Probability approximation by Clark-Ocone covariance representation

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Abstract

Based on the Stein method and a general integration by parts framework we derive various bounds on the distance between probability measures. We show that this framework can be implemented on the Poisson space by covariance identities obtained from the Clark-Ocone representation formula and derivation operators. Our approach avoids the use of the inverse of the Ornstein-Uhlenbeck operator as in the existing literature, and also applies to the Wiener space.

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1 Introduction

The Stein and Chen-Stein methods have been applied to derive bounds on distances between probability laws on the Wiener and Poisson spaces, cf. [6], [7] and [8]. The results of these papers rely on covariance representations based on the number (or Ornstein-Uhlenbeck) operator L on multiple Wiener-Poisson stochastic integrals and its inverse L^{-1}. In particular the bound

$$d_W(F, \mathcal{N}) \leq \sqrt{\mathrm{E}\left[|1 - \langle DF, -DL^{-1}F\rangle|^2\right]} \quad (1.1)$$

has been derived for centered functionals of a standard real-valued Brownian motion in [6], Theorem 3.1. Here d_W is the Wasserstein distance, $\mathcal{N}$ is a random variable distributed according to the standard Gaussian law, D is the classical Malliavin gradient and $\langle \cdot, \cdot \rangle$ is the

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usual inner product on $L^2(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+), \ell)$, with ℓ the Lebesgue measure.

Although the Ornstein-Uhlenbeck operator L has nice contractivity properties as well as an integral representation, it can be difficult to compute in practice as its eigenspaces are made of multiple stochastic integrals. Thus, although the Ornstein-Uhlenbeck operator applies particularly well to functionals based on multiple stochastic integrals, it is of a more delicate use in applications to functionals whose multiple stochastic integral expansion is not explicitly known. This is due to the fact that the operator L is expressed as the composition of a divergence and a gradient operator, on both the Poisson and Wiener spaces.

In this paper we derive bounds on distances between probability laws using covariance representations based on the Clark-Ocone representation formula. In contrast with covariance identities based on the number operator, which relies on the divergence-gradient composition, the Clark-Ocone formula only requires the computation of a gradient and a conditional expectation. In particular, in Corollary 3.4 below we show that (1.1) can be replaced by

$$d_W(F, \mathcal{N}) \leq \sqrt{\mathrm{E}[|1 - \langle D_\cdot F, \mathrm{E}[D_\cdot F \mid \mathcal{F}_\cdot]\rangle|^2]}, \tag{1.2}$$

where F is a functional of a normal martingale such that $\mathrm{E}[F] = 0$. Here, D denotes a Malliavin type gradient operator having the chain rule of derivation, and $\{\mathcal{F}_t\}_{t\geq 0}$ is the natural filtration of the normal martingale.

In case D is the classical Malliavin gradient on the Wiener space, the bound (1.2) offers an alternative to (1.1). For example, if $F = I_n(f_n)$ is a multiple stochastic integral with respect to the Brownian motion and the symmetric kernels f_n satisfy certain integrability conditions the inequality (1.2) gives

$$\begin{aligned} d_W(I_n(f_n), \mathcal{N}) &\leq |1 - n!\|f_n\|^2_{L^2(\mathbb{R}_+^n)}| \\ &+ n^2 \sqrt{\sum_{k=0}^{n-2} (k!)^2 (2(n-1) - 2k)! \binom{n-1}{k}^4 \int_0^\infty \int_0^\infty \langle g^s_{n-1,k}, g^t_{n-1,k}\rangle_{L^2(\mathbb{R}_+^{2(n-1-k)})} \, \mathrm{d}s\mathrm{d}t} \end{aligned} \tag{1.3}$$

obtained by the multiplication formula for multiple Wiener integrals, where

$$g^t_{n-1,k}(*) = (f_n(*, t)) \circ_k (f_n(*, t)\mathbf{1}_{[0,t]^{n-1}}(*)), \qquad t \in \mathbb{R}_+,$$

and the symbol $\circ_k$ denotes the canonical symmetrization of the L^2 contraction over k variables, denoted by $\otimes_k$. On the other hand, by Proposition 3.2 of [6] the inequality (1.1) yields

$$
\begin{aligned}
d_W(I_n(f_n),\mathcal{N}) &\leq |1-n!\|f_n\|^2_{L^2(\mathbb{R}^n_+)}| \\
&\quad + n\sqrt{\sum_{k=0}^{n-2}(k!)^2(2(n-1)-2k)!\binom{n-1}{k}^4 \|f_n\circ_{k+1} f_n\|^2_{L^2(\mathbb{R}_+^{2(n-k-1)})}}.
\end{aligned}
$$

However, due to its importance, the Wiener case will be the object of a more detailed analysis in a subsequent work.

Here our focus will be on the Poisson space, for which (1.2) provides an alternative to Theorem 3.1 of [7]. Several applications are considered in Section 4. This includes functionals of Poisson jump times $(T_k)_{k\geq 1}$ of the form $f(T_k)$, for which we obtain the bound

$$
d_W(f(T_k),\mathcal{N}) \leq \left\| 1 - f'(T_k)\int_0^{T_k} \mathrm{E}[f'(T_{k-h}+t)]_{h=N_t}\mathrm{d}t \right\|_{L^1(P)},
$$

cf. Proposition 4.1, and a similar result for the gamma approximation, with linear and quadratic functionals of the Poisson jump times as examples. The analogs of (1.3) for Poisson multiple stochastic integrals are treated in Proposition 4.3, and comparisons with the results of [7] are discussed.

This paper is organized as follows. In Section 2 we present a general framework for bounds on probability distances based on an abstract integration by parts formula. Next in Section 3 we show that the conditions of this integration by parts setting can be satisfied under the existence of a Clark-Ocone type stochastic representation formula. In Section 4 we apply this general setting to a Clark-Ocone formula stated with a derivation operator on the Poisson space, and consider several examples, including multiple stochastic integrals and other functionals of jump times. In Section 5 we consider the total variation distance between a normalized Poisson compound sum and the standard Gaussian distribution.

We close this section by quoting Stein's lemmas for normal and gamma approximations. The following lemma on normal approximation can be traced back to Stein's contribution [12], see also the recent survey [5], and [6].

Lemma 1.1 *Let $h:\mathbb{R}\to[0,1]$ be a continuous function. The functional equation*

$$
f'(x) = xf(x) + h(x) - \mathrm{E}[h(\mathcal{N})], \qquad x\in\mathbb{R},
$$

has a solution $f_h\in\mathcal{C}_b^1(\mathbb{R})$ given by

$$
f_h(x) = \mathrm{e}^{x^2/2}\int_{-\infty}^x (h(a)-\mathrm{E}[h(\mathcal{N})])\mathrm{e}^{-a^2/2}\,\mathrm{d}a,
$$

with the bounds

$$|f_h(x)| \leq \sqrt{\frac{\pi}{2}} \quad and \quad |f_h'(x)| \leq 2, \qquad x \in \mathbb{R}.$$

The next lemma on the gamma approximation can be found in e.g. Lemma 1.3-(ii) of [6]. In the sequel we denote by $\Gamma(\nu/2)$ a random variable distributed according to the gamma law with parameters $(\nu/2, 1)$, $\nu > 0$.

Lemma 1.2 *Let $h : \mathbb{R} \to \mathbb{R}$ be a twice differentiable function such that*

$$|h(x)| \leq c\mathrm{e}^{ax}, \qquad x > -\nu,$$

for some $c > 0$ and $a < 1/2$. Then, letting $\Gamma_\nu := 2\Gamma(\nu/2) - \nu$, the functional equation

$$2(x+\nu)f'(x) = xf(x) + h(x) - \mathrm{E}[h(\Gamma_\nu)], \quad x > -\nu, \tag{1.4}$$

has a solution f_h which is bounded and differentiable on $(-\nu, \infty)$, and such that

$$\|f_h\|_\infty \leq 2\|h'\|_\infty \quad and \quad \|f_h'\|_\infty \leq \|h''\|_\infty.$$

2 General results

2.1 Integration by parts

The main results of this paper will be derived under the abstract integration by parts (IBP) formula (2.1) below. Let $\mathcal{T}$ denote a subset of $\mathcal{C}^1(\mathbb{R})$ containing the constant functions. Given F and G two real-valued random variables defined on a probability space $(\Omega, \mathcal{F}, P)$ and $A \in \mathcal{F}$ an event with $P(A) > 0$, we let

$$\mathrm{Cov}_A(F, G) := \mathrm{E}[FG \,|\, A] - \mathrm{E}[F \,|\, A]\mathrm{E}[G \,|\, A]$$

denote the covariance of F and G given A. The following general Assumption 2.1 says that the integration by parts formula with weights W_1 and W_2 holds for a random variable F given A on $\mathcal{T}$.

Assumption 2.1 *Given F a random variable, we assume that there exist two real-valued random variables $W_1 \in L^1(P(\cdot \,|\, A))$ and W_2 such that*

$$\mathrm{E}[W_2\phi'(F) \,|\, A] = \mathrm{Cov}_A(\phi(F), W_1), \tag{2.1}$$

for any $\phi \in \mathcal{T}$ such that $\phi(F)$, $W_1\phi(F)$, and $W_2\phi'(F) \in L^1(P(\cdot \,|\, A))$.

In particular, we note that if the IBP formula (2.1) with weights W_1 and W_2 holds on $\mathcal{T}$ for the random variable F given A, then W_1 is centered with respect to $P(\cdot \mid A)$ if and only if we have

$$\mathrm{E}[W_1\phi(F) \mid A] = \mathrm{E}[W_2\phi'(F) \mid A], \qquad \phi \in \mathcal{T},$$

as follows by taking $\phi = 1$ identically. An implementation of this formula on the Poisson space will be provided in Section 4 via the Clark-Ocone representation formula.

2.2 Normal approximation

Total variation distance

The total variation distance between two real-valued random variables Z_1 and Z_2 with laws P_{Z_1} and P_{Z_2} is defined by

$$d_{TV}(Z_1, Z_2) := \sup_{C\in\mathcal{B}(\mathbb{R})} |P_{Z_1}(C) - P_{Z_2}(C)| = \sup_{C\in\mathcal{B}_b(\mathbb{R})} |P_{Z_1}(C) - P_{Z_2}(C)|,$$

where $\mathcal{B}(\mathbb{R})$ and $\mathcal{B}_b(\mathbb{R})$ stand for the families of Borel and bounded Borel subsets of $\mathbb{R}$, respectively. The following bounds on the total variation distance $d_{TV}(F \mid A\,,\,\mathcal{N})$ between the law of F given A and the law of $\mathcal{N}$ hold under Assumption 2.1.

Theorem 2.1 *Let $A \in \mathcal{F}$ be such that $P(A) > 0$ and assume that the IBP formula* (2.1) *holds for F given A on $\mathcal{T} = \mathcal{C}_b^1(\mathbb{R})$. Then*

1. *If $W_2 = 1$ and W_1 is $P(\cdot \mid A)$-centered we have*

$$d_{TV}(F \mid A\,,\,\mathcal{N}) \le \sqrt{\frac{\pi}{2}}\,\mathrm{E}[|W_1 - F| \mid A]. \tag{2.2}$$

2. *If $W_1 = F$ is $P(\cdot \mid A)$-centered we have*

$$d_{TV}(F \mid A\,,\,\mathcal{N}) \le 2\,\mathrm{E}[|1 - W_2| \mid A]. \tag{2.3}$$

Proof. 1) Take $C \in \mathcal{B}_b(\mathbb{R})$ and let $a > 0$ be such that $C \subset [-a, a]$. Consider a sequence of continuous functions $h_n : \mathbb{R} \longrightarrow [0, 1]$, $n \ge 1$, such that $\lim_{n\to\infty} h_n(x) = \mathbb{1}_C(x)$, μ-a.e. where $\mu(\mathrm{d}x) = (\mathrm{d}x + P_{F \mid A}(\mathrm{d}x))_{|[-a,a]}$ (restriction to $[-a, a]$ of the sum of the Lebesgue measure and the law of F given A), cf. [11] or Corollary 1.10 of [2]. Lemma 1.1 and the integration by parts formula (2.1) show that for any $n \ge 1$ we have

$$|\mathrm{E}[h_n(F)\mathbb{1}_A] - \mathrm{E}[h_n(\mathcal{N})]P(A)| = |\mathrm{E}[(f'_{h_n}(F) - F f_{h_n}(F))\mathbb{1}_A]| \tag{2.4}$$

$$
\begin{aligned}
&= |\mathrm{E}[f_{h_n}(F)(W_1 - F)\mathbb{1}_A]| \\
&\leq \sqrt{\frac{\pi}{2}}\mathrm{E}[|W_1 - F|\mathbb{1}_A].
\end{aligned}
$$

Dividing first this inequality by $P(A) > 0$ and then taking the limit as n goes to infinity, the Dominated Convergence Theorem shows that

$$
|P(F \in C \mid A) - P(\mathcal{N} \in C)| \leq \sqrt{\frac{\pi}{2}}\,\mathrm{E}[|W_1 - F| \mid A],
$$

for any $C \in \mathcal{B}_b(\mathbb{R})$. The claim follows taking the supremum over all bounded Borel sets.

