{i-1}} u_{t_i} \cdots D_{t_0} u_{t_1} u_{t_0}\rangle \\
&\quad dt_0 \cdots dt_k \\
&= \mathrm{trace}\,(Du)^{k+1} + \sum_{i=0}^{k-2} \frac{1}{k-i} \int_0^\infty \cdots \int_0^\infty \\
&\quad \langle D_{t_i} \langle D^\dagger_{t_{k-1}} u_{t_k}, D_{t_k} u_{t_{k+1}} \cdots D_{t_{i+1}} u_{t_{i+2}} D_{t_k} u_{t_{i+1}}\rangle, D_{t_{i-1}} u_{t_i} \cdots D_{t_0} u_{t_1} u_{t_0}\rangle \\
&\quad dt_0 \cdots dt_k \\
&= \mathrm{trace}\,(Du)^{k+1} + \sum_{i=0}^{k-2} \frac{1}{k-i} \langle (Du)^i u, D\mathrm{trace}\,(Du)^{k-i}\rangle.
\end{aligned}
$$

□

Proof of Theorem 5.5.1. We decompose

$$
\begin{aligned}
&\mathbb{E}[(\delta(u))^{n+1}] = \mathbb{E}[\langle u, D(\delta(u))^n\rangle] \\
&= n\,\mathbb{E}[(\delta(u))^{n-1}\langle u, D\delta(u)\rangle] \\
&= \sum_{k=1}^{n} \frac{n!}{(n-k)!}\,\mathbb{E}\left[(\delta(u))^{n-k}\langle (Du)^{k-1}u, D\delta(u)\rangle\right] \\
&\quad - \sum_{k=1}^{n} \frac{n!}{(n-k)!}(n-k)\,\mathbb{E}\left[(\delta(u))^{n-k-1}\langle (Du)^{k}u, D\delta(u)\rangle\right],
\end{aligned}
$$

as a telescoping sum and then apply Lemma 5.5.3, which yields (5.5.3). □

5.6 Differential Calculus on Random Morphisms

In this section, in addition to the shift of Brownian paths by absolutely continuous functions as in (5.2.4), we consider a general class of transformations of Brownian motion and its associated differential calculus. The main result Corollary 5.6.5 of this section will be applied in Section 5.7 to construct another example of a gradient operator satisfying the assumptions of Chapter 3. Here we work with a d-dimensional Brownian motion $(B_t)_{t\in\mathbb{R}_+}$ as in Section 2.14. Let

$$U : \mathcal{S}(\mathbb{R}_+;\mathbb{R}^d) \longrightarrow L^2(\Omega\times\mathbb{R}_+;\mathbb{R}^d)$$

be a random linear operator such that $Uf \in L^2(\Omega\times\mathbb{R}_+;\mathbb{R}^d)$ is adapted for all f in a space $\mathcal{S}(\mathbb{R}_+;\mathbb{R}^d)$ of functions dense in $L^2(\mathbb{R}_+;\mathbb{R}^d)$.

The operator U is extended by linearity to the algebraic tensor product $\mathcal{S}(\mathbb{R}_+;\mathbb{R}^d)\otimes\mathcal{S}$, in this case Uf is not necessarily adapted if $f\in\mathcal{S}(\mathbb{R}_+;\mathbb{R}^d)\otimes\mathcal{S}$.

Definition 5.6.1. *Let $(h(t))_{t\in\mathbb{R}_+} \in L^2(\Omega\times\mathbb{R}_+;\mathbb{R}^d)$ be a square-integrable process, and let the transformation*

$$\Lambda(U,h) : \mathcal{S} \longrightarrow L^2(\Omega\times\mathbb{R}_+;\mathbb{R}^d)$$

be defined as

$$\Lambda(U,h)F$$
$$= f\left(I_1(Uu_1) + \int_0^\infty \langle u_1(t), h(t)\rangle dt, \ldots, I_1(Uu_n) + \int_0^\infty \langle u_n(t), h(t)\rangle dt\right),$$

for $F\in\mathcal{S}$ of the form

$$F = f(I_1(u_1),\ldots,I_1(u_n)),$$

$u_1,\ldots,u_n \in \mathcal{S}(\mathbb{R}_+;\mathbb{R}^d)$, $f\in\mathcal{C}_b^\infty(\mathbb{R}^n;\mathbb{R})$.

In the particular case where

$$U : \mathcal{S}(\mathbb{R}_+;\mathbb{R}^d) \longrightarrow L^2(\Omega\times\mathbb{R}_+;\mathbb{R}^d)$$

is given as

$$[Uf](t) = V(t)f(t), \qquad t\in\mathbb{R}_+,$$

by an adapted family of random endomorphisms

$$V(t) : \mathbb{R}^d \longrightarrow \mathbb{R}^d, \qquad t\in\mathbb{R}_+,$$

this definition states that $\Lambda(U,h)F$ is the evaluation of F on the perturbed process of differential $V^*(t)dB(t) + h(t)dt$ instead of $dB(t)$, where

$$V^*(t) : \mathbb{R}^d \longrightarrow \mathbb{R}^d$$

denotes the dual of $V(t) : \mathbb{R}^d \longrightarrow \mathbb{R}^d$, $t \in \mathbb{R}_+$.

We are going to define $\Lambda(U, h)$ on the space $\mathcal{S}$ of smooth functionals. For this we need to show that the definition of $\Lambda(U, h)F$ is independent of the particular representation

$$F = f(I_1(u_1), \ldots, I_1(u_n)), \qquad u_1, \ldots, u_n \in H,$$

chosen for $F \in \mathcal{S}$.

Lemma 5.6.2. *Let $F, G \in \mathcal{S}$ be written as*

$$F = f\left(I_1(u_1), \ldots, I_1(u_n)\right), \qquad u_1, \ldots, u_n \in \mathcal{S}(\mathbb{R}_+; \mathbb{R}^d), \quad f \in \mathcal{C}^1(\mathbb{R}^n; \mathbb{R}),$$

and

$$G = g\left(I_1(v_1), \ldots, I_1(v_m)\right), \qquad v_1, \ldots, v_m \in \mathcal{S}(\mathbb{R}_+; \mathbb{R}^d), \quad g \in \mathcal{C}^1(\mathbb{R}^m; \mathbb{R}).$$

If $F = G$ $\mathbb{P}$-a.s. then $\Lambda(U, h)F = \Lambda(U, h)G$, $\mathbb{P}$-a.s.

Proof. Let $e_1, \ldots, e_k \in \mathcal{S}(\mathbb{R}_+; \mathbb{R}^d)$ be orthonormal vectors that generate $u_1, \ldots, u_n$, $v_1, \ldots, v_m$. Assume that u_i and v_i are written as

$$u_i = \sum_{j=1}^k \alpha_i^j e_j \quad \text{and} \quad v_i = \sum_{j=1}^k \beta_i^j e_j, \qquad i = 1, \ldots, n,$$

in the basis $e_1, \ldots, e_k$. Then F and G are also represented as

$$F = \tilde{f}\left(I_1(e_1), \ldots, I_1(e_k)\right),$$

and $G = \tilde{g}\left(I_1(e_1), \ldots, I_1(e_k)\right)$, where the functions $\tilde{f}$ and $\tilde{g}$ are defined by

$$\tilde{f}(x_1, \ldots, x_k) = f\left(\sum_{j=1}^k \alpha_1^j x_j, \ldots, \sum_{j=1}^k \alpha_n^j x_j\right), \qquad x_1, \ldots, x_k \in \mathbb{R},$$

and

$$\tilde{g}(x_1, \ldots, x_k) = f\left(\sum_{j=1}^k \beta_1^j x_j, \ldots, \sum_{j=1}^k \beta_n^j x_j\right), \qquad x_1, \ldots, x_k \in \mathbb{R}.$$

Since $F = G$ and $I_1(e_1), \ldots, I_1(e_k)$ are independent, we have $\tilde{f} = \tilde{g}$ *a.e.*, hence everywhere, and by linearity,

$$\Lambda(U, h)F = \tilde{f}\left(I_1(Ue_1) + \int_0^\infty \langle e_1(t), h(t)\rangle dt, \ldots, I_1(Ue_k) + \int_0^\infty \langle e_k(t), h(t)\rangle dt\right),$$

and

$$\Lambda(U,h)G$$
$$= \tilde{g}\left(I_1(Ue_1) + \int_0^\infty \langle e_1(t), h(t)\rangle dt, \ldots, I_1(Ue_k) + \int_0^\infty \langle e_k(t), h(t)\rangle dt\right),$$

hence $\Lambda(U,h)F = \Lambda(U,h)G$. □

Moreover, $\Lambda(U,h)$ is linear and multiplicative:

$$\Lambda(U,h)f(F_1,\ldots,F_n) = f(\Lambda(U,h)F_1,\ldots,\Lambda(U,h)F_n),$$

$F_1,\ldots,F_n \in \mathcal{S}$, $f \in \mathcal{C}_b^1(\mathbb{R}^n;\mathbb{R})$.

Definition 5.6.3. *Let $(U_\varepsilon)_{\varepsilon\in[0,1]}$ be a family of linear operators*

$$U_\varepsilon : \mathcal{S}(\mathbb{R}_+;\mathbb{R}^d) \longrightarrow L^2(\Omega\times\mathbb{R}_+;\mathbb{R}^d),$$

such that

i) U_0 is the identity of $\mathcal{S}(\mathbb{R}_+;\mathbb{R}^d)$, i.e. we have $U_0 f = f$, $\mathbb{P}$-a.s., $f \in \mathcal{S}(\mathbb{R}_+;\mathbb{R}^d)$.

ii) for any $f \in \mathcal{S}(\mathbb{R}_+;\mathbb{R}^d)$, $U_\varepsilon f \in L^2(\Omega\times\mathbb{R}_+;\mathbb{R}^d)$ and is adapted, $\varepsilon \in [0,1]$,

iii) the family $(U_\varepsilon)_{\varepsilon\in[0,1]}$ admits a derivative at $\varepsilon = 0$ in the form of an operator

$$\mathcal{L} : \mathcal{S}(\mathbb{R}_+;\mathbb{R}^d) \longrightarrow L^2(\Omega\times\mathbb{R}_+;\mathbb{R}^d),$$

such that

$$((U_\varepsilon f - f)/\varepsilon)_{\varepsilon\in[0,1]}$$

converges in $L^2(\Omega\times\mathbb{R}_+;\mathbb{R}^d)$ to $\mathcal{L}f = (\mathcal{L}_t f)_{t\in\mathbb{R}_+}$ as ε goes to zero, $f \in \mathcal{S}(\mathbb{R}_+;\mathbb{R}^d)$.

Let $h \in L^2(\Omega\times\mathbb{R}_+;\mathbb{R}^d)$ be a square-integrable adapted process.

The operator $\mathcal{L}$ is extended by linearity to $\mathcal{S}(\mathbb{R}_+;\mathbb{R}^d)\otimes\mathcal{S}$. The family $(U_\varepsilon)_{\varepsilon\in[0,1]}$ needs not have the semigroup property. The above assumptions imply that $\mathcal{L}DF \in \mathrm{Dom}(\delta)$, $F \in \mathcal{S}$, with

$$\delta(\mathcal{L}DF) = \sum_{i=1}^n \partial_i f(I_1(u_1),\ldots,I_1(u_n))\delta(\mathcal{L}u_i) \qquad (5.6.1)$$
$$- \sum_{i,j=1}^n \langle u_i, \mathcal{L}u_j\rangle_{L^2(\mathbb{R}_+;\mathbb{R}^d)} \partial_i\partial_j f(I_1(u_1),\ldots,I_1(u_n)),$$

for $F = f(I_1(u_1),\ldots,I_1(u_n))$, where we used Relation (5.2.3). We now compute on $\mathcal{S}$ the derivative at $\varepsilon = 0$ of one-parameter families

$$\Lambda(U_\varepsilon, \varepsilon h) : \mathcal{S} \longrightarrow L^2(\Omega), \qquad \varepsilon \in \mathbb{R},$$

of transformations of Brownian functionals. Let the linear operator trace be defined on the algebraic tensor product $H \otimes H$ as

$$\mathrm{trace}\, u \otimes v = (u, v)_H, \qquad u, v \in H.$$

Proposition 5.6.4. *For $F \in \mathcal{S}$, we have in $L^2(\Omega)$:*

$$\frac{d}{d\varepsilon}\Lambda(U_\varepsilon, \varepsilon h)F_{|\varepsilon=0} = \int_0^\infty \langle h_0(t), D_t F\rangle dt + \delta(\mathcal{L}DF) + \mathrm{trace}\,(\mathrm{Id}_H \otimes \mathcal{L})DDF. \tag{5.6.2}$$

Proof. Let $A : \mathcal{S} \longrightarrow \mathcal{S}$ be defined by

$$AF = \delta(\mathcal{L}DF) + \mathrm{trace}\,(\mathrm{Id}_H \otimes \mathcal{L})DDF + \int_0^\infty \langle h_0(t), D_t F\rangle dt, \qquad F \in \mathcal{S}.$$

For $F = I_1(u)$, $u \in \mathcal{S}(\mathbb{R}_+; \mathbb{R}^d)$, we have

$$\begin{aligned}
\frac{d}{d\varepsilon}\Lambda(U_\varepsilon, \varepsilon h)F_{|\varepsilon=0} &= \int_0^\infty \langle h_0(t), u(t)\rangle dt + I_1(\mathcal{L}u) \\
&= \int_0^\infty \langle h_0(t), D_t F\rangle dt + \delta(\mathcal{L}DF) + \mathrm{trace}\,(\mathrm{Id}_H \otimes \mathcal{L})DDF \\
&= AF
\end{aligned}$$

since $DDF = 0$. From (5.6.1), for $F_1, \ldots, F_n \in \mathcal{S}$ and $f \in \mathcal{C}_b^\infty(\mathbb{R}^n; \mathbb{R})$ we have

$$\begin{aligned}
&Af(F_1, \ldots, F_n) = \\
&\quad \delta(\mathcal{L}Df(F_1, \ldots, F_n)) + \mathrm{trace}\,(\mathrm{Id}_H \otimes \mathcal{L})DDf(F_1, \ldots, F_n) \\
&\quad + \int_0^\infty \langle h_0(t), D_t f(F_1, \ldots, F_n)\rangle dt \\
&= \sum_{i=1}^n \delta\left(\partial_i f(F_1, \ldots, F_n)\mathcal{L}DF_i\right) \\
&\quad + \sum_{i=1}^n \partial_i f(F_1, \ldots, F_n)\mathrm{trace}\,(\mathrm{Id}_H \otimes \mathcal{L})DDF_i \\
&\quad + \sum_{i,j=1}^n \partial_i \partial_j f(F_1, , \ldots, F_n) \int_0^\infty \langle \mathcal{L}_s DF_i, D_s F_j\rangle ds \\
&\quad + \sum_{i=1}^n \partial_i f(F_1, \ldots, F_n) \int_0^\infty \langle h_0(t), D_t F_i\rangle dt \\
&= \sum_{i=1}^n \partial_i f(F_1, \ldots, F_n)\delta(\mathcal{L}DF_i) + \sum_{i=1}^n \partial_i f(F_1, \ldots, F_n)\mathrm{trace}\,(\mathrm{Id}_H \otimes \mathcal{L})DDF_i
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{i=1}^n \partial_i f(F_1,\ldots,F_n) \int_0^\infty \langle h_0(t), D_t F_i\rangle dt \\
&= \sum_{i=1}^n \partial_i f(F_1,\ldots,F_n) \left(\delta(\mathcal{L}DF_i) + \mathrm{trace}\,(\mathrm{Id}_H \otimes \mathcal{L}) DDF_i + \int_0^\infty \langle h_0(t), D_t F_i\rangle dt \right) \\
&= \sum_{i=1}^n \partial_i f(F_1,\ldots,F_n) AF_i.
\end{aligned}
$$

Hence for $F_1 = I_1(u_1), \ldots, F_n = I_1(u_n) \in \mathcal{S}$ and $f \in \mathcal{C}_b^\infty(\mathbb{R}^n;\mathbb{R})$:

$$
\begin{aligned}
Af(F_1,\ldots,F_n) &= \sum_{i=1}^n \partial_i f(F_1,\ldots,F_n) AF_i \\
&= \sum_{i=1}^n \partial_i f(F_1,\ldots,F_n) \left(\frac{d}{d\varepsilon} \Lambda(U_\varepsilon, \varepsilon h) F_i \right)_{|\varepsilon=0} \\
&= \left(\frac{d}{d\varepsilon} \Lambda(U_\varepsilon, \varepsilon h) f(F_1,\ldots,F_n) \right)_{|\varepsilon=0}.
\end{aligned}
$$

Consequently, Relation (5.6.2) holds on $\mathcal{S}$. □

Corollary 5.6.5. *Assume that $\mathcal{L} : L^2(\mathbb{R}_+;\mathbb{R}^d) \longrightarrow L^2(\Omega \times \mathbb{R}_+;\mathbb{R}^d)$ is antisymmetric as an endomorphism of $L^2(\mathbb{R}_+;\mathbb{R}^d)$, $\mathbb{P}$-a.s., we have in $L^2(\Omega)$:*

$$
\frac{d}{d\varepsilon} \Lambda(U_\varepsilon, \varepsilon h) F_{|\varepsilon=0} = \int_0^\infty \langle h_0(t), D_t F\rangle dt + \delta(\mathcal{L}DF), \qquad F \in \mathcal{S}.
$$

Proof. Since $\mathcal{L}$ is antisymmetric, we have for any symmetric tensor $u \otimes u \in \mathcal{S}(\mathbb{R}_+;\mathbb{R}^d) \otimes \mathcal{S}(\mathbb{R}_+;\mathbb{R}^d)$:

$$
\mathrm{trace}\,(\mathrm{Id}_H \otimes \mathcal{L}) u \otimes u = \mathrm{trace}\, u \otimes \mathcal{L}u = \langle u, \mathcal{L}u\rangle_H = -\langle \mathcal{L}u, u\rangle_H = 0.
$$

Hence the term $\mathrm{trace}\,(\mathrm{Id}_H \otimes \mathcal{L}) DDF$ of Proposition 5.6.4 vanishes $\mathbb{P}$-a.s. since DDF is a symmetric tensor. □

5.7 Riemannian Brownian Motion

In this section we mention another example of a gradient operator satisfying the Clark formula Assumption 3.2.1 of Chapter 3. As an application we derive concentration inequalities on path space using the method of covariance representation. This section is not self-contained and we refer to [42], [43], [46], [87] for details on Riemannian Brownian motion.

Let $(B(t))_{t\in\mathbb{R}_+}$ denote a $\mathbb{R}^d$-valued Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$, generating the filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$. Let M be a Riemannian manifold of dimension d whose Ricci curvature is uniformly bounded from below, and let $O(M)$ denote the bundle of orthonormal frames over M. The Levi-Civita parallel transport defines d canonical horizontal vector fields $A_1, \ldots, A_d$ on $O(M)$, and the Stratonovich stochastic differential equation

$$\begin{cases} dr(t) = \displaystyle\sum_{i=1}^{d} A_i(r(t)) \circ dx^i(t), & t \in \mathbb{R}_+, \\ r(0) = (m_0, r_0) \in O(M), \end{cases}$$

defines an $O(M)$-valued process $(r(t))_{t\in\mathbb{R}_+}$. Let $\pi : O(M) \longrightarrow M$ be the canonical projection, let

$$\gamma(t) = \pi(r(t)), \qquad t \in \mathbb{R}_+,$$

be the Brownian motion on M and let the Itô parallel transport along $(\gamma(t))_{t\in\mathbb{R}_+}$ is defined as

$$t_{t\leftarrow 0} = r(t)r_0^{-1} : T_{m_0}M \simeq \mathbb{R}^d \longrightarrow T_{\gamma(t)}M, \qquad t \in [0,T].$$

Let $\mathcal{C}_0(\mathbb{R}_+;\mathbb{R}^d)$ denote the space of continuous $\mathbb{R}^d$-valued functions on $\mathbb{R}_+$ vanishing at the origin. Let also $\mathbf{P}(M)$ denote the set of continuous paths on M starting at m_0, let

$$\begin{aligned} I : \mathcal{C}_0(\mathbb{R}_+;\mathbb{R}^d) &\longrightarrow \mathbf{P}(M) \\ (\omega(t))_{t\in\mathbb{R}_+} &\longmapsto I(\omega) = (\gamma(t))_{t\in\mathbb{R}_+} \end{aligned}$$

be the Itô map, and let ν denote the image measure on $\mathbf{P}(M)$ of the Wiener measure $\mathbb{P}$ by I. In order to simplify the notation we write F instead of $F \circ I$, for random variables and stochastic processes. Let Ω_r denote the curvature tensor and $\mathrm{ric}_r : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ the Ricci tensor of M at the frame $r \in O(M)$. Given an adapted process $(z_t)_{t\in\mathbb{R}_+}$ with absolutely continuous trajectories, we let $(\hat{z}(t))_{t\in\mathbb{R}_+}$ be defined by

$$\dot{\hat{z}}(t) = \dot{z}(t) + \frac{1}{2}\mathrm{ric}_{r(t)} z(t), \qquad t \in \mathbb{R}_+, \qquad \hat{z}(0) = 0. \tag{5.7.1}$$

We recall that $z \longmapsto \hat{z}$ can be inverted, i.e. there exists a process $(\tilde{z}_t)_{t\in\mathbb{R}_+}$ such that $\hat{\tilde{z}} = z$, cf. Section 3.7 of [46]. Finally, let $Q_{t,s} : \mathbb{R}^d \longrightarrow \mathbb{R}^d$, be defined as

$$\frac{dQ_{t,s}}{dt} = -\frac{1}{2}\mathrm{ric}_{r(t)} Q_{t,s}, \qquad Q_{s,s} = \mathrm{Id}_{T_{m_0}}, \qquad 0 \le s \le t,$$

and let

$$q(t,z) = -\int_0^t \Omega_{r(s)}(\circ dB(s), z(s)), \qquad t \in \mathbb{R}_+,$$

where $\circ dB(s)$ denotes the Stratonovich differential. Let $Q^*_{t,s}$ be the adjoint of $Q_{t,s}$, let $\mathbb{H} = L^2(\mathbb{R}_+, \mathbb{R}^d)$, and let $\mathbf{H} = L^\infty(\mathbf{P}(M), \mathbb{H}; d\nu)$. Let finally $\mathcal{C}_c^\infty(M^n)$ denote the space of infinitely differentiable functions with compact support in M^n.

In the sequel we endow $\mathbf{P}(M)$ with the σ-algebra $\mathcal{F}^P$ on $\mathbf{P}(M)$ generated by subsets of the form

$$\{\gamma \in \mathbf{P}(M) \ : \ (\gamma(t_1), \ldots, \gamma(t_n)) \in B_1 \times \cdots \times B_n\},$$

where $0 \leq t_1 < \cdots < t_n$, $B_1, \ldots, B_n \in \mathcal{B}(M)$, $n \geq 1$.

Let

$$\mathcal{S}(\mathbf{P}(M); \mathbb{R}) = \{F = f(\gamma(t_1), \ldots, \gamma(t_n)) \ : \ f \in \mathcal{C}_b^\infty(M^n; \mathbb{R}), \\ 0 \leq t_1 \leq \cdots \leq t_n \leq 1, \ n \geq 1\},$$

and

$$\mathcal{U}(\mathbf{P}(M) \times \mathbb{R}_+; \mathbb{R}^d) = \left\{ \sum_{k=1}^n F_k \int_0^\cdot u_k(s) ds \ : \ F_1, \ldots, F_n \in \mathcal{S}(\mathbf{P}(M); \mathbb{R}), \right. \\ \left. u_1, \ldots, u_n \in L^2(\mathbb{R}_+; \mathbb{R}^d), \ n \geq 1 \right\}$$

In the following, the space $L^2(\mathbf{P}(M), \mathcal{F}^P, \nu)$ will be simply denoted by $L^2(\mathbf{P}(M))$. Note that the spaces $\mathcal{S}(\mathbf{P}(M); \mathbb{R})$ and $\mathcal{U}(\mathbf{P}(M) \times \mathbb{R}_+; \mathbb{R}^d)$ are dense in $L^2(\mathbf{P}(M); \mathbb{R})$ and in $L^2(\mathbf{P}(M) \times \mathbb{R}_+; \mathbb{R}^d)$ respectively. The following definition of the intrisic gradient on $\mathbf{P}(M)$ can be found in [46].

Definition 5.7.1. *Let $\hat{D} : L^2(\mathbf{P}(M); \mathbb{R}) \longrightarrow L^2(\mathbf{P}(M) \times \mathbb{R}_+; \mathbb{R}^d)$ be the gradient operator defined as*

$$\hat{D}_t F = \sum_{i=1}^n t_{0 \leftarrow t_i} \nabla_i^M f(\gamma(t_1), \ldots, \gamma(t_n)) \mathbf{1}_{[0,t_i]}(t), \qquad t \in \mathbb{R}_+,$$

for $F \in \mathcal{S}(\mathbf{P}(M); \mathbb{R})$ of the form $F = f(\gamma(t_1), \ldots, \gamma(t_n))$, where ∇_i^M denotes the gradient on M applied to the i-th variable of f.

Given an adapted vector field $(Z(t))_{t \in \mathbb{R}_+}$ on M with $Z(t) \in T_{\gamma(t)}M$, $t \in \mathbb{R}_+$, we let $z(t) = t_{0 \leftarrow t} Z(t)$, $t \in \mathbb{R}_+$, and assume that $\dot{z}(t)$ exists, $\forall t \in \mathbb{R}_+$. Let

$$\nabla Z(t) = \lim_{\varepsilon \to 0} \frac{t_{t \leftarrow t+\varepsilon} Z(t+\varepsilon) - Z(t)}{\varepsilon}.$$

Then

$$\dot{z}(t) = t_{0\leftarrow t}\nabla Z(t), \qquad t \in \mathbb{R}_+.$$

let Ω_r denote the curvature tensor of M and let $\mathrm{ric}_r : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ denote the Ricci tensor at the frame $r \in O(M)$, and let the process $(\hat{z}(t))_{t\in\mathbb{R}_+}$ be defined by

$$\begin{cases} \dot{\hat{z}}(t) = \dot{z}(t) + \dfrac{1}{2}\mathrm{ric}_{r(t)} z(t), \qquad t \in \mathbb{R}_+, \\ \\ \hat{z}(0) = 0. \end{cases} \tag{5.7.2}$$

As a consequence of Corollary 5.6.5 we obtain the following relation between the gradient $\hat{D}$ and the operators D and δ, cf. Theorem 2.3.8 and Theorem 2.6 of [29].

Corollary 5.7.2. *Assume that the Ricci curvature of M is uniformly bounded, and let $z \in \mathcal{U}(\mathbf{P}(M) \times \mathbb{R}_+; \mathbb{R}^d)$ be adapted. We have*

$$\int_0^\infty \langle \hat{D}_t F, \dot{z}(t)\rangle dt = \int_0^\infty \langle D_t F, \dot{\hat{z}}(t)\rangle dt + \delta(q(\cdot, z)D.F), \tag{5.7.3}$$

$F \in \mathcal{S}(\mathbf{P}(M); \mathbb{R})$, where $q(t, z) : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ is defined as

$$q(t, z) = -\int_0^t \Omega_{r(s)}(\circ dB(s), z(s)), \qquad t \in \mathbb{R}_+.$$

Proof. We let $V_\varepsilon(t) = \exp(\varepsilon q(t, z))$, $t \in \mathbb{R}_+$, $\varepsilon \in \mathbb{R}$. Then from Proposition 3.5.3 of [46] we have

$$\int_0^\infty \langle \hat{D}F, \dot{z}(t)\rangle dt = \frac{d}{d\varepsilon}\Lambda(U_\varepsilon, \varepsilon\dot{\hat{z}})F_{|\varepsilon=0}.$$

Since the Ricci curvature of M is bounded, we have $\dot{\hat{z}} \in L^2(\mathbb{R}_+; L^\infty(W; \mathbb{R}))$ from (5.7.2). Moreover, from Theorem 2.2.1 of [46], $\varepsilon \longmapsto \Lambda(U_\varepsilon, 0)r(t)$ is differentiable in $L^2(W; \mathbb{R})$, hence continuous, $\forall t \in \mathbb{R}_+$. Consequently, from (5.7.2) and by construction of $\mathcal{U}(\mathbf{P}(M) \times \mathbb{R}_+; \mathbb{R}^d)$, $\varepsilon \longmapsto \Lambda(U_\varepsilon, 0)\dot{\hat{z}}$ is continuous in $L^2(W \times \mathbb{R}_+; \mathbb{R}^d)$ and we can apply Corollary 5.6.5 with $\mathcal{L}_t = q(t, z)$ to obtain (5.7.3). □

If $u \in \mathcal{U}(\mathbf{P}(M) \times \mathbb{R}_+; \mathbb{R}^d)$ is written as $u = \displaystyle\sum_{i=1}^n G_i z_i$, z_i deterministic, $G_i \in \mathcal{S}(\mathbf{P}(M); \mathbb{R})$, $i = 1, \ldots, n$, we let

$$\mathrm{trace}\, q(t, D_t u) = \sum_{i=1}^n q(t, z_i) D_t G_i.$$

Given $u \in \mathcal{U}(\mathbf{P}(M) \times \mathbb{R}_+; \mathbb{R}^d)$ written as $u = \sum_{i=1}^n G_i z_i$, z_i deterministic, $G_i \in \mathcal{S}(\mathbf{P}(M); \mathbb{R})$, $i = 1, \ldots, n$, we let

$$\hat{u} = \sum_{i=1}^{n} G_i \hat{z}_i.$$

We now recall the inversion of $z \longmapsto \hat{z}$ by the method of variation of constants described in Section 3.7 of [46]. Let $\mathrm{Id}_{\gamma(t)}$ denote the identity of $T_{\gamma(t)}M$. We have

$$\dot{z}(t) = \dot{\tilde{z}}(t) + \frac{1}{2}\mathrm{ric}_{r(t)}\tilde{z}(t), \qquad t \in \mathbb{R}_+,$$

where $(\tilde{z}(t))_{t\in\mathbb{R}_+}$ is defined as

$$\tilde{z}(t) = \int_0^t Q_{t,s}\dot{z}(s)ds, \qquad t \in \mathbb{R}_+,$$

and $Q_{t,s} : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ satisfies

$$\frac{dQ_{t,s}}{dt} = -\frac{1}{2}\mathrm{ric}_{r(t)}Q_{t,s}, \qquad Q_{s,s} = \mathrm{Id}_{\gamma(0)}, \qquad 0 \le s \le t.$$

Let also the process $(\hat{Z}(t))_{t\in\mathbb{R}_+}$ be defined by

$$\begin{cases} \nabla\hat{Z}(t) = \nabla Z(t) + \dfrac{1}{2}\mathrm{Ric}_{\gamma(t)}Z(t), \qquad t \in \mathbb{R}_+, \\ \\ \hat{Z}(0) = 0, \end{cases}$$

with $\hat{z}(t) = \tau_{0\leftarrow t}\hat{Z}(t)$, $t \in \mathbb{R}_+$. In order to invert the mapping $Z \longmapsto \hat{Z}$, let

$$\tilde{Z}(t) = \int_0^t R_{t,s}\nabla Z(s)ds, \qquad t \in \mathbb{R}_+,$$

where $R_{t,s} : T_{\gamma(s)}M \longrightarrow T_{\gamma(t)}M$ is defined by the equation

$$\nabla_t R_{t,s} = -\frac{1}{2}\mathrm{Ric}_{\gamma(t)}R_{t,s}, \qquad R_{s,s} = \mathrm{Id}_{\gamma(s)}, \qquad 0 \le s \le t,$$

∇_t denotes the covariant derivative along $(\gamma(t))_{t\in\mathbb{R}_+}$, and

$$\mathrm{Ric}_m : T_mM \longrightarrow T_mM$$

denotes the Ricci tensor at $m \in M$, with the relation

$$\mathrm{ric}_{r(t)} = t_{0\leftarrow t} \circ \mathrm{Ric}_{\gamma(t)} \circ t_{t\leftarrow 0}.$$

Then we have

$$\begin{cases} \nabla Z(t) = \nabla\tilde{Z}(t) + \frac{1}{2}\mathrm{Ric}_{\gamma(t)}\tilde{Z}(t), \qquad t \in \mathbb{R}_+, \\ \\ Z(0) = 0. \end{cases}$$

We refer to [46] for the next definition.

Definition 5.7.3. *The damped gradient*

$$\tilde{D} : L^2(\mathbf{P}(M);\mathbb{R}) \longrightarrow L^2(\mathbf{P}(M)\times\mathbb{R}_+;\mathbb{R}^d)$$

is defined as

$$\tilde{D}_t F = \sum_{i=1}^n \mathbf{1}_{[0,t_i]}(t) Q^*_{t_i,t} t_{0\leftarrow t_i} \nabla_i^M f(\gamma(t_1),\ldots,\gamma(t_n)), \qquad t\in\mathbb{R}_+,$$

for $F\in\mathcal{S}(\mathbf{P}(M);\mathbb{R})$ *of the form* $F = f(\gamma(t_1),\ldots,\gamma(t_n))$, *where*

$$Q^*_{t,s} : \mathbb{R}^d \longrightarrow \mathbb{R}^d$$

denotes the adjoint of $Q_{t,s} : \mathbb{R}^d \longrightarrow \mathbb{R}^d$, $0\le s<t$.

Given $f\in\mathcal{C}_c^\infty(M^n)$ we also have

$$\tilde{D}_t F = \sum_{i=1}^n \mathbf{1}_{[0,t_i]}(t) t_{0\leftarrow t} R^*_{t_i,t} \nabla_i^M f(\gamma(t_1),\ldots,\gamma(t_n)), \qquad t\in\mathbb{R}_+,$$

where $R^*_{t_i,t} : T_{\gamma(t_i)} \longrightarrow T_{\gamma(t)}$ is the adjoint of $R_{t_i,t} : T_{\gamma(t)} \longrightarrow T_{\gamma(t_i)}$.

Proposition 5.7.4. *We have for* $z\in\mathcal{U}(\mathbf{P}(M)\times\mathbb{R}_+;\mathbb{R}^d)$*:*

$$\int_0^\infty \langle \tilde{D}_t F, \dot{z}(t)\rangle dt = \int_0^\infty \langle \hat{D}_t F, \dot{\tilde{z}}(t)\rangle dt, \qquad F\in\mathcal{S}(\mathbf{P}(M);\mathbb{R}). \tag{5.7.4}$$

Proof. We compute

$$\begin{aligned}\int_0^\infty \langle \tilde{D}_t F, \dot{z}(t)\rangle dt &= \sum_{i=1}^n \int_0^{t_i} \langle Q^*_{t_i,s} t_{0\leftarrow t_i} \nabla_i^M f(\gamma(t_1),\ldots,\gamma(t_n)), \dot{z}(s)\rangle ds \\ &= \sum_{i=1}^n \int_0^{t_i} \langle t_{0\leftarrow t_i} \nabla_i^M f(\gamma(t_1),\ldots,\gamma(t_n)), Q_{t_i,s}\dot{z}(s)\rangle dt \\ &= \int_0^\infty \langle \hat{D}_t F, \dot{\tilde{z}}(t)\rangle dt, \qquad F\in\mathcal{S}(\mathbf{P}(M);\mathbb{R}).\end{aligned}$$

□

We also have

$$\int_0^\infty \langle \tilde{D}_t F, \dot{\tilde{z}}(t)\rangle dt = \int_0^\infty \langle \hat{D}_t F, \dot{z}(t)\rangle dt, \qquad F\in\mathcal{S}(\mathbf{P}(M);\mathbb{R}).$$

Taking expectation on both sides of (5.7.3) and (5.7.4) it follows that the processes DF and $\tilde{D}F$ have the same adapted projections:

$$\mathbb{E}[D_t F \mid \mathcal{F}_t] = \mathbb{E}[\tilde{D}_t F \mid \mathcal{F}_t], \qquad t\in\mathbb{R}_+, \tag{5.7.5}$$

$F = f(\gamma(t_1), \ldots, \gamma(t_n))$. Using this relation and the Clark formula

$$F = \mathbb{E}[F] + \int_0^\infty \mathbb{E}[D_t F \mid \mathcal{F}_t] \cdot dB(t),$$

on the Wiener space, cf. Proposition 5.2.7 we obtain the expression of the Clark formula on path space, i.e. Assumption 3.2.1 is satisfied by $\tilde{D}$.

Proposition 5.7.5. *Let* $F \in \mathrm{Dom}\,(\tilde{D})$*, then*

$$F = \mathbb{E}[F] + \int_0^\infty \mathbb{E}[\tilde{D}_t F \mid \mathcal{F}_t] \cdot dB(t).$$

The following covariance identity is then a consequence of Proposition 3.4.1.

Proposition 5.7.6. *Let* $F, G \in \mathrm{Dom}\,(\tilde{D})$*, then*

$$\mathrm{Cov}\,(F,G) = \mathbb{E}\left[\int_0^\infty \tilde{D}_t F \cdot \mathbb{E}[\tilde{D}_t G \mid \mathcal{F}_t]\; dt\right]. \tag{5.7.6}$$

From Proposition 3.6.1 we obtain a concentration inequality on the path space.

Lemma 5.7.7. *Let* $F \in \mathrm{Dom}\,(\tilde{D})$*. If* $\|\tilde{D}F\|_{L^2(\mathbb{R}_+, L^\infty(\mathrm{P}(M)))} \leq C$*, for some* $C > 0$*, then*

$$\nu(F - \mathbb{E}[F] \geq x) \leq \exp\left(-\frac{x^2}{2C\|\tilde{D}F\|_{\mathbf{H}}}\right), \qquad x \geq 0. \tag{5.7.7}$$

In particular, $\mathbb{E}[e^{\lambda F^2}] < \infty$*, for* $\lambda < (2C\|\tilde{D}F\|_{\mathbf{H}})^{-1}$*.*

5.8 Time Changes on Brownian Motion

In this section we study the transformations given by time changes on Brownian motion, in connection with the operator $\nabla^\ominus$ of Definition 4.8.1.

Proposition 5.8.1. *On the Wiener space,* $\nabla^\ominus$ *satisfies the relation*

$$\nabla_t^\ominus(FG) = F\nabla_t^\ominus G + G\nabla_t^\ominus F - D_t F D_t G, \qquad t \in \mathbb{R}_+. \tag{5.8.1}$$

Proof. We will show by induction using the derivation property (5.2.3) of D that for all $k \geq 1$,

$$\begin{aligned}\nabla_t^\ominus(I_n(f^{\otimes n})I_1(g)^k) = I_n(f^{\otimes n})\nabla_t^\ominus\left(I_1(g)^k\right) + I_1(g)^k\nabla_t^\ominus I_n(f^{\otimes n})\\ -D_t I_n(f^{\otimes n}) D_t\left(I_1(g)^k\right),\end{aligned} \tag{5.8.2}$$

$t \in \mathbb{R}_+$. We have

$$\nabla^\ominus\left(I_n(f^{\otimes n})I_1(g)\right) = \nabla^\ominus\left(I_{n+1}(f^{\otimes n} \circ g) + n\langle f, g\rangle_{L^2(\mathbb{R}_+)} I_{n-1}(f^{\otimes(n-1)})\right)$$

$$\begin{aligned}
&= -I_{n+1}((g'\mathbf{1}_{[t,\infty)}) \circ f^{\otimes n}) - nI_{n+1}((f'\mathbf{1}_{[t,\infty)}) \circ g \circ f^{\otimes(n-1)}) \\
&\quad -n(n-1)\langle f,g\rangle_{L^2(\mathbb{R}_+)} I_{n-1}((f'\mathbf{1}_{[t,\infty)}) \circ f^{\otimes(n-2)}) \\
&= -nI_{n+1}((f'\mathbf{1}_{[t,\infty)}) \circ f^{\otimes(n-1)} \circ g) - n\langle g,(f'\mathbf{1}_{[t,\infty)})\rangle_{L^2(\mathbb{R}_+)} I_{n-1}(f^{\otimes(n-1)}) \\
&\quad -I_{n+1}(f^{\otimes n} \circ (g'\mathbf{1}_{[t,\infty)})) - n(n-1)\langle f,g\rangle_{L^2(\mathbb{R}_+)} I_{n-1}(f^{\otimes(n-2)} \circ (f'\mathbf{1}_{[t,\infty)})) \\
&\quad -n\langle f, g'\mathbf{1}_{[t,\infty)}\rangle_{L^2(\mathbb{R}_+)} I_{n-1}(f^{\otimes(n-1)}) \\
&\quad +n\langle f'\mathbf{1}_{[t,\infty)}, g\rangle_{L^2(\mathbb{R}_+)} I_{n-1}(f^{\otimes(n-1)}) + n\langle g'\mathbf{1}_{[t,\infty)}, f\rangle_{L^2(\mathbb{R}_+)} I_{n-1}(f^{\otimes(n-1)}) \\
&= -nI_n((f'\mathbf{1}_{[t,\infty)}) \circ f^{\otimes(n-1)}) I_1(g) - I_n(f^{\otimes n}) I_1(g'\mathbf{1}_{[t,\infty)}) \\
&\quad -nf(t)g(t)I_{n-1}(f^{\otimes(n-1)}) \\
&= I_1(g)\nabla_t^{\ominus} I_n(f^{\otimes n}) + I_n(f^{\otimes n})\nabla_t^{\ominus} I_1(g) - D_t I_1(g) D_t I_n(f^{\otimes n}), \qquad t \in \mathbb{R}_+,
\end{aligned}$$

which shows (5.8.2) for $k = 1$. Next, assuming that (5.8.2) holds for some $k \geq 1$, we have

$$\begin{aligned}
\nabla_t^{\ominus}(I_n(f^{\otimes n})I_1(g)^{k+1}) &= I_1(g)\nabla_t^{\ominus}(I_n(f^{\otimes n})I_1(g)^k) + I_n(f^{\otimes n})I_1(g)^k \nabla_t^{\ominus} I_1(g) \\
&\quad -D_t I_1(g) D_t(I_1(g)^k I_n(f^{\otimes n})) \\
&= I_1(g)\left(I_1(g)^k \nabla_t^{\ominus} I_n(f^{\otimes n}) + I_n(f^{\otimes n})\nabla_t^{\ominus}\left(I_1(g)^k\right)\right. \\
&\quad \left. -D_t\left(I_1(g)^k\right) D_t I_n(f^{\otimes n})\right) \\
&\quad +I_n(f^{\otimes n})I_1(g)^k \nabla_t^{\ominus} I_1(g) - D_t I_1(g)\left(I_1(g)^k D_t I_n(f^{\otimes n})\right. \\
&\quad \left. +I_n(f^{\otimes n}) D_t\left(I_1(g)^k\right)\right) \\
&= I_1(g)^{k+1}\nabla_t^{\ominus} I_n(f^{\otimes n}) + I_n(f^{\otimes n})\nabla_t^{\ominus}\left(I_1(g)^{k+1}\right) \\
&\quad -D_t\left(I_1(g)^{k+1}\right) D_t I_n(f^{\otimes n}),
\end{aligned}$$

$t \in \mathbb{R}_+$, which shows that (5.8.2) holds at the rank $k+1$. □

Definition 5.8.2. *Let $h \in L^2(\mathbb{R}_+)$, with $\|h\|_{L^\infty(\mathbb{R}_+)} < 1$, and*

$$\nu_h(t) = t + \int_0^t h(s)ds, \qquad t \in \mathbb{R}_+.$$

We define a mapping $\mathcal{T}_h : \Omega \to \Omega$, $t, \varepsilon \in \mathbb{R}_+$, as

$$\mathcal{T}_h(\omega) = \omega \circ \nu_h^{-1}, \qquad h \in L^2(\mathbb{R}_+), \ \sup_{x \in \mathbb{R}_+} | h(x) | < 1.$$

The transformation $\mathcal{T}_h$ acts on the trajectory of $(B_s)_{s\in\mathbb{R}_+}$ by change of time, or by perturbation of its predictable quadratic variation. Although $\mathcal{T}_h$ is not absolutely continuous, the functional $F \circ \mathcal{T}_h$ is well-defined for $F \in \mathcal{S}$, since elements of $\mathcal{S}$ can be defined trajectory by trajectory.

