Splitting of Poisson noise and Lévy processes on real Lie algebras

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Abstract

The compensated Poisson noise is expressed as a composite sum (splitting) of creation and annihilation operators, whose probabilistic interpretation relies on time changes. We construct an Itô table for this decomposition and obtain continuous and discrete time realizations of Lévy processes on the finite difference algebra fd and on sl$_2$, e.g. the space-time dual of the Poisson process (compensated gamma process), and the continuous binomial process.

1   Introduction

Quantum stochastic calculus shows that the Gaussian white noise $\dot{W}_t$ is not an elementary object, but a composite sum of annihilation and creation densities:

$$\dot{W}_t = a_t^- + a_t^+.$$

The importance of a_t^- and a_t^+ lies not only in the construction of new non-commutative stochastic processes, but also on their probabilistic interpretation and links with stochastic calculus. Namely,

i) a_t^- and a_t^+ are mutually adjoint,

ii) a_t^- identifies to the white noise (space) derivative,

iii) $a_t^+ dt$ is an anticipative (Hitsuda-Skorokhod) differential,

iv) the commutator $[a_s^-, a_t^+] = \delta(t-s)$ is closely linked to the quadratic variation of Brownian motion.

It is natural to ask whether the Poisson noise a_t° is itself elementary or composite, in particular if it has a decomposition into a sum of creation and annihilation, which would differ from the product decomposition $a_t^\circ = a_t^+ a_t^-$.

In this paper, the compensated Poisson noise a_t° is written as a composite sum:

$$a_t^\circ = a_t^\ominus + a_t^\oplus. \tag{1.1}$$

Letting $\tilde{a}_t^- = a_t^- + a_t^\ominus$ and $\tilde{a}_t^+ = a_t^+ + a_t^\oplus$ and denoting by $(N_t - t)_{t \in \mathbf{R}_+}$ the compensated Poisson process, we also have

$$\overbrace{(N_t - t)}^{\cdot} = \tilde{a}_t^- + \tilde{a}_t^+,$$

where

i) $\tilde{a}_t^-$ and $\tilde{a}_t^+$ are mutually adjoint,

ii) $\tilde{a}_t^-$ identifies to a time derivative with respect to the Poisson process jump times,

iii) $\tilde{a}_t^+ dt$ is an anticipative (Skorokhod type) differential,

iv) a modification of the commutator $[\tilde{a}_t^-, \tilde{a}_t^+]$ gives the quadratic variation noise $\dot{N}_t$ of the Poisson process.

The discrete time compensated gamma process can be realized as the space-time dual of the Poisson process

$$k - T_k = N_{T_k} - T_k,$$

where T_k denotes the k-th jump time of $(N_t)_{t \in \mathbf{R}_+}$. Our first step is to use the decomposition of $N_{T_k} - T_k$ provided by $a_t^\ominus$ and $a_t^\oplus$ to construct discrete time realizations of the compensated gamma process and of the continuous binomial process (with independent, hyperbolic cosine distributed increments). The proofs use the general framework of [1] for Lévy processes on sl_2.

The second step is to construct continuous time realizations of these processes. In [11], a realization of the gamma (or exponential) process of [3] has been constructed in terms of a_t^-, a_t^+, a_t° and appropriate cocycles. We will obtain a different realization of this process and of the continuous binomial process in continuous time, using the

operators $a_t^\ominus$ and $a_t^\oplus$.

Due to the non-differentiability of N_t as a function of its jump times, the powers of $\tilde{a}_t^-$ and $\tilde{a}_t^+$ lead to infinities. Moreover, $\tilde{A}_t^- = \int_0^t \tilde{a}_s^- ds$ and $\tilde{A}_t^+ = \int_0^t \tilde{a}_s^+ ds$ are not adapted processes, thus quantum stochastic calculus can not be constructed in the usual way for the decomposition (1.1). Using a renormalized commutator table for which $[\tilde{A}_t^-, \tilde{A}_t^+]$ is the quadratic variation N_t of $(N_t - t)_{t\in\mathbf{R}_+}$ and $\tilde{A}_t^-$, $\tilde{A}_t^+$ are treated as adapted processes, we obtain a representation of the finite difference algebra in relation to the Poisson process. In this sense, $t \mapsto \tilde{A}_t^- + \tilde{A}_t^+$ and $t \mapsto i(\tilde{A}_t^- - \tilde{A}_t^+)$ respectively become continuous time realizations of the gamma process of [4] and of the continuous binomial process, cf. Sect. 5.4.1 of [8], which are constructed as Lévy processes on the finite difference algebra fd.

We proceed as follows. In Sect. 2 we recall the definitions and properties of the gradient operator $\tilde{D}$ to be used in this paper, with its associated integration by parts formula. In Sect. 3 we obtain the expression of $a_t^\ominus$ and $a_t^\oplus$ on the symmetric Fock space and interpret the integration by parts on Poisson space as a second quantization of the classical integration by parts on $\mathbf{R}_+$. In Sect. 4 we construct in discrete time the gamma and continuous binomial processes. In Sects. 5 and 6, a redefined operator composition is introduced in order to cancel infinities and state a closed commutator table for the noises $a_t^\ominus$ and $a_t^\oplus$. This allows one to construct continuous time realizations of the gamma and continuous binomial processes in Sect. 7.

2   Integration by parts on Poisson space

Let $(N_t)_{t\in\mathbf{R}_+}$ be a right-continuous standard Poisson process with jump times $(T_k)_{k\geq 1}$, and $T_0 = 0$, i.e.

$$N_t = \sum_{k=1}^\infty \mathbf{1}_{[T_k,\infty)}(t), \quad t \in \mathbb{R}_+.$$

Of particular interest in the analysis on Poisson space is a version of the gradient operator $\tilde{D}$ of [5], [7], defined on the space

$$\mathcal{S} = \{F = f(T_1, \ldots, T_n) \ : \ f \in \mathcal{C}_b^\infty(\mathbb{R}_+^n)\}$$

of smooth functionals of jump times.

Definition 2.1 *Let*

$$\tilde{D}_t F = -\sum_{k=1}^{k=n} \mathbf{1}_{[0,T_k]}(t)\partial_k f(T_1, \ldots, T_n), \quad t \in \mathbb{R}_+, \quad F \in \mathcal{S}.$$

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In variational terms, i.e. by infinitesimal time changes on the Poisson process paths, we have:

$$\langle \tilde{D}F, u \rangle = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}(f(T_1 + \varepsilon \tilde{u}(T_1), \ldots, T_n + \varepsilon \tilde{u}(T_n)) - f(T_1, \ldots, T_n))$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product in $L^2(\mathbb{R}_+)$, and

$$\tilde{u}(t) = -\int_0^t u(s)ds, \quad t \in \mathbb{R}_+.$$

Let $\mathcal{F}_{(a,b]}$ denote the σ-algebra generated by $\{N_t - N_s, \ a < s < t \le b\}$, and let $(\mathcal{F}_t)_{t \in \mathbb{R}_+} = (\mathcal{F}_{(0,t]})_{t \in \mathbb{R}_+}$ denote the filtration generated by $(N_t)_{t \in \mathbb{R}_+}$. Then the operator $\tilde{D}$ satisfies the property

$$\mathbf{1}_{(t,\infty)}(s)\tilde{D}_s G = 0, \quad s \in \mathbb{R}_+, \tag{2.1}$$

if $G \in \mathcal{S}$ is $\mathcal{F}_t$-measurable.

Proposition 2.1 *The operator $\tilde{D}$ admits a closable adjoint $\tilde{\delta}$ which satisfies*

$$E[\langle \tilde{D}F, u \rangle] = E[F\tilde{\delta}(u)],$$

and

$$\tilde{\delta}(Fu) = F\int_0^\infty u(t)d(N_t - t) - \langle \tilde{D}F, u \rangle, \quad F \in \mathcal{S}, \ u \in L^2(\mathbb{R}_+), \tag{2.2}$$

for all square-integrable adapted process $u \in L^2(\Omega \times \mathbb{R}_+)$.