2) By (2.4) and the integration by parts formula, for any $n \geq 1$, we have

$$
\begin{aligned}
|\mathrm{E}[h_n(F)\mathbb{1}_A] - \mathrm{E}[h_n(\mathcal{N})]P(A)| &= |\mathrm{E}[f'_{h_n}(F)(1 - W_2)\mathbb{1}_A]| \\
&\leq 2\mathrm{E}[|1 - W_2|\mathbb{1}_A].
\end{aligned}
$$

The claim follows arguing exactly as in case (1) above. □

Wasserstein distance

The Wasserstein distance between the laws of Z_1 and Z_2 is defined by

$$
d_W(Z_1, Z_2) := \sup_{h \in \mathrm{Lip}(1)} |\mathrm{E}[h(Z_1)] - \mathrm{E}[h(Z_2)]|,
$$

where Lip(1) denotes the class of real-valued Lipschitz functions with Lipschitz constant less than or equal to 1. We have the following upper bound for the Wasserstein distance between a centered random variable F and $\mathcal{N}$.

Theorem 2.2 *Assume that the IBP formula* (2.1) *holds for F given A with $W_1 = F$, on the space $\mathcal{T}$ of twice differentiable functions whose first derivative is bounded by* 1 *and whose second derivative is bounded by* 2*. Then we have*

$$
d_W(F \mid A\,, \mathcal{N}) \leq \mathrm{E}[|1 - W_2| \mid A], \tag{2.5}
$$

provided F is $P(\cdot \mid A)$-centered.

Proof. Using the bound (2.33) in [7] and the IBP formula (2.1), we have

$$
\begin{aligned}
d_W(F \mid A, \mathcal{N}) &\leq \sup_{\phi \in \mathcal{T}} |\mathrm{E}[\phi'(F) - F\phi(F) \mid A]| \\
&= \sup_{\phi \in \mathcal{T}} |\mathrm{E}[\phi'(F)(1 - W_2) \mid A]| \leq \mathrm{E}[|1 - W_2| \mid A].
\end{aligned}
$$

□

2.3 Gamma approximation

Here we use the distance

$$d_{\mathcal{H}}(Z_1, Z_2) := \sup_{h\in\mathcal{H}} |\mathrm{E}[h(Z_1)] - \mathrm{E}[h(Z_2)]|, \tag{2.6}$$

where

$$\mathcal{H} := \{h \in \mathcal{C}_b^2(\mathbb{R}) : \ \max\{\|h\|_\infty, \|h'\|_\infty, \|h''\|_\infty\} \le 1\}.$$

The following upper bound for the $d_{\mathcal{H}}$-distance between the centered random variable F given A and a centered gamma random variable holds under the IBP formula (2.1) of Assumption 2.1.

Theorem 2.3 *Let F be a $P(\cdot \mid A)$-centered, a.s. $(-\nu, \infty)$-valued random variable. Given $A \in \mathcal{F}$ such that $P(A) > 0$, assume that the IBP formula (2.1) holds for F given A on $\mathcal{T} = \mathcal{C}_b^1(\mathbb{R})$ with $W_1 = F$. Then we have*

$$d_{\mathcal{H}}(F \mid A\,,\, \Gamma_\nu) \le \mathrm{E}[|2(F + \nu) - W_2| \mid A], \tag{2.7}$$

where the random variable Γ_ν is defined in Lemma 1.2.

Proof. Let $h \in \mathcal{H}$ be arbitrarily fixed. Since h is bounded above by 1, there exist $c > 0$ and $a < 1/2$ such that $|h(x)| \le c\mathrm{e}^{ax}$, $\forall\ x > -\nu$ (take $c > 1$ and $0 < a < 1/2$ so small that $1 < c\mathrm{e}^{-a\nu}$). Let f_h be solution of (1.4) (its existence is guaranteed by Lemma 1.2). By the IBP formula (2.1) on $\mathcal{C}_b^1(\mathbb{R})$ for the centered random variable F given A with $W_1 = F$, we have

$$\begin{aligned}|\mathrm{E}[h(F)\mathbb{1}_A] - \mathrm{E}[h(\Gamma_\nu)]P(A)| &= |\mathrm{E}[(2(F+\nu)f_h'(F) - Ff_h(F))\mathbb{1}_A]| \\ &= |\mathrm{E}\,[(2(F+\nu)f_h'(F) - W_2 f_h'(F))\mathbb{1}_A]\,| \\ &\le \|h''\|_\infty \mathrm{E}[|2(F+\nu) - W_2|\mathbb{1}_A].\end{aligned}$$

The claim follows by dividing the above inequality by $P(A) > 0$ and then taking the supremum over all functions $h \in \mathcal{H}$. □

3 Integration by parts via the Clark-Ocone formula

In this section we consider an implementation of the the IBP formula (2.1) of Assumption 2.1, based on the Clark-Ocone formula for a real-valued normal martingale $(M_t)_{t\ge0}$ defined on a probability space $(\Omega, \mathcal{F}, P)$, generating a right-continuous filtration $(\mathcal{F}_t)_{t\geq 0}$. In other words, $(M_t)_{t\geq 0}$ is a square integrable martingale with respect to the natural filtration $\mathcal{F}_t = \sigma(M_s : 0 \leq s \leq t)$, such that $\mathrm{E}[|M_t - M_s|^2 \,|\, \mathcal{F}_s] = t - s$, $0 \leq s < t$, and the filtration is right-continuous. Let ℓ be the Lebesgue measure on $\mathbb{R}_+$. In this section we assume the existence of a gradient operator

$$D : \mathrm{Dom}(D) \subset L^2(\Omega, \mathcal{F}, P) \longrightarrow L^2(\Omega \times \mathbb{R}_+, \mathcal{F} \otimes \mathcal{B}(\mathbb{R}_+), P \otimes \ell)$$

with domain $\mathrm{Dom}(D)$, defined by $DF = (D_t F)_{t\geq 0}$ and satisfying the following properties:

(i) D satisfies the Clark-Ocone representation formula

$$F = \mathrm{E}[F] + \int_0^\infty \mathrm{E}[D_t F \,|\, \mathcal{F}_t] \, \mathrm{d}M_t, \qquad F \in \mathrm{Dom}(D), \tag{3.1}$$

(ii) D satisfies the chain rule of derivation

$$D_t \phi(F) = \phi'(F) D_t F, \qquad F \in \mathrm{Dom}(D), \tag{3.2}$$

for all $\phi \in \mathcal{T} \subseteq \mathcal{C}^1(\mathbb{R})$, cf. e.g. § 3.6 of [10] (here $\mathcal{T}$ contains the constant functions).

This condition will be satisfied in both the Wiener and Poisson settings of Section 4. In addition we will assume that for any $F \in \mathrm{Dom}(D)$ and $\phi \in \mathcal{T}$ we have $\phi(F) \in \mathrm{Dom}(D)$. From (3.1) the gradient operator D satisfies the following covariance identity, cf. e.g. Proposition 3.4.1 in [10], p. 121.

Lemma 3.1 *For any $F, G \in \mathrm{Dom}(D)$ we have*

$$\mathrm{Cov}(F, G) = \mathrm{E}\left[\int_0^\infty \mathrm{E}[D_t F \,|\, \mathcal{F}_t] D_t G \, \mathrm{d}t\right]. \tag{3.3}$$

3.1 Integration by parts

We now implement the IBP formula (2.1) for functionals in the domain of D, based on the Clark-Ocone representation formula (3.1). Note that IBP formulas of the form (2.1) can also be obtained by the Ornstein-Uhlenbeck semigroup, cf. e.g. Proposition 2.1 of [3].

Proposition 3.2 *If $F, G \in \mathrm{Dom}(D)$ and $\phi'(F)\varphi_{F,G}(F) \in L^1(P)$ for any $\phi \in \mathcal{T}$, then the IBP formula* (2.1) *holds on $\mathcal{T}$ for F with $W_1 = G$ and $W_2 = \varphi_{F,G}(F)$, i.e.*

$$\mathrm{E}[\varphi_{F,G}(F)\phi'(F)] = \mathrm{Cov}(\phi(F), G)$$

where $\varphi_{F,G}$ *is the function*

$$\varphi_{F,G}(z) := \mathrm{E}\left[\int_0^\infty D_t F \, \mathrm{E}[D_t G \mid \mathcal{F}_t] \, \mathrm{d}t \,\middle|\, F = z\right], \qquad z \in \mathbb{R}. \tag{3.4}$$

Proof. By Lemma 3.1 and the properties of the gradient operator, for any $\phi \in \mathcal{T}$ and $F, G \in \mathrm{Dom}(D)$, we have

$$\begin{aligned}\mathrm{Cov}(\phi(F), G) &= \mathrm{E}\left[\int_0^\infty \mathrm{E}[D_t G \mid \mathcal{F}_t] \, D_t \phi(F) \, \mathrm{d}t\right] \\ &= \mathrm{E}\left[\phi'(F) \int_0^\infty D_t F \, \mathrm{E}[D_t G \mid \mathcal{F}_t] \, \mathrm{d}t\right] \\ &= \mathrm{E}\left[\mathrm{E}\left[\phi'(F) \int_0^\infty D_t F \, \mathrm{E}[D_t G \mid \mathcal{F}_t] \, \mathrm{d}t \,\middle|\, F\right]\right] \\ &= \mathrm{E}[\phi'(F) \varphi_{F,G}(F)]. \end{aligned} \tag{3.5}$$

□

3.2 Normal and gamma approximation

We now apply Theorems 2.1 and 2.2 using the Clark-Ocone formula (3.3). For any $F \in \mathrm{Dom}(D)$, we define

$$\varphi_F(z) := \varphi_{F,F}(z) = \mathrm{E}\left[\int_0^\infty D_t F \, \mathrm{E}[D_t F \mid \mathcal{F}_t] \, \mathrm{d}t \,\middle|\, F = z\right], \qquad z \in \mathbb{R} \tag{3.6}$$

and note that by Jensen's inequality

$$\|\varphi_F(F)\|_{L^1(P)} \le \|DF\|^2_{L^2(P \otimes \ell)} = \mathrm{E}\left[\int_0^\infty |D_t F|^2 \, \mathrm{d}t\right] < \infty, \qquad F \in \mathrm{Dom}(D).$$

The next proposition follows as a simple consequence of Theorems 2.1, 2.2, 2.3 and Proposition 3.2 and uses the definition (2.6) of the distance $d_{\mathcal{H}}$.

Proposition 3.3 *For any* $F \in \mathrm{Dom}(D)$ *such that* $\mathrm{E}[F] = 0$*, we have*

$$d_{TV}(F, \mathcal{N}) \le 2\mathrm{E}[|1 - \varphi_F(F)|]$$

and

$$d_W(F, \mathcal{N}) \le \mathrm{E}[|1 - \varphi_F(F)|],$$

where φ_F *is defined in* (3.6)*. If moreover* F *is a.s.* $(-\nu, \infty)$*-valued then we have*

$$d_{\mathcal{H}}(F \, , \, \Gamma_\nu) \le \mathrm{E}[|2(F + \nu) - \varphi_F(F)|].$$

Letting $\langle\cdot,\cdot\rangle$ denote the usual inner product on $L^2(\mathbb{R}_+)$, from Proposition 3.3 we also have the following corollary:

Corollary 3.4 *For any $F \in \mathrm{Dom}(D)$ such that $\mathrm{E}[F] = 0$, we have*

$$\begin{aligned} d_{TV}(F, \mathcal{N}) &\leq 2\|1 - \langle D_\cdot F, \mathrm{E}[D_\cdot F \mid \mathcal{F}_\cdot]\rangle\|_{L^2(P)} \\ &\leq 2|1 - \|F\|^2_{L^2(P)}| + 2\|\langle D_\cdot F, \mathrm{E}[D_\cdot F \mid \mathcal{F}_\cdot]\rangle - \mathrm{E}[\langle D_\cdot F, \mathrm{E}[D_\cdot F \mid \mathcal{F}_\cdot]\rangle]\|_{L^2(P)} \end{aligned}$$

and

$$\begin{aligned} d_W(F, \mathcal{N}) &\leq \|1 - \langle D_\cdot F, \mathrm{E}[D_\cdot F \mid \mathcal{F}_\cdot]\rangle\|_{L^2(P)} \\ &\leq |1 - \|F\|^2_{L^2(P)}| + \|\langle D_\cdot F, \mathrm{E}[D_\cdot F \mid \mathcal{F}_\cdot]\rangle - \mathrm{E}[\langle D_\cdot F, \mathrm{E}[D_\cdot F \mid \mathcal{F}_\cdot]\rangle]\|_{L^2(P)}. \end{aligned}$$

If moreover F is a.s. $(-\nu, \infty)$-valued then we have

$$\begin{aligned} d_{\mathcal{H}}(F, \Gamma_\nu) &\leq \|2(F+\nu) - \langle D_\cdot F, \mathrm{E}[D_\cdot F \mid \mathcal{F}_\cdot]\rangle\|_{L^2(P)} \\ &\leq \|2(F+\nu) - \|F\|^2_{L^2(P)}\|_{L^2(P)} + \|\langle D_\cdot F, \mathrm{E}[D_\cdot F \mid \mathcal{F}_\cdot]\rangle - \mathrm{E}[\langle D_\cdot F, \mathrm{E}[D_\cdot F \mid \mathcal{F}_\cdot]\rangle]\|_{L^2(P)}. \end{aligned}$$

Proof. The first inequality follows by Proposition 3.3 and the Cauchy-Schwarz inequality. The second inequality follows by the triangle inequality noticing that by the Itô isometry and the Clark-Ocone formula we have

$$\begin{aligned} \mathrm{E}[\langle D_\cdot F, \mathrm{E}[D_\cdot F \mid \mathcal{F}_\cdot]\rangle] &= \mathrm{E}\left[\int_0^\infty D_t F\, \mathrm{E}[D_t F \,|\, \mathcal{F}_t]\, \mathrm{d}t\right] \\ &= \mathrm{E}\left[\int_0^\infty \mathrm{E}[D_t F \,|\, \mathcal{F}_t]^2\, \mathrm{d}t\right] \\ &= \mathrm{E}\left[\left(\int_0^\infty \mathrm{E}[D_t F \,|\, \mathcal{F}_t]\, \mathrm{d}M_t\right)^2\right] \\ &= \|F\|^2_{L^2(P)}. \end{aligned}$$

The counterpart of this statement for the Wasserstein and $d_{\mathcal{H}}$ distances is proved similarly.

