Introduction to Stochastic Finance with Market Examples, Second Edition

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Solutions Manual

Chapter 1

Exercise 1.1 The payoff C is that of a *put* option, whose strike price $K = \$3$ can be determined by trial and error.

Exercise 1.2 Each of the two possible scenarios yields one equation:

$$\begin{cases} 5\alpha + \beta = 0 \\ \\ 2\alpha + \beta = 6, \end{cases} \qquad \text{with solution} \qquad \begin{cases} \alpha = -2 \\ \\ \beta = +10. \end{cases}$$

The hedging strategy at $t = 0$ is to **shortsell** $-\alpha = +2$ units of the asset S priced $S_0 = 4$, and to put $\beta = \$10$ on the savings account. The price $V_0 = \alpha S_0 + \beta$ of the initial portfolio at time $t = 0$ is

$$V_0 = \alpha S_0 + \beta = -2 \times 4 + 10 = \$2,$$

which yields the price of the claim at time $t = 0$. In order to hedge then option, one should:

i) At time $t = 0$,

   a. Charge the \$2 option price.
   b. Shortsell $-\alpha = +2$ units of the stock priced $S_0 = 4$, which yields \$8.
   c. Put $\beta = \$8 + \$2 = \$10$ on the savings account.

ii) At time $t = 1$,

   a. If $S_1 = \$5$, spend \$10 from savings to buy back $-\alpha = +2$ stocks.
   b. If $S_1 = \$2$, spend \$4 from savings to buy back $-\alpha = +2$ stocks, and deliver a \$10 - \$4 = \$6 payoff.

Pricing the option by the expected value $\mathbb{E}^*[C]$ yields the equality

$$\begin{aligned} \$2 &= \mathbb{E}^*[C] \\ &= 0 \times \mathbb{P}^*(C = 0) + 6 \times \mathbb{P}^*(C = 6) \\ &= 0 \times \mathbb{P}^*(S_1 = 2) + 6 \times \mathbb{P}^*(S_1 = 5) \\ &= 6 \times q^*, \end{aligned}$$

hence the risk-neutral probability measure $\mathbb{P}^*$ is given by

$$p^* = \mathbb{P}^*(S_1 = 5) = \frac{2}{3} \quad \text{and} \quad q^* = \mathbb{P}^*(S_1 = 2) = \frac{1}{3}.$$

Exercise 1.3

a) Each of the stated conditions yields one equation, *i.e.*

$$\begin{cases} 4\alpha + \beta = 1 \\ \\ 5\alpha + \beta = 3, \end{cases} \qquad \text{with solution} \qquad \begin{cases} \alpha = 2 \\ \\ \beta = -7. \end{cases}$$

Therefore, the portfolio allocation at $t = 0$ consists to purchase $\alpha = 2$ unit of the asset S priced $S_0 = 4$, and to borrow $-\beta = \$7$ in cash.

We can check that the price $V_0 = \alpha S_0 + \beta$ of the initial portfolio at time $t = 0$ is

$$V_0 = \alpha S_0 + \beta = 2 \times 4 - 7 = \$1.$$

b) This loss is expressed as

$$\alpha \times \$2 + \beta = 2 \times 2 - 7 = -\$3.$$

Note that the \$1 received when selling the option is not counted here because it has already been fully invested into the portfolio.

Exercise 1.4

a) i) Does this model allow for arbitrage? Yes | ✓ No |

ii) If this model allows for arbitrage opportunities, how can they be realized? By shortselling | By borrowing on savings | ✓ N.A. |

b) i) Does this model allow for arbitrage? Yes | No | ✓

ii) If this model allows for arbitrage opportunities, how can they be realized? By shortselling | By borrowing on savings | N.A. | ✓

c) i) Does this model allow for arbitrage? Yes | ✓ No |

ii) If this model allows for arbitrage opportunities, how can they be realized? By shortselling | ✓ By borrowing on savings | N.A. |

Exercise 1.5

a) We need to search for possible risk-neutral probability measure(s) $\mathbb{P}^*$ such that $\mathbb{E}^*\left[S_1^{(1)}\right] = (1+r)S_0^{(1)}$. Letting

$$\begin{cases} p^* = \mathbb{P}^*\big(S_1^{(1)} = S_0^{(1)}(1+a)\big) = \mathbb{P}^*\left(\dfrac{S_1^{(1)} - S_0^{(1)}}{S_0^{(1)}} = a\right), \\ \\ \theta^* = \mathbb{P}^*\big(S_1^{(1)} = S_0^{(1)}(1+b)\big) = \mathbb{P}^*\left(\dfrac{S_1^{(1)} - S_0^{(1)}}{S_0^{(1)}} = b\right), \\ \\ q^* = \mathbb{P}^*\big(S_1^{(1)} = (1+c)S_0^{(1)}\big) = \mathbb{P}^*\left(\dfrac{S_1^{(1)} - S_0^{(1)}}{S_0^{(1)}} = c\right), \end{cases}$$

We have

$$\begin{cases} (1+a)p^*S_0^{(1)} + (1+b)\theta^*S_0^{(1)} + (1+c)q^*S_0^{(1)} = (1+r)S_0^{(1)} \\ \\ p^* + \theta^* + q^* = 1, \end{cases}$$

from which we obtain

$$\begin{cases} p^*a + \theta^*b + q^*c = r, \\ \\ p^* + \theta^* + q^* = 1. \end{cases} \quad \Longrightarrow \quad \begin{cases} p^* = \dfrac{(1-\theta^*)c + \theta^*b - r}{c-a} \in (0,1), \\ \\ q^* = \dfrac{r - (1-\theta^*)a - \theta^*b}{c-a} \in (0,1). \end{cases}$$

In order for p^* and q^* to belong to the interval $(0,1)$, we should have

$$\begin{cases} 0 < (1-\theta^*)c + \theta^*b - r < c - a, \\ \\ 0 < r - (1-\theta^*)a - \theta^*b < c - a, \end{cases}$$

i.e.

$$\begin{cases} \dfrac{r-c}{b-c} < \theta^* < \dfrac{r-a}{b-c}, \\ \\ \dfrac{r-c}{b-a} < \theta^* < \dfrac{r-a}{b-a}. \end{cases}$$

Therefore, there exists an infinity of risk-neutral probability measures depending on the value of

$$\theta^* \in \left(\max\left(\frac{r-c}{b-c}, \frac{r-c}{b-a}\right), \min\left(\frac{r-a}{b-c}, \frac{r-a}{b-a}\right)\right),$$

in which case the market is without arbitrage but not complete. This is the case when $a < r < c$.

b) Hedging a claim with possible payoff values C_a, C_b, C_c would require to solve

$$\begin{cases} (1+a)\xi^{(1)}S_0^{(1)} + (1+r)\xi^{(0)}S_0^{(0)} = C_a \\ (1+b)\xi^{(1)}S_0^{(1)} + (1+r)\xi^{(0)}S_0^{(0)} = C_b \\ (1+c)\xi^{(1)}S_0^{(1)} + (1+r)\xi^{(0)}S_0^{(0)} = C_c, \end{cases}$$

for $\xi^{(0)}$ and $\xi^{(1)}$, which is not possible in general due to the existence of three conditions with only two unknowns.

Exercise 1.6

a) The risk-neutral condition $\mathbb{E}^*[R_1] = 0$ reads

$$b\mathbb{P}^*(R_1 = b) + 0 \times \mathbb{P}^*(R_1 = 0) + (-b) \times (R_1 = -b) = bp^* - bq^* = 0,$$

hence

$$p^* = q^* = \frac{1-\theta^*}{2},$$

since $p^* + q^* + \theta^* = 1$.

b) We have

$$\begin{aligned} \mathrm{Var}^*\left[\frac{S_1^{(1)} - S_0^{(1)}}{S_0^{(1)}}\right] &= \mathbb{E}^*\left[R_1^2\right] - (\mathbb{E}^*[R_1])^2 \\ &= \mathbb{E}^*\left[R_1^2\right] \\ &= b^2\mathbb{P}^*(R_1 = b) + 0^2 \times \mathbb{P}^*(R_1 = 0) + (-b)^2 \times (R_1 = -b) \\ &= b^2(p^* + q^*) \\ &= b^2(1-\theta^*) \\ &= \sigma^2, \end{aligned}$$

hence $\theta^* = 1 - \sigma^2/b^2$, and therefore

$$p^* = q^* = \frac{1-\theta^*}{2} = \frac{\sigma^2}{2b^2},$$

provided that $\sigma^2 \leq b^2$.

Exercise 1.7

a) We denote the risk-neutral measure by $p^* = \mathbb{P}^*(S_1 = 2)$, $q^* = \mathbb{P}^*(S_1 = 1)$.

i) | Yes | | No | ✓ | *Comment*: No loss is possible, while a 100% profit is possible with non-zero probability 1/3.

ii) | Yes \| | | No \| ✓ | *Comment*: The (unique) risk-neutral measure $(p^*, q^*) = (0, 1)$ is given by

$$\$2 \times p^* + \$1 \times q^* = \$1 \times (1+r) = \$1 \quad \text{and} \quad p^* + q^* = 1,$$

and is not equivalent to $\mathbb{P}$ given by $(p, q) = (1/3, 2/3)$.

b) We denote the risk-neutral measure by $p^* = \mathbb{P}^*(S_1 = 2)$, $\theta^* = \mathbb{P}^*(S_1 = 1)$, $q^* = \mathbb{P}^*(S_1 = 0)$.

i) | Yes \| ✓ | | No \| | *Comment*: The risk-neutral measure (p^*, θ^*, q^*) is given by the equations

$$\$2 \times p^* + \$1 \times \theta^* + \$0 \times q^* = \$1 \times (1+r) = \$1 \ \text{ and } \ p^* + \theta^* + q^* = 1, \tag{S.1.1}$$

which clearly admit solutions, see (biv) below.

ii) | Yes \| ✓ | | No \| | *Comment*: Realizing arbitrage would mean building a portfolio achieving no strictly negative return with probability one, which is impossible since the probability of 100% loss is $\mathbb{P}(S_1 = 0) = 1 - 1/4 - 1/9 = 23/36 > 0$.

iii) | Yes \| | | No \| ✓ | *Comment*: Examples of claims that cannot be attained can be easily constructed in this market. For example, the claim $\mathbb{1}_{\{S_1>0\}}$ cannot be attained since there is no portfolio allocation (α, β) satisfying

$$\begin{cases} \$2 \times \alpha + \beta = \$1 \\ \$1 \times \alpha + \beta = \$1 \\ \$0 \times \alpha + \beta = \$0. \end{cases}$$

iv) | Yes \| | | No \| ✓ | *Comment*: The risk-neutral measure is clearly not unique, as for example

$$(p^*, \theta^*, q^*) = (1/4, 1/2, 1/4) \quad \text{and} \quad (p^*, \theta^*, q^*) = (1/3, 1/3, 1/3)$$

are both solutions of (S.1.1).

Exercise 1.8

a) The possible values of R are a and b.
b) We have

$$\mathbb{E}^*[R] = a\mathbb{P}^*(R = a) + b\mathbb{P}^*(R = b)$$

$$= a\frac{b-r}{b-a} + b\frac{r-a}{b-a}$$
$$= r.$$

c) By Theorem 1.6, there do not exist arbitrage opportunities in this market since from Question (b) there exists a risk-neutral probability measure $\mathbb{P}^*$ whenever $a < r < b$.

d) The risk-neutral probability measure is unique hence the market model is complete by Theorem 1.13.

e) Taking

$$\eta = \frac{\alpha(1+b) - \beta(1+a)}{\pi_1(b-a)} \quad \text{and} \quad \xi = \frac{\beta - \alpha}{S_0(b-a)},$$

we check that

$$\begin{cases} \eta\pi_1 + \xi S_0(1+a) = \alpha & \text{if } R = a, \\ \\ \eta\pi_1 + \xi S_0(1+b) = \beta & \text{if } R = b, \end{cases}$$

which shows that

$$\eta\pi_1 + \xi S_1 = C$$

in both cases $R = a$ and $R = b$.

f) We have

$$\begin{aligned} \pi_0(C) &= \eta\pi_0 + \xi S_0 \\ &= \frac{\alpha(1+b) - \beta(1+a)}{(1+r)(b-a)} + \frac{\beta - \alpha}{b-a} \\ &= \frac{\alpha(1+b) - \beta(1+a) - (1+r)(\alpha - \beta)}{(1+r)(b-a)} \\ &= \frac{\alpha b - \beta a - r(\alpha - \beta)}{(1+r)(b-a)}. \end{aligned} \tag{S.1.2}$$

g) We have

$$\begin{aligned} \mathbb{E}^*[C] &= \alpha\mathbb{P}^*(R = a) + \beta\mathbb{P}^*(R = b) \\ &= \alpha\frac{b-r}{b-a} + \beta\frac{r-a}{b-a}. \end{aligned} \tag{S.1.3}$$

h) Comparing (S.1.2) and (S.1.3) above, we do obtain

$$\pi_0(C) = \frac{1}{1+r}\,\mathbb{E}^*[C]$$

i) The initial value $\pi_0(C)$ of the portfolio is interpreted as the arbitrage-free price of the option contract and it equals the expected value of the discounted payoff.

j) We have

$$C = (K - S_1)^+ = (11 - S_1)^+ = \begin{cases} 11 - S_1 & \text{if } K > S_1, \\ \\ 0 & \text{if } K \leq S_1. \end{cases}$$

k) We have $S_0 = 1$, $a = 8$, $b = 11$, $\alpha = 2$, $\beta = 0$, hence

$$\xi = \frac{\beta - \alpha}{S_0(b - a)} = \frac{0 - 2}{11 - 8} = -\frac{2}{3},$$

and

$$\eta = \frac{\alpha(1 + b) - \beta(1 + a)}{\pi_1(b - a)} = \frac{24}{3 \times 1.05}.$$

l) The arbitrage-free price $\pi_0(C)$ of the contingent claim with payoff C is

$$\pi_0(C) = \eta\pi_0 + \xi S_0 = 6.952.$$

Exercise 1.9 Letting R denote the price of one right, it will require $10R/3$ to purchase one stock at €6.35, hence absence of arbitrage tells us that

$$\frac{10}{3}R + 6.35 = 8,$$

from which it follows that

$$R = \frac{3}{10}(8 - 6.35) = €0.495.$$

Note that the actual share right was quoted at €0.465 according to market data. See also Exercise 17.8 for the pricing of convertible bonds.

Exercise 1.10 Let $a := (152 - 180)/180 = -7/45$ and $b := (203 - 180)/180 = 23/180$ denote the potential market returns, with $r = 0.03$. From the strike price K and the risk-neutral probabilities

$$p_r^* = \frac{r - a}{b - a} = 0.6549 \quad \text{and} \quad q_r^* = \frac{b - r}{b - a} = 0.3451,$$

the price of the option at the beginning of the year is given from Proposition 1.16 as the discounted expected value

$$\frac{1}{1 + r}\mathbb{E}^*[(K - S_1)^+] = \frac{1}{1 + r}\left(p_r^*(K - 203)^+ + q_r^*(K - 152)^+\right).$$

Equating this price with the intrinsic value $(K - 180)^+$ of the put option yields the equation

$$(K - 180)^+ = \frac{1}{1 + r}\left(p_r^*(K - 203)^+ + q_r^*(K - 152)^+\right)$$

which requires $K > 180$ (the case $K \leq 152$ is not considered because both the option price and option payoff vanish in this case). Hence we consider the equation

$$K - 180 = \frac{1}{1+r}\left(p_r^*(K-203)^+ + q_r^*(K-152)^+\right),$$

with the following cases.

i) If $K \in [180, 203]$ we get

$$(1+r)(K-180) = q_r^*(K-152),$$

hence

$$K = \frac{(1+r)180 - q_r^*152}{1+r-q_r^*} = \frac{(1+r)180 - q_r^*152}{p_r^*+r} = 194.11.$$

ii) If $K \geq 203$ we find

$$K = \frac{180(1+r) - 203p_r^* - 152q_r^*}{r} < 203,$$

which is out of range and leads to a contradiction.

We note that the above formula

$$K = \frac{(1+r)180 - q_r^*152}{p_r^*+r} = \frac{28b - 180a + r(180(b-a)+152)}{(b+1-a)r - a}$$

yields a decreasing function $K(r)$ of r in the interval $[0, 100\%]$, although the function is not monotone over $\mathbb{R}_+$.

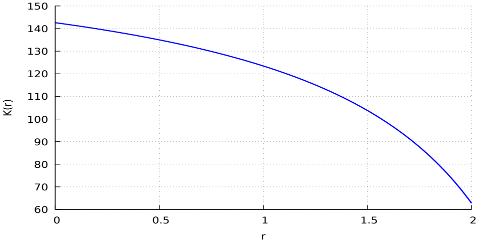


Fig. S.1: Strike price as a function of risk-free rate r.

Chapter 2

Exercise 2.1 Let $m := \$2,550$ denote the amount invested each year.

a) By (2.1), the value of the plan after $N = 10$ years becomes

$$m\sum_{k=1}^{N}(1+r)^k = m(1+r)\frac{(1+r)^N-1}{r},$$

which in turns becomes

$$(1+r)^N m\sum_{k=1}^{N}(1+r)^k = m(1+r)^{N+1}\frac{(1+r)^N-1}{r},$$

after N additional years without further contributions to the plan. Equating

$$A = 30835 = m(1+r)^{N+1}\frac{(1+r)^N-1}{r}$$

shows that

$$\frac{(1+r)^{2N+1}-(1+r)^{N+1}}{r} = \frac{A}{m},$$

with $m = 2550$, or

$$\frac{(1+r)^{21}-(1+r)^{11}}{r} = \frac{30835}{2550} \simeq 12.09215,$$

hence $r \simeq 1.23\%$ according to Figure S.2, which is typical of an annual fixed deposit interest rate.

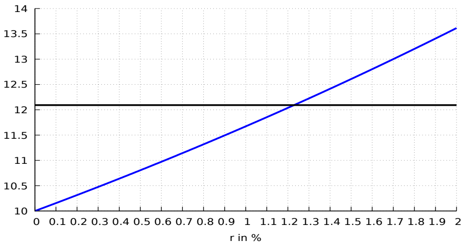


Fig. S.2: Graph of $r \longmapsto ((1+r)^{21}-(1+r)^{11})/r$.

In the hypothesis $r = 3.25\%$ we would find

$$A = m(1+r)^{N+1}\frac{(1+r)^N-1}{r} = 42040.42.$$

b) Taking $N = 10$, $m = 2,550$ and $r = 0.0325$, we find

$$\begin{aligned}
A_{2N} :&= m(1+r)^N \sum_{k=1}^{N} (1+r)^{N-k+1} \\
&= m(1+r)^N \sum_{k=1}^{N} (1+r)^k \\
&= m(1+r)^{N+1} \frac{(1+r)^N - 1}{r} \\
&= \$42,040.42.
\end{aligned}$$

c) In this case, with $N = 10$, $m = 2,550$ and $r = 0.0325$, we find

$$A_{2N} = A_N = m \sum_{k=1}^{N} (1+r)^{N-k+1} = m(1+r)\frac{(1+r)^N - 1}{r} = \$30,532.79.$$

Exercise 2.2

a) Let $m := \$3,581$ denote the amount invested each year. After multiplying (2.1) by $(1+r)^N$ in order to account for the compounded interest from year 11 until year 20, we get the equality

$$A = m(1+r)^{N+1} \frac{(1+r)^N - 1}{r}$$

shows that

$$(1+r)^{21} - (1+r)^{11} = r \frac{50862}{3581} \simeq 14.2033r,$$

showing that $r \simeq 2.28\%$ according to Figure S.3.

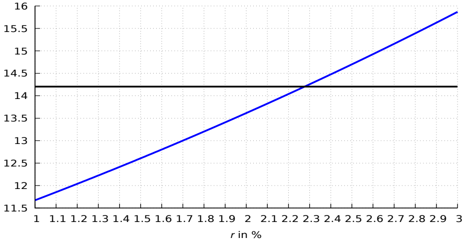


Fig. S.3: Graph of $r \longmapsto ((1+r)^{21} - (1+r)^{11})/r$.

b) Taking $N = 10$, $m = 3,581$ and $r = 0.0325$, we find

$$\begin{aligned} A_{2N} :&= m(1+r)^N \sum_{k=1}^{N} (1+r)^{N-k+1} \\ &= m(1+r)^N \sum_{k=1}^{N} (1+r)^k \\ &= m(1+r)^{N+1} \frac{(1+r)^N - 1}{r} \\ &= \$59,037.94. \end{aligned}$$

c) In this case, we find

$$A_{2N} = m \sum_{k=1}^{N} (1+r)^{N-k+1} = m(1+r)\frac{(1+r)^N - 1}{r} = \$42,877.61.$$

Exercise 2.3

a) We find $m = \$10,000$.
b) Denoting by A_k the amount owed by the borrower at the beginning of year n^o $k = 1, 2, \ldots, N$, the amount $A1 = A$ can be decomposed at the beginning of the first year as

$$A_1 = m + (A_1 - m),$$

where $A_1 - m$ is subject to interests at the rate $r = 2\%$ *i.e.* at the end of the first year there remains $A_2 = (A_1 - m)(1 + r)$ to be refunded. Similarly, the amount A_2 due at the beginning of the second year can be decomposed as $A_2 = m - (A_2 - m)$, *i.e.* at the end of the second year there remains

$$\begin{aligned} (A_2 - m)(1+r) &= ((A_1 - m)(1+r) - m)(1+r) \\ &= A_1(1+r)^2 - m(1+r)^2 - m(1+r) \end{aligned}$$

to be refunded. After repeating the argument, we find that at the end of year k there remains

$$\begin{aligned} (1+r)^k A_1 - m \sum_{l=1}^{k} (1+r)^l &= (1+r)^k A_1 - m(1+r)\frac{1-(1+r)^k}{1-(1+r)} \\ &= (1+r)^k A_1 + m(1+r)\frac{1-(1+r)^k}{r} \end{aligned}$$

to be refunded. At the end of year N, the loan will be completely repaid if hence $A_N = 0$, which reads

$$(1+r)^{N-1}A + m\frac{1-(1+r)^N}{r} = 0,$$

and yields

$$m = \frac{(1+r)^{N-1}rA}{(1+r)^N - 1} = \frac{rA}{(1+r)(1-(1+r)^{-N})}.$$

Taking $N = 10$, $A = 100,000$ and $r = 0.02$, we find

$$m = \frac{rA}{(1+r)(1-(1+r)^{-N})} = \frac{0.02 \times 100,000}{1.02 \times (1 - 1.02^{-10})} = \$10,914.36.$$

c) In this case, amount remaining on the account at the end of the first year is $(A-m)(1+r)$, and at the end of the second year it becomes $((A-m)(1+r)-m)(1+r)$. After repeating the argument, we find that at the end of year k there remains

$$\left((1+r)^{k-1}A - m\sum_{l=0}^{k-1}(1+r)^l\right)(1+r) = (1+r)^k A - m(1+r)\frac{(1+r)^k - 1}{r}$$

on the account. Therefore, what is left at the end of year N is

$$(1+r)^N A - m(1+r)\frac{(1+r)^N - 1}{r}.$$

Taking $N = 10$, $A = 100,000$ and $r = 0.02$, we find

$$1.02^{10} \times 100,000 - 10,000 \times 1.02 \times \frac{1.02^{10} - 1}{0.02} = \$10,212.29.$$

Exercise 2.4

a) By (2.1), the the discounted value of the loan after N months is

$$m(1+r)^{-N}\sum_{k=0}^{N-1}(1+r)^k = m(1+r)^{-N}\frac{1-(1+r)^N}{1-(1+r)} = m\frac{1-(1+r)^{-N}}{r},$$

which should match $A = \$3,000$ with $m = \$275$ and $N = 12$, hence as in Proposition 2.1 we have

$$\frac{1-(1+r)^{-12}}{r} = \frac{A}{m} = 10.909090909,$$

see Equation (2.2), hence

$$r \simeq 1.49767\% \quad \text{per month}.$$

b) The yearly interest rate is given by

$$(1+r)^N - 1 = (1+r)^{12} - 1 = 1.0149767^{12} - 1 \simeq 19.529\% \quad \text{per year.}$$

<u>*Remark:*</u> Computing the interest rate as

$$\frac{12 \times \$275}{\$3000} - 1 = 0.1 = 10\%$$

is not correct because this implicitly means that the $12 \times \$275 = \$3,300$ are repaid as one lump sum at the end of the $12th$ month, which is not the case.

c) The analysis of replies to Question (c) shows that "All of the above" was the most popular answer, followed by "Block".

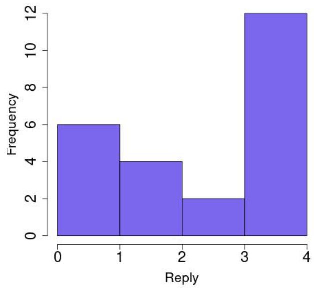


Fig. S.4: Histogram of replies to Question c).

Exercise 2.5 We check that for any $\mathbb{P}^*$ of the form

$$\mathbb{P}^*(R_t = -1) := p^*, \quad \mathbb{P}^*(R_t = 0) := 1 - 2p^*, \quad \mathbb{P}^*(R_t = 1) := p^*,$$

we have

$$\mathbb{E}^*[S_1] = S_0(2p^* + 1 - 2p^*) = S_0,$$

and similarly

$$\mathbb{E}^*[S_2 \mid S_1] = S_1(2p^* + (1 - 2p^*)) = S_1,$$

hence the probability measure $\mathbb{P}^*$ is risk-neutral.