Proposition 5.8.3. *We have for $F \in \mathcal{S}$*

$$\int_0^\infty h(t)\left(\nabla_t^\ominus + \frac{1}{2}D_tD_t\right)Fdt = -\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}(F\circ\mathcal{T}_{\varepsilon h} - F).$$

Proof. We first notice that as a consequence of Proposition 5.8.1, the operator

$$\nabla_t^\ominus + \frac{1}{2}D_tD_t$$

$t \in \mathbb{R}_+$, has the derivation property. Indeed, by Proposition 5.8.1 we have

$$\begin{aligned}\nabla_t^\ominus(FG) + \frac{1}{2}D_tD_t(FG) &= F\nabla_t^\ominus G + G\nabla_t^\ominus F - D_tFD_tG \\ &\quad + \frac{1}{2}\left(FD_tD_tG + GD_tD_tF + 2D_tFD_tG\right) \\ &= F\left(\nabla_t^\ominus G + \frac{1}{2}D_tD_tG\right) + G\left(\nabla_t^\ominus F + \frac{1}{2}D_tD_tF\right).\end{aligned}$$

Moreover, $\mathcal{T}_{\varepsilon h}$ is multiplicative, hence we only need to treat the particular case of $F = I_1(f)$. We have

$$\begin{aligned}I_1(f)\circ\mathcal{T}_{\varepsilon h} - I_1(f) &= \int_0^\infty f(s)dB(\nu_{\varepsilon h}^{-1}(s)) - I_1(f) \\ &= \int_0^\infty f(\nu_{\varepsilon h}(s))dB_s - \int_0^\infty f(s)dB_s \\ &= \int_0^\infty \left(f\left(t + \varepsilon\int_0^t h(s)ds\right) - f(t)\right)dB_t.\end{aligned}$$

After division by $\varepsilon > 0$, this converges in $L^2(\Omega)$ as $\varepsilon \to 0$ to

$$\begin{aligned}\int_0^\infty f'(t)\int_0^t h(s)dsdB_t &= \int_0^\infty h(t)\int_t^\infty f'(s)dB_sdt \\ &= -\int_0^\infty h(t)\nabla_t^\ominus I_1(f)dt \\ &= -\int_0^\infty h(t)\left(\nabla_t^\ominus + \frac{1}{2}D_tD_t\right)I_1(f)dt.\end{aligned}$$

□

5.9 Notes and References

Proposition 5.2.1 is usually taken as a definition of the Malliavin derivative D, see for example [96]. The relation between multiple Wiener integrals and Hermite polynomials originates in [136]. Corollary 5.2.4 can be found in Lemma 1.2 of [95] and in [100]. Finding the probabilistic interpretation of D for normal martingales other than the Brownian motion or the Poisson process, e.g. for the Azéma martingales, is still an open problem. In relation to

Proposition 5.6.1, see [29] for a treatment of transformations called Euclidean motions, in which case the operator $V(t) : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ is chosen to be an isometry and h is adapted, so that $\Lambda(U,h)$ is extended by quasi-invariance of the Wiener measure, see also [64]. Corollary 5.5.2 recovers and extend the sufficient conditions for the invariance of the Wiener measure under random rotations given in [149], i.e. the Skorohod integral $\delta(Rh)$ to has a Gaussian law when $h \in H = L^2(\mathbb{R}_+, \mathbb{R}^d)$ and R is a random isometry of H. We refer to [44], [46] for the Clark formula and the construction of gradient and divergence operators on Riemannian path space, to [24] for the logarithmic Sobolev inequality, and to [62] for the corresponding deviation results stated in Section 5.7.

Chapter 6
Analysis on the Poisson space

In this chapter we give the definition of the Poisson measure on a space of configurations of a metric space X, and we construct an isomorphism between the Poisson measure on X and the Poisson process on $\mathbb{R}_+$. From this we obtain the probabilistic interpretation of the gradient D as a finite difference operator and the relation between Poisson multiple stochastic integrals and Charlier polynomials. Using the gradient and divergence operators we also derive an integration by parts characterization of Poisson measures, and other results such as deviation and concentration inequalities on the Poisson space.

6.1 Poisson Random Measures

Let X be a σ-compact metric space (i.e. X can be partitioned into a countable union of compact metric spaces) with a diffuse Radon measure σ. The space of configurations of X is the set of Radon measures

$$\Omega^X := \left\{ \omega = \sum_{k=0}^{n} \epsilon_{x_k} \ : \ (x_k)_{k=0}^{n} \subset X, \ n \in \mathbb{N} \cup \{\infty\} \right\}, \tag{6.1.1}$$

where ϵ_x denotes the Dirac measure at $x \in X$, i.e.

$$\epsilon_x(A) = \mathbf{1}_A(x), \qquad A \in \mathcal{B}(X),$$

and Ω defined in (6.1.1) is restricted to locally finite configurations.

The configuration space Ω^X is endowed with the vague topology and its associated σ-algebra denoted by $\mathcal{F}^X$, cf. [4]. When X is compact we will consider Poisson functionals of the form

$$F(\omega) = f_0 \mathbf{1}_{\{\omega(X)=0\}} + \sum_{n=1}^{\infty} \mathbf{1}_{\{\omega(X)=n\}} f_n(x_1, \ldots, x_n), \qquad (6.1.2)$$

where $f_n \in L^1(X^n, \sigma^{\otimes n})$ is symmetric in n variables, $n \geq 1$. As an example,

$$F(\omega) := \omega(A), \qquad \omega \in \Omega,$$

is represented using the symmetric functions

$$f_n(x_1, \ldots, x_n) = \sum_{k=1}^{n} \mathbf{1}_A(x_k), \qquad n \geq 1.$$

Our construction of the Poisson measure is inspired by that of [94].

Definition 6.1.1. *In case X is precompact (i.e. X has a compact closure), let $\mathcal{F}^X$ denote the σ-field generated by all functionals F of the form* (6.1.2), *and let π_σ^X denote the probability measure on $(\Omega^X, \mathcal{F}^X)$ defined via*

$$\mathbb{E}_{\pi_\sigma^X}[F] = \mathrm{e}^{-\sigma(X)} f_0 + \mathrm{e}^{-\sigma(X)} \sum_{n=1}^{\infty} \frac{1}{n!} \int_X \cdots \int_X f_n(x_1, \ldots, x_n) \sigma(dx_1) \cdots \sigma(dx_n), \qquad (6.1.3)$$

for all non-negative F of the form (6.1.2).

For example, for A a compact subset of X, the mapping $\omega \longmapsto \omega(A)$ has the Poisson distribution with parameter $\sigma(A)$ under π_σ^X. Indeed we have

$$\mathbf{1}_{\{\omega(A)=k\}} = \sum_{n=k}^{\infty} \mathbf{1}_{\{\omega(X)=n\}} f_n(x_1, \ldots, x_n),$$

with

$$f_n(x_1, \ldots, x_n) = \frac{1}{k!(n-k)!} \sum_{\eta \in \Sigma_n} \mathbf{1}_{A^k}(x_{\eta(1)}, \ldots, x_{\eta(k)}) \mathbf{1}_{(X \setminus A)^{n-k}}(x_{\eta(k+1)}, \ldots, x_{\eta(n)}),$$

hence

$$\begin{aligned} \pi_\sigma^X(\omega(A) = k) &= \mathbb{E}_{\pi_\sigma^X}[\mathbf{1}_{\{\omega(A)=k\}}] \\ &= \mathrm{e}^{-\sigma(X)} \sum_{n=1}^{\infty} \frac{1}{k!(n-k)!} \sigma(A)^k \sigma(X \setminus A)^{n-k} \\ &= \mathrm{e}^{-\sigma(A)} \frac{\sigma(A)^k}{k!}. \end{aligned} \qquad (6.1.4)$$

The above construction is then extended to σ-compact X in the next definition.

Definition 6.1.2. *In case X is σ-compact we consider a countable partition $X = \bigcup_{n\in\mathbb{N}} X_n$ in compact subsets, and let*

$$\Omega^X = \prod_{n=0}^{\infty} \Omega^{X_n}, \qquad \mathcal{F}^X = \bigotimes_{n=0}^{\infty} \mathcal{F}^{X_n}, \qquad \pi_\sigma^X = \bigotimes_{n=0}^{\infty} \pi_\sigma^{X_n}. \tag{6.1.5}$$

Note that π_σ^X in Definition 6.1.2 is independent of the choice of partition made for X in (6.1.5).

The argument leading to Relation (6.1.4) can be extended to n variables.

Proposition 6.1.3. *Let $A_1, \ldots, A_n$ be compact disjoint subsets of X. Under the measure π_σ^X on $(\Omega^X, \mathcal{F}^X)$, the $\mathbb{N}^n$-valued vector*

$$\omega \longmapsto (\omega(A_1), \ldots, \omega(A_n))$$

has independent components with Poisson distributions of respective parameters

$$\sigma(A_1), \ldots, \sigma(A_n).$$

Proof. Consider a disjoint partition $A_1 \cup \cdots \cup A_n$ of X and

$$\begin{aligned} F(\omega) &= \mathbf{1}_{\{\omega(A_1)=k_1\}} \cdots \mathbf{1}_{\{\omega(A_n)=k_n\}} \\ &= \mathbf{1}_{\{\omega(X)=k_1+\cdots+k_n\}} f_n(x_1^1, \ldots, x_{k_1}^1, \ldots, x_1^n, \ldots, x_{k_n}^n), \end{aligned}$$

where

$$\begin{aligned} & f_n(x_1, \ldots, x_N) \\ &= \sum_{\eta\in\Sigma_N} \frac{1}{k_1! \cdots k_n!} \mathbf{1}_{A_1}(x_{\eta(1)}, \ldots, x_{\eta(k_1)}) \cdots \mathbf{1}_{A_n}(x_{\eta(k_1+\ldots+k_{n-1}+1)}, \ldots, x_{\eta(N)}) \end{aligned}$$

is the symmetrization in $N = k_1 + \cdots + k_n$ variables of the function

$$\begin{aligned} & (x_1^1, \ldots, x_{k_1}^1, \ldots, x_1^n, \ldots, x_{k_n}^n) \\ & \qquad \longmapsto \frac{(k_1 + \cdots + k_n)!}{k_1! \cdots k_n!} \mathbf{1}_{A_1}(x_1^1, \ldots, x_{k_1}^1) \cdots \mathbf{1}_{A_n}(x_1^n, \ldots, x_{k_n}^n), \end{aligned}$$

hence

$$\begin{aligned} \pi_\sigma^X(\omega(A_1) = k_1, \ldots, \omega(A_n) = k_n) &= \mathbb{E}_{\pi_\sigma^X}[F] \\ &= \mathrm{e}^{-(\sigma(A_1)+\cdots+\sigma(A_n))} \frac{\sigma(A_1)^{k_1} \cdots \sigma(A_n)^{k_n}}{k_1! \cdots k_n!}. \end{aligned}$$

□

When X is compact, the conditional distribution of $\omega = \{x_1, \ldots, x_n\}$ given that $\omega(X) = n$ is given by the formula

$$\pi_\sigma^X(\{x_1,\ldots,x_n\}\subset A^n \mid \omega(X)=n)=\left(\frac{\sigma(A)}{\sigma(X)}\right)^n,$$

which follows from taking $f_n = \mathbf{1}_{A^n}$ in (6.1.2), and extends to symmetric Borel subsets of X^n.

In the next proposition we compute the Fourier transform of π_σ^X via the Poisson stochastic integral

$$\int_X f(x)\omega(dx)=\sum_{x\in\omega} f(x), \qquad f\in L^1(X,\sigma).$$

In the sequel we will drop the index X in π_σ^X.

Proposition 6.1.4. *Let $f\in L^1(X,\sigma)$. We have*

$$\mathbb{E}_{\pi_\sigma}\left[\exp\left(i\int_X f(x)\omega(dx)\right)\right]=\exp\left(\int_X(\mathrm{e}^{if(x)}-1)\sigma(dx)\right). \tag{6.1.6}$$

Proof. We first assume that X is compact. We have

$$\begin{aligned}
&\mathbb{E}_{\pi_\sigma}\left[\exp\left(i\int_X f(x)\omega(dx)\right)\right]\\
&=\mathrm{e}^{-\sigma(X)}\sum_{n=0}^\infty\frac{1}{n!}\int_X\cdots\int_X \mathrm{e}^{i(f(x_1)+\cdots+f(x_n))}\sigma(dx_1)\cdots\sigma(dx_n).\\
&=\mathrm{e}^{-\sigma(X)}\sum_{n=0}^\infty\frac{1}{n!}\left(\int_X\mathrm{e}^{if(x)}\sigma(dx)\right)^n\\
&=\exp\left(\int_X(\mathrm{e}^{if(x)}-1)\sigma(dx)\right).
\end{aligned}$$

The extension to the σ-compact case is done using (6.1.5). □

We have

$$\begin{aligned}
\mathbb{E}\left[\int_X f(x)\omega(dx)\right]&=-i\frac{d}{d\varepsilon}\mathbb{E}_{\pi_\sigma}\left[\exp\left(i\varepsilon\int_X f(x)\omega(dx)\right)\right]_{|\varepsilon=0}\\
&=-i\frac{d}{d\varepsilon}\exp\left(\int_X(\mathrm{e}^{i\varepsilon f(x)}-1)\sigma(dx)\right)_{|\varepsilon=0}\\
&=\int_X f(x)\sigma(dx), \qquad f\in L^1(X,\sigma),
\end{aligned}$$

and similarly by second order derivation we obtain

$$\mathbb{E}\left[\left(\int_X f(x)(\omega(dx)-\sigma(dx))\right)^2\right]=\int_X|f(x)|^2\sigma(dx), \quad f\in L^2(X,\sigma). \tag{6.1.7}$$

Both formulae can also be proved on simple functions of the form $f=\sum_{i=1}^n\alpha_i\mathbf{1}_{A_i}$, and then extended to measurable functions under appropriate integrability conditions.

The formula (6.1.6) can be extended as

$$\mathbb{E}_{\pi_\sigma}\left[\exp\left(i\int_{\mathbb{R}^d} f(x)(\omega(dx)-\sigma(dx))\right)\right] = \exp\left(\int_{\mathbb{R}^d}(\mathrm{e}^{if(x)} - if(x) - 1)\sigma(dx)\right),$$

by assuming that

$$\int_{\{y\in X\ :\ |f(y)|>1\}} |f(y)|\sigma(dy) < \infty \quad \text{and} \quad \int_{\{y\in X\ :\ |f(y)|\le 1\}} |f(y)|^2\sigma(dy) < \infty,$$

from the inequality

$$|\mathrm{e}^{it} - it\mathbf{1}_{\{|t|\le 1\}} - 1| \le 2(1\wedge |t|^2), \qquad t\in\mathbb{R}. \tag{6.1.8}$$

Similarly, taking $X = \mathbb{R}^d$, assuming that

$$\int_{\mathbb{R}^d}(1\wedge |y|^2_{\mathbb{R}^d})\sigma(dy) < \infty,$$

where $|\cdot|_{\mathbb{R}^d}$ is the Euclidean norm on $\mathbb{R}^d$, and using (6.1.8), the vector of single Poisson stochastic integrals

$$F = \left(\int_{\{|y|_{\mathbb{R}^d}\le 1\}} y_k\,(\omega(dy)-\sigma(dy)) + \int_{\{|y|_{\mathbb{R}^d}>1\}} y_k\,\omega(dy)\right)_{1\le k\le n} \tag{6.1.9}$$

has the characteristic function

$$\varphi_F(u) = \mathbb{E}[\mathrm{e}^{i\langle F,u\rangle_{\mathbb{R}^d}}] = \exp\left(\int_{\mathbb{R}^d}(\mathrm{e}^{i\langle y,u\rangle} - 1 - i\langle y,u\rangle\mathbf{1}_{\{|y|_{\mathbb{R}^d}\le 1\}})\sigma(dy)\right), \tag{6.1.10}$$

$u\in\mathbb{R}^d$. Relation (6.1.10) is called the Lévy-Khintchine formula, and the random vector

$$F = (F_1,\ldots,F_n)$$

is said to have an n-dimensional infinitely divisible distribution with Lévy measure $\sigma(dx)$.

Denote by $\pi^X_{\sigma,\alpha}$ the thinning of order $\alpha\in(0,1)$ of the Poisson measure π^X_σ, i.e. $\pi^X_{\sigma,\alpha}$ is obtained by removing, resp. keeping, independently each configuration point with probability α, resp. $1-\alpha$.

The next proposition is a classical result on the thinning of Poisson measure.

Proposition 6.1.5. *Let $\alpha\in(0,1)$. We have $\pi^X_{\sigma,\alpha} = \pi^X_{\alpha\sigma}$, i.e. $\pi^X_{\sigma,\alpha}$ is the Poisson measure with intensity $\alpha\sigma(dx)$ on Ω^X.*

Proof. It suffices to treat the case where X is compact. Using the fact that given n configuration points, the number of points removed by thinning has a binomial distribution with parameter (n, α), we obtain

$$\begin{aligned}
&\mathbb{E}_{\pi_{\sigma,\alpha}}\left[\exp\left(i\int_X f(x)\omega(dx)\right)\right] \\
&= \mathrm{e}^{-\sigma(X)}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{\alpha^k}{n!}\binom{n}{k}\left(\int_X \mathrm{e}^{if(x)}\sigma(dx)\right)^k(\sigma(X))^{n-k}(1-\alpha)^{n-k} \\
&= \mathrm{e}^{-\sigma(X)}\sum_{n=0}^{\infty}\frac{1}{n!}\left(\alpha\int_X \mathrm{e}^{if(x)}\sigma(dx)+(1-\alpha)\sigma(X)\right)^n \\
&= \mathrm{e}^{\alpha\int_X(\mathrm{e}^{if(x)}-1)\sigma(dx)}.
\end{aligned}$$

In terms of probabilities, for all compact $A \in \mathcal{B}(X)$ we have

$$\begin{aligned}
\pi^X_{\sigma,\alpha}(\omega(A)=n) &= \mathrm{e}^{-\sigma(A)}\sum_{k=n}^{\infty}\frac{\sigma(A)^k}{k!}\alpha^n(1-\alpha)^{k-n}\binom{k}{n} \\
&= \mathrm{e}^{-\sigma(A)}\frac{(\alpha\sigma(A))^n}{n!}\sum_{k=n}^{\infty}\frac{\sigma(A)^{k-n}}{(k-n)!}(1-\alpha)^{k-n} \\
&= \mathrm{e}^{-\alpha\sigma(A)}\frac{(\alpha\sigma(A))^n}{n!}.
\end{aligned}$$

□

Remark 6.1.6. *The construction of Poisson measures with a diffuse intensity measure can be extended to not necessarily diffuse intensities.*

Proof. In case σ is not diffuse, we can identify the atoms $(x_k)_{k\in\mathbb{N}}$ of σ which are at most countably infinite with masses $(\mu(x_k))_{k\in\mathbb{N}}$. Next we choose a family $(X_k, \nu_k)_{k\in\mathbb{N}}$ of measure spaces such that ν_k is diffuse and $\nu_k(X_k) = \mu(\{x_k\})$, $k \in \mathbb{N}$. Letting

$$\bar{X} := (X \setminus \{x_0, x_1, \ldots\}) \cup \bigcup_{k=0}^{\infty} X_k$$

and

$$\bar{\mu} := \mu + \sum_{k=0}^{\infty} \nu_k,$$

then $\bar{\mu}$ is a diffuse measure on $\bar{X}$. Next, letting

$$f(x) = x\mathbf{1}_{X\setminus\{x_0,x_1,\ldots\}}(x) + \sum_{k=1}^{\infty} x_k \mathbf{1}_{X_k}(x), \qquad x \in \bar{X},$$

we note that

$$\int_{\bar{X}} f(x)(\omega(dx) - \bar{\mu}(dx))$$

has an infinitely divisible law with Lévy measure μ since

$$\int_X (e^{iux} - iux - 1)\mu(dx) = \int_{\bar{X}} (e^{iuf(x)} - iuf(x) - 1)\bar{\mu}(dx).$$

□

More generally, functionals on the Poisson space on (X, μ) of the form

$$F(\omega) = f(\omega(A_1), \ldots, \omega(A_n))$$

can be constructed on the Poisson space on $(\bar{X}, \bar{\mu})$ as

$$\tilde{F}(\omega) = f(\omega(B_1), \ldots, \omega(B_n)),$$

with

$$B_i = (A_i \setminus \{x_0, x_2, \ldots\}) \bigcup \left(\bigcup_{k \ : \ x_k \in A_i} X_k \right).$$

Poisson random measures on a metric space X can be constructed from the Poisson process on $\mathbb{R}_+$ by identifying X with $\mathbb{R}_+$. More precisely we have the following result, see e.g. [36], p. 192.

Proposition 6.1.7. *There exists a measurable map*

$$\tau : X \to \mathbb{R}_+,$$

a.e. bijective, such that $\lambda = \tau_* \sigma$, i.e. the Lebesgue measure is the image of σ by τ.

We denote by $\tau_* \omega$ the image measure of ω by τ, i.e. $\tau_* : \Omega^X \to \Omega$ maps

$$\omega = \sum_{i=1}^{\infty} \epsilon_{x_i} \qquad \text{to} \qquad \tau_* \omega = \sum_{i=1}^{\infty} \epsilon_{\tau(x_i)}. \tag{6.1.11}$$

We have, for $A \in \mathcal{F}^X$:

$$\begin{aligned} \tau_* \omega(A) &= \#\{x \in \omega \ : \ \tau(x) \in A\} \\ &= \#\{x \in \omega \ : \ x \in \tau^{-1}(A)\} \\ &= \omega(\tau^{-1}(A)). \end{aligned}$$

Proposition 6.1.8. *The application $\tau_* : \Omega^X \to \Omega$ maps the Poisson measure π_σ on Ω^X to the Poisson measure π_λ on Ω.*

Proof. It suffices to check that for all families $A_1, \ldots, A_n$ of disjoint Borel subsets of X and $k_1, \ldots, k_n \in \mathbb{N}$, we have

$$\pi_\sigma(\{\omega \in \Omega^X \ : \ \tau_* \omega(A_1) = k_1, \ldots, \tau_* \omega(A_n) = k_n\})$$

$$
\begin{aligned}
&= \prod_{i=1}^n \pi_\sigma(\{\tau_*\omega(A_i) = k_i\}) \\
&= \prod_{i=1}^n \pi_\sigma(\{\omega(\tau^{-1}(A_i)) = k_i\}) \\
&= \exp\left(-\sum_{i=1}^n \sigma(\tau^{-1}(A_i))\right) \prod_{i=1}^n \frac{(\sigma(\tau^{-1}(A_i)))^{k_i}}{k_i!} \\
&= \exp\left(-\sum_{i=1}^n \lambda(A_i)\right) \prod_{i=1}^n \frac{(\lambda(A_i))^{k_i}}{k_i!} \\
&= \prod_{i=1}^n \pi_\lambda(\{\omega(A_i) = k_i\}) \\
&= \pi_\lambda(\{\omega(A_1) = k_1, \ldots, \omega(A_n) = k_n\}).
\end{aligned}
$$

□

Clearly, $F \longmapsto F \circ \tau^*$ defines an isometry from $L^p(\Omega) \to L^p(\Omega^X)$, $p \geq 1$, and similarly we get that $\int_X f \circ \tau(x)\ \tau_*\omega(dx)$ has same distribution as $\int_0^\infty f(t)\omega(dt)$, since

$$
\begin{aligned}
\mathbb{E}_{\pi_\sigma}\left[\exp\left(i\int_X f(\tau(x))\tau_*\omega(dx)\right)\right] &= \exp\left(\int_X (\mathrm{e}^{if(\tau(x))} - 1)\sigma(dx)\right) \\
&= \exp\left(\int_0^\infty (\mathrm{e}^{if(t)} - 1)\lambda(dt)\right) \\
&= \mathbb{E}_{\pi_\lambda}\left[\exp\left(i\int_0^\infty f(t)\omega(dt)\right)\right].
\end{aligned}
$$

Using the measurable bijection $\tau : X \to \mathbb{R}_+$ and the relation

$$
D_t F(\omega) = \left(D_{\tau^{-1}(t)} F \circ (\tau^*)^{-1}\right) \circ \tau^*(\omega), \quad t \in \mathbb{R}_+, \quad \omega \in \Omega,
$$

we can also restate Proposition 4.7.3 for a Poisson measure on X.

Corollary 6.1.9. *Let $F \in \mathrm{Dom}\,(D)$ be such that $DF \leq K$, a.s., for some $K \geq 0$, and $\|DF\|_{L^\infty(\Omega, L^2(X))} < \infty$. Then*

$$
\begin{aligned}
\mathbb{P}(F - \mathbb{E}[F] \geq x) &\leq \exp\left(-\frac{\|DF\|^2_{L^\infty(\Omega,L^2(X))}}{K^2} g\left(\frac{xK}{\|DF\|^2_{L^\infty(\Omega,L^2(X))}}\right)\right) \\
&\leq \exp\left(-\frac{x}{2K}\log\left(1 + \frac{xK}{\|DF\|^2_{L^\infty(\Omega,L^2(X))}}\right)\right),
\end{aligned}
$$

with $g(u) = (1+u)\log(1+u) - u$, $u \geq 0$. If $K = 0$ (decreasing functionals) we have

$$\mathbb{P}(F - \mathbb{E}[F] \geq x) \leq \exp\left(-\frac{x^2}{2\|DF\|^2_{L^\infty(\Omega,L^2(X))}}\right). \tag{6.1.12}$$

In particular if $F = \int_X f(y)\omega(dy)$ we have $\|DF\|_{L^\infty(\Omega,L^2(X))} = \|f\|_{L^2(X)}$ and if $f \leq K$, a.s., then

$$\mathbb{P}\left(\int_X f(y)(\omega(dy) - \sigma(dy)) \geq x\right)$$
$$\leq \exp\left(-\frac{\int_X f^2(y)\sigma(dy)}{K^2} g\left(\frac{xK}{\int_X f^2(y)\sigma(dy)}\right)\right).$$

If $f \leq 0$, a.s., then

$$\mathbb{P}\left(\int_X f(y)(\omega(dy) - \sigma(dy)) \geq x\right) \leq \exp\left(-\frac{x^2}{2\int_X f^2(y)\sigma(dy)}\right).$$

This result will be recovered in Section 6.9, cf. Proposition 6.9.3 below.

6.2 Multiple Poisson Stochastic Integrals

We start by considering the particular case of the Poisson space

$$\Omega = \left\{\omega = \sum_{k=1}^{n} \epsilon_{t_k} \ : \ 0 \leq t_1 < \cdots < t_n, \ n \in \mathbb{N} \cup \{\infty\}\right\}$$

on $X = \mathbb{R}_+$, where we dropped the upper index $\mathbb{R}_+$ in $\Omega^{\mathbb{R}_+}$, with intensity measure

$$\nu(dx) = \lambda dx, \qquad \lambda > 0.$$

In this case the configuration points can be arranged in an ordered fashion and the Poisson martingale of Section 2.3 can be constructed as in the next proposition.

Proposition 6.2.1. *The Poisson process $(N_t)_{t\in\mathbb{R}_+}$ of Definition 2.3.1 can be constructed as*

$$N_t(\omega) = \omega([0,t]), \qquad t \in \mathbb{R}_+.$$

Proof. Clearly, the paths of $(N_t)_{t\in\mathbb{R}_+}$ are piecewise continuous, càdlàg (i.e. continuous on the right with left limits), with jumps of height equal to one. Moreover, by definition of the Poisson measure on Ω, the vector $(N_{t_1} - N_{t_0}, \ldots, N_{t_n} - N_{t_{n-1}})$ of Poisson process increments, $0 \leq t_0 < t_1 < \cdots < t_n$, is made of independent, Poisson distributed, random variables with parameters $\lambda(t_1 - t_0), \ldots, \lambda(t_n - t_{n-1})$. Hence the law of $(N_t)_{t\in\mathbb{R}_+}$ coincides with that of the standard Poisson process defined, cf. Corollary 2.3.5 in Section 2.3. □

In other words, every configuration $\omega \in \Omega$ can be viewed as the ordered sequence $\omega = (T_k)_{k\geq 1}$ of jump times of $(N_t)_{t\in\mathbb{R}_+}$ on $\mathbb{R}_+$.

Applying Corollary 2.5.11 and using induction on $n \geq 1$ yields the following result.

Proposition 6.2.2. *Let $f_n : \mathbb{R}_+^n \longmapsto \mathbb{R}$ be continuous with compact support in $\mathbb{R}_+^n$. Then we have the $\mathbb{P}(d\omega)$-almost sure equality*

$$I_n(f_n)(\omega) = n! \int_0^\infty \int_0^{t_n^-} \cdots \int_0^{t_2^-} f_n(t_1,\ldots,t_n)(\omega(dt_1) - dt_1)\cdots(\omega(dt_n) - dt_n).$$

The above formula can also be written as

$$I_n(f_n) = n! \int_0^\infty \int_0^{t_n^-} \cdots \int_0^{t_2^-} f_n(t_1,\ldots,t_n) d(N_{t_1} - t_1)\cdots d(N_{t_n} - t_n),$$

and by symmetry of f_n in n variables we have

$$I_n(f_n) = \int_{\Delta_n} f_n(t_1,\ldots,t_n)(\omega(dt_1) - dt_1)\cdots(\omega(dt_n) - dt_n),$$

with

$$\Delta_n = \{(t_1,\ldots,t_n) \in \mathbb{R}_+^n \ : \ t_i \neq t_j, \ \forall i \neq j\}.$$

Using the mappings $\tau : X \to \mathbb{R}_+$ of Proposition 6.1.7 and

$$\tau_* : \Omega^X \to \Omega$$

defined in (6.1.11), we can extend the construction of multiple Poisson stochastic integrals to the setting of an abstract set X of indices.

Definition 6.2.3. *For all $f_n \in \mathcal{C}_c(\Delta_n)$, let*

$$I_n^X(f_n)(\omega) := I_n(f_n \circ \tau^{-1})(\tau_*\omega). \tag{6.2.1}$$

Letting

$$\Delta_n^X = \{(x_1,\ldots,x_n) \in X^n \ : \ x_i \neq x_j, \ \forall i \neq j\},$$

we have

$$\begin{aligned} I_n^X(f_n)(\omega) &= I_n(f_n \circ \tau^{-1})(\tau_*\omega) \qquad (6.2.2)\\ &= \int_{\Delta_n} f_n(\tau^{-1}(t_1),\ldots,\tau^{-1}(t_n))(\tau_*\omega(dt_1) - \lambda dt_1)\cdots(\tau_*\omega(dt_n) - \lambda dt_n)\\ &= \int_{\Delta_n^X} f_n(x_1,\ldots,x_n)(\omega(dx_1) - \sigma(dx_1))\cdots(\omega(dx_n) - \sigma(dx_n)), \end{aligned}$$

and for $g_{n+1} \in \mathcal{C}_c(\Delta_{n+1})$,

$$I_{n+1}^X(g_{n+1})$$

$$= \int_{\Delta_{n+1}^X} g_n(x_1,\ldots,x_n,x)(\omega(dx)-\sigma(dx))(\omega(dx_1)-\sigma(dx_1))\cdots$$
$$\cdots(\omega(dx_n)-\sigma(dx_n))$$
$$= \int_X \int_{\Delta_n^X} \mathbf{1}_{\{x\notin\{x_1,\ldots,x_n\}\}} g_n(x_1,\ldots,x_n,x)(\omega(dx_1)-\sigma(dx_1))\cdots$$
$$\cdots(\omega(dx_n)-\sigma(dx_n))(\omega(dx)-\sigma(dx))$$
$$= \int_{\Delta_n^X} I_n^X(g_{n+1}(*,x))(\omega\setminus\{x\})(\omega(dx)-\sigma(dx)), \tag{6.2.3}$$

where $\omega\setminus\{x\}$ denotes the configuration $\omega\in\Omega^X$ after removal of the configuration point $x\in X$, if x belongs to ω, i.e.

$$\omega\setminus\{x\} := \sum_{\substack{k=0\\ x_k\neq x}}^n \epsilon_{x_k} = \sum_{k=0}^n \mathbf{1}_{\{x\neq x_k\}}\epsilon_{x_k}, \qquad x\in X.$$

The integral $I_n^X(f_n)$ extends to symmetric functions in $f_n\in L^2(X)^{\circ n}$ via the following isometry formula.

Proposition 6.2.4. *For all symmetric functions $f_n\in L^2(X,\sigma)^{\circ n}$, $g_m\in L^2(X,\sigma)^{\circ m}$, we have*

$$\mathbb{E}_{\pi_\sigma}\left[I_n^X(f_n)I_m^X(g_m)\right] = n!\mathbf{1}_{\{n=m\}}\langle f_n,g_m\rangle_{L^2(X,\sigma)^{\circ n}}. \tag{6.2.4}$$

Proof. Denoting $f_n(\tau^{-1}(x_1),\ldots,\tau^{-1}(x_n))$ by $f_n\circ\tau^{-1}(x_1,\ldots,x_n)$ we have

$$\begin{aligned}
\mathbb{E}_{\pi_\sigma}[I_n^X(f_n)I_m^X(g_m)] &= \mathbb{E}_{\pi_\sigma}[I_n(f_n\circ\tau^{-1})(\tau_*\omega)I_m(g_m\circ\tau^{-1})(\tau_*\omega)]\\
&= \mathbb{E}_{\pi_\lambda}[I_n(f_n\circ\tau^{-1})I_m(g_m\circ\tau^{-1})]\\
&= n!\mathbf{1}_{\{n=m\}}\langle f_n\circ\tau^{-1},g_m\circ\tau^{-1}\rangle_{L^2(\mathbb{R}_+^n,\lambda^{\otimes n})}\\
&= n!\mathbf{1}_{\{n=m\}}\langle f_n,g_m\rangle_{L^2(X^n,\sigma^{\otimes n})}.
\end{aligned}$$

□

We have the following multiplication formula, in which we again use the convention $I_0^X(f_0)=f_0$ for $f_0\in\mathbb{R}$.

Proposition 6.2.5. *We have for $u,v\in L^2(X,\sigma)$ such that $uv\in L^2(X,\sigma)$:*

$$I_1^X(u)I_n^X(v^{\otimes n}) \tag{6.2.5}$$
$$= I_{n+1}^X(v^{\otimes n}\circ u) + nI_n^X((uv)\circ v^{\otimes(n-1)}) + n\langle u,v\rangle_{L^2(X,\sigma)}I_{n-1}^X(v^{\otimes(n-1)}).$$

Proof. This result can be proved by direct computation from (6.2.1). Alternatively it can be proved first for $X=\mathbb{R}_+$ using stochastic calculus or directly from Proposition 4.5.1 with $\phi_t=1$, $t\in\mathbb{R}_+$:

$$I_1(u\circ\tau^{-1})I_n((v\circ\tau^{-1})^{\otimes n})$$
$$= I_{n+1}((v\circ\tau^{-1})^{\otimes n}\circ u\circ\tau^{-1}) + nI_n((u\circ\tau^{-1}v\circ\tau^{-1})\circ(v\circ\tau^{-1})^{\otimes(n-1)})$$
$$+n\langle u\circ\tau^{-1},v\circ\tau^{-1}\rangle_{L^2(\mathbb{R}_+,\lambda)}I_{n-1}((v\circ\tau^{-1})^{\otimes(n-1)}),$$

and then extended to the general setting of metric spaces using the mapping

$$\tau_* : \Omega^X \to \Omega$$

of Proposition 6.1.8 and Relation (6.2.2). □

Similarly using the mapping $\tau_* : \Omega^X \to \Omega$ and Proposition 4.5.6 we have

$$I_n(f_n)I_m(g_m) = \sum_{s=0}^{2(n\wedge m)} I_{n+m-s}(h_{n,m,s}),$$

$f_n \in L^2(X,\sigma)^{\circ n}$, $g_m \in L^2(X,\sigma)^{\circ m}$, where

$$h_{n,m,s} = \sum_{s\leq 2i \leq 2(s\wedge n\wedge m)} i! \binom{n}{i}\binom{m}{i}\binom{i}{s-i} f_n \circ_i^{s-i} g_m,$$

and $f_n \circ_k^l g_m$, $0 \leq l \leq k$, is the symmetrization of

$$(x_{l+1},\ldots,x_n,y_{k+1},\ldots,y_m) \longmapsto \int_{X^l} f_n(x_1,\ldots,x_n)g_m(x_1,\ldots,x_k,y_{k+1},\ldots,y_m)\sigma(dx_1)\cdots\sigma(dx_l)$$

in $n+m-k-l$ variables.

Given $f_{k_1} \in L^2(X,\sigma)^{k_1},\ldots,f_{k_d} \in L^2(X,\sigma)^{k_d}$ with disjoint supports we have

$$I_n(f_{k_1} \circ \cdots \circ f_{k_d}) = \prod_{i=1}^{d} I_{k_i}(f_{k_i}), \tag{6.2.6}$$

for $n = k_1 + \cdots + k_d$.

Remark 6.2.6. *Relation (6.2.5) implies that the linear space generated by*

$$\{I_n(f_1 \otimes \cdots \otimes f_n) \ : \ f_1,\ldots,f_n \in \mathcal{C}_c^\infty(X), \ n \in \mathbb{N}\},$$

coincides with the space of polynomials in first order integrals of the form $I_1(f)$, $f \in \mathcal{C}_c^\infty(X)$.

Next we turn to the relation between multiple Poisson stochastic integrals and the Charlier polynomials.

Definition 6.2.7. *Let the Charlier polynomial of order $n \in \mathbb{N}$ and parameter $t \geq 0$ be defined by*

$$C_0(k,t) = 1, \quad C_1(k,t) = k - t, \quad C_2(k,t) = k^2 - k(2t+1) + t^2,$$

$k \in \mathbb{R}$, $t \in \mathbb{R}_+$, and the recurrence relation

$$C_{n+1}(k,t) = (k-n-t)C_n(k,t) - ntC_{n-1}(k,t), \qquad n \geq 1. \tag{6.2.7}$$

Let

$$p_k(t) = e^{-t}\frac{t^k}{k!}, \qquad k \in \mathbb{N}, \quad t \in \mathbb{R}_+, \tag{6.2.8}$$

denote the Poisson probability density, which satisfies the finite difference differential equation

$$\frac{\partial p_k}{\partial t}(t) = -\varDelta p_k(t), \tag{6.2.9}$$

where $\varDelta$ is the difference operator

$$\varDelta f(k) := f(k) - f(k-1), \qquad k \in \mathbb{N}.$$

Let also

$$\psi_\lambda(k,t) = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} C_n(k,t), \qquad \lambda \in (-1,1),$$

denote the generating function of Charlier polynomials.

Proposition 6.2.8. *For all $k \in \mathbb{Z}$ and $t \in \mathbb{R}_+$ we have the relations*

$$C_n(k,t) = \frac{(-1)^n}{p_k(t)} t^n (\varDelta)^n p_k(t), \tag{6.2.10}$$

$$C_n(k,t) = \frac{t^n}{p_k(t)} \frac{\partial^n p_k}{\partial t^n}(t), \tag{6.2.11}$$

$$C_n(k+1,t) - C_n(k,t) = -\frac{\partial C_n}{\partial t}(k,t), \tag{6.2.12}$$

$$C_n(k+1,t) - C_n(k,t) = nC_{n-1}(k,t), \tag{6.2.13}$$

$$C_{n+1}(k,t) = kC_n(k-1,t) - tC_n(k,t), \tag{6.2.14}$$

and the generating function $\psi_\lambda(k,t)$ satisfies

$$\psi_\lambda(k,t) = \mathrm{e}^{-\lambda t}(1+\lambda)^k, \tag{6.2.15}$$

$\lambda, t > 0$, $k \in \mathbb{N}$.

Proof. By the Definition 6.2.8 of $p_k(t)$ it follows that

$$\frac{(-1)^n}{p_k(t)} t^n (\varDelta)^n p_k(t)$$

satisfies the recurrence relation (6.2.7), i.e.

$$\frac{(-1)^{n+1}}{p_k(t)} t^{n+1} (\Delta)^{n+1} p_k(t)$$
$$= (k-n-t) \frac{(-1)^n}{p_k(t)} t^n (\Delta)^n p_k(t) - nt \frac{(-1)^{n-1}}{p_k(t)} t^{n-1} (\Delta)^{n-1} p_k(t),$$

as well as its initial conditions, hence (6.2.10) holds. Relation (6.2.11) then follows from Equation (6.2.9). On the other hand, the process

$$(C_n(N_t,t))_{t\in\mathbb{R}_+} = (I_n(\mathbf{1}_{[0,t]}^{\otimes n}))_{t\in\mathbb{R}_+}$$

is a martingale from Lemma 2.7.2 and can using Itô's formula Proposition 2.12.1 it can be written as

$$\begin{aligned}
C_n(N_t,t) &= I_n(\mathbf{1}_{[0,t]}^{\otimes n}) \\
&= C_n(0,0) + \int_0^t (C_n(N_{s^-}+1,s) - C_n(N_{s^-},s)) d(N_s - s) \\
&\quad + \int_0^t \left((C_n(N_{s^-}+1,s) - C_n(N_{s^-},s)) + \frac{\partial C_n}{\partial s}(N_s,s) \right) ds \\
&= n \int_0^t I_{n-1}(\mathbf{1}_{[0,s]}^{\otimes (n-1)}) d(N_s - s) \\
&= n \int_0^t C_{n-1}(N_{s^-},s) d(N_s - s)
\end{aligned}$$

where the last integral is in the Stieltjes sense of Proposition 2.5.10, hence Relations (6.2.12) and (6.2.13) hold. Next, Relation (6.2.14) follows from (6.2.11) and (6.2.9) as

$$\begin{aligned}
C_{n+1}(k,t) &= \frac{t^{n+1}}{p_k(t)} \frac{\partial^{n+1} p_k}{\partial t^{n+1}}(t) \\
&= -\frac{t^{n+1}}{p_k(t)} \frac{\partial^n p_k}{\partial t^n}(t) + \frac{t^{n+1}}{p_k(t)} \frac{\partial^n p_{k-1}}{\partial t^n}(t) \\
&= -t \frac{t^n}{p_k(t)} \frac{\partial^n p_k}{\partial t^n}(t) + k \frac{t^n}{p_{k-1}(t)} \frac{\partial^n p_{k-1}}{\partial t^n}(t) \\
&= -tC_n(k,t) + kC_n(k-1,t).
\end{aligned}$$

Finally, using Relation (6.2.14) we have

$$\begin{aligned}
\frac{\partial \psi_\lambda}{\partial \lambda}(k,t) &= \sum_{n=1}^\infty \frac{\lambda^{n-1}}{(n-1)!} C_n(k,t) \\
&= \sum_{n=0}^\infty \frac{\lambda^n}{n!} C_{n+1}(k,t)
\end{aligned}$$

$$\begin{aligned}
&= -t \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} C_n(k-1,t) + k \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} C_n(k,t) \\
&= -t \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} C_n(k-1,t) + k \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} C_n(k,t) \\
&= -t\psi_\lambda(k,t) + k\psi_\lambda(k-1,t),
\end{aligned}$$

$\lambda \in (-1,1)$, hence the generating function $\psi_\lambda(k,t)$ satisfies the differential equation

$$\frac{\partial \psi_\lambda}{\partial \lambda}(k,t) = -\lambda\psi_\lambda(k,t) + k\psi_\lambda(k-1,t), \quad \psi_0(k,t) = 1, \quad k \geq 1,$$

which yields (6.2.15) by induction on k. □

We also have

$$\frac{\partial^k p_k}{\partial t^k}(t) = (-\Delta)^k p_k(t).$$

The next proposition links the Charlier polynomials with multiple Poisson stochastic integrals.

Proposition 6.2.9. *The multiple Poisson stochastic integral of the function*

$$\mathbf{1}_{A_1}^{\otimes k_1} \circ \cdots \circ \mathbf{1}_{A_d}^{\otimes k_d}$$

satisfies

$$I_n(\mathbf{1}_{A_1}^{\otimes k_1} \circ \cdots \circ \mathbf{1}_{A_d}^{\otimes k_d})(\omega) = \prod_{i=1}^{d} C_{k_i}(\omega(A_i), \sigma(A_i)), \tag{6.2.16}$$

provided that $A_1, \ldots, A_d$ are mutually disjoint compact subsets of X and $n = k_1 + \cdots + k_d$.