□

In this work our main focus will be on the Poisson space, and in Section 4, we shall compare the upper bound on the Wasserstein distance with the bound obtained in [7] on the Poisson space.

4 Analysis on the Poisson space

In this section we apply the results of Section 3 to functionals of a standard Poisson process $(N_t)_{t\geq 0}$ with jump times $(T_k)_{k\geq 1}$ defined on an underlying probability space $(\Omega, \mathcal{F}, P)$. We let

$$\mathcal{S} = \{F = f(T_1, \dots, T_d) \ : \ d \geq 1, \ f \in \mathcal{C}_p^1(\mathbb{R}_+^d)\},$$

where $\mathcal{C}_p^1(\mathbb{R}_+^d)$ denotes the space of continuously differentiable functions such that f and its partial derivatives have polynomial growth, i.e. for any $i \in \{0, 1, \dots, d\}$ there exist $\alpha_j^{(i)} \geq 0$, $j = 1, \dots, d$, such that

$$\sup_{(x_1,\dots,x_d)\in\mathbb{R}_+^d} |x_1^{-\alpha_1^{(i)}} \dots x_d^{-\alpha_d^{(i)}} \partial_i f(x_1, \dots, x_d)| < \infty,$$

where $\partial_0 f := f$. Given $F = f(T_1, \dots, T_d) \in \mathcal{S}$, we consider the gradient on the Poisson space defined as

$$D_t F = -\sum_{k=1}^{d} \mathbb{1}_{[0,T_k]}(t) \partial_k f(T_1, \dots, T_d), \quad t \geq 0 \tag{4.1}$$

(see e.g. Definition 7.2.1 in [10] p. 256). We recall that the gradient operator

$$D : \mathcal{S} \subset L^2(\Omega, \mathcal{F}, P) \longrightarrow L^2(\Omega \times \mathbb{R}_+, \mathcal{F} \otimes \mathcal{B}(\mathbb{R}_+), P \otimes \ell)$$

is closable (see [10] p. 259). We shall continue to denote by D its minimal closed extension, whose domain $\mathrm{Dom}(D)$ coincides with the completion of $\mathcal{S}$ with respect to the norm

$$\|F\|_{1,2} = \|F\|_{L^2(P)} + \|DF\|_{L^2(P\otimes\ell)}.$$

By Proposition 7.2.8 in [10] p. 262 the operator D satisfies the Clark-Ocone representation formula, i.e. for any $F \in \mathrm{Dom}(D)$ we have

$$F = \mathrm{E}[F] + \int_0^\infty \mathrm{E}[D_t F \,|\, \mathcal{F}_t]\, (\mathrm{d}N_t - \mathrm{d}t), \tag{4.2}$$

where $(\mathcal{F}_t)_{t\geq 0}$ is the filtration generated by $(N_t)_{t\geq 0}$. We note that the gradient D satisfies the chain rule on the set $\mathcal{T}$ of real-valued functions which have polynomial growth and are continuously differentiable with bounded derivative, i.e. for any $g \in \mathcal{T}$ and $F \in \mathrm{Dom}(D)$ we have $g(F) \in \mathrm{Dom}(D)$ and $Dg(F) = g'(F)DF$, cf. Lemma 6.1 in the Appendix.

Before turning to some concrete examples of Poisson functionals we note that identifying the process $\mathrm{E}[D_t F \,|\, \mathcal{F}_t]$ in (4.2) amounts to finding the predictable representation of the random variable F. For example if $F = X_T$ is the terminal value of the solution $(X_t)_{t\in[0,T]}$ to the stochastic differential equation

$$X_t = X_0 + \int_0^t \sigma(s, X_{s^-})(\mathrm{d}N_s - \mathrm{d}s), \tag{4.3}$$

where $\sigma : [0,T]\times\Omega \longrightarrow \mathbb{R}$ is a measurable function, then we immediately have $\mathrm{E}[D_t F \,|\, \mathcal{F}_t] = \sigma(t, X_{t^-})$ and Corollary 3.4 shows that e.g.

$$\begin{aligned} d_{TV}(X_T, \mathcal{N}) &\leq 2 \left\| 1 - \int_0^T \sigma(t, X_t) D_t X_T \,\mathrm{d}t \right\|_{L^2(P)} \\ &\leq 2 \left| 1 - \|X_T\|^2_{L^2(P)} \right| + 2 \left\| \int_0^T (\sigma(t, X_t) D_t X_T - \mathrm{E}[\sigma(t, X_t) D_t X_T]) \,\mathrm{d}t \right\|_{L^2(P)}, \end{aligned}$$

provided the terminal value X_T belongs to $\mathrm{Dom}(D)$ and $\mathrm{E}[X_0] = 0$. In particular, the domain condition can be achieved under a usual Lipschitz condition on $\sigma(\cdot, x)$, $x \in \mathbb{R}$, and a usual sub-linear growth condition on $\sigma(t, \cdot)$, $t \in [0,T]$. We refer the reader to e.g. Proposition 3.2 of [4] for an explicit solution of (4.3) which is suitable for D-differentiation when $\sigma(t,x)$ vanishes at $t = T$, for any $x \in \mathbb{R}$.

4.1 Approximation of Poisson jump times functionals

Proposition 4.1 *Let $f \in \mathcal{C}_p^1(\mathbb{R}_+)$ be such that $\mathrm{E}[f(T_k)] = 0$, $k \geq 1$. Then*

$$d_{TV}(f(T_k), \mathcal{N}) \leq 2 \left\| 1 - f'(T_k) \int_0^{T_k} \mathrm{E}[f'(T_{k-h} + t)]_{h=N_t} \mathrm{d}t \right\|_{L^1(P)} \tag{4.4}$$

and

$$d_W(f(T_k), \mathcal{N}) \leq \left\| 1 - f'(T_k) \int_0^{T_k} \mathrm{E}[f'(T_{k-h} + t)]_{h=N_t} \mathrm{d}t \right\|_{L^1(P)}. \tag{4.5}$$

If moreover $f(T_k) > -\nu$ a.s. we have

$$d_{\mathcal{H}}(f(T_k)\,,\, \Gamma_\nu) \leq \left\| 2(f(T_k) + \nu) - f'(T_k) \int_0^{T_k} \mathrm{E}[f'(T_{k-h} + t)]_{h=N_t} \mathrm{d}t \right\|_{L^1(P)}. \tag{4.6}$$

Proof. We have $f(T_k) \in \mathrm{Dom}(D)$ and $D_t f(T_k) = f_k'(T_k) D_t T_k = -f_k'(T_k) \mathbb{1}_{[0,T_k]}(t)$, $t \geq 0$. By the formula in [10] p. 261 we have

$$\mathrm{E}[D_t f(T_k) \,|\, \mathcal{F}_t] = -\int_t^\infty f'(x) p_{k-1-N_t}(x - t) \,\mathrm{d}x,$$

where $p_k(t) = P(N_t = k)$. So

$$\varphi_{f(T_k)}(f(T_k)) = \mathrm{E}\left[\int_0^\infty D_t f(T_k) \mathrm{E}[D_t f(T_k) \,|\, \mathcal{F}_t] \,\mathrm{d}t \,\middle|\, f(T_k) \right]$$

$$
\begin{aligned}
&= -\mathrm{E}\left[f'(T_k) \int_0^{T_k} \mathrm{E}[D_t f(T_k) \,|\, \mathcal{F}_t] \,\mathrm{d}t \,\middle|\, f(T_k) \right] \\
&= \mathrm{E}\left[f'(T_k) \int_0^{T_k} \int_t^{\infty} f'(x) p_{k-1-N_t}(x-t) \,\mathrm{d}x\mathrm{d}t \,\middle|\, f(T_k) \right] \\
&= \mathrm{E}\left[f'(T_k) \int_0^{T_k} \int_0^{\infty} f'(x+t) p_{k-1-N_t}(x) \,\mathrm{d}x\mathrm{d}t \,\middle|\, f(T_k) \right] \\
&= \mathrm{E}\left[f'(T_k) \int_0^{T_k} \mathrm{E}[f'(T_{k-h}+t)]_{h=N_t} \mathrm{d}t \,\middle|\, f(T_k) \right]. \qquad (4.7)
\end{aligned}
$$

Finally, by Proposition 3.3 we deduce

$$
\begin{aligned}
d_{TV}(f(T_k), \mathcal{N}) &\leq 2\mathrm{E}[|1 - \varphi_{f(T_k)}(f(T_k))|] \\
&= 2\mathrm{E}\left[\left| \mathrm{E}\left[1 - f'(T_k) \int_0^{T_k} \mathrm{E}[f'(T_{k-h}+t)]_{h=N_t} \mathrm{d}t \,\middle|\, f(T_k) \right] \right| \right] \\
&\leq 2 \left\| 1 - f'(T_k) \int_0^{T_k} \mathrm{E}[f'(T_{k-h}+t)]_{h=N_t} \mathrm{d}t \right\|_{L^1(P)}.
\end{aligned}
$$

The inequalities concerning d_W and $d_{\mathcal{H}}$ can be proved similarly. □

Example - Linear Poisson jump times functionals

Proposition 4.1 can be applied to linear functionals of Poisson jump times. Consider first the normal approximation. Take e.g. $f(x) = (x-k)/\sqrt{k}$, i.e. $f(T_k) = (T_k - k)/\sqrt{k}$, $k \geq 1$, and note that T_k/k is gamma distributed with mean 1 and variance $1/k$. All hypotheses of Proposition 4.1 are satisfied and we have

$$
\left\| 1 - f'(T_k) \int_0^{T_k} \mathrm{E}[f'(T_{k-h}+t)]_{h=N_t} \mathrm{d}t \right\|_{L^1(P)} = \left\| \frac{T_k}{k} - 1 \right\|_{L^1(P)} \leq \sqrt{\mathrm{Var}(T_k/k)} = \frac{1}{\sqrt{k}},
$$

where the latter inequality follows by the Cauchy-Schwarz inequality. So (4.4) and (4.5) recovers the classical Berry-Esséen bound. For the gamma approximation we take e.g. $\nu := 2k$ and $f(x) = 2(x-k)$, $k \geq 1$. In such a case Γ_ν has the same law of $f(T_k)$ and we check that $d_{\mathcal{H}}(f(T_k), \Gamma_\nu) = 0$. Indeed,

$$
\left\| 2(f(T_k)+\nu) - f'(T_k) \int_0^{T_k} \mathrm{E}[f'(T_{k-h}+t)]_{h=N_t} \mathrm{d}t \right\|_{L^1(P)} = \left\| 4T_k - 4T_k \right\|_{L^1(P)} = 0.
$$

Example - Quadratic Poisson jump times functionals

Proposition 4.1 can also be applied to quadratic functionals of Poisson jump times. Consider first the normal approximation, take e.g.

$$
f(x) = \alpha x^2 - \beta, \quad \text{with} \quad \alpha = \frac{1}{2k^{3/2}} \quad \text{and} \quad \beta = -\frac{k+1}{2\sqrt{k}}, \qquad k \geq 1.
$$

Recall that if X is gamma distributed with parameters a and b, then $\mathrm{E}[X^k] = (a+k-1)(a+k-2)\cdots(a+1)a/b^k$, $k \geq 1$. One easily sees that all the assumptions of Proposition 4.1 are satisfied, and we find

$$\begin{aligned} & \left\| 1 - f'(T_k) \int_0^{T_k} \mathrm{E}[f'(T_{k-h}+t)]_{h=N_t} \mathrm{d}t \right\|_{L^1(P)} \qquad (4.8) \\ \leq \ & \frac{1}{k} + \sqrt{\frac{4}{k} + \frac{10}{k^2} + \frac{6}{k^3}} + \frac{1}{2}\sqrt{1+\frac{1}{k}} \left(\sqrt{\frac{4}{3k} - \frac{2}{k^2} + \frac{2}{3k^3}} + \sqrt{\frac{4}{k} + \frac{10}{k^2} + \frac{6}{k^3}} + \frac{2}{k} \right), \end{aligned}$$

cf. the Appendix. Note that this upper bound is asymptotically equivalent to $(3 + \frac{1}{\sqrt{3}})/\sqrt{k}$ as $k \to \infty$, and so we recover the Berry-Esséen bound.