Proof. The proof consists in several steps, cf. e.g. [12]:

i) For $u \in L^2(\mathbb{R}_+)$ and $F \in \mathcal{S}$, the relation

$$E\left[\int_0^\infty \tilde{D}_t Fu(t)dt\right] = E\left[F\int_0^\infty u(t)d(N_t - t)\right]$$

follows by integration by parts on $(T_1, \ldots, T_n)$ in the simplex $\Delta_n = \{(t_1, \ldots, t_n) \in \mathbb{R}_+^n \ : \ 0 \le t_1 < \cdots < t_n\}$.

ii) The operator $\tilde{D}$ has the derivation property and this implies that it has a closable adjoint $\tilde{\delta}$ which satisfies the classical divergence formula (2.2).

iii) From the property (2.1), Relation (2.2) extends to simple predictable processes, and by density to all square-integrable predictable processes in $L^2(\Omega \times \mathbb{R}_+)$. Finally we remark that the spaces of predictable and adapted processes coincide in $L^2(\Omega \times \mathbb{R}_+)$.

$\square$

In particular we have for all square-integrable adapted process $u \in L^2(\Omega \times \mathbb{R}_+)$:

$$E\left[\int_0^\infty \tilde{D}_t F u(t) dt\right] = E\left[F \int_0^\infty u(t) d(N_t - t)\right],$$

and

$$\tilde{\delta}(u) = \int_0^\infty u(t) d(N_t - t).$$

The scalar product between $\tilde{D}F$ and u is taken as an integral with respect to dt, and as a consequence, $\tilde{D}_t$ can be used to define a quantum noise.

Definition 2.2 *Given an adapted process $u \in L^2(\Omega \times \mathbb{R}_+)$, the operators $\tilde{a}^-(u)$, $\tilde{a}^+(u)$ and $\tilde{a}^+(u)$ are defined on $\mathcal{S}$ as*

$$\tilde{a}^-(u)F = \langle u, \tilde{D}F\rangle, \quad \tilde{a}^+(u)F = \tilde{\delta}(uF), \quad \tilde{a}^\circ(u)F = \tilde{\delta}(u\tilde{D}F), \qquad F \in \mathcal{S}.$$

We will use the generic notation a_t to denote operator densities, and A_t to denote operator processes, i.e.

$$A_t = a(\mathbf{1}_{[0,t]}) = \int_0^t a_s ds, \qquad t \in \mathbb{R}_+.$$

Relation (2.2) implies that the compensated Poisson stochastic integral $\int_0^\infty u(t) d(N_t - t)$ can be decomposed in a sum of annihilation and creation operators as

$$\int_0^\infty u(t) d(N_t - t) = \tilde{a}^-(u) + \tilde{a}^+(u), \tag{2.3}$$

where $u \in L^2(\Omega \times \mathbb{R}_+)$ is adapted.

3 Second quantization of the integration by parts on $\mathbb{R}_+$

Let

$$\Gamma(L^2(\mathbb{R}_+; \mathbb{C})) = \bigoplus_{n=0}^\infty L^2(\mathbb{R}_+; \mathbb{C})^{\circ n},$$

denote the symmetric Fock space on $L^2(\mathbb{R}_+; \mathbb{C})$, where "$\circ$" denotes the symmetric tensor product. In practice we will only consider real-valued functionals to simplify the notation. All statements can be transferred to the complex case.

In this section we express the integration by parts formula (2.2) as the second quantization of the usual integration by parts on $\mathbb{R}_+$:

$$\int_0^\infty \tilde{u}(t)v(t)dt = \int_0^\infty u(t)v(t)dt - \int_0^\infty \tilde{v}(t)u(t)dt, \quad u,v \in \mathcal{C}_c^\infty(\mathbb{R}_+),$$

cf. Prop. 3.1. This allows in particular to determine the action of $\tilde{a}^-(u)$ and $\tilde{a}^+(u)$ on $\Gamma(L^2(\mathbb{R}_+;\mathbb{C}))$, which will be essential to determine the commutator tables of Sect. 6. Let Ψ_f, $f \in L^2(\mathbb{R}_+)$, denote the exponential vector

$$\Psi_f = \sum_{n=0}^\infty \frac{1}{n!} f^{\circ n},$$

with

$$\langle \Psi_f, \Psi_g \rangle = e^{\langle f,g \rangle}, \quad f,g \in L^2(\mathbb{R}_+).$$

where the notation $\langle \cdot, \cdot \rangle$ is also used to denote the complex scalar product on $\Gamma(L^2(\mathbb{R}_+;\mathbb{C}))$. We recall the definition of the annihilation and creation operators $a^-(u)$, $a^+(u)$ and conservation operator $a^\circ(u)$ on the symmetric Fock space $\Gamma(L^2(\mathbb{R}_+;\mathbb{C}))$ as

$$a^-(u)\Psi_f = \langle u, f \rangle \Psi_f, \quad a^+(u)\Psi_f = u \circ \Psi_f, \quad \text{and} \quad a^\circ(u)\Psi_f = (uf) \circ \Psi_f,$$

or in terms of matrix elements:

$$\langle a^-(u)\Psi_f, \Psi_g \rangle = \langle u, f \rangle \langle \Psi_f, \Psi_g \rangle = \langle \Psi_f, a^+(u)\Psi_g \rangle,$$

and

$$\langle a^\circ(u)\Psi_f, \Psi_g \rangle = \langle uf, g \rangle \langle \Psi_f, \Psi_g \rangle.$$

From now on we work in the Poisson probabilistic interpretation of $\Gamma(L^2(\mathbb{R}_+;\mathbb{C}))$, i.e.

$$\Psi_f = \exp\left(-\int_0^\infty f(s)ds\right)\prod_{k=1}^\infty (1 + f(T_k)), \quad f \in \mathcal{C}_c^\infty(\mathbb{R}_+).$$

Definition 3.1 *Let $a^\ominus(u)$ and $a^\oplus(u)$ be defined on $\mathcal{E}$ as*

$$\langle a^\ominus(u)\Psi_f, \Psi_g \rangle = \langle \tilde{u}f', g \rangle \langle \Psi_f, \Psi_g \rangle = \langle \Psi_f, a^\oplus(u)\Psi_g \rangle, \quad f,g \in \mathcal{C}_c^\infty(\mathbb{R}_+),$$

where $\tilde{u}$ is the function $\tilde{u}(t) = -\int_0^t u(s)ds$, $t \in \mathbb{R}_+$, and $u \in L^2(\mathbb{R}_+)$.

We have

$$a^\ominus(u)\Psi_f = (\tilde{u}f') \circ \Psi_f, \quad \text{and} \quad a^\oplus(u)\Psi_f = (uf - \tilde{u}f') \circ \Psi_f.$$

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Proposition 3.1 *The sum of $a^{\ominus}(u)$ and $a^{\oplus}(u)$ is the conservation or number operator $a^{\circ}(u)$:*

$$a^{\ominus}(u) + a^{\oplus}(u) = a^{\circ}(u).$$

Proof. We have

$$
\begin{aligned}
\langle a^{\ominus}(u)\Psi_f, \Psi_g\rangle + \langle a^{\oplus}(u)\Psi_f, \Psi_g\rangle &= \langle \tilde{u}f', g\rangle\langle\Psi_f, \Psi_g\rangle + \langle \tilde{u}g', f\rangle\langle\Psi_f, \Psi_g\rangle \\
&= \langle uf, g\rangle\langle\Psi_f, \Psi_g\rangle.
\end{aligned}
$$

$\square$

The operator $\tilde{a}^{-}(u)$ is closable and its closed domain contains the linear space $\mathcal{E}$ generated by the exponential vectors Ψ_f, $f \in \mathcal{C}_c^{\infty}(\mathbb{R}_+)$.

Proposition 3.2 *We have the decompositions on $\mathcal{E}$:*

$$\tilde{a}^{-}(u) = a^{-}(u) + a^{\ominus}(u) \quad and \quad \tilde{a}^{+}(u) = a^{+}(u) + a^{\oplus}(u), \qquad u \in L^2(\mathbb{R}_+).$$

Proof. Let $f, g \in \mathcal{C}_c^{\infty}(\mathbb{R}_+)$ with support in $[0, t]$, and

$$F = \prod_{k=1}^{\infty}(1 + f(T_k)), \qquad G = \prod_{k=1}^{\infty}(1 + g(T_k)).$$

We have

$$
\begin{aligned}
\langle \tilde{a}^{-}(u)F, G\rangle &= \sum_{n=1}^{\infty} E\left[\tilde{u}(T_n)f'(T_n)\prod_{\substack{k=1\\k\neq n}}^{\infty}(1+f(T_k))\prod_{l=1}^{\infty}(1+g(T_l))\right] \\
&= \sum_{m=1}^{\infty}\sum_{n=1}^{n=m}\frac{e^{-t}}{m!}\langle \tilde{u}f', 1+g\rangle_{L^2([0,t])}(\langle 1+f, 1+g\rangle_{L^2([0,t])})^{m-1} \\
&= \langle \tilde{u}f', 1+g\rangle_{L^2([0,t])}\sum_{m=0}^{\infty}\frac{e^{-t}}{m}(\langle 1+f, 1+g\rangle_{L^2([0,t])})^{m} \\
&= \langle \tilde{u}f', 1+g\rangle_{L^2([0,t])}\langle F, G\rangle \\
&= \langle u, f\rangle\langle F, G\rangle + \langle \tilde{u}f', g\rangle\langle F, G\rangle.
\end{aligned}
$$