Exercise 2.6

a) In order to check for arbitrage opportunities we look for a risk-neutral probability measure $\mathbb{P}^*$ which should satisfy

$$\mathbb{E}^*\left[S_{k+1}^{(1)} \mid \mathcal{F}_k\right] = (1+r)S_k^{(1)}, \qquad k = 0, 1, \ldots, N-1,$$

with $r = 0$. Rewriting $\mathbb{E}^*\left[S_{k+1}^{(1)} \mid \mathcal{F}_k\right]$ as

$$\begin{aligned}
&\mathbb{E}^*\left[S_{k+1}^{(1)} \mid \mathcal{F}_k\right] \\
&= (1-b)S_k^{(1)}\mathbb{P}^*(R_{k+1} = -b \mid \mathcal{F}_k) + S_k^{(1)}\mathbb{P}^*(R_{k+1} = 0 \mid \mathcal{F}_k) \\
&\quad + (1+b)S_k^{(1)}\mathbb{P}^*(R_{k+1} = b \mid \mathcal{F}_k) \\
&= (1-b)S_k^{(1)}\mathbb{P}^*(R_{k+1} = -b) + S_k^{(1)}\mathbb{P}^*(R_{k+1} = 0) + (1+b)S_k^{(1)}\mathbb{P}^*(R_{k+1} = b),
\end{aligned}$$

$k = 0, 1, \ldots, N-1$, it follows that any risk-neutral probability measure $\mathbb{P}^*$ should satisfy the equations

$$\begin{cases}
(1+b)S_k^{(1)}\mathbb{P}^*(R_{k+1} = b) + S_k^{(1)}\mathbb{P}^*(R_{k+1} = 0) + (1-b)S_k^{(1)}\mathbb{P}^*(R_{k+1} = -b) = S_k^{(1)} \\
\\
\mathbb{P}^*(R_{k+1} = b) + \mathbb{P}^*(R_{k+1} = 0) + \mathbb{P}^*(R_{k+1} = -b) = 1,
\end{cases}$$

$k = 0, 1, \ldots, N-1$, *i.e.*

$$\begin{cases}
b\mathbb{P}^*(R_k = b) - b\mathbb{P}^*(R_k = -b) = 0, \\
\\
\mathbb{P}^*(R_k = b) + \mathbb{P}^*(R_k = -b) = 1 - \mathbb{P}^*(R_k = 0),
\end{cases}$$

$k = 1, 2, \ldots, N$, with solution

$$\mathbb{P}^*(R_k = b) = \mathbb{P}^*(R_k = -b) = \frac{1-\theta^*}{2},$$

$k = 1, 2, \ldots, N$.

b) We have

$$\begin{aligned}
\mathbb{E}^*\left[\frac{S_{k+1}^{(1)} - S_k^{(1)}}{S_k^{(1)}} \middle| \mathcal{F}_k\right] &= \frac{1}{S_k^{(1)}}\,\mathbb{E}^*\left[S_{k+1}^{(1)} - S_k^{(1)} \mid \mathcal{F}_k\right] \\
&= \frac{1}{S_k^{(1)}}\left(\mathbb{E}^*\left[S_{k+1}^{(1)} \mid \mathcal{F}_k\right] - \mathbb{E}^*\left[S_k^{(1)} \mid \mathcal{F}_k\right]\right) \\
&= \frac{1}{S_k^{(1)}}\left(\mathbb{E}^*\left[S_{k+1}^{(1)} \mid \mathcal{F}_k\right] - S_k^{(1)}\right) \\
&= 0,
\end{aligned}$$

and

$$\mathrm{Var}^*\left[\frac{S_{k+1}^{(1)} - S_k^{(1)}}{S_k^{(1)}} \middle| \mathcal{F}_k\right]$$

$$
\begin{aligned}
&= \mathbb{E}^*\left[\left(\frac{S_{k+1}^{(1)} - S_k^{(1)}}{S_k^{(1)}}\right)^2 \middle| \mathcal{F}_k\right] - \left(\mathbb{E}^*\left[\frac{S_{k+1}^{(1)} - S_k^{(1)}}{S_k^{(1)}} \middle| \mathcal{F}_k\right]\right)^2 \\
&= \mathbb{E}^*\left[\left(\frac{S_{k+1}^{(1)} - S_k^{(1)}}{S_k^{(1)}}\right)^2 \middle| \mathcal{F}_k\right] \\
&= b^2 \mathbb{P}_\sigma^*(R_{k+1} = -b \mid \mathcal{F}_k) + b^2 \mathbb{P}_\sigma^*(R_{k+1} = b \mid \mathcal{F}_k) \\
&= b^2 \frac{1 - \mathbb{P}_\sigma^*(R_{k+1} = 0)}{2} + b^2 \frac{1 - \mathbb{P}_\sigma^*(R_{k+1} = 0)}{2} \\
&= b^2(1 - \theta^*) \\
&= \sigma^2,
\end{aligned}
$$

$k = 0, 1, \ldots, N-1$, hence

$$
\mathbb{P}_\sigma^*(R_k = 0) = \theta^* = 1 - \frac{\sigma^2}{b^2},
$$

and therefore

$$
\mathbb{P}_\sigma^*(R_k = b) = \mathbb{P}_\sigma^*(R_k = -b) = \frac{1 - \mathbb{P}_\sigma^*(R_k = 0)}{2} = \frac{\sigma^2}{2b^2},
$$

$k = 0, 1, \ldots, N-1$, under the condition $0 < \sigma^2 < b^2$.

Exercise 2.7

a) The possible values of R_t are a and b.
b) We have

$$
\begin{aligned}
\mathbb{E}^*[R_{t+1} \mid \mathcal{F}_t] &= a\mathbb{P}^*(R_{t+1} = a \mid \mathcal{F}_t) + b\mathbb{P}^*(R_{t+1} = b \mid \mathcal{F}_t) \\
&= a\frac{b-r}{b-a} + b\frac{r-a}{b-a} = r.
\end{aligned}
$$

c) Letting $p^* = (r-a)/(b-a)$ and $q^* = (b-r)/(b-a)$ we have

$$
\begin{aligned}
\mathbb{E}^*[S_{t+k} \mid \mathcal{F}_t] &= \sum_{i=0}^{k} (p^*)^i (q^*)^{k-i} \binom{k}{i} (1+b)^i (1+a)^{k-i} S_t \\
&= S_t \sum_{i=0}^{k} \binom{k}{i} \left(p^*(1+b)\right)^i \left(q^*(1+a)\right)^{k-i} \\
&= S_t \left(p^*(1+b) + q^*(1+a)\right)^k \\
&= S_t \left(\frac{r-a}{b-a}(1+b) + \frac{b-r}{b-a}(1+a)\right)^k \\
&= (1+r)^k S_t.
\end{aligned}
$$

Assuming that the formula holds for $k=1$, its extension to $k \geq 2$ can also be proved recursively from the tower property of conditional expectations, as follows:

$$\begin{aligned}
\mathbb{E}^*[S_{t+k} \mid \mathcal{F}_t] &= \mathbb{E}^*[\mathbb{E}^*[S_{t+k} \mid \mathcal{F}_{t+k-1}] \mid \mathcal{F}_t] \\
&= (1+r)\,\mathbb{E}^*[S_{t+k-1} \mid \mathcal{F}_t] \\
&= (1+r)\,\mathbb{E}^*[\mathbb{E}^*[S_{t+k-1} \mid \mathcal{F}_{t+k-2}] \mid \mathcal{F}_t] \\
&= (1+r)^2\,\mathbb{E}^*[S_{t+k-2} \mid \mathcal{F}_t] \\
&= (1+r)^2\,\mathbb{E}^*[\mathbb{E}^*[S_{t+k-2} \mid \mathcal{F}_{t+k-3}] \mid \mathcal{F}_t] \\
&= (1+r)^3\,\mathbb{E}^*[S_{t+k-3} \mid \mathcal{F}_t] \\
&= \cdots \\
&= (1+r)^{k-2}\,\mathbb{E}^*[S_{t+2} \mid \mathcal{F}_t] \\
&= (1+r)^{k-2}\,\mathbb{E}^*[\mathbb{E}^*[S_{t+2} \mid \mathcal{F}_{t+1}] \mid \mathcal{F}_t] \\
&= (1+r)^{k-1}\,\mathbb{E}^*[S_{t+1} \mid \mathcal{F}_t] \\
&= (1+r)^k S_t.
\end{aligned}$$

Exercise 2.8

a) We check that

$$\begin{aligned}
\mathbb{E}^*[R_{t+1} \mid \mathcal{F}_t] &= a\mathbb{P}^*(R_{t+1} = a \mid \mathcal{F}_t) + b\mathbb{P}^*(R_{t+1} = b \mid \mathcal{F}_t) \\
&= a\frac{b-r}{b-a} + b\frac{r-a}{b-a} = r.
\end{aligned}$$

b) We have

$$\begin{aligned}
\mathbb{E}^*\left[\widetilde{S}_{t+1} \,\middle|\, \mathcal{F}_t\right] &= \frac{1}{A_0(1+r)^{t+1}}\mathbb{E}^*[S_{t+1} \mid \mathcal{F}_t] \\
&= \frac{1}{A_0(1+r)^{t+1}}\big((1+a)S_t\mathbb{P}^*(R_{t+1} = a \mid \mathcal{F}_t) + (1+b)S_t\mathbb{P}^*(R_{t+1} = b \mid \mathcal{F}_t)\big) \\
&= \frac{1}{A_0(1+r)^{t+1}}\left((1+a)S_t\frac{b-r}{b-a} + (1+b)S_t\frac{r-a}{b-a}\right) \\
&= \widetilde{S}_t\frac{b-a+(b-a)r}{(1+r)(b-a)} \\
&= \widetilde{S}_t, \qquad t = 0, 1, \ldots, N-1.
\end{aligned}$$

c) We have

$$\mathbb{E}^*\left[(S_N)^\beta\right] = S_0\,\mathbb{E}^*\left[\left(\prod_{k=1}^N (1+R_k)\right)^\beta\right]$$

$$= S_0 \, \mathbb{E}^* \left[\prod_{k=1}^{N} (1+R_k)^\beta \right]$$
$$= S_0 \prod_{k=1}^{N} \mathbb{E}^* \left[(1+R_k)^\beta \right],$$

after using the independence of the returns $(R_k)_{k=1,2,\ldots,N}$, with

$$\mathbb{E}^* \left[(1+R_k)^\beta \right] = (1+a)^\beta \frac{b-r}{b-a} + (1+b)^\beta \frac{r-a}{b-a}, \quad k = 0,1,\ldots,N,$$

hence we find

$$\mathbb{E}^* \left[(S_N)^\beta \right] = S_0^\beta \left((1+a)^\beta \frac{b-r}{b-a} + (1+b)^\beta \frac{r-a}{b-a} \right)^N .$$

d) We have

$$\begin{aligned} \mathbb{P}^* \big(S_t \geq \alpha A_t \ \text{ for some } t \in \{0,1,\ldots,N\} \big) &= \mathbb{P}^* \left(\max_{t=0,1,\ldots,n} \frac{S_t}{A_t} \geq \alpha \right) \\ &\leq \frac{\mathbb{E}\left[(M_N)^\beta \right]}{\alpha^\beta} \\ &= \left(\frac{S_0}{(1+r)^N \alpha A_0} \right)^\beta \left((1+a)^\beta \frac{b-r}{b-a} + (1+b)^\beta \frac{r-a}{b-a} \right)^N , \end{aligned}$$

since the discounted price process

$$(M_t)_{t=0,1,\ldots,N} := \left(\frac{S_t}{A_t} \right)_{t=0,1,\ldots,N}$$

is a nonnegative martingale by part (b).

e) Since $(M_t)_{t=0,1,\ldots,N}$ is a nonnegative martingale, we have

$$\begin{aligned} \mathbb{E}[S_{t+1} \mid \mathcal{F}_t] &= \mathbb{E}[M_{t+1} A_{t+1} \mid \mathcal{F}_t] \\ &= A_{t+1} \, \mathbb{E}[M_{t+1} \mid \mathcal{F}_t] \\ &= A_{t+1} M_t \\ &\geq A_t M_t \\ &= S_t, \qquad t = 0,1,\ldots,N-1, \end{aligned}$$

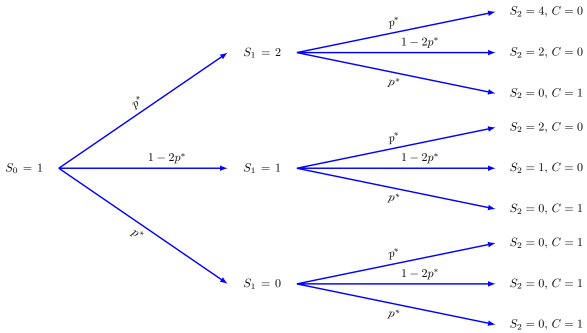
because $r \geq 0$, hence $(S_t)_{t=0,1,\ldots,N}$ is a nonnegative *sub*martingale. Therefore, we have

$$\mathbb{P}^* \left(\max_{t=0,1,\ldots,n} S_t \geq x \right) \leq \frac{\mathbb{E}\left[(M_N)^\beta \right]}{x^\beta}$$

$$\leq \left(\frac{S_0}{x}\right)^\beta \left((1+a)^\beta \frac{b-r}{b-a} + (1+b)^\beta \frac{r-a}{b-a}\right)^N.$$

Chapter 3

Exercise 3.1 (Exercise 2.5 continued). We consider the following trinomial tree.



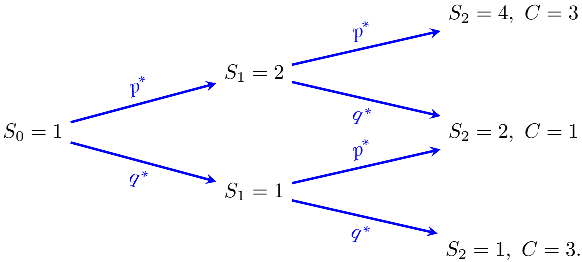
At time $t = 0$, we find

$$\begin{aligned}
\pi_0(C) &= \frac{1}{(1+r)^2}\,\mathbb{E}^*[(K-S_2)^+] \\
&= p^*(p^* + (1-2p^*) + p^*) + (1-2p^*)p^* + (p^*)^2 \\
&= p^* + (1-2p^*)p^* + (p^*)^2 \\
&= 2p^* - (p^*)^2.
\end{aligned}$$

At time $t = 1$, we find

$$\begin{aligned}
\pi_1(C) &= \frac{1}{1+r}\,\mathbb{E}^*[(K-S_2)^+ \mid S_1] \\
&= \begin{cases} p^* & \text{if } S_1 = 2S_0, \\ \\ p^* & \text{if } S_1 = S_0, \\ \\ 1 & \text{if } S_1 = 0. \end{cases}
\end{aligned}$$

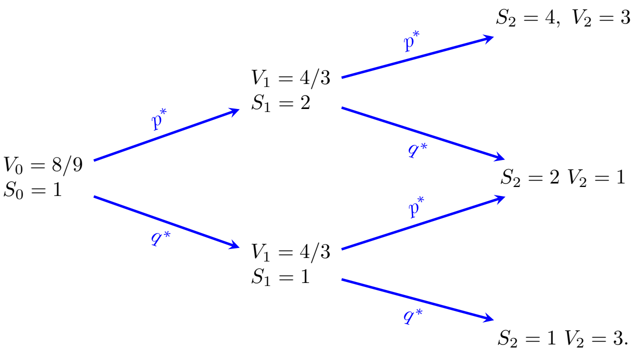
Exercise 3.2 We have $p^* = (r-a)/(b-a) = 1/2$ and $q^* = (b-r)/(b-a) = 1/2$, and the following underlying asset price tree:



We first price, and then hedge. At time $t = 1$, by Theorem 3.5 we have

$$\pi_1(C) = V_1 = \begin{cases} \dfrac{3p^* + q^*}{1+r} = \dfrac{4}{3} & \text{if } S_1 = 2 \\[2ex] \dfrac{p^* + 3q^*}{1+r} = \dfrac{4}{3} & \text{if } S_1 = 1, \end{cases} \quad \text{and } \pi_0(C) = V_0 = \frac{4}{3}\frac{p^* + q^*}{1+r} = \frac{8}{9}.$$

This leads to the following option pricing tree:



Regarding hedging, if $S_1 = 2$ the condition $\bar{\xi}_2 \cdot \overline{S}_2 = \xi_2 S_2 + \eta_2 A_2 = V_2$ reads

$$S_1 = 2 \Longrightarrow \begin{cases} 4\xi_2 + \eta_2(1+r)^2 = 3 \\[2ex] 2\xi_2 + \eta_2(1+r)^2 = 1, \end{cases}$$

hence $(\xi_2, \eta_2) = (1, -4/9)$. On the other hand, if $S_1 = 1$ we have

$$S_1 = 1 \Longrightarrow \begin{cases} 2\xi_2 + \eta_2(1+r)^2 = 1 \\ \\ \xi_2 + \eta_2(1+r)^2 = 3, \end{cases}$$

hence $(\xi_2, \eta_2) = (-2, 20/9)$. Finally, at time $t = 0$ with $S_0 = 1$ the condition $\bar{\xi}_1 \cdot \overline{S}_1 = \xi_1 S_1 + \eta_1 A_1 = V_1$ yields

$$\begin{cases} 2\xi_1 + \eta_1(1+r) = \dfrac{4}{3} \\ \\ \xi_1 + \eta_1(1+r) = \dfrac{4}{3}, \end{cases}$$

hence $(\xi_1, \eta_1) = (0, 8/9)$. The results can be summarized in the following table:

<table>
<tr><td rowspan="6">$S_0 = 1$
$V_0 = 8/9$
$\xi_1 = 0$
$\eta_1 = 8/9$</td><td rowspan="3">$S_1 = 2,\ V_1 = 4/3$
$\xi_2 = 1,\ \eta_2 = -4/9$</td><td rowspan="2">$S_2 = 4$
$V_2 = 3$</td></tr>
<tr></tr>
<tr><td rowspan="2">$S_2 = 1$
$V_2 = 3$</td></tr>
<tr><td rowspan="3">$S_1 = 1,\ V_1 = 4/3$
$\xi_2 = -2,\ \eta_2 = 20/9$</td></tr>
<tr><td rowspan="2">$S_2 = 1$
$V_2 = 3$</td></tr>
<tr></tr>
</table>

Table S.1: CRR pricing and hedging table.

In addition, it can be checked that the portfolio strategy $(\xi_k, \eta_k)_{k=1,2}$ is self-financing, as we have

$$\begin{aligned} \xi_1 S_1 + \eta_1 A_1 &= \frac{8}{9} \times \frac{3}{2} \\ &= \begin{cases} 2 - \dfrac{4}{9} \times \dfrac{3}{2} \\ \\ -2 + \dfrac{20}{9} \times \dfrac{3}{2} \end{cases} \\ &= \xi_2 S_1 + \eta_2 A_1. \end{aligned}$$

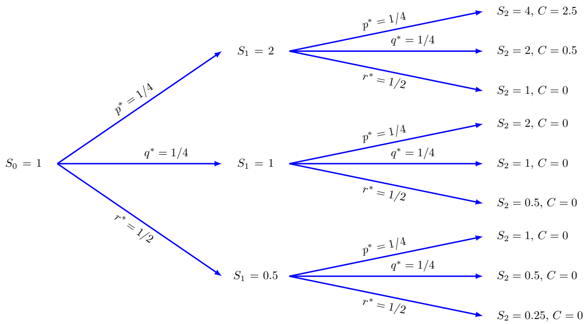
Exercise 3.3

a) We have

$$\begin{aligned} \mathbb{E}^*[S_{t+1} \mid \mathcal{F}_t] &= \mathbb{E}^*[S_{t+1} \mid S_t] \\ &= \frac{S_t}{2} \mathbb{P}^*(R_t = -0.5) + S_t \mathbb{P}^*(R_t = 0) + 2S_t \mathbb{P}^*(R_t = 1) \\ &= \left(\frac{r^*}{2} + q^* + 2p^* \right) S_t \\ &= S_t, \qquad t = 0, 1, \end{aligned}$$

with $r = 0$.

b) We have the following graph:



c) The down-an-out barrier call option is priced at time $t = 0$ as

$$V_0 = \mathbb{E}^*[C] = 2.5 \times (p^*)^2 + 0.5 \times p^* q^* = \frac{3}{16}.$$

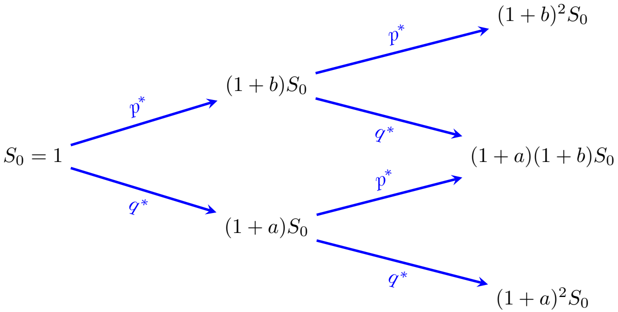
At time $t = 1$ we have

$$V_1 = 2.5 \times p^* + 0.5 \times q^* = 2.5 \times \frac{1}{4} + 0.5 \times \frac{1}{4} = \frac{3}{4}$$

if $S_1 = 2$, and $V_1 = 0$ in both cases $S_1 = 1$ and $S_1 = 0.5$.

d) This market is not complete, and not every contingent claim is attainable, because the risk-neutral probability measure $\mathbb{P}^*$ is not unique, for example $(r^*, q^*, p^*) = (1/4, 5/8, 1/8)$ and $(r^*, q^*, p^*) = (1/2, 1/4, 1/4)$ are both risk-neutral probability measures.

Exercise 3.4 The CRR model can be described by the following binomial tree.



a) By the formulas

$$\begin{aligned} V_1 &= \frac{1}{1+r}\,\mathbb{E}^*[V_2 \mid \mathcal{F}_1] = \frac{1}{1+r}\,\mathbb{E}^*[V_2 \mid S_1] \\ &= \frac{S_0(1+b)^2 - 8}{1+r}\mathbb{P}^*(S_2 = S_0(1+b)^2 \mid S_1) \\ &= p^*\,\frac{(S_0(1+b)^2 - 8)}{1+r}\mathbb{1}_{\{S_1 = S_0(1+b)\}}, \end{aligned}$$

and

$$\begin{aligned} V_0 &= \frac{1}{1+r}\,\mathbb{E}^*[V_1 \mid \mathcal{F}_0] \\ &= \frac{1}{1+r}\left(p^*\,\frac{(S_0(1+b)^2 - 8)}{1+r} \times \mathbb{P}^*(S_1 = S_0(1+b)) + 0 \times \mathbb{P}^*(S_1 = S_0(1+a))\right) \\ &= (p^*)^2\,\frac{(S_0(1+b)^2 - 8)}{(1+r)^2}, \end{aligned}$$

we find the table

<table>
<tr><td rowspan="3">$S_0 = 1$
$V_0 = 1/16$</td><td rowspan="2">$S_1 = 3,\ V_1 = 1/4$</td><td>$S_2 = 9$
$V_2 = 1$</td></tr>
<tr><td>$S_2 = 3$
$V_2 = 0$</td></tr>
<tr><td>$S_1 = 1,\ V_1 = 0$</td><td>$S_2 = 1$
$V_2 = 0$</td></tr>
</table>

Table S.2: CRR pricing tree.

Note that we could also directly compute V_0 from

$$V_0 = \frac{1}{(1+r)^2} \mathbb{E}^*[V_2 \mid \mathcal{F}_0].$$

b) When $S_1 = S_0(1+b)$, the equation $\xi_2 S_2 + \eta_2 A_2 = V_2$ reads

$$\begin{cases} \xi_2 S_0(1+b)^2 + \eta_2 A_0(1+r)^2 = S_0(1+b)^2 - 8 \\ \\ \xi_2 S_0(1+b)(1+a) + \eta_2 A_0(1+r)^2 = 0, \end{cases}$$

which yields

$$\xi_2 = \frac{S_0(1+b)^2 - 8}{S_0(b-a)(1+b)} \quad \text{and} \quad \eta_2 = -\frac{(S_0(1+b)^2 - 8)(1+a)}{(b-a)A_0(1+r)^2}. \tag{S.3.4}$$

When $S_1 = S_0(1+a)$, the equation $\xi_2 S_2 + \eta_2 A_2 = V_2$ reads

$$\begin{cases} \xi_2 S_0(1+a)^2 + \eta_2 A_0(1+r)^2 = 0 \\ \\ \xi_2 S_0(1+b)^2 + \eta_2 A_0(1+r)^2 = 0, \end{cases}$$

which has the unique solution $(\xi_2, \eta_2) = (0, 0)$. Next, the equation $\xi_1 S_1 + \eta_1 A_1 = V_1$ reads

$$\begin{cases} \xi_1 S_0(1+b) + \eta_1 A_0(1+r) = \dfrac{p^*(S_0(1+b)^2 - 8)}{1+r}, \\ \\ \xi_1 S_0(1+a) + \eta_1 A_0(1+r) = 0, \end{cases}$$

which yields

$$\xi_1 = p^* \frac{S_0(1+b)^2 - 8}{S_0(b-a)(1+r)} \quad \text{and} \quad \eta_1 = -p^* \frac{(1+a)(S_0(1+b)^2 - 8)}{(b-a)A_0(1+r)^2}. \tag{S.3.5}$$

This can be summarized in the following table:

<table>
<tr><td rowspan="4">$S_0 = 1$
$V_0 = 1/16$
$\xi_1 = 1/8$
$\eta_1 = -1/16$</td><td rowspan="2">$S_1 = 3$, $V_1 = 1/4$

$\xi_2 = 1/6$, $\eta_2 = -1/8$</td><td>$S_2 = 9$
$V_2 = 1$</td></tr>
<tr><td rowspan="2">$S_2 = 3$
$V_2 = 0$</td></tr>
<tr><td rowspan="2">$S_1 = 1$, $V_1 = 0$

$\xi_2 = 0$, $\eta_2 = 0$</td></tr>
<tr><td>$S_2 = 1$
$V_2 = 0$</td></tr>
</table>

Table S.3: CRR pricing and hedging tree.

When $S_1 = S_0(1+a)$ at time $t = 1$ the option price is $V_1 = 0$ and the hedging strategy is to cut all positions: $\xi_2 = \eta_2 = 0$. On the other hand, if $S_1 = S_0(1+b)$ then there is a chance of being in the money at maturity and we need to increase our position in the underlying asset from $\xi_1 = 1/8$ to $\xi_2 = 1/6$.