4.2 Approximation of multiple Poisson stochastic integrals

We present some applications of Corollary 3.4 to Poisson functionals. For $n \geq 1$, we denote by

$$I_n(f_n) = n! \int_0^\infty \int_0^{t_n^-} \cdots \int_0^{t_2^-} f_n(t_1, \ldots, t_n)\, (\mathrm{d}N_{t_1} - \mathrm{d}t_1) \cdots (\mathrm{d}N_{t_n} - \mathrm{d}t_n)$$

the multiple Poisson stochastic integral of the symmetric function $f_n \in L^2(\mathbb{R}_+^n)$ with $I_n(f_n) = I_n(\tilde{f}_n)$ when f_n is not symmetric, where $\tilde{f}_n$ denotes the symmetrization of f_n in n variables (see e.g. Section 6.2 in [10]). As a convention we identify $L^2(\mathbb{R}_+^0)$ with $\mathbb{R}$, and let

$$I_0(f_0) = f_0, \qquad f_0 \in L^2(\mathbb{R}_+^0).$$

Moreover, we shall adopt the usual convention $\sum_{k=i}^{j} = 0$ if $i > j$. Let the space $S_n^{1,2}$ of weakly differentiable functions be defined as the completion of the symmetric functions $f_n \in \mathcal{C}_c^1([0,\infty)^n)$ under the norm

$$\begin{aligned} \|f_n\|_{1,2} &= \|f_n\|_{L^2(\mathbb{R}_+^n)} + \sqrt{\int_0^\infty \cdots \int_0^\infty \int_t^\infty |\partial_1 f_n(s_1, \ldots, s_n)|^2 \, \mathrm{d}s_1 \mathrm{d}t \mathrm{d}s_2 \cdots \mathrm{d}s_n} \\ &= \|f_n\|_{L^2(\mathbb{R}_+^n)} + \sqrt{\int_0^\infty \cdots \int_0^\infty \int_0^\infty |\partial_1 f_{n[t}(s_1, \ldots, s_n)|^2 \, \mathrm{d}s_1 \mathrm{d}t \mathrm{d}s_2 \cdots \mathrm{d}s_n} \qquad (4.9) \end{aligned}$$

where $\partial_i f_{n[t}(s_1, \ldots, s_n) = \partial_i f_n(s_1, \ldots, s_n) \mathbb{1}_{[t,\infty)}(s_i)$. The next lemma is proved in the Appendix, cf. Proposition 8 of [9] or Proposition 7.7.2 page 279 of [10].

Lemma 4.2 *For any function $f_n \in S_n^{1,2}$ symmetric in its n variables we have $I_n(f_n) \in$* $\mathrm{Dom}(D)$ *with*

$$D_t I_n(f_n) = n I_{n-1}(f_n(*, t)) - n I_n(\partial_1 f_{n[t}), \qquad t \in \mathbb{R}_+, \qquad (4.10)$$

and

$$\|DI_n(f_n)\|^2_{L^2(P\otimes\ell)} = n^2(n-1)!\int_0^\infty\cdots\int_0^\infty |f_n(t_1,\ldots,t_n)|^2\,\mathrm{d}t_1\cdots\mathrm{d}t_n + n^2 n!\int_0^\infty\cdots\int_0^\infty\int_t^\infty |\partial_1 f_n(t_1,\ldots,t_n)|^2\,\mathrm{d}t_1\mathrm{d}t\mathrm{d}t_2\cdots\mathrm{d}t_n.$$

We recall the multiplication formula for multiple Poisson stochastic integrals, cf. e.g. Proposition 4.5.6 of [10]. For symmetric functions $f_n \in L^2(\mathbb{R}_+^n)$ and $g_m \in L^2(\mathbb{R}_+^m)$, we define $f_n \otimes_k^l g_m$, $0 \le l \le k$, to be the function

$$(x_{l+1},\ldots,x_n,y_{k+1},\ldots,y_m) \mapsto \int_{\mathbb{R}_+^l} f_n(x_1,\ldots,x_n)g_m(x_1,\ldots,x_k,y_{k+1},\ldots,y_m)\,\mathrm{d}x_1\cdots\mathrm{d}x_l$$

of $n+m-k-l$ variables. We denote by $f_n \circ_k^l g_m$ the symmetrization in $n+m-k-l$ variables of $f_n \otimes_k^l g_m$, $0 \le l \le k$. Note that if $k=l$ then $\otimes_k := \otimes_k^k$ is the classical L^2 contraction over k variables and $\circ_k := \circ_k^k$ is the canonical symmetrization of $\otimes_k$. We have

$$I_n(f_n)I_m(g_m) = \sum_{k=0}^{2(n\wedge m)} I_{n+m-k}(h_{n,m,k})$$

if the functions

$$h_{n,m,k} = \sum_{k\le 2i\le 2(k\wedge n\wedge m)} i!\binom{n}{i}\binom{m}{i}\binom{i}{k-i} f_n \circ_i^{k-i} g_m$$

belong to $L^2(\mathbb{R}_+^{n+m-k})$, $0 \le k \le 2(n\wedge m)$. In particular, letting $\mathbb{1}_{\{(t_1,\ldots,t_n)<t\}}$ denote the function $\mathbb{1}_{[0,t]^n}(t_1,\ldots,t_n)$, for any symmetric function $f_n \in S_n^{1,2}$, we have

$$I_{n-1}(f_n(*,t))I_{n-1}(f_n(*,t)\mathbb{1}_{\{*<t\}}) = \sum_{k=0}^{2n-2} I_{2n-2-k}(g^{(1,t)}_{n-1,n-1,k}) \tag{4.11}$$

if the functions

$$g^{(1,t)}_{n-1,n-1,k} = \sum_{k\le 2i\le 2(k\wedge(n-1))} i!\left|\binom{n-1}{i}\right|^2\binom{i}{k-i} f_n(*,t)\circ_i^{k-i} f_n(*,t)\mathbb{1}_{\{*<t\}} \tag{4.12}$$

belong to $L^2(\mathbb{R}_+^{2n-2-k})$, $0 \le k \le 2n-2$, and

$$I_n(\partial_1 f_{n[t})I_{n-1}(f_n(*,t)\mathbb{1}_{\{*<t\}}) = \sum_{k=0}^{2n-2} I_{2n-1-k}(g^{(2,t)}_{n,n-1,k}) \tag{4.13}$$

if the functions

$$g^{(2,t)}_{n,n-1,k} = \sum_{k\le 2i\le 2(k\wedge(n-1))} i!\binom{n}{i}\binom{n-1}{i}\binom{i}{k-i}\partial_1 f_{n[t}\circ_i^{k-i} f_n(*,t)\mathbb{1}_{\{*<t\}}$$

belong to $L^2(\mathbb{R}_+^{2n-1-k})$, $0 \le k \le 2n-2$. Part (2) of the next proposition proposes an alternative to the Gamma bound of Theorem 2.6 of [8].

Proposition 4.3 1) *For any symmetric function* $f_n \in S_n^{1,2}$ *such that*

$$g^{(1,t)}_{n-1,n-1,k} \in L^2(\mathbb{R}_+^{2n-2-k}), \quad 0 \leq k \leq 2n-2$$

and

$$g^{(2,t)}_{n,n-1,k} \in L^2(\mathbb{R}_+^{2n-1-k}), \quad 0 \leq k \leq 2n-2,$$

we have

$$\begin{aligned} d_{TV}(I_n(f_n), \mathcal{N}) &\leq 2|1 - n!\|f_n\|^2_{L^2(\mathbb{R}_+^n)}| \\ &+ 2n^2 \left(\sum_{k=0}^{2n-3} (2n-2-k)! \int_{(0,\infty)^2} \langle g^{(1,t)}_{n-1,n-1,k} - g^{(2,t)}_{n,n-1,k+1}, g^{(1,s)}_{n-1,n-1,k} - g^{(2,s)}_{n,n-1,k+1} \rangle_{L^2(\mathbb{R}_+^{2n-2-k})} \, \mathrm{d}s\mathrm{d}t \right. \\ &\left. + \int_0^\infty \int_0^\infty (2n-1)! \langle g^{(2,t)}_{n,n-1,0}, g^{(2,s)}_{n,n-1,0} \rangle_{L^2(\mathbb{R}_+^{2n-1})} \, \mathrm{d}s\mathrm{d}t \right)^{1/2} \end{aligned}$$

and

$$\begin{aligned} d_{W}(I_n(f_n), \mathcal{N}) &\leq |1 - n!\|f_n\|^2_{L^2(\mathbb{R}_+^n)}| \\ &+ n^2 \left(\sum_{k=0}^{2n-3} (2n-2-k)! \int_{(0,\infty)^2} \langle g^{(1,t)}_{n-1,n-1,k} - g^{(2,t)}_{n,n-1,k+1}, g^{(1,s)}_{n-1,n-1,k} - g^{(2,s)}_{n,n-1,k+1} \rangle_{L^2(\mathbb{R}_+^{2n-2-k})} \, \mathrm{d}s\mathrm{d}t \right. \\ &\left. + \int_0^\infty \int_0^\infty (2n-1)! \langle g^{(2,t)}_{n,n-1,0}, g^{(2,s)}_{n,n-1,0} \rangle_{L^2(\mathbb{R}_+^{2n-1})} \, \mathrm{d}s\mathrm{d}t \right)^{1/2}. \end{aligned}$$

2) *If moreover* $I_n(f_n)$ *is a.s.* $(-\nu, \infty)$*-valued then we have*

$$\begin{aligned} d_{\mathcal{H}}(I_n(f_n), \Gamma_\nu) &\leq \sqrt{4\nu^2 + n!\|f_n\|^4_{L^2(\mathbb{R}_+^n)} + 4n!(1-\nu)\|f_n\|^2_{L^2(\mathbb{R}_+^n)}} \\ &+ n^2 \left(\sum_{k=0}^{2n-3} (2n-2-k)! \int_{(0,\infty)^2} \langle g^{(1,t)}_{n-1,n-1,k} - g^{(2,t)}_{n,n-1,k+1}, g^{(1,s)}_{n-1,n-1,k} - g^{(2,s)}_{n,n-1,k+1} \rangle_{L^2(\mathbb{R}_+^{2n-2-k})} \, \mathrm{d}s\mathrm{d}t \right. \\ &\left. + \int_0^\infty \int_0^\infty (2n-1)! \langle g^{(2,t)}_{n,n-1,0}, g^{(2,s)}_{n,n-1,0} \rangle_{L^2(\mathbb{R}_+^{2n-1})} \, \mathrm{d}s\mathrm{d}t \right)^{1/2}. \end{aligned}$$

Proof. By Lemma 4.2 we have $I_n(f_n) \in \mathrm{Dom}(D)$ and

$$D_t I_n(f_n) = n I_{n-1}(f_n(*, t)) - n I_n(\partial_1 f_{n[t}), \qquad t \in \mathbb{R}_+.$$