Using the Poisson probabilistic interpretations

$$\Psi_f = F\exp\left(-\int_0^{\infty}f(s)ds\right) \quad and \quad \Psi_g = G\exp\left(-\int_0^{\infty}g(s)ds\right)$$

of Ψ_f and Ψ_g, we obtain

$$\langle \tilde{a}^{-}(u)\Psi_f, \Psi_g\rangle = \langle a^{-}(u)\Psi_f, \Psi_g\rangle + \langle a^{\ominus}(u)\Psi_f, \Psi_g\rangle.$$

By duality we immediately obtain the relation $\tilde{a}^{+}(u) = a^{+}(u) + a^{\oplus}(u)$. $\square$

4 Realizations of Lévy processes in discrete time

The space-time dual of the Poisson process, i.e. the discrete time compensated gamma process $(T_k - k)_{k \geq 1}$, can be realized by integration of $\mathbf{1}_{[0,T_k]}$ with respect to the compensated Poisson process $(N_t - t)_{t \in \mathbf{R}_+}$, as

$$k - T_k = N_{T_k} - T_k = \int_0^\infty \mathbf{1}_{[0,T_k]}(t) d(N_t - t).$$

Therefore it is natural to study the discrete time quantum stochastic processes

$$\tilde{A}_{T_k}^- = \tilde{a}^-(\mathbf{1}_{[0,T_k]}), \qquad \tilde{A}_{T_k}^+ = \tilde{a}^+(\mathbf{1}_{[0,T_k]}), \qquad \tilde{A}_{T_k}^\circ = \tilde{a}^\circ(\mathbf{1}_{[0,T_k]}),$$

$$A_{T_k}^- = a^-(\mathbf{1}_{[0,T_k]}), \qquad A_{T_k}^+ = a^+(\mathbf{1}_{[0,T_k]}), \qquad A_{T_k}^\circ = a^\circ(\mathbf{1}_{[0,T_k]}),$$

$$A_{T_k}^\ominus = a^\ominus(\mathbf{1}_{[0,T_k]}), \qquad A_{T_k}^\oplus = a^\oplus(\mathbf{1}_{[0,T_k]}), \qquad k \geq 1,$$

as well as their increments, e.g. $\tilde{a}^-(\mathbf{1}_{(T_{k-1},T_k]})$ and $\tilde{a}^+(\mathbf{1}_{(T_{k-1},T_k]})$, $k \geq 1$. (For adapted $u \in L^2(\Omega \times \mathbb{R}_+)$ we define $a^\ominus(u)$ and $a^\oplus(u)$ on $\mathcal{S}$ as $a^\ominus(u) = \tilde{a}^-(u) - a^-(u)$ and $a^\oplus(u) = \tilde{a}^+(u) - a^+(u)$, which is consistent with Prop. 3.2). Let

$$Q_{T_k} = q(\mathbf{1}_{[0,T_k]}) = A_{T_k}^- + A_{T_k}^+, \qquad P_{T_k} = p(\mathbf{1}_{[0,T_k]}) = i(A_{T_k}^- - A_{T_k}^+),$$

$$\tilde{Q}_{T_k} = \tilde{q}(\mathbf{1}_{[0,T_k]}) = \tilde{A}_{T_k}^- + \tilde{A}_{T_k}^+, \qquad \tilde{P}_{T_k} = \tilde{p}(\mathbf{1}_{[0,T_k]}) = i(\tilde{A}_{T_k}^- - \tilde{A}_{T_k}^+),$$

$$Q_{T_k}^\circ = q(\mathbf{1}_{[0,T_k]}) = A_{T_k}^\ominus + A_{T_k}^\oplus = A_{T_k}^\circ, \qquad P_{T_k}^\circ = p^\circ(\mathbf{1}_{[0,T_k]}) = i(A_{T_k}^\ominus - A_{T_k}^\oplus), \quad k \geq 1.$$

Proposition 4.1 *We have the decomposition*

$$k - T_k = \int_0^\infty \mathbf{1}_{[0,T_k]}(t) d(N_t - t) = \tilde{A}_{T_k}^- + \tilde{A}_{T_k}^+ = \tilde{Q}_{T_k} = Q_{T_k} + Q_{T_k}^\circ, \quad k \geq 1.$$

Proof. It suffices to rewrite (2.2) or (2.3) for the predictable process $u = \mathbf{1}_{[0,T_k]}$. $\square$

It has been noticed in [13] that $\{-\frac{i}{2}(\tilde{Q}_{T_k} + 2\tilde{A}_{T_k}^\circ), -\frac{i}{2}\tilde{P}_{T_k}, i(\frac{k}{2} + \tilde{A}_{T_k}^\circ)\}$ generates the Segal-Shale-Weil representation of sl_2. In fact the triple $\{\tilde{A}_{T_k}^-, \tilde{A}_{T_k}^+, \tilde{A}_{T_k}^\circ\}$ can also be used to generate the representation $\{M, B^-, B^+\}$ of sl_2, with

$$[B^-, B^+] = M, \quad [M, B^-] = -2B^-, \quad [M, B^+] = 2B^+.$$

Proposition 4.2 *A family $(j_{kl})_{0 \leq k < l}$ of representations of sl_2 is defined by*

$$j_{kl}(M) = (l - k)\mathbf{1} + 2\tilde{a}^\circ(\mathbf{1}_{(T_k,T_l]})$$

$$j_{kl}(B^-) = -e^{-i\varphi}(\tilde{a}^-(\mathbf{1}_{(T_k,T_l]}) + \tilde{a}^\circ(\mathbf{1}_{(T_k,T_l]})),$$

$$j_{kl}(B^+) = -e^{+i\varphi}(\tilde{a}^+(\mathbf{1}_{(T_k,T_l]}) + \tilde{a}^\circ(\mathbf{1}_{(T_k,T_l]})), \quad 1 \leq k < l.$$

Proof. Let $u \in L^2(\Omega \times \mathbb{R}_+)$ be adapted. We have for $F \in \mathcal{S}$:

$$\tilde{a}^-(u)F = \int_0^\infty u(t)\tilde{D}_t F dt, \quad \tilde{a}^+(u)F = F \int_0^\infty u(t)d(N_t - t) - \int_0^\infty u(t)\tilde{D}_t F dt.$$

Moreover, from Relation (2.2) applied to the process $u\mathbf{1}_{[0,T_k]}$ and to the random variable $\partial_k f(T_1, \ldots, T_n)$, we obtain

$$\tilde{a}^\circ(u)F = \int_0^\infty u(t)\tilde{D}_t F d(N_t - t) - \sum_{k=1}^n \int_0^\infty u(t)\mathbf{1}_{[0,T_k]}(t)\tilde{D}_t \partial_k f(T_1, \ldots, T_n)dt.$$

Letting $\tau_k = T_k - T_{k-1}$, $k \geq 1$, denote k-th interjump time of the Poisson process, we deduce the representations

$$\tilde{a}^-(\mathbf{1}_{(T_{k-1},T_k]})f(\tau_1, \ldots, \tau_n) = -\tau_k \partial_k f(\tau_1, \ldots, \tau_n),$$
$$\tilde{a}^+(\mathbf{1}_{(T_{k-1},T_k]})f(\tau_1, \ldots, \tau_n) = (1 - \tau_k)f(\tau_1, \ldots, \tau_n) + \tau_k \partial_k f(\tau_1, \ldots, \tau_n),$$
$$\tilde{a}^\circ(\mathbf{1}_{(T_{k-1},T_k]})f(\tau_1, \ldots, \tau_n) = -(1 - \tau_k)\partial_k f(\tau_1, \ldots, \tau_n) - \tau_k \partial_k^2 f(\tau_1, \ldots, \tau_n),$$