Note that the self-financing condition

$$\xi_1 S_1 + \eta_1 A_1 = \xi_2 S_1 + \eta_2 A_1, \tag{S.3.6}$$

is verified. For example when $S_1 = S_0(1+a)$ we have

$$\frac{1}{8} \times S_1 - \frac{1}{16} A_1 = 0 \times S_1 + 0 \times A_1 = 0,$$

while when $S_1 = S_0(1+b)$ we find

$$\frac{1}{8} \times S_1 - \frac{1}{16} A_1 = \frac{1}{6} \times S_1 - \frac{1}{8} \times A_1 = \frac{1}{4}.$$

c) We can also use the self-financing condition (S.3.6) to recover (S.3.5) by rewriting the system of equations as

$$\begin{cases} \xi_1 S_0(1+b) + \eta_1 A_0(1+r) = \xi_2 S_0(1+b) + \eta_2 A_0(1+r) \\ \\ \xi_1 S_0(1+a) + \eta_1 A_0(1+r) = 0, \end{cases}$$

with (ξ_2, η_2) given by (S.3.4), which recovers

$$V_1 = \xi_1 S_1 + \eta_1 A_1 = \begin{cases} \dfrac{3}{8} - \dfrac{2}{16} = \dfrac{1}{4} & \text{if } S_1 = 3, \\ \\ \dfrac{1}{8} - \dfrac{2}{16} = 0 & \text{if } S_1 = 1. \end{cases}$$

Exercise 3.5

a) We build a portfolio based at times $t = 0, 1$ on α_{t+1} units of stock and $\$\beta_{t+1}$ in cash. When $S_1 = 2$, we should have

$$\begin{cases} 4\alpha_2 + \beta_2 = 0 \\ 2\alpha_2 + \beta_2 = 1, \end{cases}$$

hence $(\alpha_2, \beta_2) = (-1/2, 2)$. On the other hand, when $S_1 = 1$ we should have

$$\begin{cases} 2\alpha_1 + \beta_1 = 1 \\ \alpha_1 + \beta_1 = 0, \end{cases}$$

hence $(\alpha_2, \beta_2) = (1, -1)$.

b) When $S_1 = 2$, the price of the claim at $t = 1$ is

$$\alpha_2 S_1 + \beta_2 = (-1/2) \times 2 + 2 \times 1 = 1.$$

When $S_1 = 1$, the price of the claim at $t = 1$ is $\alpha_2 S_1 + \beta_2 = 1 \times 1 - 1 \times 1 = 0$.

c) At time $t = 1$ we build a portfolio using α_1 units of stock and $\$\beta_1$ in cash. We should have

$$\begin{cases} 2\alpha_1 + \beta_1 = 1 \\ \alpha_1 + \beta_1 = 0, \end{cases}$$

hence $(\alpha_1, \beta_1) = (1, -1)$.

d) The price of the claim C at time $t = 0$ is $\alpha_1 S_0 + \beta_1 = 1 \times 1 + (-1) \times 1 = 0$.

e) The probabilities $(p^*, q^*) = ((r-a)/(b-a), (b-r)/(b-a)) = (0, 1)$ are clearly risk-neutral in the sense of Definition 2.12, as they yield

$$\mathbb{E}^*[S_2 \mid S_1] = S_1 \qquad \text{and} \qquad \mathbb{E}^*[S_1 \mid S_0] = S_0.$$

with the risk-free rate $r = 0$. However, this does not form a risk-neutral probability measure $\mathbb{P}^*$ equivalent to $\mathbb{P}$ in the sense of Definition 2.14 when $q = \mathbb{P}(R_1 = 0) = \mathbb{P}(R_2 = 0) > 0$.

In case $(p, q) = (0, 1)$, the probabilities $(p^*, q^*) = (0, 1)$ would yield an equivalent risk-neutral probability measure.

f) According to Theorem 2.15 this model allows for arbitrage opportunities as the unique available risk-neutral probability measure $\mathbb{P}^*$ are not be equivalent to the historical probability measure $\mathbb{P}$ when $q = \mathbb{P}(R_1 = 0) = \mathbb{P}(R_2 = 0) > 0$. In this case, arbitrage opportunities are easily implemented by purchasing the option at the price 0 of part (d) while receiving a strictly positive payoff at maturity. More generally, arbitrage opportunities exist when the underlying price may increase with nonzero probability, without a possibility of strict decrease.

In case $(p, q) = (0, 1)$, no arbitrage is possible as prices remain constant.

Exercise 3.6 We have the model-free answer

$$\begin{aligned}
\pi_k(C) &= \frac{1}{(1+r)^{N-k}} \mathbb{E}^*[h(S_N) \mid \mathcal{F}_k] \\
&= \frac{1}{(1+r)^{N-k}} \mathbb{E}^*[\alpha + \beta S_N \mid \mathcal{F}_k] \\
&= \frac{\alpha}{(1+r)^{N-k}} + \frac{\beta}{(1+r)^{N-k}} \mathbb{E}^*[S_N \mid \mathcal{F}_k] \\
&= \frac{\alpha}{(1+r)^N} A_k + \beta S_k, \qquad k = 0, 1, \ldots, N.
\end{aligned}$$

The hedging portfolio strategy is to hold β units of the underlying asset priced S_k and $\alpha/(1+r)^N$ units of the riskless asset priced $A_k = (1+r)^k$ at time $k = 0, 1, \dots, N$.

Note that in the particular case of the CRR model, this answer is also compatible with (3.19)-(3.20).

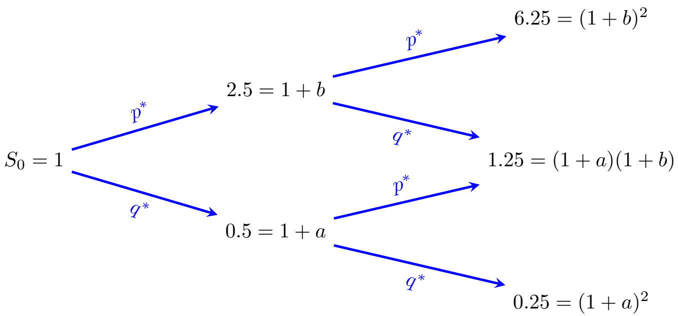
Exercise 3.7 Call-put parity.

a) The relation $(x-K)^+ = x - K + (K-x)^+$ can be verified by successively checking the cases $x \leq K$ and $x \geq K$.
b) Respectively denoting by $C(k)$ and $P(k)$ the call and put prices at time $k = 0, 1, \dots, N$, by part (a) we have

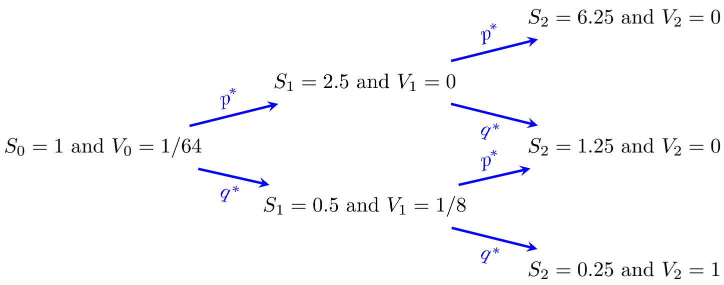
$$
\begin{aligned}
C(k) &= (1+r)^{-(N-k)}\, \mathbb{E}^*\left[(S_N - K)^+ \,\middle|\, \mathcal{F}_k\right] \\
&= (1+r)^{-(N-k)}\, \mathbb{E}^*\left[S_N - K + (K - S_N)^+ \,\middle|\, \mathcal{F}_k\right] \\
&= (1+r)^{-(N-k)}\, \mathbb{E}^*[S_N \mid \mathcal{F}_k] - (1+r)^{-(N-k)} K \\
&\quad + (1+r)^{-(N-k)}\, \mathbb{E}^*\left[(K - S_N)^+ \,\middle|\, \mathcal{F}_k\right] \\
&= S_k - (1+r)^{-(N-k)} K + (1+r)^{-(N-k)}\, \mathbb{E}^*\left[(K - S_N)^+ \,\middle|\, \mathcal{F}_k\right] \\
&= S_k - (1+r)^{-(N-k)} K + P(k).
\end{aligned}
$$

Exercise 3.8

a) Taking $q^* = 1 - p^* = 1/4$, we find the binary tree



b) We find the binary tree



and the table

<table>
<tr><td rowspan="6">$S_0 = 1$
$V_0 = 1/64$</td><td rowspan="3">$S_1 = 2.5,\ V_1 = 0$</td><td rowspan="2">$S_2 = 6.25$
$V_2 = 0$</td></tr>
<tr></tr>
<tr><td rowspan="2">$S_2 = 1.25$
$V_2 = 0$</td></tr>
<tr><td rowspan="3">$S_1 = 0.5,\ V_1 = 1/8$</td></tr>
<tr><td rowspan="2">$S_2 = 0.25$
$V_2 = 1$</td></tr>
<tr></tr>
</table>

Table S.4: CRR pricing tree.

c) Here, we compute the hedging strategy from the option prices. When $S_1 = S_0(1+b)$ we clearly have $\xi_2 = \eta_2 = 0$. When $S_1 = S_0(1+a)$, the equation $\xi_2 S_2 + \eta_2 A_2 = V_2$ reads

$$\begin{cases} \xi_2 S_0(1+a)^2 + \eta_2(1+r)^2 = (K - S_0(1+a)^2) \\ \\ \xi_2 S_0(1+b)(1+a) + \eta_2(1+r)^2 = 0 \end{cases}$$

hence

$$\xi_2 = -\frac{(K - S_0(1+a)^2)}{S_0(b-a)(1+a)} \quad \text{and} \quad \eta_2 = \frac{(K - S_0(1+a)^2)(1+b)}{S_0(b-a)(1+r)^2}.$$

Next, at time $t = 1$ the equation $\xi_1 S_1 + \eta_1 A_1 = V_1$ reads

$$\begin{cases} \xi_1 S_0(1+a) + \eta_1(1+r) = S_0 \dfrac{q^*(K - (1+a)(1+b))}{1+r}, \\ \\ \xi_1 S_0(1+b) + \eta_1(1+r) = 0 \end{cases}$$

which yields

$$\xi_1 = -\frac{q^*(K - S_0(1+a)(1+b))}{S_0(b-a)(1+r)} \quad \text{and} \quad \eta_1 = \frac{q^*(K - S_0(1+a)(1+b))(1+b)}{S_0(b-a)(1+r)^2}.$$

This can be summarized in the following table:

<table>
<tr><td rowspan="6">$S_0 = 1$
$V_0 = 1/64$
$\xi_1 = -1/16$
$\eta_1 = 5/64$</td><td rowspan="3">$S_1 = 2.5$, $V_1 = 0$

$\xi_2 = 0$, $\eta_2 = 0$</td><td rowspan="2">$S_2 = 6.25$
$V_2 = 0$</td></tr>
<tr></tr>
<tr><td rowspan="2">$S_2 = 1.25$
$V_2 = 0$</td></tr>
<tr><td rowspan="3">$S_1 = 0.5$, $V_1 = 1/8$

$\xi_2 = -1$, $\eta_2 = 5/16$</td></tr>
<tr><td rowspan="2">$S_2 = 0.25$
$V_2 = 1$</td></tr>
<tr></tr>
</table>

Table S.5: CRR pricing and hedging tree.

If $S_1 = S_0(1+a)$ then there is a chance of being in the money at maturity and we need to short sell further by decreasing ξ_1 from $\xi_1 = -1/16$ to $\xi_2 = -1$. Note that the self-financing condition

$$\xi_1 S_1 + \eta_1 A_1 = \xi_2 S_1 + \eta_2 A_1$$

is satisfied.

Exercise 3.9

a) The binary call option can be priced under the risk-neutral probability measure $\mathbb{P}^*$ as

$$\begin{aligned}\pi_0(C) &= \frac{1}{1+r}\mathbb{E}^*[C] \\ &= \frac{1}{1+r}\mathbb{E}^*[\mathbb{1}_{[K,\infty)}(S_N)] \\ &= \frac{1}{1+r}\mathbb{P}^*(S_N \geq K) \\ &= \frac{p^*}{1+r},\end{aligned}$$

with $p^* := \mathbb{P}^*(S_N \geq K)$.

b) Investing $\$p^*$ by purchasing one binary call option yields a potential net return of

$$\begin{cases} \dfrac{\$1 - p^*}{p^*} = \dfrac{\$1}{p^*} - 1 & \text{if } S_N \geq K, \\ \\ \dfrac{\$0 - p^*}{p^*} = -100\% & \text{if } S_N < K. \end{cases}$$

c) The corresponding expected return is

$$p^* \times \left(\frac{1}{p^*} - 1\right) + (1 - p^*) \times (-1) = 0.$$

d) The corresponding expected return is

$$p^* \times 0.86 + (1 - p^*) \times (-1) = p^* \times 1.86 - 1,$$

which will be *negative* if

$$p^* < \frac{1}{1.86} \simeq 0.538.$$

That means, the expected gain can be negative even if

$$0.538 > p^* = \mathbb{P}^*(S_N \geq K) > 0.5.$$

Similarly, the expected gain

$$(1 - p^*) \times 0.86 + p^* \times (-1) = 0.86 - p^* \times 1.86,$$

on binary put options will be *negative* if $1 - p^* > 1/1.86$, *i.e.* if

$$p^* > \frac{0.86}{1.86} \simeq 0.462.$$

That means, the expected gain can be negative even if $1 - 0.462 > \mathbb{P}^*(S_N < K) > 0.5$. In conclusion, the average gains of both call and put options will be negative if $p^* \in (0.462, 0.538)$.

Note that the average of call and put option gains will still be negative, as

$$\frac{p^* \times 1.86 - 1}{2} + \frac{0.86 - p^* \times 1.86}{2} = \frac{0.86 - 1}{2} < 0.$$

Exercise 3.10

a) Based on the price map of the put spread collar option:

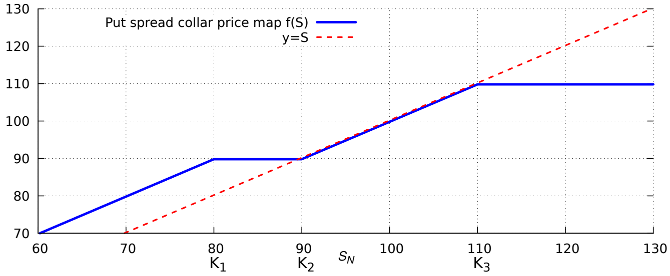


Fig. S.5: Put spread collar price map.

we deduce the following payoff function graph of the put spread collar option in the next Figure S.6.

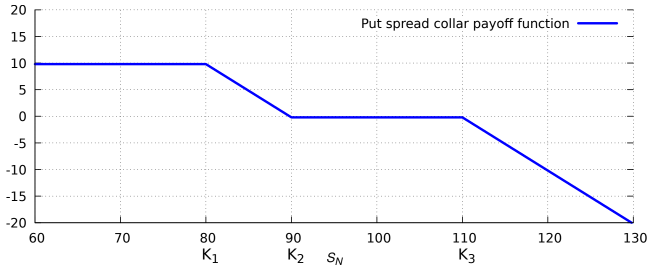


Fig. S.6: Put spread collar payoff function.

b) The payoff function can be written as

$$\begin{aligned}&-(K_1 - x)^+ + (K_2 - x)^+ - (x - K_3)^+ \\ &\qquad = -(80 - x)^+ + (90 - x)^+ - (x - 110)^+,\end{aligned}$$

see also https://optioncreator.com/stp7xy2.

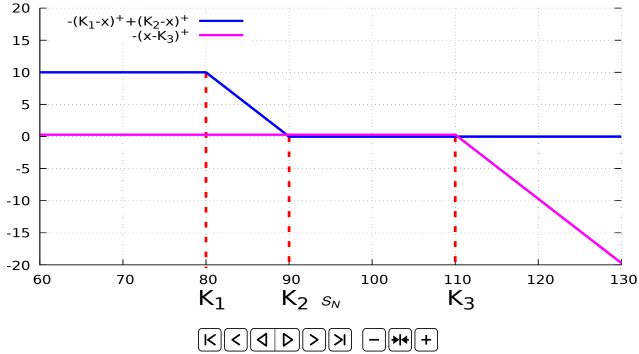


Fig. S.7: Put spread collar payoff as a combination of call and put option payoffs.*

Hence this collar option payoff can be realized by

1. issuing (or shorting/selling) one *put option* with strike price $K_1 = 80$, and
2. purchasing and holding one *put option* with strike price $K_2 = 90$, and
3. issuing (or shorting/selling) one *call option* with strike price $K_3 = 110$.

Exercise 3.11

a) Based on the price map of the call spread collar option:

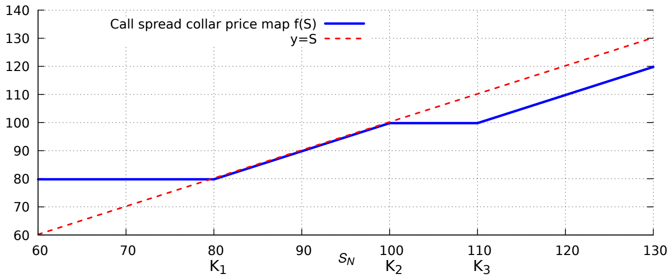


Fig. S.8: Call spread collar price map.

we deduce the following payoff function graph of the call spread collar option in the next Figure S.9.

* The animation works in Acrobat Reader on the entire pdf file.

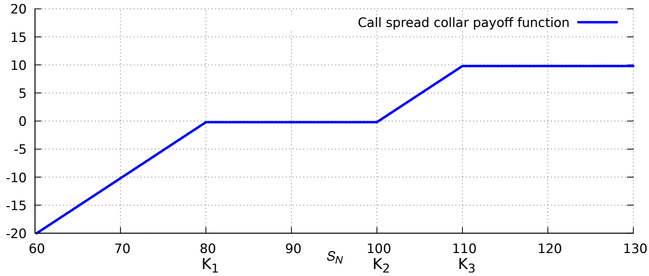


Fig. S.9: Call spread collar payoff function.

b) The payoff function can be written as

$$
\begin{aligned}
&-(K_1 - x)^+ + (x - K_2)^+ - (x - K_3)^+ \\
&\qquad\qquad = -(80 - x)^+ + (x - 100)^+ - (x - 110)^+,
\end{aligned}
$$

see also https://optioncreator.com/st3e4cz.

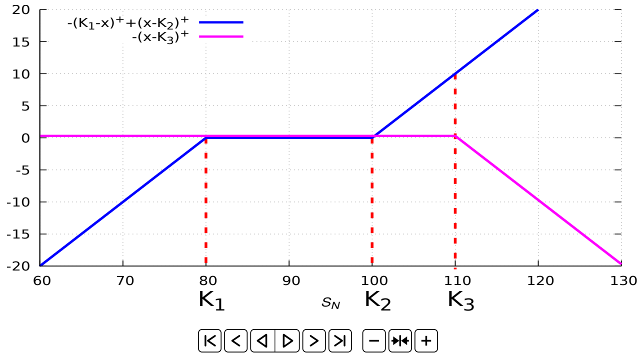


Fig. S.10: Call spread collar payoff as a combination of call and put option payoffs.[*]

Hence this collar option payoff can be realized by

1. issuing (or shorting/selling) one *put option* with strike price $K_1 = 80$, and
2. purchasing and holding one *call option* with strike price $K_2 = 100$, and

[*] The animation works in Acrobat Reader on the entire pdf file.

3. issuing (or shorting/selling) one *call option* with strike price $K_3 = 110$.

Exercise 3.12 We have

$$\begin{aligned}
&\mathbb{E}^*\left[\phi\left(\frac{S_1+\cdots+S_N}{N}\right)\right] \leq \mathbb{E}^*\left[\frac{\phi(S_1)+\cdots+\phi(S_N)}{N}\right] && \text{since } \phi \text{ is convex,}\\
&= \frac{\mathbb{E}^*[\phi(S_1)]+\cdots+\mathbb{E}^*[\phi(S_N)]}{N}\\
&= \frac{\mathbb{E}^*[\phi(\mathbb{E}^*[S_N \mid \mathcal{F}_1])]+\cdots+\mathbb{E}^*[\phi(\mathbb{E}^*[S_N \mid \mathcal{F}_N])]}{N} && \text{because } (S_n)_{n\in\mathbb{N}} \text{ is a martingale,}\\
&\leq \frac{\mathbb{E}^*[\mathbb{E}^*[\phi(S_N) \mid \mathcal{F}_1]]+\cdots+\mathbb{E}^*[\mathbb{E}^*[\phi(S_N) \mid \mathcal{F}_N]]}{N} && \text{by Jensen's inequality,}\\
&= \frac{\mathbb{E}^*[\phi(S_N)]+\cdots+\mathbb{E}^*[\phi(S_N)]}{N} && \text{by the tower property.}\\
&= \mathbb{E}^*[\phi(S_N)].
\end{aligned}$$

The above argument is implicitly using the fact that a convex function $\phi(S_n)$ of a martingale $(S_n)_{n\in\mathbb{N}}$ is itself a *sub*martingale, as

$$\phi(S_k) = \phi(\mathbb{E}^*[S_N \mid \mathcal{F}_k]) \leq \mathbb{E}^*[\phi(S_N) \mid \mathcal{F}_k], \quad k = 1,2,\ldots,N.$$

Exercise 3.13 (Exercise 2.7 continued).

a) The condition $V_N = C$ reads

$$\begin{cases} \eta_N\pi_N + \xi_N(1+a)S_{N-1} = (1+a)S_{N-1} - K \\ \\ \eta_N\pi_N + \xi_N(1+b)S_{N-1} = (1+b)S_{N-1} - K, \end{cases}$$

from which we deduce the (static) hedging strategy $\xi_N = 1$ and $\eta_N = -K(1+r)^{-N}/\pi_0$.

b) We have

$$\begin{cases} \eta_{N-1}\pi_{N-1} + \xi_{N-1}(1+a)S_{N-2} = \eta_N\pi_{N-1} + \xi_N(1+a)S_{N-2} \\ \\ \eta_{N-1}\pi_{N-1} + \xi_{N-1}(1+b)S_{N-2} = \eta_N\pi_{N-1} + \xi_N(1+b)S_{N-2}, \end{cases}$$

which yields $\xi_{N-1} = \xi_N = 1$ and $\eta_{N-1} = \eta_N = -K(1+r)^{-N}/\pi_0$. Similarly, solving the self-financing condition

$$\begin{cases} \eta_t\pi_t + \xi_t(1+a)S_{t-1} = \eta_{t+1}\pi_t + \xi_{t+1}(1+a)S_{t-1} \\ \\ \eta_t\pi_t + \xi_t(1+b)S_{t-1} = \eta_{t+1}\pi_t + \xi_{t+1}(1+b)S_{t-1} \end{cases}$$

at time t yields

$$\xi_t = 1 \quad \text{and} \quad \eta_t = -(1+r)^{-N}\frac{K}{\pi_0}, \qquad t = 1,2,\ldots,N.$$

c) We have

$$\begin{aligned}\pi_t(C) &= V_t \\ &= \eta_t \pi_t + \xi_t S_t \\ &= S_t - K(1+r)^{-N} \frac{\pi_t}{\pi_0} \\ &= S_t - K(1+r)^{-(N-t)}.\end{aligned}$$

d) For all $t = 0, 1, \ldots, N$ we have

$$\begin{aligned}(1+r)^{-(N-t)} \mathbb{E}^*[C \mid \mathcal{F}_t] &= (1+r)^{-(N-t)} \mathbb{E}^*[S_N - K \mid \mathcal{F}_t], \\ &= (1+r)^{-(N-t)} \mathbb{E}^*[S_N \mid \mathcal{F}_t] - (1+r)^{-(N-t)} \mathbb{E}^*[K \mid \mathcal{F}_t] \\ &= (1+r)^{-(N-t)}(1+r)^{N-t} S_t - K(1+r)^{-(N-t)} \\ &= S_t - K(1+r)^{-(N-t)} \\ &= V_t = \pi_t(C).\end{aligned}$$

For a future contract expiring at time N we take $K = S_0(1+r)^N$ and the contract is usually quoted at time t using the forward price $(1+r)^{N-t}(S_t - K(1+r)^{N-t}) = (1+r)^{N-t} S_t - K = (1+r)^{N-t} S_t - S_0(1+r)^N$, or simply using $(1+r)^{N-t} S_t$. Future contracts are "marked to market" at each time step $t = 1, 2, \ldots, N$ via a positive or negative cash flow exchange $(1+r)^{N-t} S_t - (1+r)^{N-t+1} S_{t-1}$ from the seller to the buyer, ensuring that the absolute difference $|(1+r)^{N-t} S_t - K|$ has been credited to the buyer's account if it is positive, or to the seller's account if it is negative.

Exercise 3.14

a) We write

$$V_N = \begin{cases} \xi_N S_{N-1}(1+1/2) + \eta_N = (S_{N-1}(1+1/2))^2 \\ \\ \xi_N S_{N-1}(1-1/2) + \eta_N = (S_{N-1}(1-1/2))^2, \end{cases}$$

which yields

$$\begin{cases} \xi_N = 2S_{N-1} \\ \\ \eta_N = -3(S_{N-1})^2/4. \end{cases}$$

b) i) We have

$$\begin{aligned}\mathbb{E}^*[(S_N)^2 \mid \mathcal{F}_{N-1}] &= p^*(S_{N-1})^2(1+1/2)^2 + (1-p^*)(S_{N-1})^2(1-1/2)^2 \\ &= \frac{1}{2}(S_{N-1})^2 \left((1+1/2)^2 + (1-1/2)^2\right) \\ &= 5(S_{N-1})^2/4.\end{aligned}$$

ii) We have

$$\begin{aligned}
\xi_{N-1}S_{N-1}+\eta_{N-1}A_0 &= \begin{cases} \xi_{N-1}S_{N-2}(1+1/2)+\eta_{N-1} \\[2ex] \xi_{N-1}S_{N-2}(1-1/2)+\eta_{N-1} \end{cases} \\
&= V_{N-1} \\
&= 5(S_{N-1})^2/4 \\
&= \begin{cases} 5(S_{N-2}(1+1/2))^2/4 \\[2ex] 5(S_{N-2}(1-1/2))^2/4, \end{cases}
\end{aligned}$$

hence

$$\begin{cases} \xi_{N-1} = 5S_{N-2}/2 \\[2ex] \eta_{N-1} = -15(S_{N-2})^2/16. \end{cases}$$

iii) We have

$$\begin{aligned}
\xi_{N-1}S_{N-1}+\eta_{N-1}A_0 &= 5S_{N-2}S_{N-1}/2-15(S_{N-2})^2/16 \\
&= \begin{cases} 5(S_{N-2})^2(1+1/2)/2-15(S_{N-2})^2/16 \\[2ex] 5(S_{N-2})^2(1-1/2)/2-15(S_{N-2})^2/16 \end{cases} \\
&= \begin{cases} 15(S_{N-2})^2/4-15(S_{N-2})^2/16 \\[2ex] 5(S_{N-2})^2-15(S_{N-2})^2/16 \end{cases} \\
&= \begin{cases} 45(S_{N-2})^2/16 \\[2ex] 5(S_{N-2})^2/16, \end{cases}
\end{aligned}$$

and on the other hand,

$$\begin{aligned}
\xi_N S_{N-1}+\eta_N A_0 &= 2(S_{N-1})^2-3(S_{N-1})^2/4 \\
&= \begin{cases} 2(S_{N-2})^2(1+1/2)^2-3(S_{N-2})^2(1+1/2)^2/4 \\[2ex] 2(S_{N-2})^2(1-1/2)^2-3(S_{N-2})^2(1-1/2)^2/4 \end{cases} \\
&= \begin{cases} 45(S_{N-2})^2/16 \\[2ex] 5(S_{N-2})^2/16. \end{cases}
\end{aligned}$$

Remark: We could also determine (ξ_{N-1},η_{N-1}) as in Proposition 3.12, from (ξ_N,η_N) and the self-financing condition

$$\xi_{N-1}S_{N-1}+\eta_{N-1}A_0 = \xi_N S_{N-1}+\eta_N A_{N-1},$$

as

$$\xi_{N-1}S_{N-1}+\eta_{N-1}A_0=\begin{cases}\xi_{N-1}S_{N-2}(1+1/2)+\eta_{N-1}\\ \\ \xi_{N-1}S_{N-2}(1-1/2)+\eta_{N-1}\end{cases}$$
$$=\xi_N S_{N-1}+\eta_N A_0$$
$$=2(S_{N-1})^2-3(S_{N-1})^2/4$$
$$=\begin{cases}2(S_{N-2})^2(1+1/2)^2-3(S_{N-2})^2(1+1/2)^2/4\\ \\ 2(S_{N-2})^2(1-1/2)^2-3(S_{N-2})^2(1-1/2)^2/4,\end{cases}$$

which recovers $\xi_{N-1}=5S_{N-2}/2$ and $\eta_{N-1}=-15(S_{N-2})^2/16$.