So by Lemma 2.7.2 p. 88 of [10] and the definition of $\partial_1 f_{n[t}$, we have

$$\mathrm{E}[D_t I_n(f_n) \,|\, \mathcal{F}_t] = n I_{n-1}(f_n(*, t) \mathbb{1}_{\{* < t\}}) - n I_n(\partial_1 f_{n[t} \mathbb{1}_{[0,t]^n}) = n I_{n-1}(f_n(*, t) \mathbb{1}_{\{* < t\}}), \quad (4.14)$$

since $f_n(*,t)\mathbb{1}_{\{*<t\}} = 0$, $t \in \mathbb{R}_+$. Combining this with the multiplication formulas (4.11) and (4.13) for multiple Poisson stochastic integrals, we deduce

$$
\begin{aligned}
&\langle D.I_n(f_n), \mathrm{E}[D.I_n(f_n) \mid \mathcal{F}.]\rangle \\
&= n^2 \int_0^\infty I_{n-1}(f_n(*,t)) I_{n-1}(f_n(*,t)\mathbb{1}_{\{*<t\}})\,\mathrm{d}t - n^2 \int_0^\infty I_n(\partial_1 f_{n[t]} I_{n-1}(f_n(*,t)\mathbb{1}_{\{*<t\}})\,\mathrm{d}t \\
&= n^2 \sum_{k=0}^{2n-2} \int_0^\infty I_{2n-2-k}(g^{(1,t)}_{n-1,n-1,k})\,\mathrm{d}t - n^2 \sum_{k=0}^{2n-2} \int_0^\infty I_{2n-1-k}(g^{(2,t)}_{n,n-1,k})\,\mathrm{d}t \\
&= n^2 \sum_{k=0}^{2n-2} \int_0^\infty I_{2n-2-k}(g^{(1,t)}_{n-1,n-1,k})\,\mathrm{d}t - n^2 \sum_{k=1}^{2n-2} \int_0^\infty I_{2n-1-k}(g^{(2,t)}_{n,n-1,k})\,\mathrm{d}t \\
&\quad - n^2 \int_0^\infty I_{2n-1}(g^{(2,t)}_{n,n-1,0})\,\mathrm{d}t \\
&= n^2 \sum_{k=0}^{2n-2} \int_0^\infty I_{2n-2-k}(g^{(1,t)}_{n-1,n-1,k})\,\mathrm{d}t - n^2 \sum_{s=0}^{2n-3} \int_0^\infty I_{2n-2-s}(g^{(2,t)}_{n,n-1,s+1})\,\mathrm{d}t \\
&\quad - n^2 \int_0^\infty I_{2n-1}(g^{(2,t)}_{n,n-1,0})\,\mathrm{d}t \\
&= n^2 \int_0^\infty I_0(g^{(1,t)}_{n-1,n-1,2n-2})\,\mathrm{d}t + n^2 \sum_{k=0}^{2n-3} \int_0^\infty I_{2n-2-k}(g^{(1,t)}_{n-1,n-1,k} - g^{(2,t)}_{n,n-1,k+1})\,\mathrm{d}t \\
&\quad - n^2 \int_0^\infty I_{2n-1}(g^{(2,t)}_{n,n-1,0})\,\mathrm{d}t \\
&= n^2 (n-1)! \int_0^\infty f_n(*,t) \circ^{n-1}_{n-1} f_n(*,t)\mathbb{1}_{\{*<t\}}\,\mathrm{d}t \\
&\quad + n^2 \sum_{k=0}^{2n-3} \int_0^\infty I_{2n-2-k}(g^{(1,t)}_{n-1,n-1,k} - g^{(2,t)}_{n,n-1,k+1})\,\mathrm{d}t - n^2 \int_0^\infty I_{2n-1}(g^{(2,t)}_{n,n-1,0})\,\mathrm{d}t \\
&= n^2 (n-1)! \int_0^\infty \int_{[0,t]^{n-1}} f_n(t_1, \ldots t_{n-1}, t) f_n(t_1, \ldots t_{n-1}, t)\,\mathrm{d}t_1 \cdots \mathrm{d}t_{n-1}\mathrm{d}t \\
&\quad + n^2 \sum_{k=0}^{2n-3} \int_0^\infty I_{2n-2-k}(g^{(1,t)}_{n-1,n-1,k} - g^{(2,t)}_{n,n-1,k+1})\,\mathrm{d}t - n^2 \int_0^\infty I_{2n-1}(g^{(2,t)}_{n,n-1,0})\,\mathrm{d}t.
\end{aligned}
$$

Using the first equality above and the isometry formula for multiple Poisson stochastic integrals (see Proposition 2.7.1 p. 87 in [10]) we have

$$
\begin{aligned}
&\mathrm{E}[\langle D.I_n(f_n), \mathrm{E}[D.I_n(f_n) \mid \mathcal{F}.]\rangle] \\
&= n^2 (n-1)! \int_0^\infty \int_{[0,t]^{n-1}} f_n(t_1, \ldots t_{n-1}, t) f_n(t_1, \ldots t_{n-1}, t)\,\mathrm{d}t_1 \cdots \mathrm{d}t_{n-1}\mathrm{d}t.
\end{aligned}
$$

Hence

$$
\langle DI_n(f_n), \mathrm{E}[DI_n(f_n) \mid \mathcal{F}.]\rangle - \mathrm{E}[\langle DI_n(f_n), \mathbb{E}[DI_n(f_n) \mid \mathcal{F}.]\rangle]
$$

$$= \quad n^2 \sum_{k=0}^{2n-3} \int_0^\infty I_{2n-2-k}(g^{(1,t)}_{n-1,n-1,k} - g^{(2,t)}_{n,n-1,k+1})\,\mathrm{d}t - n^2 \int_0^\infty I_{2n-1}(g^{(2,t)}_{n,n-1,0})\,\mathrm{d}t.$$

We conclude by Corollary 3.4, noticing that $I_n(f_n)$ is a centered random variable (see [10] pp. 87-88), $\|I_n(f_n)\|^2_{L^2(P)} = n!\|f_n\|^2_{L^2(\mathbb{R}^n_+)}$, for any $\nu > 0$

$$\|2(I_n(f_n)+\nu) - n!\|f_n\|^2_{L^2(\mathbb{R}^n_+)}\|^2_{L^2(P)} = n!\|f_n\|^4_{L^2(\mathbb{R}^n_+)} + 4n!(1-\nu)\|f_n\|^2_{L^2(\mathbb{R}^n_+)} + 4\nu^2,$$

and

$$\begin{aligned}
&\| \sum_{k=0}^{2n-3} \int_0^\infty I_{2n-2-k}(g^{(1,t)}_{n-1,n-1,k} - g^{(2,t)}_{n,n-1,k+1})\,\mathrm{d}t - \int_0^\infty I_{2n-1}(g^{(2,t)}_{n,n-1,0})\,\mathrm{d}t\|^2_{L^2(P)} \\
&= \mathrm{E}\left[\left(\sum_{k=0}^{2n-3} \int_0^\infty I_{2n-2-k}(g^{(1,t)}_{n-1,n-1,k} - g^{(2,t)}_{n,n-1,k+1})\,\mathrm{d}t\right)^2\right] \\
&\quad +\mathrm{E}\left[\left(\int_0^\infty I_{2n-1}(g^{(2,t)}_{n,n-1,0})\,\mathrm{d}t\right)^2\right] \\
&\quad -2\sum_{k=0}^{2n-3} \mathrm{E}\left[\int_0^\infty \int_0^\infty I_{2n-2-k}(g^{(1,t)}_{n-1,n-1,k} - g^{(2,t)}_{n,n-1,k+1}) I_{2n-1}(g^{(2,s)}_{n,n-1,0})\,\mathrm{d}s\,\mathrm{d}t\right] \\
&= \mathrm{E}\left[\left(\sum_{k=0}^{2n-3} \int_0^\infty I_{2n-2-k}(g^{(1,t)}_{n-1,n-1,k} - g^{(2,t)}_{n,n-1,k+1})\,\mathrm{d}t\right)^2\right] \\
&\quad +\mathrm{E}\left[\left(\int_0^\infty I_{2n-1}(g^{(2,t)}_{n,n-1,0})\,\mathrm{d}t\right)^2\right] \\
&= \sum_{k=0}^{2n-3} \mathrm{E}\left[\left(\int_0^\infty I_{2n-2-k}(g^{(1,t)}_{n-1,n-1,k} - g^{(2,t)}_{n,n-1,k+1})\,\mathrm{d}t\right)^2\right] \\
&\quad +\mathrm{E}\left[\left(\int_0^\infty I_{2n-1}(g^{(2,t)}_{n,n-1,0})\,\mathrm{d}t\right)^2\right] \\
&= \sum_{k=0}^{2n-3} \mathrm{E}\left[\int_0^\infty \int_0^\infty I_{2n-2-k}(g^{(1,t)}_{n-1,n-1,k} - g^{(2,t)}_{n,n-1,k+1}) I_{2n-2-k}(g^{(1,s)}_{n-1,n-1,k} - g^{(2,s)}_{n,n-1,k+1})\,\mathrm{d}s\mathrm{d}t\right] \\
&\quad +\mathrm{E}\left[\int_0^\infty \int_0^\infty I_{2n-1}(g^{(2,t)}_{n,n-1,0}) I_{2n-1}(g^{(2,s)}_{n,n-1,0})\,\mathrm{d}s\mathrm{d}t\right] \\
&= \sum_{k=0}^{2n-3} (2n-2-k)! \int_0^\infty \int_0^\infty \langle g^{(1,t)}_{n-1,n-1,k} - g^{(2,t)}_{n,n-1,k+1}, g^{(1,s)}_{n-1,n-1,k} - g^{(2,s)}_{n,n-1,k+1}\rangle_{L^2(\mathbb{R}^{2n-2-k}_+)}\,\mathrm{d}s\mathrm{d}t \\
&\quad +(2n-1)! \int_0^\infty \int_0^\infty \langle g^{(2,t)}_{n,n-1,0}, g^{(2,s)}_{n,n-1,0}\rangle_{L^2(\mathbb{R}^{2n-1}_+)}\,\mathrm{d}s\mathrm{d}t.
\end{aligned}$$

□

Single Poisson stochastic integrals

In the particular case $n = 1$, the space $S_1^{1,2}$ is the completion of $\mathcal{C}_c^1([0,\infty))$ under the norm

$$\|f\|_{1,2} = \|f\|_{L^2(\mathbb{R}_+)} + \|\|f'_{[\cdot}\|_{L^2(\mathbb{R}_+)}\|_{L^2(\mathbb{R}_+)} := \sqrt{\int_0^\infty |f(t)|^2\,\mathrm{d}t} + \sqrt{\int_0^\infty \int_t^\infty |f'(s)|^2\,\mathrm{d}s\mathrm{d}t}$$

$$= \sqrt{\int_0^\infty |f(t)|^2\,\mathrm{d}t} + \sqrt{\int_0^\infty s|f'(s)|^2\,\mathrm{d}s} \tag{4.15}$$

where $f'_{[t}(s) := f'(s)\mathbb{1}_{[t,\infty)}(s)$ and we have $I_1(f) \in \mathrm{Dom}(D)$ with

$$D_t I_1(f) = f(t) - I_1(\mathbf{1}_{[t,\infty)}f'), \qquad t \in \mathbb{R}_+,$$

and

$$\|DI_1(f)\|^2_{L^2(P\otimes\ell)} = \int_0^\infty |f(t)|^2\,\mathrm{d}t + \int_0^\infty \int_t^\infty |f'(s)|^2\,\mathrm{d}s\mathrm{d}t.$$

The following result is a simple consequence of Proposition 4.3 for $n = 1$.

Corollary 4.4 *For any $f \in S_1^{1,2}$, we have*

$$d_{TV}(I_1(f), \mathcal{N}) \le 2|1 - \|f\|^2_{L^2(\mathbb{R}_+)}| + 2\sqrt{\int_0^\infty |f'(t)|^2 \left(\int_0^t f(z)\,\mathrm{d}z\right)^2 \mathrm{d}t}$$

and

$$d_W(I_1(f), \mathcal{N}) \le |1 - \|f\|^2_{L^2(\mathbb{R}_+)}| + \sqrt{\int_0^\infty |f'(t)|^2 \left(\int_0^t f(z)\,\mathrm{d}z\right)^2 \mathrm{d}t}.$$

If moreover $I_1(f)$ is a.s. $(-\nu,\infty)$-valued then we have

$$d_{\mathcal{H}}(I_1(f), \Gamma_\nu) \le \sqrt{4\nu^2 + \|f\|^4_{L^2(\mathbb{R}_+)} + 4(1-\nu)\|f\|^2_{L^2(\mathbb{R}_+)}} + \sqrt{\int_0^\infty |f'(t)|^2 \left(\int_0^t f(z)\,\mathrm{d}z\right)^2 \mathrm{d}t}.$$

Note that Corollary 3.4 of [7] states that

$$d_W(I_1(f), \mathcal{N}) \le |1 - \|f\|^2_{L^2(\mathbb{R}_+)}| + \|f\|^3_{L^3(\mathbb{R}_+)},$$

for any $f \in L^2(\mathbb{R}_+)$, which shows that

$$I_1(f_k) \longrightarrow \mathcal{N} \quad \text{in law}$$

provided $\|f_k\|_{L^2(\mathbb{R}_+)} \longrightarrow 1$ and $\|f_k\|_{L^3(\mathbb{R}_+)} \longrightarrow 0$ as k goes to infinity. Next we consider a couple of examples for comparison with Corollary 4.4.