$f \in \mathcal{C}_c^\infty(\mathbb{R}_+^n)$. In other terms, the operators $\tilde{a}^\circ(\mathbf{1}_{(T_{k-1},T_k]})$, $\tilde{a}^+(\mathbf{1}_{(T_{k-1},T_k]})$, $\tilde{a}^-(\mathbf{1}_{(T_{k-1},T_k]})$ act on smooth functions in $L^2(\mathbb{R}_+; \mathbb{C}, e^{-\tau}d\tau)$ as

$$\tilde{a}^\circ = -(1 - \tau)\partial_\tau - \tau\partial_\tau^2, \quad \tilde{a}^+ = (1 - \tau) + \tau\partial_\tau, \quad \tilde{a}^- = -\tau\partial_\tau, \qquad (4.1)$$

and the eigenvectors of $\tilde{a}^\circ$ are the Laguerre polynomials. The following relations are then easily proved by algebraic computation using the representation (4.1), cf. [13]:

$$[\tilde{a}^-(\mathbf{1}_{(T_{k-1},T_k]}), \tilde{a}^+(\mathbf{1}_{(T_{l-1},T_l]})] = \mathbf{1}_{\{k=l\}} - \tilde{a}^-(\mathbf{1}_{(T_{k-1},T_k]}) - \tilde{a}^+(\mathbf{1}_{(T_{l-1},T_l]}),$$
$$[\tilde{a}^\circ(\mathbf{1}_{(T_{k-1},T_k]}), \tilde{a}^+(\mathbf{1}_{(T_{l-1},T_l]})] = (\tilde{a}^\circ(\mathbf{1}_{(T_{k-1},T_k]}) + \tilde{a}^+(\mathbf{1}_{(T_{k-1},T_k]}))\mathbf{1}_{\{k=l\}},$$
$$[\tilde{a}^-(\mathbf{1}_{(T_{k-1},T_k]}), \tilde{a}^\circ(\mathbf{1}_{(T_{l-1},T_l]})] = (\tilde{a}^\circ(\mathbf{1}_{(T_{k-1},T_k]}) + \tilde{a}^-(\mathbf{1}_{(T_{k-1},T_k]}))\mathbf{1}_{\{k=l\}}.$$

These relations imply

$$[\tilde{a}^-(\mathbf{1}_{[0,T_k]}), \tilde{a}^+(\mathbf{1}_{[0,T_l]})] = k \wedge l - \tilde{a}^-(\mathbf{1}_{[0,T_{k\wedge l}]}) - \tilde{a}^+(\mathbf{1}_{[0,T_{k\wedge l}]}),$$
$$[\tilde{a}^\circ(\mathbf{1}_{[0,T_k]}), \tilde{a}^+(\mathbf{1}_{[0,T_l]})] = \tilde{a}^\circ(\mathbf{1}_{[0,T_{k\wedge l}]}) + \tilde{a}^+(\mathbf{1}_{[0,T_{k\wedge l}]}),$$
$$[\tilde{a}^-(\mathbf{1}_{[0,T_k]}), \tilde{a}^\circ(\mathbf{1}_{[0,T_l]})] = \tilde{a}^\circ(\mathbf{1}_{[0,T_{k\wedge l}]}) + \tilde{a}^-(\mathbf{1}_{[0,T_{k\wedge l}]}),$$

which in turn yields the desired representation. $\qquad\qquad\square$

We use the notation and definitions of [1] regarding Lévy processes on Lie algebras. The family $(j_{lk})_{0 \leq l < k}$ also defines a representation of the current algebra

$$\mathbf{g}^{\mathrm{N}} = \left\{ \sum_{i=1}^{i=n} X_i \mathbf{1}_{(i-1,i]} \ : \ X_i \in \mathbf{g}, \ i = 1, \ldots, n, \ 0 \leq t_0 < \cdots < t_n, \ n \geq 1 \right\},$$

and we have $j_{kl}(B^-)^* = j_{kl}(B^+)$ and $j_{kl}(M)^* = j_{kl}(M)$. Moreover, $j_{kl}(B^-)\mathbf{1} = 0$ and $j_{kl}(T)\mathbf{1} = (l-k)\mathbf{1}$, where $\mathbf{1}$ denotes the vacuum state in $\Gamma(L^2(\mathbb{R}_+; \mathbb{C}))$. Hence j_{kl} has same law as the process of Example 3.1 in [1].

Proposition 4.3 *i) The process*

$$\tilde{A}^-_{T_k} + \tilde{A}^+_{T_k} = Q_{T_k} + Q^\circ_{T_k} = \tilde{Q}_{T_k} = A^-_{T_k} + A^+_{T_k} + A^\circ_{T_k}, \qquad k \geq 1,$$

is a discrete time compensated gamma process.

ii) The process

$$i(\tilde{A}^-_{T_k} - \tilde{A}^+_{T_k}) = P_{T_k} + P^\circ_{T_k} = \tilde{P}_{T_k} = i(A^-_{T_k} - A^+_{T_k} + A^\ominus_{T_k} - A^\oplus_{T_k}), \qquad k \geq 1,$$

is a discrete time continuous binomial process.

Proof.

i) We have

$$T_k = k - \tilde{a}^-(\mathbf{1}_{[0,T_k]}) - \tilde{a}^-(\mathbf{1}_{[0,T_k]}),$$

which obviously has a gamma law with parameter $k \geq 1$, since T_k is the k-th jump time of a standard Poisson process. This result follows independently for $\varphi = 0$ from the relation

$$k\mathbf{1} - (\tilde{a}^-(\mathbf{1}_{[0,T_k]}) + \tilde{a}^-(\mathbf{1}_{[0,T_k]})) = j_{0,k}(T) = j_{0,k}(B^- + B^+ + M)$$

and the analysis of Example 4.2 of [1] which shows that $j_{0,k}(B^- + B^+ + M)$ has a gamma law with parameter $k \geq 1$.

ii) From [13],

$$i(\tilde{a}^-(\mathbf{1}_{[0,T_k]}) - \tilde{a}^+(\mathbf{1}_{[0,T_k]}))$$

has characteristic function $u \mapsto (\cosh u)^{-k}$. Writing for $\varphi = \pi/2$:

$$i(\tilde{a}^-(\mathbf{1}_{[0,T_k]}) - \tilde{a}^+(\mathbf{1}_{[0,T_k]})) = j_{0,k}(B^- + B^+),$$

it follows from [1] that its density is

$$x \mapsto \frac{2^{k-1}}{\pi(k-1)!} \left| \Gamma\left(\frac{k+ix}{2}\right) \right|^2.$$

$\square$

The density of $i(\tilde{a}^-(1_{(T_{k-1},T_k]}) - \tilde{a}^+(1_{(T_{k-1},T_k]}))$ is the hyperbolic cosine density

$$\frac{1}{\cosh((\pi x)/2)},$$

cf. p. 502 of [9], which implies in particular the relation

$$\frac{1}{\pi}\left|\Gamma\left(\frac{1+ix}{2}\right)\right|^2 = \frac{1}{\cosh((\pi x)/2)}, \quad x \in \mathbb{R}.$$

Moreover it follows in general from Example 4.2 in [1] that for $|\beta| > 1$,

$$-k/2 - \mathrm{sgn}(\beta)(c-1/c)^{-1}j_{0,k}(\beta M + B^- + B^+)$$

has a negative binomial law with parameter (c^2, k), with $c = \beta\mathrm{sgn}(\beta) - \sqrt{\beta^2 - 1}$. In discrete time this result has been obtained in [13] with two other parameterizations which are now shown to be part of the more general framework of [1].

Proposition 4.4 *([13]) Let $s \in \mathbb{R}$.*

i) The operator

$$\tilde{A}^\circ_{T_k} + is\tilde{A}^+_{T_k} - is\tilde{A}^-_{T_k} + s^2 T_k$$

has a negative binomial law with parameter $(s^2/(1+s^2), k)$.

ii)

$$e^{-2s}\tilde{A}^\circ_{T_k} - \frac{1}{2}\sinh(2s)\left(\tilde{A}^-_{T_k} + \tilde{A}^+_{T_k}\right) + k\sinh(s)^2$$

has a negative binomial law with parameter $(\tanh^2(s), k)$.