Proof. We have

$$I_0(\mathbf{1}_A^{\otimes 0}) = 1 = C_0(\omega(A), \sigma(A)),$$

and

$$I_1(\mathbf{1}_A)(\omega) = \omega(A) - \sigma(A) = C_1(\omega(A), \sigma(A)).$$

On the other hand, by Proposition 6.2.5 we have the recurrence relation

$$\begin{aligned}
&I_1(\mathbf{1}_B) I_k(\mathbf{1}_A^{\otimes k}) \\
&= I_{k+1}(\mathbf{1}_A^{\otimes k} \circ \mathbf{1}_B) + k I_k(\mathbf{1}_{A\cap B} \circ \mathbf{1}_A^{\otimes (k-1)}) + k\sigma(A\cap B) I_{k-1}(\mathbf{1}_A^{\otimes (k-1)}),
\end{aligned}$$

which, when $A = B$, coincides with the Relation (6.2.7) that defines the Charlier polynomials, hence by induction on $k \in \mathbb{N}$ we obtain

$$I_k(\mathbf{1}_A^{\otimes k})(\omega) = C_k(\omega(A)).$$

Finally from (6.2.6) we have

$$I_n(\mathbf{1}_{A_1}^{\otimes k_1} \circ \cdots \circ \mathbf{1}_{A_d}^{\otimes k_d}) = \prod_{i=1}^{d} I_{k_i}(\mathbf{1}_{A_i}^{\otimes k_i})$$
$$= \prod_{i=1}^{d} C_{k_i}(\omega(A_i), \sigma(A_i)),$$

which shows (6.2.16). □

In this way we recover the orthogonality properties of the Charlier polynomials with respect to the Poisson distribution, with $t = \sigma(A)$:

$$\begin{aligned} \langle C_n(\cdot,t), C_m(\cdot,t)\rangle_{\ell^2(\mathbb{N},p_\cdot(t))} &= \mathrm{e}^{-t}\sum_{k=0}^{\infty} \frac{t^k}{k!} C_n(k,t)C_m(k,t) \\ &= \mathbb{E}[C_n(\omega(A),t)C_m(\omega(A),t)] \\ &= \mathbb{E}[I_n(\mathbf{1}_A^{\otimes n})I_m(\mathbf{1}_A^{\otimes m})] \\ &= \mathbf{1}_{\{n=m\}} n! t^n. \end{aligned} \tag{6.2.17}$$

The next lemma is the Poisson space version of Lemma 5.1.6.

Lemma 6.2.10. *Let F of the form*

$$F(\omega) = g(\omega(A_1), \ldots, \omega(A_k))$$

where

$$\sum_{l_1,\ldots,l_k=0}^{\infty} |g(l_1,\ldots,l_k)|^2 p_{l_1}(\sigma(A_1)) \cdots p_{l_k}(\sigma(A_k)) < \infty, \tag{6.2.18}$$

and $A_1, \ldots, A_k$ are compact disjoint subsets of X. Then F admits the chaos expansion

$$F = \sum_{n=0}^{\infty} I_n(f_n),$$

where for all $n \geq 1$, $I_n(f_n)$ can be written as a linear combination

$$I_n(f_n)(\omega) = P_n(\omega(A_1), \ldots, \omega(A_n), \sigma(A_1), \ldots \sigma(A_n))$$

of multivariate Charlier polynomials

$$C_{l_1}(\omega(A_1), \sigma(A_1)) \cdots C_{l_k}(\omega(A_k), \sigma(A_k)).$$

Proof. We decompose g satisfying (6.2.18) as an orthogonal series

$$g(i_1, \ldots, i_k) = \sum_{n=0}^{\infty} P_n(i_1, \ldots, i_n, \sigma(A_1), \ldots, \sigma(A_n)),$$

where

$$P_n(i_1, \ldots, i_k, \sigma(A_1), \ldots, \sigma(A_k)) \\ = \sum_{l_1+\cdots+l_k=n} \alpha_{l_1,\ldots,l_k} C_{l_1}(i_1, \sigma(A_1)) \cdots C_{l_k}(i_k, \sigma(A_k))$$

is a linear combination of multivariate Charlier polynomials of degree n which is identified to the multiple stochastic integral

$$\begin{aligned} P_n(\omega(A_1), \ldots, \omega(A_n), \sigma(A_1), \ldots \sigma(A_n)) \\ &= \sum_{l_1+\cdots+l_k=n} \alpha_{l_1,\ldots,l_k} I_n(\mathbf{1}_{A_1}^{\otimes k_1} \circ \cdots \circ \mathbf{1}_{A_d}^{\otimes k_d}) \\ &= I_n\left(\sum_{l_1+\cdots+l_k=n} \alpha_{l_1,\ldots,l_k} \mathbf{1}_{A_1}^{\otimes k_1} \circ \cdots \circ \mathbf{1}_{A_d}^{\otimes k_d} \right) \\ &= I_n(f_n), \end{aligned}$$

by Proposition 6.2.9. □

6.3 Chaos Representation Property

The following expression of the exponential vector

$$\xi(u) = \sum_{k=0}^{\infty} \frac{1}{n!} I_n(u^{\otimes n})$$

is referred to as the Doléans exponential.

Proposition 6.3.1. *For all $u \in L^2(X)$ we have*

$$\xi(u) = \exp\left(\int_X u(x)(\omega(dx) - \sigma(dx)) \right) \prod_{x\in\omega} ((1 + u(x))\mathrm{e}^{-u(x)}).$$

Proof. The case $X = \mathbb{R}_+$ is treated in Proposition 2.13.1, in particular when $\phi_t = 1$, $t \in \mathbb{R}_+$, and the extension to X a metric space is obtained using the isomorphism $\tau : X \to \mathbb{R}_+$ of Proposition 6.1.7. □

In particular, from Proposition 6.2.9 the exponential vector $\xi(\lambda\mathbf{1}_A)$ satisfies

$$\xi(\lambda\mathbf{1}_A) = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} C_n(\omega(A), \sigma(A))$$

$$= e^{-\lambda\sigma(A)}(1+\lambda)^{\omega(A)}$$
$$= \psi_\lambda(\omega(A), \sigma(A)).$$

Next we show that the Poisson measure has the chaos representation property, i.e. every square-integrable functional on Poisson space has an orthogonal decomposition in a series of multiple stochastic integrals.

Proposition 6.3.2. *Every square-integrable random variable $F \in L^2(\Omega^X, \pi_\sigma)$ admits the Wiener-Poisson decomposition*

$$F = \sum_{n=0}^{\infty} I_n(f_n)$$

in series of multiple stochastic integrals.

Proof. A modification of the proof of Theorem 4.1 in [52], cf. also Theorem 1.3 of [70], shows that the linear space spanned by

$$\left\{ e^{-\int_X u(x)\sigma(dx)} \prod_{x\in\omega}(1+u(x)) \ : \ u \in \mathcal{C}_c(X) \right\}$$

is dense in $L^2(\Omega^X)$. This concludes the proof since this space is contained in the closure of $\mathcal{S}$ in $L^2(\Omega^X)$. □

As a corollary, the standard Poisson process $(N_t)_{t\in\mathbb{R}_+}$ has the chaos representation property.

As in the Wiener case, cf. Relation (5.1.8), Proposition 6.3.2 implies that any $F \in L^2(\Omega)$ has a chaos decomposition

$$F = \sum_{n=0}^{\infty} I_n(g_n),$$

where

$$I_n(g_n) = \sum_{d=1}^{n} \sum_{k_1+\cdots+k_d=n} \frac{1}{k_1!\cdots k_d!} I_n(u_1^{\otimes k_1}\circ\cdots\circ u_d^{\otimes k_d})\, \mathbb{E}[F I_n(u_1^{\otimes k_1}\circ\cdots\circ u_d^{\otimes k_d})], \tag{6.3.1}$$

for any orthonormal basis $(u_n)_{n\in\mathbb{N}}$ of $L^2(X,\sigma)$, which completes the statement of Lemma 6.2.10.

Consider now the compound Poisson process

$$Y_t = \sum_{k=1}^{N_t} Y_k,$$

of Definition 2.4.1, where $(Y_k)_{k\geq 1}$ is an i.i.d. sequence of random variables with distribution μ on $\mathbb{R}^d$ and $(N_t)_{t\in\mathbb{R}_+}$ is a Poisson process with intensity $\lambda > 0$, can be constructed as

$$Y_t = \int_0^t \int_{\mathbb{R}^d} x\omega(ds, dx), \tag{6.3.2}$$

by taking $X = \mathbb{R}_+ \times \mathbb{R}^d$ and $\sigma(ds, dx) = \lambda ds\mu(dx)$. The compensated compound Poisson process

$$X_t = \left(\sum_{k=1}^{N_t} Y_k\right) - \lambda t\,\mathbb{E}[Y_1], \qquad t \in \mathbb{R}_+,$$

of Section 2.4 has the chaos representation property if and only if Y_k is a.s. constant, i.e. when $(M_t)_{t\in\mathbb{R}_+}$ is the compensated Poisson martingale, cf. Section 2.10 and Proposition 4.2.4.

Next we turn to some practical computations of chaos expansions in the Poisson case. In particular, from (6.2.17) we deduce the orthogonal expansion

$$\mathbf{1}_{\{N_t - N_s = n\}} = \sum_{k=0}^{\infty} \frac{1}{k!(t-s)^k} \langle \mathbf{1}_{\{n\}}, C_k^{t-s} \rangle_{\ell^2(\mathbb{Z}, p^{t-s})} C_k^{t-s}(N_t - N_s),$$

$0 \leq s \leq t$, $n \in \mathbb{N}$, hence from (6.2.11):

$$\mathbf{1}_{\{N_t - N_s = n\}} = \sum_{k=0}^{\infty} \frac{1}{k!} p_n^{(k)}(t-s) I_k(\mathbf{1}_{[s,t]}^{\otimes k}), \tag{6.3.3}$$

$0 \leq s \leq t$, $n \in \mathbb{N}$.

From (6.3.3) we obtain for $s = 0$ and $n \geq 1$:

$$\begin{aligned} \mathbf{1}_{[T_n,\infty[}(t) &= \mathbf{1}_{\{N_t \geq n\}} \\ &= \sum_{k=0}^{\infty} \sum_{l \geq n} \frac{1}{k!} p_l^{(k)}(t) I_k(\mathbf{1}_{[0,t]}^{\otimes k}) \\ &= \sum_{k=0}^{\infty} \frac{1}{k!} \partial^k P_n(t) I_k(\mathbf{1}_{[0,t]}^{\otimes k}), \end{aligned} \tag{6.3.4}$$

where

$$P_n(t) = \int_0^t p_{n-1}(s) ds, \qquad t \in \mathbb{R}_+, \tag{6.3.5}$$

is the distribution function of T_n and $p_n(s)$ is defined in (2.3.1).

More generally, we have the following result.

Proposition 6.3.3. *Let $f \in \mathcal{C}_b^1(\mathbb{R}_+)$. We have*

$$f(T_n) = -\sum_{k=0}^{\infty} \frac{1}{k!} I_k \left(\int_{t_1 \vee \cdots \vee t_k}^{\infty} f'(s) P_n^{(k)}(s) ds \right), \qquad (6.3.6)$$

where $t_1 \vee \cdots \vee t_n = \max(t_1, \ldots, t_n)$, $t_1, \ldots, t_n \in \mathbb{R}_+$.

Proof. We have

$$\begin{aligned} f(T_n) &= -\int_0^\infty f'(s) \mathbf{1}_{[T_n,\infty)}(s) ds \\ &= -\int_0^\infty f'(s) \sum_{k=0}^{\infty} \frac{1}{k!} P_n^{(k)}(s) I_k(\mathbf{1}_{[0,s]}^{\otimes k}) ds \\ &= -\sum_{k=0}^{\infty} \frac{1}{k!} \int_0^\infty f'(s) P_n^{(k)}(s) I_k(\mathbf{1}_{[0,s]}^{\otimes k}) ds \\ &= -\sum_{k=0}^{\infty} \int_0^\infty f'(s) P_n^{(k)}(s) \int_0^s \int_0^{t_k} \cdots \int_0^{t_2} d\tilde{N}_{t_1} \cdots d\tilde{N}_{t_k} ds \\ &= -\sum_{k=0}^{\infty} \int_0^\infty f'(s) P_n^{(k)}(s) \int_0^\infty \int_0^{t_k} \cdots \int_0^{t_2} \mathbf{1}_{[0,s]}(t_1 \vee \cdots \vee t_k) d\tilde{N}_{t_1} \cdots d\tilde{N}_{t_k} ds \\ &= -\sum_{k=0}^{\infty} \int_0^\infty \left(\int_{t_k}^\infty f'(s) P_n^{(k)}(s) ds \int_0^{t_k} \cdots \int_0^{t_2} d\tilde{N}_{t_1} \cdots d\tilde{N}_{t_{k-1}}, \right) d\tilde{N}_{t_k}. \end{aligned}$$

□

Note that Relation (6.3.6) can be rewritten after integration by parts on $\mathbb{R}_+$ as

$$\begin{aligned} &f(T_n) \qquad (6.3.7) \\ &= \sum_{k=0}^{\infty} \frac{1}{k!} I_k \left(f(t_1 \vee \cdots \vee t_k) P_n^{(k)}(t_1 \vee \cdots \vee t_k) + \int_{t_1 \vee \cdots \vee t_k}^{\infty} f(s) P_n^{(k+1)}(s) ds \right), \end{aligned}$$

and then extends to all $f \in L^2(\mathbb{R}_+, t^{n-1} \mathrm{e}^{-t} dt)$.

Next we state a result for smooth functions of a finite number of jump times. As a convention, if $k_1 \geq 0, \ldots, k_d \geq 0$ satisfy $k_1 + \cdots + k_d = n$, we define

$$(t_1^1, \ldots, t_{k_1}^1, t_1^2, \ldots, t_{k_2}^2, \ldots, t_1^d, \ldots, t_{k_d}^d)$$

as

$$(t_1^1, \ldots, t_{k_1}^1, t_1^2, \ldots, t_{k_2}^2, \ldots, t_1^d, \ldots, t_{k_d}^d) = (t_1, \ldots, t_n).$$

The next result extends Proposition 6.3.3 to the multivariate case. Its proof uses only Poisson-Charlier orthogonal expansions instead of using Proposition 4.2.5 and the gradient operator D.

Proposition 6.3.4. *Let $n_1, \ldots, n_d \in \mathbb{N}$ with $1 \leq n_1 < \cdots < n_d$, and let $f \in \mathcal{C}_c^d(\Delta_d)$. The chaos expansion of $f(T_{n_1}, \ldots, T_{n_d})$ is given as*

$$f(T_{n_1}, \ldots, T_{n_d}) = (-1)^d \sum_{n=0}^{\infty} I_n(\mathbf{1}_{\Delta_n} h_n),$$

where

$$h_n(t_1, \ldots, t_n) = \tag{6.3.8}$$

$$\sum_{\substack{k_1 + \cdots + k_d = n \\ k_1 \geq 0, \ldots, k_d \geq 0}} \int_{t_{k_d}^d}^{\infty} \cdots \int_{t_{k_i}^i}^{t_1^{i+1}} \cdots \int_{t_{k_1}^1}^{t_1^2} \partial_1 \cdots \partial_d f(s_1, \ldots, s_d) K_{s_1, \ldots, s_d}^{k_1, \ldots, k_d} ds_1 \cdots ds_d,$$

with, for $0 = s_0 \leq s_1 \leq \cdots \leq s_d$ and $k_1, \ldots k_d \in \mathbb{N}$:

$$K_{s_1, \ldots, s_d}^{k_1, \ldots, k_d} = \sum_{\substack{m_1 \geq n_1, \ldots, m_d \geq n_d \\ 0 = m_0 \leq m_1 \leq \cdots \leq m_d}} p_{m_1 - m_0}^{(k_1)}(s_1 - s_0) \cdots p_{m_d - m_{d-1}}^{(k_d)}(s_d - s_{d-1}).$$

Proof. Let $0 = s_0 \leq s_1 \leq \cdots \leq s_d$, and $n_1, \ldots, n_d \in \mathbb{N}$. We have from (6.3.3) and (6.2.16):

$$\begin{aligned}
\prod_{i=1}^{d} \mathbf{1}_{\{N_{s_i} - N_{s_{i-1}} = n_i\}} &= \sum_{n=0}^{\infty} \sum_{\substack{k_1 + \cdots + k_d = n \\ k_1 \geq 0, \ldots, k_d \geq 0}} \frac{1}{k_1! \cdots k_d!} \\
&\quad \prod_{i=1}^{d} p_{m_i - m_{i-1}}^{(k_i)}(s_i - s_{i-1}) I_{k_1}(\mathbf{1}_{[s_0, s_1]}^{\otimes k_1}) \cdots I_{k_d}(\mathbf{1}_{[s_{d-1}, s_d]}^{\otimes k_d}) \\
&= \sum_{n=0}^{\infty} \sum_{\substack{k_1 + \cdots + k_d = n \\ k_1 \geq 0, \ldots, k_d \geq 0}} \frac{1}{k_1! \cdots k_d!} \\
&\quad \prod_{i=1}^{d} p_{m_i - m_{i-1}}^{(k_i)}(s_i - s_{i-1}) I_n(\mathbf{1}_{[s_0, s_1]}^{\otimes k_1} \circ \cdots \circ \mathbf{1}_{[s_{d-1}, s_d]}^{\otimes k_d}),
\end{aligned}$$

where the last equality used the assumption $s_1 \leq \cdots \leq s_d$. Now, with $0 = m_0 \leq m_1 \leq \cdots \leq m_d$,

$$\begin{aligned}
\mathbf{1}_{[T_{m_1}, T_{m_1+1}[}(s_1) \cdots \mathbf{1}_{[T_{m_d}, T_{m_d+1}[}(s_d) &= \mathbf{1}_{\{N_{s_1} = m_1\}} \cdots \mathbf{1}_{\{N_{s_d} = m_d\}} \\
&= \mathbf{1}_{\{N_{s_1} - N_{s_0} = m_1 - m_0\}} \cdots \mathbf{1}_{\{N_{s_d} - N_{s_{d-1}} = m_d - m_{d-1}\}}
\end{aligned}$$

$$= \sum_{n=0}^{\infty} \sum_{\substack{k_1 + \cdots + k_d = n \\ k_1 \geq 0, \ldots, k_d \geq 0}} \frac{1}{k_1! \cdots k_d!}$$

$$\prod_{i=1}^{d} p_{m_i - m_{i-1}}^{(k_i)}(s_i - s_{i-1}) I_n(\mathbf{1}_{[s_0,s_1]}^{\otimes k_1} \circ \cdots \circ \mathbf{1}_{[s_{d-1},s_d]}^{\otimes k_d}).$$

Given that $s_1 \leq \cdots \leq s_d$, for any $i < j$ the conditions $s_i \in [T_{m_i}, T_{m_{i+1}})$ and $s_j \in [T_{m_j}, T_{m_{j+1}})$ imply $m_i \leq m_j$, hence

$$\prod_{i=1}^{d} \mathbf{1}_{[T_{n_i},\infty)}(s_i) = \sum_{\substack{m_1 \geq n_1, \ldots, m_d \geq n_d \\ 0 = m_0 \leq m_1 \leq \cdots \leq m_d}} \mathbf{1}_{[T_{m_1}, T_{m_1+1}[}(s_1) \cdots \mathbf{1}_{[T_{m_d}, T_{m_d+1}[}(s_d)$$

$$= \sum_{\substack{m_1 \geq n_1, \ldots, m_d \geq n_d \\ 0 = m_0 \leq m_1 \leq \cdots \leq m_d}} \mathbf{1}_{\{N_{s_1} = m_1\}} \cdots \mathbf{1}_{\{N_{s_d} = m_d\}}$$

$$= \sum_{\substack{m_1 \geq n_1, \ldots, m_d \geq n_d \\ 0 = m_0 \leq m_1 \leq \cdots \leq m_d}} \mathbf{1}_{\{N_{s_1} - N_{s_0} = m_1 - m_0\}} \cdots \mathbf{1}_{\{N_{s_d} - N_{s_{d-1}} = m_d - m_{d-1}\}}$$

$$= \sum_{n=0}^{\infty} \sum_{\substack{k_1 + \cdots + k_d = n \\ k_1 \geq 0, \ldots, k_d \geq 0}} \frac{1}{k_1! \cdots k_d!}$$

$$\sum_{\substack{m_1 \geq n_1, \ldots, m_d \geq n_d \\ 0 = m_0 \leq m_1 \leq \cdots \leq m_d}} p_{m_1 - m_0}^{(k_1)}(s_1 - s_0) \cdots p_{m_d - m_{d-1}}^{(k_d)}(s_d - s_{d-1})$$

$$I_n(\mathbf{1}_{[s_0,s_1]}^{\otimes k_1} \circ \cdots \circ \mathbf{1}_{[s_{d-1},s_d]}^{\otimes k_d})$$

$$= \sum_{n=0}^{\infty} \sum_{\substack{k_1 + \cdots + k_d = n \\ k_1 \geq 0, \ldots, k_d \geq 0}} \frac{1}{k_1! \cdots k_d!} K_{s_1,\ldots,s_d}^{k_1,\ldots,k_d} I_n(\mathbf{1}_{[s_0,s_1]}^{\otimes k_1} \circ \cdots \circ \mathbf{1}_{[s_{d-1},s_d]}^{\otimes k_d}).$$

Given $f \in \mathcal{C}_c^d(\Delta_d)$, using the identity

$$f(T_{n_1}, \ldots, T_{n_d})$$
$$= (-1)^d \int_0^\infty \cdots \int_0^\infty \mathbf{1}_{[T_{n_1},\infty[}(s_1) \cdots \mathbf{1}_{[T_{n_d},\infty[}(s_d) \frac{\partial^d}{\partial_1 \cdots \partial_d} f(s_1, \ldots, s_d) ds_1 \cdots ds_d$$
$$= (-1)^d \int_{\Delta_d} \mathbf{1}_{[T_{n_1},\infty[}(s_1) \cdots \mathbf{1}_{[T_{n_d},\infty[}(s_d) \frac{\partial^d}{\partial_1 \cdots \partial_d} f(s_1, \ldots, s_d) ds_1 \cdots ds_d,$$

we get

$$f(T_{n_1}, \ldots, T_{n_d}) = (-1)^d \sum_{n=0}^{\infty} \sum_{\substack{k_1 + \cdots + k_d = n \\ k_1 \geq 0, \ldots, k_d \geq 0}} \frac{1}{k_1! \cdots k_d!}$$

$$\int_{\Delta_d} \frac{\partial^d}{\partial_1 \cdots \partial_d} f(s_1, \ldots, s_d) K_{s_1,\ldots,s_d}^{k_1,\ldots,k_d} I_n(\mathbf{1}_{[s_0,s_1]}^{\otimes k_1} \circ \cdots \circ \mathbf{1}_{[s_{d-1},s_d]}^{\otimes k_d}) ds_1 \cdots ds_d.$$

From (6.2.16), we have for $s_1 \leq \cdots \leq s_d$ and $k_1 \geq 0, \ldots, k_d \geq 0$:

$$\begin{aligned} & I_n\left(\mathbf{1}_{[s_0,s_1]}^{\otimes k_1} \circ \cdots \circ \mathbf{1}_{[s_{d-1},s_d]}^{\otimes k_d}\right) \\ & = k_1! \cdots k_d! \int_0^\infty \int_0^{t_{k_d}^d} \cdots \int_0^{t_2^1} \mathbf{1}_{[s_0,s_1]}^{\otimes 2}(t_1^1, t_{k_1}^1) \cdots \mathbf{1}_{[s_{d-1},s_d]}^{\otimes 2}(t_1^d, t_{k_d}^d) \\ & \quad d\tilde{N}_{t_1^1} \cdots d\tilde{N}_{t_{k_d}^d}, \end{aligned}$$

hence by exchange of deterministic and stochastic integrals we obtain

$$\begin{aligned} f(T_{n_1}, \ldots, T_{n_d}) = (-1)^d \sum_{n=0}^\infty \sum_{\substack{k_1 + \cdots + k_d = n \\ k_1 \geq 0, \ldots, k_d \geq 0}} \\ I_n\left(\mathbf{1}_{\Delta_n} \int_{t_{k_d}^d}^\infty \int_{t_{k_{d-1}}^{d-1}}^{t_1^d} \cdots \int_{t_{k_2}^2}^{t_1^3} \int_{t_{k_1}^1}^{t_1^2} \frac{\partial^d f}{\partial s_1 \cdots \partial s_d}(s_1, \ldots, s_d) K_{s_1,\ldots,s_d}^{k_1,\ldots,k_d} ds_1 \cdots ds_d\right). \end{aligned}$$

□

Remarks

i) All expressions obtained above for $f(T_1, \ldots, T_d)$, $f \in \mathcal{C}_b^\infty(\Delta_d)$, extend to $f \in L^2(\Delta_d, \mathrm{e}^{-s_d} ds_1 \cdots ds_d)$, i.e. to square-integrable $f(T_1, \ldots, T_d)$, by repeated integrations by parts.

ii) Chaotic decompositions on the Poisson space on the compact interval $[0, 1]$ as in [83] or [84] can be obtained by considering the functional $f(1 \wedge T_1, \ldots, 1 \wedge T_d)$ instead of $f(T_1, \ldots, T_d)$.

6.4 Finite Difference Gradient

In this section we study the probabilistic interpretation and the extension to the Poisson space on X of the operators D and δ defined in Definitions 4.1.1 and 4.1.2.

Let the spaces $\mathcal{S}$ and $\mathcal{U}$ of Section 3.1 be taken equal to

$$\mathcal{S} = \left\{ \sum_{k=0}^n I_k(f_k) \ : \ f_k \in L^4(X)^{\circ k}, \ k = 0, \ldots, n, \ n \in \mathbb{N} \right\},$$

and

$$\mathcal{U} = \left\{ \sum_{k=0}^{n} I_k(g_k(*,\cdot)) \ : \ g_k \in L^2(X)^{\circ k} \otimes L^2(X), \ k = 0,\ldots,n, \ n \in \mathbb{N} \right\}.$$

Definition 6.4.1. *Let the linear, unbounded, closable operators*

$$D^X : L^2(\Omega^X, \pi_\sigma) \to L^2(\Omega^X \times X, \mathbb{P} \otimes \sigma)$$

and

$$\delta^X : L^2(\Omega^X \times X, \mathbb{P} \otimes \sigma) \to L^2(\Omega^X, \mathbb{P})$$

be defined on $\mathcal{S}$ and $\mathcal{U}$ respectively by

$$D_x^X I_n(f_n) := n I_{n-1}(f_n(*,x)), \tag{6.4.1}$$

$\pi_\sigma(d\omega) \otimes \sigma(dx)$-a.e., $n \in \mathbb{N}$, $f_n \in L^2(X,\sigma)^{\circ n}$, and

$$\delta^X(I_n(f_{n+1}(*,\cdot))) := I_{n+1}(\tilde{f}_{n+1}), \tag{6.4.2}$$

$\pi_\sigma(d\omega)$-a.s., $n \in \mathbb{N}$, $f_{n+1} \in L^2(X,\sigma)^{\circ n} \otimes L^2(X,\sigma)$.

In particular we have

$$\delta^X(f) = I_1(f) = \int_X f(x)(\omega(dx) - \sigma(dx)), \qquad f \in L^2(X,\sigma), \tag{6.4.3}$$

and

$$\delta^X(\mathbf{1}_A) = \omega(A) - \sigma(A), \qquad A \in \mathcal{B}(X), \tag{6.4.4}$$

and the Skorohod integral has zero expectation:

$$\mathbb{E}[\delta^X(u)] = 0, \qquad u \in \mathrm{Dom}\,(\delta^X). \tag{6.4.5}$$

In case $X = \mathbb{R}_+$ we simply write D and δ instead of $D^{\mathbb{R}_+}$ and $\delta^{\mathbb{R}_+}$.

Note that using the mapping τ of Proposition 6.1.7 we have the relations

$$(D_{\tau(x)}F) \circ \tau^* = D_x^X(F \circ \tau^*), \qquad \pi_\sigma(d\omega) \otimes \sigma(dx) - a.e.$$

and

$$\delta(u_{\tau(\cdot)}) \circ \tau^* = \delta^X(u_\cdot \circ \tau^*), \qquad \pi_\sigma(d\omega) - a.e.$$

From these relations and Proposition 4.1.4 we have the following proposition.

Proposition 6.4.2. *For any $u \in \mathcal{U}$ we have*

$$D_x^X \delta^X(u) = u(x) + \delta^X(D_x^X u). \tag{6.4.6}$$

Let $\mathrm{Dom}\,(D^X)$ denote the set of functionals $F : \Omega^X \to \mathbb{R}$ with the expansion

$$F = \sum_{n=0}^{\infty} I_n(f_n)$$

such that

$$\sum_{n=1}^{\infty} n!n\|f_n\|^2_{L^2(X^n,\sigma^{\otimes n})} < \infty,$$

and let Dom (δ^X) denote the set of processes $u : \Omega^X \times X \to \mathbb{R}$ with the expansion

$$u(x) = \sum_{n=0}^{\infty} I_n(f_{n+1}(*, x)), \qquad x \in X,$$

such that

$$\sum_{n=0}^{\infty} (n+1)!\|\tilde{f}_{n+1}\|^2_{L^2(X^{n+1},\sigma^{\otimes(n+1)})} < \infty.$$

The following duality relation can be obtained by transfer from Proposition 4.1.3 using Proposition 6.1.8. Here we also provide a direct proof.

Proposition 6.4.3. *The operators D^X and δ^X satisfy the duality relation*

$$\mathbb{E}[\langle D^X F, u\rangle_{L^2(X,\sigma)}] = \mathbb{E}[F\delta^X(u)], \tag{6.4.7}$$

$F \in \text{Dom}\,(D^X)$, $u \in \text{Dom}\,(\delta^X)$.

Proof. The proof is identical to those of Propositions 1.8.2 and 4.1.3, and follows from the isometry formula (6.2.4). We consider $F = I_n(f_n)$ and $u_x = I_m(g_{m+1}(*, x))$, $x \in X$, $f_n \in L^2(X)^{\circ n}$, $g_{m+1} \in L^2(X)^{\circ m} \otimes L^2(X)$. We have

$$\begin{aligned}
\mathbb{E}[F\delta^X(u)] &= \mathbb{E}[I_{m+1}(\tilde{g}_{m+1})I_n(f_n)] \\
&= n!\mathbf{1}_{\{n=m+1\}}\langle f_n, \tilde{g}_n\rangle_{L^2(X^n)} \\
&= \mathbf{1}_{\{n-1=m\}} \int_{X^n} f_n(x_1, \ldots, x_{n-1}, x) g_n(x_1, \ldots, x_{n-1}, x) \\
&\quad \sigma(dx_1)\cdots\sigma(dx_{n-1})\sigma(dx) \\
&= n\mathbf{1}_{\{n-1=m\}} \int_0^{\infty} \mathbb{E}[I_{n-1}(f_n(*, t))I_{n-1}(g_n(*, t))]dt \\
&= \mathbb{E}[\langle D^X_\cdot I_n(f_n), I_m(g_{m+1}(*, \cdot))\rangle_{L^2(X,\sigma)}] \\
&= \mathbb{E}[\langle D^X F, u\rangle_{L^2(X,\sigma)}].
\end{aligned}$$

Again, we may alternatively use the mapping $\tau : X \to \mathbb{R}$ to prove this proposition from Proposition 4.1.3. □

Propositions 3.1.2 and 6.4.3 show in particular that D^X is closable.

The next lemma gives the probabilistic interpretation of the gradient D^X.

Lemma 6.4.4. *For any F of the form*

$$F = f(I_1(u_1), \ldots, I_1(u_n)), \tag{6.4.8}$$

with $u_1, \ldots, u_n \in \mathcal{C}_c(X)$*, and* f *is a bounded and continuous function, or a polynomial on* $\mathbb{R}^n$*, we have* $F \in \mathrm{Dom}\,(D^X)$ *and*

$$D_x^X F(\omega) = F(\omega \cup \{x\}) - F(\omega), \qquad \mathbb{P} \otimes \sigma(d\omega, dx) - a.e., \tag{6.4.9}$$

where as a convention we identify $\omega \in \Omega^X$ *with its support.*

Proof. We start by assuming that $u_1 = \mathbf{1}_{A_1}, \ldots, u_n = \mathbf{1}_{A_n}$, where $A_1, \ldots, A_n$ are compact disjoint measurable subsets of X. In this case the proposition clearly holds for f polynomial from Proposition 6.2.9 and Relations (6.2.13), (6.2.16), which imply

$$\begin{aligned}
&D_x^X I_n(\mathbf{1}_{A_1}^{\otimes k_1} \circ \cdots \circ \mathbf{1}_{A_d}^{\otimes k_d})(\omega) \\
&= \sum_{i=1}^d \mathbf{1}_{A_i}(x) k_i I_{k_i-1}(\mathbf{1}_{A_i}^{\otimes k_i-1})(\omega) \prod_{j\neq i} I_{k_j}(\mathbf{1}_{A_j}^{\otimes k_j})(\omega) \\
&= \sum_{i=1}^d \mathbf{1}_{A_i}(x) k_i C_{k_i-1}(\omega(A_i), \sigma(A_i)) \prod_{j\neq i} C_{k_j}(\omega(A_j), \sigma(A_j)) \\
&= \sum_{i=1}^d \mathbf{1}_{A_i}(x)(C_{k_i}(\omega(A_i)+1, \sigma(A_i)) - C_{k_i}(\omega(A_i), \sigma(A_i))) \\
&\quad \prod_{j\neq i} C_{k_j}(\omega(A_j), \sigma(A_j)) \\
&= \prod_{i=1}^n C_{k_i}(\omega(A_i) + \mathbf{1}_{A_i}(x), \sigma(A_i)) - \prod_{i=1}^n C_{k_i}(\omega(A_i), \sigma(A_i)) \\
&= I_n(\mathbf{1}_{A_1}^{\otimes k_1} \circ \cdots \circ \mathbf{1}_{A_d}^{\otimes k_d})(\omega \cup \{x\}) - I_n(\mathbf{1}_{A_1}^{\otimes k_1} \circ \cdots \circ \mathbf{1}_{A_d}^{\otimes k_d})(\omega),
\end{aligned}$$

by (6.2.13).

If $f \in \mathcal{C}_b(\mathbb{R}^n)$, from Lemma 6.2.10 the functional

$$F := f(I_1(\mathbf{1}_{A_1}), \ldots, I_1(\mathbf{1}_{A_n}))$$

has the chaotic decomposition

$$F = \mathbb{E}[F] + \sum_{k=1}^\infty I_k(g_k),$$

where $I_k(g_k)$ is a polynomial in $\omega(A_1), \ldots, \omega(A_n)$. Let now

$$Q_k := \mathbb{E}[F] + \sum_{l=1}^{k} I_l(g_l), \qquad k \geq 1.$$

The sequence $(Q_k)_{k\in\mathbb{N}} \subset \mathcal{S}$ consists in polynomial functionals converging to F in $L^2(\Omega^X)$. By the Abel transformation of sums

$$\begin{aligned}\sum_{k=0}^{\infty}(f(k+1)-f(k))C_n(k,\lambda)\frac{\lambda^k}{k!} &= \sum_{k=1}^{\infty} f(k)(kC_n(k-1,\lambda)-\lambda C_n(k,\lambda))\frac{\lambda^{k-1}}{k!} \\ &= \frac{1}{\lambda}\sum_{k=1}^{\infty} f(k)C_{n+1}(k,\lambda)\frac{\lambda^k}{k!}, \qquad (6.4.10)\end{aligned}$$

with $C_n(0,\lambda) = (-\lambda)^n$, $n \in \mathbb{N}$, we get, with $\lambda = \sigma(A_i)$ and $l = k_1 + \cdots + k_d$,

$$\begin{aligned}&\mathbb{E}\left[I_l\left(\mathbf{1}_{A_1}^{\otimes k_1}\circ\cdots\circ\mathbf{1}_{A_d}^{\otimes k_d})\right)\times\right. \\ &\qquad (f(I_1(\mathbf{1}_{A_1}),\ldots,I_1(\mathbf{1}_{A_i})+1,\ldots,I_1(\mathbf{1}_{A_d})) - f(I_1(\mathbf{1}_{A_1}),\ldots,I_1(\mathbf{1}_{A_d})))] \\ &= \frac{1}{\sigma(A_i)}\,\mathbb{E}[f(I_1(\mathbf{1}_{A_1}),\ldots,I_1(\mathbf{1}_{A_d}))I_{l+1}(\mathbf{1}_{A_1}^{\otimes k_1}\circ\cdots\circ\mathbf{1}_{A_d}^{\otimes k_d}\circ\mathbf{1}_{A_i})] \\ &= \frac{1}{\sigma(A_i)}\,\mathbb{E}[I_{l+1}(g_{l+1})I_{l+1}(\mathbf{1}_{A_1}^{\otimes k_1}\circ\cdots\circ\mathbf{1}_{A_d}^{\otimes k_d}\circ\mathbf{1}_{A_i})] \\ &= \frac{(l+1)!}{\sigma(A_i)}\langle g_{l+1},\mathbf{1}_{A_1}^{\otimes k_1}\circ\cdots\circ\mathbf{1}_{A_d}^{\otimes k_d}\circ\mathbf{1}_{A_i}\rangle_{L^2(X^l,\sigma^{\otimes l})} \\ &= \frac{1}{\sigma(A_i)}\,\mathbb{E}[\langle D^X I_{l+1}(g_{l+1}),\mathbf{1}_{A_i}\rangle_{L^2(X,\sigma)}I_l(\mathbf{1}_{A_1}^{\otimes k_1}\circ\cdots\circ\mathbf{1}_{A_d}^{\otimes k_d})].\end{aligned}$$

Hence the projection of

$$f(I_1(\mathbf{1}_{A_1}),\ldots,I_1(\mathbf{1}_{A_i})+1,\ldots,I_1(\mathbf{1}_{A_d})) - f(I_1(\mathbf{1}_{A_1}),\ldots,I_1(\mathbf{1}_{A_d}))$$

on the chaos $\mathcal{H}_l$ of order $l \in \mathbb{N}$ is

$$\frac{1}{\sigma(A_i)}\langle D^X I_{l+1}(g_{l+1}),\mathbf{1}_{A_i}\rangle_{L^2(X,\sigma)},$$

and we have the chaotic decomposition

$$\begin{aligned}&f(I_1(\mathbf{1}_{A_1}),\ldots,I_1(\mathbf{1}_{A_i})+1,\ldots,I_1(\mathbf{1}_{A_d})) - f(I_1(\mathbf{1}_{A_1}),\ldots,I_1(\mathbf{1}_{A_d})) \\ &\qquad = \frac{1}{\sigma(A_i)}\sum_{k=1}^{\infty}\langle D^X I_k(g_k),\mathbf{1}_{A_i}\rangle_{L^2(X,\sigma)},\end{aligned}$$

where the series converges in $L^2(\Omega^X)$. Hence

$$\sum_{i=1}^{n}\mathbf{1}_{A_i}(f(I_1(\mathbf{1}_{A_1}),\ldots,I_1(\mathbf{1}_{A_i})+1,\ldots,I_1(\mathbf{1}_{A_d})) - f(I_1(\mathbf{1}_{A_1}),\ldots,I_1(\mathbf{1}_{A_d})))$$

$$
\begin{aligned}
&= \sum_{i=1}^{n} \frac{1}{\sigma(A_i)} \mathbf{1}_{A_i} \sum_{k=1}^{\infty} \langle D^X I_k(g_k), \mathbf{1}_{A_i} \rangle_{L^2(X,\sigma)} \\
&= \sum_{k=1}^{\infty} D^X I_k(g_k) \\
&= \lim_{n\to\infty} D^X Q_n,
\end{aligned}
$$

which shows that $(D^X Q_k)_{k\in\mathbb{N}}$ converges in $L^2(\Omega^X \times X)$ to

$$
\sum_{i=1}^{n} \mathbf{1}_{A_i} (f(I_1(\mathbf{1}_{A_1}),\ldots,I_1(\mathbf{1}_{A_i})+1,\ldots,I_1(\mathbf{1}_{A_d})) - f(I_1(\mathbf{1}_{A_1}),\ldots,I_1(\mathbf{1}_{A_d}))).
$$

The proof is concluded by the closability of D^X and approximation of functions in $\mathcal{C}_c(X)$ by linear combination of indicator functions. □

Definition 6.4.5. *Given a mapping $F : \Omega^X \longrightarrow \mathbb{R}$, let*

$$
\varepsilon_x^+ F : \Omega^X \longrightarrow \mathbb{R} \qquad \textit{and} \qquad \varepsilon_x^- F : \Omega^X \longrightarrow \mathbb{R},
$$

$x \in X$, be defined by

$$
(\varepsilon_x^- F)(\omega) = F(\omega \backslash x), \qquad \textit{and} \qquad (\varepsilon_x^+ F)(\omega) = F(\omega \cup x), \qquad \omega \in \Omega^X.
$$

Note that Relation (6.4.9) can be written as

$$
D_x^X F = \varepsilon_x^+ F - F, \qquad x \in X. \tag{6.4.11}
$$

On the other hand, the result of Lemma 6.4.4 is clearly verified on simple functionals. For instance when $F = I_1(u)$ is a single Poisson stochastic integral, we have

$$
\begin{aligned}
D_x^X I_1(u)(\omega) &= I_1(u)(\omega \cup \{x\}) - I_1(u)(\omega) \\
&= \int_X u(y)(\omega(dy) + \epsilon_x(dy) - \sigma(dy)) - \int_X u(y)(\omega(dy) - \sigma(dy)) \\
&= \int_X u(y)\epsilon_x(dy) \\
&= u(x).
\end{aligned}
$$

Corollary 6.4.6. *For all bounded F random variables and all measurable $A \in \mathcal{B}(X)$, $0 < \sigma(A) < \infty$, we have*

$$
\mathbb{E}\left[\int_A F(\omega \cup \{x\})\sigma(dx)\right] = \mathbb{E}[F\omega(A)]. \tag{6.4.12}
$$

Proof. From Proposition 6.4.3, Lemma 6.4.4, and Relation (6.4.4) we have

$$
\mathbb{E}\left[\int_A F(\omega \cup \{x\})\sigma(dx)\right] = \mathbb{E}\left[\int_X \mathbf{1}_A(x) D_x F \sigma(dx)\right] + \sigma(A)\,\mathbb{E}[F]
$$

$$= \mathbb{E}[F\delta^X(\mathbf{1}_A)] + \sigma(A)\,\mathbb{E}[F]$$
$$= \mathbb{E}[F\omega(A)].$$

□

Hence as in [154] we get that the law of the mapping $(x,\omega) \longmapsto \omega \cup \{x\}$ under $\mathbf{1}_A(x)\sigma(dx)\pi_\sigma(d\omega)$ is absolutely continuous with respect to π_σ. In particular, $(\omega,x) \longmapsto F(\omega\cup\{x\})$ is well-defined, $\pi_\sigma \otimes \sigma$-a.e., and this justifies the extension of Lemma 6.4.4 in the next proposition.

Proposition 6.4.7. *For any $F \in \mathrm{Dom}\,(D^X)$ we have*

$$D_x^X F(\omega) = F(\omega \cup \{x\}) - F(\omega),$$

$\pi_\sigma(d\omega) \times \sigma(dx)$-a.e.

Proof. There exists a sequence $(F_n)_{n\in\mathbb{N}}$ of functionals of the form (6.4.8), such that $(D^X F_n)_{n\in\mathbb{N}}$ converges everywhere to $D^X F$ on a set A_F such that $(\pi_\sigma\otimes\sigma)(A_F^c) = 0$. For each $n \in \mathbb{N}$, there exists a measurable set $B_n \subset \Omega^X \times X$ such that $(\pi_\sigma \otimes \sigma)(B_n^c) = 0$ and

$$D_x^X F_n(\omega) = F_n(\omega \cup \{x\}) - F_n(\omega), \qquad (\omega, x) \in B_n.$$

Taking the limit as n goes to infinity on $(\omega,x) \in A_F \cap \bigcap_{n=0}^\infty B_n$, we get

$$D_x^X F(\omega) = F(\omega \cup \{x\}) - F(\omega), \qquad \pi_\sigma(d\omega) \times \sigma(dx) - a.e.$$

□

Proposition 6.4.7 also allows us to recover the annihilation property (6.4.1) of D^X, i.e.:

$$D_x^X I_n(f_n) = nI_{n-1}(f_n(*,x)), \qquad \sigma(dx) - a.e. \tag{6.4.13}$$

Indeed, using the relation

$$\mathbf{1}_{\Delta_n}(x_1,\ldots,x_n)\epsilon_x(dx_i)\epsilon_x(dx_j) = 0, \qquad i,j = 1,\ldots,n,$$

we have for $f_n \in L^2(X,\sigma)^{\circ n}$:

$$D_x^X I_n(f_n) = D_x^X \int_{\Delta_n} f_n(x_1,\ldots x_n)(\omega(dx_1) - \sigma(dx_1))\cdots(\omega(dx_n) - \sigma(dx_n))$$
$$= \int_{\Delta_n} f_n(x_1,\ldots,x_n)\prod_{i=1}^n(\omega(dx_i) - \sigma(dx_i) + (1-\omega(\{x\}))\epsilon_x(dx_i))$$
$$- \int_{\Delta_n} f_n(x_1,\ldots,x_n)(\omega(dx_1) - \sigma(dx_1))\cdots(\omega(dx_n) - \sigma(dx_n))$$
$$= (1-\omega(\{x\}))\sum_{i=1}^n \int_{\Delta_{n-1}} f_n(x_1,\ldots,x,\ldots,x_n) \prod_{1\le k\ne i\le n}(\omega(dx_k) - \sigma(dx_k))$$

$$= (1 - \omega(\{x\})) \sum_{i=1}^{n} I_{n-1}(f_n(\underbrace{\cdots}_{i-1}, x, \underbrace{\cdots}_{n-i})), \qquad x \in X.$$

Hence we have for $f_n \in \mathcal{C}_c(X^n)$:

$$D_x^X I_n(f_n) = \mathbf{1}_{\{x \notin \omega\}} \sum_{i=1}^{n} I_{n-1}(f_n(\underbrace{\cdots}_{i-1}, x, \underbrace{\cdots}_{n-i})), \qquad x \in X,$$

and since f_n is symmetric,

$$D_x^X I_n(f_n) = \mathbf{1}_{\{x \notin \omega\}} n I_{n-1}(f_n(*, x)), \qquad x \in X,$$

from which we recover (6.4.13) since σ is diffuse.