Example - Single Poisson stochastic integrals with specific kernels

1. Take $g_k(t) = (2/k)^{1/2}\mathrm{e}^{-t/k}$ $t \geq 0$, $k \geq 1$. We shall show later on that $g_k \in S_1^{1,2}$. We have $\|g_k\|_{L^2(\mathbb{R}_+)} = 1$ and

$$\left(\int_0^\infty |g_k'(t)|^2 \left(\int_0^t g_k(z)\,\mathrm{d}z\right)^2 \mathrm{d}t\right)^{1/2} = \frac{2}{k}\left(\int_0^\infty \mathrm{e}^{-2t/k}|1-\mathrm{e}^{-t/k}|^2\mathrm{d}t\right)^{1/2}$$

$$= \frac{2}{k}\left(\int_0^\infty (\mathrm{e}^{-2t/k} - 2\mathrm{e}^{-3t/k} + \mathrm{e}^{-4t/k})\mathrm{d}t\right)^{1/2} = \sqrt{\frac{1}{3k}},$$

hence by Corollary 4.4 we get

$$d_W(I_1(g_k), \mathcal{N}) \leq \sqrt{\frac{1}{3k}},$$

while Corollary 3.4 of [7] yields

$$d_W(I_1(g_k), \mathcal{N}) \leq \int_0^\infty |g_k(t)|^3\,\mathrm{d}t = \sqrt{\frac{8}{3}}\sqrt{\frac{1}{3k}}$$

and $\sqrt{8/3} > 1$.

To check that $g_k \in S_1^{1,2}$ it suffices to verify that $g \in S_1^{1,2}$, where $g(t) = \mathrm{e}^{-t}$, $t \geq 0$. Let $\{\chi_k\}_{k\geq 1}$ be a sequence of functions in $\mathcal{C}_c^1([0,\infty))$ with $0 \leq \chi_k(t) \leq 1$, for any $t \geq 0$, $\chi_k(t) = 1$, for any $t \in [0,k]$, and $\sup_{k\geq 1, t\geq 0}|\chi_k'(t)| < \infty$. Then one may easily see that $\{\chi_k g\}_{k\geq 1}$ is a sequence in $\mathcal{C}_c^1([0,\infty))$ converging to g in the norm $\|\cdot\|_{1,2}$.

2. Take

$$g_k(t) = \frac{2}{(k+\frac{1}{2})^{1/2}}\mathbb{1}_{[0,2^{-1}(k+2^{-1})]}(t)\cos(2\pi t), \qquad t \geq 0, \quad k \geq 1.$$

Note that g_k is continuous and piecewise differentiable (with a piecewise continuous derivative) and so g_k is weakly differentiable. We shall show later on that $g_k \in S_1^{1,2}$. We have

$$\|g_k\|_{L^2(\mathbb{R}_+)} = \frac{2}{(k+\frac{1}{2})^{1/2}}\sqrt{\int_0^{2^{-1}(k+2^{-1})} |\cos(2\pi t)|^2\,\mathrm{d}t} = 1,$$

and

$$\left(\int_0^\infty |g_k'(t)|^2 \left(\int_0^t g_k(z)\,\mathrm{d}z\right)^2 \mathrm{d}t\right)^{1/2}$$

$$= \frac{4\pi}{(k+\frac{1}{2})^{1/2}}\left(\int_0^{2^{-1}(k+2^{-1})} |\sin(2\pi t)|^2 \left(\int_0^t g_k(z)\,\mathrm{d}z\right)^2 \mathrm{d}t\right)^{1/2}$$

$$
\begin{aligned}
&= \frac{8\pi}{k+\frac{1}{2}} \left(\int_0^{2^{-1}(k+2^{-1})} |\sin(2\pi t)|^2 \left(\int_0^t \cos(2\pi z)\,\mathrm{d}z \right)^2 \mathrm{d}t \right)^{1/2} \\
&= \frac{4}{k+2^{-1}} \left(\int_0^{2^{-1}(k+2^{-1})} |\sin(2\pi t)|^4 \,\mathrm{d}t \right)^{1/2} \\
&= \frac{4}{k+2^{-1}} \left(\frac{3}{4} \int_0^{2^{-1}(k+2^{-1})} |\sin(2\pi t)|^2 \,\mathrm{d}t \right)^{1/2} \\
&= \frac{2\sqrt{3}}{k+2^{-1}} \left(2^{-1}(k+2^{-1}) - \int_0^{2^{-1}(k+2^{-1})} |\cos(2\pi t)|^2 \,\mathrm{d}t \right)^{1/2} \\
&= \frac{\sqrt{3}}{k+2^{-1}} \sqrt{k+\frac{1}{2}} = \frac{\sqrt{3}}{\sqrt{k+2^{-1}}}
\end{aligned}
$$

hence by Corollary 4.4 we get

$$
d_W(I_1(f_k), \mathcal{N}) \leq \frac{\sqrt{3}}{\sqrt{k+2^{-1}}},
$$

whereas by Corollary 3.4 of [7] we have

$$
d_W(I_1(f_k), \mathcal{N}) \leq \int_0^\infty |g_k(t)|^3 \,\mathrm{d}t = \frac{16}{3\pi} \frac{1}{\sqrt{k+2^{-1}}}.
$$

Note that $16/(3\pi) < \sqrt{3}$.

To check that $g_k \in S_1^{1,2}$ it suffices to verify that $g \in S_1^{1,2}$, where $g(t) = \mathbb{1}_{[0,\pi/2]}(t)\cos t$, $t \geq 0$. Let ρ be a smooth probability density on $[0,\infty)$ with support in $[0,1]$, $\rho_k(t) = k\rho(kt)$ and

$$
G_k(t) := g * \rho_k(t) = \int_0^t \rho_k(t-s) g(s)\,\mathrm{d}s, \quad t \geq 0.
$$

Then one may easily see that $\{G_k\}_{k\geq 1}$ is a sequence in $\mathcal{C}_c^1([0,\infty))$ converging to g in the norm $\|\cdot\|_{1,2}$.

Double Poisson stochastic integrals

For the case of double Poisson stochastic integrals we have the following corollary.

Corollary 4.5 *For any symmetric function $f \in S_2^{1,2}$, we have*

$$
\begin{aligned}
d_{TV}(I_2(f), \mathcal{N}) &\leq 2|1 - 2\|f\|^2_{L^2(\mathbb{R}_+)}| \\
&+8\Bigg(2! \int_{(0,\infty)^2} \int_0^\infty f(x,t)f(x,s)\,\mathrm{d}x \int_0^{t\wedge s} f(y,t)f(y,s)\,\mathrm{d}y\mathrm{d}s\mathrm{d}t
\end{aligned}
$$

$$+\int_{(0,\infty)^2}\int_0^{t\wedge s}|f(x,t)f(x,s)|^2\,\mathrm{d}x\mathrm{d}s\mathrm{d}t$$
$$+3!\int_{(0,\infty)^2}\int_{t\vee s}^{\infty}\int_0^{\infty}|\partial_1 f(x,y)|^2\,\mathrm{d}x\mathrm{d}y\int_0^{t\wedge s}f(z,t)f(z,s)\,\mathrm{d}z\mathrm{d}s\mathrm{d}t\Bigg)^{1/2}$$

and

$$d_W(I_2(f),\mathcal{N})\leq|1-2\|f\|^2_{L^2(\mathbb{R}_+)}|$$
$$+4\Bigg(2!\int_{(0,\infty)^2}\int_0^{\infty}f(x,t)f(x,s)\,\mathrm{d}x\int_0^{t\wedge s}f(y,t)f(y,s)\,\mathrm{d}y\mathrm{d}s\mathrm{d}t$$
$$+\int_{(0,\infty)^2}\int_0^{t\wedge s}|f(x,t)f(x,s)|^2\,\mathrm{d}x\mathrm{d}s\mathrm{d}t$$
$$+3!\int_{(0,\infty)^2}\int_{t\vee s}^{\infty}\int_0^{\infty}|\partial_1 f(x,y)|^2\,\mathrm{d}x\mathrm{d}y\int_0^{t\wedge s}f(z,t)f(z,s)\,\mathrm{d}z\mathrm{d}s\mathrm{d}t\Bigg)^{1/2}.$$

If moreover $I_2(f)$ is a.s. $(-\nu,\infty)$-valued then we have

$$d_{\mathcal{H}}(I_2(f),\Gamma_\nu)\leq(2\|f\|^4_{L^2(\mathbb{R}_+)}+8(1-\nu)\|f\|^2_{L^2(\mathbb{R}_+)}+4\nu^2)^{1/2}$$
$$+4\Bigg(2!\int_{(0,\infty)^2}\int_0^{\infty}f(x,t)f(x,s)\,\mathrm{d}x\int_0^{t\wedge s}f(y,t)f(y,s)\,\mathrm{d}y\mathrm{d}s\mathrm{d}t$$
$$+\int_{(0,\infty)^2}\int_0^{t\wedge s}|f(x,t)f(x,s)|^2\,\mathrm{d}x\mathrm{d}s\mathrm{d}t$$
$$+3!\int_{(0,\infty)^2}\int_{t\vee s}^{\infty}\int_0^{\infty}|\partial_1 f(x,y)|^2\,\mathrm{d}x\mathrm{d}y\int_0^{t\wedge s}f(z,t)f(z,s)\,\mathrm{d}z\mathrm{d}s\mathrm{d}t\Bigg)^{1/2}.$$

Proof. It follows after a direct computation taking $n=2$ in Proposition 4.3 and noticing that, for any $f\in S_2^{1,2}$,

$$g^{(1,t)}_{1,1,0}(x,y)=f(x,t)f(y,t)\mathbb{1}_{(0,t)}(y),\qquad g^{(1,t)}_{1,1,1}(x)=|f(x,t)|^2\mathbb{1}_{(0,t)}(x),\qquad g^{(1,t)}_{1,1,2}=\int_0^t|f(x,t)|^2\,\mathrm{d}x$$

and

$$g^{(2,t)}_{2,1,0}(x,y,z)=\partial_1 f(x,y)f(z,t)\mathbb{1}_{[t,\infty)}(x)\mathbb{1}_{(0,t)}(z),\qquad g^{(2,t)}_{2,1,1}\equiv g^{(2,t)}_{2,1,2}\equiv 0.$$

□

5 Normal approximation of the compound Poisson distribution

In this section we present an application of formula (2.2) to the compound Poisson distribution. Let $(Z_k)_{k\geq 1}$ be a sequence of real-valued i.i.d. random variables independent of a

Poisson distributed random variable N_n with parameter $n \geq 1$. We assume that Z_1 has moments of any order and that its distribution has a continuously differentiable density $p_{Z_1}(z)$ with respect to the Lebesgue measure, such that $\lim_{z\to\pm\infty} |z|^p p_{Z_1}(z) = 0$ for all $p \geq 1$. We also assume that $\frac{\mathrm{d}}{\mathrm{d}z} \log p_{Z_1}(z) = p'_{Z_1}(z)/p_{Z_1}(z)$ has at most polynomial growth. Consider the sequence

$$F_n := \frac{\sum_{k=1}^{N_n} Z_k - n\mathrm{E}[Z_1]}{\sqrt{n\mathrm{E}[Z_1^2]}}, \quad n \geq 1.$$

It is well-known that $F_n \longrightarrow \mathcal{N}$, in law, as $n \longrightarrow \infty$. In the following we are going to upper bound the total variation distance between F_n and $\mathcal{N}$. The following lemma applies the IBP formula of [1] to each $F_n \,|\, N_n = m$, $m, n \geq 1$.

Lemma 5.1 *Let $m, n \geq 1$ be fixed integers. Under the foregoing assumptions on the law of the jump amplitude Z_1, we have that the IBP formula* (2.1) *holds on $\mathcal{C}_b^1(\mathbb{R})$ for $F_n \,|\, N_n = m$ with $P(\cdot \,|\, N_n = m)$-centered*

$$W_1 = W_n^{(m)} := -\frac{\sqrt{n\mathrm{E}[Z_1^2]}}{m} \sum_{k=1}^{m} q_{Z_1}(Z_k) \quad \textit{where} \quad q_{Z_1}(z) := \frac{p'_{Z_1}(z)}{p_{Z_1}(z)} \mathbb{1}_{\{p_{Z_1}(z)>0\}}$$

and $W_2 = 1$.

Proof. See Theorem 3.1 and Section 4 in [1]. □

We have the following bound for the total variation distance.