Proof. We use the relation

$$\tilde{a}^\circ(1_{[0,T_k]}) + is\tilde{a}^+(1_{[0,T_k]}) - is\tilde{a}^-(1_{[0,T_k]}) + s^2 T_k$$
$$= -k/2 - \mathrm{sgn}(\beta(s))(c(s) - 1/c(s))^{-1}j_{0,k}\left(\beta(s)M + B^- + B^+\right),$$

with

$$c(s) = \frac{|s|}{\sqrt{1+s^2}}, \quad \beta(s) = \frac{-(1+2s^2)}{2s\sqrt{1+s^2}}, \quad (c(s) - 1/c(s))^{-1} = -|s|\sqrt{1+s^2},$$

$e^{i\varphi(s)} = (-s^2 + is)(s\sqrt{1+s^2})^{-1}$, and the fact that from Example 4.2 of [1],

$$-k/2 - \mathrm{sgn}(\beta)(c(s) - 1/c(s))^{-1}j_{0,k}\left(\beta(s)M + B^- + B^+\right)$$

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has a negative binomial law with parameter $(c^2(s), k) = (s^2/(1 + s^2), k)$. The second property is proved similarly, since

$$e^{-2s}\tilde{a}^\circ(\mathbf{1}_{[0,T_k]}) - \frac{1}{2}\sinh(2s)\left(\tilde{a}^+(\mathbf{1}_{[0,T_k]}) + \tilde{a}^-(\mathbf{1}_{[0,T_k]})\right) + k\sinh(s)^2$$
$$= -\frac{k}{2} - \text{sgn}(\beta(s))(c(s) - 1/c(s))^{-1}j_{0,k}\left(\beta(s)M + B^- + B^+\right),$$

with $\varphi(s) = 0$ and

$$c(s) = \tanh(|s|), \quad (c(s) - 1/c(s))^{-1} = -\frac{\sinh(2|s|)}{2} \quad \text{and} \quad \beta(s) = \frac{1}{2}\coth(s) + \frac{1}{2}\tanh(s).$$

$\square$

The above results can be closely related to quantum stochastic calculus on the boson Fock space. Let $\alpha_x^- = \partial_x$, $\alpha_y^- = \partial_y$, $\alpha_x^+ = x - \partial_x$, $\alpha_y^+ = y - \partial_y$ denote the annihilation and creation operators on the two-dimensional boson Fock space $\Gamma(\mathbb{C}e_1 \oplus \mathbb{C}e_2) \simeq L^2\left(\mathbb{R}^2; \mathbb{C}, \frac{1}{2\pi}e^{-\frac{1}{2}(x^2+y^2)}\right)$. The operators $\tilde{a}^+, \tilde{a}^-, \tilde{a}^\circ$ can be identified to operators on $\Gamma(\mathbb{C}e_1 \oplus \mathbb{C}e_2)$, acting on the exponential variable τ written as $\tau = (x^2 + y^2)/2$, where x, y are independent standard Gaussian variables. Under this identification,

$$\tilde{a}^\circ = \frac{1}{2}(\alpha_x^+\alpha_x^- + \alpha_y^+\alpha_y^-),$$
$$\tilde{a}^+ = -\frac{1}{2}((\alpha_x^+)^2 + \alpha_x^+\alpha_x^- + (\alpha_y^+)^2 + \alpha_y^+\alpha_y^-),$$
$$\tilde{a}^- = -\frac{1}{2}((\alpha_x^-)^2 + \alpha_x^+\alpha_x^- + (\alpha_y^-)^2 + \alpha_y^+\alpha_y^-),$$
$$P = \frac{i}{2}((\alpha_x^+)^2 - (\alpha_x^-)^2 + (\alpha_y^+)^2 - (\alpha_y^-)^2).$$

One can check also that

$$\tau = \frac{1}{2}((\alpha_x^+ + \alpha_x^-)^2 + (\alpha_y^+ + \alpha_y^-)^2) = \frac{1}{2}(x^2 + y^2)$$

is a discrete square of Gaussian white noise. This gives the possibility of defining gamma and continuous binomial random variables with parameter $1/2$ instead of 1. It is natural to ask whether this discrete time analysis with parameter 1 can give representation of laws of parameter strictly smaller than 1, with classical commutation relation. It turns out that this it is possible, e.g. for the gamma law with parameter

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1/2, simply by letting

$$\tilde{a}^{\circ} = \frac{1}{2}\alpha_x^+ \alpha_x^-,$$

$$\tilde{a}^+ = -\frac{1}{2}((\alpha_x^+)^2 + \alpha_x^+ \alpha_x^-),$$

$$\tilde{a}^- = -\frac{1}{2}((\alpha_x^-)^2 + \alpha_x^+ \alpha_x^-),$$

$$P = \frac{i}{2}((\alpha_x^+)^2 - (\alpha_x^-)^2),$$

i.e. each distributions is split into two independent halves with parameter $1/2$. The approach proposed in Sect. 6 will allow to go below this limit.

5   Annihilation and creation densities

The annihilation and creation densities associated to $a^-(u)$ and $a^+(u)$ are defined from

$$\langle a_t^- \Psi_f, \Psi_g \rangle = f(t)\langle \Psi_f, \Psi_g \rangle = \langle \Psi_f, a_t^+ \Psi_g \rangle,$$

i.e.

$$a_t^- \Psi_f = f(t) \otimes \Psi_f \quad \text{and} \quad a_t^+ \Psi_f = \delta(t - \cdot) \circ \Psi_f, \quad f \in \mathcal{C}_c^\infty(\mathbb{R}_+).$$

The creation density a_t^+ is meaningful only in distribution sense, a property which is linked to the non-differentiability of Brownian paths. Similarly, a creation density can be defined for $a^\ominus(u)$ and $a^\oplus(u)$, from

$$
\begin{aligned}
\langle a_t^\ominus \Psi_f, \Psi_g \rangle &= -\int_t^\infty f'(s)g(s)ds \langle \Psi_f, \Psi_g \rangle \\
&= \left(f(t)g(t) + \int_t^\infty f(s)g'(s)ds \right) \langle \Psi_f, \Psi_g \rangle = \langle \Psi_f, a_t^\oplus \Psi_g \rangle, \quad t \in \mathbb{R}_+,
\end{aligned}
$$

i.e.

$$a_t^\ominus \Psi_f = -(\mathbf{1}_{[t,\infty)} f') \circ \Psi_f \quad \text{and} \quad a_t^\oplus \Psi_g = (g(t)\delta(t - \cdot) + \mathbf{1}_{[t,\infty)} g') \circ \Psi_g.$$

The sum of annihilation and creation densities $a_t^\ominus$ and $a_t^\oplus$ equals again the conservation density:

$$a_t^\ominus + a_t^\oplus = a_t^\circ = a_t^+ a_t^-.$$

Due to the fact that $\tilde{D}$ satisfies the property $\tilde{D}_s F = 0$, $s > t$, if $F \in \mathcal{S}$ is $\mathcal{F}_t$-measurable, the stochastic integration of adapted operator processes with respect to $dA_t^\ominus$ and $dA_t^\oplus$ can be developed, cf. [15]. However if $0 < s < a$ and $F \in \mathcal{S}$ is

$\mathcal{F}_{(a,b]}$-measurable we do not have $\tilde{D}_s F = 0$, in particular $\tilde{a}_t^-$ and $\tilde{a}_t^+$ are not quantum noises in the sense of e.g. [6] and the processes $\tilde{A}_t^-$ and $\tilde{A}_t^+$ are not adapted operator processes. The following Itô table has been proved in [14] in the sense of non-adapted quantum stochastic calculus:

$\cdot$	dA_t^-	$dA_t^\ominus$	$dA_t^\oplus$	dA_t^+
dA_t^+	0	0	0	0
$dA_t^\oplus$	0	0	0	0
$dA_t^\ominus$	0	0	dA_t°	dA_t^+
dA_t^-	0	0	dA_t^-	dt

with commutator table

$[\cdot,\cdot]$	dA_t^-	$dA_t^\ominus$	$dA_t^\oplus$	dA_t^+
dA_t^-	0	0	dA_t^-	dt
$dA_t^\ominus$	0	0	dA_t°	dA_t^+
$dA_t^\oplus$	$-dA_t^-$	$-dA_t^\circ$	0	0
dA_t^+	$-dt$	$-dA_t^+$	0	0

(5.1)

In adapted quantum stochastic calculus, the commutator table

$$\{[A_t^\varepsilon, A_t^\eta], \qquad \varepsilon, \eta = -, +, \circ\},$$

is immediately deduced from the commutator table

$$\{[dA_t^\varepsilon, dA_t^\eta] \qquad \varepsilon, \eta = -, +, \circ\},$$

from the relation

$$d[A_t^\varepsilon, A_t^\eta] = [dA_t^\varepsilon, dA_t^\eta], \qquad \varepsilon, \eta = -, +, \circ,$$

which follows from the adaptedness of A_t^-, A_t^+, A_t°. However, here the processes $A_t^\ominus$ and $A_t^\oplus$ are not adapted and we only have

$$
\begin{aligned}
d[A_t^\varepsilon, A_t^\eta] &= A_t^\varepsilon dA_t^\eta - (dA_t^\eta)A_t^\varepsilon + A_t^\eta dA_t^\varepsilon - (dA_t^\varepsilon)A_t^\eta + dA_t^\varepsilon \cdot dA_t^\eta - dA_t^\eta \cdot dA_t^\varepsilon \\
&= [A_t^\varepsilon, dA_t^\eta] + [A_t^\eta, dA_t^\varepsilon] + [dA_t^\varepsilon, dA_t^\eta], \quad \varepsilon, \eta = \ominus, \oplus.
\end{aligned}
$$

Thus for $\varepsilon, \eta = \ominus, \oplus$, the commutator table of processes does not follow from the commutator table of differentials (5.1).