Exercise 3.15

a) By Theorem 2.19 this model admits a unique risk-neutral probability measure $\mathbb{P}^*$ because $a<r<b$, and from (2.16) we have

$$\mathbb{P}^*(R_t=a)=\frac{b-r}{b-a}=\frac{0.07-0.05}{0.07-0.02},$$

and

$$\mathbb{P}(R_t=b)=\frac{r-a}{b-a}=\frac{0.05-0.02}{0.07-0.02},$$

$t=1,2,\ldots,N$.

b) There are no arbitrage opportunities in this model, due to the existence of a risk-neutral probability measure.

c) This market model is complete because the risk-neutral probability measure is unique.

d) We have

$$C=(S_N)^2,$$

hence

$$\widetilde{C}=\frac{(S_N)^2}{(1+r)^N}=h(X_N),$$

with

$$h(x)=x^2(1+r)^N. \tag{S.3.7}$$

Now we have

$$\widetilde{V}_t=\tilde{v}(t,X_t),$$

where the function $v(t,x)$ is given from Proposition 3.8 as

$$\tilde{v}(t,x)=\sum_{k=0}^{N-t}\binom{N-t}{k}(p^*)^k(q^*)^{N-t-k}h\left(x\left(\frac{1+b}{1+r}\right)^k\left(\frac{1+a}{1+r}\right)^{N-t-k}\right).$$

Using (S.3.7) and the binomial theorem, we find

$$
\begin{aligned}
\tilde{v}(t,x) &= x^2(1+r)^N \sum_{k=0}^{N-t} \binom{N-t}{k} \\
&\quad \times (p^*)^k (q^*)^{N-t-k} \left(\frac{1+b}{1+r}\right)^{2k} \left(\frac{1+a}{1+r}\right)^{2(N-t-k)} \\
&= x^2(1+r)^N \sum_{k=0}^{N-t} \binom{N-t}{k} \\
&\quad \times \left(\frac{(r-a)(1+b)^2}{(b-a)(1+r)^2}\right)^k \left(\frac{(b-r)(1+a)^2}{(b-a)(1+r)^2}\right)^{N-t-k} \\
&= x^2(1+r)^N \left(\frac{(r-a)(1+b)^2}{(b-a)(1+r)^2} + \frac{(b-r)(1+a)^2}{(b-a)(1+r)^2}\right)^{N-t} \\
&= \frac{x^2\left((r-a)(1+b)^2+(b-r)(1+a)^2\right)^{N-t}}{(1+r)^{N-2t}(b-a)^{N-t}} \\
&= \frac{x^2\left((r-a)(1+2b+b^2)+(b-r)(1+2a+a^2)\right)^{N-t}}{(1+r)^{N-2t}(b-a)^{N-t}} \\
&= \frac{x^2\left(r(1+2b+b^2)-a(1+2b+b^2)+b(1+2a+a^2)-r(1+2a+a^2)\right)^{N-t}}{(1+r)^{N-2t}(b-a)^{N-t}} \\
&= x^2 \frac{(1+r(a+b+2)-ab)^{N-t}}{(1+r)^{N-2t}}.
\end{aligned}
$$

e) We have

$$
\begin{aligned}
\xi_t^1 &= \frac{v\left(t, \frac{1+b}{1+r}X_{t-1}\right) - v\left(t, \frac{1+a}{1+r}X_{t-1}\right)}{X_{t-1}(b-a)/(1+r)} \\
&= X_{t-1} \frac{\left(\frac{1+b}{1+r}\right)^2 - \left(\frac{1+a}{1+r}\right)^2}{(b-a)/(1+r)} \frac{(1+r(a+b+2)-ab)^{N-t}}{(1+r)^{N-2t}} \\
&= S_{t-1}(a+b+2)\frac{(1+r(a+b+2)-ab)^{N-t}}{(1+r)^{N-t}}, \qquad t=1,2,\ldots,N,
\end{aligned}
$$

representing the quantity of the risky asset to be present in the portfolio at time t. On the other hand we have

$$
\begin{aligned}
\xi_t^0 &= \frac{V_t - \xi_t^1 X_t}{X_t^0} \\
&= \frac{V_t - \xi_t^1 X_t}{\pi_0} \\
&= X_t(1+r(a+b+2)-ab)^{N-t} \frac{X_t - X_{t-1}(a+b+2)/(1+r)}{\pi_0(1+r)^{N-2t}}
\end{aligned}
$$

$$= S_t(1+r(a+b+2)-ab)^{N-t}\frac{S_t - S_{t-1}(a+b+2)}{\pi_0(1+r)^N}$$
$$= -(S_{t-1})^2(1+r(a+b+2)-ab)^{N-t}\frac{(1+a)(1+b)}{\pi_0(1+r)^N},$$

$t = 1, 2, \ldots, N$.

f) Let us check that the portfolio is self-financing. We have

$$\begin{aligned}
\bar{\xi}_{t+1} \cdot \overline{S}_t &= \xi^0_{t+1} S^0_t + \xi^1_{t+1} S^1_t \\
&= -(S_t)^2(1+r(a+b+2)-ab)^{N-t-1}\frac{(1+a)(1+b)}{\pi_0(1+r)^N} S^0_t \\
&\quad +(S_t)^2(a+b+2)\frac{(1+r(a+b+2)-ab)^{N-t-1}}{(1+r)^{N-t-1}} \\
&= (S_t)^2\frac{(1+r(a+b+2)-ab)^{N-t-1}}{(1+r)^{N-t}} \\
&\quad \times ((a+b+2)(1+r)-(1+a)(1+b)) \\
&= \frac{1}{(1+r)^{N-3t}}(X_t)^2(1+r(a+b+2)-ab)^{N-t} \\
&= (1+r)^t V_t \\
&= \bar{\xi}_t \cdot \overline{S}_t, \qquad t = 1, 2, \ldots, N.
\end{aligned}$$

Exercise 3.16

a) We have

$$\begin{aligned}
V_t &= \xi_t S_t + \eta_t \pi_t \\
&= \xi_t(1+R_t)S_{t-1} + \eta_t(1+r)\pi_{t-1}.
\end{aligned}$$

b) We have

$$\begin{aligned}
\mathbb{E}^*[R_t|\mathcal{F}_{t-1}] &= a\mathbb{P}^*(R_t = a \mid \mathcal{F}_{t-1}) + b\mathbb{P}^*(R_t = b \mid \mathcal{F}_{t-1}) \\
&= a\frac{b-r}{b-a} + b\frac{r-a}{b-a} \\
&= b\frac{r}{b-a} - a\frac{r}{b-a} \\
&= r.
\end{aligned}$$

c) By the result of Question (a), we have

$$\begin{aligned}
\mathbb{E}^*[V_t \mid \mathcal{F}_{t-1}] &= \mathbb{E}^*[\xi_t(1+R_t)S_{t-1} \mid \mathcal{F}_{t-1}] + \mathbb{E}^*[\eta_t(1+r)\pi_{t-1} \mid \mathcal{F}_{t-1}] \\
&= \xi_t S_{t-1}\,\mathbb{E}^*[1+R_t \mid \mathcal{F}_{t-1}] + (1+r)\,\mathbb{E}^*[\eta_t\pi_{t-1} \mid \mathcal{F}_{t-1}] \\
&= (1+r)\xi_t S_{t-1} + (1+r)\eta_t\pi_{t-1} \\
&= (1+r)\xi_{t-1} S_{t-1} + (1+r)\eta_{t-1}\pi_{t-1}
\end{aligned}$$

$$= (1+r)V_{t-1},$$

where we used the self-financing condition.

d) We have

$$\begin{aligned}
V_{t-1} &= \frac{1}{1+r}\,\mathbb{E}^*[V_t \mid \mathcal{F}_{t-1}] \\
&= \frac{3}{1+r}\mathbb{P}^*(R_t = a \mid \mathcal{F}_{t-1}) + \frac{8}{1+r}\mathbb{P}^*(R_t = b \mid \mathcal{F}_{t-1}) \\
&= \frac{1}{1+0.15}\left(3\frac{0.25-0.15}{0.25-0.05} + 8\frac{0.15-0.05}{0.25-0.05}\right) \\
&= \frac{1}{1.15}\left(\frac{3}{2} + \frac{8}{2}\right) \\
&= 4.78.
\end{aligned}$$

Problem 3.17 CRR model with transaction costs.

a) i) In the event of an increase in the stock position ξ_t, the corresponding cost of purchase $(1+\lambda)(\xi_{t+1} - \xi_t)S_t > 0$ has to be *deducted from* the savings account value $\eta_t A_t$, which becomes updated as

$$\eta_{t+1}A_t = \eta_t A_t - (1+\lambda)(\xi_{t+1} - \xi_t)S_t,$$

hence we have

$$\eta_{t+1}A_t + (1+\lambda)\xi_{t+1}S_t = \eta_t A_t + (1+\lambda)\xi_t S_t.$$

ii) In the event of a decrease in the stock position ξ_t, the corresponding sale profit $(\xi_t - \xi_{t+1})(1-\lambda)S_t > 0$ has to be *added to* from the savings account value $\eta_t A_t$, which becomes updated as

$$\eta_{t+1}A_t = \eta_t A_t + (\xi_t - \xi_{t+1})(1-\lambda)S_t,$$

hence we have

$$\eta_{t+1}A_t + \xi_{t+1}(1-\lambda)S_t = \eta_t A_t + \xi_t(1-\lambda)S_t.$$

b) We have:

i) If $\xi_{t+1}(\beta S_{t-1}) > \xi_t(S_{t-1})$,

$$(\xi_t(S_{t-1}) - \xi_{t+1}(\beta S_{t-1}))\beta^{\uparrow}\widetilde{S}_{t-1} = (\eta_{t+1}(\beta S_{t-1}) - \eta_t(S_{t-1}))\rho.$$

ii) If $\xi_{t+1}(\beta S_{t-1}) < \xi_t(S_{t-1})$,

$$(\xi_t(S_{t-1}) - \xi_{t+1}(\beta S_{t-1}))\,\beta_{\downarrow}\widetilde{S}_{t-1} = (\eta_{t+1}(\beta S_{t-1}) - \eta_t(S_{t-1}))\rho,$$

and

iii) If $\xi_{t+1}(\alpha S_{t-1}) > \xi_t(S_{t-1})$,

$$(\xi_t(S_{t-1}) - \xi_{t+1}(\alpha S_{t-1}))\, \alpha^{\uparrow} \widetilde{S}_{t-1} = \rho\eta_{t+1}(\alpha S_{t-1}) - \rho\eta_t(S_{t-1}).$$

iv) If $\xi_{t+1}(\alpha S_{t-1}) < \xi_t(S_{t-1})$,

$$(\xi_t(S_{t-1}) - \xi_{t+1}(\alpha S_{t-1}))\, \alpha_{\downarrow} \widetilde{S}_{t-1} = \rho\eta_{t+1}(\alpha S_{t-1}) - \rho\eta_t(S_{t-1}).$$

c) We find

$$g_\beta(\xi_t(S_{t-1}), \xi_{t+1}(\beta S_{t-1}))\big(\xi_t(S_{t-1}) - \xi_{t+1}(\beta S_{t-1})\big)\widetilde{S}_{t-1} = \rho\eta_{t+1}(\beta S_{t-1}) - \rho\eta_t(S_{t-1}).$$

and

$$g_\alpha(\xi_t(S_{t-1}), \xi_{t+1}(\alpha S_{t-1}))\big(\xi_t(S_{t-1}) - \xi_{t+1}(\alpha S_{t-1})\big)\widetilde{S}_{t-1} = \rho\eta_{t+1}(\alpha S_{t-1}) - \rho\eta_t(S_{t-1}).$$

d) The equation is

$$\begin{aligned}\widetilde{S}_{t-1} g_\beta(\xi_t(S_{t-1}), \xi_{t+1}(\beta S_{t-1}))\big(\xi_t(S_{t-1}) - \xi_{t+1}(\beta S_{t-1})\big)& \\ -\widetilde{S}_{t-1} g_\alpha(\xi_t(S_{t-1}), \xi_{t+1}(\alpha S_{t-1}))\big(\xi_t(S_{t-1}) - \xi_{t+1}(\alpha S_{t-1})\big)& \\ = \rho\eta_{t+1}(\beta S_{t-1}) - \rho\eta_{t+1}(\alpha S_{t-1}),&\end{aligned}$$

which can be rewritten as

$$f(x, S_{t-1}) = 0 \tag{S.3.8}$$

at $x = \xi_t(S_{t-1})$. The function

$$x \mapsto f(x, S_{t-1})$$

is continuous by construction, and its derivative is the function

$$x \mapsto g_\beta(x, \xi_{t+1}(\beta S_{t-1})) - g_\alpha(x, \xi_{t+1}(\alpha S_{t-1})),$$

which can only take four values $\beta^{\uparrow} - \alpha^{\uparrow}$, $\beta^{\uparrow} - \alpha_{\downarrow}$, $\beta_{\downarrow} - \alpha^{\uparrow}$, $\beta_{\downarrow} - \alpha_{\downarrow}$, which are all strictly positive due to the conditions

$$\begin{cases} \alpha^{\uparrow} := \alpha(1+\lambda) < \beta(1-\lambda) =: \beta_{\downarrow}, \\ \alpha_{\downarrow} := \alpha(1-\lambda) < \beta(1-\lambda) := \beta_{\downarrow}, \\ \alpha^{\uparrow} := \alpha(1+\lambda) < \beta(1+\lambda) =: \beta^{\uparrow}. \end{cases}$$

Hence $x \mapsto f(x, S_{t-1})$ is strictly increasing, and we have

$$\lim_{x\to-\infty} f(x, S_{t-1}) = -\infty \quad \text{and} \quad \lim_{x\to\infty} f(x, S_{t-1}) = \infty.$$

Therefore, Equation (S.3.8) admits a unique solution $x = \xi_t(S_{t-1})$.

e) We have

$$\xi_t(S_{t-1}) = \rho \frac{\eta_{t+1}(\beta S_{t-1}) - \eta_{t+1}(\alpha S_{t-1})}{\widetilde{S}_{t-1}\big(g_\beta(\xi_t(S_{t-1}),\xi_{t+1}(\beta S_{t-1})) - g_\alpha(\xi_t(S_{t-1}),\xi_{t+1}(\alpha S_{t-1}))\big)}$$
$$+ \frac{\xi_{t+1}(\beta S_{t-1}) g_\beta(\xi_t(S_{t-1}),\xi_{t+1}(\beta S_{t-1})) - \xi_{t+1}(\alpha S_{t-1}) g_\alpha(\xi_t(S_{t-1}),\xi_{t+1}(\alpha S_{t-1}))}{g_\beta(\xi_t(S_{t-1}),\xi_{t+1}(\beta S_{t-1})) - g_\alpha(\xi_t(S_{t-1}),\xi_{t+1}(\alpha S_{t-1}))},$$

and

$$\eta_t(S_{t-1})$$
$$= \widetilde{S}_{t-1} \frac{g_\alpha(\xi_t(S_{t-1}),\xi_{t+1}(\alpha S_{t-1})) g_\beta(\xi_t(S_{t-1}),\xi_{t+1}(\beta S_{t-1})) \xi_{t+1}(\beta S_{t-1})}{\rho g_\alpha(\xi_t(S_{t-1}),\xi_{t+1}(\alpha S_{t-1})) - \rho g_\beta(\xi_t(S_{t-1}),\xi_{t+1}(\beta S_{t-1}))}$$
$$- \widetilde{S}_{t-1} \frac{g_\beta(\xi_t(S_{t-1}),\xi_{t+1}(\beta S_{t-1})) g_\alpha(\xi_t(S_{t-1}),\xi_{t+1}(\alpha S_{t-1})) \xi_{t+1}(\alpha S_{t-1})}{\rho g_\alpha(\xi_t(S_{t-1}),\xi_{t+1}(\alpha S_{t-1})) - \rho g_\beta(\xi_t(S_{t-1}),\xi_{t+1}(\beta S_{t-1}))}$$
$$+ \frac{g_\alpha(\xi_t(S_{t-1}),\xi_{t+1}(\alpha S_{t-1})) \eta_{t+1}(\beta S_{t-1}) - g_\beta(\xi_t(S_{t-1}),\xi_{t+1}(\beta S_{t-1})) \eta_{t+1}(\alpha S_{t-1})}{g_\alpha(\xi_t(S_{t-1}),\xi_{t+1}(\alpha S_{t-1})) - g_\beta(\xi_t(S_{t-1}),\xi_{t+1}(\beta S_{t-1}))}.$$

f) i) In case $f\big(\xi_{t+1}(\alpha S_{t-1}), S_{t-1}\big) \geq 0$ we have $\xi_t(S_{t-1}) \leq \xi_{t+1}(\alpha S_{t-1})$ because f is increasing, hence

$$g_\alpha(\xi_t(S_{t-1}),\xi_{t+1}(\alpha S_{t-1})) = \alpha^\uparrow(1+\lambda)\alpha.$$

ii) In case $f\big(\xi_{t+1}(\alpha S_{t-1}), S_{t-1}\big) < 0$ we have $\xi_t(S_{t-1}) > \xi_{t+1}(\alpha S_{t-1})$ because f is increasing, hence

$$g_\alpha(\xi_t(S_{t-1}),\xi_{t+1}(\alpha S_{t-1})) = \alpha_\downarrow = (1-\lambda)\alpha.$$

Note that in case $f\big(\xi_{t+1}(\alpha S_{t-1}), S_{t-1}\big) = 0$ we have $\xi_t(S_{t-1}) = \xi_{t+1}(\alpha S_{t-1})$ hence there is no transaction from S_{t-1} to αS_{t-1}. Similarly,

iii) If $f\big(\xi_{t+1}(\beta S_{t-1}), S_{t-1}\big) \geq 0$ then $\xi_t(S_{t-1}) \leq \xi_{t+1}(\beta S_{t-1})$, hence

$$g_\beta(\xi_t(S_{t-1}),\xi_{t+1}(\beta S_{t-1})) = \beta^\uparrow(1+\lambda)\beta.$$

iv) If $f\big(\xi_{t+1}(\beta S_{t-1}), S_{t-1}\big) < 0$ then $\xi_t(S_{t-1}) > \xi_{t+1}(\beta S_{t-1})$, hence

$$g_\beta(\xi_t(S_{t-1}),\xi_{t+1}(\beta S_{t-1})) = \beta_\downarrow = (1-\lambda)\beta.$$

Note that in case $f\big(\xi_{t+1}(\beta S_{t-1}), S_{t-1}\big) = 0$ we have $\xi_t(S_{t-1}) = \xi_{t+1}(\beta S_{t-1})$ hence there is no transaction from S_{t-1} to βS_{t-1}.

g) With the parameters $N = 2$, $K = \$2$, $S_0 = 8$, $\rho = 1$, $\alpha = 0.5$, $\beta = 2$, and the transaction cost rate $\lambda = 12.5\%$, we find the tree of asset prices

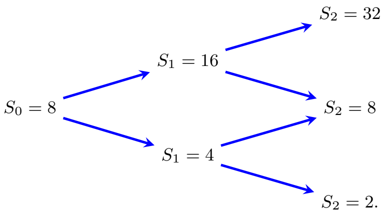


Fig. S.11: Tree of market prices with $N = 2$.

At maturity time N we use the equations (4.4.4)-(4.4.5) of Proposition 4.11, which read here

$$\xi_N\big(S_{N-1}\big) = \frac{h(\beta S_{N-1}) - h(\alpha S_{N-1})}{(\beta-\alpha)S_{N-1}}, \tag{4.4.4}$$

where $h(t,x) = (x-K)^+$, and

$$\eta_N(S_{N-1}) = \frac{\beta h(\alpha S_{N-1}) - \alpha h(\beta S_{N-1})}{(\beta-\alpha)A_N}, \tag{4.4.5}$$

as the evaluation of the terminal payoff is not affected by bid/ask prices. This yields

$$(\eta_2(16), \xi_2(16)) = (-2, 1) \quad \text{and} \quad (\eta_2(4), \xi_2(4)) = (-2, 1).$$

In this case we check that $f(\xi_2(16), S_0) = f(1,8) = 0$ and $f(\xi_2(4), S_0) = f(1,8) = 0$, which yields the hedging strategy $\xi_1(8) = \xi_2(16) = \xi_2(4) = 1$ and $\eta_1(8) = \eta_1(15) = \eta_1(4) = -2$ as the portfolio is self-financing. This static hedging involves no transaction costs and gives the initial price $V_0 = 8 \times 1 - 2 \times 1 = \$6$.

Due to the simplicity of the case $K = \$2$, we now consider the case $K = \$4$. In this case, (4.4.4) and (4.4.5) give

$$(\eta_2(16), \xi_2(16)) = (-4, 1) \quad \text{and} \quad (\eta_2(4), \xi_2(4)) = (-4/3, 2/3),$$

which yields

$$\begin{aligned} f(\xi_2(16), S_0) &= f(1,8) \\ &= g_\beta(1,1)(1-1) - g_\alpha(1,2/3)(1-2/3) - \frac{-4-(-4/3)}{8} \\ &= -\frac{1}{3}\alpha_\downarrow + \frac{1}{3} \end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{3}(1-\lambda)\alpha + \frac{1}{3} \\
&= -\frac{1}{3} \times 0.875 \times \frac{1}{2} + \frac{1}{3} \\
&> 0,
\end{aligned}
$$

hence

$$
g_\beta(\xi_1(S_0), \xi_2(\beta S_0)) = \beta_\uparrow = (1+\lambda)\beta.
$$

We also have

$$
\begin{aligned}
&f(\xi_2(4), S_0) = f(2/3, 8) \\
&= g_\beta(2/3, 1)(2/3-1) - g_\alpha(2/3, 2/3)(2/3-2/3) - \frac{-4-(-4/3)}{8} \\
&= -\frac{1}{3}\beta_\uparrow + \frac{1}{3} \\
&= -\frac{1}{3}(1+\lambda)\beta + \frac{1}{3} \\
&= -\frac{2}{3} \times 1.125 + \frac{1}{3} \\
&< 0,
\end{aligned}
$$

hence

$$
g_\alpha(\xi_1(S_0), \xi_2(\alpha S_0)) = \alpha_\downarrow = (1-\lambda)\alpha.
$$

Therefore, we find

$$
\begin{aligned}
\xi_1(S_0) &= \rho \frac{\eta_2(\beta S_0) - \eta_2(\alpha S_0)}{\widetilde{S}_0\big(g_\beta(\xi_1(S_0), \xi_2(\beta S_0)) - g_\alpha(\xi_1(S_0), \xi_2(\alpha S_0))\big)} \\
&\quad + \frac{\xi_2(\beta S_0) g_\beta(\xi_1(S_0), \xi_2(\beta S_0)) - \xi_2(\alpha S_0) g_\alpha(\xi_1(S_0), \xi_2(\alpha S_0))}{g_\beta(\xi_1(S_0), \xi_2(\beta S_0)) - g_\alpha(\xi_1(S_0), \xi_2(\alpha S_0))} \\
&= \rho \frac{\eta_2(\beta S_0) - \eta_2(\alpha S_0)}{\widetilde{S}_0\big((1+\lambda)\beta - (1-\lambda)\alpha\big)} + \frac{\xi_2(\beta S_0)(1+\lambda)\beta - \xi_2(\alpha S_0)(1-\lambda)\alpha}{(1+\lambda)\beta - (1-\lambda)\alpha} \\
&= \frac{-4-(-4/3)}{8\big(1.125 \times 2 - 0.875 \times 0.5\big)} + \frac{1.125 \times 2 - (2/3) \times 0.875 \times 0.5}{2 \times 1.125 - 0.5 \times 0.875} \\
&= 0.8965,
\end{aligned}
$$

and

$$
\begin{aligned}
\eta_1(S_0) = \widetilde{S}_0 & \frac{g_\alpha(\xi_1(S_0), \xi_2(\alpha S_0)) g_\beta(\xi_1(S_0), \xi_2(\beta S_0)) \xi_2(\beta S_0)}{\rho g_\alpha(\xi_1(S_0), \xi_2(\alpha S_0)) - \rho g_\beta(\xi_1(S_0), \xi_2(\beta S_0))} \\
& - \widetilde{S}_0 \frac{g_\beta(\xi_1(S_0), \xi_1(\beta S_0)) g_\alpha(\xi_1(S_0), \xi_2(\alpha S_0)) \xi_2(\alpha S_0)}{\rho g_\alpha(\xi_1(S_0), \xi_2(\alpha S_0)) - \rho g_\beta(\xi_1(S_0), \xi_2(\beta S_0))} \\
& + \frac{g_\alpha(\xi_1(S_0), \xi_2(\alpha S_0)) \eta_2(\beta S_0) - g_\beta(\xi_1(S_0), \xi_2(\beta S_0)) \eta_2(\alpha S_0)}{g_\alpha(\xi_1(S_0), \xi_2(\alpha S_0)) - g_\beta(\xi_1(S_0), \xi_2(\beta S_0))}
\end{aligned}
$$

$$= \widetilde{S}_0 \frac{(1-\lambda)\alpha(1+\lambda)\beta\xi_2(\beta S_0)}{\rho(1-\lambda)\alpha - \rho(1+\lambda)\beta} - \widetilde{S}_0 \frac{(1+\lambda)\beta(1-\lambda)\alpha\xi_2(\alpha S_0)}{(1-\lambda)\alpha\rho - (1+\lambda)\beta\rho}$$
$$+ \frac{(1-\lambda)\alpha\eta_2(\beta S_0) - (1+\lambda)\beta\eta_2(\alpha S_0)}{(1-\lambda)\alpha - (1+\lambda)\beta}$$
$$= 8\frac{0.875 \times 0.5 \times 1.125 \times 2 - 1.125 \times 2 \times 0.875 \times 0.5 \times 2/3}{0.875 \times 0.5 - 1.125 \times 2}$$
$$+ \frac{0.875 \times 0.5 \times (-4) - 1.125 \times 2 \times (-4/3)}{0.875 \times 0.5 - 1.125 \times 2}$$
$$= -2.1379.$$

This leads to the initial option price

$$V_0 = 0.8965 \times 8 - 2.1379 = 5.0345.$$

Remark: Note that with $\lambda = 0$ and $K = 4$ we would find

$$\xi_1(S_0) = \frac{8}{9} = 0.88, \quad \eta_1(S_0) = -\frac{20}{9} = -2.22, \quad \text{and} \quad V_0 = \frac{44}{9} = 4.88.$$

Therefore, the presence of transaction costs increases the price of the option, and requires a higher stock position and a higher level of debt.

h) Please refer to the attached **IPython notebook**.*

Remark: Transaction costs in the CRR model were originally introduced in Boyle and Vorst (1992). The present solution is based on the method of Mel′nikov and Petrachenko (2005), which originally also takes into account different borrowing and lending rates $\rho^{\uparrow} = 1 + r^{\uparrow}$ and $\rho_{\downarrow} = 1 + r_{\downarrow}$, which can be regarded as bid/ask prices for the riskless asset, and can also represent transaction costs.