Proposition 6.4.7 implies that D^X satisfies the following finite difference product rule.

Proposition 6.4.8. *We have for $F, G \in \mathcal{S}$:*

$$D_x^X(FG) = F D_x^X G + G D_x^X F + D_x^X F D_x^X G, \tag{6.4.14}$$

$\mathbb{P}(d\omega)d\sigma(x)$*-a.e.*

Proof. This formula can be proved either from Propositions 4.5.2 and 6.1.8 with $\phi_t = 1$, $t \in \mathbb{R}_+$, when $X = \mathbb{R}_+$, or directly from (6.4.9):

$$\begin{aligned} D_x^X(FG)(\omega) &= F(\omega \cup \{x\})G(\omega \cup \{x\}) - F(\omega)G(\omega) \\ &= F(\omega)(G(\omega \cup \{x\}) - G(\omega)) + G(\omega)(F(\omega \cup \{x\}) - F(\omega)) \\ &\quad + (F(\omega \cup \{x\}) - F(\omega))(G(\omega \cup \{x\}) - G(\omega)) \\ &= F(\omega)D_x^X G(\omega) + G(\omega)D_x^X F(\omega) + D_x^X F(\omega)D_x^X G(\omega), \end{aligned}$$

$d\mathbb{P} \times \sigma(dx)$-a.e. □

As a consequence of Proposition 6.4.7 above, when $X = \mathbb{R}_+$ the Clark formula Proposition 4.2.3 takes the following form when stated on the Poisson space.

Proposition 6.4.9. *Assume that $X = \mathbb{R}_+$. For any $F \in \mathrm{Dom}(D)$ we have*

$$F = \mathbb{E}[F] + \int_0^\infty \mathbb{E}[F(\omega \cup \{t\}) - F(\omega) \mid \mathcal{F}_t] d(N_t - t).$$

In case $X = \mathbb{R}_+$ the finite difference operator $D : L^2(\Omega) \longrightarrow L^2(\Omega \times \mathbb{R}_+)$ can be written as

$$D_t F = \mathbf{1}_{\{N_t < n\}}(f(T_1, \ldots, T_{N_t}, t, T_{N_t+1}, \ldots, T_{n-1}) - f(T_1, \ldots, T_n)), \tag{6.4.15}$$

$t \in \mathbb{R}_+$, for $F = f(T_1, \ldots, T_n)$, hence $\mathbb{E}[D_t F | \mathcal{F}_t]$ can be computed via the following lemma.

Lemma 6.4.10. *Let $X = \mathbb{R}_+$ and $\sigma(dx) = dx$. For any F of the form $F = f(T_1, \ldots, T_n)$ we have*

$$\begin{aligned}
&\mathbb{E}[D_t F|\mathcal{F}_t] \\
&= \mathbf{1}_{\{N_t<n\}} \int_t^\infty \mathrm{e}^{-(s_n-t)} \int_t^{s_n} \cdots \int_t^{s_{N_t+3}} \\
&\Big(f(T_1, \ldots, T_{N_t}, t, s_{N_t+2}, \ldots, s_n) - \int_t^{s_{N_t+2}} f(T_1, \ldots, T_{N_t}, s_{N_t+1}, \ldots, s_n) ds_{N_t+1} \Big) \\
&\quad ds_{N_t+2} \cdots ds_n .
\end{aligned}$$

Proof. By application of Proposition 2.3.6 we have

$$\begin{aligned}
&\mathbb{E}[D_t F|\mathcal{F}_t] \\
&= \mathbf{1}_{\{N_t<n\}} \mathbb{E}[f(T_1, \ldots, T_{N_t}, t, T_{N_t+1}, \ldots, T_{n-1}) - f(T_1, \ldots, T_n)|\mathcal{F}_t] \\
&= \mathbf{1}_{\{N_t<n\}} \int_t^\infty \mathrm{e}^{-(s_n-t)} \int_t^{s_n} \cdots \int_t^{s_{N_t+3}} f(T_1, \ldots, T_{N_t}, t, s_{N_t+2}, \ldots, s_n) \\
&\quad ds_{N_t+2} \cdots ds_n \\
&\quad -\mathbf{1}_{\{N_t<n\}} \int_t^\infty \mathrm{e}^{-(s_n-t)} \int_t^{s_n} \cdots \int_t^{s_{N_t+2}} f(T_1, \ldots, T_{N_t}, s_{N_t+1}, \ldots, s_n) \\
&\quad ds_{N_t+1} \cdots ds_n \\
&= \mathbf{1}_{\{N_t<n\}} \int_t^\infty \mathrm{e}^{-(s_n-t)} \int_t^{s_n} \cdots \int_t^{s_{N_t+3}} \\
&\Big(f(T_1, \ldots, T_{N_t}, t, s_{N_t+2}, \ldots, s_n) - \int_t^{s_{N_t+2}} f(T_1, \ldots, T_{N_t}, s_{N_t+1}, \ldots, s_n) ds_{N_t+1} \Big) \\
&\quad ds_{N_t+2} \cdots ds_n .
\end{aligned}$$

□

6.5 Divergence Operator

The adjoint δ^X of D^X satisfies the following divergence formula.

Proposition 6.5.1. *Let $u : X \times \Omega^X \longrightarrow \mathbb{R}$ and $F : \Omega^X \longrightarrow \mathbb{R}$ such that $u(\cdot, \omega)$, $D^X_\cdot F(\omega)$, and $u(\cdot, \omega) D^X_\cdot F(\omega) \in L^1(X, \sigma)$, $\omega \in \Omega^X$. We have*

$$F \delta^X(u) = \delta^X(uF) + \langle u, D^X F \rangle_{L^2(X,\sigma)} + \delta^X(u D^X F). \tag{6.5.1}$$

The relation also holds if the series and integrals converge, or if $F \in \mathrm{Dom}\,(D^X)$ and $u \in \mathrm{Dom}\,(\delta^X)$ is such that $u D^X F \in \mathrm{Dom}\,(\delta^X)$.

Proof. Relation (6.5.1) follows by duality from Proposition 6.4.8, or from Proposition 4.5.6 and Proposition 6.1.8. □

In the next proposition, Relation (6.5.2) can be seen as a generalization of (6.2.14) in Proposition 6.2.8:

$$C_{n+1}(k,t) = kC_n(k-1,t) - tC_n(k,t),$$

which is recovered by taking $u = \mathbf{1}_A$ and $t = \sigma(A)$. The following statement provides a connection between the Skorohod integral and the Poisson stochastic integral.

Proposition 6.5.2. *For all* $u \in \mathrm{Dom}\,(\delta^X)$ *we have*

$$\delta^X(u) = \int_X u_x(\omega \setminus \{x\})(\omega(dx) - \sigma(dx)). \tag{6.5.2}$$

Proof. The statement clearly holds by (6.4.3) when $g \in L^2(X,\sigma)$ is deterministic. Next we show using (6.5.1) that the identity also holds for a process of the form $u = gI_n(f_n)$, $g \in L^2(X,\sigma)$, by induction on the order of the multiple stochastic integral $F = I_n(f_n)$. From (6.5.1) we have

$$\begin{aligned}
&\delta^X(gF) = -\delta^X(gD^XF) + F\delta^X(g) - \langle D^XF, g\rangle_{L^2(X,\sigma)} \\
&= -\int_X g(x)D_x^XF(\omega\backslash\{x\})\omega(dx) + \int_X g(x)D_x^XF(\omega\backslash\{x\})\sigma(dx) \\
&\quad +F\delta^X(g) - \langle D^XF, g\rangle_{L^2(X,\sigma)} \\
&= -\int_X g(x)F(\omega)\omega(dx) + \int_X g(x)F(\omega\backslash\{x\})\omega(dx) + \langle D^XF(\omega), g\rangle_{L^2(X,\sigma)} \\
&\quad +F\delta^X(g) - \langle D^XF, g\rangle_{L^2(X,\sigma)} \\
&= -F(\omega)\int_X g(x)\sigma(dx) + \int_X g(x)F(\omega\backslash\{x\})\omega(dx) \\
&= \int_X g(x)F(\omega\backslash\{x\})\omega(dx) - \int_X g(x)F(\omega\backslash\{x\})\sigma(dx).
\end{aligned}$$

We used the fact that since σ is diffuse on X, for $u : X \times \Omega^X \longrightarrow \mathbb{R}$ we have

$$u_x(\omega\backslash\{x\}) = u_x(\omega), \qquad \sigma(dx) - a.e., \quad \omega \in \Omega^X,$$

hence

$$\int_X u_x(\omega\backslash\{x\})\sigma(dx) = \int_X u_x(\omega)\sigma(dx), \qquad \omega \in \Omega^X, \tag{6.5.3}$$

and

$$\delta^X(u) = \int_X u_x(\omega\backslash\{x\})\omega(dx) - \int_X u_x(\omega)\sigma(dx), \qquad u \in \mathrm{Dom}\,(\delta^X).$$

□

Note that (6.5.1) can also be recovered from (6.5.2) using a simple trajectorial argument. For $x \in \omega$ we have

$$\begin{aligned}
\varepsilon_x^- D_x^X F(\omega) &= \varepsilon_x^-\varepsilon_x^+ F(\omega) - \varepsilon_x^- F(\omega) \\
&= F(\omega) - \varepsilon_x^- F(\omega) \\
&= F(\omega) - F(\omega\backslash\{x\}),
\end{aligned}$$

hence

$$\begin{aligned}
&\delta^X(uD^XF)(\omega)\\
&= \int_X u_x(\omega\backslash\{x\})D_x^XF(\omega\backslash\{x\})\omega(dx) - \int_X u_x(\omega\backslash\{x\})D_x^XF(\omega\backslash\{x\})\sigma(dx)\\
&= \int_X u_x(\omega\backslash\{x\})F(\omega)\omega(dx) - \int_X u_x(\omega\backslash\{x\})F(\omega\backslash\{x\})\omega(dx)\\
&\quad -\langle D^XF(\omega), u(\omega)\rangle_{L^2(X,\sigma)}\\
&= F(\omega)\delta^X(u)(\omega) - \delta^X(uF)(\omega) - \langle D^XF(\omega), u(\omega)\rangle_{L^2(X,\sigma)},
\end{aligned}$$

since from Relation (6.5.3),

$$\begin{aligned}
F(\omega)\int_X u_x(\omega\backslash\{x\})\sigma(dx) &= F(\omega)\int_X u_x(\omega)\sigma(dx)\\
&= \int_X F(\omega\backslash\{x\})u_x(\omega\backslash\{x\})\sigma(dx).
\end{aligned}$$

Relation (6.5.2) can also be proved using Relation (6.2.3).

In case $X = \mathbb{R}_+$, Proposition 6.5.2 yields Proposition 2.5.10 since u_{t^-} does not depend on the presence of a jump at time t. On the other hand, Proposition 4.3.4 can be written as follows.

Proposition 6.5.3. *When $X = \mathbb{R}_+$, for any square-integrable adapted process $(u_t)_{t\in\mathbb{R}_+} \in L^2_{ad}(\Omega\times\mathbb{R}_+)$ we have*

$$\delta(u) = \int_0^\infty u_t d(N_t - t).$$

The following is the Skorohod isometry on the Poisson space, which follows here from Proposition 6.5.1, or from Propositions 4.3.1 and 6.1.8.

Proposition 6.5.4. *For $u : \Omega^X \times X \to \mathbb{R}$ measurable and sufficiently integrable we have*

$$\mathbb{E}_{\pi_\sigma}\left[|\delta^X(u)|^2\right] = \mathbb{E}\left[\|u\|^2_{L^2(X,\sigma)}\right] + \mathbb{E}\left[\int_X\int_X D_x^Xu(y)D_y^Xu(x)\sigma(dx)\sigma(dy)\right]. \tag{6.5.4}$$

Proof. Applying Proposition 6.4.2, Proposition 6.5.1 and Relation (6.4.5) we have

$$\begin{aligned}
&\mathbb{E}_{\pi_\sigma}\left[|\delta^X(u)|^2\right]\\
&= \mathbb{E}_{\pi_\sigma}\left[\delta^X(u\delta^X(u)) + \langle u, D^X\delta^X(u)\rangle_{L^2(X,\sigma)} + \delta^X(uD^X\delta^X(u))\right]\\
&= \mathbb{E}_{\pi_\sigma}\left[\langle u, D^X\delta^X(u)\rangle_{L^2(X,\sigma)}\right]\\
&= \mathbb{E}_{\pi_\sigma}\left[\|u\|^2_{L^2(X,\sigma)} + \int_X u(x)\delta^X(D_x^Xu)\sigma(dx)\right]\\
&= \mathbb{E}_{\pi_\sigma}\left[\|u\|^2_{L^2(X,\sigma)} + \int_X\int_X D_y^Xu(x)D_x^Xu(y)\sigma(dx)\sigma(dy)\right].
\end{aligned}$$

□

The Skorohod isometry (6.5.4) shows that δ^X is continuous on the subspace $\mathbb{L}_{1,2}$ of $L^2(X \times \Omega^X)$ defined by the norm

$$\|u\|_{1,2}^2 = \|u\|_{L^2(\Omega^X \times X))}^2 + \|D^X u\|_{L^2(\Omega^X \times X^2)}^2.$$

Recall that the moment $\mathbb{E}_\lambda[Z^n]$ of order n of a Poisson random variable Z with intensity λ can be written as

$$\mathbb{E}_\lambda[Z^n] = T_n(\lambda)$$

where $T_n(\lambda)$ is the Touchard polynomial of order n, defined by $T_0(\lambda) = 1$ and the recurrence relation

$$T_n(\lambda) = \lambda \sum_{k=0}^{n} \binom{n}{k} T_k(\lambda), \qquad n \geq 1. \tag{6.5.5}$$

Replacing the Touchard polynomial $T_n(\lambda)$ by its centered version $\tilde{T}_n(\lambda)$ defined by $\tilde{T}_0(\lambda) = 1$, i.e.

$$\tilde{T}_{n+1}(\lambda) = \sum_{k=0}^{n-1} \binom{n}{k} \lambda^{n-k+1} \tilde{T}_k(\lambda), \qquad n \geq 0, \tag{6.5.6}$$

gives the moments of the *centered* Poisson random variable with intensity $\lambda > 0$ is

$$\tilde{T}_n(\lambda) = \mathbb{E}[(Z - \lambda)^n], \qquad n \geq 0.$$

The next proposition extends the Skorohod isometry of Proposition 6.5.4 to higher order moments, and recovers Proposition 6.5.4 in case $n = 1$.

Proposition 6.5.5. *We have, for $u : \Omega^X \times X \to \mathbb{R}$ a sufficiently integrable process,*

$$\mathbb{E}[(\delta^X(u))^{n+1}] = \sum_{k=0}^{n-1} \binom{n}{k} \mathbb{E}\left[\int_X (u_t)^{n-k+1} (\delta^X(u))^k \sigma(dt)\right]$$
$$+ \sum_{k=1}^{n} \binom{n}{k} \mathbb{E}\left[\int_X (u_t)^{n-k+1} ((\delta^X((I + D_t^X)u))^k - (\delta^X(u))^k) \sigma(dt)\right],$$

for all $n \geq 1$.

Proof. Using the relation

$$\begin{aligned} D_t^X (\delta^X(u))^n &= \varepsilon_t^+ (\delta^X(u))^n - (\delta^X(u))^n \\ &= (\varepsilon_t^+ \delta^X(u))^n - (\delta^X(u))^n \\ &= (\delta^X(u) + D_t^X \delta^X(u))^n - (\delta^X(u))^n \end{aligned}$$

$$= (\delta^X(u) + u_t + \delta^X(D_t^X u))^n - (\delta^X(u))^n, \qquad t \in X,$$

that follows from Proposition 6.4.2 and Relation (6.4.11), we get, applying the duality relation of Proposition 6.4.3,

$$\begin{aligned}
\mathbb{E}[(\delta^X(u))^{n+1}] &= \mathbb{E}\left[\int_X u(t) D_t^X (\delta^X(u))^n \sigma(dt)\right] \\
&= \mathbb{E}\left[\int_X u_t((\delta^X(u) + u_t + \delta^X(D_t^X u))^n - (\delta^X(u))^n)\sigma(dt)\right] \\
&= \sum_{k=0}^{n} \binom{n}{k} \mathbb{E}\left[\int_X (u_t)^{n-k+1} (\delta^X((I + D_t^X)u))^k \sigma(dt)\right] \\
&\quad - \mathbb{E}\left[(\delta^X(u))^n \int_X u_t \sigma(dt)\right] \\
&= \sum_{k=0}^{n-1} \binom{n}{k} \mathbb{E}\left[\int_X (u_t)^{n-k+1} (\delta^X(u))^k \sigma(dt)\right] \\
&\quad + \sum_{k=1}^{n} \binom{n}{k} \mathbb{E}\left[\int_X (u_t)^{n-k+1} ((\delta^X((I + D_t^X)u))^k - (\delta^X(u))^k)\sigma(dt)\right].
\end{aligned}$$

□

Clearly, the moments of the compensated Poisson stochastic integral

$$\int_X f(t)(\omega(dt) - \sigma(dt))$$

of $f \in L^2(X, \sigma)$ satisfy the recurrence identity

$$\begin{aligned}
&\mathbb{E}\left[\left(\int_X f(t)(\omega(dt) - \sigma(dt))\right)^{n+1}\right] \\
&= \sum_{k=0}^{n-1} \binom{n}{k} \int_X (f(t))^{n-k+1} \sigma(dt)\, \mathbb{E}\left[\left(\int_X f(t)(\omega(dt) - \sigma(dt))\right)^k\right],
\end{aligned}$$

which is analog to Relation (6.5.6) for the centered Touchard polynomials and recovers in particular the isometry formula (6.1.7) for $n = 1$. Similarly we can show that

$$\begin{aligned}
&\mathbb{E}\left[\left(\int_X f(s)\omega(ds)\right)^{n+1}\right] \\
&\qquad = \sum_{k=0}^{n} \binom{n}{k} \mathbb{E}\left[\left(\int_X f(s)\omega(ds)\right)^k \int_X (f(t))^{n+1-k} \sigma(dt)\right],
\end{aligned} \tag{6.5.7}$$

which is analog to (6.5.5).

6.6 Characterization of Poisson Measures

The duality relation (6.4.7) satisfied by the operators D^X and δ^X can also be used to characterize Poisson measures.

Proposition 6.6.1. *Let π be a probability measure on Ω^X such that $h \in \mathcal{C}_c^\infty(X)$, $I_1(h)$ has finite moments of all orders under π. Assume that*

$$\mathbb{E}_\pi\left[\delta^X(u)\right] = 0, \tag{6.6.1}$$

for all $u \in \mathcal{U}$, or equivalently

$$\mathbb{E}_\pi\left[\langle D^X F, h\rangle_{L^2(X,\sigma)}\right] = \mathbb{E}_\pi\left[F\delta^X(h)\right], \tag{6.6.2}$$

$F \in \mathcal{S}$, $h \in \mathcal{C}_c^\infty(X)$. Then π is the Poisson measure π_σ with intensity σ.

Proof. First, we note that from Remark 6.2.6, if $I_1(h)$ has finite moments of all orders under π, for all $h \in \mathcal{C}_c^\infty(X)$, then $\delta^X(u)$ is integrable under π for all $u \in \mathcal{U}$. Next we show that (6.6.1) and (6.6.2) are equivalent. First we note that Relation (6.6.1) implies (6.6.2) from (6.5.1). The proof of the converse statement is done for u of the form $u = hF$, $F = I_n(f^{\otimes n})$, $f, h \in \mathcal{C}_c^\infty(X)$, by induction on the degree $n \in \mathbb{N}$ of $I_n(f^{\otimes n})$. The implication clearly holds when $n = 0$. Next, assuming that

$$\mathbb{E}_\pi\left[\delta^X(hI_n(f^{\otimes n}))\right] = 0, \qquad f, h \in \mathcal{C}_c^\infty(X),$$

for some $n \geq 0$, the Kabanov multiplication formula (6.2.5) shows that

$$\begin{aligned}\delta^X(hI_{n+1}(f^{\otimes(n+1)})) &= \delta^X(h)I_{n+1}(f^{\otimes(n+1)}) - \langle h, D^X I_{n+1}(f^{\otimes(n+1)})\rangle_{L^2(X,\sigma)}\\ &\quad -(n+1)\delta^X((hf)I_n(f^{\otimes n})),\end{aligned}$$

hence from Relation (6.6.1) applied at the rank n we have

$$\begin{aligned}\mathbb{E}_\pi\left[\delta^X(hI_{n+1}(f^{\otimes n+1}))\right] &= \mathbb{E}_\pi\left[\delta^X(h)I_{n+1}(f^{\otimes(n+1)})\right]\\ &\quad - \mathbb{E}_\pi\left[\langle h, D^X I_{n+1}(f^{\otimes(n+1)})\rangle_{L^2(X,\sigma)}\right]\\ &\quad -(n+1)\,\mathbb{E}_\pi\left[\delta^X((hf)I_n(f^{\otimes n}))\right]\\ &= \mathbb{E}_\pi\left[\delta^X(h)I_{n+1}(f^{\otimes(n+1)})\right]\\ &\quad - \mathbb{E}_\pi\left[\langle h, D^X I_{n+1}(f^{\otimes(n+1)})\rangle_{L^2(X,\sigma)}\right]\\ &= 0,\end{aligned}$$

by (6.6.2).

Next we show that $\pi = \pi_\sigma$. We have for $h \in \mathcal{C}_c^\infty(X)$ and $n \geq 1$, using (6.6.2):

$$\begin{aligned}
\mathbb{E}_\pi\left[\left(\int_X h(x)\omega(dx)\right)^n\right] &= \mathbb{E}_\pi\left[\delta^X(h)\left(\int_X h(x)\omega(dx)\right)^{n-1}\right] \\
&\quad + \left(\int_X h(x)\sigma(dx)\right)\mathbb{E}_\pi\left[\left(\int_X h(x)\omega(dx)\right)^{n-1}\right] \\
&= \mathbb{E}_\pi\left[\left\langle h, D^X\left(\int_X h(x)\omega(dx)\right)^{n-1}\right\rangle_{L^2(X,\sigma)}\right] \\
&\quad + \left(\int_X h(x)\sigma(dx)\right)\mathbb{E}_\pi\left[\left(\int_X h(x)\omega(dx)\right)^{n-1}\right] \\
&= \mathbb{E}_\pi\left[\int_X h(x)\left(h(x) + \int_X h(y)(\omega\backslash\{x\})(dy)\right)^{n-1}\sigma(dx)\right] \\
&= \mathbb{E}_\pi\left[\int_X h(x)\left(h(x) + \int_X h(y)\omega(dy)\right)^{n-1}\sigma(dx)\right] \\
&= \sum_{k=0}^{n-1}\binom{n-1}{k}\left(\int_X h^{n-k}(x)\sigma(dx)\right)\mathbb{E}_\pi\left[\left(\int_X h(x)\omega(dx)\right)^k\right].
\end{aligned}$$

This induction relation coincides with (6.5.7) and characterizes the moments of $\omega \longmapsto \int_X h(x)\omega(dx)$ under π_σ, hence the moments of $\omega \longmapsto \int_X h(x)\omega(dx)$ under π are that of a Poisson random variable with intensity $\int_X h(x)\sigma(dx)$. By dominated convergence this implies

$$\mathbb{E}_\pi\left[\exp\left(iz\int_X h(x)\omega(dx)\right)\right] = \exp\int_X (\mathrm{e}^{izh} - 1)d\sigma,$$

$z \in \mathbb{R}$, $h \in \mathcal{C}_c^\infty(X)$, hence $\pi = \pi_\sigma$. □

This proposition can be modified as follows.

Proposition 6.6.2. *Let π be a probability measure on Ω^X such that $\delta^X(u)$ is integrable, $u \in \mathcal{U}$. Assume that*

$$\mathbb{E}_\pi\left[\delta^X(u)\right] = 0, \qquad u \in \mathcal{U}, \tag{6.6.3}$$

or equivalently

$$\mathbb{E}_\pi\left[\langle D^X F, u\rangle_{L^2(X,\sigma)}\right] = \mathbb{E}_\pi\left[F\delta^X(u)\right], \qquad F \in \mathcal{S}, \quad u \in \mathcal{U}. \tag{6.6.4}$$

Then π is the Poisson measure π_σ with intensity σ.

Proof. Clearly, (6.6.3) implies (6.6.4) as in the proof of Proposition 6.6.1. The implication (6.6.4) ⇒ (6.6.3) follows in this case by taking $F = 1$. Denoting the characteristic function of $\omega \longmapsto \int_X h(x)\omega(dx)$ by

$$\psi(z) = \mathbb{E}_\pi\left[\exp\left(iz\int_X h(y)\omega(dy)\right)\right],$$

$z \in \mathbb{R}$, we have:

$$\begin{aligned}
\frac{d\psi}{dz}(z) &= i\,\mathbb{E}_\pi \left[\int_X h(y)\omega(dy) \exp\left(iz \int_X h(y)\omega(dy) \right) \right] \\
&= i\,\mathbb{E}_\pi \left[\delta^X(h) \exp\left(iz \int_X h(y)\omega(dy) \right) \right] \\
&\quad + i\,\mathbb{E}_\pi \left[\int_X h(y)\sigma(dy) \exp\left(iz \int_X h(y)\omega(dy) \right) \right] \\
&= i\,\mathbb{E}_\pi \left[\left\langle h, D^X \exp\left(iz \int_X h(y)\omega(dy) \right) \right\rangle_{L^2(X,\sigma)} \right] + i\psi(z) \int_X h(y)\sigma(dy) \\
&= i\langle h, \mathrm{e}^{izh} - 1 \rangle_{L^2(X,\sigma)} \mathbb{E}_\pi \left[\exp\left(iz \int_X h(y)\omega(dy) \right) \right] + i\psi(z) \int_X h(y)\sigma(dy) \\
&= i\psi(z) \langle h, \mathrm{e}^{izh} \rangle_{L^2(X,\sigma)}, \qquad z \in \mathbb{R}.
\end{aligned}$$

We used the relation

$$D_x^X \exp\left(iz \int_X h(y)\omega(dy) \right) = (\mathrm{e}^{izh(x)} - 1) \exp\left(iz \int_X h(y)\omega(dy) \right), \qquad x \in X,$$

that follows from Proposition 6.4.7. With the initial condition $\psi(0) = 1$ we obtain

$$\psi(z) = \exp \int_X (\mathrm{e}^{izh(y)} - 1)\sigma(dy), \qquad z \in \mathbb{R}.$$

□

Corollary 6.6.3. *Let π be a probability measure on Ω^X such that $I_n(f^{\otimes n})$ is integrable under π, $f \in \mathcal{C}_c^\infty(X)$. The relation*

$$\mathbb{E}_\pi \left[I_n(f^{\otimes n}) \right] = 0, \tag{6.6.5}$$

holds for all $f \in \mathcal{C}_c^\infty(X)$ and $n \geq 1$, if and only if π is the Poisson measure π_σ with intensity σ.

Proof. If (6.6.5) holds then by polarization and the Definition 6.4.2 we get

$$\mathbb{E}_\pi \left[\delta^X(g \otimes I_n(f_1 \otimes \cdots \otimes f_n)) \right] = 0,$$

$g, f_1, \ldots, f_n \in \mathcal{C}_c^\infty(X)$, $n \geq 0$, and from Remark 6.2.6 we have

$$\mathbb{E}_\pi[\delta^X(u)] = 0, \qquad u \in \mathcal{U},$$

hence $\pi = \pi_\sigma$ from Proposition 6.6.1. □

6.7 Clark Formula and Lévy Processes

In this section we extend the construction of the previous section to the case of Lévy processes, and state a Clark formula in this setting. Let $X = \mathbb{R}_+ \times \mathbb{R}^d$ and consider a random measure of the form

$$X(dt, dx) = \delta_0(dx)dB_t + \omega(dt, dx) - \sigma(dx)dt,$$

where $\omega(dt, dx) - \sigma(dx)dt$ is a compensated Poisson random measure on $\mathbb{R}^d \setminus \{0\} \times \mathbb{R}_+$ of intensity $\sigma(dx)dt$, and $(B_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion independent of $N(dt, dx)$. The underlying probability space is denoted by $(\Omega, \mathcal{F}, \mathbb{P})$, where $\mathcal{F}$ is generated by X. We define the filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ generated by X as

$$\mathcal{F}_t = \sigma(X(ds, dx) \ : x \in \mathbb{R}^d, \ s \leq t).$$

The integral of a square-integrable $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$-adapted process $u \in L^2(\Omega) \otimes L^2(\mathbb{R}^d \times \mathbb{R}_+)$ with respect to $X(dt, dx)$ is written as

$$\int_{\mathbb{R}^d\times\mathbb{R}_+} u(t, x)X(dt, dx),$$

with the isometry

$$\mathbb{E}\left[\left(\int_{\mathbb{R}^d\times\mathbb{R}_+} u(t, x)X(dt, dx)\right)^2\right] = \mathbb{E}\left[\int_{\mathbb{R}^d\times\mathbb{R}_+} |u(t, x)|^2\tilde{\sigma}(dx)dt\right], \quad (6.7.1)$$

with

$$\tilde{\sigma}(dx) = \delta_0(dx) + \sigma(dx).$$

The multiple stochastic integral $I_n(h_n)$ of $h_n \in L^2(\mathbb{R}^d \times \mathbb{R}_+)^{\circ n}$ can be defined by induction with

$$\begin{aligned} I_1(h) &= \int_{\mathbb{R}^d\times\mathbb{R}_+} h(t, x)X(dt, dx) \\ &= \int_0^\infty h(0, t)dB_t + \int_{\mathbb{R}^d\setminus\{0\}\times\mathbb{R}_+} h(t, x)(\omega(dt, dx) - \sigma(dx)dt), \end{aligned}$$

$h \in L^2(\mathbb{R}^d \times \mathbb{R}_+)$, and

$$I_n(h_n) = n\int_{\mathbb{R}^d\times\mathbb{R}_+} I_{n-1}(\pi^n_{t,x}h_n)X(dt, dx),$$

where

$$\pi^n_{t,x} : L^2(\mathbb{R}^d \times \mathbb{R}_+)^{\circ n} \longrightarrow L^2(\mathbb{R}^d \times \mathbb{R}_+)^{\circ(n-1)} \quad (6.7.2)$$

is defined by

$$\left[\pi_{t,x}^n h_n\right](x_1,t_1,\ldots,x_{n-1},t_{n-1}) \\ = h_n(x_1,t_1,\ldots,x_{n-1},t_{n-1},t,x)\mathbf{1}_{[0,t]}(t_1)\cdots\mathbf{1}_{[0,t]}(t_{n-1}),$$

for $x_1,\ldots,x_{n-1},x \in \mathbb{R}^d$ and $t_1,\ldots,t_{n-1},t \in \mathbb{R}_+$.

As in (6.1.10) the characteristic function of $I_1(h)$ is given by

$$\mathbb{E}\left[e^{izI_1(h)}\right] \\ = \exp\left(-\frac{z^2}{2}\int_0^\infty h(0,t)^2 dt + \int_{\mathbb{R}^d\setminus\{0\}\times\mathbb{R}_+}(e^{izh(t,x)} - 1 - izh(t,x))\sigma(dx)dt\right).$$

The isometry property

$$\mathbb{E}\left[|I_n(h_n)|^2\right] = n!\|h_n\|^2_{L^2(\mathbb{R}^d\times\mathbb{R}_+)^{\otimes n}},$$

follows from Relation (6.7.1).

From Proposition 5.1.5 and Proposition 6.3.2, every $F \in L^2(\Omega)$ admits a decomposition

$$F = \mathbb{E}[F] + \sum_{n\geq 1}\frac{1}{n!}I_n(f_n) \tag{6.7.3}$$

into a series of multiple stochastic integrals, with $f_n \in L^2(\mathbb{R}^d\times\mathbb{R}_+)^{\circ n}$, $n \geq 1$.

The next proposition is a version of the Clark predictable representation formula for Lévy processes.

Proposition 6.7.1. *For $F \in L^2(\Omega)$, we have*

$$F = \mathbb{E}[F] + \int_{\mathbb{R}^d\times\mathbb{R}_+}\mathbb{E}[D_{t,x}^X F \mid \mathcal{F}_t]X(dt,dx). \tag{6.7.4}$$

Proof. Let

$$\tilde{\Delta}_n = \left\{((x_1,t_1),\ldots,(x_n,t_n)) \in (\mathbb{R}^d\times\mathbb{R}_+)^n \ : \ t_1 < \cdots < t_n\right\}.$$

From (6.7.3) we have for $F \in \mathcal{S}$:

$$\begin{aligned} F &= \mathbb{E}[F] + \sum_{n\geq 1} I_n(f_n\mathbf{1}_{\tilde{\Delta}_n}) \\ &= \mathbb{E}[F] + \sum_{n\geq 1}\int_{\mathbb{R}^d\times\mathbb{R}_+} I_{n-1}(f_n(\cdot,t,x)\mathbf{1}_{\tilde{\Delta}_n}(\cdot,t,x))X(dt,dx) \\ &= \mathbb{E}[F] + \int_{\mathbb{R}^d\times\mathbb{R}_+}\sum_{n=0}^\infty \mathbb{E}[I_n(f_{n+1}(\cdot,t,x)\mathbf{1}_{\tilde{\Delta}_n}) \mid \mathcal{F}_t]X(dt,dx) \\ &= \mathbb{E}[F] + \int_{\mathbb{R}^d\times\mathbb{R}_+}\mathbb{E}[D_{t,x}^X F \mid \mathcal{F}_t]X(dt,dx) \end{aligned}$$

The extension of this statement to $F \in L^2(\Omega)$ is a consequence of the fact that the adapted projection of $D^X F$ extends to a continuous operator from $L^2(\Omega)$ into the space of adapted processes in $L^2(\Omega) \otimes L^2(\mathbb{R}^d \times \mathbb{R}_+)$. For

$$F = \sum_{n=0}^{\infty} I_n(f_n) \in \mathcal{S}$$

and

$$u = \sum_{n=0}^{\infty} I_n(u_{n+1}) \in \mathcal{U}, \qquad u_{n+1} \in L^2(\mathbb{R}^d \times \mathbb{R}_+)^{\circ n} \otimes L^2(\mathbb{R}^d \times \mathbb{R}_+), \quad n \in \mathbb{N},$$

we can extend the continuity argument of Proposition 3.2.6 as follows:

$$\begin{aligned}
&\left| \mathbb{E}\left[\int_{\mathbb{R}^d \times \mathbb{R}_+} u(t,x)\, \mathbb{E}[D^X_{t,x} F \mid \mathcal{F}_t] \sigma(dx) dt \right] \right| \\
&\leq \sum_{n=0}^{\infty} (n+1)! \left| \int_{\mathbb{R}^d \times \mathbb{R}_+} \langle f_{n+1}(\cdot, t, x) \mathbf{1}_{[0,t]}(\cdot), u_{n+1}(\cdot, t, x) \rangle_{L^2(\mathbb{R}^d \times \mathbb{R}_+)^{\otimes n}} \sigma(dx) dt \right| \\
&\leq \sum_{n=0}^{\infty} (n+1)! \| f_{n+1} \|_{L^2(\mathbb{R}^d \times \mathbb{R}_+)} \| u_{n+1} \|_{L^2(\mathbb{R}^d \times \mathbb{R}_+)^{\otimes (n+1)}} \\
&\leq \left(\sum_{n=0}^{\infty} n! \| f_n \|^2_{L^2(\mathbb{R}^d \times \mathbb{R}_+)^{\otimes n}} \sum_{n=0}^{\infty} n! \| u_{n+1} \|^2_{L^2(\mathbb{R}^d \times \mathbb{R}_+)^{\otimes (n+1)}} \right)^{1/2} \\
&\leq \| F \|_{L^2(\Omega)} \| u \|_{L^2(\Omega) \otimes L^2(\mathbb{R}^d \times \mathbb{R}_+)}.
\end{aligned}$$

□

Note that Relation (6.7.4) can be written as

$$\begin{aligned}
F = \mathbb{E}[F] &+ \int_0^{\infty} \mathbb{E}[D^X_{t,0} F \mid \mathcal{F}_t] dB_t \\
&+ \int_{\mathbb{R}^d \times \mathbb{R}_+} \mathbb{E}[D^X_{t,x} F \mid \mathcal{F}_t] (\omega(dt, dx) - \sigma(dx) dt).
\end{aligned}$$

6.8 Covariance identities

The Ornstein-Uhlenbeck semi-group satisfies

$$P_t I_n(f_n) = \mathrm{e}^{-nt} I_n(f_n), \qquad f_n \in L^2(X)^{\circ n}, \ n \in \mathbb{N}.$$

We refer to [144] for the construction of the Ω^X-valued diffusion process associated to $(P_t)_{t \in \mathbb{R}_+}$. Here we shall only need the existence of the probability density kernel associated to $(P_t)_{t \in \mathbb{R}_+}$.

Lemma 6.8.1. *In case σ is finite on X we have*

$$P_t F(\omega) = \int_{\Omega^X \times \Omega^X} F(\tilde{\omega} \cup \hat{\omega}) q_t(\omega, d\tilde{\omega}, d\hat{\omega}), \quad \omega \in \Omega^X, \tag{6.8.1}$$

where $q_t(\omega, d\tilde{\omega}, d\hat{\omega})$ is the probability kernel on $\Omega^X \times \Omega^X$ defined by

$$q_t(\omega, d\tilde{\omega}, d\hat{\omega}) = \sum_{\omega' \subset \omega} \frac{|\omega|!}{|\omega'|!|\omega \backslash \omega'|!} (\mathrm{e}^{-t})^{|\omega'|} (1 - \mathrm{e}^{-t})^{|\omega - \omega'|} \epsilon_{\omega'}(d\tilde{\omega}) \pi_{(1-\mathrm{e}^{-t})\sigma}(d\hat{\omega}).$$

Here, $\pi_{(1-\mathrm{e}^{-t})\sigma}$ is the thinned Poisson measure with intensity $(1-\mathrm{e}^{-t})\sigma(dx)$, ϵ_ω denote the Dirac measure at $\omega \in \Omega^X$ and $|\omega| = \omega(X) \in \mathbb{N} \backslash \{0\} \cup \{+\infty\}$ represents the (π_σ-a.s. finite) cardinal of $\omega \in \Omega^X$.

Proof. We consider random functionals of the form

$$F = \mathrm{e}^{-\int_X u(x)\sigma(dx)} \prod_{x \in \omega} (1 + u(x)) = \sum_{k=0}^{\infty} \frac{1}{n!} I_n(u^{\otimes n}),$$

cf. Proposition 6.3.1, for which we have

$$P_t F = \sum_{n=0}^{\infty} \frac{1}{n!} \mathrm{e}^{-nt} I_n(u^{\otimes n}) = \exp\left(-\mathrm{e}^{-t} \int_X u(x)\sigma(dx)\right) \prod_{x \in \omega} (1 + \mathrm{e}^{-t} u(x)),$$

and

$$\begin{aligned}
&\int_{\Omega^X \times \Omega^X} F(\hat{\omega} \cup \tilde{\omega}) \mathbb{P}(\omega, d\tilde{\omega}, d\hat{\omega}) \\
&= \exp\left(-\int_X u(x)\sigma(dx)\right) \int_{\Omega^X} \int_{\Omega^X} \sum_{\omega' \subset \omega} \\
&\qquad \prod_{x \in \hat{\omega} \cup \tilde{\omega}} (1 + u(x)) \frac{|\omega|!}{|\omega'|!|\omega \backslash \omega'|!} \mathrm{e}^{-t|\omega'|} (1 - \mathrm{e}^{-t})^{|\omega - \omega'|} \epsilon_\omega(d\tilde{\omega}) \pi_{(1-\mathrm{e}^{-t})\sigma}(d\hat{\omega}) \\
&= \exp\left(-\mathrm{e}^{-t} \int_X u(x)\sigma(dx)\right) \\
&\qquad \int_{\Omega^X} \sum_{\omega' \subset \omega} \prod_{x \in \tilde{\omega}} (1 + u(x)) \frac{|\omega|!}{|\omega'|!|\omega \backslash \omega'|!} \mathrm{e}^{-t|\omega'|} (1 - \mathrm{e}^{-t})^{|\omega - \omega'|} \epsilon_\omega(d\tilde{\omega}) \\
&= \exp\left(-\mathrm{e}^{-t} \int_X u(x)\sigma(dx)\right) \\
&\qquad \sum_{\omega' \subset \omega} \prod_{x \in \omega'} (1 + u(x)) \frac{|\omega|!}{|\omega'|!|\omega \backslash \omega'|!} \mathrm{e}^{-t|\omega'|} (1 - \mathrm{e}^{-t})^{|\omega - \omega'|} \\
&= \exp\left(-\mathrm{e}^{-t} \int_X u(x)\sigma(dx)\right) \\
&\qquad \sum_{\omega' \subset \omega} \frac{|\omega|!}{|\omega'|!|\omega \backslash \omega'|!} \prod_{x \in \omega'} \mathrm{e}^{-t}(1 + u(x)) \prod_{x \in \omega \backslash \omega'} (1 - \mathrm{e}^{-t})
\end{aligned}$$

$$= \exp\left(-e^{-t}\int_X u(x)\sigma(dx)\right)\prod_{x\in\omega}(1+e^{-t}u(x))$$
$$= P_tF(\omega).$$

□

The semi-group P_t can be rewritten as

$$P_tF(\omega) = \sum_{\omega'\subset\omega}\frac{|\omega|!}{|\omega'|!|\omega\backslash\omega'|!}e^{-t|\omega'|}(1-e^{-t})^{|\omega-\omega'|}\int_{\Omega^X}F(\omega'\cup\hat\omega)\pi_{(1-e^{-t})\sigma}(d\hat\omega).$$

Again, Lemma 6.8.1 and Jensen's inequality (9.3.1) imply

$$\|P_tu\|_{L^2(\mathbb{R}_+)}\le\|u\|_{L^2(\mathbb{R}_+)}, \qquad u\in L^2(\Omega^X\times\mathbb{R}_+),$$

a.s., hence

$$\|P_tu\|_{L^\infty(\Omega^X,L^2(\mathbb{R}_+))}\le\|u\|_{L^\infty(\Omega^X,L^2(\mathbb{R}_+))},$$

$t\in\mathbb{R}_+$, $u\in L^2(\Omega^X\times\mathbb{R}_+)$. A covariance identity can be written using the Ornstein-Uhlenbeck semi-group, in the same way as in Proposition 4.4.1.

Proposition 6.8.2. *We have the covariance identity*

$$\mathrm{Cov}\,(F,G) = \mathbb{E}\left[\int_0^\infty\int_X e^{-s}D_x^XFP_sD_x^XG\sigma(dx)ds\right], \tag{6.8.2}$$

$F,G\in\mathrm{Dom}\,(D^X)$.