The classical commutation relations

$$[a^-(u), a^\circ(v)] = a^-(uv), \quad [a^\circ(u), a^+(v)] = a^+(uv), \quad [a^-(u), a^+(v)] = \langle u, v \rangle$$

are extended to include $a^\ominus(u)$ and $a^\oplus(u)$ in the following proposition.

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Proposition 5.1 *We have the commutator table, for $u, v \in \mathcal{C}_c^\infty(\mathbb{R}_+)$:*

$[\cdot, \cdot]$	$a^-(v)$	$a^\ominus(v)$	$a^\oplus(v)$	$a^+(v)$
$a^-(u)$	0	$a^-(\tilde{v}u' - uv)$	$a^-(\tilde{v}u')$	$\langle u, v \rangle$
$a^\ominus(u)$	$a^-(uv - \tilde{u}v')$	0	$a^\circ(\tilde{u}v')$	$a^+(\tilde{u}v')$
$a^\oplus(u)$	$-a^-(\tilde{u}v')$	$-a^\circ(\tilde{v}u')$	0	$a^+(uv - \tilde{u}v')$
$a^+(u)$	$-\langle v, u \rangle$	$-a^+(\tilde{v}u')$	$a^+(\tilde{v}u' - uv)$	0

Proof. We compute

$$[a^\ominus(u), a^+(v)]\Psi_f = (\tilde{u}v') \circ \Psi_f, \tag{5.2}$$

$$[a^\oplus(u), a^+(v)]\Psi_f = ((uv - \tilde{u}v')f) \circ \Psi_f, \tag{5.3}$$

$$[a^-(u), a^\ominus(v)]\Psi_f = \langle \tilde{v}u, f' \rangle \otimes \Psi_f = \langle vu - \tilde{v}u', f \rangle \otimes \Psi_f, \tag{5.4}$$

$$[a^-(u), a^\oplus(v)]\Psi_f = \langle \tilde{v}u', f \rangle \otimes \Psi_f = \langle uv, f \rangle \otimes \Psi_f - \langle \tilde{v}u, f' \rangle \otimes \Psi_f, \tag{5.5}$$

$$[a^\ominus(u), a^\oplus(v)]\Psi_f = (\tilde{u}v'f) \circ \Psi_f = (\tilde{u}(vf)' - \tilde{u}vf') \circ \Psi_f. \tag{5.6}$$

$\square$

The commutator table in terms of matrix elements

$\langle [\cdot, \cdot]\Psi_f, \Psi_g \rangle$	$a^-(v)$	$a^\ominus(v)$	$a^\oplus(v)$	$a^+(v)$
$a^-(u)$	0	$\langle \tilde{v}u, f' \rangle$	$\langle \tilde{v}u', f \rangle$	$\langle u, v \rangle$
$a^\ominus(u)$	$-\langle \tilde{u}v, f' \rangle$	0	$\langle \tilde{u}v'f, g \rangle$	$\langle \tilde{u}v', g \rangle$
$a^\oplus(u)$	$-\langle \tilde{u}v', f \rangle$	$-\langle \tilde{v}u'f, g \rangle$	0	$\langle \tilde{u}v, g' \rangle$
$a^+(u)$	$-\langle v, u \rangle$	$-\langle \tilde{v}u', g \rangle$	$-\langle \tilde{v}u, g' \rangle$	0

can also be rewritten as

$[\cdot, \cdot]$	$a^-(v)$	$a^\ominus(v)$	$a^\oplus(v)$	$a^+(v)$
$a^-(u)$	0	$\langle \tilde{v}u, f' \rangle$	$\langle uv, f \rangle - \langle \tilde{v}u, f' \rangle$	$\langle u, v \rangle$
$a^\ominus(u)$	$-\langle \tilde{u}v, f' \rangle$	0	$\langle \tilde{u}(vf)', g \rangle - \langle \tilde{u}vf', g \rangle$	$\langle \tilde{u}v', g \rangle$
$a^\oplus(u)$	$\langle \tilde{u}v, f' \rangle - \langle uv, f \rangle$	$\langle \tilde{v}uf', g \rangle - \langle \tilde{v}(uf)', g \rangle$	0	$\langle uv, g \rangle - \langle \tilde{u}v', g \rangle$
$a^+(u)$	$-\langle v, u \rangle$	$-\langle \tilde{v}u', g \rangle$	$\langle \tilde{v}u', g \rangle - \langle uv, g \rangle$	0

This gives a formal expression for the commutator densities:

$$[a_s^\ominus, a_t^+]\Psi_f = \mathbf{1}_{[s,\infty)}(\cdot)\delta'_\cdot(t) \circ \Psi_f,$$

$$[a_s^\oplus, a_t^+]\Psi_f = \delta(s-t) \otimes \delta(s-\cdot) \circ \Psi_f - (\mathbf{1}_{[s,\infty)}(\cdot)\delta'_\cdot(t)) \circ \Psi_f,$$

$$[a_s^\ominus, a_t^-]\Psi_f = \mathbf{1}_{[s,\infty)}(t)f'(t) \otimes \Psi_f,$$

$$[a^\oplus, a_t^-]\Psi_f = -f(t)\delta(s-t) \otimes \Psi_f - \mathbf{1}_{[s,\infty)}(t)f'(t) \otimes \Psi_f,$$

$$[a_s^\ominus, a_t^\oplus]\Psi_f = -f(t) \otimes (\mathbf{1}_{[s,\infty)}(\cdot)\delta'_\cdot(t)) \circ \Psi_f + \mathbf{1}_{[s,\infty)}(t)f'(t) \otimes \delta(t-\cdot) \circ \Psi_f.$$

These relations are only formal, and it is the goal of the next section to remove the infinities and give a meaning to these relations. In fact, the compositions on $\mathcal{S}$ of

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operators such as $\tilde{A}_t^+ \tilde{A}_t^+ = \tilde{a}^+(\mathbf{1}_{[0,t]})\tilde{a}^+(\mathbf{1}_{[0,t]})$ and $\tilde{A}_t^- \tilde{A}_t^+ = \tilde{a}^-(\mathbf{1}_{[0,t]})\tilde{a}^+(\mathbf{1}_{[0,t]})$ are not defined in $L^2(\Omega)$, due to the non-differentiability of N_t in the jump times T_k, $k \geq 1$. The redefinition introduced in the next section will be it make it possible to formulate a closed commutator table and to treat $A_t^\ominus$ and $A_t^\oplus$ as if they were adapted processes, or as if $[A_t^\varepsilon, dA_t^\eta] = 0$.

6 Renormalized commutator table

It has been shown in [2] that quantum stochastic calculus for powers of annihilation and creation densities can make sense through a renormalization procedure. For example we have formally

$$[a_t^{-2}, a_s^{+2}] = 2(\delta(s-t))^2 + 4\delta(s-t)a_t^\circ,$$

and this relation is given sense by renormalizing the square $(\delta(s-t))^2$ of the Dirac distribution as $(\delta(s-t))^2 = c\delta(s-t)$ where c is an arbitrary constant, which gives

$$[a_t^{-2}, a_s^{+2}] = 2c\delta(s-t)^2 + 4\delta(s-t)a_t^\circ.$$

The annihilation and creation densities $a_t^\ominus$ and $a_t^\oplus$ are in fact already higher powers of noise, since their sum is a_t°, and their composition itself leads to infinities. In this section we will define a modified composition of operators for which $A_t^\ominus$ and $A_t^\oplus$ behave as adapted processes, i.e. the commutator table associated to this composition is identical to (5.1). Consider the integration by parts

$$\langle \tilde{u}v', f \rangle = \langle uv, f \rangle - \langle \tilde{u}v, f' \rangle, \qquad u, v, f \in \mathcal{C}_c^\infty(\mathbb{R}_+), \tag{6.1}$$