Problem 3.18 CRR model with dividends (1).

a) Denoting $\widehat{S}_2$ the asset price at time 2 before the dividend is paid at the rate α, we find that the ex-dividend asset price S_2 after dividend payment is

$$S_2 = \widehat{S}_2 - \alpha\widehat{S}_2,$$

hence

$$\begin{aligned} V_2 &= \xi_2 S_2 + \eta_2 A_2 + \alpha\xi_2\widehat{S}_2 \\ &= \xi_2 S_2 + \eta_2 A_2 + \alpha\xi_2\frac{S_2}{1-\alpha} \\ &= \xi_2\frac{S_2}{1-\alpha} + \eta_2 A_2. \end{aligned}$$

* Right-click to save as attachment (may not work on 🍎).

b) Denoting $\widehat{S}_1$ the asset price at time 1 before the dividend is paid at the rate α, we find that the ex-dividend asset price S_1 after dividend payment is

$$S_1 = \widehat{S}_1 - \alpha \widehat{S}_1,$$

hence

$$\begin{aligned} V_1 &= \xi_1 S_1 + \eta_1 A_1 + \alpha \xi_1 \widehat{S}_1 \\ &= \xi_1 S_1 + \eta_1 A_1 + \alpha \xi_1 \frac{S_1}{1-\alpha} \\ &= \xi_1 \frac{S_1}{1-\alpha} + \eta_1 A_1. \end{aligned}$$

c) If $S_1 = 3$ we have

$$V_2 = \xi_2 \frac{S_2}{1-\alpha} + \eta_2 A_2 = \begin{cases} \dfrac{9\xi_2}{1-\alpha} + \eta_2 2^2 = \$1 & \text{if } S_2 = 9, \\ \\ \dfrac{3\xi_2}{1-\alpha} + \eta_2 2^2 = 0 & \text{if } S_2 = 3, \end{cases}$$

hence $(\xi_2, \eta_2) = ((1-\alpha)/6, -1/8)$.

If $S_1 = 1$ we have

$$V_2 = \xi_2 \frac{S_2}{1-\alpha} + \eta_2 A_2 = \begin{cases} \dfrac{3\xi_2}{1-\alpha} + \eta_2 2^2 = 0 & \text{if } S_2 = 3, \\ \\ \dfrac{\xi_2}{1-\alpha} + \eta_2 2^2 = 0 & \text{if } S_2 = 1, \end{cases}$$

hence $(\xi_2, \eta_2) = (0, 0)$.

d) We have

$$\begin{cases} V_1 = \xi_2 S_1 + 2\eta_2 = 3 \times \dfrac{1-\alpha}{6} - 2 \times \dfrac{1}{8} = \dfrac{1-2\alpha}{4} & \text{if } S_1 = 3, \\ \\ V_1 = \xi_2 S_1 + 2\eta_2 = 0 \times 1 + 0 \times 2 = 0 & \text{if } S_1 = 1. \end{cases}$$

e) We have

$$V_1 = \xi_1 \frac{S_1}{1-\alpha} + \eta_1 A_1 = \begin{cases} \dfrac{3\xi_1}{1-\alpha} + 2\eta_1 = \dfrac{1-2\alpha}{4} & \text{if } S_1 = 3, \\ \\ \dfrac{\xi_1}{1-\alpha} + 2\eta_1 = 0 & \text{if } S_1 = 1, \end{cases}$$

hence $(\xi_1, \eta_1) = ((\alpha-1)(2\alpha-1)/8, (2\alpha-1)/16)$.

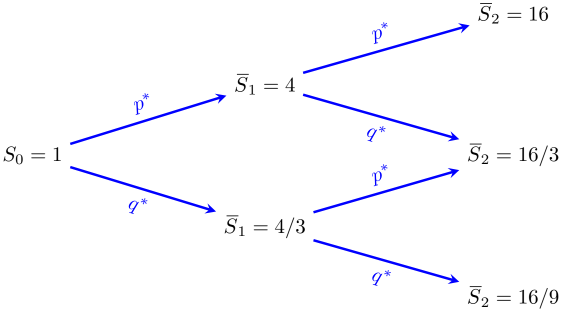
f) At time $k = 0$ we have

$$V_0 = \xi_1 S_0 + \eta_1 = \frac{(\alpha-1)(2\alpha-1)}{8} + \frac{2\alpha-1}{16} = \frac{(2\alpha-1)^2}{16}.$$

g) Multiplying the prices $(S_k)_{k=1,2}$ of the original tree by

$$\left(\frac{1}{1-\alpha}\right)^k = \left(\frac{1}{1-1/4}\right)^k = \left(\frac{4}{3}\right)^k,$$

we find the prices $(\overline{S}_k)_{k=1,2} = (S_k/(1-\alpha)^k)_{k=1,2}$ as in the following tree:



h) The market returns found in Question (g) are $\overline{a} = 1/3$ and $\overline{b} = 3$, with $r = 1\%$. Therefore we have

$$p^* = \frac{r-a}{b-a} = \frac{1-1/3}{3-1/3} = \frac{1}{4} \quad \text{and} \quad q^* = \frac{3-1}{3-1/3} = \frac{b-r}{b-a} = \frac{3}{4}.$$

i) If $S_1 = 3$ we have

$$\frac{1}{1+r}\mathbb{E}^*\left[(S_2-K)^+ \mid \overline{S}_1 = 3\right] = \$1 \times \frac{p^*}{2} = \frac{1}{8},$$

which coincides with

$$V_1 = \xi_2 S_1 + 2\eta_2 = \frac{3}{8} - \frac{2}{8} = \frac{1}{8}.$$

If $S_1 = 1$ we have

$$\frac{1}{1+r}\mathbb{E}^*\left[(S_2-K)^+ \mid \overline{S}_1 = 1\right] = 0,$$

which coincides with

$$V_1 = \xi_2 S_1 + 2\eta_2 = 0.$$

j) At time $k = 0$ we have

$$\frac{1}{(1+r)^2}\,\mathbb{E}^*\left[(S_2-K)^+\right]=\frac{(p^*)^2}{(1+r)^2}=\frac{1}{64},$$

which coincides with

$$V_0=\xi_1 S_0+\eta_1=\frac{3}{64}-\frac{1}{32}=\frac{1}{64}.$$

We also have

$$\frac{1}{1+r}\,\mathbb{E}^*\left[V_1\right]=\frac{p^*}{1+r}\times\frac{1}{8}=\frac{1}{8}\times\frac{1}{8}=\frac{1}{64}.$$

Problem 3.19 CRR model with dividends (2).

a) We have

$$\begin{aligned}S_k^{(1)}&=\left\{\begin{array}{ll}(1+b)(1-\alpha)S_{k-1}^{(1)} & \text{if } R_k=b\\ \\ (1+a)(1-\alpha)S_{k-1}^{(1)} & \text{if } R_k=a\end{array}\right\}\\&=(1+R_k)(1-\alpha)S_{k-1}^{(1)},\qquad k=1,2,\ldots,N,\end{aligned}$$

and

$$S_k^{(1)}=S_0^{(1)}\prod_{i=1}^{k}(1+R_i),\qquad k=0,1,\ldots,N,$$

with the binary tree

$$\begin{tikzcd}
 & (1+b)(1-\alpha)S_0^{(1)} \\
S_0^{(1)} \arrow[ur] \arrow[dr] & \\
 & (1+a)(1-\alpha)S_0^{(1)}
\end{tikzcd}$$

b) The asset price before dividend payment is $S_k^{(1)}/(1-\alpha)$, hence the dividend amount is

$$\frac{S_k^{(1)}}{1-\alpha}-S_k^{(1)}=\frac{\alpha S_k^{(1)}}{1-\alpha},$$

therefore, the dividend value represents a percentage $\alpha/(1-\alpha)$ of the ex-dividend price $S_k^{(1)}$. Moreover, the return of the risky asset satisfies the following relation

$$\mathbb{E}^*\left[\frac{S_{k+1}^{(1)}}{1-\alpha}\,\middle|\,\mathcal{F}_k\right] = \mathbb{E}^*\left[(1+R_k)S_k^{(1)}\,\middle|\,\mathcal{F}_k\right]$$
$$= \frac{r-a}{b-a}(1+b)S_k^{(1)} + \frac{b-r}{b-a}(1+a)S_k^{(1)}$$
$$= (1+r)S_k^{(1)}, \qquad k = 0,1,\ldots,N-1.$$

c) When reinvesting the dividend amount $\dfrac{\alpha}{1-\alpha}\xi_k S_k^{(1)}$ into the new portfolio allocation, we have

$$\begin{aligned} V_k &= \xi_{k+1}S_k^{(1)} + \eta_{k+1}S_k^{(0)} \\ &= \xi_k S_k^{(1)} + \eta_k S_k^{(0)} + \frac{\alpha}{1-\alpha}\xi_k S_k^{(1)} \\ &= \xi_k \frac{S_k^{(1)}}{1-\alpha} + \eta_k S_k^{(0)}, \end{aligned}$$

at times $k = 1,2,\ldots,N-1$. Moreover, at time N we will similarly have

$$V_N = \xi_N S_N^{(1)} + \frac{\alpha}{1-\alpha}\xi_N S_N^{(1)} + \eta_N S_N^{(0)} = \xi_N \frac{S_N^{(1)}}{1-\alpha} + \eta_N S_N^{(0)},$$

therefore the self-financing condition reads

$$V_k = \xi_k \frac{S_k^{(1)}}{1-\alpha} + \eta_k S_k^{(0)}, \qquad k = 1,2,\ldots,N. \tag{S.3.9}$$

d) By the self-financing condition (S.3.9) we have

$$\begin{aligned} \widetilde{V}_k - \widetilde{V}_{k-1} &= \xi_{k+1}\frac{S_k^{(1)}}{S_k^{(0)}} + \eta_{k+1} - \xi_k \frac{S_{k-1}^{(1)}}{S_{k-1}^{(0)}} - \eta_k \\ &= \frac{\xi_k S_k^{(1)}}{S_k^{(0)}(1-\alpha)} + \eta_k - \xi_k \frac{S_{k-1}^{(1)}}{S_{k-1}^{(0)}} - \eta_k \\ &= \xi_k\left(\frac{S_k^{(1)}}{S_k^{(0)}(1-\alpha)} - \frac{S_{k-1}^{(1)}}{S_{k-1}^{(0)}}\right), \qquad k = 1,2,\ldots,N, \end{aligned}$$

which allows us to conclude from Question (b) that

$$\begin{aligned} \mathbb{E}^*\left[\widetilde{V}_k \mid \mathcal{F}_{k-1}\right] - \widetilde{V}_{k-1} &= \mathbb{E}^*\left[\widetilde{V}_k - \widetilde{V}_{k-1} \mid \mathcal{F}_{k-1}\right] \\ &= \mathbb{E}^*\left[\xi_k \times \left(\frac{S_k^{(1)}}{S_k^{(0)}(1-\alpha)} - \frac{S_{k-1}^{(1)}}{S_{k-1}^{(0)}}\right)\,\middle|\,\mathcal{F}_{k-1}\right] \end{aligned}$$

$$
\begin{aligned}
&= \xi_k \, \mathbb{E}^* \left[\frac{S_k^{(1)}}{S_k^{(0)}(1-\alpha)} - \frac{S_{k-1}^{(1)}}{S_{k-1}^{(0)}} \,\Big|\, \mathcal{F}_{k-1} \right] \\
&= \frac{\xi_k}{S_k^{(0)}} \left(\mathbb{E}^* \left[\frac{S_k^{(1)}}{(1-\alpha)} \,\Big|\, \mathcal{F}_{k-1} \right] - (1+r) S_{k-1}^{(1)} \right) \\
&= 0, \qquad k = 1, 2, \ldots, N,
\end{aligned}
$$

therefore $\big(\widetilde{V}_k\big)_{k=0,1,\ldots,N-1}$ is a martingale under $\mathbb{P}^*$.

e) Assuming that the portfolio strategy attains the claim C we have $C = V_N$ and $\widetilde{C} = \widetilde{V}_N$, hence by the martingale property of $\big(\widetilde{V}_k\big)_{k=0,1,\ldots,N-1}$ under $\mathbb{P}^*$, we find

$$
\widetilde{V}_k = \mathbb{E}^* \big[\widetilde{V}_N \,\big|\, \mathcal{F}_k \big] = \mathbb{E}^* \big[\widetilde{C} \,\big|\, \mathcal{F}_k \big], \qquad k = 0, 1, \ldots, N,
$$

which shows that

$$
V_k = \frac{1}{(1+r)^{N-k}} \, \mathbb{E}^*[C \mid \mathcal{F}_k], \qquad k = 0, 1, \ldots, N,
$$

f) By a binomial probability computation, we have

$$
\begin{aligned}
V_k &= \frac{1}{(1+r)^{N-k}} \, \mathbb{E}^*[h(S_N) \mid \mathcal{F}_k] \\
&= \frac{1}{(1+r)^{N-k}} \, \mathbb{E}^* \left[h\left(x \prod_{l=t+1}^{N} (1+R_l) \right) \,\Big|\, \mathcal{F}_k \right]_{x = S_k^{(1)}}, \\
&= \frac{1}{(1+r)^{N-k}} \\
&\quad \times \sum_{l=0}^{N-k} \binom{N-k}{l} (p^*)^l (q^*)^{N-k-l} h\big(S_k^{(1)} (1+b)^k (1+a)^{N-k-l} (1-\alpha)^{N-k} \big) \\
&= C_0\big(k, S_k^{(1)} (1-\alpha)^{N-k}, N, a, b, r\big).
\end{aligned}
$$

g) We "absorb" the dividend rate α into new market returns by taking $a_\alpha, b_\alpha, r_\alpha$ such that

$$
1 + a_\alpha = (1+a)(1-\alpha), \quad 1 + b_\alpha = (1+b)(1-\alpha), \quad 1 + r_\alpha = (1+r)(1-\alpha),
$$

i.e.

$$
a_\alpha = -\alpha + a(1-\alpha), \quad b_\alpha = -\alpha + b(1-\alpha), \quad r_\alpha = -\alpha + r(1-\alpha).
$$

As a consequence, we have

$$
V_k = \frac{1}{(1+r)^{N-k}}
$$

$$\times \sum_{l=0}^{N-k} \binom{N-k}{l} (p^*)^l (q^*)^{N-k-l} h\big(S_k^{(1)}(1+b)^k(1+a)^{N-k-l}(1-\alpha)^{N-k}\big)$$

$$= \frac{(1-\alpha)^{N-k}}{(1+r_\alpha)^{N-k}} \sum_{l=0}^{N-k} \binom{N-k}{l} (p^*)^k (q^*)^{N-k-l} h\big(S_k^{(1)}(1+b_\alpha)^k(1+a_\alpha)^{N-k-l}\big)$$

$$= (1-\alpha)^{N-k} C_0\big(k, S_k^{(1)}, N, a_\alpha, b_\alpha, r_\alpha\big),$$

where

$$p^* := \mathbb{P}^*(R_k = b) = \frac{r_\alpha - a_\alpha}{b_\alpha - a_\alpha} = \frac{r-a}{b-a} > 0,$$

and

$$q^* := \mathbb{P}^*(R_k = a) = \frac{b_\alpha - r_\alpha}{b_\alpha - a_\alpha} = \frac{b-r}{b-a} > 0,$$

$k = 1, 2, \ldots, N$.

h) We have

$$\widetilde{V}_k = \frac{1}{(1+r)^N} C_\alpha\big(k, S_k^{(1)}, N, a_\alpha, b_\alpha, r_\alpha\big), \quad k = 0, 1, \ldots, N,$$

hence by the martingale property we have

$$\begin{aligned}
\widetilde{V}_k &= \frac{1}{(1+r)^k} C_\alpha\big(k, S_k^{(1)}, N, a_\alpha, b_\alpha, r_\alpha\big) \\
&= \mathbb{E}^*\big[\widetilde{V}_{k+1} \mid \mathcal{F}_k\big] \\
&= \frac{1}{(1+r)^{k+1}} \mathbb{E}^*\Big[C_\alpha\big(k+1, S_{k+1}^{(1)}, N, a_\alpha, b_\alpha, r_\alpha\big) \Big| \mathcal{F}_k\Big] \\
&= \frac{1}{(1+r)^{k+1}} \Big(p^* C_\alpha\big(k+1, S_k^{(1)}(1+b_\alpha), N, a_\alpha, b_\alpha, r_\alpha\big) \\
&\quad + q^* C_\alpha\big(k+1, S_k^{(1)}(1+a_\alpha), N, a_\alpha, b_\alpha, r_\alpha\big)\Big).
\end{aligned}$$

This yields

$$\begin{aligned}
&(1+r) C_\alpha\big(k, S_k^{(1)}, N, a_\alpha, b_\alpha, r_\alpha\big) \\
&= p^* C_\alpha\big(k+1, S_k^{(1)}(1+b_\alpha), N, a_\alpha, b_\alpha, r_\alpha\big) + q^* C_\alpha\big(k+1, S_k^{(1)}(1+a_\alpha), N, a_\alpha, b_\alpha, r_\alpha\big).
\end{aligned}$$

i) We find the equations

$$\begin{cases}
\eta_k S_k^{(0)} + \xi_k(1+a_\alpha) S_{k-1}^{(1)} = C_\alpha\big(k, (1+a_\alpha) S_{k-1}^{(1)}, N, a_\alpha, b_\alpha, r_\alpha\big) \\[2ex]
\eta_k S_k^{(0)} + \xi_k(1+b_\alpha) S_{k-1}^{(1)} = C_\alpha\big(k, (1+b_\alpha) S_{k-1}^{(1)}, N, a_\alpha, b_\alpha, r_\alpha\big),
\end{cases}$$

which imply

$$
\begin{aligned}
\xi_k &= \frac{C_\alpha\big(k,(1+b_\alpha)S^{(1)}_{k-1},N,a_\alpha,b_\alpha,r_\alpha\big)-C_\alpha\big(k,(1+a_\alpha)S^{(1)}_{k-1},N,a_\alpha,b_\alpha,r_\alpha\big)}{(b_\alpha-a_\alpha)S^{(1)}_{k-1}} \\
&= (1-\alpha)^{N-k}\frac{C_0\big(k,(1+b_\alpha)S^{(1)}_{k-1},N,a_\alpha,b_\alpha,r_\alpha\big)}{(b_\alpha-a_\alpha)S^{(1)}_{k-1}} \\
&\quad -(1-\alpha)^{N-k}\frac{C_0\big(k,(1+a_\alpha)S^{(1)}_{k-1},N,a_\alpha,b_\alpha,r_\alpha\big)}{(b_\alpha-a_\alpha)S^{(1)}_{k-1}},
\end{aligned}
$$

and

$$
\begin{aligned}
\eta_k &= \frac{(1+b_\alpha)C_\alpha\big(k,(1+b_\alpha)S^{(1)}_{k-1},N,a_\alpha,b_\alpha,r_\alpha\big)}{(b_\alpha-a_\alpha)S^{(0)}_{k-1}} \\
&\quad - \frac{(1+a_\alpha)C_\alpha\big(k,(1+a_\alpha)S^{(1)}_{k-1},N,a_\alpha,b_\alpha,r_\alpha\big)}{(b_\alpha-a_\alpha)S^{(0)}_{k-1}} \\
&= (1-\alpha)^{N-k}\frac{(1+b_\alpha)C_0\big(k,(1+b_\alpha)S^{(1)}_{k-1},N,a_\alpha,b_\alpha,r_\alpha\big)}{(b_\alpha-a_\alpha)S^{(0)}_{k-1}} \\
&\quad -(1-\alpha)^{N-k}\frac{(1+a_\alpha)C_0\big(k,(1+a_\alpha)S^{(1)}_{k-1},N,a_\alpha,b_\alpha,r_\alpha\big)}{(b_\alpha-a_\alpha)S^{(0)}_{k-1}},
\end{aligned}
$$

$k=1,2,\ldots,N$.

j) *A possible answer*: We have

$$
\begin{aligned}
\xi_k = \frac{1}{(b-a)S^{(1)}_{k-1}} \sum_{l=0}^{N-k} \binom{N-k}{l} (p^*)^k (q^*)^{N-k-l} \\
\times \Big(h\big((1-\alpha)^{N-k}S^{(1)}_k(1+b)^{k+1}(1+a)^{N-k-l}\big) \\
-h\big((1-\alpha)^{N-k}S^{(1)}_k(1+b)^{k}(1+a)^{N-k-l+1}\big)\Big)
\end{aligned}
$$

and

$$
\begin{aligned}
\eta_k = \frac{1}{(b-a)S^{(1)}_{k-1}} \sum_{l=0}^{N-k} \binom{N-k}{l} (p^*)^k (q^*)^{N-k-l} \\
\times \Big((1+b)h\big((1-\alpha)^{N-k}S^{(1)}_k(1+b)^{k}(1+a)^{N-k-l+1}\big) \\
-(1+a)h\big((1-\alpha)^{N-k}S^{(1)}_k(1+b)^{k+1}(1+a)^{N-k-l}\big)\Big),
\end{aligned}
$$

$k=1,2,\ldots,N$. Differentiation with respect to α of the general term inside the above summations yields respectively

$$(1+a)yh'((1+a)y) - (1+b)yh'((1+b)y) \tag{S.3.10}$$

for ξ_k, and

$$(1+b)yh'((1+b)y) - (1+a)yh'((1+a)y), \tag{S.3.11}$$

for η_k, with $y := (1-\alpha)^{N-k} S_k^{(1)} (1+b)^k (1+a)^{N-k-l}$ and $a < b$.

We note that the sign of the above quantities (S.3.10)-(S.3.11) depends on whether the function $x \longmapsto xh'(x)$ is non-decreasing, which is the case for example for the payoff functions $h(x) = (x-K)^+$ and $h(x) = (K-x)^+$ of both European call and put options.

In particular, when the function $x \longmapsto xh'(x)$ is non-decreasing, the amount invested on the risky (resp. riskless) asset will be lower (resp. higher) in the presence of a higher dividend.

We also note that the expected return

$$p^*(1+b)(1-\alpha) + q^*(1+a)(1-\alpha) = r(1-\alpha)$$

and the variance

$$\begin{aligned} &p^*(1+b)^2(1-\alpha)^2 + q^*(1+a)^2(1-\alpha)^2 - r^2(1-\alpha)^2 \\ &= (1-\alpha)^2\big(p^*(1+b)^2 + q^*(1+a)^2 - r^2\big) \end{aligned}$$

of returns are lower in the presence of dividends.