Proof. By the chaos representation property Proposition 6.3.2, orthogonality of multiple integrals of different orders, and continuity of P_s, $s\in\mathbb{R}_+$, on $L^2(\Omega^X,\mathbb{P})$, it suffices to prove the identity for $F=I_n(f_n)$ and $G=I_n(g_n)$. We have

$$\begin{aligned}
\mathbb{E}_{\pi_\sigma}[I_n(f_n)I_n(g_n)] &= n!\langle f_n,g_n\rangle_{L^2(X,\sigma)^{\circ n}}\\
&= n!\int_{X^n}f_n(x_1,\ldots,x_n)g_n(x_1,\ldots,x_n)\sigma(dx_1)\cdots\sigma(dx_n)\\
&= n!\int_X\int_{X^{(n-1)}}f_n(x,y)g_n(x,y)\,\sigma^{\otimes(n-1)}(dx)\,\sigma(dy)\\
&= n\int_X\mathbb{E}_{\pi_\sigma}[I_{n-1}(f_n(\cdot,y))I_{n-1}(g_n(\cdot,y))]\,\sigma(dy)\\
&= \frac{1}{n}\mathbb{E}_{\pi_\sigma}\left[\int_X D_y^XI_n(f_n)D_y^XI_n(g_n)\,\sigma(dy)\right]\\
&= \mathbb{E}_{\pi_\sigma}\left[\int_0^\infty e^{-ns}\int_X D_y^XI_n(f_n)D_y^XI_n(g_n)\sigma(dy)ds\right]\\
&= \mathbb{E}_{\pi_\sigma}\left[\int_0^\infty e^{-s}\int_X D_y^XI_n(f_n)P_sD_y^XI_n(g_n)\sigma(dy)ds\right].
\end{aligned}$$

□

The above identity can be rewritten using the integral representation (6.8.1) of P_t, to extend Proposition 3 of [59]:

Corollary 6.8.3. *We have*

$$\mathrm{Cov}\,(F,G) = \int_0^1 \int_X \int_{\Omega^X\times\Omega^X} \sum_{\omega'\subset\omega} D_x^X F(\omega)(G(\omega'\cup\hat\omega\cup\{x\}) - G(\omega'\cup\hat\omega)) \\ \frac{|\omega|!}{|\omega'|!|\omega\backslash\omega'|!}\alpha^{|\omega'|}(1-\alpha)^{|\omega-\omega'|}\pi_{(1-\alpha)\sigma}(d\hat\omega)\pi_\sigma(d\omega)\sigma(dx)d\alpha, \quad (6.8.3)$$

$F, G \in \mathrm{Dom}\,(D^X)$.

Proof. From (6.8.1) and (6.8.2) we have

$$\mathrm{Cov}\,(F,G) = \mathbb{E}\left[\int_0^\infty \int_X \mathrm{e}^{-s}(D_x^X F)(P_s D_x^X G)\sigma(dx)ds\right] \\ = \int_0^\infty \int_X \int_{\Omega^X\times\Omega^X} \sum_{\omega'\subset\omega} \mathrm{e}^{-s} D_x^X F(\omega)(G(\omega'\cup\hat\omega\cup\{x\}) - G(\omega'\cup\hat\omega)) \\ \frac{|\omega|!}{|\omega'|!|\omega\backslash\omega'|!}\mathrm{e}^{-s|\omega'|}(1-\mathrm{e}^{-s})^{|\omega-\omega'|}\pi_{(1-\mathrm{e}^{-s})\sigma}(d\hat\omega)\pi_\sigma(d\omega)\sigma(dx)ds.$$

We conclude the proof by applying the change of variable $\alpha = e^{-s}$. □

In other terms, denoting by ω_α the thinning of ω with parameter $\alpha \in (0,1)$, and by $\hat\omega_{1-\alpha}$ an independent Poisson random measure with intensity $(1-\alpha)\sigma(dx)$, we can rewrite (6.8.3) as

$$\mathrm{Cov}\,(F,G) = \mathbb{E}\left[\int_0^1 \int_X D_x^X F(\omega) D_x^X G(\omega_\alpha\cup\hat\omega_{1-\alpha})\sigma(dx)d\alpha\right]. \quad (6.8.4)$$

This statement also admits a direct proof using characteristic functions. Let

$$\varphi(t) = \mathbb{E}\left[\mathrm{e}^{it|\omega|}\right] = \mathrm{e}^{\sigma(X)(\mathrm{e}^{it}-1)}, \quad t \geq 0,$$

and let $\varphi_\alpha(s,t)$ denote the characteristic function of

$$(|\omega|, |\omega_\alpha\cup\hat\omega_{1-\alpha}|) = (|\omega|, |\omega_\alpha| + |\hat\omega_{1-\alpha}|),$$

i.e.

$$\varphi_\alpha(s,t) = \mathbb{E}\left[\exp\left(is|\omega| + it|\omega_\alpha\cup\hat\omega_{1-\alpha}|\right)\right] \\ = \int_{\Omega^X}\int_{\Omega^X\times\Omega^X} \exp\left(is|\omega| + it|\tilde\omega\cup\hat\omega|\right)\pi_\sigma(d\omega)p_{-\log\alpha}(\omega, d\tilde\omega, d\hat\omega).$$

Since from Proposition 6.1.5 the thinning of order α of a Poisson random measure of intensity $\sigma(dx)$ is itself a Poisson random measure with intensity $\alpha\sigma(dx)$, we have

$$\begin{aligned}
&\varphi_\alpha(s,t)\\
&= \int_{\Omega^X}\int_{\Omega^X\times\Omega^X} \exp\left(is(|\tilde\omega|+|\omega-\tilde\omega|)+it(|\tilde\omega|+|\hat\omega|)\right)\pi_\sigma(d\omega)q_{-\log\alpha}(\omega,d\tilde\omega,d\hat\omega)\\
&= \int_{\Omega^X}\exp\left(it|\hat\omega|\right)\pi_{(1-\alpha)\sigma}(d\hat\omega)\\
&\times\int_{\Omega^X}\exp\left(is(|\omega'|+|\omega-\omega'|)+it|\omega'|\right)\sum_{\omega'\subset\omega}\frac{|\omega|!}{|\omega'|!|\omega\backslash\omega'|!}\alpha^{|\omega'|}(1-\alpha)^{|\omega-\omega'|}\pi_\sigma(d\omega)\\
&= (\varphi(t))^{1-\alpha}\mathrm{e}^{-\sigma(X)}\sum_{n=0}^\infty\frac{(\sigma(X))^n}{n!}\sum_{k=0}^n\mathrm{e}^{is(k+(n-k))+itk}\frac{n!}{k!(n-k)!}\alpha^k(1-\alpha)^{n-k}\\
&= (\varphi(t))^{1-\alpha}\mathrm{e}^{-\sigma(X)}\sum_{k,l=0}^\infty\frac{(\alpha\sigma(X))^k}{k!}\frac{((1-\alpha)\sigma(X))^l}{l!}\mathrm{e}^{is(k+l)+itk}\\
&= (\varphi(t))^{1-\alpha}(\varphi(t+s))^\alpha(\varphi(s))^{1-\alpha}\\
&= (\varphi_0(s,t))^\alpha(\varphi_1(s,t))^{1-\alpha}.
\end{aligned}$$

Relation (6.8.4) also shows that

$$\begin{aligned}
&\mathrm{Cov}\,(\mathrm{e}^{is|\omega|},\mathrm{e}^{it|\omega|}) = \varphi_1(s,t)-\varphi_0(s,t)\\
&= \int_0^1\frac{d\varphi_\alpha}{d\alpha}(s,t)d\alpha\\
&= \int_0^1\frac{d}{d\alpha}((\varphi(t))^{1-\alpha}(\varphi(t+s))^\alpha(\varphi(s))^{1-\alpha})d\alpha\\
&= \int_0^1\log\left(\frac{\varphi(s+t)}{\varphi(s)\varphi(t)}\right)\varphi_\alpha(s,t)d\alpha\\
&= \int_0^1(\mathrm{e}^{it}-1)(\mathrm{e}^{is}-1)\varphi_\alpha(s,t)d\alpha\\
&= \int_0^1\int_{\Omega^X}\int_{\Omega^X}\int_X D_x^X\mathrm{e}^{is|\omega|}(D_x^X\mathrm{e}^{it|\cdot|})(\omega_\alpha\cup\hat\omega_{1-\alpha})\sigma(dx)\mathbb{P}(d\omega)\mathbb{P}(d\tilde\omega)d\alpha.
\end{aligned}$$

6.9 Deviation Inequalities

Using the covariance identity of Proposition 6.8.2 and the representation of Lemma 6.8.1 we now present a general deviation result for Poisson functionals. In this proposition and the following ones, the supremum on Ω^X can be taken as an essential supremum with respect to π_σ.

Proposition 6.9.1. *Let $F\in\mathrm{Dom}\,(D^X)$ be such that $\mathrm{e}^{sF}\in\mathrm{Dom}\,(D^X)$, $0\le s\le t_0$, for some $t_0>0$. Then*

$$\pi_\sigma(F-\mathbb{E}[F]\ge x)\le\exp\left(\min_{0<t<t_0}\left(-tx+\int_0^t h(s)\,ds\right)\right),\qquad x>0,$$

where

$$h(s) = \sup_{(\omega,\omega')\in\Omega^X\times\Omega^X} \left| \int_X (\mathrm{e}^{sD_y^X F(\omega)} - 1)\, D_y^X F(\omega')\sigma(dy) \right|, \quad s \in [0, t_0). \tag{6.9.1}$$

If moreover h is nondecreasing and finite on $[0, t_0)$ then

$$\pi_\sigma(F - \mathbb{E}[F] \geq x) \leq \exp\left(- \int_0^x h^{-1}(s) ds \right), \qquad 0 < x < h(t_0^-), \tag{6.9.2}$$

where h^{-1} is the left-continuous inverse of h:

$$h^{-1}(x) = \inf\{t > 0 \ : \ h(t) \geq x\}, \qquad 0 < x < h(t_0^-).$$

Proof. We start by deriving the following inequality for F a centered random variable:

$$\mathbb{E}[F\mathrm{e}^{sF}] \leq h(s)\, \mathbb{E}[\mathrm{e}^{sF}], \quad 0 \leq s \leq t_0. \tag{6.9.3}$$

This follows from (6.8.2). Indeed, using the integral representation (6.8.1) of the Ornstein-Uhlenbeck semi-group $(P_t)_{t\in\mathbb{R}_+}$ for $P_v D_y^X F(\omega)$, we have,

$$\begin{aligned}
\mathbb{E}[F\mathrm{e}^{sF}] &= \mathbb{E}\left[\int_0^\infty \mathrm{e}^{-v} \int_X D_y^X \mathrm{e}^{sF} P_v D_y^X F \sigma(dy) dv \right] \\
&= \int_{\Omega^X} \int_0^\infty \mathrm{e}^{-v} \int_X (\mathrm{e}^{sD_y^X F(\omega)} - 1) \mathrm{e}^{sF(\omega)} \\
&\quad \times \int_{\Omega^X\times\Omega^X} D_y^X F(\omega' \cup \tilde{\omega}) q_v(\omega, d\omega', d\tilde{\omega}) \sigma(dy) dv \pi_\sigma(d\omega) \\
&\leq \int_{\Omega^X} \int_0^\infty \mathrm{e}^{-v} \mathrm{e}^{sF(\omega)} \\
&\quad \times \int_{\Omega^X\times\Omega^X} \left| \int_X (\mathrm{e}^{sD_y^X F(\omega)} - 1) D_y^X F(\omega' \cup \tilde{\omega}) \sigma(dy) \right| q_v(\omega, d\omega', d\tilde{\omega}) dv \pi_\sigma(d\omega) \\
&\leq \sup_{(\omega,\omega')\in\Omega^X\times\Omega^X} \left| \int_X (\mathrm{e}^{sD_y^X F(\omega)} - 1) D_y^X F(\omega') \sigma(dy) \right| \mathbb{E}\left[\mathrm{e}^{sF} \int_0^\infty \mathrm{e}^{-v} dv \right] \\
&= \sup_{(\omega,\omega')\in\Omega^X\times\Omega^X} \left| \int_X (\mathrm{e}^{sD_y^X F(\omega)} - 1) D_y^X F(\omega') \sigma(dy) \right| \mathbb{E}\left[\mathrm{e}^{sF} \right],
\end{aligned}$$

which yields (6.9.3). In the general case, we let $L(s) = \mathbb{E}\left[\exp(s(F - \mathbb{E}[F]))\right]$ and obtain:

$$\frac{L'(s)}{L(s)} \leq h(s), \quad 0 \leq s \leq t_0,$$

which using Chebychev's inequality gives:

$$\pi_\sigma(F - \mathbb{E}[F] \geq x) \leq \exp\left(-tx + \int_0^t h(s) ds \right). \tag{6.9.4}$$

Using the relation $\displaystyle \frac{d}{dt}\left(\int_0^t h(s)\, ds - tx \right) = h(t) - x$, we can then optimize as follows:

$$\begin{aligned}\min_{0<t<t_0}\left(-tx+\int_0^t h(s)\,ds\right) &= \int_0^{h^{-1}(x)} h(s)\,ds - xh^{-1}(x)\\ &= \int_0^x s\,dh^{-1}(s) - xh^{-1}(x)\\ &= -\int_0^x h^{-1}(s)\,ds, \end{aligned} \qquad (6.9.5)$$

hence

$$\pi_\sigma(F-\mathbb{E}[F]\geq x)\leq \exp\left(-\int_0^x h^{-1}(s)ds\right), \qquad 0<x<h(t_0^-).$$

□

In the sequel we derive several corollaries from Proposition 6.9.1 and discuss possible choices for the function h, in particular for vectors of random functionals.

Proposition 6.9.2. *Let $F:\Omega^X\to\mathbb{R}$ and let $K:X\to\mathbb{R}_+$ be a function such that*

$$D_y^X F(\omega)\leq K(y), \qquad y\in X, \quad \omega\in\Omega^X. \qquad (6.9.6)$$

Then

$$\pi_\sigma(F-\mathbb{E}[F]\geq x)\leq \exp\left(\min_{t>0}\left(-tx+\int_0^t h(s)\,ds\right)\right), \qquad x>0,$$

where

$$h(t)=\sup_{\omega\in\Omega^X}\int_X \frac{e^{tK(y)}-1}{K(y)}|D_y^X F(\omega)|^2\sigma(dy), \quad t>0. \qquad (6.9.7)$$

If moreover h is finite on $[0,t_0)$ then

$$\pi_\sigma(F-\mathbb{E}[F]\geq x)\leq \exp\left(-\int_0^x h^{-1}(s)ds\right), \qquad 0<x<h(t_0^-). \qquad (6.9.8)$$

If $K(y)=0$, $y\in X$, we have:

$$\pi_\sigma(F-\mathbb{E}[F]\geq x)\leq \exp\left(-\frac{x^2}{2\tilde{\alpha}^2}\right), \qquad x>0,$$

with

$$\tilde{\alpha}^2=\sup_{\omega\in\Omega^X}\int_X (D_y^X F(\omega))^2\sigma(dy).$$

Proof. Let $F_n=\max(-n,\min(F,n))$, $n\geq 1$. Since when K is $\mathbb{R}_+$-valued the condition $D_y^X F_n(\omega)\leq K(y)$, $\omega\in\Omega^X$, $y\in X$, is satisfied we may apply Proposition 6.9.1 to F_n to get

$$\begin{aligned}
h(t) &= \sup_{(\omega,\omega')\in\Omega^X\times\Omega^X} \left| \int_X \frac{\mathrm{e}^{tD_y^X F_n(\omega)}-1}{D_y^X F_n(\omega)} D_y^X F_n(\omega) D_y^X F_n(\omega')\,\sigma(dy)\right| \\
&\leq \sup_{(\omega,\omega')\in\Omega^X\times\Omega^X} \int_X \frac{\mathrm{e}^{tK(y)}-1}{K(y)} |D_y^X F_n(\omega)|\,|D_y^X F_n(\omega')|\,\sigma(dy) \\
&\leq \frac{1}{2} \sup_{(\omega,\omega')\in\Omega^X\times\Omega^X} \int_X \frac{\mathrm{e}^{tK(y)}-1}{K(y)} (|D_y^X F_n(\omega)|^2 + |D_y^X F_n(\omega')|^2)\,\sigma(dy) \\
&\leq \sup_{\omega\in\Omega^X} \int_X \frac{\mathrm{e}^{tK(y)}-1}{K(y)} |D_y^X F_n(\omega)|^2\,\sigma(dy) \\
&\leq \sup_{\omega\in\Omega^X} \int_X \frac{\mathrm{e}^{tK(y)}-1}{K(y)} |D_y^X F(\omega)|^2\,\sigma(dy),
\end{aligned}$$

from which the conclusion follows after letting n tend to infinity. □

Part of the next corollary recovers a result of [155], see also [61].

Corollary 6.9.3. *Let $F \in L^2(\Omega^X, \pi_\sigma)$ be such that $D^X F \leq K$, $\pi_\sigma \otimes \sigma$-a.e., for some $K \in \mathbb{R}$, and $\|D^X F\|_{L^\infty(\Omega^X, L^2(X,\sigma))} \leq \tilde{\alpha}$. We have for $x > 0$:*

$$\pi_\sigma(F - \mathbb{E}[F] \geq x) \leq \mathrm{e}^{x/K} \left(1 + \frac{xK}{\tilde{\alpha}^2}\right)^{-\frac{x}{K} - \frac{\tilde{\alpha}^2}{K^2}}, \qquad x > 0, \tag{6.9.9}$$

and for $K = 0$:

$$\pi_\sigma(F - \mathbb{E}[F] \geq x) \leq \exp\left(-\frac{x^2}{2\tilde{\alpha}^2}\right), \qquad x > 0. \tag{6.9.10}$$

Proof. If $K \geq 0$, let us first assume that F is a bounded random variable. The function h in (6.9.7) is such that

$$\begin{aligned}
h(t) &\leq \frac{\mathrm{e}^{tK}-1}{K} \|D^X F\|^2_{L^\infty(\Omega^X, L^2(X,\sigma))} \\
&\leq \tilde{\alpha}^2 \frac{\mathrm{e}^{tK}-1}{K}, \qquad t > 0.
\end{aligned}$$

Applying (6.9.4) with $\tilde{\alpha}^2(\mathrm{e}^{tK}-1)/K$ gives

$$\pi_\sigma(F - \mathbb{E}[F] \geq x) \leq \exp\left(-tx + \frac{\tilde{\alpha}^2}{K^2}(\mathrm{e}^{tK} - tK - 1)\right).$$

Optimizing in t with $t = K^{-1}\log(1 + Kx/\tilde{\alpha}^2)$ (or using directly (6.9.2) with the inverse $K^{-1}\log\left(1 + Kt/\tilde{\alpha}^2\right)$) we have

$$\pi_\sigma(F - \mathbb{E}[F] \geq x) \leq \exp\left(\frac{x}{K} - \left(\frac{x}{K} + \frac{\tilde{\alpha}^2}{K^2}\right)\log\left(1 + \frac{xK}{\tilde{\alpha}^2}\right)\right),$$

which yields (6.9.10) and (6.9.9), depending on the value of K. For unbounded F, apply the above to $F_n = \max(-n, \min(F, n))$ with $|D^X F_n| \leq |D^X F|$, $n \geq 1$. Then (6.9.9) follows since, as n goes to infinity, F_n converges to F in $L^2(\Omega^X)$, $D^X F_n$ converges to $D^X F$ in $L^2(\Omega^X, L^2(X, \sigma))$, and $D^X F_n \leq K$, $n \geq 1$. The same argument applies if $K = 0$. □

As an example if F is the Poisson stochastic integral

$$F = \int_X f(x)(\omega(dx) - \sigma(dx)),$$

where $f \in L^2(X, \sigma)$ is upper bounded by $K > 0$, then

$$\pi_\sigma(F - \mathbb{E}[F] \geq x) \leq e^{x/K} \left(1 + \frac{xK}{\tilde{\alpha}^2}\right)^{-\frac{x}{K} - \frac{\tilde{\alpha}^2}{K^2}}, \qquad x > 0,$$

where

$$\tilde{\alpha}^2 = \int_X |f(x)|^2 \sigma(dx).$$

Corollary 6.9.3 yields the following result, which recovers Corollary 1 of [57].

Corollary 6.9.4. *Let*

$$F = (F_1, \ldots, F_n) = \left(\int_{\{|y|_2 \leq 1\}} y_k \, (\omega(dy) - \sigma(dy)) + \int_{\{|y|_2 > 1\}} y_k \, \omega(dy) \right)_{1 \leq k \leq n} \tag{6.9.11}$$

be an infinitely divisible random variable in $\mathbb{R}^n$ with Lévy measure σ. Assume that $X = \mathbb{R}^n$ and $\sigma(dx)$ has bounded support, let

$$K = \inf\{r > 0 \ : \ \sigma(\{x \in X \ : \ \|x\| > r\}) = 0\},$$

and $\tilde{\alpha}^2 = \int_{\mathbb{R}^n} \|y\|^2 \sigma(dy)$. For any Lipschitz (c) function $f : \mathbb{R}^n \to \mathbb{R}$ with respect to a given norm $\| \cdot \|$ on $\mathbb{R}^n$, we have

$$\pi_\sigma(f(F) - \mathbb{E}[f(F)] \geq x) \leq e^{x/(cK)} \left(1 + \frac{xK}{c\tilde{\alpha}^2}\right)^{-x/(cK) - \tilde{\alpha}^2/K^2}, \quad x > 0.$$

Proof. The representation (6.9.11) shows that

$$\begin{aligned} |D_x^X f(F)(\omega)| &= |f(F(\omega \cup \{x\})) - f(F(\omega))| \\ &\leq c\|F(\omega \cup \{x\}) - F(\omega)\| \\ &= c\|x\|. \end{aligned} \tag{6.9.12}$$

We conclude the proof by an application of Corollary 6.9.3. □

6.10 Notes and References

Early statements of the Clark formula on the Poisson space can be found in [133], [134] and [135]. See also [1] for a white noise version of this formula on Poisson space. The Clark formula for Lévy processes has been considered in [1], [112], [102], [141], and applied to quadratic hedging in incomplete markets driven by jump processes in [102]. The construction of stochastic analysis on the Poisson space using difference operators has been developed in [70], [35], [99], [104], cf. [98] for the Definition 6.2.3 of Poisson multiple stochastic integrals. The Kabanov [72] multiplication formula has been extended to Azéma martingales in [120]. Symmetric difference operators on the Poisson space have also been introduced in [104]. The study of the characterization of Poisson measures by integration by parts has been initiated in [90], see also [127], Relation (6.4.12) is also known as the Mecke characterization of Poisson measures. Proposition 6.5.5 is useful to study the invariance of Poisson measures under random transformations, cf. [118]. The deviation inequalities presented in this chapter are based on [23]. On the Poisson space, explicit computations of chaos expansions can be carried out from Proposition 4.2.5 (cf. [70] and [142]) using the iterated difference operator $D^X_{t_1} \cdots D^X_{t_n} F$, but may be complicated by the recursive computation of finite differences, cf. [83]. A direct calculation using only the operator D can also be found in [84], for a Poisson process on a bounded interval, see also [114] for the chaos decomposition of Proposition 6.3.4. See [103] for a characterization of anticipative integrals with respect to the compensated Poisson process.

Chapter 7
Local Gradients on the Poisson space

We study a class of local gradient operators on Poisson space that have the derivation property. This allows us to give another example of a gradient operator that satisfies the hypotheses of Chapter 3, this time for a discontinuous process. In particular we obtain an anticipative extension of the compensated Poisson stochastic integral and other expressions for the Clark predictable representation formula. The fact that the gradient operator satisfies the chain rule of derivation has important consequences for deviation inequalities, computation of chaos expansions, characterizations of Poisson measures, and sensitivity analysis. It also leads to the definition of an infinite dimensional geometry under Poisson measures.

7.1 Intrinsic Gradient on Configuration Spaces

Let X be a Riemannian manifold with volume element σ, cf. e.g. [15]. We denote by T_xX the tangent space at $x \in X$, and let

$$TX = \bigcup_{x \in X} T_xX$$

denote the tangent bundle to X. Assume we are given a differential operator L defined on $\mathcal{C}_c^1(X)$ with adjoint L^*, satisfying the duality relation

$$\langle Lu, V\rangle_{L^2(X,\sigma;TX)} = \langle u, L^*V\rangle_{L^2(X,\sigma)}, \qquad u \in \mathcal{C}_c^1(X), \quad V \in \mathcal{C}_c^1(X, TX).$$

In the sequel, L will be mainly chosen equal to the gradient ∇^X on X. We work on the Poisson probability space $(\Omega^X, \mathcal{F}^X, \pi_\sigma^X)$ introduced in Definition 6.1.2.

Definition 7.1.1. *Given Λ a compact subset of X, we let $\mathcal{S}$ denote the set of functionals F of the form*

$$F(\omega) = f_0 \mathbf{1}_{\{\omega(\Lambda)=0\}} + \sum_{n=1}^{\infty} \mathbf{1}_{\{\omega(\Lambda)=n\}} f_n(x_1, \ldots, x_n), \tag{7.1.1}$$

where $f_n \in \mathcal{C}_c^1(\Lambda^n)$ *is symmetric in* n *variables,* $n \geq 1$*, with the notation*

$$\omega \cap \Lambda = \{x_1, \ldots, x_n\}$$

when $\omega(\Lambda) = n$*,* $\omega \in \Omega^X$*.*

In the next definition the differential operator L on X is "lifted" to a differential operator $\hat{D}^L$ on Ω^X.

Definition 7.1.2. *The intrinsic gradient* $\hat{D}^L$ *is defined on* $F \in \mathcal{S}$ *of the form* (7.1.1) *as*

$$\hat{D}_x^L F(\omega) = \sum_{n=1}^{\infty} \mathbf{1}_{\{\omega(\Lambda)=n\}} \sum_{i=1}^{n} L_{x_i} f_n(x_1, \ldots, x_n) \mathbf{1}_{\{x_i\}}(x), \quad \omega(dx) - a.e.,$$

$\omega \in \Omega^X$.

In other words if $\omega(\Lambda) = n$ and $\omega \cap \Lambda = \{x_1, \ldots, x_n\}$ we have

$$\hat{D}_x^L F = \begin{cases} L_{x_i} f_n(x_1, \ldots, x_n), & \text{if } x = x_i \text{ for some } i \in \{1, \ldots, n\}, \\ \\ 0, & \text{if } x \notin \{x_1, \ldots, x_n\}. \end{cases}$$

Let $\mathcal{I}$ denote the space of functionals of the form

$$\mathcal{I} = \Big\{ f\left(\int_X \varphi_1(x)\omega(dx), \ldots, \int_X \varphi_n(x)\omega(dx) \right), \\ \varphi_1, \ldots, \varphi_n \in \mathcal{C}_c^{\infty}(X), \ f \in \mathcal{C}_b^{\infty}(\mathbb{R}^n), \ n \in \mathbb{N} \Big\},$$

and

$$\mathcal{U} = \left\{ \sum_{i=1}^{n} F_i u_i \ : \ u_1, \ldots, u_n \in \mathcal{C}_c^{\infty}(X), \ F_1, \ldots, F_n \in \mathcal{I}, \ n \geq 1 \right\},$$

Note that for $F \in \mathcal{I}$ of the form

$$F = f\left(\int_X \varphi_1 d\omega, \ldots, \int_X \varphi_n d\omega \right), \qquad \varphi_1, \ldots, \varphi_n \in \mathcal{C}_c^{\infty}(X),$$

we have

$$\hat{D}_x^L F(\omega) = \sum_{i=1}^{n} \partial_i f\left(\int_X \varphi_1 d\omega, \ldots, \int_X \varphi_n d\omega \right) L_x \varphi_i(x), \qquad x \in \omega.$$

The following result is the integration by parts formula satisfied by $\hat{D}^L$.

Proposition 7.1.3. *We have for $F \in \mathcal{I}$ and $V \in \mathcal{C}_c^1(X, TX)$:*

$$\mathbb{E}\left[\langle \hat{D}^L F, V\rangle_{L^2(X,d\omega;TX)}\right] = \mathbb{E}\left[F \int_X L^* V(x)\omega(dx)\right]$$

Proof. We have

$$\begin{aligned}
\mathbb{E}\left[\langle \hat{D}^L F, V\rangle_{L^2(X,d\omega;TX)}\right] &= \mathbb{E}\left[\sum_{x\in\omega}\langle \hat{D}^L_x F, V(x)\rangle_{TX}\right] \\
&= \sum_{n=1}^\infty \mathbb{E}\left[\mathbf{1}_{\{\omega(\Lambda)=n\}}\sum_{i=1}^n \langle \hat{D}^L_{x_i} F, V(x_i)\rangle_{TX}\right] \\
&= e^{-\sigma(\Lambda)}\sum_{n=1}^\infty \frac{\sigma(\Lambda)^n}{n!} \\
&\quad \sum_{i=1}^n \int_\Lambda \cdots \int_\Lambda \langle L_{x_i} f_n(x_1,\ldots,x_n), V(x_i)\rangle_{TX} \frac{\sigma(dx_1)}{\sigma(\Lambda)}\cdots\frac{\sigma(dx_n)}{\sigma(\Lambda)} \\
&= e^{-\sigma(\Lambda)}\sum_{n=1}^\infty \frac{1}{n!}\sum_{i=1}^n \int_\Lambda \cdots \int_\Lambda f_n(x_1,\ldots,x_n) L^*_{x_i} V(x_i)\sigma(dx_1)\cdots\sigma(dx_n) \\
&= e^{-\sigma(\Lambda)}\sum_{n=1}^\infty \frac{1}{n!}\int_\Lambda \cdots \int_\Lambda f_n(x_1,\ldots,x_n) \sum_{i=1}^n L^*_{x_i} V(x_i)\sigma(dx_1)\cdots\sigma(dx_n) \\
&= \mathbb{E}\left[F\int_X L^* V(x)\omega(dx)\right].
\end{aligned}$$

□

In particular when $L = \nabla^X$ is the gradient on X we write $\hat{D}$ instead of $\hat{D}^X$ and obtain the following integration by parts formula:

$$\mathbb{E}\left[\langle \hat{D} F, V\rangle_{L^2(X,d\omega;TX)}\right] = \mathbb{E}\left[F \int_X \operatorname{div}^X V(x)\omega(dx)\right], \tag{7.1.2}$$

provided ∇^X and div^X satisfy the duality relation

$$\langle \nabla^X u, V\rangle_{L^2(X,\sigma;TX)} = \langle u, \operatorname{div}^X V\rangle_{L^2(X,\sigma)},$$

$u \in \mathcal{C}_c^1(X)$, $V \in \mathcal{C}_c^1(X, TX)$.

The next result provides a relation between the gradient ∇^X on X and its lifting $\hat{D}$ on Ω, using the operators of Definition 6.4.5.

Lemma 7.1.4. *For $F \in \mathcal{I}$ we have*

$$\hat{D}_x F(\omega) = \varepsilon_x^- \nabla^X \varepsilon_x^+ F(\omega) \quad \text{on} \qquad \{(\omega, x) \in \Omega^x \times X \ : \ x \in \omega\}. \tag{7.1.3}$$

Proof. Let

$$F = f\left(\int_X \varphi_1 d\omega, \ldots, \int_X \varphi_n d\omega\right), \qquad x \in X, \quad \omega \in \Omega^X,$$

and assume that $x \in \omega$. We have

$$\begin{aligned}
\hat{D}_x F(\omega) &= \sum_{i=1}^n \partial_i f\left(\int_X \varphi_1 d\omega, \ldots, \int_X \varphi_n d\omega\right) \nabla^X \varphi_i(x) \\
&= \sum_{i=1}^n \partial_i f\left(\varphi_1(x) + \int_X \varphi_1 d(\omega \backslash x), \ldots, \varphi_n(x) + \int_X \varphi_n d(\omega \backslash x)\right) \nabla^X \varphi_i(x) \\
&= \nabla^X f\left(\varphi_1(x) + \int_X \varphi_1 d(\omega \backslash x), \ldots, \varphi_n(x) + \int_X \varphi_n d(\omega \backslash x)\right) \\
&= \nabla^X \varepsilon_x^+ f\left(\int_X \varphi_1 d(\omega \backslash x), \ldots, \int_X \varphi_n d(\omega \backslash x)\right) \\
&= \left(\nabla^X \varepsilon_x^+ F\right)(\omega \backslash \{x\}) \\
&= \varepsilon_x^- \nabla^X \varepsilon_x^+ F(\omega).
\end{aligned}$$

□

The next proposition uses the operator δ^X defined in Definition 6.4.1.

Proposition 7.1.5. *For $V \in \mathcal{C}_c^\infty(X; TX)$ and $F \in \mathcal{I}$ we have*

$$\begin{aligned}
\langle \hat{D} F(\omega), V \rangle_{L^2(X, d\omega; TX)} \\
= \langle \nabla^X DF(\omega), V \rangle_{L^2(X, \sigma; TX)} + \delta^X(\langle \nabla^X DF, V \rangle_{TX})(\omega).
\end{aligned} \tag{7.1.4}$$

Proof. This identity follows from the relation

$$\hat{D}_x F(\omega) = (\nabla_x^X D_x F)(\omega \backslash \{x\}), \qquad x \in \omega,$$

and the application to $u = \langle \nabla^X DF, V \rangle_{TX}$ of the relation

$$\delta^X(u) = \int_X u(x, \omega \backslash \{x\}) \omega(dx) - \int_X u(x, \omega) \sigma(dx),$$

cf. Relation (6.5.2) in Proposition 6.5.2. □

In addition, for $F, G \in \mathcal{I}$ we have the isometry

$$\langle \hat{D} F, \hat{D} G \rangle_{L^2_\omega(TX)} = \langle \varepsilon^- \nabla^X \varepsilon^+ F, \varepsilon^- \nabla^X \varepsilon^+ G \rangle_{L^2_\omega(TX)}, \tag{7.1.5}$$

$\omega \in \Omega^X$, as an application of Relation (7.1.3) that holds $\omega(dx)$-a.e. for fixed $\omega \in \Omega^X$.

Similarly from (7.1.5) and Proposition 6.5.2 we have the relation

$$\langle \hat{D} F, \hat{D} G \rangle_{L^2_\omega(TX)} = \delta^X\left(\langle \nabla^X DF, \nabla^X DG \rangle_{TX}\right) + \langle \nabla^X DF, \nabla^X DG \rangle_{L^2_\sigma(TX)}, \tag{7.1.6}$$

$\omega \in \Omega^X$, $F, G \in \mathcal{I}$. Taking expectations on both sides in (7.1.4) using Relation (6.4.5), we recover Relation (7.1.2) in a different way:

$$\begin{aligned}\mathbb{E}[\langle \hat{D}F(\omega), V\rangle_{L^2(X,d\omega;TX)}] &= \mathbb{E}[\langle \nabla^X DF, V\rangle_{L^2(X,\sigma;TX)}] \\ &= \mathbb{E}[F\delta^X(\operatorname{div}^X V)],\end{aligned}$$

$V \in \mathcal{C}_c^\infty(X;TX)$, $F \in \mathcal{I}$.

Definition 7.1.6. *Let $\hat{\delta}_{\pi_\sigma}$ denote the adjoint of $\hat{D}$ under π_σ, defined as*

$$\mathbb{E}_{\pi_\sigma}\left[F\hat{\delta}_{\pi_\sigma}(G)\right] = \mathbb{E}_{\pi_\sigma}\left[\langle \hat{D}F, \hat{D}G\rangle_{L^2_\omega(TX)}\right],$$

on $G \in \mathcal{I}$ such that

$$\mathcal{I} \ni F \longmapsto \mathbb{E}_{\pi_\sigma}\left[\langle \hat{D}F, \hat{D}G\rangle_{L^2_\omega(TX)}\right]$$

extends to a bounded operator on $L^2(\Omega^X, \pi_\sigma)$.

We close this section with a remark on integration by parts characterization of Poisson measures, cf. Section 6.6, using the local gradient operator instead of the finite difference operator. We now assume that $\operatorname{div}_\sigma^X$ is defined on $\nabla^X f$ for all $f \in \mathcal{C}_c^\infty(X)$, with

$$\int_X g(x)\operatorname{div}_\sigma^X \nabla^X f(x)\sigma(dx) = \int_X \langle \nabla^X g(x), \nabla^X f(x)\rangle_{T_xX}\sigma(dx),$$

$f, g \in \mathcal{C}_c^1(X)$.

As a corollary of our pointwise lifting of gradients we obtain in particular a characterization of the Poisson measure. Let

$$H_\sigma^X = \operatorname{div}_\sigma^X \nabla^X$$

denote the Laplace-Beltrami operator on X.

Corollary 7.1.7. *The isometry relation*

$$\mathbb{E}_\pi\left[\langle \hat{D}F, \hat{D}G\rangle_{L^2_\omega(TX)}\right] = \mathbb{E}_\pi\left[\langle \nabla^X DF, \nabla^X DG\rangle_{L^2_\sigma(TX)}\right], \qquad (7.1.7)$$

$F, G \in \mathcal{I}$, holds under the Poisson measure π_σ with intensity σ. Moreover, under the condition

$$\mathcal{C}_c^\infty(X) = \{H_\sigma^X f \ : \ f \in \mathcal{C}_c^\infty(X)\},$$

Relation (7.1.7) entails $\pi = \pi_\sigma$.

Proof.

i) Relations (6.4.5) and (7.1.6) show that (7.1.7) holds when $\pi = \pi_\sigma$.

ii) If (7.1.7) is satisfied, then taking $F = I_n(u^{\otimes n})$ and $G = I_1(h)$, $h, u \in \mathcal{C}_c^\infty(X)$, Relation (7.1.6) implies

$$\mathbb{E}_\pi\left[\delta((H_\sigma^X h)uI_{n-1}(u^{\otimes(n-1)}))\right] = \mathbb{E}_\pi\left[\delta\left(\langle\nabla^X DF, \nabla^X h\rangle_{TX}\right)\right] = 0, \qquad n \geq 1,$$

hence $\pi = \pi_\sigma$ from Corollary 6.6.3.

□

We close this section with a study of the intrinsic gradient $\hat{D}$ when $X = \mathbb{R}_+$. Recall that the jump times of the standard Poisson process $(N_t)_{t\in\mathbb{R}_+}$ are denoted by $(T_k)_{k\geq 1}$, with $T_0 = 0$, cf. Section 2.3. In the next definition, all $\mathcal{C}^\infty$ functions on

$$\Delta_d = \{(t_1, \ldots, t_d) \in \mathbb{R}_+^d \ : \ 0 \leq t_1 < \cdots < t_d\}$$

are extended by continuity to the closure of Δ_d.

Definition 7.1.8. *Let $\mathcal{S}_d$ denote the set of smooth random functionals F of the form*

$$F = f(T_1, \ldots, T_d), \qquad f \in \mathcal{C}_b^1(\mathbb{R}_+^d), \quad d \geq 1. \tag{7.1.8}$$

We have

$$\hat{D}_t F = \sum_{k=1}^d \mathbf{1}_{\{T_k\}}(t)\partial_k f(T_1, \ldots, T_d), \qquad dN_t - a.e.,$$

with $F = f(T_1, \ldots, T_d)$, $f \in \mathcal{C}_b^\infty(\Delta_d)$, where $\partial_k f$ is the partial derivative of f with respect to its k-th variable, $1 \leq k \leq d$.

Lemma 7.1.9. *Let $F \in \mathcal{S}_d$ and $h \in \mathcal{C}_b^1(\mathbb{R}_+)$ with $h(0) = 0$. We have the integration by parts formula*

$$\mathbb{E}\left[\langle\hat{D}F, h\rangle_{L^2(\mathbb{R}_+, d\omega)}\right] = -\mathbb{E}\left[F\left(\sum_{k=1}^d h'(T_k) - \int_0^{T_d} h'(t)dt\right)\right].$$

Proof. By integration by parts on Δ_d using Relation (2.3.4) we have, for $F \in \mathcal{S}_d$ of the form (7.1.8),

$$\mathbb{E}[\langle\hat{D}F, h\rangle_{L^2(\mathbb{R}_+, dN_t)}] = \sum_{k=1}^d \int_0^\infty \int_0^{t_d} \cdots \int_0^{t_2} \mathrm{e}^{-t_d} h(t_k)\partial_k f(t_1, \ldots, t_d)dt_1 \cdots dt_d$$

$$= \int_0^\infty \mathrm{e}^{-t_d} \int_0^{t_d} \cdots \int_0^{t_2} h(t_1)\partial_1 f(t_1, \ldots, t_d)dt_1 \cdots dt_d$$

$$+ \sum_{k=2}^d \int_0^\infty \mathrm{e}^{-t_d} \int_0^{t_d} \cdots \int_0^{t_{k+1}} h(t_k)\frac{\partial}{\partial t_k}\int_0^{t_k} \cdots \int_0^{t_2} f(t_1, \ldots, t_d)dt_1 \cdots dt_d$$

$$-\sum_{k=2}^d \int_0^\infty \mathrm{e}^{-t_d} \int_0^{t_d} \cdot \int_0^{t_{k+1}} h(t_k)$$

$$\begin{aligned}
&\int_0^{t_k}\int_0^{t_{k-2}}\cdot\int_0^{t_2} f(t_1,\ldots,t_{k-2},t_k,t_k,\ldots,t_d)dt_1\cdot\widehat{dt}_{k-1}\cdot dt_d\\
&= -\int_0^\infty \mathrm{e}^{-t_d}\int_0^{t_d}\cdots\int_0^{t_2} h'(t_1)f(t_1,\ldots,t_d)dt_1\cdots dt_d\\
&\quad+\int_0^\infty \mathrm{e}^{-t_d}\int_0^{t_d}\cdots\int_0^{t_3} h(t_2)f(t_2,t_2,\ldots,t_d)dt_2\cdots dt_d\\
&\quad-\sum_{k=2}^d\int_0^\infty \mathrm{e}^{-t_d}\int_0^{t_d}\cdot\int_0^{t_{k+1}} h'(t_k)\int_0^{t_k}\cdots\int_0^{t_2} f(t_1,\ldots,t_d)dt_1\cdots dt_d\\
&\quad+\int_0^\infty \mathrm{e}^{-t_d}h(t_d)\int_0^{t_d}\cdots\int_0^{t_2} f(t_1,\ldots,t_d)dt_1\cdots dt_d\\
&\quad+\sum_{k=2}^{d-1}\int_0^\infty \mathrm{e}^{-t_d}\int_0^{t_d}\cdot\int_0^{t_{k+2}} h(t_{k+1})\\
&\quad\int_0^{t_{k+1}}\int_0^{t_{k-1}}\cdot\int_0^{t_2} f(t_1,.,t_{k-1},t_{k+1},t_{k+1},.,t_d)dt_1\cdot\widehat{dt_k}\cdot dt_d\\
&-\sum_{k=2}^d\int_0^\infty \mathrm{e}^{-t_d}\int_0^{t_d}\cdot\int_0^{t_{k+1}} h(t_k)\int_0^{t_k}\int_0^{t_{k-2}}\cdot\int_0^{t_2} f(t_1,.,t_{k-2},t_k,t_k,.,t_d)dt_1\cdot dt_d\\
&= -\sum_{k=1}^d\int_0^\infty \mathrm{e}^{-t_d}\int_0^{t_d}\cdots\int_0^{t_{k+1}} h'(t_k)\int_0^{t_k}\cdots\int_0^{t_2} f(t_1,\ldots,t_d)dt_1\cdots dt_d\\
&\quad+\int_0^\infty \mathrm{e}^{-t_d}h(t_d)\int_0^{t_d}\cdots\int_0^{t_2} f(t_1,\ldots,t_d)dt_1\cdots dt_d\\
&= -\mathbb{E}\left[F\left(\sum_{k=1}^d h'(T_k)-\int_0^{T_d}h'(t)dt\right)\right],
\end{aligned}$$

where $\widehat{dt_k}$ denotes the absence of dt_k in the multiple integrals with respect to $dt_1\cdots dt_d$. □

As a consequence we have the following corollary which directly involves the compensated Poisson stochastic integral.