which is the integrated form of

$$\tilde{u}(t)v'(t)f(s)\delta(t-s) = u(t)v(t)f(s)\delta(t-s) + \tilde{u}(t)v(t)f'(s)\delta_t'(s) \tag{6.2}$$

with respect to ds and dt. Redefinition will consist here in removing the term

$$\tilde{u}(t)v(t)f'(s)\delta_t'(s)$$

in (6.2) (i.e. the term $-\langle \tilde{u}v, f' \rangle$ in (6.1)), or equivalently in replacing $\tilde{u}v$ with uv in (5.2)-(5.6), to get

$$[a^\ominus(u), a^+(v)]'\Psi_f = (uv) \circ \Psi_f = a^+(uv)\Psi_f,$$
$$[a^-(u), a^\ominus(v)]'\Psi_f = 0,$$
$$[a^\ominus(u), a^\oplus(v)]'\Psi_f = (uvf) \circ \Psi_f = a^\circ(uv) \circ \Psi_f,$$
$$[a^\oplus(u), a^+(v)]'\Psi_f = 0,$$
$$[a^-(u), a^\oplus(v)]'\Psi_f = \langle uv, f \rangle \otimes \Psi_f = a^-(uv)\Psi_f, \quad u \in L^2(\mathbb{R}_+) \cap L^4(\mathbb{R}_+),$$

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where the notation $[\cdot, \cdot]'$ is used to denote the redefined commutation relation. This implies the redefined composition rules

$$a^{\ominus}(u)a^{+}(v) := a^{+}(v)a^{\ominus}(u) + a^{+}(uv), \qquad (6.3)$$

$$a^{\ominus}(v)a^{-}(u) := a^{-}(u)a^{\ominus}(v), \qquad (6.4)$$

$$a^{\oplus}(u)a^{+}(v) := a^{+}(v)a^{\oplus}(u), \qquad (6.5)$$

$$a^{\oplus}(v)a^{-}(u) := a^{-}(u)a^{\oplus}(v) - a^{-}(uv), \qquad (6.6)$$

$$a^{\ominus}(u)a^{\circ}(v) := a^{\circ}(v)a^{\ominus}(u) + a^{\circ}(uv) \qquad (6.7)$$

$$a^{\oplus}(u)a^{\circ}(v) := a^{\circ}(v)a^{\oplus}(u) - a^{\circ}(uv), \qquad (6.8)$$

where the right-hand sides make sense as ordinary composition of operators. The commutator table becomes

$[\cdot, \cdot]'$	$a^{-}(v)$	$a^{\ominus}(v)$	$a^{\oplus}(v)$	$a^{+}(v)$
$a^{-}(u)$	0	0	$a^{-}(uv)$	$\langle u, v \rangle$
$a^{\ominus}(u)$	0	0	$a^{\circ}(uv)$	$a^{+}(uv)$
$a^{\oplus}(u)$	$-a^{-}(uv)$	$-a^{\circ}(vu)$	0	0
$a^{+}(u)$	$-\langle v, u \rangle$	$-a^{+}(vu)$	0	0

$\qquad (6.9)$

$u, v \in L^2(\mathbb{R}_+) \cap L^4(\mathbb{R}_+)$, or in terms of matrix elements:

$\langle [\cdot, \cdot]' \Psi_f, \Psi_g \rangle$	$a^{-}(v)$	$a^{\ominus}(v)$	$a^{\oplus}(v)$	$a^{+}(v)$
$a^{-}(u)$	0	0	$\langle uv, f \rangle$	$\langle u, v \rangle$
$a^{\ominus}(u)$	0	0	$\langle uvf, g \rangle$	$\langle uv, g \rangle$
$a^{\oplus}(u)$	$-\langle uv, f \rangle$	$-\langle vuf, g \rangle$	0	0
$a^{+}(u)$	$-\langle v, u \rangle$	$-\langle vu, g \rangle$	0	0

This implies in particular

$$d[A_t^{\varepsilon}, A_t^{\eta}]' = [dA_t^{\varepsilon}, dA_t^{\eta}], \qquad \varepsilon, \eta = \ominus, \oplus, -, +,$$

i.e. the table (6.9) is deduced from the table (5.1) as if $t \mapsto a^{\ominus}(1_{[0,t]})$ and $t \mapsto a^{\oplus}(1_{[0,t]})$ were adapted processes. Let $N(u)$ denote the (non-compensated) Poisson noise:

$$N(u) = \int_0^{\infty} u(s)ds + \tilde{a}^{-}(u) + \tilde{a}^{+}(u).$$

The commutator table implies for $u, v \in L^2(\mathbb{R}_+) \cap L^4(\mathbb{R}_+)$:

$[\cdot, \cdot]'$	$\tilde{a}^{-}(v)$	$\tilde{a}^{+}(v)$
$\tilde{a}^{-}(u)$	0	0
$\tilde{a}^{+}(u)$	0	$N(uv)$

$\qquad (6.10)$

i.e. the commutator $[\tilde{A}_t^-, \tilde{A}_t^+]$ is the quadratic variation N_t of the compensated Poisson process $N_t - t$. For a more probabilistic approach to the above redefinition, we have the commutation relation

$$[\tilde{a}^-(u), \tilde{a}^+(v)] = \sum_{k=1}^\infty \tilde{u}(T_k)v'(T_k),$$

and the above redefinition states that

$$[\tilde{a}^-(u), \tilde{a}^+(v)]' = \sum_{k=1}^\infty u(T_k)v(T_k) = \int_0^\infty u(t)v(t)dN_t.$$

In particular, the powers of the operators $a^\varepsilon(u)$, $\varepsilon = -, +, \ominus, \oplus$ are now defined as operators on $\mathcal{S}$ for $u \in \cap_{p \geq 2} L^p(\mathbb{R}_+)$.

We close this section with an interpretation of the above renormalization in terms of operator densities. In

$$[a_s^\ominus, a_t^+]\Psi_f = \mathbf{1}_{[s,\infty)}(\cdot)\delta'_.(t) \circ \Psi_f,$$
$$[a_s^\oplus, a_t^+]\Psi_f = \delta(s-t) \otimes \delta(s-\cdot) \circ \Psi_f - (\mathbf{1}_{[s,\infty)}(\cdot)\delta'_.(t)) \circ \Psi_f,$$

we replaced the second quantization

$$\mathbf{1}_{[s,\infty)}(\cdot)\delta'(t-\cdot) \circ \Psi_f$$

of $\mathbf{1}_{[s,\infty)}(\cdot)\delta'(t-\cdot)$ by

$$\delta(s-t) \otimes \delta(s-\cdot) \circ \Psi_f,$$

to obtain:

$$[a_s^\ominus, a_t^+]'\Psi_f = \delta(s-t) \otimes \delta(s-\cdot) \circ \Psi_f,$$
$$[a_s^\oplus, a_t^+]'\Psi_f = 0.$$

By duality from the last relation (or using (6.4) and (6.5)), we have

$$[a_s^\ominus, a_t^-]'\Psi_f = 0,$$

i.e. in

$$[a_s^\ominus, a_t^-]\Psi_f = \mathbf{1}_{[s,\infty)}(t)f'(t) \otimes \Psi_f,$$

we replaced $\mathbf{1}_{[s,\infty)}(t)f'(t)$ by 0. With this rule, from

$$[a^\oplus, a_t^-]\Psi_f = -f(t)\delta(s-t) \otimes \Psi_f - \mathbf{1}_{[s,\infty)}(t)f'(t) \otimes \Psi_f,$$
$$[a_s^\ominus, a_t^\oplus]\Psi_f = -f(t) \otimes (\mathbf{1}_{[s,\infty)}(\cdot)\delta'_.(t)) \circ \Psi_f + \mathbf{1}_{[s,\infty)}(t)f'(t) \otimes \delta(t-\cdot) \circ \Psi_f,$$

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we obtain

$$[a^{\oplus}, a_t^-]'\Psi_f = f(t)\delta(s-t) \otimes \Psi_f,$$
$$[a_s^{\ominus}, a_t^{\oplus}]'\Psi_f = f(t)\delta(t-s) \otimes \delta(t-\cdot) \circ \Psi_f.$$

To summarize, the commutator densities become

$$[a_s^{\ominus}, a_t^+]'\Psi_f = \delta(s-t) \otimes \delta(s-\cdot) \circ \Psi_f,$$
$$[a_s^{\oplus}, a_t^+]'\Psi_f = 0,$$
$$[a_s^{\ominus}, a_t^-]'\Psi_f = 0,$$
$$[a^{\oplus}, a_t^-]'\Psi_f = f(t)\delta(s-t) \otimes \Psi_f,$$
$$[a_s^{\ominus}, a_t^{\oplus}]'\Psi_f = f(t)\delta(t-s) \otimes \delta(t-\cdot) \circ \Psi_f.$$