Problem 3.20

a) In order to check for arbitrage opportunities we look for a risk-neutral probability measure $\mathbb{P}^*$ which should satisfy

$$\mathbb{E}^*\big[S_{k+1}^{(1)} \,\big|\, \mathcal{F}_k\big] = (1+r)S_k^{(1)}, \qquad k = 0,1,\ldots,N-1.$$

Rewriting $\mathbb{E}^*\big[S_{k+1}^{(1)} \,\big|\, \mathcal{F}_k\big]$ as

$$\begin{aligned} \mathbb{E}^*\big[S_{k+1}^{(1)} \,\big|\, \mathcal{F}_k\big] &= (1+a)S_k^{(1)}\mathbb{P}^*(R_{k+1} = a \mid \mathcal{F}_k) + S_k^{(1)}\mathbb{P}^*(R_{k+1} = 0 \mid \mathcal{F}_k) \\ &\quad + (1+b)S_k^{(1)}\mathbb{P}^*(R_{k+1} = b \mid \mathcal{F}_k) \\ &= (1+a)S_k^{(1)}\mathbb{P}^*(R_{k+1} = a) + S_k^{(1)}\mathbb{P}^*(R_{k+1} = 0) \\ &\quad + (1+b)S_k^{(1)}\mathbb{P}^*(R_{k+1} = b), \end{aligned}$$

$k = 0,1,\ldots,N-1$, it follows that any risk-neutral probability measure $\mathbb{P}^*$ should satisfy the equations

$$\begin{cases} (1+r)S_k^{(1)} = \\ \quad (1+b)S_k^{(1)}\mathbb{P}^*(R_{k+1}=b) + S_k^{(1)}\mathbb{P}^*(R_{k+1}=0) + (1+a)S_k^{(1)}\mathbb{P}^*(R_{k+1}=a), \\ \mathbb{P}^*(R_{k+1}=b) + \mathbb{P}^*(R_{k+1}=0) + \mathbb{P}^*(R_{k+1}=a) = 1, \end{cases}$$

$k = 0, 1, \ldots, N-1$, *i.e.*

$$\begin{cases} b\mathbb{P}^*(R_k = b) + a\mathbb{P}^*(R_k = a) = r, \\ \mathbb{P}^*(R_k = b) + \mathbb{P}^*(R_k = a) = 1 - \mathbb{P}^*(R_k = 0), \end{cases}$$

$k = 1, 2, \ldots, N$, with solution

$$\mathbb{P}^*(R_k = b) = \frac{r - (1 - \mathbb{P}^*(R_k = 0))a}{b - a} = \frac{r - (1 - \theta^*)a}{b - a},$$

and

$$\mathbb{P}^*(R_k = a) = \frac{(1 - \mathbb{P}^*(R_k = 0))b - r}{b - a} = \frac{(1 - \theta^*)b - r}{b - a},$$

$k = 1, 2, \ldots, N$. We check that this ternary tree model is without arbitrage if and only if there exists $\theta^* := \mathbb{P}^*(R_k = 0) \in (0, 1)$ such that

$$(1 - \theta^*)a < r < (1 - \theta^*)b, \tag{S.3.12}$$

or

$$0 < \theta^* < \min\left(\frac{r - a}{-a}, \frac{b - r}{b}\right) = \begin{cases} 1 - \dfrac{r}{b} & \text{if } r \geq 0, \\ 1 - \dfrac{r}{a} & \text{if } r \leq 0. \end{cases}$$

Condition (S.3.12) is necessary in order to have

$$\mathbb{P}^*(R_k = b) > 0 \quad \text{and} \quad \mathbb{P}^*(R_k = a) > 0,$$

and it is sufficient because it also implies

$$\mathbb{P}^*(R_k = b) = 1 - \theta^* - \mathbb{P}^*(R_k = a) \leq 1$$

and

$$\mathbb{P}^*(R_k = a) = 1 - \theta^* - \mathbb{P}^*(R_k = b) \leq 1.$$

b) We will show that this ternary tree model is without arbitrage if and only if $a < r < b$.

(i) Indeed, if the condition $a < r < b$ is satisfied there always exists $\theta \in (0, 1)$ such that

$$a < (1-\theta)a < r < (1-\theta)b < b,$$

as can be seen by taking

$$\theta \in \left(0, \min\left(\frac{r-a}{-a}, \frac{b-r}{b}\right)\right),$$

hence there exists a risk-neutral probability measure $\mathbb{P}^*_\theta$, and the market model is without arbitrage.

(ii) Conversely, if this ternary tree model is without arbitrage there exists some $\theta = \mathbb{P}^*(R_t = 0) \in (0,1)$ such that

$$(1-\theta)a < r < (1-\theta)b.$$

c) When $r \leq a < 0 < b$ the risky asset overperforms the riskless asset, therefore we can realize arbitrage by borrowing from the riskless asset to purchase the risky asset. When $a < 0 < b \leq r$ the riskless asset overperforms the risky asset, therefore we can realize arbitrage by shortselling the risky asset and save the profit of the short sale on the riskless asset.

d) Under the absence of arbitrage condition $a < r < b$, every value of $\theta \in (0,1)$ such that

$$0 < \theta < \min\left(\frac{r-a}{-a}, \frac{b-r}{b}\right)$$

satisfies

$$(1-\theta)a < r < (1-\theta)b,$$

and gives rise to a different risk-neutral probability measure, hence the risk-neutral measure is not unique and by Theorem 5.11 this ternary tree model is *not* complete.

In particular, every risk-neutral probability measure $\mathbb{P}^*_\theta$ will give rise to a different claim price

$$\pi_t^{(\theta)}(C) = \frac{1}{(1+r)^{N-t}} \mathbb{E}^*_\theta \left[C \,\middle|\, \mathcal{F}_t \right], \qquad t = 0, 1, \ldots, N.$$

e) We have

$$\begin{aligned}
&\mathrm{Var}^*\left[\frac{S^{(1)}_{k+1} - S^{(1)}_k}{S^{(1)}_k} \,\middle|\, \mathcal{F}_k\right] \\
&= \mathbb{E}^*\left[\left(\frac{S^{(1)}_{k+1} - S^{(1)}_k}{S^{(1)}_k}\right)^2 \,\middle|\, \mathcal{F}_k\right] - \left(\mathbb{E}^*\left[\frac{S^{(1)}_{k+1} - S^{(1)}_k}{S^{(1)}_k} \,\middle|\, \mathcal{F}_k\right]\right)^2
\end{aligned}$$

$$
\begin{aligned}
&= \mathbb{E}^*\left[\left(\frac{S^{(1)}_{k+1}-S^{(1)}_k}{S^{(1)}_k}\right)^2 \,\middle|\, \mathcal{F}_k\right] - r^2 \\
&= a^2\mathbb{P}^*_\sigma(R_{k+1}=a \mid \mathcal{F}_k) + b^2\mathbb{P}^*_\sigma(R_{k+1}=b \mid \mathcal{F}_k) - r^2 \\
&= a^2\frac{(1-\mathbb{P}^*_\sigma(R_{k+1}=0))b-r}{b-a} + b^2\frac{r-(1-\mathbb{P}^*_\sigma(R_{k+1}=0))a}{b-a} - r^2 \\
&= ab(\theta-1) + r(a+b) - r^2 \\
&= \sigma^2,
\end{aligned}
$$

$k = 0, 1, \ldots, N-1$, hence

$$\mathbb{P}^*_\sigma(R_k = 0) = \theta = 1 + \frac{\sigma^2 + r^2 - r(a+b)}{ab},$$

and therefore

$$\mathbb{P}^*_\sigma(R_k = b) = \frac{r-(1-\mathbb{P}^*_\sigma(R_k=0))a}{b-a} = \frac{\sigma^2 - r(a-r)}{b(b-a)},$$

and

$$\mathbb{P}^*_\sigma(R_k = a) = \frac{(1-\mathbb{P}^*_\sigma(R_k=0))b-r}{b-a} = \frac{r(b-r)-\sigma^2}{a(b-a)},$$

$k = 1, 2, \ldots, N$, under the condition

$$\sigma^2 > \max(-r(r-a), r(b-r)),$$

in addition to the condition $0 < \theta < 1$, *i.e.*

$$r(b-r) + rb < \sigma^2 < (b-r)(r-a).$$

Finally, we find

$$-r(r-a) < \sigma^2 < (b-r)(r-a),$$

if $r \in (a, 0]$, and

$$r(b-r) < \sigma^2 < (b-r)(r-a),$$

if $r \in [0, b)$.

f) In this case the ternary tree becomes a trinomial recombining tree, and the expression of the risk-neutral probability measure becomes

$$\mathbb{P}^*_\theta(R_k = b) = \frac{r(b+1) + (1-\theta)b}{b^2 + 2b},$$

and

$$\mathbb{P}^*_\theta(R_k = a) = (b+1)\frac{(1-\theta)b - r}{b^2+2b},$$

$k = 1, 2, \ldots, N$. The market model is without arbitrage if and only if there exists $\theta := \mathbb{P}^*_\theta(R_k = 0) \in (0, 1)$ such that

$$-(1-\theta)\frac{b}{b+1} < r < (1-\theta)b,$$

or

$$0 < \theta < 1 - \frac{r}{b}.$$

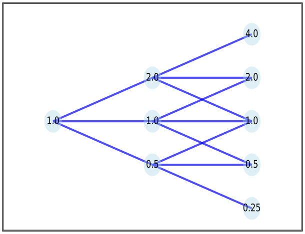
g) Using the tower property of conditional expectations, we have

$$\begin{aligned}
f\big(k, S_k^{(1)}\big) &= \frac{1}{(1+r)^{N-k}}\,\mathbb{E}^*[C \mid \mathcal{F}_k] \\
&= \frac{1}{(1+r)^{N-k}}\,\mathbb{E}^*[\mathbb{E}^*[C \mid \mathcal{F}_{k+1}] \mid \mathcal{F}_k] \\
&= \frac{1}{(1+r)^{N-k}}\,\mathbb{E}^*\left[(1+r)^{N-(k+1)} f\big(k+1, S_{k+1}^{(1)}\big) \,\big|\, \mathcal{F}_k\right] \\
&= \frac{1}{1+r}\,\mathbb{E}^*\left[f\big(k+1, S_{k+1}^{(1)}\big) \,\big|\, \mathcal{F}_k\right] \\
&= \frac{1}{1+r}\Big(f\big(k+1, S_k^{(1)}(1+a)\big)\mathbb{P}^*_\theta(R_k = a) + f\big(k+1, S_k^{(1)}\big)\mathbb{P}^*_\theta(R_k = 0) \\
&\quad + f\big(k+1, S_k^{(1)}(1+b)\big)\mathbb{P}^*_\theta(R_k = b)\Big).
\end{aligned}$$

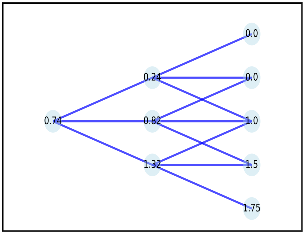
h) In this case we have $f(N, x) = (K - x)^+$.

i) See the attached code.[*†]

j) Taking $\theta = 0.5$ we find the following graph:



(a) Underlying asset prices.



(b) Put option prices.

Fig. S.12: Put option prices in the trinomial model.

[*] Download the modified (trinomial) **IPython notebook** that can be run **here** or **here**.

[†] Download the corresponding (binomial) **IPython notebook**. The Anaconda distribution can be installed from https://www.anaconda.com/distribution/ or tried online at https://jupyter.org/try.

There also exists extensions of the trinomial model to five states (pentanomial model), six states (hexanomial model), etc.

Chapter 4

Exercise 4.1 If $0 \leq s \leq t$, using the facts that $\mathbb{E}[B_t] = 0$ and $\mathbb{E}\left[B_t^2\right] = 0$, $t \geq 0$, we have

$$\begin{aligned}
\mathbb{E}[B_t B_s] &= \mathbb{E}[(B_t - B_s)B_s] + \mathbb{E}\left[B_s^2\right] \\
&= \mathbb{E}[(B_t - B_s)]\,\mathbb{E}[B_s] + \mathbb{E}\left[B_s^2\right] \\
&= 0 + s \\
&= s,
\end{aligned}$$

and similarly we obtain $\mathbb{E}[B_t B_s] = t$ when $0 \leq t \leq s$, hence in general we have

$$\mathbb{E}[B_t B_s] = \min(s,t), \qquad s,t \geq 0.$$

Exercise 4.2 We need to check whether the four properties of the definition of Brownian motion are satisfied.

a) Conditions 1-2-3 can be checked using the time shift $t \mapsto c + t$. As for Condition 4, $B_{c+t} - B_{c+s}$ clearly has the centered Gaussian distribution with variance $c + t - (c + s) = t - s$. We conclude that $(B_{c+t} - B_c)_{t\in\mathbb{R}_+}$ is a standard Brownian motion.
b) We note that B_{ct^2} is a centered Gaussian random variable with variance ct^2 - not t, hence $(B_{ct^2})_{t\in\mathbb{R}_+}$ is not a standard Brownian motion when $c \neq 1$.
c) Similarly, checking Conditions 1-2-3 does not pose any particular problem using the time change $t \mapsto t/c^2$. As for Condition 4, $B_{c+t} - B_{c+s}$ clearly has a centered Gaussian distribution with

$$\begin{aligned}
\operatorname{Var}\big(c\big(B_{t/c^2} - B_{s/c^2}\big)\big) &= c^2 \operatorname{Var}\big(B_{t/c^2} - B_{s/c^2}\big) \\
&= (t-s)c^2/c^2 \\
&= t - s.
\end{aligned}$$

As a consequence, $\big(B_{t/c^2}\big)_{t\in\mathbb{R}_+}$ is a standard Brownian motion.
d) This process does not have independent increments, hence it cannot be a Brownian motion. For example, by (4.1) we have

$$\begin{aligned}
&\mathbb{E}\left[\big(B_t + B_{t/2} - \big(B_s + B_{s/2}\big)\big)\big(B_s + B_{s/2}\big)\right] \\
&= \mathbb{E}\left[B_t B_s + B_t B_{s/2} + B_{t/2} B_s + B_{t/2} B_{s/2}\right] \\
&\quad - \mathbb{E}\left[B_s B_s + B_s B_{s/2} + B_{s/2} B_s + B_{s/2} B_{s/2}\right]
\end{aligned}$$

$$= s + \frac{s}{2} + s + \frac{s}{2} - s - \frac{s}{2} - \frac{s}{2} - \frac{s}{2}$$
$$= \frac{s}{2},$$

which differs from 0, hence the two increments are not independent. Indeed, independence of $B_t + B_{t/2} - \big(B_s + B_{s/2}\big)$ and $B_s + B_{s/2}$ would yield

$$\begin{aligned}
&\mathbb{E}\left[\big(B_t + B_{t/2} - \big(B_s + B_{s/2}\big)\big)\big(B_s + B_{s/2}\big)\right] \\
&= \mathbb{E}\left[B_t + B_{t/2} - \big(B_s + B_{s/2}\big)\right]\mathbb{E}\left[\big(B_s + B_{s/2}\big)\right] \\
&= 0.
\end{aligned}$$

Exercise 4.3 By Definition 4.5, we have

$$\int_0^T 2dB_t = 2(B_T - B_0) = 2B_T,$$

which has a Gaussian distribution with mean 0 and variance $4T$. On the other hand, by Definition 4.5 again, we have

$$\begin{aligned}
\int_0^T (2 \times \mathbb{1}_{[0,T/2]}(t) + \mathbb{1}_{(T/2,T]}(t))dB_t &= 2(B_{T/2} - B_0) + (B_T - B_{T/2}) \\
&= B_T + B_{T/2},
\end{aligned}$$

which has a Gaussian distribution with mean 0 and variance

$$\begin{aligned}
\operatorname{Var}[B_T + B_{T/2}] &= \operatorname{Var}[(B_T - B_{T/2}) + 2B_{T/2}] \\
&= \operatorname{Var}[B_T - B_{T/2}] + 4\operatorname{Var}[B_{T/2}] \\
&= \frac{T}{2} + \frac{4T}{2} \\
&= \frac{5T}{2}.
\end{aligned}$$

Equivalently, using the Itô isometry (4.8), we have

$$\begin{aligned}
&\operatorname{Var}\left[\left(\int_0^T (2 \times \mathbb{1}_{[0,T/2]}(t) + \mathbb{1}_{(T/2,T]}(t))dB_t\right)\right] \\
&= \mathbb{E}\left[\left(\int_0^T (2 \times \mathbb{1}_{[0,T/2]}(t) + \mathbb{1}_{(T/2,T]}(t))dB_t\right)^2\right] \\
&= \int_0^T \big(2 \times \mathbb{1}_{[0,T/2]}(t) + \mathbb{1}_{(T/2,T]}(t)\big)^2 dt \\
&= 4\int_0^{T/2} dt + \int_{T/2}^T dt \\
&= \frac{5T}{2}.
\end{aligned}$$

Exercise 4.4 By Proposition 4.12, the stochastic integral $\int_0^{2\pi} \sin(t)\, dB_t$ has a Gaussian distribution with mean 0 and variance

$$\int_0^{2\pi} \sin^2(t) dt = \frac{1}{2} \int_0^{2\pi} (1 - \cos(2t)) dt = \pi.$$

Exercise 4.5 By the Itô formula (4.28), we have

$$\begin{aligned} d(f(t)B_t) &= f(t)dB_t + B_t df(t) + df(t) \bullet dB_t \\ &= f(t)dB_t + B_t f'(t)dt + f'(t)dt \bullet dB_t \\ &= f(t)dB_t + B_t f'(t)dt, \end{aligned}$$

and by integration on both sides we get

$$\begin{aligned} \int_0^T f(t)dB_t + \int_0^T B_t f'(t)dt &= \int_0^T d(f(t)B_t) \\ &= f(T)B_T - f(0)B_0 \\ &= 0, \end{aligned}$$

since $f(T) = 0$ and $B_0 = 0$, hence the conclusion. Note that this result can also be obtained by integration by parts on the interval $[0, T]$, see (4.11).

Exercise 4.6

a) The stochastic integral $\int_0^1 t^2 dB_t$ is a centered Gaussian random variable with variance

$$\mathbb{E}\left[\left(\int_0^1 t^2 dB_t\right)^2\right] = \int_0^1 t^4 dt = \frac{1}{5}.$$

b) The stochastic integral $\int_0^1 t^{-1/2} dB_t$ has the variance

$$\mathbb{E}\left[\left(\int_0^1 t^{-1/2} dB_t\right)^2\right] = \int_0^1 \frac{1}{t} dt = +\infty.$$

In fact, the stochastic integral $\int_0^1 t^{-1/2} dB_t$ does not exist as a random variable in $L^2(\Omega)$ because the function $t \mapsto t^{-1/2}$ is not in $L^2([0, 1])$.

Remark. Writing Relation (4.11) with $f(t) = t^{-1/2}$ gives

$$\int_0^1 t^{-1/2} dB_t = \frac{B_T}{\sqrt{T}} + \frac{1}{2} \int_0^1 t^{-3/2} B_t dt,$$

however this is only a formal statement as f is not in $\mathcal{C}^1([0, 1])$. Informally, we can check that the term $\int_0^T t^{-3/2} B_t dt$ has the infinite variance

$$
\begin{aligned}
\mathbb{E}\left[\left(\int_0^T B_t f'(t)dt\right)^2\right] &= \mathbb{E}\left[\left(\int_0^T B_t f'(t)dt\right)\left(\int_0^T B_s f'(s)ds\right)\right] \\
&= \mathbb{E}\left[\int_0^T\int_0^T B_s B_t f'(s)f'(t)dsdt\right] \\
&= \int_0^T\int_0^T f'(s)f'(t)\,\mathbb{E}[B_sB_t]dsdt \\
&= \frac{1}{4}\int_0^T\int_0^T s^{-3/2}t^{-3/2}\min(s,t)dsdt \\
&= \frac{1}{4}\int_0^T t^{-3/2}\int_0^t s^{-5/2}dsdt + \frac{1}{4}\int_0^T t^{-5/2}\int_t^T s^{-3/2}dsdt \\
&= \frac{1}{4}\int_0^T t^{-3/2}\int_0^t s^{-5/2}dsdt + \frac{1}{4}\int_0^T t^{-5/2}\int_t^T s^{-3/2}dsdt \\
&= +\infty,
\end{aligned}
$$

where we used Relation (4.1) or the result of Exercise 4.1

Exercise 4.7

a) By Proposition 4.12, the probability distribution of X_n is Gaussian with mean zero and variance

$$
\begin{aligned}
\mathrm{Var}[X_n] &= \mathbb{E}\left[\left(\int_0^{2\pi}\sin(nt)dB_t\right)^2\right] \\
&= \int_0^{2\pi}\sin^2(nt)dt \\
&= \frac{1}{2}\int_0^{2\pi}\cos(0)dt - \frac{1}{2}\int_0^{2\pi}\cos(2nt)dt \\
&= \pi, \qquad n \geq 1.
\end{aligned}
$$

b) The random variables $(X_n)_{n\geq 1}$ have same Gaussian distribution, and they are pairwise independent as from Corollary 4.13 we have

$$
\begin{aligned}
\mathbb{E}[X_nX_m] &= \mathbb{E}\left[\int_0^{2\pi}\sin(nt)dB_t\int_0^{2\pi}\sin(mt)dB_t\right] \\
&= \int_0^{2\pi}\sin(nt)\sin(mt)dt \\
&= \frac{1}{2}\int_0^{2\pi}\cos((n-m)t)dt - \frac{1}{2}\int_0^{2\pi}\cos((n+m)t)dt \\
&= 0
\end{aligned}
$$

and the vector (X_n, X_m) is jointly Gaussian, for $n, m \geq 1$ such that $n \neq m$. Note that this condition implies independence only when the random variables have a Gaussian distribution.

Exercise 4.8 We have $X_t = f(B_t)$ with $f(x) = \sin^2 x$, $f'(x) = 2\sin x \cos x = \sin(2x)$, and $f''(x) = 2\cos(2x)$, hence

$$\begin{aligned}
dX_t &= d\sin^2(B_t) \\
&= df(B_t) \\
&= f'(B_t)dB_t + \frac{1}{2}f''(B_t)dt \\
&= \sin(2B_t)dB_t + \cos(2B_t)dt.
\end{aligned}$$

Exercise 4.9

a) Using the Itô isometry (4.16), we have

$$\begin{aligned}
\mathbb{E}\left[B_T^3\right] &= \mathbb{E}\left[\int_0^T dB_t \left(T + 2\int_0^T B_t dB_t\right)\right] \\
&= T\,\mathbb{E}\left[\int_0^T dB_t\right] + 2\,\mathbb{E}\left[\int_0^T dB_t \int_0^T B_t dB_t\right] \\
&= 2\,\mathbb{E}\left[\int_0^T B_t dt\right] \\
&= 2\int_0^T \mathbb{E}[B_t]dt \\
&= 0.
\end{aligned}$$

We also have

$$\begin{aligned}
\mathbb{E}\left[B_T^4\right] &= \mathbb{E}\left[\left(T + 2\int_0^T B_t dB_t\right)^2\right] \\
&= \mathbb{E}\left[T^2 + 4T\int_0^T B_t dB_t + 4\left(\int_0^T B_t dB_t\right)^2\right] \\
&= T^2 + 4T\,\mathbb{E}\left[\int_0^T B_t dB_t\right] + 4\,\mathbb{E}\left[\left(\int_0^T B_t dB_t\right)^2\right] \\
&= T^2 + 4\,\mathbb{E}\left[\int_0^T |B_t|^2 dt\right] \\
&= T^2 + 4\int_0^T \mathbb{E}\left[|B_t|^2\right]dt \\
&= T^2 + 4\int_0^T t dt \\
&= T^2 + 4\frac{T^2}{2} \\
&= 3T^2.
\end{aligned}$$

b) If $X \simeq \mathcal{N}(0, \sigma^2)$, we have $X \simeq B_T$ with $\sigma^2 = T$, hence the answer to Question (a) yields

$$\mathbb{E}\left[X^3\right] = 0 \quad \text{and} \quad \mathbb{E}\left[X^4\right] = 3\sigma^4.$$

We note that those moments can be recovered directly from the Gaussian probability density function as

$$\mathbb{E}\left[X^3\right] = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} x^3 \mathrm{e}^{-x^2/(2\sigma^2)} dx = 0$$

and

$$\mathbb{E}\left[X^4\right] = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} x^4 \mathrm{e}^{-x^2/(2\sigma^2)} dx = 3\sigma^4.$$

Exercise 4.10 Taking expectation on both sides of (4.39) shows that

$$\begin{aligned}
0 &= \mathbb{E}\left[(B_T)^3\right] \\
&= \mathbb{E}\left[C + \int_0^T \zeta_{t,T} dB_t\right] \\
&= C + \mathbb{E}\left[\int_0^T \zeta_{t,T} dB_t\right] \\
&= 0
\end{aligned}$$

by (4.17), hence $C = 0$. Next, applying Itô's formula to the function $f(x) = x^3$ shows that

$$\begin{aligned}
(B_T)^3 &= f(B_T) \\
&= f(B_0) + \int_0^T f'(B_t) dB_t + \frac{1}{2} \int_0^T f''(B_t) dt \\
&= 3 \int_0^T B_t^2 dB_t + 3 \int_0^T B_t dt.
\end{aligned}$$

By the integration by parts formula (4.11) applied to $f(t) = t$, we find

$$\int_0^T B_t dt = T B_T - \int_0^T t dB_t = \int_0^T (T - t) dB_t,$$

hence

$$\begin{aligned}
(B_T)^3 &= 3 \int_0^T B_t^2 dB_t + 3 \left(T B_T - \int_0^T t dB_t \right) \\
&= 3 \int_0^T (T - t + B_t^2) dB_t,
\end{aligned}$$

and we find $\zeta_{t,T} = 3(T-t+B_t^2)$, $t \in [0,T]$. This type of stochastic integral decomposition can be used for option hedging, cf. Section 7.5.

Exercise 4.11

a) We have

$$\begin{aligned}
\mathbb{E}\left[e^{\int_0^T f(s)dB_s} \Big| \mathcal{F}_t \right] &= e^{\int_0^t f(s)dB_s} \mathbb{E}\left[e^{\int_t^T f(s)dB_s} \Big| \mathcal{F}_t \right] \\
&= e^{\int_0^t f(s)dB_s} \mathbb{E}\left[e^{\int_t^T f(s)dB_s} \right] \\
&= \exp\left(\int_0^t f(s)dB_s + \frac{1}{2} \int_t^T |f(s)|^2 ds \right), \quad \text{(S.4.13)}
\end{aligned}$$

$0 \leq t \leq T$, where we used the Gaussian moment generating function $\mathbb{E}\left[e^X\right] = e^{\sigma^2/2}$ for $X \simeq \mathcal{N}(0,\sigma^2)$ and the fact that $\int_t^T f(s)dB_s \simeq \mathcal{N}\left(0, \int_t^T f^2(s)ds\right)$ by Proposition 4.12.

b) We have

$$\begin{aligned}
&\mathbb{E}\left[\exp\left(\int_0^t f(s)dB_s - \frac{1}{2}\int_0^t f^2(s)ds \right) \Big| \mathcal{F}_u \right] \\
&= \exp\left(-\frac{1}{2}\int_0^t f^2(s)ds \right) \mathbb{E}\left[\exp\left(\int_0^t f(s)dB_s \right) \Big| \mathcal{F}_u \right] \\
&= \exp\left(-\frac{1}{2}\int_0^t f^2(s)ds \right) \mathbb{E}\left[\exp\left(\int_0^u f(s)dB_s + \int_u^t f(s)dB_s \right) \Big| \mathcal{F}_u \right] \\
&= \exp\left(\int_0^u f(s)dB_s - \frac{1}{2}\int_0^t f^2(s)ds \right) \mathbb{E}\left[\exp\left(\int_u^t f(s)dB_s \right) \Big| \mathcal{F}_u \right] \\
&= \exp\left(\int_0^u f(s)dB_s - \frac{1}{2}\int_0^t f^2(s)ds \right) \mathbb{E}\left[\exp\left(\int_u^t f(s)dB_s \right) \right] \\
&= \exp\left(\int_0^u f(s)dB_s - \frac{1}{2}\int_0^t f^2(s)ds + \frac{1}{2}\int_u^t f^2(s)ds \right) \\
&= \exp\left(\int_0^u f(s)dB_s - \frac{1}{2}\int_0^u f^2(s)ds \right), \qquad 0 \leq u \leq t.
\end{aligned}$$

This result can also be obtained by directly applying (S.4.13).

c) We apply the conclusion of Question (b) to the constant function $f(t) := \sigma$, $t \geq 0$.