Corollary 7.1.10. *Let $F\in\mathcal{S}_d$ and $h\in\mathcal{C}_b^1(\mathbb{R}_+)$ with $h(0)=0$. We have the integration by parts formula*

$$\mathbb{E}[\langle\hat{D}F,h\rangle_{L^2(\mathbb{R}_+,d\omega)}] = -\mathbb{E}\left[F\int_0^\infty h'(t)d(N_t-t)\right]. \qquad (7.1.9)$$

Proof. From Lemma 7.1.9 it suffices to notice that if $k>d$,

$$\begin{aligned}
&\mathbb{E}[Fh'(T_k)] = \int_0^\infty \mathrm{e}^{-t_k}h'(t_k)\int_0^{t_k}\cdots\int_0^{t_d}\cdots\int_0^{t_2} f(t_1,\ldots,t_d)dt_1\cdots dt_k\\
&= \int_0^\infty \mathrm{e}^{-t_k}h(t_k)\int_0^{t_k}\cdots\int_0^{t_d}\cdots\int_0^{t_2} f(t_1,\ldots,t_d)dt_1\cdots dt_k\\
&\quad-\int_0^\infty \mathrm{e}^{-t_{k-1}}h(t_{k-1})\int_0^{t_{k-1}}\cdots\int_0^{t_d}\cdots\int_0^{t_2} f(t_1,\ldots,t_d)dt_1\cdots dt_{k-1}\\
&= \mathbb{E}[F(h(T_k)-h(T_{k-1}))]
\end{aligned}$$

$$= \mathbb{E}\left[F\int_{T_{k-1}}^{T_k} h'(t)dt\right],$$

in other terms the discrete-time process

$$\left(\sum_{k=1}^{n} h'(T_k) - \int_0^{T_k} h'(t)dt\right)_{k\geq 1} = \left(\int_0^{T_k} h'(t)d(N_t - t)\right)_{k\geq 1}$$

is a martingale. □

Alternatively we may also use the strong Markov property to show directly that

$$\mathbb{E}\left[F\left(\sum_{k=d+1}^{\infty} h'(T_k) - \int_{T_{d+1}}^{\infty} h'(s)ds\right)\right] = 0.$$

By linearity the adjoint $\hat{\delta}$ of $\hat{D}$ is defined on simple processes $u \in \mathcal{U}$ of the form $u = hG$, $G \in \mathcal{S}_d$, $h \in \mathcal{C}^1(\mathbb{R}_+)$, from the relation

$$\hat{\delta}(hG) = -G\int_0^{\infty} h'(t)d(N_t - t) + \langle h, \hat{D}G\rangle_{L^2(\mathbb{R}_+,dN_t)}.$$

Relation (7.1.9) implies immediately the following duality relation.

Proposition 7.1.11. *For $F \in \mathcal{S}_d$ and $h \in \mathcal{C}_c^1(\mathbb{R}_+)$ we have :*

$$\mathbb{E}\left[\langle \hat{D}F, hG\rangle_{L^2(\mathbb{R}_+,dN_t)}\right] = \mathbb{E}\left[F\hat{\delta}(hG)\right].$$

Proof. We have

$$\begin{aligned}
\mathbb{E}\left[\langle \hat{D}F, hG\rangle_{L^2(\mathbb{R}_+,dN_t)}\right] &= \mathbb{E}\left[\langle \hat{D}(FG), h\rangle_{L^2(\mathbb{R}_+,dN_t)} - F\langle \hat{D}G, h\rangle_{L^2(\mathbb{R}_+,dN_t)}\right] \\
&= \mathbb{E}\left[F(G\hat{\delta}(h) - \langle h, \hat{D}G\rangle_{L^2(\mathbb{R}_+,dN_t)})\right] \\
&= -\mathbb{E}\left[F\left(G\int_0^{\infty} h'(t)d(N_t - t) + \langle h, \hat{D}G\rangle_{L^2(\mathbb{R}_+,dN_t)}\right)\right] \\
&= \mathbb{E}\left[F\hat{\delta}(hG)\right].
\end{aligned}$$

□

7.2 Damped Gradient on the Half Line

In this section we construct an example of a gradient which, has the derivation property and, unlike $\hat{D}$, satisfies the duality Assumption 3.1.1 and the Clark formula Assumption 3.2.1 of Section 3.1. Recall that the jump times of the standard Poisson process $(N_t)_{t\in\mathbb{R}_+}$ are denoted by $(T_k)_{k\geq 1}$, with $T_0 = 0$, cf. Section 2.3.

Let

$$r(t,s) = -(s \vee t), \qquad s,t \in \mathbb{R}_+,$$

denote the Green function associated to equation

$$\begin{cases} \mathcal{L}f := -f'', \qquad f \in \mathcal{C}^\infty([0,\infty)) \\ \\ f'(0) = f'(\infty) = 0. \end{cases}$$

In other terms, given $g \in \mathcal{C}^\infty([0,\infty))$, the solution of

$$g(t) = -f''(t), \qquad f'(0) = f'(\infty) = 0,$$

is given by

$$f(t) = \int_0^\infty r(t,s)g(s)ds, \qquad t \in \mathbb{R}_+.$$

Let also

$$\begin{aligned} r^{(1)}(t,s) &= \frac{\partial r}{\partial t}(t,s) \\ &= -\mathbf{1}_{]-\infty,t]}(s), \qquad s,t \in \mathbb{R}_+, \end{aligned}$$

i.e.

$$\begin{aligned} f(t) &= \int_0^\infty r^{(1)}(t,s)g(s)ds \\ &= -\int_0^t g(s)ds, \qquad t \in \mathbb{R}_+, \end{aligned} \tag{7.2.1}$$

is the solution of

$$\begin{cases} f' = -g, \\ \\ f(0) = 0. \end{cases}$$

Let $\mathcal{S}$ denote the space of functionals of the form

$$\mathcal{I} = \left\{ F = f(T_1, \dots, T_d) \ : \ f \in \mathcal{C}_b^1(\mathbb{R}^d), \ d \geq 1 \right\},$$

and let

$$\mathcal{U} = \left\{ \sum_{i=1}^n F_i u_i \ : \ u_1, \dots, u_n \in \mathcal{C}_c(\mathbb{R}_+), \ F_1, \dots, F_n \in \mathcal{S}, \ n \geq 1 \right\}.$$

Definition 7.2.1. *Given $F \in \mathcal{S}$ of the form $F = f(T_1, \dots, T_d)$, we let*

$$\tilde{D}_s F = -\sum_{k=1}^d \mathbf{1}_{[0,T_k]}(s) \partial_k f(T_1, \dots, T_d).$$

Note that we have

$$\begin{aligned}\tilde{D}_s F &= \sum_{k=1}^{d} r^{(1)}(T_k, s)\partial_k f(T_1, \ldots, T_d) \\ &= \int_0^\infty r^{(1)}(t, s)\hat{D}_t F dN_t.\end{aligned}$$

From Proposition 2.3.6 we have the following lemma.

Lemma 7.2.2. *For F of the form $F = f(T_1, \ldots, T_n)$ we have*

$$\begin{aligned}\mathbb{E}[\tilde{D}_t F|\mathcal{F}_t] &= -\sum_{N_t<k\leq n} \mathbb{E}[\partial_k f(T_1, \ldots, T_n)|\mathcal{F}_t] \\ &= -\sum_{N_t<k\leq n} \int_t^\infty \mathrm{e}^{-(s_n-t)} \int_t^{s_n} \cdots \int_t^{s_{N_t+2}} \\ &\qquad \partial_k f(T_1, \ldots, T_{N_t}, s_{N_t+1}, \ldots, s_n) ds_{N_t+1} \cdots ds_n.\end{aligned}$$

According to Definition 3.2.2, $\mathbb{D}([a, \infty))$, $a > 0$, denotes the completion of $\mathcal{S}$ under the norm

$$\|F\|_{\mathbb{D}([a,\infty))} = \|F\|_{L^2(\Omega)} + \left(\mathbb{E}\left[\int_a^\infty |\tilde{D}_t F|^2 dt\right]\right)^{1/2},$$

i.e. $(\tilde{D}_t F)_{t\in[a,\infty)}$ is defined in $L^2(\Omega \times [a, \infty))$ for $F \in \mathbb{D}([a, \infty))$. Clearly, the stability Assumption 3.2.10 is satisfied by $\tilde{D}$ since

$$\mathbf{1}_{[0,T_k]}(t) = \mathbf{1}_{\{N_t<k\}}$$

is $\mathcal{F}_t$-measurable, $t \in \mathbb{R}_+$, $k \in \mathbb{N}$. Hence the following lemma holds as a consequence of Proposition 3.2.11. For completeness we provide an independent direct proof.

Lemma 7.2.3. *Let $T > 0$. For any $\mathcal{F}_T$-measurable random variable $F \in L^2(\Omega)$ we have $F \in \mathbb{D}_{[T,\infty)}$ and*

$$\tilde{D}_t F = 0, \qquad t \geq T.$$

Proof. In case $F = f(T_1, \ldots, T_n)$ with $f \in \mathcal{C}_c^\infty(\mathbb{R}^n)$, F does not depend on the future of the Poisson process after T, it does not depend on the k-th jump time T_k if $T_k > T$, i.e.

$$\partial_i f(T_1, \ldots, T_n) = 0 \quad \text{for} \quad T_k > T, \qquad 1 \leq k \leq i \leq n.$$

This implies

$$\partial_i f(T_1, \ldots, T_n)\mathbf{1}_{[0,T_i]}(t) = 0 \quad t \geq T \quad i = 1, \ldots, n,$$

and

$$\tilde{D}_t F = -\sum_{i=1}^{n} \partial_i f(T_1, \ldots, T_n) \mathbf{1}_{[0,T_i]}(t) = 0, \quad t \geq T.$$

Hence $\tilde{D}_t F = 0$, $t \geq T$. □

Proposition 7.2.4. *We have for $F \in \mathcal{S}$ and $u \in \mathcal{C}_c(\mathbb{R}_+)$:*

$$\mathbb{E}[\langle \tilde{D}F, u \rangle_{L^2(\mathbb{R}_+, dt)}] = \mathbb{E}\left[F \int_0^\infty u(t)(dN_t - dt)\right]. \tag{7.2.2}$$

Proof. We have, using (7.2.1),

$$\begin{aligned}\mathbb{E}\left[\langle \tilde{D}F, u \rangle_{L^2(\mathbb{R}_+, dt)}\right] &= \mathbb{E}\left[\int_0^\infty \int_0^\infty r^{(1)}(s,t) \hat{D}_s F u(t) dN_s dt\right] \\ &= -\mathbb{E}\left[\left\langle \hat{D}.F, \int_0^{\cdot} u(t) dt \right\rangle_{L^2(\mathbb{R}_+, dN_t)}\right] \\ &= \mathbb{E}\left[F \int_0^\infty u(t) d(N_t - t)\right],\end{aligned}$$

from Corollary 7.1.10. □

The above proposition can also be proved by finite dimensional integration by parts on jump times conditionally to the value of N_T, see Proposition 7.3.3 below.

The divergence operator defined next is the adjoint of $\tilde{D}$.

Definition 7.2.5. *We define $\tilde{\delta}$ on $\mathcal{U}$ by*

$$\tilde{\delta}(hG) := G \int_0^\infty h(t)(dN_t - dt) - \langle h, \tilde{D}G \rangle_{L^2(\mathbb{R}_+)},$$

$G \in \mathcal{S}$, $h \in L^2(\mathbb{R}_+)$.

The closable adjoint

$$\tilde{\delta} : L^2(\Omega \times [0,1]) \longrightarrow L^2(\Omega)$$

of $\tilde{D}$ is another example of a Skorokhod type integral on the Poisson space. Using this definition we obtain the following integration by parts formula which shows that the duality Assumption 3.1.1 is satisfies by $\tilde{D}$ and $\tilde{\delta}$.

Proposition 7.2.6. *The divergence operator*

$$\tilde{\delta} : L^2(\Omega \times \mathbb{R}_+) \longrightarrow L^2(\Omega)$$

is the adjoint of the gradient operator

$$\tilde{D} : L^2(\Omega) \to L^2(\Omega \times \mathbb{R}_+),$$

i.e. we have

$$\mathbb{E}\left[F\tilde{\delta}(u)\right] = \mathbb{E}\left[\langle \tilde{D}F, u\rangle_{L^2(\mathbb{R}_+)}\right], \qquad F \in \mathcal{S}, \quad u \in \mathcal{U}. \tag{7.2.3}$$

Proof. It suffices to note that Proposition 7.2.4 implies

$$\begin{aligned}\mathbb{E}[\langle \tilde{D}F, hG\rangle_{L^2(\mathbb{R}_+,dt)}] &= \mathbb{E}\left[\langle \tilde{D}(FG), h\rangle_{L^2(\mathbb{R}_+,dt)} - F\langle \tilde{D}G, h\rangle_{L^2(\mathbb{R}_+,dt)}\right] \\ &= \mathbb{E}\left[F\left(G\int_0^\infty h(t)d(N_t - t) - \langle h, \tilde{D}G\rangle_{L^2(\mathbb{R}_+,dt)}\right)\right],\end{aligned} \tag{7.2.4}$$

for $F, G \in \mathcal{S}$. □

As a consequence, the duality Assumption 3.1.1 of Section 3 is satisfied by $\tilde{D}$ and $\tilde{\delta}$ and from Proposition 3.1.2 we deduce that $\tilde{D}$ and $\tilde{\delta}$ are closable.

Recall that from Proposition 6.4.9, the finite difference operator

$$D_tF = \mathbf{1}_{\{N_t<n\}}(f(T_1, \ldots, T_{N_t}, t, T_{N_t+1}, \ldots, T_{n-1}) - f(T_1, \ldots, T_n)),$$

$t \in \mathbb{R}_+$, $F = f(T_1, \ldots, T_n)$, defined in Chapter 6 satisfies the Clark formula Assumption 3.2.1, i.e. by Proposition 4.2.3 applied to $\phi_t = 1$, $t \in \mathbb{R}_+$, we have

$$F = \mathbb{E}[F] + \int_0^\infty \mathbb{E}[D_tF \mid \mathcal{F}_t]d(N_t - t), \tag{7.2.5}$$

$F \in L^2(\Omega)$.

On the other hand, the gradient $\tilde{D}$ has the derivation property and for this reason it can be easier to manipulate than the finite difference operator D in recursive computations. Its drawback is that its domain is smaller than that of D, due to the differentiability conditions it imposes on random functionals.

In the next proposition we show that the adapted projections of $(D_tF)_{t\in\mathbb{R}_+}$ and $(\tilde{D}_tF)_{t\in\mathbb{R}_+}$ coincide, cf. e.g. Proposition 20 of [106], by a direct computation of conditional expectations. See also Proposition 3.1 in [2] and Proposition 5.3.2 in [22].

Proposition 7.2.7. *The adapted projections of $\tilde{D}$ and D coincide, i.e.*

$$\mathbb{E}[\tilde{D}_tF \mid \mathcal{F}_t] = \mathbb{E}[D_tF \mid \mathcal{F}_t], \qquad t \in \mathbb{R}_+.$$

Proof. We have

$$\begin{aligned}\mathbb{E}[\tilde{D}_tF|\mathcal{F}_t] &= -\sum_{k=1}^n \mathbf{1}_{[0,T_k]}(t)\,\mathbb{E}[\partial_k f(T_1, \ldots, T_n)|\mathcal{F}_t] \\ &= -\sum_{N_t<k\le n} \mathbb{E}[\partial_k f(T_1, \ldots, T_n)|\mathcal{F}_t]\end{aligned}$$

$$= -\sum_{N_t<k\le n}$$
$$\int_0^\infty e^{-(s_n-t)}\int_t^{s_n}\cdots\int_t^{s_{N_t+2}}\partial_k f(T_1,\ldots,T_{N_t},s_{N_t+1},\ldots,s_n)ds_{N_t+1}\cdots ds_n$$
$$= -\sum_{k=N_t+2}^{n}\int_t^\infty e^{-(s_n-t)}\int_t^{s_n}\cdots$$
$$\cdots\frac{\partial}{\partial s_k}\int_t^{s_k}\cdots\int_t^{s_{N_t+2}} f(T_1,\ldots,T_{N_t},s_{N_t+1},\ldots,s_n)ds_{N_t+1}\cdots ds_n$$
$$+\sum_{k=N_t+2}^{n}\int_t^\infty e^{-(s_n-t)}\int_t^{s_n}\cdots$$
$$\cdots\int_t^{s_{N_t+2}} f(T_1,\ldots,T_{N_t},s_{N_t+1},.,s_{k-2},s_k,s_k,s_{k+1},.,s_n)$$
$$ds_{N_t+1}\cdots\widehat{ds_{k-1}}\cdots ds_n$$
$$-\mathbf{1}_{\{n>N_t\}}\int_t^\infty e^{-(s_n-t)}\int_t^{s_n}\cdots\int_t^{s_{N_t+2}}\partial_{N_t+1}f(T_1,\ldots,T_{N_t},s_{N_t+1},\ldots,s_n)$$
$$ds_{N_t+1}\cdots ds_n$$
$$= -\mathbf{1}_{\{N_t<n-1\}}\int_t^\infty e^{-(s_n-t)}\frac{\partial}{\partial s_n}\int_t^{s_n}\cdots\int_t^{s_{N_t+2}} f(T_1,\ldots,T_{N_t},s_{N_t+1},\ldots,s_n)$$
$$ds_{N_t+1}\cdots ds_n$$
$$-\sum_{k=N_t+2}^{n-1}\int_t^\infty e^{-(s_n-t)}\int_t^{s_n}\cdots$$
$$\frac{\partial}{\partial s_k}\int_t^{s_k}\cdots\int_t^{s_{N_t+2}} f(T_1,\ldots,T_{N_t},s_{N_t+1},\ldots,s_n)ds_{N_t+1}\cdots ds_n$$
$$+\sum_{k=N_t+2}^{n}\int_t^\infty e^{-(s_n-t)}\int_t^{s_n}\cdots\int_t^{s_{N_t+2}}$$
$$f(T_1,\ldots,T_{N_t},s_{N_t+1},.,s_{k-2},s_k,s_k,s_{k+1},.,s_n)ds_{N_t+1}\cdots\widehat{ds_{k-1}}\cdots ds_n$$
$$-\mathbf{1}_{\{n>N_t\}}\int_t^\infty e^{-(s_n-t)}\int_t^{s_n}\cdots\int_t^{s_{N_t+2}}\partial_{N_t+1}f(T_1,\ldots,T_{N_t},s_{N_t+1},\ldots,s_n)$$
$$ds_{N_t+1}\cdots ds_n$$
$$= -\mathbf{1}_{\{N_t<n-1\}}\int_t^\infty e^{-(s_n-t)}\frac{\partial}{\partial s_n}\int_t^{s_n}\cdots\int_t^{s_{N_t+2}} f(T_1,\ldots,T_{N_t},s_{N_t+1},\ldots,s_n)$$
$$ds_{N_t+1}\cdots ds_n$$
$$-\sum_{k=N_t+2}^{n-1}\int_t^\infty e^{-(s_n-t)}\int_t^{s_n}\cdots\int_t^{s_{N_t+2}}$$
$$f(T_1,\ldots,T_{N_t},s_{N_t+1},\ldots,s_{k-1},s_{k+1},s_{k+1},\ldots,s_n)ds_{N_t+1}\cdots\widehat{ds_k}\cdots ds_n$$
$$+\sum_{k=N_t+2}^{n}\int_t^\infty e^{-(s_n-t)}\int_t^{s_n}\cdots\int_t^{s_{N_t+2}}$$

$$
\begin{aligned}
& f(T_1,\ldots,T_{N_t},s_{N_t+1},.,s_{k-2},s_k,s_k,s_{k+1},.,s_n)ds_{N_t+1}\cdots\widehat{ds_{k-1}}\cdots ds_n \\
& -\mathbf{1}_{\{n>N_t\}}\int_t^\infty \mathrm{e}^{-(s_n-t)}\int_t^{s_n}\cdots\int_t^{s_{N_t+2}}\partial_{N_t+1}f(T_1,\ldots,T_{N_t},s_{N_t+1},\ldots,s_n) \\
& ds_{N_t+1}\cdots ds_n \\
= & -\mathbf{1}_{\{N_t<n-1\}}\int_t^\infty \mathrm{e}^{-(s_n-t)}\int_t^{s_n}\cdots\int_t^{s_{N_t+2}} f(T_1,\ldots,T_{N_t},s_{N_t+1},\ldots,s_n) \\
& ds_{N_t+1}\cdots ds_n \\
& +\mathbf{1}_{\{N_t<n-1\}}\int_t^\infty \mathrm{e}^{-(s_n-t)}\int_t^{s_n}\cdots\int_t^{s_{N_t+2}} f(T_1,\ldots,T_{N_t},s_{N_t+2},s_{N_t+2},\ldots,s_n) \\
& ds_{N_t+1}\cdots ds_n \\
& -\mathbf{1}_{\{n>N_t\}}\int_t^\infty \mathrm{e}^{-(s_n-t)}\int_t^{s_n}\cdots\int_t^{s_{N_t+2}}\partial_{N_t+1}f(T_1,\ldots,T_{N_t},s_{N_t+1},\ldots,s_n) \\
& ds_{N_t+1}\cdots ds_n \\
= & -\mathbf{1}_{\{N_t<n-1\}}\int_0^\infty \mathrm{e}^{-(s_n-t)}\int_t^{s_n}\cdots\int_t^{s_{N_t+2}} f(T_1,\ldots,T_{N_t},s_{N_t+1},\ldots,s_n) \\
& ds_{N_t+1}\cdots ds_n \\
& +\mathbf{1}_{\{N_t<n-1\}}\int_t^\infty \mathrm{e}^{-(s_n-t)}\int_t^{s_n}\cdots\int_t^{s_{N_t+2}} f(T_1,\ldots,T_{N_t},s_{N_t+2},s_{N_t+2},\ldots,s_n) \\
& ds_{N_t+1}\cdots ds_n \\
& -\mathbf{1}_{\{N_t<n-1\}}\int_t^\infty \mathrm{e}^{-(s_n-t)}\int_t^{s_n}\cdots\int_t^{s_{N_t+3}} f(T_1,\ldots,T_{N_t},s_{N_t+2},s_{N_t+2},\ldots,s_n) \\
& ds_{N_t+2}\cdots ds_n \\
& +\mathbf{1}_{\{N_t<n-1\}}\int_t^\infty \mathrm{e}^{-(s_n-t)}\int_t^{s_n}\cdots\int_t^{s_{N_t+3}} f(T_1,\ldots,T_{N_t},t,s_{N_t+2},\ldots,s_n) \\
& ds_{N_t+2}\cdots ds_n \\
& -\mathbf{1}_{\{n=N_t+1\}}\int_t^\infty \mathrm{e}^{-(s_n-t)} f(T_1,\ldots,T_{n-1},s_n)ds_n \\
& +\mathbf{1}_{\{n=N_t+1\}} f(T_1,\ldots,T_{n-1},t) \\
= & -\mathbf{1}_{\{n>N_t\}}\int_t^\infty \mathrm{e}^{-(s_n-t)}\int_t^{s_n}\cdots\int_t^{s_{N_t+2}} f(T_1,\ldots,T_{N_t},s_{N_t+1},\ldots,s_n) \\
& ds_{N_t+1}\cdots ds_n \\
& +\mathbf{1}_{\{n<N_t\}}\int_t^\infty \mathrm{e}^{-(s_n-t)}\int_t^{s_n}\cdots\int_t^{s_{N_t+3}} f(T_1,\ldots,T_{N_t},t,s_{N_t+2},\ldots,s_n) \\
& ds_{N_t+2}\cdots ds_n \\
= & \ \mathbb{E}[D_tF|\mathcal{F}_t],
\end{aligned}
$$

from Lemma 6.4.10. □

As a consequence of Proposition 7.2.7 we also have

$$
\mathbb{E}[\tilde{D}_tF \mid \mathcal{F}_a] = \mathbb{E}[D_tF \mid \mathcal{F}_a], \qquad 0 \le a \le t. \tag{7.2.6}
$$

For functions of a single jump time, by Relation (2.3.6) we simply have

$$
\begin{aligned}
\mathbb{E}[\tilde{D}_t f(T_n)|\mathcal{F}_t] &= -\mathbf{1}_{\{N_t<n\}}(t)\,\mathbb{E}[f'(T_n)|\mathcal{F}_t] \\
&= -\mathbf{1}_{\{N_t<n\}}\left(\mathbf{1}_{\{N_t\ge n\}}f'(T_n) + \int_t^\infty f'(x)p_{n-1-N_t}(x-t)dx\right)
\end{aligned}
$$

$$
\begin{aligned}
&= -\int_t^\infty f'(x)p_{n-1-N_t}(x-t)dx \\
&= f(t)p_{n-1-N_t}(0) + \int_t^\infty f(x)p'_{n-1-N_t}(x-t)dx \\
&= f(t)\mathbf{1}_{\{T_{n-1}<t<T_n\}} + \int_t^\infty f(x)p'_{n-1-N_t}(x-t)dx,
\end{aligned}
$$

which coincides with

$$
\begin{aligned}
&\mathbb{E}[D_t f(T_n)|\mathcal{F}_t] \\
&= \mathbb{E}[\mathbf{1}_{\{N_t<n-1\}}(f(T_{n-1}) - f(T_n)) + \mathbf{1}_{\{N_t=n-1\}}(f(t) - f(T_n))|\mathcal{F}_t] \\
&= \mathbb{E}[(\mathbf{1}_{\{T_{n-1}>t\}}f(T_{n-1}) + \mathbf{1}_{\{T_{n-1}<t<T_n\}}f(t) - \mathbf{1}_{\{T_n>t\}}f(T_n))|\mathcal{F}_t] \\
&= \mathbf{1}_{\{T_{n-1}<t<T_n\}}f(t) + \mathbb{E}[(\mathbf{1}_{\{T_{n-1}>t\}}f(T_{n-1}) - \mathbf{1}_{\{T_n>t\}}f(T_n))|\mathcal{F}_t] \\
&= \mathbf{1}_{\{T_{n-1}<t<T_n\}}f(t) + \int_t^\infty (p_{n-2-N_t}(x-t) - p_{n-1-N_t}(x-t))f(x)dx \\
&= \mathbf{1}_{\{T_{n-1}<t<T_n\}}f(t) + \int_t^\infty f(x)p'_{n-1-N_t}(x-t)dx.
\end{aligned}
$$

As a consequence of Proposition 7.2.7 and (7.2.5) we find that $\tilde{D}$ satisfies the Clark formula, hence the Clark formula Assumption 3.2.1 is satisfied by $\tilde{D}$.

Proposition 7.2.8. *For any $F \in L^2(\Omega)$ we have*

$$
F = \mathbb{E}[F] + \int_0^\infty \mathbb{E}[\tilde{D}_t F \mid \mathcal{F}_t]d(N_t - t).
$$

In other words we have

$$
\begin{aligned}
F &= \mathbb{E}[F] + \int_0^\infty \mathbb{E}[D_t F|\mathcal{F}_t]d(N_t - t) \\
&= \mathbb{E}[F] + \int_0^\infty \mathbb{E}[\tilde{D}_t F|\mathcal{F}_t]d(N_t - t),
\end{aligned}
$$

$F \in L^2(\Omega)$.

Since the duality Assumption 3.1.1 and the Clark formula Assumption 3.2.1 are satisfied by $\tilde{D}$, it follows from Proposition 3.3.1 that the operator $\tilde{\delta}$ coincides with the compensated Poisson stochastic integral with respect to $(N_t - t)_{t\in\mathbb{R}_+}$ on the adapted square-integrable processes. This fact is stated in the next proposition with an independent proof.

Proposition 7.2.9. *The adjoint of $\tilde{D}$ extends the compensated Poisson stochastic integral, i.e. for all adapted square-integrable process $u \in L^2(\Omega \times \mathbb{R}_+)$ we have*

$$
\tilde{\delta}(u) = \int_0^\infty u_t d(N_t - t).
$$

Proof. We consider first the case where v is a cylindrical elementary predictable process $v = F\mathbf{1}_{(s,T]}(\cdot)$ with $F = f(T_1, \ldots, T_n)$, $f \in \mathcal{C}_c^\infty(\mathbb{R}^n)$. Since v is predictable, F is $\mathcal{F}_s$-measurable hence from Lemma 7.2.3 we have $\tilde{D}_t F = 0$, $s \geq t$, and

$$\tilde{D}_t v_u = 0, \quad t \geq u.$$

Hence from Definition 7.2.5 we get

$$\begin{aligned}\tilde{\delta}(v) &= F(\tilde{N}_T - \tilde{N}_t) \\ &= \int_0^\infty F \mathbf{1}_{(t,T]}(s) d\tilde{N}_s \\ &= \int_0^\infty v_s d\tilde{N}_s.\end{aligned}$$

We then use the fact that $\tilde{D}$ is linear to extend the property to the linear combinations of elementary predictable processes. The compensated Poisson stochastic integral coincides with $\tilde{\delta}$ on the predictable square-integrable processes from a density argument using the Itô isometry. □

Since the adjoint $\tilde{\delta}$ of $\tilde{D}$ extends the compensated Poisson stochastic integral, we may also use Proposition 3.3.2 to show that the Clark formula Assumption 3.2.1 is satisfied by $\tilde{D}$, and in this way we recover the fact that the adapted projections of $\tilde{D}$ and D coincide:

$$\mathbb{E}[\tilde{D}_t F \mid \mathcal{F}_t] = \mathbb{E}[D_t F \mid \mathcal{F}_t], \qquad t \in \mathbb{R}_+,$$

for $F \in L^2(\Omega)$.

7.3 Damped Gradient on a Compact Interval

In this section we work under the Poisson measure on the compact interval $[0, T]$, $T > 0$.

Definition 7.3.1. *We denote by $\mathcal{S}_c$ the space of Poisson functionals of the form*

$$F = h_n(T_1, \ldots, T_n), \qquad h_n \in \mathcal{C}_c((0,\infty)^n), \quad n \geq 1, \tag{7.3.1}$$

and by $\mathcal{S}_f$ the space of Poisson functionals of the form

$$F = f_0 \mathbf{1}_{\{N_T = 0\}} + \sum_{n=1}^m \mathbf{1}_{\{N_T = n\}} f_n(T_1, \ldots, T_n), \tag{7.3.2}$$

where $f_0 \in \mathbb{R}$ and $f_n \in \mathcal{C}^1([0,T]^n)$, $1 \leq n \leq m$, is symmetric in n variables, $m \geq 1$.

The elements of $\mathcal{S}_c$ can be written as

$$F = f_0 \mathbf{1}_{\{N_T = 0\}} + \sum_{n=1}^\infty \mathbf{1}_{\{N_T = n\}} f_n(T_1, \ldots, T_n),$$

where $f_0 \in \mathbb{R}$ and $f_n \in \mathcal{C}^1([0,T]^n)$, $1 \leq n \leq m$, is symmetric in n variables, $m \geq 1$, with the continuity condition

$$f_n(T_1,\ldots,T_n) = f_{n+1}(T_1,\ldots,T_n,T).$$

We also let

$$\mathcal{U}_c = \left\{ \sum_{i=1}^n F_i u_i \ : \ u_1,\ldots,u_n \in \mathcal{C}([0,T]),\ F_1,\ldots,F_n \in \mathcal{S}_c,\ n \geq 1 \right\},$$

and

$$\mathcal{U}_f = \left\{ \sum_{i=1}^n F_i u_i \ : \ u_1,\ldots,u_n \in \mathcal{C}([0,T]),\ F_1,\ldots,F_n \in \mathcal{S}_f,\ n \geq 1 \right\}.$$

Recall that under $\mathbb{P}$ we have, for all $F \in \mathcal{S}_f$ of the form (7.3.2):

$$\mathbb{E}[F] = \mathrm{e}^{-\lambda T} f_0 + \mathrm{e}^{-\lambda T} \sum_{n=1}^m \lambda^n \int_0^T \int_0^{t_n} \cdots \int_0^{t_2} f_n(t_1,\ldots,t_n) dt_1 \cdots dt_n.$$

Definition 7.3.2. *Let $\bar{D}$ be defined on $F \in \mathcal{S}_f$ of the form (7.3.2) by*

$$\bar{D}_t F = -\sum_{n=1}^m \mathbf{1}_{\{N_T = n\}} \sum_{k=1}^n \mathbf{1}_{[0,T_k]}(t) \partial_k f_n(T_1,\ldots,T_n).$$

If F has the form (7.3.1) we have

$$\bar{D}_t F = -\sum_{k=1}^n \mathbf{1}_{[0,T_k]}(t) \partial_k f_n(T_1,\ldots,T_n),$$

where $\partial_k f_n$ denotes the partial derivative of f_n with respect to its k-th variable as in Definition 7.2.1.

We define $\bar{\delta}$ on $u \in \mathcal{U}_f$ by

$$\bar{\delta}(Fu) := F \int_0^T u_t (dN_t - dt) - \int_0^\infty u_t \bar{D}_t F dt, \tag{7.3.3}$$

$F \in \mathcal{S}_f$, $u \in \mathcal{C}([0,T])$.

The following result shows that $\bar{D}$ and $\bar{\delta}$ also satisfy the duality Assumption 3.1.1.

Proposition 7.3.3. *The operators $\bar{D}$ and $\bar{\delta}$ satisfy the duality relation*

$$\mathbb{E}[\langle \bar{D}F, u \rangle] = \mathbb{E}[F \bar{\delta}(u)], \tag{7.3.4}$$

$F \in \mathcal{S}_f$, $u \in \mathcal{U}_f$.

Proof. By standard integration by parts we first prove (7.3.4) when $u \in \mathcal{C}([0,T])$ and F has the form (7.3.2). We have

$$\begin{aligned}
&\mathbb{E}[\langle \bar{D}F, u\rangle] \\
&= -\mathrm{e}^{-\lambda T} \sum_{n=1}^{m} \frac{\lambda^n}{n!} \sum_{k=1}^{n} \int_0^T \cdots \int_0^T \int_0^{t_k} u(s)ds \partial_k f_n(t_1,\ldots,t_n) dt_1 \cdots dt_n \\
&= \mathrm{e}^{-\lambda T} \sum_{n=1}^{m} \frac{\lambda^n}{n!} \sum_{k=1}^{n} \int_0^T \cdots \int_0^T f_n(t_1,\ldots,t_n) u(t_k) dt_1 \cdots dt_n \\
&\quad - \mathrm{e}^{-\lambda T} \sum_{n=1}^{m} \frac{\lambda^n}{(n-1)!} \int_0^T u(s)ds \int_0^T \cdots \int_0^T f_n(t_1,\ldots,t_{n-1},T) dt_1 \cdots dt_{n-1}.
\end{aligned}$$

The continuity condition

$$f_n(t_1,\ldots,t_{n-1},T) = f_{n-1}(t_1,\ldots,t_{n-1}) \tag{7.3.5}$$

yields

$$\begin{aligned}
\mathbb{E}\left[\langle \bar{D}F, u\rangle\right] &= \mathrm{e}^{-\lambda T} \sum_{n=1}^{m} \frac{\lambda^n}{n!} \int_0^T \cdots \int_0^T f_n(t_1,\ldots,t_n) \sum_{k=1}^{n} u(t_k) dt_1 \cdots dt_n \\
&\quad - \lambda \mathrm{e}^{-\lambda T} \int_0^T u(s)ds \sum_{n=0}^{m} \frac{\lambda^n}{n!} \int_0^T \cdots \int_0^T f_n(t_1,\ldots,t_n) dt_1 \cdots dt_n \\
&= \mathbb{E}\left[F \left(\sum_{k=1}^{N_T} u(T_k) - \lambda \int_0^T u(s)ds \right) \right] \\
&= \mathbb{E}\left[F \int_0^T u(t) d(N_t - dt) \right].
\end{aligned}$$

Next we define $\bar{\delta}(uG)$, $G \in \mathcal{S}_f$, by (7.3.3), with for all $F \in \mathcal{S}_f$:

$$\begin{aligned}
\mathbb{E}\left[G\langle \bar{D}F, u\rangle\right] &= \mathbb{E}\left[\langle \bar{D}(FG), u\rangle - F\langle \bar{D}G, u\rangle\right] \\
&= \mathbb{E}\left[F \left(G \int_0^T u(t) d(N_t - dt) - \langle \bar{D}G, u\rangle \right) \right] \\
&= \mathbb{E}\left[F \bar{\delta}(uG) \right],
\end{aligned}$$

which proves (7.3.4). □

Hence, the duality Assumption 3.1.1 of Section 3 is also satisfied by $\bar{D}$ and $\bar{\delta}$, which are closable from Proposition 3.1.2, with domains $\mathrm{Dom}(\bar{D})$ and $\mathrm{Dom}(\bar{\delta})$.

The stability Assumption 3.2.10 is also satisfied by $\bar{D}$ and Lemma 7.2.3 holds as well as a consequence of Proposition 3.2.11, i.e. for any $\mathcal{F}_T$-measurable random variable $F \in L^2(\Omega)$ we have

$$\tilde{D}_t F = 0, \qquad t \geq T.$$

Similarly, $\bar{\delta}$ coincides with the stochastic integral with respect to the compensated Poisson process, i.e.

$$\tilde{\delta}(u) = \int_0^\infty u_t d(N_t - t),$$

for all adapted square-integrable process $u \in L^2(\Omega \times \mathbb{R}_+)$, with the same proof as in Proposition 7.2.9.

Consequently, from Proposition 3.3.2 it follows that the Clark formula Assumption 3.2.1 is satisfied by $\bar{D}$, and the adapted projections of $\bar{D}$, $\tilde{D}$, and D coincide:

$$\begin{aligned}\mathbb{E}[\bar{D}_t F \mid \mathcal{F}_t] &= \mathbb{E}[\tilde{D}_t F \mid \mathcal{F}_t] \\ &= \mathbb{E}[D_t F \mid \mathcal{F}_t], \qquad t \in \mathbb{R}_+,\end{aligned}$$

for $F \in L^2(\Omega)$.

Note that the gradients $\tilde{D}$ and $\bar{D}$ coincide on a common domain under the continuity condition (7.3.5). In case (7.3.5) is not satisfied by F the gradient $\bar{D}F$ can still be defined in $L^2(\Omega \times [0,T])$ on $F \in \mathcal{S}_f$ while $\tilde{D}F$ exists only in distribution sense due to the presence of the indicator function $\mathbf{1}_{\{N_T=k\}} = \mathbf{1}_{\{[T_k,T_{k+1})\}}(T)$ in (7.3.2).

Yet when (7.3.5) does not hold, we still get the integration by parts

$$\begin{aligned}\mathbb{E}\left[\langle \bar{D}F, u\rangle\right] &= \mathbb{E}\left[F \sum_{k=1}^{N_T} u(T_k)\right] \qquad (7.3.6)\\ &= \mathbb{E}\left[F \int_0^T u(t) dN(t)\right], \qquad F \in \mathcal{S}_f, \quad u \in \mathcal{U}_f,\end{aligned}$$

under the additional condition

$$\int_0^T u(s) ds = 0. \qquad (7.3.7)$$

However, in this case Proposition 3.1.2 does not apply to extend $\bar{D}$ by closability from its definition on $\mathcal{S}_f$ since the condition (7.3.7) is required in the integration by parts (7.3.6).

7.4 Chaos Expansions

In this section we review the application of $\tilde{D}$ to the computation of chaos expansions when $X = \mathbb{R}_+$. As noted above the gradient $\tilde{D}$ has some properties in common with D, namely its adapted projection coincides with that of D, and in particular from Proposition 7.2.7 we have

$$\mathbb{E}[D_t F] = \mathbb{E}[\tilde{D}_t F], \qquad t \in \mathbb{R}_+.$$

In addition, since the operator $\tilde{D}$ has the derivation property it is easier to manipulate than the finite difference operator D in recursive computations.

We aim at applying Proposition 4.2.5 in order to compute the chaos expansions

$$F = \mathbb{E}[F] + \sum_{n=1}^{\infty} I_n(f_n),$$

with

$$f_n(t_1, \ldots, t_n) = \frac{1}{n!} \mathbb{E}[\tilde{D}_{t_1} \cdots \tilde{D}_{t_n} F],$$

$dt_1 \cdots dt_n d\mathbb{P}$-a.e., $n \geq 1$.

However, Proposition 4.2.5 cannot be applied since the gradient $\tilde{D}$ cannot be iterated in L^2 due to the non-differentiability of $\mathbf{1}_{[0,T_k]}(t)$ in T_k. In particular, an expression such as

$$\mathbb{E}[\tilde{D}_{t_1} \cdots \tilde{D}_{t_n} F] \tag{7.4.1}$$

makes a priori no sense and may differ from $\mathbb{E}[D_{t_1} \cdots D_{t_n} F]$ for $n \geq 2$.

Note that we have

$$\tilde{D}_{t_n} \cdots \tilde{D}_{t_1} f(T_k) = (-1)^n \mathbf{1}_{[0,T_k]}(t_n) f^{(n)}(T_k), \qquad 0 < t_1 < \cdots < t_n,$$

and

$$\begin{aligned} \mathbb{E}[\tilde{D}_{t_n} \cdots \tilde{D}_{t_1} f(T_k)] &= (-1)^n \mathbb{E}[\mathbf{1}_{[0,T_k]}(t_n) f^{(n)}(T_k)] \\ &= (-1)^n \int_{t_n}^{\infty} f^{(n)}(t) p_{k-1}(t) dt, \end{aligned}$$

$0 < t_1 < \cdots < t_n$, which differs from

$$\mathbb{E}[D_{t_n} \cdots D_{t_1} f(T_k)] = -\int_{t_n}^{\infty} f(t) P_k^{(n)}(t) dt,$$

computed in Theorem 1 of [114], where

$$P_k(t) = \int_0^t p_{k-1}(s) ds, \qquad t \in \mathbb{R}_+,$$

is the distribution function of T_k, cf. (6.3.5).

Hence on the Poisson space $\tilde{D}_{t_n} \cdots \tilde{D}_{t_1}$, $0 < t_1 < \cdots < t_n$, cannot be used in the L^2 sense as $D_{t_n} \cdots D_{t_1}$ to give the chaos decomposition of a random variable. Nevertheless we have the following proposition, see [116] for an approach to this problem gradient $\tilde{D}$ in distribution sense.

Proposition 7.4.1. *For any* $F \in \bigcap_{n=0}^{\infty} \mathrm{Dom}\,(D^n \tilde{D})$ *we have the chaos expansion*

$$F = \mathbb{E}[F] + \sum_{n\geq 1}^{\infty} \tilde{I}_n(1_{\Delta_n} f_n),$$

where

$$f_n(t_1, \ldots, t_n) = \mathbb{E}[D_{t_1} \cdots D_{t_{n-1}} \tilde{D}_{t_n} F],$$

$0 < t_1 < \cdots < t_n$, $n \geq 1$.

Proof. We apply Proposition 4.2.5 to $\tilde{D}_t F$, $t \in \mathbb{R}_+$:

$$\tilde{D}_t F = \mathbb{E}[\tilde{D}_t F] + \sum_{n=1}^{\infty} \tilde{I}_n(1_{\tilde{\Delta}_n} \mathbb{E}[D^n \tilde{D}_t F]),$$

which yields

$$\mathbb{E}[\tilde{D}_t F | \mathcal{F}_t] = \mathbb{E}[\tilde{D}_t F] + \sum_{n=1}^{\infty} \tilde{I}_n(1_{\tilde{\Delta}_{n+1}(*,t)} \mathbb{E}[D^n \tilde{D}_t F]).$$

Finally, integrating both sides with respect to $d(N_t - t)$ and using of the Clark formula Proposition 7.2.8 and the inductive definition (2.7.1) we get

$$F - \mathbb{E}[F] = \sum_{n=0}^{\infty} \tilde{I}_{n+1}(1_{\tilde{\Delta}_{n+1}} \mathbb{E}[D^n \tilde{D} F]).$$

□

The next lemma provides a way to compute the functions appearing in Proposition 7.4.1.

Lemma 7.4.2. *We have for* $f \in \mathcal{C}_c^1(\mathbb{R})$ *and* $n \geq 1$

$$D_t \tilde{D}_s f(T_n) = \tilde{D}_{s\vee t} f(T_{n-1}) - \tilde{D}_{s\vee t} f(T_n) - \mathbf{1}_{\{s<t\}} \mathbf{1}_{[T_{n-1},T_n]}(s \vee t) f'(s \vee t),$$

$s, t \in \mathbb{R}_+$.

Proof. From Relation (6.4.15) we have

$$D_t \tilde{D}_s f(T_n) = -\mathbf{1}_{[0,T_{n-1}]}(t) \left(\mathbf{1}_{[0,T_{n-1}]}(s) f'(T_{n-1}) - \mathbf{1}_{[0,T_n]}(s) f'(T_n)\right)$$

$$
\begin{aligned}
&-\mathbf{1}_{[T_{n-1},T_n]}(t)\left(\mathbf{1}_{[0,t]}(s)f'(t)-\mathbf{1}_{[0,T_n]}(s)f'(T_n)\right)\\
&=\mathbf{1}_{\{t<s\}}\left(\mathbf{1}_{[0,T_n]}(s)f'(T_n)-\mathbf{1}_{[0,T_{n-1}]}(s)f'(T_{n-1}),\right)\\
&\quad+\mathbf{1}_{\{s<t\}}\left(\mathbf{1}_{[0,T_n]}(t)f'(T_n)-\mathbf{1}_{[0,T_{n-1}]}(t)f'(T_{n-1})-\mathbf{1}_{[T_{n-1},T_n]}(t)f'(t)\right),
\end{aligned}
$$

$\mathbb{P}$-a.s. □

In the next proposition we apply Lemma 7.4.2 to the computation of the chaos expansion of $f(T_k)$.