7 Lévy processes in continuous time on sl_2 and on the finite difference algebra fd

In this section we construct continuous time realizations of the gamma and continuous binomial processes, using the commutator tables of Sect. 6. We let:

$$q(u) = a^-(u) + a^+(u), \qquad\qquad p(u) = i(a^-(u) - a^+(u)),$$
$$\tilde{q}(u) = \tilde{a}^-(u) + \tilde{a}^+(u), \qquad\qquad \tilde{p}(u) = i(\tilde{a}^-(u) - \tilde{a}^+(u)),$$
$$q^{\circ}(u) = a^{\ominus}(u) + a^{\oplus}(u) = a^{\circ}(u), \qquad p^{\circ}(u) = i(a^{\ominus}(u) - a^{\oplus}(u)),$$

and

$$Q_t = q(\mathbf{1}_{[0,t]}) = A_t^- + A_t^+, \qquad\qquad P_t = p(\mathbf{1}_{[0,t]}) = i(A_t^- - A_t^+),$$
$$\tilde{Q}_t = \tilde{q}(\mathbf{1}_{[0,t]}) = \tilde{A}_t^- + \tilde{A}_t^+, \qquad\qquad \tilde{P}_t = \tilde{p}(\mathbf{1}_{[0,t]}) = i(\tilde{A}_t^- - \tilde{A}_t^+),$$
$$Q_t^{\circ} = q^{\circ}(\mathbf{1}_{[0,t]}) = A_t^{\ominus} + A_t^{\oplus} = A_t^{\circ}, \qquad P_t^{\circ} = p^{\circ}(\mathbf{1}_{[0,t]}) = i(A_t^{\ominus} - A_t^{\oplus}), \quad t \in \mathbb{R}_+.$$

The Itô table becomes

$\cdot$	dQ_t	dQ_t°	dP_t°	dP_t
dP_t	idt	idA_t^-	dA_t^-	dt
dP_t°	idA_t^+	idQ_t°	dQ_t°	dA_t^+
dQ_t°	dA_t^+	dQ_t°	$-idQ_t^{\circ}$	$-idA_t^+$
dQ_t	dt	dA_t^-	$-idA_t^-$	$-idt$

cf. [14], with commutator table

$[\cdot,\cdot]'$	Q_t	Q_t°	P_t°	P_t
Q_t	0	$-iP_t$	$-iQ_t$	-2it
Q_t°	iP_t	0	$-2iQ_t^\circ$	$-iQ_t$
P_t°	iQ_t	$2iQ_t^\circ$	0	iP_t
P_t	$2it$	iQ_t	$-iP_t$	0

In particular, the commutator table

$[\cdot,\cdot]'$	Q_t°	$-iP_t^\circ$
Q_t°	0	$-2Q_t^\circ$
$-iP_t^\circ$	$2Q_t^\circ$	0

is a representation of the Lie algebra of upper triangular 2×2 matrices, generated respectively by

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

whereas the table

$[\cdot,\cdot]$	Q_t	$-iP_t$
Q_t	0	$-2t$
$-iP_t$	$2t$	0

(7.1)

is the well-known representation of the Heisenberg-Weyl Lie algebra hw. This shows that $\tilde{A}_t^-$ and $\tilde{A}_t^+$ can be used to construct a representation of the finite difference algebra which is isomorphic to the algebra of triangular 2×2 matrices.

Let $\{I, M, B^-, B^+\}$ be a basis of the Lie algebra gl_2 of the linear group $\mathsf{GL}(2;\mathbb{R})$, such that $\{M, B^-, B^+\}$ is a basis of the Lie algebra sl_2, i.e.

$$[B^-, B^+] = M, \quad [M, B^-] = -2B^-, \quad [M, B^+] = 2B^+.$$

We fix $\varphi \in \mathbb{R}$ and let

$$T = M + e^{-i\varphi}B^- + e^{i\varphi}B^+, \quad U = (M-I)/2 + e^{-i\varphi}B^-, \quad V = (M-I)/2 + e^{i\varphi}B^+.$$

Then $\{T, U, V\}$ is a basis of the algebra fd, called the finite difference algebra:

$$[U, V] = [T, V] = [U, T] = T.$$

The following proposition is an immediate consequence of the commutator tables (6.9) or (6.10).

Proposition 7.1 *A family* $(j_{st})_{0 \leq s < t}$ *of representations of the finite difference Lie algebra* fd *(or Lévy process on* fd*) is defined by*

$$j_{st}(T) = N(\mathbf{1}_{(s,t]}), \quad j_{st}(U) = \tilde{a}^-(\mathbf{1}_{(s,t]}), \quad j_{st}(V) = \tilde{a}^+(\mathbf{1}_{(s,t]}),$$

The family $(j_{st})_{0 \leq s < t}$ also defines a representation of the current algebra

$$\mathsf{fd}^{\mathbf{R}_+} = \left\{ \sum_{i=1}^{i=n} X_i \mathbf{1}_{(t_{i-1}, t_i]} \; : \; X_i \in \mathsf{fd}, \; i = 1, \ldots, n, \; 0 \leq t_0 < \cdots < t_n, \; n \geq 1 \right\}.$$

We have $j_{st}(U)^* = j_{st}(V)$ and $j_{st}(T)^* = j_{st}(T)$. This Lévy process is the restriction to fd of the Lévy process $\tilde{j}_{st}$ on gl_2 of Sect. 3.5 in [1] with

$$\tilde{j}_{st}(B^-)\mathbf{1} = 0 \quad \text{and} \quad \tilde{j}_{st}(T)\mathbf{1} = (t-s)\mathbf{1}.$$

The following result holds only when writing the powers of operators in normal order according to the commutator tables (6.9) or (6.10) (according to the classical commutator table, $\tilde{A}_t^- + \tilde{A}_t^+ = Q_t + Q_t^\circ = A_t^- + A_t^+ + A_t^\circ$ is a compensated Poisson process).

Proposition 7.2 *i) The process*

$$t \mapsto \tilde{A}_t^- + \tilde{A}_t^+ = Q_t + Q_t^\circ = \tilde{Q}_t = A_t^- + A_t^+ + A_t^\circ, \quad t \in \mathbb{R}_+,$$

is a realization of the compensated gamma process (cf. [3], [4]).

ii) The process

$$t \mapsto i(\tilde{A}_t^- - \tilde{A}_t^+) = P_t + P_t^\circ = \tilde{P}_t = i(A_t^- - A_t^+ + A_t^\ominus - A_t^\oplus), \quad t \in \mathbb{R}_+,$$

is the continuous binomial process, cf. Sect. 5.4.1 of [8].

Proof. We have for $\varphi = 0$:

$$t + \tilde{a}^-(\mathbf{1}_{[0,t]}) + \tilde{a}^+(\mathbf{1}_{[0,t]}) = j_{0,t}(T) = j_{0,t}(B^- + B^+ + M),$$

and Cor. 4.2 of [1] shows that $j_{0,t}(T) = j_{0,t}(B^- + B^+ + M)$ is the gamma or exponential process with density $x \mapsto \Gamma(t)^{-1} x^{t-1} e^{-x} \mathbf{1}_{[0,\infty)}(x)$ and mean t. For $\varphi = \pi/2$ we have

$$i(\tilde{a}^-(\mathbf{1}_{[0,t]}) - \tilde{a}^+(\mathbf{1}_{[0,t]})) = j_{0,t}(B^- + B^+) = ij_{0,t}(U - V)$$

and Cor. 4.2 of [1] also shows that $j_{0,t}(B^- + B^+)$ has density

$$x \mapsto \frac{2^{t-1}}{\pi \Gamma(t)} \left| \Gamma\left(\frac{t+ix}{2}\right) \right|^2,$$

cf. (1.7.2) in [10] and Sect. 4.3.5 of [16], with characteristic function $u \mapsto (\cosh u)^{-t}$.

$\square$

In order to construct a representation of sl_2 itself, the conservation operator $\tilde{a}^\circ$ has been used in discrete time in Sect. 4. In this section, only a representation of fd has been constructed in Prop. 7.1. The definition of an operator density $\tilde{a}_t^+ \tilde{a}_t^-$ for $\tilde{A}_t^\circ$ in order to construct a representation of sl_2 seems beyond the redefinition described in this paper.

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