Exercise 4.12 We have

$$\begin{aligned}
dS_t &= d\big(S_t^{(1)} - S_t^{(2)}\big) \\
&= dS_t^{(1)} - dS_t^{(2)}
\end{aligned}$$

$$= \mu S_t^{(1)} dt + \sigma_1 dW_t^{(1)} - \big(\mu S_t^{(2)} dt + \sigma_2 dW_t^{(2)}\big)$$
$$= \mu\big(S_t^{(1)} - \mu S_t^{(2)}\big) dt + \sigma_1 dW_t^{(1)} - \sigma_2 dW_t^{(2)}.$$

The process $M_t := \sigma_1 W_t^{(1)} - \sigma_2 W_t^{(2)}$ is a continuous martingale, with

$$\begin{aligned} dM_t \bullet dM_t &= \big(\sigma_1 dW_t^{(1)} - \sigma_2 dW_t^{(2)}\big) \bullet \big(\sigma_1 dW_t^{(1)} - \sigma_2 dW_t^{(2)}\big) \\ &= \sigma_1^2 dW_t^{(1)} \bullet dW_t^{(1)} - 2\sigma_1\sigma_2 dW_t^{(1)} \bullet dW_t^{(2)} + \sigma_2^2 dW_t^{(2)} \bullet dW_t^{(2)} \\ &= (\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2) dt. \end{aligned}$$

Therefore, letting

$$\sigma := \sqrt{\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2}$$

and

$$W_t := \frac{M_t}{\sigma} = \frac{\sigma_1}{\sigma} W_t^{(1)} - \frac{\sigma_2}{\sigma} W_t^{(2)},$$

by the Lévy characterization theorem, see *e.g.* Theorem IV.3.6 in Revuz and Yor (1994), the process $(W_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion with quadratic variation $dW_t \bullet dW_t = dt$, with

$$\begin{aligned} dS_t &= \mu\big(S_t^{(1)} - S_t^{(2)}\big) dt + \sigma_1 dW_t^{(1)} - \sigma_2 dW_t^{(2)} \\ &= \mu S_t dt + \sigma dW_t. \end{aligned}$$

Remark: Since $\rho \in [-1, 1]$, we have

$$\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2 \geq \sigma_1^2 - 2\sigma_1\sigma_2 + \sigma_2^2 = (\sigma_1 - \sigma_2)^2 \geq 0.$$

Exercise 4.13

a) Using (4.31), we have

$$\begin{aligned} \mathbb{E}\left[\exp\left(\beta \int_0^T B_t dB_t\right)\right] &= \mathbb{E}\left[\mathrm{e}^{\beta(B_T^2 - T)/2}\right] \\ &= \mathrm{e}^{-\beta T/2}\,\mathbb{E}\left[\mathrm{e}^{\beta(B_T)^2/2}\right] \\ &= \frac{\mathrm{e}^{-\beta T/2}}{\sqrt{2\pi T}} \int_{-\infty}^{\infty} \mathrm{e}^{\beta x^2/2} \mathrm{e}^{-x^2/(2T)} dx \\ &= \frac{\mathrm{e}^{-\beta T/2}}{\sqrt{2\pi T}} \int_{-\infty}^{\infty} \mathrm{e}^{(\beta - 1/T)x^2/2} dx \\ &= \frac{\mathrm{e}^{-\beta T/2}}{\sqrt{1 - \beta T}} \int_{-\infty}^{\infty} \frac{\mathrm{e}^{-x^2/(2/(1/T - \beta))}}{\sqrt{2\pi/(1/T - \beta)}} dx \\ &= \frac{\mathrm{e}^{-\beta T/2}}{\sqrt{1 - \beta T}}, \end{aligned}$$

for all $\beta < 1/T$, knowing that $B_T \simeq \mathcal{N}(0,T)$.

b) The function $\beta \mapsto 1/\sqrt{1-\beta T}$ can be identified as the moment generating function of the gamma distribution with shape parameter $\lambda = 1/2$, scaling parameter $1/T$ and probability density function

$$y \mapsto \frac{1}{\Gamma(1/2)}(yT)^{-1/2}\mathrm{e}^{-y/T},$$

due to the relation

$$\frac{T^{-\lambda}}{\Gamma(\lambda)}\int_0^\infty \mathrm{e}^{\beta y}y^{\lambda-1}\mathrm{e}^{-y/T}dy = \frac{1}{(1-\beta T)^\lambda}, \qquad \beta < 1/T,$$

therefore $T + \int_0^T B_t dB_t = B_T^2$ has a gamma distribution with shape parameter $1/2$ and scaling parameter T. In other words, the square $1 + \int_0^T B_t dB_t/T = B_T^2/T$ of the normal random variable $B_T/\sqrt{T} \simeq \mathcal{N}(0,1)$ has a χ^2 distribution with one degree of freedom.

Exercise 4.14

a) Letting $Y_t = \mathrm{e}^{bt}X_t$, we have

$$\begin{aligned} dY_t &= d(\mathrm{e}^{bt}X_t) \\ &= b\mathrm{e}^{bt}X_t dt + \mathrm{e}^{bt}dX_t \\ &= b\mathrm{e}^{bt}X_t dt + \mathrm{e}^{bt}(-bX_t dt + \sigma\mathrm{e}^{-bt}dB_t) \\ &= \sigma dB_t, \end{aligned}$$

hence

$$Y_t = Y_0 + \int_0^t dY_s = Y_0 + \sigma\int_0^t dB_s = Y_0 + \sigma B_t,$$

and

$$X_t = \mathrm{e}^{-bt}Y_t = \mathrm{e}^{-bt}Y_0 + \sigma\mathrm{e}^{-bt}B_t = \mathrm{e}^{-bt}X_0 + \sigma\mathrm{e}^{-bt}B_t.$$

Alternatively, we can also search for a solution X_t of the form $X_t = f(t, B_t)$, with

$$dX_t = df(t,B_t) = \frac{\partial f}{\partial t}(t,B_t)dt + \frac{\partial f}{\partial x}(t,B_t)dB_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t,B_t)dt$$

from the Itô formula. Matching this expression to the stochastic differential equation (4.40) would yield

$$\frac{\partial f}{\partial t}(t,B_t) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t,B_t)dt = -bX_t = -bf(t,B_t)$$

and

$$\frac{\partial f}{\partial x}(t, B_t) = \sigma e^{-bt},$$

hence

$$\frac{\partial f}{\partial t}(t, x) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t, x)dt = -bf(t, x) \quad \text{and} \quad \frac{\partial f}{\partial x}(t, x) = \sigma e^{-bt},$$

$x \in \mathbb{R}$, which can be solved as $f(t, x) = f(t, 0) + \sigma x e^{-bt}$ and

$$\frac{\partial f}{\partial t}(0, x) = -bf(t, 0),$$

which gives $f(t, 0) = f(0, 0)e^{-bt}$, and recovers

$$X_t = f(t, B_t) = X_0 e^{-bt} + \sigma e^{-bt} B_t, \qquad t \geq 0.$$

b) Letting $Y_t = e^{bt} X_t$, we have

$$\begin{aligned} dY_t &= d(e^{bt} X_t) \\ &= be^{bt} X_t dt + e^{bt} dX_t \\ &= be^{bt} X_t dt + e^{bt}\big(- bX_t dt + \sigma e^{-at} dB_t\big) \\ &= \sigma e^{(b-a)t} dB_t, \end{aligned}$$

hence we can solve for Y_t by integrating on both sides as

$$Y_t = Y_0 + \int_0^t dY_s = Y_0 + \sigma \int_0^t e^{(b-a)s} dB_s, \qquad t \geq 0.$$

This yields the solution

$$X_t = e^{-bt} Y_t = e^{-bt} X_0 + \sigma e^{-bt} \int_0^t e^{(b-a)s} dB_s, \qquad t \geq 0.$$

Comments:

(i) This type of computation appears anywhere *discounting* by the factor e^{-bt} is involved.

(ii) In part (b) the solution cannot take the form $X_t = f(t, B_t)$ when $a \neq b$. Indeed, solving

$$\frac{\partial f}{\partial x}(t, x) = \sigma e^{-at}$$

gives $f(t, x) = f(t, 0) + \sigma x e^{-at}$, yielding

$$\frac{\partial f}{\partial t}(t, x) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t, x) = \frac{\partial f}{\partial t}(t, 0) - a\sigma x e^{-at},$$

which cannot match $-bf(t, x)$ unless $a = b$.

(iii) The stochastic integral $\int_0^t \mathrm{e}^{(b-a)s} dB_s$ cannot be computed in closed form. It is a centered Gaussian random variable with variance

$$\int_0^t \mathrm{e}^{2(b-a)s} ds = \frac{\mathrm{e}^{2(b-a)t} - 1}{2(b-a)}$$

if $b \neq a$, and variance t if $a = b$. Using integration by parts on $[0, t]$, we may also write

$$\int_0^t \mathrm{e}^{-(a+r)s} dB_s = \mathrm{e}^{-(a+r)t} B_t + (a+r) \int_0^t \mathrm{e}^{-(a+r)s} B_s ds, \quad t \geq 0.$$

Exercise 4.15

a) Note that the stochastic integral

$$\int_0^T \frac{1}{T-s} dB_s$$

is not defined in $L^2(\Omega)$ as the function $s \mapsto 1/(T-s)$ is not in $L^2([0,T])$, and by the Itô isometry (4.8) we have

$$\mathbb{E}\left[\left(\int_0^T \frac{1}{T-s} dB_s\right)^2\right] = \int_0^T \frac{1}{(T-s)^2} ds = \left[\frac{1}{T-s}\right]_0^\infty = +\infty.$$

On the other hand, by (4.27) and (4.42) we have

$$\begin{aligned} d\left(\frac{X_t}{T-t}\right) &= \frac{dX_t}{T-t} + X_t d\left(\frac{1}{T-t}\right) + dX_t \bullet d\left(\frac{1}{T-t}\right) \\ &= \frac{dX_t}{T-t} + \frac{X_t}{(T-t)^2} dt \\ &= \sigma \frac{dB_t}{T-t}, \end{aligned}$$

hence, by integration over the time interval $[0, t]$ and using the initial condition $X_0 = 0$, we find

$$\frac{X_t}{T-t} = \frac{X_0}{T} + \int_0^t d\left(\frac{X_s}{T-s}\right) = \sigma \int_0^t \frac{dB_s}{T-s}, \qquad 0 \leq t < T.$$

b) By (4.17), we have

$$\mathbb{E}[X_t] = (T-t)\sigma\, \mathbb{E}\left[\int_0^t \frac{1}{T-s} dB_s\right] = 0, \qquad 0 \leq t < T.$$

c) By the Itô isometry (4.8), we have

$$\begin{aligned}
\mathrm{Var}[X_t] &= (T-t)^2\sigma^2 \,\mathrm{Var}\left[\int_0^t \frac{1}{T-s} dB_s\right] \\
&= (T-t)^2\sigma^2 \,\mathbb{E}\left[\left(\int_0^t \frac{1}{T-s} dB_s\right)^2\right] \\
&= (T-t)^2\sigma^2 \int_0^t \frac{1}{(T-s)^2} ds \\
&= (T-t)^2\sigma^2 \left(\frac{1}{T-t} - \frac{1}{T}\right) \\
&= \sigma^2 t \frac{T-t}{T}, \qquad 0 \le t < T.
\end{aligned}$$

d) We have

$$\lim_{t\to T} \|X_t\|_{L^2(\Omega)} = \lim_{t\to T} \mathrm{Var}[X_t] = \sigma^2 \lim_{t\to T}\left(t - \frac{t^2}{T}\right) = 0.$$

Exercise 4.16 Exponential Vašíček (1977) model (1). Applying the Itô formula (4.29) to $X_t = \mathrm{e}^{r_t} = f(r_t)$ with $f(x) = \mathrm{e}^x$, we have

$$\begin{aligned}
dX_t &= d\mathrm{e}^{r_t} \\
&= f'(r_t)dr_t + \frac{1}{2} f''(r_t) dr_t \bullet dr_t \\
&= \mathrm{e}^{r_t} dr_t + \frac{1}{2}\mathrm{e}^{r_t} dr_t \bullet dr_t \\
&= \mathrm{e}^{r_t}((a - br_t)dt + \sigma dB_t) + \frac{1}{2}\mathrm{e}^{r_t}((a - br_t)dt + \sigma dB_t)^2 \\
&= \mathrm{e}^{r_t}((a - br_t)dt + \sigma dB_t) + \frac{\sigma^2}{2}\mathrm{e}^{r_t} dt \\
&= X_t\left(a + \frac{\sigma^2}{2} - b\log(X_t)\right) dt + \sigma X_t dB_t \\
&= X_t(\widetilde{a} - \widetilde{b} f(X_t))dt + \sigma g(X_t) dB_t,
\end{aligned}$$

hence

$$\widetilde{a} = a + \frac{\sigma^2}{2} \qquad \text{and} \qquad \widetilde{b} = b$$

the functions $f(x)$ and $g(x)$ are given by $f(x) = \log x$ and $g(x) = x$. Note that this stochastic differential equation is that of the exponential Vasicek model.

Exercise 4.17 Exponential Vasicek model (2).

a) We have $Z_t = \mathrm{e}^{-at} Z_0 + \sigma \int_0^t \mathrm{e}^{-(t-s)a} dB_s$.

b) We have $Y_t = \mathrm{e}^{-at}Y_0 + \frac{\theta}{a}\big(1-\mathrm{e}^{-at}\big) + \sigma\int_0^t \mathrm{e}^{-(t-s)a}dB_s$.

c) We have $dX_t = X_t\left(\theta + \frac{\sigma^2}{2} - a\log X_t\right)dt + \sigma X_t dB_t$.

d) We have $r_t = \exp\left(\mathrm{e}^{-at}\log r_0 + \frac{\theta}{a}\big(1-\mathrm{e}^{-at}\big) + \sigma\int_0^t \mathrm{e}^{-(t-s)a}dB_s\right)$.

e) Using the Gaussian moment generating function identity $\mathbb{E}[\mathrm{e}^X] = \mathrm{e}^{\alpha^2/2}$ for $X \simeq \mathcal{N}(0,\alpha^2)$ and the variance formula (4.10), we have

$$\begin{aligned}
\mathbb{E}[r_t \mid \mathcal{F}_u] &= \mathbb{E}\left[\exp\left(\mathrm{e}^{-at}\log r_0 + \frac{\theta}{a}\big(1-\mathrm{e}^{-at}\big) + \sigma\int_0^t \mathrm{e}^{-(t-s)a}dB_s\right)\ \Big|\ \mathcal{F}_u\right]\\
&= \mathrm{e}^{\mathrm{e}^{-at}\log r_0 + \frac{\theta}{a}\left(1-\mathrm{e}^{-at}\right) + \sigma\int_0^u \mathrm{e}^{-(t-s)a}dB_s}\,\mathbb{E}\left[\exp\left(\sigma\int_u^t \mathrm{e}^{-(t-s)a}dB_s\right)\ \Big|\ \mathcal{F}_u\right]\\
&= \mathrm{e}^{\mathrm{e}^{-at}\log r_0 + \frac{\theta}{a}\left(1-\mathrm{e}^{-at}\right) + \sigma\int_0^u \mathrm{e}^{-(t-s)a}dB_s}\,\mathbb{E}\left[\exp\left(\sigma\int_u^t \mathrm{e}^{-(t-s)a}dB_s\right)\right]\\
&= \exp\left(\mathrm{e}^{-at}\log r_0 + \frac{\theta}{a}\big(1-\mathrm{e}^{-at}\big) + \sigma\int_0^u \mathrm{e}^{-(t-s)a}dB_s + \frac{\sigma^2}{2}\int_u^t \mathrm{e}^{-2(t-s)a}ds\right)\\
&= \exp\left(\mathrm{e}^{-at}\log r_0 + \frac{\theta}{a}\big(1-\mathrm{e}^{-at}\big) + \sigma\int_0^u \mathrm{e}^{-(t-s)a}dB_s + \frac{\sigma^2}{4a}\big(1-\mathrm{e}^{-2(t-u)a}\big)\right)\\
&= \exp\left(\mathrm{e}^{-(t-u)a}\left(\mathrm{e}^{-au}\log r_0 + \frac{\theta}{a}\big(1-\mathrm{e}^{-au}\big) + \sigma\int_0^u \mathrm{e}^{-(u-s)a}dB_s\right)\right.\\
&\qquad\left. + \frac{\theta}{a}\big(1-\mathrm{e}^{-(t-u)a}\big) + \frac{\sigma^2}{4a}\big(1-\mathrm{e}^{-2(t-u)a}\big)\right)\\
&= \exp\left(\mathrm{e}^{-(t-u)a}\log r_u + \frac{\theta}{a}\big(1-\mathrm{e}^{-(t-u)a}\big) + \frac{\sigma^2}{4a}\big(1-\mathrm{e}^{-2(t-u)a}\big)\right)\\
&= r_u^{\mathrm{e}^{-(t-u)a}}\exp\left(\frac{\theta}{a}\big(1-\mathrm{e}^{-(t-u)a}\big) + \frac{\sigma^2}{4a}\big(1-\mathrm{e}^{-2(t-u)a}\big)\right).
\end{aligned}$$

In particular, for $u = 0$ we find

$$\mathbb{E}[r_t] = r_0^{\mathrm{e}^{-at}}\exp\left(+\frac{\theta}{a}\big(1-\mathrm{e}^{-at}\big) + \frac{\sigma^2}{4a}\big(1-\mathrm{e}^{-2at}\big)\right).$$

f) Similarly, we have

$$\begin{aligned}
\mathbb{E}[r_t^2 \mid \mathcal{F}_u] &= \mathbb{E}\left[\exp\left(2\mathrm{e}^{-at}\log r_0 + \frac{2\theta}{a}\big(1-\mathrm{e}^{-at}\big) + 2\sigma\int_0^t \mathrm{e}^{-(t-s)a}dB_s\right)\ \Big|\ \mathcal{F}_u\right]\\
&= \mathrm{e}^{2\mathrm{e}^{-at}\log r_0 + \frac{2\theta}{a}\left(1-\mathrm{e}^{-at}\right) + 2\sigma\int_0^u \mathrm{e}^{-(t-s)a}dB_s}\,\mathbb{E}\left[\exp\left(2\sigma\int_u^t \mathrm{e}^{-(t-s)a}dB_s\right)\ \Big|\ \mathcal{F}_u\right]\\
&= \mathrm{e}^{2\mathrm{e}^{-at}\log r_0 + \frac{2\theta}{a}\left(1-\mathrm{e}^{-at}\right) + 2\sigma\int_0^u \mathrm{e}^{-(t-s)a}dB_s}\,\mathbb{E}\left[\exp\left(2\sigma\int_u^t \mathrm{e}^{-(t-s)a}dB_s\right)\right]
\end{aligned}$$

$$
\begin{aligned}
&= \exp\left(2\mathrm{e}^{-at}\log r_0 + \frac{2\theta}{a}(1-\mathrm{e}^{-at}) + 2\sigma\int_0^u \mathrm{e}^{-(t-s)a}dB_s + 2\sigma^2\int_u^t \mathrm{e}^{-2(t-s)a}ds\right)\\
&= \exp\left(2\mathrm{e}^{-at}\log r_0 + \frac{2\theta}{a}(1-\mathrm{e}^{-at}) + 2\sigma\int_0^u \mathrm{e}^{-(t-s)a}dB_s + \frac{\sigma^2}{a}(1-\mathrm{e}^{-2(t-u)a})\right)\\
&= \exp\left(2\mathrm{e}^{-(t-u)a}\left(2\mathrm{e}^{-au}\log r_0 + \frac{2\theta}{a}(1-\mathrm{e}^{-au}) + 2\sigma\int_0^u \mathrm{e}^{-(u-s)a}dB_s\right)\right.\\
&\qquad \left. + \frac{2\theta}{a}(1-\mathrm{e}^{-(t-u)a}) + \frac{\sigma^2}{a}(1-\mathrm{e}^{-2(t-u)a})\right)\\
&= \exp\left(2\mathrm{e}^{-(t-u)a}\log r_u + \frac{2\theta}{a}(1-\mathrm{e}^{-(t-u)a}) + \frac{\sigma^2}{a}(1-\mathrm{e}^{-2(t-u)a})\right)\\
&= r_u^{2\mathrm{e}^{-(t-u)a}}\exp\left(\frac{2\theta}{a}(1-\mathrm{e}^{-(t-u)a}) + \frac{\sigma^2}{a}(1-\mathrm{e}^{-2(t-u)a})\right),
\end{aligned}
$$

hence

$$
\begin{aligned}
&\mathrm{Var}[r_t \mid \mathcal{F}_u] = \mathbb{E}[r_t^2 \mid \mathcal{F}_u] - (\mathbb{E}[r_t \mid \mathcal{F}_u])^2\\
&= r_u^{2\mathrm{e}^{-(t-u)a}}\exp\left(\frac{2\theta}{a}(1-\mathrm{e}^{-(t-u)a}) + \frac{\sigma^2}{a}(1-\mathrm{e}^{-2(t-u)a})\right)\\
&\quad - r_u^{2\mathrm{e}^{-(t-u)a}}\exp\left(\frac{2\theta}{a}(1-\mathrm{e}^{-(t-u)a}) + \frac{\sigma^2}{2a}(1-\mathrm{e}^{-2(t-u)a})\right)\\
&= r_u^{2\mathrm{e}^{-(t-u)a}}\exp\left(\frac{2\theta}{a}(1-\mathrm{e}^{-(t-u)a}) + \frac{\sigma^2}{a}(1-\mathrm{e}^{-2(t-u)a})\right)\\
&\qquad \times\left(1-\exp\left(-\frac{\sigma^2}{2a}(1-\mathrm{e}^{-2(t-u)a})\right)\right).
\end{aligned}
$$

g) We find $\lim_{t\to\infty}\mathbb{E}[r_t] = r_0\exp\left(\frac{\theta}{a}+\frac{\sigma^2}{4a}\right)$ and

$$
\begin{aligned}
\lim_{t\to\infty}\mathrm{Var}[r_t] &= \exp\left(\frac{2\theta}{a}+\frac{\sigma^2}{a}\right)\left(1-\exp\left(-\frac{\sigma^2}{2a}\right)\right)\\
&= \exp\left(\frac{2\theta}{a}\right)\left(\exp\left(\frac{\sigma^2}{a}\right)-1\right).
\end{aligned}
$$

Exercise 4.18 Cox-Ingersoll-Ross (CIR) model.

a) We have

$$
r_t = r_0 + \int_0^t(\alpha-\beta r_s)ds + \sigma\int_0^t\sqrt{r_s}dB_s, \qquad t\geq 0. \tag{S.4.14}
$$

b) Taking expectations on both sides of (S.4.14) and using the fact that the expectation of the stochastic integral with respect to Brownian motion is zero, we find

$$\begin{aligned}
u(t) &= \mathbb{E}[r_t] \\
&= \mathbb{E}\left[r_0 + \int_0^t (\alpha - \beta r_s) ds + \sigma \int_0^t \sqrt{r_s} dB_s\right] \\
&= \mathbb{E}\left[r_0 + \int_0^t (\alpha - \beta r_s) ds\right] \\
&= r_0 + \mathbb{E}\left[\int_0^t (\alpha - \beta r_s) ds\right] \\
&= r_0 + \int_0^t (\alpha - \beta\, \mathbb{E}[r_s]) ds \\
&= r_0 + \int_0^t (\alpha - \beta u(s)) ds,
\end{aligned}$$

which yields the differential equation $u'(t) = \alpha - \beta u(t)$. Letting $w(t) := \mathrm{e}^{\beta t} u(t)$, we have

$$w'(t) = \beta \mathrm{e}^{\beta t} u(t) + \mathrm{e}^{\beta t} u'(t) = \alpha \mathrm{e}^{\beta t},$$

hence

$$\begin{aligned}
\mathbb{E}[r_t] &= u(t) \\
&= \mathrm{e}^{-\beta t} w(t) \\
&= \mathrm{e}^{-\beta t}\left(w(0) + \alpha \int_0^t \mathrm{e}^{\beta s} ds\right) \\
&= \mathrm{e}^{-\beta t}\left(u(0) + \frac{\alpha}{\beta}(\mathrm{e}^{\beta t} - 1)\right) \\
&= \mathrm{e}^{-\beta t} r_0 + \frac{\alpha}{\beta}(1 - \mathrm{e}^{-\beta t}), \qquad t \geq 0. \qquad \text{(S.4.15)}
\end{aligned}$$

c) By applying Itô's formula (4.29) to

$$r_t^2 = f\left(r_0 + \int_0^t (\alpha - \beta r_s) ds + \sigma \int_0^t \sqrt{r_s} dB_s\right),$$

with $f(x) = x^2$, we find

$$\begin{aligned}
d(r_t)^2 &= f'(r_t) dr_t + \frac{1}{2} f''(r_t) dr_t \bullet dr_t \\
&= 2 r_t dr_t + dr_t \bullet dr_t \\
&= r_t(\sigma^2 + 2\alpha - 2\beta r_t) dt + 2\sigma r_t^{3/2} dB_t
\end{aligned}$$

or, in integral form,

$$r_t^2 = r_0^2 + \int_0^t r_s(\sigma^2 + 2\alpha - 2\beta r_s) ds + 2\sigma \int_0^t r_s^{3/2} dB_s, \quad t \geq 0. \qquad \text{(S.4.16)}$$

d) Taking again the expectation on both sides of (S.4.16), we find

$$\begin{aligned}
v(t) &= \mathbb{E}[r_t^2] \\
&= \mathbb{E}\left[r_0^2 + \int_0^t r_s(\sigma^2 + 2\alpha - 2\beta r_s)ds + 2\sigma \int_0^t r_s^{3/2} dB_s\right] \\
&= r_0^2 + \mathbb{E}\left[\int_0^t r_s(\sigma^2 + 2\alpha - 2\beta r_s)ds\right] \\
&= r_0^2 + \int_0^t (\sigma^2\, \mathbb{E}[r_s] + 2\alpha\, \mathbb{E}[r_s] - 2\beta\, \mathbb{E}[r_s^2])ds \\
&= v(0) + \int_0^t (\sigma^2 u(s) + 2\alpha u(s) - 2\beta v(s))ds,
\end{aligned}$$

and after differentiation with respect to t this yields the differential equation

$$v'(t) = (\sigma^2 + 2\alpha)u(t) - 2\beta v(t), \qquad t \geq 0.$$

By (S.4.15) we find

$$v'(t) = (\sigma^2 + 2\alpha)\left(\frac{\alpha}{\beta} + \left(r_0 - \frac{\alpha}{\beta}\right)\mathrm{e}^{-\beta t}\right) - 2\beta v(t), \qquad t \geq 0.$$