Proposition 7.4.3. *For $k \geq 1$, the chaos expansion of $f(T_k)$ is given as*

$$f(T_k)=\mathbb{E}[f(T_k)]+\sum_{n\geq 1}\frac{1}{n!}I_n(f_n^k),$$

where $f_n^k(t_1,\ldots,t_n)=\alpha_n^k(f)(t_1\vee\cdots\vee t_n)$, $t_1,\ldots,t_n\in\mathbb{R}_+$, and

$$
\begin{aligned}
\alpha_n^k(f)(t)&=-\int_t^\infty f'(s)\partial^{n-1}p_k(s)ds, &\quad(7.4.2)\\
&=f(t)\partial^{n-1}p_k(t)+\langle f,\mathbf{1}_{[t,\infty[}\partial^n p_k\rangle_{L^2(\mathbb{R}_+)}, \qquad t\in\mathbb{R}_+, \qquad n\geq 1,
\end{aligned}
$$

where the derivative f' in (7.4.2) is taken in the distribution sense.

We note the relation

$$\frac{d\alpha_n^k(f)}{dt}(t)=\alpha_n^k(f')(t)-\alpha_{n+1}^k(f)(t), \quad t\in\mathbb{R}_+.$$

From this proposition it is clearly seen that $f(T_n)\mathbf{1}_{[0,t]}(T_n)$ is $\mathcal{F}_{[0,t]}$-measurable, and that $f(T_n)\mathbf{1}_{[t,\infty[}(T_n)$ is not $\mathcal{F}_{[t,\infty[}$-measurable.

Proof. of Proposition 7.4.3. Let us first assume that $f\in\mathcal{C}_c^1(\mathbb{R}_+)$. We have

$$
\begin{aligned}
f_1^k(t)&=\mathbb{E}[\tilde{D}_t f(T_k)]\\
&=-\mathbb{E}[\mathbf{1}_{[0,T_k]}(t)f'(T_k)]\\
&=-\int_t^\infty p_k(s)f'(s)ds.
\end{aligned}
$$

Now, from Lemma 7.4.2, for $n\geq 2$ and $0\leq t_1<\cdots<t_n$,

$$D_{t_1}\cdots D_{t_{n-1}}\tilde{D}_{t_n}f(T_k)=D_{t_1}\cdots D_{t_{n-2}}(\tilde{D}_{t_n}f(T_{k-1})-\tilde{D}_{t_n}f(T_k)),$$

hence taking expectations on both sides and using Proposition 7.4.1 we have

$$f_n^k(t_1,\ldots,t_n)=f_{n-1}^{k-1}(t_1,\ldots,t_{n-2},t_n)-f_{n-1}^k(t_1,\ldots,t_{n-2},t_n),$$

and we can show (4.3.3) by induction, for $n\geq 2$:

$$f_n^k(t_1,\ldots,t_n)=f_{n-1}^{k-1}(t_1,\ldots,t_{n-2},t_n)-f_{n-1}^k(t_1,\ldots,t_{n-2},t_n),$$

$$= -\int_{t_n}^{\infty} f'(s) \frac{\partial^{n-2} p_{k-1}}{\partial s^{n-2}}(s) ds + \int_{t_n}^{\infty} f'(s) \frac{\partial^{n-2} p_{k-1}}{\partial s^{n-2}}(s) ds$$
$$= -\int_{0}^{\infty} f'(s) \frac{\partial^{n-1} p_{k-1}}{\partial s^{n-1}} p_k(s) ds.$$

The conclusion is obtained by density of the $\mathcal{C}_c^1$ functions in $L^2(\mathbb{R}_+, p_k(t)dt)$, $k \geq 1$. □

7.5 Covariance Identities and Deviation Inequalities

Next we present a covariance identity for the gradient $\tilde{D}$, as an application of Theorem 3.4.4.

Corollary 7.5.1. *Let $n \in \mathbb{N}$ and $F, G \in \bigcap_{k=1}^{n+1} I\!D(\Delta_k)$. We have*

$$\begin{aligned} \text{Cov}\,(F, G) = & \sum_{k=1}^{n} (-1)^{k+1}\, \mathbb{E}\left[\int_{\Delta_k} (\tilde{D}_{t_k} \cdots \tilde{D}_{t_1} F)(\tilde{D}_{t_k} \cdots \tilde{D}_{t_1} G) dt_1 \cdots dt_k\right] \\ & + (-1)^n\, \mathbb{E}\left[\int_{\Delta_{n+1}} \mathbb{E}\left[\tilde{D}_{t_{n+1}} \cdots \tilde{D}_{t_1} F \mid \mathcal{F}_{t_{n+1}}\right]\right. \\ & \left. \times\, \mathbb{E}\left[\tilde{D}_{t_{n+1}} \cdots \tilde{D}_{t_1} G \mid \mathcal{F}_{t_{n+1}}\right] dt_1 \cdots dt_{n+1}\right]. \end{aligned} \tag{7.5.1}$$

In particular,

$$\text{Cov}\,(T_m, f(T_1, \ldots, T_m)) = \sum_{i=1}^{m} \mathbb{E}[T_i \partial_i f(T_1, \ldots, T_m)].$$

From the well-known fact that exponential random variables

$$(\tau_k)_{k\geq 1} := (T_k - T_{k-1})_{k\geq 1}$$

can be constructed as the half sums of squared independent Gaussian random variables we define a mapping Θ which sends Poisson functionals to Wiener functionals, cf. [107]. Given $F = f(\tau_1, \ldots, \tau_n)$ a Poisson functional, let ΘF denote the Gaussian functional defined by

$$\Theta F = f\left(\frac{X_1^2 + Y_1^2}{2}, \ldots, \frac{X_n^2 + Y_n^2}{2}\right),$$

where $X_1, \ldots, X_n$, $Y_1, \ldots, Y_n$, denote two independent collections of standard Gaussian random variables. The random variables $X_1, \ldots, X_n$, $Y_1, \ldots, Y_n$, may be constructed as Brownian single stochastic integrals on the Wiener space W. In the next proposition we let D denote the gradient operator of Chapter 5 on the Wiener space.

Proposition 7.5.2. *The mapping $\Theta : L^p(\Omega) \to L^p(W)$ is an isometry. Further, it satisfies the intertwining relation*

$$2\Theta|\tilde{D}F|^2_{L^2(\mathbb{R}_+)} = |D\Theta F|^2_{L^2(\mathbb{R}_+)}, \tag{7.5.2}$$

Proof. The proposition follows from the fact that F and ΘF have same distribution since the half sum of two independent Gaussian squares has an exponential distribution. Relation (7.5.2) follows by a direct calculation. □

Proposition 3.6.1 applies in particular to the damped gradient operator $\tilde{D}$:

Corollary 7.5.3. *Let $F \in \mathrm{Dom}\,(\tilde{D})$. We have*

$$\mathbb{P}(F - \mathbb{E}[F] \geq x) \leq \exp\left(-\frac{x^2}{2\|\tilde{D}F\|^2_{L^2(\mathbb{R}_+,L^\infty(\Omega))}}\right), \qquad x > 0.$$

In particular if F is $\mathcal{F}_T$ measurable and $\|\tilde{D}F\|_\infty \leq K$ then

$$\mathbb{P}(F - \mathbb{E}[F] \geq x) \leq \exp\left(-\frac{x^2}{2K^2T}\right), \qquad x \geq 0.$$

As an example we may consider $F = f(\tau_1, \ldots, \tau_n)$ with

$$\sum_{k=1}^n \tau_k |\partial_k f(\tau_1, \ldots, \tau_n)|^2 \leq K^2, \qquad a.s.$$

Applying Corollary 4.7.4 to ΘF, where Θ is the mapping defined in Definition 7.5.2 and using Relation (7.5.2) yields the following deviation result for the damped gradient $\tilde{D}$ on Poisson space.

Corollary 7.5.4. *Let $F \in \mathrm{Dom}\,(\tilde{D})$. Then*

$$\mathbb{P}(F - \mathbb{E}[F] \geq x) \leq \exp\left(-\frac{x^2}{4\|\tilde{D}F\|^2_{L^\infty(\Omega,L^2(\mathbb{R}_+))}}\right).$$

The above result can also be obtained via logarithmic Sobolev inequalities, i.e. by application of Corollary 2.5 of [80] to Theorem 0.7 in [5] (or Relation (4.4) in [80] for a formulation in terms of exponential random variables). A sufficient condition for the exponential integrability of F is $\||\tilde{D}F|_{L^2(\mathbb{R}_+)}\|_\infty < \infty$, cf. Theorem 4 of [107].

7.6 Some Geometric Aspects of Poisson Analysis

In this section we use the operator $\tilde{D}$ to endow the configuration space on $\mathbb{R}_+$ with a (flat) differential structure.

We start by recalling some elements of differential geometry. Let M be a Riemannian manifold with volume measure dx, covariant derivative ∇, and exterior derivative d. Let ∇_μ^* and d_μ^* denote the adjoints of ∇ and d under a measure μ on M of the form $\mu(dx) = \mathrm{e}^{\phi(x)}dx$. The Weitzenböck formula under the measure μ states that

$$\mathrm{d}_\mu^*\mathrm{d} + \mathrm{d}\mathrm{d}_\mu^* = \nabla_\mu^*\nabla + R - \mathrm{Hess}\ \phi,$$

where R denotes the Ricci tensor on M. In terms of the de Rham Laplacian $H_R = \mathrm{d}_\mu^*\mathrm{d} + \mathrm{d}\mathrm{d}_\mu^*$ and of the Bochner Laplacian $H_B = \nabla_\mu^*\nabla$ we have

$$H_R = H_B + R - \mathrm{Hess}\ \phi. \tag{7.6.1}$$

In particular the term Hess ϕ plays the role of a curvature under the measure μ. The differential structure on $\mathbb{R}$ can be lifted to the space of configurations on $\mathbb{R}_+$. Here, $\mathcal{S}$ is defined as in Definition 7.1.8, and $\mathcal{U}$ denotes the space of smooth processes of the form

$$u(\omega, x) = \sum_{i=1}^n F_i(\omega)h_i(x), \qquad (\omega, x) \in \Omega \times \mathbb{R}_+, \tag{7.6.2}$$

$h_i \in \mathcal{C}_c^\infty(\mathbb{R}_+)$, $F_i \in \mathcal{S}$, $i = 1, \ldots, n$. The differential geometric objects to be introduced below have finite dimensional counterparts, and each of them has a stochastic interpretation. The following table describes the correspondence between geometry and probability.

Notation	Geometry	Probability
Ω	manifold	probability space
ω	element of Ω	point measure on $\mathbb{R}_+$
$\mathcal{C}_c^\infty(\mathbb{R}_+)$	tangent vectors to Ω	test functions on $\mathbb{R}_+$
σ	Riemannian metric on Ω	Lebesgue measure
d	gradient on Ω	stochastic gradient $\tilde{D}$
$\mathcal{U}$	vector field on Ω	stochastic process
$\mathrm{d}u$	exterior derivative of $u \in \mathcal{U}$	two-parameter process
$\{\cdot,\cdot\}$	bracket of vector fields on Ω	bracket on $\mathcal{U} \times \mathcal{U}$
R	curvature tensor on Ω	trilinear mapping on $\mathcal{U}$
d^*	divergence on Ω	stochastic integral operator

We turn to the definition of a covariant derivative ∇_u in the direction $u \in L^2(\mathbb{R}_+)$, first for a vector field $v \in \mathcal{C}_c^\infty(\mathbb{R}_+)$ as

$$\nabla_u v(t) = -\dot{v}(t) \int_0^t u_s ds, \qquad t \in \mathbb{R}_+,$$

where $\dot{v}(t)$ denotes the derivative of $v(t)$, and then for a vector field

$$v = \sum_{i=1}^n F_i h_i \in \mathcal{U}$$

in the next definition.

Definition 7.6.1. *Given $u \in \mathcal{U}$ and $v = \sum_{i=1}^n F_i h_i \in \mathcal{U}$, let $\nabla_u v$ be defined as*

$$\nabla_u v(t) = \sum_{i=1}^n h_i(t) \tilde{D}_u F_i - F_i \dot{h}_i(t) \int_0^t u_s ds, \qquad t \in \mathbb{R}_+, \tag{7.6.3}$$

where

$$\tilde{D}_u F = \langle \tilde{D} F, u \rangle_{L^2(\mathbb{R}_+)}, \qquad F \in \mathcal{S}.$$

We have

$$\nabla_{uF}(vG) = Fv\tilde{D}_u G + FG\nabla_u v, \qquad u, v \in \mathcal{C}_c^\infty(\mathbb{R}_+), \quad F, G \in \mathcal{S}. \tag{7.6.4}$$

We also let, by abuse of notation,

$$(\nabla_s v)(t) := \sum_{i=1}^{n} h_i(t)\tilde{D}_s F_i - F_i \dot{h}_i(t)\mathbf{1}_{[0,t]}(s),$$

for $s, t \in \mathbb{R}_+$, in order to write

$$\nabla_u v(t) = \int_0^\infty u_s \nabla_s v_t ds, \qquad t \in \mathbb{R}_+, \quad u, v \in \mathcal{U}.$$

The following is the definition of the Lie-Poisson bracket.

Definition 7.6.2. *The Lie bracket* $\{u, v\}$ *of* $u, v \in \mathcal{C}_c^\infty(\mathbb{R}_+)$ *is defined as the unique element of* $\mathcal{C}_c^\infty(\mathbb{R}_+)$ *satisfying*

$$(\tilde{D}_u \tilde{D}_v - \tilde{D}_v \tilde{D}_u)F = \tilde{D}_w F, \qquad F \in \mathcal{S}.$$

The bracket $\{\cdot, \cdot\}$ is extended to $u, v \in \mathcal{U}$ via

$$\{Ff, Gg\}(t) = FG\{f, g\}(t) + g(t)F\tilde{D}_f G - f(t)G\tilde{D}_g F, \qquad t \in \mathbb{R}_+, \quad (7.6.5)$$

$f, g \in \mathcal{C}_c^\infty(\mathbb{R}_+)$, $F, G \in \mathcal{S}$. Given this definition we are able to prove the vanishing of the associated torsion term.

Proposition 7.6.3. *The Lie bracket* $\{u, v\}$ *of* $u, v \in \mathcal{U}$ *satisfies*

$$\{u, v\} = \nabla_u v - \nabla_v u, \qquad (7.6.6)$$

i.e. the connection defined by ∇ *has a vanishing torsion*

$$T(u, v) = \nabla_u v - \nabla_v u - \{u, v\} = 0, \qquad u, v \in \mathcal{U}.$$

Proof. For all $u, v \in \mathcal{C}_c^\infty(\mathbb{R}_+)$ we have

$$\begin{aligned}
(\tilde{D}_u \tilde{D}_v - \tilde{D}_v \tilde{D}_u)T_n &= -\tilde{D}_u \int_0^{T_n} v_s ds + \tilde{D}_v \int_0^{T_n} u_s ds \\
&= v_{T_n} \int_0^{T_n} u_s ds - u_{T_n} \int_0^{T_n} v_s ds \\
&= \int_0^{T_n} \left(\dot{v}(t) \int_0^t u_s ds - \dot{u}(t) \int_0^t v_s ds \right) dt \\
&= \tilde{D}_{\nabla_u v - \nabla_v u} T_n.
\end{aligned}$$

Since $\tilde{D}$ is a derivation, this shows that

$$(\tilde{D}_u \tilde{D}_v - \tilde{D}_v \tilde{D}_u)F = \tilde{D}_{\nabla_u v - \nabla_v u} F$$

for all $F \in \mathcal{S}$, hence

$$\tilde{D}_{\{u,v\}} = \tilde{D}_u \tilde{D}_v - \tilde{D}_v \tilde{D}_u = \tilde{D}_{\nabla_u v - \nabla_v u}, \qquad u, v \in \mathcal{C}_c^\infty(\mathbb{R}_+),$$

which shows that (7.6.6) holds for $u, v \in \mathcal{C}_c^\infty(\mathbb{R}_+)$. The extension to $u, v \in \mathcal{U}$ follows from (7.6.4) and (7.6.5). □

Similarly we show the vanishing of the associated curvature.

Proposition 7.6.4. *The Riemannian curvature tensor R of ∇ vanishes on $\mathcal{U}$, i.e.*

$$R(u,v)h := [\nabla_u, \nabla_v]h - \nabla_{\{u,v\}}h = 0, \qquad u,v,h \in \mathcal{U}.$$

Proof. We have, letting $\tilde{u}(t) = -\int_0^t u_s ds$, $t \in \mathbb{R}_+$:

$$[\nabla_u, \nabla_v]h = \tilde{u}\dot{\overbrace{\nabla_v h}} - \tilde{v}\dot{\overbrace{\nabla_u h}} = \tilde{u}\ \dot{\overbrace{\tilde{v}\dot{h}}}\ - \tilde{v}\ \dot{\overbrace{\tilde{u}\dot{h}}} = -\tilde{u}v\dot{h} + \tilde{v}u\dot{h},$$

and

$$\nabla_{\{u,v\}}h = \nabla_{\tilde{u}\dot{v} - \tilde{v}\dot{u}}h = (\widetilde{\tilde{u}\dot{v} - \tilde{v}\dot{u}})\dot{h} = (u\tilde{v} - v\tilde{u})\dot{h},$$

hence $R(u,v)h = 0$, $h, u, v \in \mathcal{C}_c^\infty(\mathbb{R}_+)$. The extension of the result to $\mathcal{U}$ follows again from (7.6.4) and (7.6.5). □

Clearly, the bracket $\{\cdot,\cdot\}$ is antisymmetric, i.e.:

$$\{u,v\} = -\{v,u\}, \qquad u,v \in \mathcal{C}_c^\infty(\mathbb{R}_+).$$

Proposition 7.6.5. *The bracket $\{\cdot,\cdot\}$ satisfies the Jacobi identity*

$$\{\{u,v\},w\} + \{w,\{u,v\}\} + \{v,\{u,w\}\} = 0, \qquad u,v,w \in \mathcal{C}_c^\infty(\mathbb{R}_+),$$

hence $\mathcal{U}$ is a Lie algebra under $\{\cdot,\cdot\}$.

Proof. The vanishing of $R(u,v)$ in Proposition 7.6.4 shows that

$$[\nabla_u, \nabla_v] = \nabla_{\{u,v\}}h, \qquad u,v \in \mathcal{U},$$

hence

$$\begin{aligned}
&\nabla_{\{\{u,v\},w\}} + \nabla_{\{w,\{u,v\}\}} + \nabla_{\{v,\{u,w\}\}} \\
&= [\nabla_{\{u,v\}}, \nabla_w] + [\nabla_w, \nabla_{\{u,v\}}] + [\nabla_v, \nabla_{\{u,w\}}] \\
&= 0, \qquad u,v,h \in \mathcal{U}.
\end{aligned}$$

□

However, $\{\cdot,\cdot\}$ does not satisfy the Leibniz identity, thus it can not be considered as a Poisson bracket.

The exterior derivative $\tilde{D}u$ of a smooth vector field $u \in \mathcal{U}$ is defined from

$$\langle \tilde{D}u, h_1 \wedge h_2\rangle_{L^2(\mathbb{R}_+)\wedge L^2(\mathbb{R}_+)} = \langle \nabla_{h_1}u, h_2\rangle_{L^2(\mathbb{R}_+)} - \langle \nabla_{h_2}u, h_1\rangle_{L^2(\mathbb{R}_+)},$$

$h_1, h_2 \in \mathcal{U}$, with the norm

$$\|\tilde{D}u\|^2_{L^2(\mathbb{R}_+)\wedge L^2(\mathbb{R}_+)} := 2\int_0^\infty \int_0^\infty (\tilde{D}u(s,t))^2 dsdt, \tag{7.6.7}$$

where

$$\tilde{D}u(s,t) = \frac{1}{2}(\nabla_s u_t - \nabla_t u_s), \qquad s,t \in \mathbb{R}_+, \quad u \in \mathcal{U}.$$

The next result is analog to Proposition 4.1.4.

Lemma 7.6.6. *We have the commutation relation*

$$\tilde{D}_u \tilde{\delta}(v) = \tilde{\delta}(\nabla_u v) + \langle u, v\rangle_{L^2(\mathbb{R}_+)}, \tag{7.6.8}$$

$u, v \in \mathcal{C}_c^\infty(\mathbb{R}_+)$, *between* $\tilde{D}$ *and* δ.

Proof. We have

$$\begin{aligned}
\tilde{D}_u \tilde{\delta}(v) &= -\sum_{k=1}^\infty \dot{v}(T_k) \int_0^{T_k} u_s ds \\
&= -\tilde{\delta}\left(\dot{v}. \int_0^{\cdot} u_s ds\right) - \int_0^\infty \dot{v}(t) \int_0^t u_s dsdt \\
&= \tilde{\delta}(\nabla_u v) + \langle u, v\rangle_{L^2(\mathbb{R}_+)},
\end{aligned}$$

by (7.6.3). □

As an application we obtain a Skorohod type isometry for the operator $\tilde{\delta}$.

Proposition 7.6.7. *We have for* $u \in \mathcal{U}$:

$$\mathbb{E}\left[|\tilde{\delta}(u)|^2\right] = \mathbb{E}\left[\|u\|^2_{L^2(\mathbb{R}_+)}\right] + \mathbb{E}\left[\int_0^\infty \int_0^\infty \nabla_s u_t \nabla_t u_s dsdt\right]. \tag{7.6.9}$$

Proof. Given $u = \sum_{i=1}^n h_i F_i \in \mathcal{U}$ we have

$$\begin{aligned}
&\mathbb{E}\left[\tilde{\delta}(h_i F_i)\tilde{\delta}(h_j F_j)\right] = \mathbb{E}\left[F_i \tilde{D}_{h_i} \tilde{\delta}(h_j F_j)\right] \\
&= \mathbb{E}\left[F_i \tilde{D}_{h_i}(F_j \tilde{\delta}(h_j) - \tilde{D}_{h_j} F_j)\right] \\
&= \mathbb{E}\left[(F_i F_j \tilde{D}_{h_i} \tilde{\delta} h_j + F_i \tilde{\delta}(h_j) \tilde{D}_{h_i} F_j - F_i \tilde{D}_{h_i} \tilde{D}_{h_j} F_j)\right] \\
&= \mathbb{E}\left[(F_i F_j \langle h_i, h_j\rangle_{L^2(\mathbb{R}_+)} + F_i F_j \tilde{\delta}(\nabla_{h_i} h_j) + F_i \tilde{\delta}(h_j) \tilde{D}_{h_i} F_j - F_i \tilde{D}_{h_i} \tilde{D}_{h_j} F_j)\right] \\
&= \mathbb{E}\left[(F_i F_j \langle h_i, h_j\rangle_{L^2(\mathbb{R}_+)} + \tilde{D}_{\nabla_{h_i} h_j}(F_i F_j) + \tilde{D}_{h_j}(F_i \tilde{D}_{h_i} F_j) - F_i \tilde{D}_{h_i} \tilde{D}_{h_j} F_j)\right] \\
&= \mathbb{E}\Big[(F_i F_j \langle h_i, h_j\rangle_{L^2(\mathbb{R}_+)} + \tilde{D}_{\nabla_{h_i} h_j}(F_i F_j) + \tilde{D}_{h_j} F_i \tilde{D}_{h_i} F_j \\
&\quad + F_i(\tilde{D}_{h_j} \tilde{D}_{h_i} F_j - \tilde{D}_{h_i} \tilde{D}_{h_j} F_j))\Big] \\
&= \mathbb{E}\Big[(F_i F_j \langle h_i, h_j\rangle_{L^2(\mathbb{R}_+)} + \tilde{D}_{\nabla_{h_i} h_j}(F_i F_j) + \tilde{D}_{h_j} F_i \tilde{D}_{h_i} F_j
\end{aligned}$$

$$
\begin{aligned}
&+F_i \tilde{D}_{\nabla_{h_j} h_i - \nabla_{h_i} h_j} F_j)\Big] \\
&= \mathbb{E}\left[(F_i F_j \langle h_i, h_j \rangle_{L^2(\mathbb{R}_+)} + F_j \tilde{D}_{\nabla_{h_i} h_j} F_i + F_i \tilde{D}_{\nabla_{h_j} h_i} F_j + \tilde{D}_{h_j} F_i \tilde{D}_{h_i} F_j) \right] \\
&= \mathbb{E}\Big[F_i F_j \langle h_i, h_j \rangle_{L^2(\mathbb{R}_+)} + F_j \int_0^\infty \tilde{D}_s F_i \int_0^\infty \nabla_t h_j(s) h_i(t) dt ds \\
&\quad + F_i \int_0^\infty \tilde{D}_t F_j \int_0^\infty \nabla_s h_i(t) h_j(s) ds dt \\
&\quad + \int_0^\infty h_i(t) \tilde{D}_t F_j dt \int_0^\infty h_j(s) \tilde{D}_s F_i ds \Big],
\end{aligned}
$$

where we used the commutation relation (7.6.8). □

Proposition (7.6.7) is a version of the Skorohod isometry for the operator $\tilde{\delta}$ and it differs from from Propositions 4.3.1 and 6.5.4 which apply to finite difference operators on the Poisson space.

Finally we state a Weitzenböck type identity on configuration space under the form of the commutation relation

$$
\tilde{D}\tilde{\delta} + \tilde{\delta}\tilde{D} = \nabla^* \nabla + \mathrm{Id}_{L^2(\mathbb{R}_+)},
$$

i.e. the Ricci tensor under the Poisson measure is the identity $\mathrm{Id}_{L^2(\mathbb{R}_+)}$ on $L^2(\mathbb{R}_+)$ by comparison with (7.6.1).

Theorem 7.6.8. *We have for $u \in \mathcal{U}$:*

$$
\begin{aligned}
&\mathbb{E}\left[|\tilde{\delta}(u)|^2 \right] + \mathbb{E}\left[\|\tilde{D}u\|^2_{L^2(\mathbb{R}_+) \wedge L^2(\mathbb{R}_+)} \right] \\
&\qquad = \mathbb{E}\left[\|u\|^2_{L^2(\mathbb{R}_+)} \right] + \mathbb{E}\left[\|\nabla u\|^2_{L^2(\mathbb{R}_+) \otimes L^2(\mathbb{R}_+)} \right].
\end{aligned}
\tag{7.6.10}
$$

Proof. Relation (7.6.10) for $u = \displaystyle\sum_{i=1}^n h_i F_i \in \mathcal{U}$ follows from Relation (7.6.7) and Proposition 7.6.7. □

7.7 Chaos Interpretation of Time Changes

In this section we study the Poisson probabilistic interpretation of the operators introduced in Section 4.8. We refer to Section 5.8 for their interpretation on the Wiener space. We now prove that $\nabla^\ominus + D$ is identified to the operator $\tilde{D}$ under the Poisson identification of Φ and $L^2(B)$.

Lemma 7.7.1. *On the Poisson space, $\nabla^\ominus$ satisfies the relation*

$$
\nabla_t^\ominus (FG) = F \nabla_t^\ominus G + G \nabla_t^\ominus F - D_t F D_t G, \qquad t \in \mathbb{R}_+, \ F, G \in \mathcal{S}. \tag{7.7.1}
$$

Proof. We will use the multiplication formula for multiple Poisson stochastic integrals of Proposition 6.2.5:

$$I_n(f^{\circ n})I_1(g) = I_{n+1}(f^{\circ n} \circ g) + n\langle f, g\rangle I_{n-1}(f^{\circ n-1}) + nI_n((fg) \circ f^{\circ n-1}),$$

$f, g \in L^4(\mathbb{R}_+)$. We first show that

$$\nabla_t^{\ominus} (I_n(f^{\circ n})I_1(g)) = I_n(f^{\circ n})\nabla_t^{\ominus} I_1(g) + I_1(g)\nabla_t^{\ominus} I_n(f^{\circ n}) - D_t I_1(g) D_t I_n(f^{\circ n}),$$

$t \in \mathbb{R}_+$, when $f, g \in \mathcal{C}_c^1(\mathbb{R}_+)$ and $\langle f, f\rangle_{L^2(\mathbb{R}_+)} = 1$. Indeed, we have

$$\begin{aligned}
&I_n(f^{\circ n})\nabla_t^{\ominus} I_1(g) + I_1(g)\nabla_t^{\ominus} I_n(f^{\circ n})\\
&= -I_n(f^{\circ n})I_1(g'1_{[t,\infty)}) - nI_1(g)I_n((f'1_{[t,\infty)}) \circ f^{\circ(n-1)})\\
&= -n\Big(I_{n+1}((f'1_{[t,\infty)}) \circ f^{\circ(n-1)} \circ g) + (n-1)I_n((fg) \circ (f'1_{[t,\infty)}) \circ f^{\circ(n-2)})\\
&\quad +I_n((gf'1_{[t,\infty)}) \circ f^{\circ(n-1)}) + \langle f'1_{[t,\infty)}, g\rangle_{L^2(\mathbb{R}_+)} I_{n-1}(f^{\circ(n-1)})\\
&\quad +(n-1)\langle f, g\rangle_{L^2(\mathbb{R}_+)} I_{n-1}((f'1_{[t,\infty)}) \circ f^{\circ(n-2)})\Big)\\
&\quad -I_{n+1}((g'1_{[t,\infty)}) \circ f^{\circ n}) - nI_n((g'1_{[t,\infty)}f) \circ f^{\circ(n-1)})\\
&\quad -n\langle g'1_{[t,\infty)}, f\rangle_{L^2(\mathbb{R}_+)} I_{n-1}(f^{\circ(n-1)})\\
&= -nI_{n+1}((f'1_{[t,\infty)}) \circ f^{\circ(n-1)} \circ g) - I_{n+1}((g'1_{[t,\infty)}) \circ f^{\circ n})\\
&\quad -n(n-1)I_n((f'1_{[t,\infty)}) \circ (fg) \circ f^{\circ(n-2)})\\
&\quad -nI_n((gf'1_{[t,\infty)}) \circ f^{\circ(n-1)}) - nI_n((fg'1_{[t,\infty)}) \circ f^{\circ(n-1)})\\
&\quad +nf(t)g(t)I_{n-1}(f^{\circ(n-1)}) - n(n-1)\langle f, g\rangle_{L^2(\mathbb{R}_+)} I_{n-1}((f'1_{[t,\infty)}) \circ f^{\circ(n-2)})\\
&= \nabla_t^{\ominus}\Big(I_{n+1}(f^{\circ n} \circ g) + nI_n(f^{\circ(n-1)} \circ (fg)) + n\langle f, g\rangle_{L^2(\mathbb{R}_+)} I_{n-1}(f^{\circ(n-1)})\Big)\\
&\quad +nf(t)g(t)I_{n-1}(f^{\circ(n-1)})\\
&= \nabla_t^{\ominus}(I_n(f^{\circ n})I_1(g)) + D_t I_1(g) D_t I_n(f^{\circ n}), \qquad f, g \in \mathcal{C}_c^1(\mathbb{R}_+).
\end{aligned}$$

We now make use of the multiplication formula for Poisson stochastic integrals to prove the result on $\mathcal{S}$ by induction. Assume that (7.7.1) holds for $F = I_n(f^{\circ n})$ and $G = I_1(g)^k$ for some $k \geq 1$. Then, using the product rule Proposition 4.5.2 or Proposition 6.4.8 for the operator D_t we have

$$\begin{aligned}
&\nabla_t^{\ominus}(I_n(f^{\circ n})I_1(g)^{k+1})\\
&= I_1(g)\nabla_t^{\ominus}(I_n(f^{\circ n})I_1(g)^k) + I_n(f^{\circ n})I_1(g)^k \nabla_t^{\ominus} I_1(g)\\
&\quad -D_t I_1(g) D_t(I_1(g)^k I_n(f^{\circ n}))\\
&= I_1(g)\left(I_1(g)^k \nabla_t^{\ominus} I_n(f^{\circ n}) + I_n(f^{\circ n})\nabla_t^{\ominus}\left(I_1(g)^k\right) - D_t\left(I_1(g)^k\right) D_t I_n(f^{\circ n})\right)\\
&\quad +I_n(f^{\circ n})I_1(g)^k \nabla_t^{\ominus} I_1(g) - D_t I_1(g)\left(I_1(g)^k D_t I_n(f^{\circ n})\right.\\
&\quad \left.+I_n(f^{\circ n})D_t\left(I_1(g)^k\right)\right) - D_t I_1(g) D_t I_1(g)^k D_t I_n(f^{\circ n})\\
&= I_1(g)^{k+1}\nabla_t^{\ominus} I_n(f^{\circ n}) + I_n(f^{\circ n})\nabla_t^{\ominus}\left(I_1(g)^{k+1}\right) - D_t\left(I_1(g)^{k+1}\right) D_t I_n(f^{\circ n}),
\end{aligned}$$

$t \in \mathbb{R}_+$. □

Proposition 7.7.2. *We have the identity*

$$\tilde{D} = D + \nabla^{\ominus}$$

on the space $\mathcal{S}$.

Proof. Lemma 7.7.1 shows that $(\nabla^{\ominus} + D)$ is a derivation operator since

$$\begin{aligned}(\nabla_t^{\ominus} + D_t)(FG) &= \nabla_t^{\ominus}(FG) + D_t(FG) \\ &= F\nabla_t^{\ominus}G + G\nabla_t^{\ominus}F - D_tFD_tG + D_t(FG) \\ &= F(\nabla_t^{\ominus} + D_t)G + G(\nabla_t^{\ominus} + D_t)F, \qquad F, G \in \mathcal{S}.\end{aligned}$$

Thus it is sufficient to show that

$$(D_t + \nabla_t^{\ominus})f(T_k) = \tilde{D}f(T_k), \qquad k \geq 1, \quad f \in \mathcal{C}_b^1(\mathbb{R}). \tag{7.7.2}$$

Letting $\pi_{[t}$ denote the projection

$$\pi_{[t}f = f1_{[t,\infty)}, \qquad f \in L^2(\mathbb{R}_+),$$

we have

$$\begin{aligned}&(D_t + \nabla_t^{\ominus})f(T_k) = (D_t + \nabla_t^{\ominus})\sum_{n\in\mathbb{N}}\frac{1}{n!}I_n(f_n^k) \\ &= \sum_{n\geq 1}\frac{1}{(n-1)!}I_{n-1}(f_n^k(\cdot,t)) - \sum_{n\geq 1}\frac{1}{(n-1)!}I_n(\pi_{[t}\otimes \mathrm{Id}^{\otimes(n-1)}\partial_1 f_n^k) \\ &= \sum_{n\in\mathbb{N}}\frac{1}{n!}I_n\left(f_{n+1}^k(\cdot,t) - n\pi_{[t}\otimes \mathrm{Id}^{\otimes(n-1)}\partial_1 f_n^k\right),\end{aligned}$$

where $\mathrm{Id} : L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+)$ is the identity operator. Now,

$$\begin{aligned}&f_{n+1}^k(t, t_1, \ldots, t_n) - n\pi_{[t}\otimes \mathrm{Id}^{\otimes(n-1)}\partial_1 f_n^k(t_1, \ldots, t_n) \\ &\quad = \alpha_{n+1}^k(f)(t_1 \vee \cdots \vee t_n \vee t) \\ &\qquad -\mathbf{1}_{\{t<t_1\vee\cdots\vee t_n\}}\left(\alpha_n^k(f') + \alpha_{n+1}^k(f)\right)(t_1 \vee \cdots \vee t_n) \\ &\quad = \alpha_{n+1}^k(f)\mathbf{1}_{\{t_1\vee\cdots\vee t_n<t\}} - \alpha_n^k(f')(t_1 \vee \cdots \vee t_n)\mathbf{1}_{\{t_1\vee\cdots\vee t_n>t\}} \\ &\quad = \alpha_n^k(-f'_{[t})(t_1 \vee \cdots \vee t_n),\end{aligned}$$

which coincides with n-th term, in the chaos expansion of $-\mathbf{1}_{[0,T_k]}f'(T_k)$ by Proposition 7.4.3, $k \in \mathbb{N}$, $n \geq 1$. Hence Relation (7.7.2) holds and we have $D + \nabla^{\ominus} = \tilde{D}$. □

Since both δ and $\tilde{\delta} = \delta + \nabla^{\oplus}$ coincide with the Itô integral on adapted processes, it follows that $\nabla^{\oplus}$ vanishes on adapted processes. By duality this implies that the adapted projection of $\nabla^\ominus$ is zero, hence by Proposition 7.7.2, $\tilde{D}$ is written as a perturbation of D by a gradient process with vanishing adapted projection.

7.8 Notes and References

The notion of lifting of the differential geometry on a Riemannian manifold X to a differential geometry on Ω^X has been introduced in [4], and the integration by parts formula (7.1.2) can be found been obtained therein, cf. also [17]. In Corollary 7.1.7, our pointwise lifting of gradients allows us to recover Theorem 5-2 of [4], page 489, as a particular case by taking expectations in Relation (7.1.5). See [21], [97], [110], for the locality of $\tilde{D}$ and $\tilde{\delta}$. See [3] and [32] for another approaches to the Weitzenböck formula on configuration spaces under Poisson measures. The proof of Proposition 7.6.7 is based on an argument of [45] for path spaces over Lie groups. The gradient $\tilde{D}$ is called "damped" in reference to [46], cf. Section 5.7. The gradient $\tilde{D}$ of Definition 7.2.1 is a modification of the gradient introduced in [25], see also [38]. However, the integration by parts formula of [25] deals with processes of zero integral only, as in (7.3.6). A different version of the gradient $\bar{D}$, which solves the closability issue mentioned at the end of Section 7.3, has been used for sensitivity analysis in [75], [121], [122]. The combined use of D^n and $\tilde{D}$ for the computation of the chaos expansion of the jump time T_d, $d \geq 1$, and the Clark representation formula for $\tilde{D}$ can be found in [106]. The construction of $\tilde{D}$ and D can also be extended to arbitrary Poisson processes with adapted intensities, cf. [34], [108], [109].

Chapter 8
Option Hedging in Continuous Time

Here we review some applications to mathematical finance of the tools introduced in the previous chapters. We construct a market model with jumps in which exponential normal martingales are used to model random prices. We obtain pricing and hedging formulas for contingent claims, extending the classical Black-Scholes theory to other complete markets with jumps.

8.1 Market Model

Let $(M_t)_{t\in\mathbb{R}_+}$ be a martingale having the chaos representation property of Definition 2.8.1 and angle bracket given by $d\langle M, M\rangle_t = \alpha_t^2 dt$. By a modification of Proposition 2.10.2, $(M_t)_{t\in[0,T]}$ satisfies the structure equation

$$d[M, M]_t = \alpha_t^2 dt + \phi_t dM_t.$$

When $(\phi_t)_{t\in[0,T]}$ is deterministic, $(M_t)_{t\in[0,T]}$ is alternatively a Brownian motion or a compensated Poisson martingale, depending on the vanishing of $(\phi_t)_{t\in[0,T]}$.

Let $r : \mathbb{R}_+ \longrightarrow \mathbb{R}$ and $\sigma : \mathbb{R}_+ \longrightarrow (0,\infty)$ be deterministic non negative bounded functions. We assume that $1 + \sigma_t\phi_t > 0$, $t \in [0, T]$. Let $(A_t)_{t\in\mathbb{R}_+}$ denote the price of the riskless asset, given by

$$\frac{dA_t}{A_t} = r_t dt, \qquad A_0 = 1, \qquad t \in \mathbb{R}_+, \tag{8.1.1}$$

i.e.

$$A_t = A_0 \exp\left(\int_0^t r_s ds\right), \qquad t \in \mathbb{R}_+.$$

For $t > 0$, let $(S^x_{t,u})_{u\in[t,T]}$ be the price process with risk-neutral dynamics given by

$$dS^x_{t,u} = r_u S^x_{t,u} du + \sigma_u S^x_{t,u^-} dM_u, \quad u \in [t,T], \quad S^x_{t,t} = x,$$

cf. Relation (2.13.5). Recall that when $(\phi_t)_{t\in\mathbb{R}_+}$ is deterministic we have

$$S^x_{t,T} = x \exp\left(\int_t^T \sigma_u \alpha_u i_u dB_u + \int_t^T (r_u - \phi_u \lambda_u \sigma_u - \frac{1}{2} i_u \sigma_u^2 \alpha_u^2) du \right) \times \prod_{k=1+N_t}^{N_T} (1 + \sigma_{T_k} \phi_{T_k}), \quad (8.1.2)$$

$0 \le t \le T$, with $S_t = S^1_{0,t}$, $t \in [0,T]$. Figure 8.1 shows a sample path of $(S_t)_{t\in[0,T]}$ when the function $(\phi_t)_{t\in[0,T]}$, plotted above the horizontal axis, takes values in $\{0,1\}$ with $S_0 = 10$, $\sigma_t = 10$, and $\alpha_t = 1$, $t \in [0,T]$.

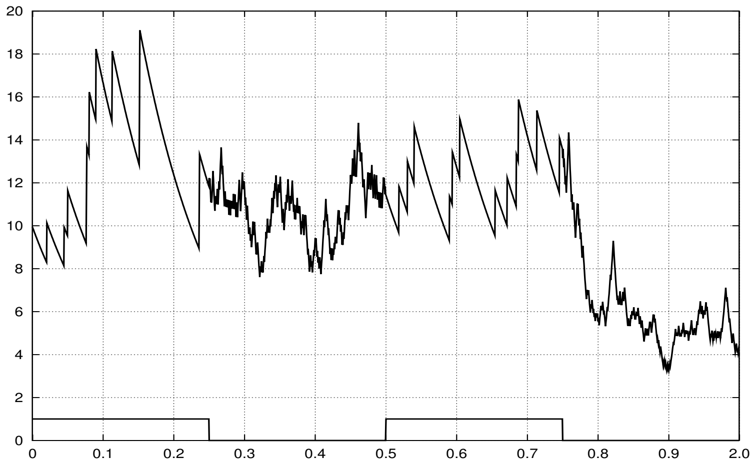

Fig. 8.1: Sample trajectory of $(S_t)_{t\in[0,T]}$

Let η_t and ζ_t be the numbers of units invested at time t, respectively in the assets $(S_t)_{t\in\mathbb{R}_+}$ and $(A_t)_{t\in\mathbb{R}_+}$. The value of the portfolio V_t at time t is given by

$$V_t = \zeta_t A_t + \eta_t S_t, \qquad t \in \mathbb{R}_+. \quad (8.1.3)$$

Definition 8.1.1. *The portfolio V_t is said to be self-financing if*

$$dV_t = \zeta_t dA_t + \eta_t dS_t. \quad (8.1.4)$$

The self-financing condition can be written as

$$A_t d\zeta_t + S_t d\eta_t = 0, \qquad 0 \le t \le T$$

under the approximation $d\langle S_t, \eta_t\rangle \simeq 0$.

Let also

$$\tilde{V}_t = V_t \exp\left(-\int_0^t r_s ds\right) \quad \text{and} \quad \tilde{S}_t = S_t \exp\left(-\int_0^t r_s ds\right)$$

denote respectively the discounted portfolio price and underlying asset price.

Lemma 8.1.2. *The following statements are equivalent:*

i) the portfolio V_t is self-financing,

ii) we have

$$\tilde{V}_t = \tilde{V}_0 + \int_0^t \sigma_u \eta_u \tilde{S}_u dM_u, \qquad t \in \mathbb{R}_+, \tag{8.1.5}$$

iii) we have

$$V_t = V_0 \exp\left(\int_0^t r_u du\right) + \int_0^t \sigma_u \eta_u \exp\left(\int_u^t r_u du\right) S_u dM_u, \tag{8.1.6}$$

$t \in \mathbb{R}_+$.