Proposition 5.2 *Under the foregoing assumptions on the law of the jump amplitude Z_1, we have*

$$d_{TV}(F_n, \mathcal{N}) \leq \mathrm{e}^{-n} + \sqrt{\frac{\pi}{2}} \|(1 - R_n)\mathbb{1}_{\{N_n \geq 1\}}\|_{L^2(P)}, \quad n \geq 1$$

where

$$R_n = -\frac{n}{N_n} \frac{\mathrm{E}[Z_1^2] \sum_{k=1}^{N_n} q_{Z_1}(Z_k)}{\sum_{k=1}^{N_n} Z_k - n\mathrm{E}[Z_1]}.$$

Proof. For any $n \geq 1$, we have

$$\begin{aligned}
d_{TV}(F_n, \mathcal{N}) &= \sup_{C\in\mathcal{B}(\mathbb{R})} |P(F_n \in C) - P(\mathcal{N} \in C)| \\
&= \sup_{C\in\mathcal{B}(\mathbb{R})} \Big| \sum_{m\geq 0} [P(F_n \in C \,|\, N_n = m)P(N_n = m) - P(\mathcal{N} \in C)P(N_n = m)] \Big| \\
&\leq \sum_{m\geq 0} d_{TV}(F_n \,|\, N_n = m, \mathcal{N})P(N_n = m) \\
&\leq \mathrm{e}^{-n} + \sqrt{\frac{\pi}{2}} \sum_{m\geq 1} \mathrm{E}[|W_n^{(m)} - F_n| \,|\, N_n = m]P(N_n = m) \qquad (5.1)
\end{aligned}$$

$$
\begin{aligned}
&= \mathrm{e}^{-n} + \sqrt{\frac{\pi}{2}}\mathrm{E}[|W_n^{(N_n)} - F_n|\mathbb{1}_{\{N_n\geq 1\}}] \\
&= \mathrm{e}^{-n} + \sqrt{\frac{\pi}{2}}\mathrm{E}\left[|F_n|\left|1 - \frac{W_n^{(N_n)}}{F_n}\right|\mathbb{1}_{\{N_n\geq 1\}}\right] \\
&\leq \mathrm{e}^{-n} + \sqrt{\frac{\pi}{2}}\|F_n\|_{L^2(P)}\left\|\left(1 - \frac{W_n^{(N_n)}}{F_n}\right)\mathbb{1}_{\{N_n\geq 1\}}\right\|_{L^2(P)} \\
&= \mathrm{e}^{-n} + \sqrt{\frac{\pi}{2}}\left\|\left(1 - \frac{W_n^{(N_n)}}{F_n}\right)\mathbb{1}_{\{N_n\geq 1\}}\right\|_{L^2(P)},
\end{aligned}
$$

where the inequality (5.1) follows by (2.2). □

Example - Normal approximation of Poisson compound sums with Gaussian addends

Suppose that Z_1 is a standard Gaussian random variable. Then all the hypotheses of Proposition 5.2 are satisfied and $R_n = n/N_n$. So, for any $n \geq 1$,

$$
d_{TV}(F_n, \mathcal{N}) \leq \mathrm{e}^{-n} + \sqrt{\frac{\pi}{2}}\left\|\left(1 - \frac{n}{N_n}\right)\mathbb{1}_{\{N_n\geq 1\}}\right\|_{L^2(P)}. \tag{5.2}
$$

We have

$$
\begin{aligned}
\left\|\left(1 - \frac{n}{N_n}\right)\mathbb{1}_{\{N_n\geq 1\}}\right\|_{L^2(P)} &= \mathrm{e}^{-n/2}\sqrt{\sum_{m\geq 1}\left(1 - \frac{n}{m}\right)^2\frac{n^m}{m!}} \\
&= \sqrt{1 - \mathrm{e}^{-n} - 2n\mathrm{e}^{-n}S_1(n) + n^2\mathrm{e}^{-n}S_2(n)},
\end{aligned} \tag{5.3}
$$

where

$$
S_1(n) := \sum_{m\geq 1}\frac{n^m}{m\, m!}
$$

and

$$
S_2(n) := \sum_{m\geq 1}\frac{n^m}{m^2\, m!}.
$$

Note that these two series converge (by e.g. the ratio test), but their sum do not have a closed form. After some manipulations, one can realize that $S_1(n)$ and $S_2(n)$ have the following series expansion at $n = \infty$:

$$
S_1(n) = \mathrm{e}^n(n^{-1} + n^{-2} + O(n^{-3})) + \log n^{-1} - i\pi - \gamma + O(n^{-7})
$$

and

$$
S_2(n) = \mathrm{e}^n(n^{-2}+3n^{-3}+O(n^{-4}))-2^{-1}\log^2(n^{-1})+(\gamma+i\pi)\log n^{-1}+5\pi^2/12-i\gamma\pi-\gamma^2/2+O(n^{-6})
$$

where γ denotes the Euler-Mascheroni constant. So

$$n(1-\mathrm{e}^{-n}-2n\mathrm{e}^{-n}S_1(n)+n^2\mathrm{e}^{-n}S_2(n))=1+\omega(n),$$

where $\omega(n)$ is a suitable function converging to zero as $n\longrightarrow\infty$. Therefore, combining (5.2) and (5.3) we get a Berry-Esseen type upper bound for F_n, which is asymptotically equivalent to $\sqrt{\frac{\pi}{2n}}$, as $n\longrightarrow\infty$.

6 Appendix

Lemma 6.1 *Let $g:\mathbb{R}\longrightarrow\mathbb{R}$ be with polynomial growth and continuously differentiable with bounded derivative and $F\in\mathrm{Dom}(D)$, we have $g(F)\in\mathrm{Dom}(D)$ and $Dg(F)=g'(F)DF$.*

Proof. For any $F\in\mathcal{S}$ we clearly have $g(F)\in\mathcal{S}$ and the claim is an immediate consequence of the action of D_t on $\mathcal{S}$ and the chain rule for the derivative. Now, take $F\in\mathrm{Dom}(D)$. Then there exists a sequence $(F^{(n)})_{n\geq 1}\subset\mathcal{S}$ such that $F^{(n)}\longrightarrow F$ in $L^2(P)$ and $DF^{(n)}\longrightarrow DF$ in $L^2(P\otimes\ell)$. Since the claim holds for functionals in $\mathcal{S}$, for $g:\mathbb{R}\longrightarrow\mathbb{R}$ with polynomial growth and continuously differentiable with bounded derivative, we have $Dg(F^{(n)})=g'(F^{(n)})DF^{(n)}$, $n\geq 1$. By the convergence in $L^2(P)$ we have that there exists a subsequence $\{n'\}$ of $\{n\}$ such that $F^{(n')}\longrightarrow F$ a.s.. The claim follows if we check that $Dg(F^{(n')})\longrightarrow g'(F)DF$ in $L^2(P\otimes\ell)$. By the boundedness of g', for a positive constant $C>0$ we have

$$\begin{aligned}
&\|Dg(F^{(n')})-g'(F)DF\|_{L^2(P\otimes\ell)}=\|g'(F^{(n')})DF^{(n')}-g'(F)DF\|_{L^2(P\otimes\ell)}\\
&\quad\leq\ \|g'(F^{(n')})DF^{(n')}-g'(F^{(n')})DF\|_{L^2(P\otimes\ell)}+\|g'(F^{(n')})DF-g'(F)DF\|_{L^2(P\otimes\ell)}\\
&\quad\leq\ C\|DF^{(n')}-DF\|_{L^2(P\otimes\ell)}+\left(\mathrm{E}\left[\int_0^\infty|g'(F^{(n')})-g'(F)|^2|D_tF|^2\,\mathrm{d}t\right]\right)^{1/2}.
\end{aligned}$$

This latter quantity tend to zero as $n'\longrightarrow\infty$. Indeed, the first term goes to zero since $DF_n\longrightarrow DF$ in $L^2(P\otimes\ell)$; the second term goes to zero by the Dominated Convergence Theorem since for some constant $C>0$ we have $|g'(F^{(n')})-g'(F)||D_tF|\leq C|D_tF|$, $P\otimes\ell$-a.e, $DF\in L^2(P\otimes\ell)$ and $g'(F^{(n')})\longrightarrow g'(F)$ a.s. by the continuity of g'. □

Proof of Lemma 4.2. Let $f_n\in\mathcal{C}_c^1([0,\infty)^n)$ be a symmetric function and define the sequence

$$F^{(m)}=\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\sum_{1\leq l_1\neq\cdots\neq l_k\leq m}\int_0^\infty\cdots\int_0^\infty f_n(T_{l_1},\ldots,T_{l_k},s_{k+1},\ldots,s_n)\,\mathrm{d}s_{k+1}\cdots\mathrm{d}s_n$$

$$
\begin{aligned}
= \quad & I_n(f_n \mathbb{1}_{[0,T_m]^n}) \\
& + \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \sum_{1 \le l_1 \ne \cdots \ne l_k \le m} \int_{\mathbb{R}_+^{n-k} \setminus [0,T_m]^{n-k}} f_n(T_{l_1}, \ldots, T_{l_k}, s_{k+1}, \ldots, s_n) \, \mathrm{d}s_{k+1} \cdots \mathrm{d}s_n,
\end{aligned}
\tag{6.1}
$$

$m \ge 1$. We have $(F^{(m)})_{m \ge 1} \subset \mathcal{S}$ and $F^{(m)}$ converges to $F = I_n(f_n)$ in $L^2(P)$. Indeed, by the isometry formula for multiple Poisson stochastic integrals (see e.g. Proposition 6.2.4 in [10]), we have

$$
\begin{aligned}
\mathrm{E}[|F - I_n(f_n \mathbb{1}_{[0,T_m]^n})|^2] &= \mathrm{E}\left[\left(I_n(f_n(1 - \mathbb{1}_{[0,T_m]^n}))\right)^2\right] \qquad (6.2) \\
&= n! \mathrm{E}\left[\int_0^\infty \cdots \int_0^\infty |f_n(t_1, \ldots, t_n)|^2 (1 - \mathbb{1}_{[0,T_m]^n}(t_1, \ldots, t_n)) \, \mathrm{d}t_1 \cdots \mathrm{d}t_n\right]
\end{aligned}
$$

and this latter term tends to 0 as m goes to infinity by the Dominated Convergence Theorem. Moreover, each of the remaining terms

$$
\int_{\mathbb{R}_+^{n-k} \setminus [0,T_m]^{n-k}} f_n(T_{l_1}, \ldots, T_{l_k}, s_{k+1}, \ldots, s_n) \, \mathrm{d}s_{k+1} \cdots \mathrm{d}s_n
$$

in (6.1) is bounded and tends to 0 a.s. as m goes to infinity, since f_n has compact support. Next, by (4.1) we have

$$
\begin{aligned}
D_t F^{(m)} = & - \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \sum_{1 \le l_1 \ne \cdots \ne l_k \le m} \\
& \times \sum_{i=1}^{k} \mathbb{1}_{[0,T_{l_i}]}(t) \int_0^\infty \cdots \int_0^\infty \partial_i f_n(T_{l_1}, \ldots, T_{l_k}, s_{k+1}, \ldots, s_n) \, \mathrm{d}s_{k+1} \cdots \mathrm{d}s_n \\
= \quad & - \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \sum_{1 \le l_1 \ne \cdots \ne l_k \le m} \\
& \int_0^\infty \cdots \int_0^\infty \sum_{i=1}^{n} \partial_i f_{n[t}(T_{l_1}, \ldots, T_{l_k}, s_{k+1}, \ldots, s_n) \, \mathrm{d}s_{k+1} \cdots \mathrm{d}s_n \\
& + \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \sum_{1 \le l_1 \ne \cdots \ne l_k \le m} \\
& \int_0^\infty \cdots \int_0^\infty \sum_{i=k+1}^{n} \partial_i f_{n[t}(T_{l_1}, \ldots, T_{l_k}, s_{k+1}, \ldots, s_n) \, \mathrm{d}s_{k+1} \cdots \mathrm{d}s_n \\
= \quad & - \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \sum_{1 \le l_1 \ne \cdots \ne l_k \le m} \\
& \int_0^\infty \cdots \int_0^\infty \sum_{i=1}^{n} \partial_i f_{n[t}(T_{l_1}, \ldots, T_{l_k}, s_{k+1}, \ldots, s_n) \, \mathrm{d}s_{k+1} \cdots \mathrm{d}s_n
\end{aligned}
$$

$$
\begin{aligned}
& -\sum_{k=0}^{n-1}(-1)^{n-k}\binom{n}{k}\sum_{1\le l_1\neq\cdots\neq l_k\le m}\sum_{i=k+1}^{n} \\
& \int_0^\infty\cdots\int_0^\infty f_n(T_{l_1},\ldots,T_{l_k},s_{k+1},\ldots,s_{i-1},t,s_{i+1},\ldots s_n)\,\mathrm{d}s_{k+1}\cdots\mathrm{d}s_{i-1}\mathrm{d}s_{i+1}\cdots\mathrm{d}s_{n-1} \\
= & -\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\sum_{1\le l_1\neq\cdots\neq l_k\le m}\int_0^\infty\cdots\int_0^\infty\sum_{i=1}^{n}\partial_i f_{n[t}(T_{l_1},\ldots,T_{l_k},s_{k+1},\ldots,s_n)\,\mathrm{d}s_{k+1}\cdots\mathrm{d}s_n \\
& -\sum_{k=0}^{n-1}(-1)^{n-k}\binom{n}{k}(n-k)\sum_{1\le l_1\neq\cdots\neq l_k\le m} \\
& \int_0^\infty\cdots\int_0^\infty f_n(T_{l_1},\ldots,T_{l_k},t,z_1,\ldots z_{n-k-1})\,\mathrm{d}z_1\cdots\mathrm{d}z_{n-k-1} \\
= & -\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\sum_{1\le l_1\neq\cdots\neq l_k\le m}\int_0^\infty\cdots\int_0^\infty\sum_{i=1}^{n}\partial_i f_{n[t}(T_{l_1},\ldots,T_{l_k},s_{k+1},\ldots,s_n)\,\mathrm{d}s_{k+1}\cdots\mathrm{d}s_n \\
& -n\sum_{k=0}^{n-1}(-1)^{n-k}\binom{n-1}{k}\sum_{1\le l_1\neq\cdots\neq l_k\le m} \\
& \int_0^\infty\cdots\int_0^\infty f_n(T_{l_1},\ldots,T_{l_k},t,z_1,\ldots,z_{n-k-1})\,\mathrm{d}z_1\cdots\mathrm{d}z_{n-k-1} \\
= & \; nI_{n-1}(f_n(*,t)\mathbb{1}_{[0,T_m]^{n-1}})-nI_n(\partial_1 f_{n[t}\mathbb{1}_{[0,T_m]^n}) \\
& -\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\sum_{1\le l_1\neq\cdots\neq l_k\le m} \qquad (6.3) \\
& \int_{\mathbb{R}_+^{n-k}\setminus[0,T_m]^{n-k}}\sum_{i=1}^{n}\partial_i f_{n[t}(T_{l_1},\ldots,T_{l_k},s_{k+1},\ldots,s_n)\,\mathrm{d}s_{k+1}\cdots\mathrm{d}s_n \\
& +n\sum_{k=0}^{n-1}(-1)^{n-1-k}\binom{n-1}{k}\sum_{1\le l_1\neq\cdots\neq l_k\le m} \qquad (6.4) \\
& \int_{\mathbb{R}_+^{n-1-k}\setminus[0,T_m]^{n-1-k}} f_n(T_{l_1},\ldots,T_{l_k},t,z_1,\ldots,z_{n-1-k})\,\mathrm{d}z_1\cdots\mathrm{d}z_{n-k-1}, \qquad t\in\mathbb{R}_+.
\end{aligned}
$$