Proof. First, note that (8.1.5) is clearly equivalent to (8.1.6). Next, the self-financing condition (8.1.4) shows that

$$\begin{aligned} dV_t &= \zeta_t dA_t + \eta_t dS_t \\ &= \zeta_t A_t r_t dt + \eta_t r_t S_t dt + \sigma_t \eta_t S_t dM_t \\ &= r_t V_t dt + \sigma_t \eta_t S_t dM_t, \end{aligned}$$

$t \in \mathbb{R}_+$, hence

$$\begin{aligned} d\tilde{V}_t &= d\left(\exp\left(-\int_0^t r_s ds\right) V_t\right) \\ &= -r_t \exp\left(-\int_0^t r_s ds\right) V_t dt + \exp\left(-\int_0^t r_s ds\right) dV_t \\ &= \exp\left(-\int_0^t r_s ds\right) \sigma_t \eta_t S_t dM_t, \qquad t \in \mathbb{R}_+, \end{aligned}$$

i.e. (8.1.5) holds. Conversely, if (8.1.5) is satisfied we have

$$\begin{aligned} dV_t &= d(A_t \tilde{V}_t) \\ &= \tilde{V}_t dA_t + A_t d\tilde{V}_t \\ &= \tilde{V}_t A_t r_t dt + \sigma_t \eta_t S_t dM_t \\ &= V_t r_t dt + \sigma_t \eta_t S_t dM_t \end{aligned}$$

$$= \zeta_t A_t r_t dt + \eta_t S_t r_t dt + \sigma_t \eta_t S_t dM_t$$
$$= \zeta_t dA_t + \eta_t dS_t,$$

hence the portfolio is self-financing. □

8.2 Hedging by the Clark Formula

In the next proposition we compute a self-financing hedging strategy leading to an arbitrary square-integrable random variable F, using the Clark formula Proposition 4.2.3.

Proposition 8.2.1. *Given $F \in L^2(\Omega)$, let*

$$\eta_t = \frac{\exp\left(-\int_t^T r_s ds\right)}{\sigma_t S_t} \mathbb{E}[D_t F | \mathcal{F}_t], \tag{8.2.1}$$

$$\zeta_t = \frac{\exp\left(-\int_t^T r_u du\right) \mathbb{E}[F|\mathcal{F}_t] - \eta_t S_t}{A_t}, \qquad t \in [0,T]. \tag{8.2.2}$$

Then the portfolio $(\eta_t, \zeta_t)_{t\in[0,T]}$ is self-financing and yields a hedging strategy leading to F, i.e. letting

$$V_t = \zeta_t A_t + \eta_t S_t, \qquad 0 \le t \le T,$$

we have

$$V_t = \exp\left(-\int_t^T r_u du\right) \mathbb{E}[F|\mathcal{F}_t], \tag{8.2.3}$$

$0 \le t \le T$*. In particular we have $V_T = F$ and*

$$V_0 = \exp\left(-\int_0^T r_u du\right) \mathbb{E}[F].$$

Proof. Applying (8.2.2) at $t = 0$ we get

$$\mathbb{E}[F] \exp\left(-\int_0^T r_u du\right) = V_0,$$

hence from (8.2.2), the definition (8.2.1) of η_t and the Clark formula we obtain

$$\begin{aligned} V_t &= \zeta_t A_t + \eta_t S_t \\ &= \exp\left(-\int_t^T r_u du\right) \mathbb{E}[F|\mathcal{F}_t] \\ &= \exp\left(-\int_t^T r_u du\right) \left(\mathbb{E}[F] + \int_0^t \mathbb{E}[D_u F|\mathcal{F}_u] dM_u\right) \end{aligned}$$

$$= V_0 \exp\left(\int_0^t r_u du\right) + \exp\left(-\int_t^T r_u du\right) \int_0^t \mathbb{E}[D_u F|\mathcal{F}_u] dM_u$$
$$= V_0 \exp\left(\int_0^t r_u du\right) + \int_0^t \eta_u \sigma_u S_u \exp\left(\int_u^t r_s ds\right) dM_u, \qquad 0 \le t \le T,$$

and from Lemma 8.1.2 this also implies that the portfolio $(\eta_t, \zeta_t)_{t\in[0,T]}$ is self-financing. □

The above proposition shows that there always exists a hedging strategy starting from

$$V_0 = \mathbb{E}[F] \exp\left(-\int_0^T r_u du\right).$$

Conversely, since there exists a hedging strategy leading to

$$\tilde{V}_T = F \exp\left(-\int_0^T r_u du\right),$$

then by (8.1.5), $(\tilde{V}_t)_{t\in[0,T]}$ is necessarily a martingale with initial value

$$\tilde{V}_0 = \mathbb{E}[\tilde{V}_T] = \mathbb{E}[F] \exp\left(-\int_0^T r_u du\right).$$

We now consider the hedging of European call option with payoff $F = (S_T - K)^+$ using the Clark formula in the setting of deterministic structure equations. In this case the next proposition allows us to compute the hedging strategy appearing in (8.2.1).

Proposition 8.2.2. *Assume that $\phi_t \ge 0$, $t \in [0,T]$. Then for $0 \le t \le T$ we have*

$$\mathbb{E}[D_t (S_T - K)^+ | \mathcal{F}_t] = \mathbb{E}\Big[i_t \sigma_t S^x_{t,T} \mathbf{1}_{[K,\infty)}(S^x_{t,T}) + \frac{j_t}{\phi_t} (\sigma_t \phi_t S^x_{t,T} - (K - S^x_{t,T})^+) \mathbf{1}_{[\frac{K}{1+\sigma_t},\infty)}(S^x_{t,T}) \Big]_{x=S_t}.$$

Proof. By Lemma 4.6.2, using Definition 4.6.1 and Relation (4.6.4) we have, for any $F \in \mathcal{S}$,

$$D_t F = D^B_t F + \frac{j_t}{\phi_t} (T^\phi_t F - F), \quad t \in [0,T]. \tag{8.2.4}$$

We have $T^\phi_t S_T = (1 + \sigma_t \phi_t) S_T$, $t \in [0,T]$, and the chain rule $D^B f(F) = f'(F) D^B F$, cf. Relation (5.2.1), holds for $F \in \mathcal{S}$ and $f \in \mathcal{C}^2_b(\mathbb{R})$. Since $\mathcal{S}$ is an algebra for deterministic $(\phi_t)_{t\in[0,T]}$, we may approach $x \longmapsto (x-K)^+$ by polynomials on compact intervals and proceed e.g. as in [101], p. 5-13. By dominated convergence, $F = (S_T - K)^+ \in \mathrm{Dom}\,(D)$ and (8.2.4) becomes

$$D_t(S_T-K)^+ = i_t\sigma_t S_T \mathbf{1}_{[K,\infty)}(S_T) + \frac{j_t}{\phi_t}((1+\sigma_t\phi_t)S_T-K)^+ - (S_T-K)^+),$$

$0 \le t \le T$. The Markov property of $(S_t)_{t\in[0,T]}$ implies

$$\mathbb{E}\left[D_t^B(S_T-K)^+|\mathcal{F}_t\right] = i_t\sigma_t\,\mathbb{E}\left[S_{t,T}^x\mathbf{1}_{[K,\infty)}(S_{t,T}^x)\right]_{x=S_t},$$

and

$$\begin{aligned}
&\frac{j_t}{\phi_t}\,\mathbb{E}[(T_t^\phi S_T-K)^+ - (S_T-K)^+|\mathcal{F}_t] \\
&= \frac{j_t}{\phi_t}\,\mathbb{E}\left[((1+\sigma_t\phi_t)S_{t,T}^x-K)^+ - (S_{t,T}^x-K)^+\right]_{x=S_t} \\
&= \frac{j_t}{\phi_t}\,\mathbb{E}\left[((1+\sigma_t\phi_t)S_{t,T}^x-K)\mathbf{1}_{[\frac{K}{1+\sigma_t\phi_t},\infty)}(S_{t,T}^x)\right]_{x=S_t} \\
&\quad -\frac{j_t}{\phi_t}\,\mathbb{E}\left[(S_{t,T}^x-K)^+\mathbf{1}_{[K,\infty)}(S_{t,T}^x)\right]_{x=S_t} \\
&= \frac{j_t}{\phi_t}\,\mathbb{E}\left[\sigma_t\phi_t S_{t,T}^x\mathbf{1}_{[\frac{K}{1+\sigma_t\phi_t},\infty)}(S_{t,T}^x) + (S_{t,T}^x-K)\mathbf{1}_{[\frac{K}{1+\sigma_t\phi_t},K]}(S_{t,T}^x)\right]_{x=S_t} \\
&= \frac{j_t}{\phi_t}\,\mathbb{E}\left[\sigma_t\phi_t S_{t,T}^x\mathbf{1}_{[\frac{K}{1+\sigma_t\phi_t},\infty)}(S_{t,T}^x) - (K-S_{t,T}^x)^+\mathbf{1}_{[\frac{K}{1+\sigma_t\phi_t},\infty]}(S_{t,T}^x)\right]_{x=S_t} \\
&= \frac{j_t}{\phi_t}\,\mathbb{E}\left[(\sigma_t\phi_t S_{t,T}^x - (K-S_{t,T}^x)^+)\mathbf{1}_{[\frac{K}{1+\sigma_t\phi_t},\infty)}(S_{t,T}^x)\right]_{x=S_t}.
\end{aligned}$$

□

If $(\phi_t)_{t\in[0,T]}$ is not constrained to be positive then

$$\begin{aligned}
&\mathbb{E}[D_t(S_T-K)^+|\mathcal{F}_t] = i_t\sigma_t\,\mathbb{E}\left[S_{t,T}^x\mathbf{1}_{[K,\infty)}(S_{t,T}^x)\right]_{x=S_t} \\
&\quad +\frac{j_t}{\phi_t}\,\mathbb{E}\left[\sigma_t\phi_t S_{t,T}^x\mathbf{1}_{[\frac{K}{1+\sigma_t\phi_t},\infty)}(S_{t,T}^x) + (S_{t,T}^x-K)\mathbf{1}_{[\frac{K}{1+\sigma_t\phi_t},K]}(S_{t,T}^x)\right]_{x=S_t},
\end{aligned}$$

with the convention $\mathbf{1}_{[b,a]} = -\mathbf{1}_{[a,b]}$, $0 \le a < b \le T$. Proposition 8.2.2 can also be proved using Lemma 3.7.2 and the Itô formula (2.12.4).

In the sequel we assume that $(\phi_t)_{t\in\mathbb{R}_+}$ is deterministic and

$$dM_t = i_t dB_t + \phi_t(dN_t - \lambda_t dt), \qquad t\in\mathbb{R}_+, \qquad M_0 = 0,$$

as in Relation (2.10.5).

Next we compute

$$\exp\left(-\int_0^T r_s ds\right)\mathbb{E}\left[(S_T-K)^+\right]$$

in terms of the Black-Scholes function

$$\mathrm{BS}(x,T;r,\sigma^2;K) = \mathrm{e}^{-rT}\,\mathbb{E}[(x\mathrm{e}^{rT-\sigma^2T/2+\sigma W_T}-K)^+],$$

where W_T is a centered Gaussian random variable with variance T.

Proposition 8.2.3. *The expectation*

$$\exp\left(-\int_0^T r_s ds\right)\mathbb{E}\left[(S_T-K)^+\right]$$

can be computed as

$$\exp\left(-\int_0^T r_s ds\right)\mathbb{E}\left[(S_T-K)^+\right]$$
$$= \exp\left(-\Gamma_0(T)\right)\sum_{k=0}^\infty \frac{1}{k!}\int_0^T\cdots\int_0^T$$
$$BS\left(S_0\exp\left(-\int_0^T \phi_s\gamma_s\sigma_s ds\right)\prod_{i=1}^k\left(1+\sigma_{t_i}\phi_{t_i}\right),T;R_T,\frac{\Gamma_0(T)}{T};K\right)$$
$$\gamma_{t_1}\cdots\gamma_{t_k}dt_1\cdots dt_k.$$

Proof. Similarly to Proposition 3.7.3 we have

$$\mathbb{E}\left[\mathrm{e}^{-TR_T}(S_T-K)^+\right] = \sum_{k=0}^\infty \mathbb{E}\left[\mathrm{e}^{-TR_T}(S_T-K)^+|N_T=k\right]\mathbb{P}(N_T=k),$$

with

$$\mathbb{P}(N_T=k) = \exp(-\Gamma_0(T))\frac{(\Gamma_0(T))^k}{k!},\qquad k\in\mathbb{N}.$$

Conditionally to $\{N_T=k\}$, the jump times $(T_1,\ldots,T_k)$ have the law

$$\frac{k!}{(\Gamma_0(T))^k}1_{\{0<t_1<\cdots<t_k<T\}}\gamma_{t_1}\cdots\gamma_{t_k}dt_1\cdots dt_k,$$

since the process $(N_{\Gamma_0^{-1}(T)_t})_{t\in\mathbb{R}_+}$ is a standard Poisson process. Hence, conditionally to

$$\{N(\Gamma_0^{-1}(\Gamma_0(T)))=k\}=\{N_T=k\},$$

its jump times $(\Gamma_0(T_1),\ldots,\Gamma_0(T_k))$ have a uniform law on $[0,\Gamma_0(T)]^k$. We then use the fact that $(\tilde{B}_t)_{t\in\mathbb{R}_+}$ and $(N_t)_{t\in\mathbb{R}_+}$ are also independent under $\mathbb{P}$ since $(r_t)_{t\in\mathbb{R}_+}$ is deterministic, and the identity in law

$$S_T \overset{law}{=} S_0X_T\exp\left(-\int_0^T\phi_s\lambda_s\sigma_s ds\right)\prod_{k=1}^{N_T}\left(1+\sigma_{T_k}\phi_{T_k}\right),$$

where

$$X_T = \exp\left(TR_T - \Gamma_0(T)/2 + \left(\frac{\Gamma_0(T)}{T}\right)^{1/2} W_T\right),$$

and W_T is independent of N. □

8.3 Black-Scholes PDE

As in the standard Black-Scholes model, it is possible to determine the hedging strategy in terms of the Delta of the price in the case $(r_t)_{t\in\mathbb{R}_+}$ is deterministic.

Let the function $C(t,x)$ be defined by

$$\begin{aligned} C(t,S_t) &= V_t \\ &= \exp\left(-\int_t^T r_u du\right) \mathbb{E}[(S_T-K)^+ \mid \mathcal{F}_t] \\ &= \exp\left(-\int_t^T r_u du\right) \mathbb{E}[(S_T-K)^+ \mid S_t], \qquad t \in \mathbb{R}_+. \end{aligned}$$

cf. (8.2.3). An application of the Itô formula leads to

$$\begin{aligned} dC(t,S_t) = &\left(\frac{\partial C}{\partial t} + r_t S_t \frac{\partial C}{\partial x} + \frac{1}{2}\frac{\partial^2 C}{\partial x^2} i_t \alpha_t^2 S_t^2 \sigma_t^2 + \lambda_t \Theta C\right)(t,S_t)dt \\ &+ S_t\sigma_t \frac{\partial C}{\partial x}(t,S_t)dM_t + (C(t,S_t(1+\sigma_t\phi_t)) - C(t,S_t))(dN_t - \lambda_t dt) \end{aligned} \quad (8.3.1)$$

where

$$\Theta C(t,S_t) = C(t,S_t(1+\sigma_t\phi_t)) - C(t,S_t) - \frac{\partial C}{\partial x}(t,S_t)S_t\sigma_t\phi_t.$$

The process

$$\begin{aligned} \widetilde{C}_t &:= C(t,S_t)\exp\left(-\int_0^t r_s ds\right) \\ &= \exp\left(-\int_0^T r_u du\right) \mathbb{E}[(S_T-K)^+ \mid \mathcal{F}_t] \\ &= \tilde{V}_t \end{aligned}$$

is a martingale from Propositions 2.5.8 and 8.2.1-(ii), with

$$\begin{aligned} d\widetilde{C}_t &= \eta_t d\widetilde{S}_t \\ &= \sigma_t \eta_t S_t dM_t \end{aligned} \quad (8.3.2)$$

$$= i_t\sigma_t\eta_t S_t dB_t + \sigma_t\phi_t\eta_t S_t(dN_t - \lambda_t dt),$$

from Lemma 8.1.2. Therefore, by identification of (8.3.1) and (8.3.2),

$$\begin{cases} r_t C(t,S_t) = \left(\frac{\partial C}{\partial t} + r_t S_t \frac{\partial C}{\partial x} + \frac{1}{2} i_t \alpha_t^2 S_t^2 \sigma_t^2 \frac{\partial^2 C}{\partial x^2} + \lambda_t \Theta C\right)(t,S_t), \\ \\ \eta_t\sigma_t S_t dM_t = S_t\sigma_t \frac{\partial C}{\partial x}(t,S_t) dM_t + \left(C(t,S_t(1+\sigma_t\phi_t)) - C(t,S_t)\right)(dN_t - \lambda_t dt). \end{cases}$$

Therefore, by identification of the Brownian and Poisson parts,

$$\begin{cases} i_t\eta_t S_t\sigma_t = i_t S_t\sigma_t \frac{\partial C}{\partial x}(t,S_t) \\ \\ j_t\eta_t S_t\sigma_t\phi_t = C(t,S_t(1+\sigma_t\phi_t)) - C(t,S_t). \end{cases} \tag{8.3.3}$$

The term $\Theta C(t,S_t)$ vanishes on the set

$$\{t \in \mathbb{R}_+ \ : \ \phi_t = 0\} = \{t \ : \ i(t) = 1\}.$$

Therefore, (8.3.3) reduces to

$$\eta_t = \frac{\partial C}{\partial x}(t,S_t),$$

i.e. the process $(\eta_t)_{t\in\mathbb{R}_+}$ is equal to the usual Delta (8.3) on $\{t \in \mathbb{R}_+ \ : \ i_t = 1\}$, and to

$$\eta_t = \frac{C(t,S_t(1+\phi_t\sigma_t)) - C(t,S_t)}{S_t\phi_t\sigma_t}$$

on the set $\{t \in \mathbb{R}_+ \ : \ i_t = 0\}$.

Proposition 8.3.1. *The Black-Scholes PDE for the price of a European call option is written as*

$$\frac{\partial C}{\partial t}(t,x) + r_t x \frac{\partial C}{\partial x}(t,x) + \frac{1}{2}\alpha_t^2 x^2 \sigma_t^2 \frac{\partial^2 C}{\partial x^2}(t,x) = r_t C(t,x),$$

on $\{t : \phi_t = 0\}$, and as

$$\frac{\partial C}{\partial t}(t,x) + r_t x \frac{\partial C}{\partial x}(t,x) + \lambda_t \Theta C(t,x) = r_t C(t,x),$$

on the set $\{t \in \mathbb{R}_+ \ : \ \phi_t \neq 0\}$, under the terminal condition $C(T,x) = (x-K)^+$.

8.4 Asian Options and Deterministic Structure

The price at time t of an Asian option is defined as

$$\mathbb{E}\left[e^{-\int_t^T r_s ds} \left(\frac{1}{T} \int_0^T S_u du - K \right)^+ \Big| \mathcal{F}_t \right].$$

The next proposition provides a replicating hedging strategy for Asian options in the case of a deterministic structure equation. Following [78], page 91, and [14], we define the auxiliary process

$$Y_t = \frac{1}{S_t} \left(\frac{1}{T} \int_0^t S_u du - K \right), \quad t \in [0, T]. \tag{8.4.1}$$

Proposition 8.4.1. *There exists a measurable function $\tilde{C}$ on $\mathbb{R}_+ \times \mathbb{R}$ such that $\tilde{C}(t, \cdot)$ is $\mathcal{C}^1$ for all $t \in \mathbb{R}_+$, and*

$$S_t \tilde{C}(t, Y_t) = \mathbb{E}\left[\left(\frac{1}{T} \int_0^T S_u du - K \right)^+ \Big| \mathcal{F}_t \right].$$

Moreover, the replicating portfolio for an Asian option with payoff

$$\left(\frac{1}{T} \int_0^T S_u du - K \right)^+$$

is given by (8.1.3) *and*

$$\eta_t = \frac{1}{\sigma_t} e^{-\int_t^T r_s ds} \Big(\tilde{C}(t, Y_t) \sigma_t \tag{8.4.2}$$
$$+ (1 + \sigma_t \phi_t) \left(\frac{j_t}{\phi_t} \left(\tilde{C}\left(t, \frac{Y_t}{1 + \sigma_t \phi_t} \right) - \tilde{C}(t, Y_t) \right) - i_t \sigma_t Y_t \partial_2 \tilde{C}(t, Y_t) \right) \Big).$$

Proof. With the above notation, the price at time t of the Asian option becomes

$$\mathbb{E}\left[e^{-\int_t^T r_s ds} S_T (Y_T)^+ \Big| \mathcal{F}_t \right].$$

For $0 \leq s \leq t \leq T$, we have

$$d(S_t Y_t) = \frac{1}{T} d\left(\int_0^t S_u du - K \right) = \frac{S_t}{T} dt,$$

hence

$$\frac{S_t Y_t}{S_s} = Y_s + \frac{1}{T} \int_s^t \frac{S_u}{S_s} du.$$

Since S_u / S_t is independent of S_t by (8.1.2), we have, for any sufficiently integrable payoff function H,

$$\mathbb{E}\left[H\left(S_T Y_T\right)\Big|\mathcal{F}_t\right] = \mathbb{E}\left[H\left(S_t Y_t + \frac{1}{T}\int_t^T S_u du\right)\Big|\mathcal{F}_t\right]$$
$$= \mathbb{E}\left[H\left(xy + \frac{x}{T}\int_t^T \frac{S_u}{S_t} du\right)\right]_{y=Y_t,\ x=S_t}.$$

Let $C \in \mathcal{C}_b^2(\mathbb{R}_+ \times \mathbb{R}^2)$ be defined as

$$C(t,x,y) = \mathbb{E}\left[H\left(xy + \frac{x}{T}\int_t^T \frac{S_u}{S_t} du\right)\right],$$

i.e.

$$C(t,S_t,Y_t) = \mathbb{E}\left[H\left(S_T Y_T\right)\Big|\mathcal{F}_t\right].$$

When $H(x) = \max(x,0)$, since for any $t \in [0,T]$, S_t is positive and $\mathcal{F}_t$-measurable, and S_u/S_t is independent of $\mathcal{F}_t$, $u \geq t$, we have:

$$\begin{aligned}
\mathbb{E}\left[H\left(S_T Y_T\right)\Big|\mathcal{F}_t\right] &= \mathbb{E}\left[S_T (Y_T)^+\Big|\mathcal{F}_t\right] \\
&= S_t\,\mathbb{E}\left[\left(Y_T\frac{S_T}{S_t}\right)^+\Big|\mathcal{F}_t\right] \\
&= S_t\,\mathbb{E}\left[\left(Y_t + \frac{1}{T}\int_t^T \frac{S_u}{S_t}du\right)^+\Big|\mathcal{F}_t\right] \\
&= S_t\,\mathbb{E}\left[\left(y + \frac{1}{T}\int_t^T \frac{S_u}{S_t}du\right)^+\right]_{y=Y_t} \\
&= S_t\tilde{C}(t,Y_t),
\end{aligned}$$

with

$$\tilde{C}(t,y) = \mathbb{E}\left[\left(y + \frac{1}{T}\int_t^T \frac{S_u}{S_t}du\right)^+\right].$$

We now proceed as in [14], which deals with the sum of a Brownian motion and a Poisson process. From the expression of $1/S_t$ given by (8.1.2) we have

$$d\left(\frac{1}{S_t}\right) = \frac{1}{S_{t^-}}\left(\left(-r_t + \frac{\alpha_t^2\sigma_t^2}{1+\sigma_t\phi_t}\right)dt - \frac{\sigma_t}{1+\sigma_t\phi_t}dM_t\right),$$

hence by (2.12.4), Itô's formula and the definition (8.4.1) of Y_t, we have

$$dY_t = Y_t\left(-r_t + \frac{\alpha_t^2\sigma_t^2}{1+\sigma_t\phi_t}\right)dt + \frac{1}{T}dt - \frac{Y_{t^-}\sigma_t}{1+\sigma_t\phi_t}dM_t.$$

Assuming that $H \in \mathcal{C}_b^2(\mathbb{R})$ and applying Lemma 3.7.2 we get

$$\begin{aligned}
&\mathbb{E}\left[D_t H\left(S_T Y_T\right)\Big|\mathcal{F}_t\right] = L_t C(t, S_t, Y_t) \\
&= i_t\left(\sigma_t S_{t^-}\partial_2 C(t, S_t, Y_t) - \frac{Y_t\sigma_t}{1+\sigma_t\phi_t}\partial_3 C(t, S_t, Y_t)\right) \qquad (8.4.3) \\
&\quad + \frac{j_t}{\phi_t}\left(C\left(t, S_{t^-} + \sigma_t S_{t^-}, Y_{t^-} - \frac{Y_t\sigma_t}{1+\sigma_t\phi_t}\right) - C(t, S_{t^-}, Y_{t^-})\right),
\end{aligned}$$

where L_t is given by (2.12.5). Next, given a family $(H_n)_{n\in\mathbb{N}}$ of $\mathcal{C}_b^2$ functions, such that $|H_n(x)| \le x^+$ and $|H_n'(x)| \le 2$, $x \in \mathbb{R}$, $n \in \mathbb{N}$, and converging pointwise to $x \to x^+$, by dominated convergence (8.4.3) holds for $C(t, x, y) = x\tilde{C}(t, y)$ and we obtain:

$$\begin{aligned}
&\mathbb{E}\left[D_t\left(\frac{1}{T}\int_0^T S_u du - K\right)^+ \Bigg|\mathcal{F}_t\right] = i_t\tilde{C}(t, Y_t)\sigma_t S_t \\
&+ S_t\left(\frac{j_t}{\phi_t}\left(\tilde{C}\left(t, \frac{Y_t}{1+\sigma_t\phi_t}\right) - \tilde{C}(t, Y_t)\right) - i_t\sigma_t Y_t\partial_2\tilde{C}(t, Y_t)\right) \\
&+ S_t\sigma_t\phi_t\left(\frac{j_t}{\phi_t}\left(\tilde{C}\left(t, \frac{Y_t}{1+\sigma_t\phi_t}\right) - \tilde{C}(t, Y_t)\right) - i_t\sigma_t Y_t\partial_2\tilde{C}(t, Y_t)\right).
\end{aligned}$$

□

As a particular case we consider the Brownian motion model, i.e. $\phi_t = 0$, for all $t \in [0, T]$, so $i_t = 1$, $j_t = 0$ for all $t \in [0, T]$, and we are in the Brownian motion model. In this case we have

$$\begin{aligned}
\eta_t &= \mathrm{e}^{-\int_t^T r_s ds}\left(-Y_t\partial_2\tilde{C}(t, Y_t) + \tilde{C}(t, Y_t)\right) \\
&= \mathrm{e}^{-\int_t^T r_s ds}\left(S_t\frac{\partial}{\partial x}\tilde{C}\left(t, \frac{1}{x}\left(\frac{1}{T}\int_0^t S_u du - K\right)\right)_{|x=S_t} + \tilde{C}(t, Y_t)\right) \\
&= \frac{\partial}{\partial x}\left(x\mathrm{e}^{-\int_t^T r_s ds}\tilde{C}\left(t, \frac{1}{x}\left(\frac{1}{T}\int_0^t S_u du - K\right)\right)\right)_{|x=S_t}, \quad t \in [0, T],
\end{aligned}$$

which can be denoted informally as a partial derivative with respect to S_t.

8.5 Notes and References

See e.g. [78] and [139] for standard references on stochastic finance, and [101] for a presentation of the Malliavin calculus applied to continuous markets. The use of normal martingales in financial modelling has been first considered in [37]. The material on Asian options is based on [74] and [13]. Hedging strategies for Lookback options have been computed in [16] using the Clark-Ocone formula.

Chapter 9
Appendix

This appendix shortly reviews some notions used in the preceding chapters. It does not aim at completeness and is addressed to the non-probabilistic reader, who is referred to standard texts, e.g. [71], [123] for more details.

9.1 Measurability

Given a sequence $(Y_n)_{n\in\mathbb{N}}$ of random variables, a random variable F on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is said to be $\mathcal{F}_n$-measurable if it can be written as a function

$$F = f_n(Y_0, \ldots, Y_n)$$

of $Y_0, \ldots, Y_n$, where $f_n : \mathbb{R}^{n+1} \to \mathbb{R}$. This defines a filtration $(\mathcal{F}_n)_{n\geq -1}$ as

$$\mathcal{F}_{-1} = \{\emptyset, \Omega\}$$

and

$$\mathcal{F}_n = \sigma(Y_0, \ldots, Y_n), \qquad n \geq 0,$$

where $\sigma(Y_0, \ldots, Y_n)$ is the smallest σ-algebra making $Y_0, \ldots, Y_n$ measurable.

The space of $\mathcal{F}_n$-measurable random variables is denoted by $L^0(\Omega, \mathcal{F}_n, \mathbb{P})$.

9.2 Gaussian Random Variables

A random variable X is Gaussian with mean μ and variance σ^2 if and only if its characteristic function satisfies

$$\mathbb{E}[e^{i\alpha X}] = e^{i\alpha\mu - \alpha^2\sigma^2/2}, \qquad \alpha \in \mathbb{R}.$$

From e.g. Corollary 16.1 of [71] we have the following.

Proposition 9.2.1. *Let $X_1, \ldots, X_n$ be an orthogonal family of centered Gaussian variables, i.e.*

$$\mathbb{E}[X_i X_j] = 0, \qquad 1 \leq i \neq j \leq n.$$

Then $(X_1, \ldots, X_n)$ is a vector of independent random variables.

9.3 Conditional Expectation

Consider $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space and $\mathcal{G} \subset \mathcal{F}$ a sub σ-algebra of $\mathcal{F}$. The conditional expectation $\mathbb{E}[F \mid \mathcal{G}]$ of $F \in L^2(\Omega, \mathcal{F}, \mathbb{P})$ given $\mathcal{G}$ can be defined as the orthogonal projection of F on $L^2(\Omega, \mathcal{G}, \mathbb{P})$ for the scalar product $\langle F, G \rangle := \mathbb{E}[FG]$, hence it satisfies

$$\mathbb{E}[G(F - \mathbb{E}[F \mid \mathcal{G}])] = 0, \qquad G \in L^2(\Omega, \mathcal{G}, \mathbb{P}).$$

The conditional expectaction has the following properties

a) $\mathbb{E}[\mathbb{E}[F \mid \mathcal{F}] \mid \mathcal{G}] = \mathbb{E}[F \mid \mathcal{G}]$ if $\mathcal{G} \subset \mathcal{F}$.

b) $\mathbb{E}[GF \mid \mathcal{G}] = G\,\mathbb{E}[F \mid \mathcal{G}]$ if G is $\mathcal{G}$-measurable and sufficiently integrable.

c) $\mathbb{E}[f(X, Y) \mid \mathcal{F}] = \mathbb{E}[f(X, y)]_{y=Y}$ if X, Y are independent and Y is $\mathcal{F}$-measurable.

Property (a) is referred to as the *tower property*. It shows in particular that $\mathbb{E}[\mathbb{E}[F \mid \mathcal{F}]] = \mathbb{E}[F]$ when $\mathcal{G} = \{\emptyset, \Omega\}$.

Property (b) also shows that $\mathbb{E}[F \mid \mathcal{F}] = F$ when F is $\mathcal{F}$-measurable.

Property (c) shows in particular that $\mathbb{E}[F \mid \mathcal{F}] = \mathbb{E}[F]$ when F is independent of $\mathcal{F}$.

The Jensen inequality states that for φ any convex function $\varphi : \mathbb{R}^d \to \mathbb{R}^d$ we have

$$\varphi(\mathbb{E}[F]) \leq \mathbb{E}[\varphi(F)]. \tag{9.3.1}$$

9.4 Martingales in Discrete Time

Consider $(\mathcal{F}_n)_{n\in\mathbb{N}}$ an increasing family of sub σ-algebra of $\mathcal{F}$. A discrete time square-integrable martingale with respect to $(\mathcal{F}_n)_{n\in\mathbb{N}}$ is a family $(M_n)_{n\in\mathbb{N}}$ of random variables such that

i) $M_n \in L^2(\Omega, \mathcal{F}_n, \mathbb{P})$, $n \in \mathbb{N}$,

ii) $\mathbb{E}[M_{n+1} \mid \mathcal{F}_n] = M_n$, $n \in \mathbb{N}$.

Examples of martingales can be constructed from the tower property Property (a) above by letting $M_n := \mathbb{E}[F \mid \mathcal{F}_n]$, $n \in \mathbb{N}$, where $F \in L^2(\Omega)$ is a given square-integrable random variables.

The process

$$(Y_0 + \cdots + Y_n)_{n\geq 0}$$

is a martingale with respect to its own filtration defined as

$$\mathcal{F}_{-1} = \{\emptyset, \Omega\}$$

and

$$\mathcal{F}_n = \sigma(Y_0, \ldots, Y_n), \qquad n \geq 0,$$

if and only if the sequence $(Y_n)_{n\in\mathbb{N}}$ satisfies

$$\mathbb{E}[Y_n \mid \mathcal{F}_{n-1}] = 0, \qquad n \in \mathbb{N}.$$

This property is satisfied in particular by classical random walks with independent increments, for which $(Y_n)_{n\in\mathbb{N}}$ is a sequence of independent centered random variables.

Given $F \in L^2(\Omega)$, the tower property shows that $(\mathbb{E}[F \mid \mathcal{F}_n])_{n\in\mathbb{N}}$ is a martingale, and we have the following proposition.

Proposition 9.4.1. *Let $F \in L^2(\Omega)$. Then $(\mathbb{E}[F \mid \mathcal{F}_n])_{n\in\mathbb{N}}$ converges to F a.s.*

Proof. This is a consequence of the martingale convergence theorem, cf. e.g. Theorem 27.1 in [71]. □

9.5 Martingales in Continuous Time

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ a filtration, i.e. an increasing family of sub σ-algebras of $\mathcal{F}$. We assume that $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ is continuous on the right, i.e.

$$\mathcal{F}_t = \bigcap_{s>t} \mathcal{F}_s, \qquad t \in \mathbb{R}_+.$$

Definition 9.5.1. *A stochastic process* $(M_t)_{t\in\mathbb{R}_+}$ *such that* $\mathbb{E}[|M_t|^2] < \infty$, $t \in \mathbb{R}_+$, *is called an* $\mathcal{F}_t$*-martingale if*

$$\mathbb{E}[M_t|\mathcal{F}_s] = M_s, \quad 0 \le s < t.$$

The martingale $(M_t)_{t\in\mathbb{R}_+}$ is said to be square-integrable when $\mathbb{E}[|M_t|^2] < \infty$, $t \in \mathbb{R}_+$.

A process $(X_t)_{t\in\mathbb{R}_+}$ is said to have independent increments if $X_t - X_s$ is independent of $\sigma(X_u \ : \ 0 \le u \le s)$, $0 \le s < t$.

Proposition 9.5.2. *Every integrable process* $(X_t)_{t\in\mathbb{R}_+}$ *with centered independent increments is a martingale with respect to the filtration*

$$\mathcal{F}_t := \sigma(X_u \ : \ u \le t), \quad t \in \mathbb{R}_+,$$

it generates.

9.6 Markov Processes

Let $\mathcal{C}_0(\mathbb{R}^n)$ denote the class of continuous functions tending to 0 at infinity. Recall that f is said to tend to 0 at infinity if for all $\varepsilon > 0$ there exists a compact subset K of $\mathbb{R}^n$ such that $|f(x)| \le \varepsilon$ for all $x \in \mathbb{R}^n \setminus K$.

Definition 9.6.1. *An* $\mathbb{R}^n$*-valued stochastic process, i.e. a family* $(X_t)_{t\in\mathbb{R}_+}$ *of random variables on* $(\Omega, \mathcal{F}, \mathbb{P})$*, is a Markov process if for all* $t \in \mathbb{R}_+$ *the* σ*-fields*

$$\mathcal{F}_t^+ := \sigma(X_s \ : \ s \ge t)$$

and

$$\mathcal{F}_t := \sigma(X_s \ : \ 0 \le s \le t).$$

are conditionally independent given X_t.

This condition can be restated by saying that for all $A \in \mathcal{F}_t^+$ and $B \in \mathcal{F}_t$ we have

$$\mathbb{P}(A \cap B \mid X_t) = \mathbb{P}(A \mid X_t)\mathbb{P}(B \mid X_t),$$

cf. Chung [27]. This definition naturally entails that:

i) $(X_t)_{t\in\mathbb{R}_+}$ is adapted with respect to $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$, i.e. X_t is $\mathcal{F}_t$-measurable, $t \in \mathbb{R}_+$, and

ii) X_u is conditionally independent of $\mathcal{F}_t$ given X_t, for all $u \geq t$, i.e.

$$\mathbb{E}[f(X_u) \mid \mathcal{F}_t] = \mathbb{E}[f(X_u) \mid X_t], \qquad 0 \leq t \leq u,$$

for any bounded measurable function f on $\mathbb{R}^n$.

Processes with independent increments provide simple examples of Markov processes.

The transition kernel $\mu_{s,t}$ associated to $(X_t)_{t\in\mathbb{R}_+}$ is defined as

$$\mu_{s,t}(x, A) = \mathbb{P}(X_t \in A \mid X_s = x) \qquad 0 \leq s \leq t.$$

The transition operator $(P_{s,t})_{0\leq s\leq t}$ associated to $(X_t)_{t\in\mathbb{R}_+}$ is defined as

$$P_{s,t}f(x) = \mathbb{E}[f(X_t) \mid X_s = x] = \int_{\mathbb{R}^n} f(y)\mu_{s,t}(x, dy), \qquad x \in \mathbb{R}^n.$$

Letting $p_{s,t}(x)$ denote the density of $X_t - X_s$ we have

$$\mu_{s,t}(x, A) = \int_A p_{s,t}(y - x)dy, \qquad A \in \mathcal{B}(\mathbb{R}^n),$$

and

$$P_{s,t}f(x) = \int_{\mathbb{R}^n} f(y)p_{s,t}(y - x)dy.$$

Next we assume that $(X_t)_{t\in\mathbb{R}_+}$ is time homogeneous, i.e. $\mu_{s,t}$ depends only on the difference $t - s$, and we will denote it by μ_{t-s}. In this case the family $(P_{0,t})_{t\in\mathbb{R}_+}$ is denoted by $(P_t)_{t\in\mathbb{R}_+}$ and defines a transition semigroup associated to $(X_t)_{t\in\mathbb{R}_+}$, with

$$P_tf(x) = \mathbb{E}[f(X_t) \mid X_0 = x] = \int_{\mathbb{R}^n} f(y)\mu_t(x, dy), \qquad x \in \mathbb{R}^n.$$

It satisfies the semigroup property

$$\begin{aligned}
P_tP_sf(x) &= \mathbb{E}[P_sf(X_t) \mid X_0 = x] \\
&= \mathbb{E}[\mathbb{E}[f(X_{t+s}) \mid X_s] \mid X_0 = x]] \\
&= \mathbb{E}[\mathbb{E}[f(X_{t+s}) \mid \mathcal{F}_s] \mid X_0 = x]] \\
&= \mathbb{E}[f(X_{t+s}) \mid X_0 = x] \\
&= P_{t+s}f(x), \qquad x \in \mathbb{R}^n.
\end{aligned}$$

9.7 Tensor Products of L^2 Spaces

Let (X, μ) and (Y, ν) denote measure spaces. Given $f \in L^2(X, \mu)$ and $g \in L^2(Y, \nu)$, the tensor product $f \otimes g$ of f by g is the function in $L^2(X \times Y, \mu \otimes \nu)$ defined by

$$(f \otimes g)(x, y) = f(x)g(y).$$

In particular, the tensor product $f_n \otimes g_m$ of two functions $f_n \in L^2(X, \sigma)^{\otimes n}$, $g_m \in L^2(X, \sigma)^{\otimes m}$, satisfies

$$f_n \otimes g_m(x_1, \ldots, x_n, y_1, \ldots, y_m) = f_n(x_1, \ldots, x_n) g_m(y_1, \ldots, y_m),$$

$(x_1, \ldots, x_n, y_1, \ldots, y_m) \in X^{n+m}$. Given $f_1, \ldots, f_n \in L^2(X, \mu)$, the symmetric tensor product $f_1 \circ \cdots \circ f_n$ is defined as the symmetrization of $f_1 \otimes \cdots \otimes f_n$, i.e.

$$(f_1 \circ \cdots \circ f_n)(t_1, \ldots, t_n) = \frac{1}{n!} \sum_{\sigma \in \Sigma_n} f_1(t_{\sigma(1)}) \cdots f_n(t_{\sigma(n)}), \qquad t_1, \ldots, t_n \in X, \tag{9.7.1}$$

where Σ_n denotes the set of permutations of $\{1, \ldots, n\}$. Let now $L^2(X)^{\circ n}$ denote the subspace of $L^2(X)^{\otimes n} = L^2(X^n)$ made of symmetric functions f_n in n variables. As a convention, $L^2(X)^{\circ 0}$ is identified to $\mathbb{R}$. From (9.7.1), the symmetric tensor product can be extended as an associative operation on $L^2(X)^{\circ n}$.

The tensor power of order n of $L^2([0, T], \mathbb{R}^d)$, $n \in \mathbb{N}$, $d \in \mathbb{N}^*$, is

$$L^2([0, T], \mathbb{R}^d)^{\otimes n} \simeq L^2([0, T]^n, (\mathbb{R}^d)^{\otimes n}).$$

For $n = 2$ we have $(\mathbb{R}^d)^{\otimes 2} = \mathbb{R}^d \otimes \mathbb{R}^d \simeq \mathcal{M}_{d,d}(\mathbb{R})$ (the linear space of square $d \times d$ matrices), hence

$$L^2([0, T], \mathbb{R}^d)^{\otimes 2} \simeq L^2([0, T]^2, \mathcal{M}_{d,d}(\mathbb{R})).$$

More generally, the tensor product $(\mathbb{R}^d)^{\otimes n}$ is isomorphic to $\mathbb{R}^{d^n}$. The generic element of $L^2([0, T], \mathbb{R}^d)^{\otimes n}$ is denoted by

$$f = (f^{(i_1, \ldots, i_n)})_{1 \le i_1, \ldots, i_n \le d},$$

with $f^{(i_1, \ldots, i_n)} \in L^2([0, T]^n)$.

9.8 Closability of Linear Operators

The notion of closability for operators in normed linear spaces consists in some minimal hypotheses ensuring that a densely defined linear operator can be consistently extended to a larger domain.

Definition 9.8.1. *A linear operator $T : \mathcal{S} \to H$ from a normed linear space $\mathcal{S}$ into a normed linear space H is said to be closable on H if for every sequence $(F_n)_{n\in\mathbb{N}} \subset \mathcal{S}$ such that $F_n \to 0$ and $TF_n \to U$ in H, one has $U = 0$.*

The following proposition is proved by the linearity of T.

Proposition 9.8.2. *Assume that T is closable. If $(F_n)_{n\in\mathbb{N}}$ and $(G_n)_{n\in\mathbb{N}}$ converge to $F \in \mathrm{Dom}\,(T)$ and $(TF_n)_{n\in\mathbb{N}}$ and $(TG_n)_{n\in\mathbb{N}}$ converge respectively to U and V in H, then $U = V$.*

Proof. Indeed, under the above assumptions, $(T(F_n - G_n))_{n\in\mathbb{N}}$ converges to $U - V$, hence $U = V$ by the closability condition. □

Next we define the domain of a closable operator.

Definition 9.8.3. *Given a closable operator $T : \mathcal{S} \to H$, let $\mathrm{Dom}\,(T)$ denote the space of functionals F for which there exists a sequence $(F_n)_{n\in\mathbb{N}}$ converging to F and such that $(TF_n)_{n\in\mathbb{N}}$ converges to $G \in H$.*

It follows from Proposition 9.8.2 that the extension of T to $\mathrm{Dom}\,(T)$ is well-defined if T is closable, as in the following definition.

Definition 9.8.4. *Given $T : \mathcal{S} \to H$ a closable operator and $F \in \mathrm{Dom}\,(T)$, we let*

$$TF = \lim_{n\to\infty} TF_n,$$

where $(F_n)_{n\in\mathbb{N}}$ denotes any sequence converging to F and such that $(TF_n)_{n\in\mathbb{N}}$ converges in H.

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Index

Notation

$\mathcal{C}_b^1(\mathbb{R})$ denotes the space of continuously differentiable functions on $\mathbb{R}$ which are bounded together with their first derivative.

$\mathcal{C}_c^1(\mathbb{R})$ denotes the space of continuously differentiable functions on $\mathbb{R}$ which have compact support in $\mathbb{R}$.

This volume gives a unified presentation of stochastic analysis for continuous and discontinuous stochastic processes, in both discrete and continuous time. It is mostly self-contained and accessible to graduate students and researchers having already received a basic training in probability. The simultaneous treatment of continuous and jump processes is done in the framework of normal martingales; that includes the Brownian motion and compensated Poisson processes as specific cases. In particular, the basic tools of stochastic analysis (chaos representation, gradient, divergence, integration by parts) are presented in this general setting. Applications are given to functional and deviation inequalities and mathematical finance.