To conclude we note that, as in (6.2), $nI_{n-1}(f_n(*,t)\mathbf{1}_{[0,T_m]^{n-1}})-nI_n(\partial_1 f_{n[t}\mathbf{1}_{[0,T_m]^n})$ converges in $L^2(P\otimes\ell)$ to

$$
D_tF := nI_{n-1}(f_n(*,t)) - nI_n(\partial_1 f_{n[t}), \qquad t\in\mathbb{R}_+, \tag{6.5}
$$

as m goes to infinity by the isometry formula for multiple Poisson stochastic integrals, and the two terms in (6.3) and (6.4) converge to 0 since $f_n\in\mathcal{C}_c^1([0,\infty)^n)$.

In order to complete the proof of the first part of the lemma by closability, given $f_n\in S_n^{1,2}$ we choose a sequence $(f_n^{(m)})_{m\in\mathbb{N}}\subset\mathcal{C}_c^1([0,\infty)^n)$ converging to f_n for the norm (4.9) and we define the sequence of functionals $(F^{(m)})_{m\geq 1}$ in $\mathrm{Dom}(D)$ by

$$F^{(m)} := I_n(f_n^{(m)}), \qquad m \geq 1.$$

Then we note that by the isometry formula for multiple Poisson stochastic integrals and the convergence of $f_n^{(m)}$ to f_n in $L^2(\mathbb{R}_+)$, we have $I_n(f_n^{(m)}) \to I_n(f_n)$ in $L^2(P)$ as $m \to \infty$. Moreover, $D_t F$ defined by (4.10) and (6.5) satisfies

$$\begin{aligned}
&\mathrm{E}\left[\int_0^\infty |D_t F - D_t F^{(m)}|^2 \, \mathrm{d}t\right] \\
\leq\ & 2n^2 \mathrm{E}\left[\int_0^\infty \left| I_n(\partial_1 f_{n[t]}) - I_n(\partial_1 f_{n[t}^{(m)}) \right|^2 \, \mathrm{d}t\right] \\
& +2n^2 \mathrm{E}\left[\int_0^\infty |I_{n-1}(f_n(*,t)) - I_{n-1}(f_n^{(m)}(*,t))|^2 \, \mathrm{d}t\right] \\
=\ & 2n^2 n! \int_0^\infty \cdots \int_0^\infty \int_t^\infty |\partial_1 f_n(s_1,\ldots,s_n) - \partial_1 f_n^{(m)}(s_1,\ldots,s_n)|^2 \, \mathrm{d}s_1 \mathrm{d}t \mathrm{d}s_2 \cdots \mathrm{d}s_n \\
& +2n^2 (n-1)! \int_0^\infty \cdots \int_0^\infty |f_n(s_1,\ldots,s_n) - f_n^{(m)}(s_1,\ldots,s_n)|^2 \, \mathrm{d}s_1 \cdots \mathrm{d}s_n.
\end{aligned}$$

So $DF^{(m)}$ converges to DF in $L^2(P \otimes \ell)$ by the convergence of $f_n^{(m)}$ to f_n with respect to the norm $\|\cdot\|_{1,2}$. Finally, using again the isometry formula for multiple Poisson stochastic integrals, we have

$$\begin{aligned}
\mathrm{E}\left[\int_0^\infty |D_t F|^2 \, \mathrm{d}t\right] &= n^2 \mathrm{E}\left[\int_0^\infty \left(I_{n-1}(f_n(*,t)) - I_n(\partial_1 f_{n[t]})\right)^2 \, \mathrm{d}t\right] \\
&= n^2 \mathrm{E}\left[\int_0^\infty \left(I_{n-1}(f_n(*,t))\right)^2 \, \mathrm{d}t\right] + n^2 \mathrm{E}\left[\int_0^\infty \left(I_n(\partial_1 f_{n[t]})\right)^2 \, \mathrm{d}t\right] \\
&\quad -2n^2 \mathrm{E}\left[\int_0^\infty I_{n-1}(f_n(*,t)) I_n(\partial_1 f_{n[t]}) \, \mathrm{d}t\right] \\
&= n^2 (n-1)! \int_0^\infty \cdots \int_0^\infty |f_n(t_1,\ldots,t_n)|^2 \, \mathrm{d}t_1 \cdots \mathrm{d}t_n \\
&\quad +n^2 n! \int_0^\infty \cdots \int_0^\infty \int_t^\infty |\partial_1 f_n(t_1,\ldots,t_n)|^2 \, \mathrm{d}t_1 \mathrm{d}t \mathrm{d}t_2 \cdots \mathrm{d}t_n.
\end{aligned}$$

□

Proof of (4.8). We have

$$\begin{aligned}
f'(T_k) \int_0^{T_k} \mathrm{E}[f'(T_{k-h} + t)]_{h=N_t} \mathrm{d}t &= 4\alpha^2 T_k \int_0^{T_k} (t + \mathrm{E}[T_{k-h}]_{h=N_t}) \, \mathrm{d}t \\
&= 4\alpha^2 T_k \int_0^{T_k} (t + k - N_t) \, \mathrm{d}t
\end{aligned}$$

$$= 2\alpha^2 T_k^3 + 4k\alpha^2 T_k^2 - 4\alpha^2 T_k \int_0^{T_k} N_t \, \mathrm{d}t$$

$$= 2\alpha^2 T_k^3 + 4k\alpha^2 T_k^2 - 4\alpha^2 T_k \sum_{h=1}^{k} (h-1)(T_h - T_{h-1}).$$

Therefore

$$\begin{aligned}
&\left\| 1 - f'(T_k) \int_0^{T_k} \mathrm{E}[f'(T_{k-h} + t)]_{h=N_t} \mathrm{d}t \right\|_{L^1(P)} \\
&\quad \leq \mathrm{E}[|1 - 4k\alpha^2 T_k^2|] + 2\alpha^2 \mathrm{E}\left[T_k \left| 2 \sum_{h=1}^{k} (h-1)(T_h - T_{h-1}) - T_k^2 \right| \right] \\
&\quad \leq \mathrm{E}[|1 - 4k\alpha^2 T_k^2|] + 2\alpha^2 \|T_k\|_{L^2(P)} \left\| 2 \sum_{h=1}^{k} (h-1)(T_h - T_{h-1}) - T_k^2 \right\|_{L^2(P)} \\
&\quad = \mathrm{E}\left[\left| 1 - \frac{T_k^2}{k^2} \right| \right] + \frac{1}{2} \sqrt{\frac{k+1}{k}} \left\| \frac{2}{k^2} \sum_{h=1}^{k} (h-1)(T_h - T_{h-1}) - \frac{T_k^2}{k^2} \right\|_{L^2(P)}.
\end{aligned}$$

We shall provide an upper bound for both these addends. We have

$$\begin{aligned}
\mathrm{E}\left[\left| 1 - \frac{T_k^2}{k^2} \right| \right] &= \mathrm{E}\left[\left| \frac{T_k^2 - (k+1)k}{k^2} + \frac{1}{k} \right| \right] \\
&\leq \frac{1}{k} + \frac{1}{k^2} \sqrt{\mathrm{Var}(T_k^2)} \\
&= \frac{1}{k} + \sqrt{\frac{4}{k} + \frac{10}{k^2} + \frac{6}{k^3}}.
\end{aligned}$$

Now, consider the other term. We have

$$\begin{aligned}
&\frac{2}{k^2} \sum_{h=1}^{k} (h-1)(T_h - T_{h-1}) - \frac{T_k^2}{k^2} \\
&= \frac{1}{k^2}((k+1)k - T_k^2) + \frac{2}{k^2} \sum_{h=1}^{k} (h-1)(T_h - T_{h-1} - 1) + \frac{2}{k^2} \sum_{h=1}^{k} (h-1) - \frac{(k+1)k}{k^2},
\end{aligned}$$

hence

$$\begin{aligned}
\left\| \frac{2}{k^2} \sum_{h=1}^{k} (h-1)(T_h - T_{h-1}) - \frac{T_k^2}{k^2} \right\|_{L^2(P)} &\leq \left\| \frac{2 \sum_{h=1}^{k} (h-1)(T_h - T_{h-1} - 1)}{k^2} \right\|_{L^2(P)} \\
&\quad + \left\| \frac{T_k^2 - (k+1)k}{k^2} \right\|_{L^2(P)} + \frac{\left| 2 \sum_{h=1}^{k} (h-1) - (k+1)k \right|}{k^2} \\
&\leq \left\| \frac{2 \sum_{h=1}^{k} (h-1)(T_h - T_{h-1} - 1)}{k^2} \right\|_{L^2(P)} + \left\| \frac{T_k^2 - (k+1)k}{k^2} \right\|_{L^2(P)} + \frac{2}{k}.
\end{aligned}$$

Note that

$$\Big\|\frac{2\sum_{h=1}^{k}(h-1)(T_h-T_{h-1}-1)}{k^2}\Big\|_{L^2(P)} = \sqrt{\mathrm{Var}\left(\frac{2\sum_{h=1}^{k}(h-1)(T_h-T_{h-1})}{k^2}\right)}$$
$$= \frac{2}{k^2}\sqrt{\sum_{h=1}^{k}|h-1|^2} = \sqrt{\frac{4}{3k}-\frac{2}{k^2}+\frac{2}{3k^3}},$$

and

$$\Big\|\frac{T_k^2-(k+1)k}{k^2}\Big\|_{L^2(P)} = \frac{1}{k^2}\sqrt{\mathrm{Var}\,(T_k^2)} = \sqrt{\frac{4}{k}+\frac{10}{k^2}+\frac{6}{k^3}}.$$

Collecting all these inequalities leads to (4.8). □

References

[1] V. Bally, M.-P. Bavouzet-Morel, and M. Messaoud. Integration by parts formula for locally smooth laws and applications to sensitivity computations. *Ann. Appl. Probab.*, 17(1):33–66, 2007.

[2] F. Hirsch and G. Lacombe. *Elements of functional analysis*, volume 192 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1999.

[3] C. Houdré and N. Privault. Concentration and deviation inequalities in infinite dimensions via covariance representations. *Bernoulli*, 8(6):697–720, 2002.

[4] J. León, J.L. Solé, and J. Vives. A pathwise approach to backward and forward stochastic differential equations on the Poisson space. *Stochastic Anal. Appl.*, 19(5):821–839, 2001.

[5] I. Nourdin. Lectures on Gaussian approximations with Malliavin calculus. Preprint, 2012.

[6] I. Nourdin and G. Peccati. Stein's method on Wiener chaos. *Probab. Theory Related Fields*, 145(1-2):75–118, 2009.

[7] G. Peccati, J. L. Solé, M. S. Taqqu, and F. Utzet. Stein's method and normal approximation of Poisson functionals. *Ann. Probab.*, 38(2):443–478, 2010.

[8] G. Peccati and C. Thäle. Gamma limits and U-statistics on the Poisson space. Preprint arXiv:1301.7289, 2013.

[9] N. Privault. A calculus on Fock space and its probabilistic interpretations. *Bull. Sci. Math.*, 123(2):97–114, 1999.

[10] N. Privault. *Stochastic Analysis in Discrete and Continuous Settings*, volume 1982 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 309 pp., 2009.

[11] W. Rudin. *Real and Complex Analysis*. Mc graw-Hill, 1974.

[12] C. Stein. Estimation of the mean of a multivariate normal distribution. *Ann. Stat.*, 9(6):1135–1151, 1981.