Looking for a solution of the form

$$v(t) = c_0 + c_1\mathrm{e}^{-\beta t} + c_2\mathrm{e}^{-2\beta t}, \qquad t \geq 0,$$

we find

$$\begin{aligned}
&v'(t) = -\beta c_1\mathrm{e}^{-\beta t} - 2\beta c_2\mathrm{e}^{-2\beta t} \\
&= (\sigma^2 + 2\alpha)\left(\frac{\alpha}{\beta} + \left(r_0 - \frac{\alpha}{\beta}\right)\mathrm{e}^{-\beta t}\right) - 2\beta(c_0 + c_1\mathrm{e}^{-\beta t} + c_2\mathrm{e}^{-2\beta t}) \\
&= \frac{\alpha}{\beta}(\sigma^2 + 2\alpha) + (\sigma^2 + 2\alpha)\left(r_0 - \frac{\alpha}{\beta}\right)\mathrm{e}^{-\beta t} - 2\beta c_0 - 2\beta c_1\mathrm{e}^{-\beta t} - 2\beta c_2\mathrm{e}^{-\beta t},
\end{aligned}$$

$t \geq 0$, hence

$$\begin{cases}
0 = \dfrac{\alpha}{\beta}(\sigma^2 + 2\alpha) - 2\beta c_0, \\[2ex]
-\beta c_1 = (\sigma^2 + 2\alpha)\left(r_0 - \dfrac{\alpha}{\beta}\right) - 2\beta c_1,
\end{cases}$$

and

$$c_0 = \frac{\alpha}{2\beta^2}(\sigma^2 + 2\alpha), \qquad c_1 = \frac{\sigma^2 + 2\alpha}{\beta}\left(r_0 - \frac{\alpha}{\beta}\right),$$

with

$$r_0^2 = v(0) = c_0 + c_1 + c_2,$$

which yields

$$
\begin{aligned}
c_2 &= r_0^2 - c_0 - c_1 \\
&= r_0^2 - \frac{\alpha}{2\beta^2}(\sigma^2 + 2\alpha) - \frac{\sigma^2 + 2\alpha}{\beta}\left(r_0 - \frac{\alpha}{\beta}\right) \\
&= r_0^2 - (\sigma^2 + 2\alpha)\left(\frac{r_0}{\beta} - \frac{\alpha}{2\beta^2}\right),
\end{aligned}
$$

and

$$
\begin{aligned}
\mathbb{E}[r_t^2] &= v(t) \\
&= c_0 + c_1 \mathrm{e}^{-\beta t} + c_2 \mathrm{e}^{-2\beta t} \\
&= \frac{\alpha}{2\beta^2}(\sigma^2 + 2\alpha) + \frac{\sigma^2 + 2\alpha}{\beta}\left(r_0 - \frac{\alpha}{\beta}\right)\mathrm{e}^{-\beta t} \\
&\quad + \left(r_0^2 - (\sigma^2 + 2\alpha)\left(\frac{r_0}{\beta} - \frac{\alpha}{2\beta^2}\right)\right)\mathrm{e}^{-2\beta t}, \qquad t \geq 0.
\end{aligned}
$$

e) We have

$$
\begin{aligned}
\mathrm{Var}[r_t] &= \mathbb{E}[r_t^2] - (\mathbb{E}[r_t])^2 \\
&= \frac{\alpha}{2\beta^2}(\sigma^2 + 2\alpha) + \frac{\sigma^2 + 2\alpha}{\beta}\left(r_0 - \frac{\alpha}{\beta}\right)\mathrm{e}^{-\beta t} \\
&\quad + \left(r_0^2 - (\sigma^2 + 2\alpha)\left(\frac{r_0}{\beta} - \frac{\alpha}{2\beta^2}\right)\right)\mathrm{e}^{-2\beta t} \\
&\quad - \left(\frac{\alpha}{\beta} + \left(r_0 - \frac{\alpha}{\beta}\right)\mathrm{e}^{-\beta t}\right)^2 \\
&= \frac{\alpha}{2\beta^2}(\sigma^2 + 2\alpha) + \frac{\sigma^2 + 2\alpha}{\beta}\left(r_0 - \frac{\alpha}{\beta}\right)\mathrm{e}^{-\beta t} \\
&\quad + \left(r_0^2 - (\sigma^2 + 2\alpha)\left(\frac{r_0}{\beta} - \frac{\alpha}{2\beta^2}\right)\right)\mathrm{e}^{-2\beta t} \\
&\quad - \left(\frac{\alpha}{\beta}\right)^2 - \frac{2\alpha}{\beta}\left(r_0 - \frac{\alpha}{\beta}\right)\mathrm{e}^{-\beta t} - \left(r_0^2 - 2r_0\frac{\alpha}{\beta} + \left(\frac{\alpha}{\beta}\right)^2\right)\mathrm{e}^{-2\beta t} \\
&= r_0\frac{\sigma^2}{\beta}\left(\mathrm{e}^{-\beta t} - \mathrm{e}^{-2\beta t}\right) + \frac{\alpha\sigma^2}{2\beta^2} - \frac{\alpha\sigma^2}{\beta^2}\mathrm{e}^{-\beta t} + \frac{\alpha\sigma^2}{2\beta^2}\mathrm{e}^{-2\beta t} \\
&= r_0\frac{\sigma^2}{\beta}\left(\mathrm{e}^{-\beta t} - \mathrm{e}^{-2\beta t}\right) + \frac{\alpha\sigma^2}{2\beta^2}\left(1 - \mathrm{e}^{-\beta t}\right)^2, \quad t \geq 0.
\end{aligned}
$$

Problem 4.19

a) The Itô formula *cannot* be applied to the function $f(x) := (x - K)^+$ because it is not (twice) differentiable.
b) The function $x \mapsto f_\varepsilon(x)$ can be plotted as follows with $K = 1$.

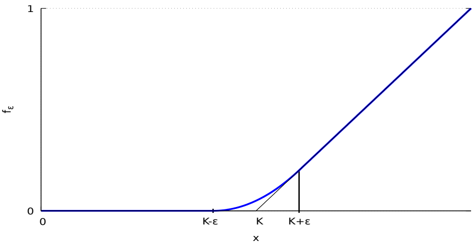


Fig. S.13: Graph of the function $x \mapsto f_\varepsilon(x)$.

We note that f_ε converges *uniformly* on $\mathbb{R}$ to the function $x \mapsto (x-K)^+$ as we have

$$0 \leq f_\varepsilon(x) - (x-K)^+ \leq \frac{\varepsilon}{4}, \qquad x \in \mathbb{R}. \tag{S.4.17}$$

c) Applying the Itô formula to the function f_ε we find

$$\begin{aligned} f_\varepsilon(B_T) &= f_\varepsilon(B_0) + \int_0^T f'_\varepsilon(B_t)dB_t + \frac{1}{2}\int_0^T f''_\varepsilon(B_t)dt \\ &= f_\varepsilon(B_0) + \int_0^T f'_\varepsilon(B_t)dB_t + \frac{1}{4\varepsilon}\int_0^T \mathbb{1}_{(K-\varepsilon,K+\varepsilon)}(B_t)dt, \end{aligned}$$

and to conclude it suffices to note that

$$\ell\left(\left\{t \in [0,T] \ : \ K-\varepsilon < B_t < K+\varepsilon\right\}\right) = \int_0^T \mathbb{1}_{(K-\varepsilon,K+\varepsilon)}(B_t)dt.$$

d) The derivative $f'_\varepsilon(x)$ of $f_\varepsilon(x)$ is given by

$$f'_\varepsilon(x) := \begin{cases} 1 & \text{if } x > K+\varepsilon, \\ \\ \dfrac{1}{2\varepsilon}(x-K+\varepsilon) & \text{if } K-\varepsilon < x < K+\varepsilon, \\ \\ 0 & \text{if } x < K-\varepsilon. \end{cases}$$

Hence we have

$$\begin{aligned} \|\mathbb{1}_{[K,\infty)}(\cdot) - f'_\varepsilon(\cdot)\|^2_{L^2(\mathbb{R}_+)} &= \int_0^\infty \left(\mathbb{1}_{[K,\infty)}(x) - f'_\varepsilon(x)\right)^2 dx \\ &= \int_{K-\varepsilon}^{K+\varepsilon} \left(1 + |f'_\varepsilon(x)|^2\right) dx \\ &\leq 2\varepsilon + \frac{1}{4\varepsilon^2}\int_{K-\varepsilon}^{K+\varepsilon} \left(x-K+\varepsilon\right)^2 dx \\ &= 2\varepsilon + \frac{1}{12\varepsilon^2}\left[\left(x-K+\varepsilon\right)^3\right]_{K-\varepsilon}^{K+\varepsilon} \end{aligned}$$

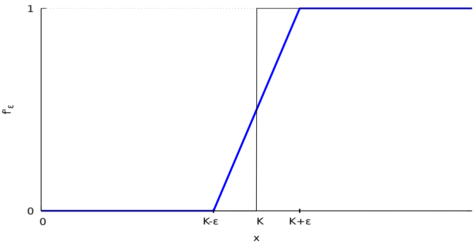


Fig. S.14: Graph of the derivative $x \mapsto f_\varepsilon'(x)$.

$$= 2\varepsilon + \frac{2\varepsilon}{3}.$$

e) i) We have

$$\begin{aligned}\mathbb{E}\left[\int_0^T \big(\mathbb{1}_{[K,\infty)}(B_t) - f_\varepsilon'(B_t)\big)^2 dt\right] &= \int_0^T \mathbb{E}\left[\big(\mathbb{1}_{[K,\infty)}(B_t) - f_\varepsilon'(B_t)\big)^2\right] dt \\ &= \int_0^T \int_{-\infty}^\infty \big(\mathbb{1}_{[K,\infty)}(x) - f_\varepsilon'(x)\big)^2 \mathrm{e}^{-x^2/(2t)} dx \frac{1}{\sqrt{2\pi t}} dt \\ &\leq \int_0^T \int_{K-\varepsilon}^{K+\varepsilon} \big(\mathbb{1}_{[K,\infty)}(x) - f_\varepsilon'(x)\big)^2 \mathrm{e}^{-(K-\varepsilon)^2/(2t)} dx \frac{1}{\sqrt{2\pi t}} dt \\ &\leq \|\mathbb{1}_{[K,\infty)}(\cdot) - f_\varepsilon'(\cdot)\|_{L^2(\mathbb{R}_+)}^2 \int_0^T \mathrm{e}^{-(K-\varepsilon)^2/(2t)} \frac{1}{\sqrt{2\pi t}} dt \\ &\leq \left(2\varepsilon + \frac{2\varepsilon}{3}\right) \int_0^T \mathrm{e}^{-(K-\varepsilon)^2/(2t)} \frac{1}{\sqrt{2\pi t}} dt,\end{aligned}$$

where

$$\int_0^T \mathrm{e}^{-(K-\varepsilon)^2/(2t)} \frac{1}{\sqrt{2\pi t}} dt < \int_0^T \mathrm{e}^{-K^2/(8t)} \frac{1}{\sqrt{2\pi t}} dt < \infty,$$

for $\varepsilon < K/2$, hence $\lim_{\varepsilon \to 0} \mathbb{E}\left[\int_0^T \big(\mathbb{1}_{[K,\infty)}(B_t) - f_\varepsilon'(B_t)\big)^2 dt\right] = 0$, and by the Itô isometry

$$\mathbb{E}\left[\left(\int_0^\infty \big(\mathbb{1}_{[K,\infty)}(B_t) - f_\varepsilon(B_t)\big) dB_t\right)^2\right] = \mathbb{E}\left[\int_0^\infty \big(\mathbb{1}_{[K,\infty)}(B_t) - f_\varepsilon(B_t)\big)^2 dt\right]$$

we find that

$$\lim_{\varepsilon \to 0} \mathbb{E}\left[\left(\int_0^\infty \mathbb{1}_{[K,\infty)}(B_t) dB_t - \int_0^\infty f_\varepsilon(B_t) dB_t\right)^2\right] = 0,$$

which shows that $\int_0^\infty f_\varepsilon(B_t)dB_t$ converges to $\int_0^\infty \mathbb{1}_{[K,\infty)}(B_t)dB_t$ in $L^2(\Omega)$ as ε tends to zero.

ii) By (S.4.17) we have

$$\mathbb{E}\left[\left((B_T-K)^+ - f_\varepsilon(B_T)\right)^2\right] \leq \frac{\varepsilon}{4},$$

hence $f_\varepsilon(B_T)$ converges to $(B_T-K)^+$ in $L^2(\Omega)$.

iii) Similarly, $f_\varepsilon(B_0)$ converges to $(B_0-K)^+$ for any fixed value of B_0.

As a consequence of (ei), (eii) and (eiii) above, the equation (4.47) shows that

$$\frac{1}{2\varepsilon}\ell\left(\left\{t \in [0,T] \ : \ K-\varepsilon < B_t < K+\varepsilon\right\}\right)$$

admits a limit in $L^2(\Omega)$ as ε tends to zero, and this limit is denoted by $\mathcal{L}^K_{[0,T]}$. The formula (4.48) is known as the *Tanaka formula*.

Problem 4.20

a) We have

$$\begin{aligned}
0 &\leq \mathbb{E}[(X-\varepsilon)^+] \\
&= \frac{1}{\sqrt{2\pi\sigma^2}}\int_\varepsilon^\infty (x-\varepsilon)\mathrm{e}^{-x^2/(2\sigma^2)}dx \\
&= \frac{1}{\sqrt{2\pi\sigma^2}}\int_\varepsilon^\infty x\mathrm{e}^{-x^2/(2\sigma^2)}dx - \frac{\varepsilon}{\sqrt{2\pi\sigma^2}}\int_\varepsilon^\infty \mathrm{e}^{-x^2/(2\sigma^2)}dx \\
&= -\frac{\sigma^2}{\sqrt{2\pi\sigma^2}}\left[\mathrm{e}^{-x^2/(2\sigma^2)}\right]_\varepsilon^\infty - \varepsilon\mathbb{P}(X \geq \varepsilon) \\
&= \frac{\sigma^2}{\sqrt{2\pi\sigma^2}}\mathrm{e}^{-\varepsilon x^2/(2\sigma^2)} - \varepsilon\mathbb{P}(X \geq \varepsilon),
\end{aligned}$$

which leads to the conclusion.

b) We have

$$\begin{aligned}
&\mathbb{P}(X \in dx \mid X+Y=z) = \frac{\mathbb{P}(X \in dx \text{ and } X+Y \in dz)}{\mathbb{P}(X+Y \in dz)} \\
&= \frac{\mathbb{P}(X \in dx \text{ and } Y \in (dz)-x)}{\mathbb{P}(X+Y \in dz)} \\
&= \frac{\sqrt{2\pi(\alpha^2+\beta^2)}}{2\pi\alpha\beta}\frac{\mathrm{e}^{-x^2/(2\alpha^2)-(z-x)^2/(2\beta^2)}}{\mathrm{e}^{-z^2/(2(\alpha^2+\beta^2))}}dx \\
&= \frac{\sqrt{1/\beta^2+1/\alpha^2}}{\sqrt{2\pi}}\mathrm{e}^{-(x^2(1+\beta^2/\alpha^2)+(x^2+z^2-2xz)(1+\alpha^2/\beta^2)-z^2)/(2(\alpha^2+\beta^2))}dx \\
&= \frac{\sqrt{1/\beta^2+1/\alpha^2}}{\sqrt{2\pi}}\mathrm{e}^{-(x^2(2+\beta^2/\alpha^2+\alpha^2/\beta^2)+z^2\alpha^2/\beta^2-2xz(1+\alpha^2/\beta^2))/(2(\alpha^2+\beta^2))}dx
\end{aligned}$$

$$= \frac{\sqrt{1/\beta^2 + 1/\alpha^2}}{\sqrt{2\pi}} \mathrm{e}^{-(x(\beta/\alpha+\alpha/\beta)-z\alpha/\beta)^2/(2(\alpha^2+\beta^2))} dx$$
$$= \frac{\sqrt{1/\beta^2 + 1/\alpha^2}}{\sqrt{2\pi}} \mathrm{e}^{-(x((\alpha^2+\beta^2)/(\alpha\beta))-z\alpha/\beta)^2/(2(\alpha^2+\beta^2))} dx$$
$$= \frac{\sqrt{1/\beta^2 + 1/\alpha^2}}{\sqrt{2\pi}} \mathrm{e}^{-(x-z\alpha^2/(\alpha^2+\beta^2))^2/(2/(1/\alpha^2+1/\beta^2))} dx.$$

c) Given that $B_u = x$ we decompose

$$B_v = (B_v - B_{(u+v)/2}) + (B_{(u+v)/2} - B_u) + x,$$

and apply the result of Question (b) by taking

$$X = B_{(u+v)/2} - B_u \qquad \text{and} \qquad Y = B_v - B_{(u+v)/2},$$

i.e.

$$\alpha^2 = \beta^2 = \frac{v-u}{2} \quad \text{and} \quad z = y - x,$$

which shows that the distribution of $B_{(u+v)/2} = x + X$ given that $B_u = x$ and $B_v = y$ is Gaussian $\mathcal{N}\left(\frac{x+y}{2}, \frac{v-u}{4}\right)$ with mean

$$x + \frac{\alpha^2 z}{\alpha^2+\beta^2} = x + \frac{y-x}{2} = \frac{x+y}{2} \quad \text{and variance} \quad \frac{\alpha^2\beta^2}{\alpha^2+\beta^2} = \frac{v-u}{4}.$$

d) Four linear interpolations are displayed in Figure S.15.

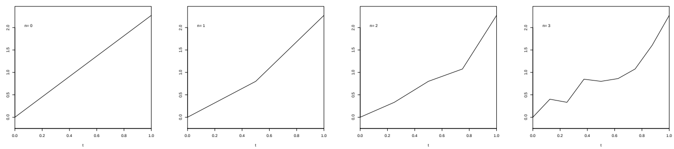


Fig. S.15: Samples of linear interpolations.

e) Clearly, the statement is true for $n = 0$ because $Z_1^{(0)}$ and B_1 have the same $\mathcal{N}(0,1)$ distribution. Next, assuming that it holds at the rank n, we note that the terms appearing in the sequence

$$Z^{(n+1)} = \left(0, Z^{(n+1)}_{1/2^{n+1}}, Z^{(n+1)}_{2/2^{n+1}}, Z^{(n+1)}_{3/2^{n+1}}, Z^{(n+1)}_{4/2^{n+1}}, \ldots, Z^{(n+1)}_1\right).$$

can be written for any $k = 0, 1, \ldots, 2^n - 1$ as

$$\left(\ldots, Z^{(n+1)}_{2k/2^{n+1}}, \frac{Z^{(n+1)}_{2k/2^{n+1}} + Z^{(n+1)}_{(2k+2)/2^{n+1}}}{2} + \mathcal{N}\left(0, 1/2^{n+2}\right), Z^{(n+1)}_{(2k+2)/2^{n+1}}, \ldots\right)$$
$$= \left(\ldots, Z^{(n)}_{k/2^n}, \frac{Z^{(n)}_{2k/2^{n+1}} + Z^{(n)}_{(2k+2)/2^{n+1}}}{2} + \mathcal{N}\left(0, 1/2^{n+2}\right), Z^{(n)}_{(k+1)/2^n}, \ldots\right)$$
$$= \left(\ldots, Z^{(n)}_{k/2^n}, \mathcal{N}\left(\frac{Z^{(n)}_{2k/2^{n+1}} + Z^{(n)}_{(2k+2)/2^{n+1}}}{2}, \frac{1}{2^{n+2}}\right), Z^{(n)}_{(k+1)/2^n}, \ldots\right). \tag{S.4.18}$$

On the other hand, the result of Question (c) shows that given that $B_{2k/2^{n+1}} = x$ and $B_{(2k+2)/2^{n+1}} = y$, the distribution of $B_{(2k+1)/2^{n+1}}$ is

$$\mathcal{N}\left(\frac{B_{2k/2^{n+1}} + B_{(2k+2)/2^{n+1}}}{2}, \frac{(2k+2)/2^{n+1} - (2k+2)/2^{n+1}}{4}\right)$$
$$= \mathcal{N}\left(\frac{B_{2k/2^{n+1}} + B_{(2k+2)/2^{n+1}}}{2}, \frac{1}{2^{n+2}}\right). \tag{S.4.19}$$

Given that $Z^{(n)}$ and $B^{(n)}$ have same distribution, we conclude by comparing (S.4.18) and (S.4.19) that $Z^{(n+1)}$ and $B^{(n+1)}$ also have same distribution.

f) We have

$$\mathbb{P}\left(\sup_{t\in[0,1]} |Z^{(n+1)}_t - Z^{(n)}_t| \geq \varepsilon_n\right)$$
$$= \mathbb{P}\Big(\max_{k=0,1,\ldots,2^n-1} |Z^{(n+1)}_{(2k+1)/2^{n+1}} - Z^{(n)}_{(2k+1)/2^{n+1}}| \geq \varepsilon_n\Big)$$
$$\leq \mathbb{P}\left(\bigcup_{k=0,1,\ldots,2^n-1} \left\{|Z^{(n+1)}_{(2k+1)/2^{n+1}} - Z^{(n)}_{(2k+1)/2^{n+1}}| \geq \varepsilon_n\right\}\right)$$
$$\leq \sum_{k=0}^{2^n-1} \mathbb{P}\big(|Z^{(n+1)}_{(2k+1)/2^{n+1}} - Z^{(n)}_{(2k+1)/2^{n+1}}| \geq \varepsilon_n\big)$$
$$= 2^n \mathbb{P}\big(|Z^{(n+1)}_{1/2^{n+1}} - Z^{(n)}_{1/2^{n+1}}| \geq \varepsilon_n\big)$$
$$= 2^n \mathbb{P}\left(\left|Z^{(n+1)}_{1/2^{n+1}} - \frac{Z^{(n)}_{0/2^n} + Z^{(n)}_{1/2^n}}{2}\right| \geq \varepsilon_n\right).$$

g) Since

$$Z^{(n+1)}_{1/2^{n+1}} = \frac{Z^{(n)}_0 + Z^{(n)}_{1/2^n}}{2} + \mathcal{N}(0, 1/2^{n+2}) = Z^{(n)}_{1/2^{n+1}} + \mathcal{N}(0, 1/2^{n+2}),$$

we have

$$\begin{aligned}
\mathbb{P}\left(\sup_{t\in[0,1]}|Z_t^{(n+1)}-Z_t^{(n)}|\geq\varepsilon_n\right) &\leq 2^n\mathbb{P}\big(|Z_{1/2^{n+1}}^{(n+1)}-Z_{1/2^{n+1}}^{(n)}|\geq\varepsilon_n\big) \\
&= 2^n\mathbb{P}\left(\left|Z_{1/2^{n+1}}^{(n+1)}-\frac{Z_{0/2^n}^{(n)}+Z_{1/2^n}^{(n)}}{2}\right|\geq\varepsilon_n\right) \\
&= 2^n\mathbb{P}\left(\left|\mathcal{N}(0,1/2^{n+2})\right|\geq\varepsilon_n\right) \\
&\leq \frac{2^{n/2}}{\varepsilon_n\sqrt{2\pi}}\mathrm{e}^{-\varepsilon_n^2 2^{n+1}},
\end{aligned}$$

where we applied the bound of Question (a) to the Gaussian random variable

$$Z_{1/2^{n+1}}^{(n+1)}-\frac{Z_{0/2^n}^{(n)}+Z_{1/2^n}^{(n)}}{2}\simeq\mathcal{N}(0,1/2^{n+2}).$$

h) We have

$$\begin{aligned}
\sum_{n\geq 0}\mathbb{P}\big(\|Z^{(n+1)}-Z^{(n)}\|_\infty\geq 2^{-n/4}\big) &= \sum_{n\geq 0}\mathbb{P}\left(\sup_{t\in[0,1]}|Z_t^{(n+1)}-Z_t^{(n)}|\geq\varepsilon_n\right) \\
&\leq \sum_{n\geq 0}\frac{2^{n/2}}{\varepsilon_n\sqrt{2\pi}}\mathrm{e}^{-\varepsilon_n^2 2^{n+1}} \\
&= \frac{1}{\sqrt{2\pi}}\sum_{n\geq 0}2^{3n/4}\mathrm{e}^{-2^{1+n/2}}<\infty,
\end{aligned}$$

since

$$\lim_{n\to\infty}\frac{2^{3(n+1)/4}\mathrm{e}^{-2^{1+(n+1)/2}}}{2^{3n/4}\mathrm{e}^{-2^{1+n/2}}}=2^{3/4}\lim_{n\to\infty}\mathrm{e}^{-2^{1+n/2}(\sqrt{2}-1)}=0.$$

Hence the Borel-Cantelli lemma shows that

$$\mathbb{P}\big(\|Z^{(n+1)}-Z^{(n)}\|_\infty\geq 2^{-n/4}\text{ for infinitely many }n\big)=0,$$

therefore we have

$$\mathbb{P}\big(\|Z^{(n+1)}-Z^{(n)}\|_\infty<2^{-n/4}\text{ except for finitely many }n\big)=1.$$

i) The result of Question (h) shows that with probability one we have

$$\lim_{p,q\to\infty}\|Z^{(p)}-Z^{(q)}\|_\infty=\lim_{p,q\to\infty}\left\|\sum_{n=q}^{p-1}Z^{(n+1)}-Z^{(n)}\right\|_\infty$$

$$\leq \lim_{p,q\to\infty} \sum_{n=q}^{p-1} \|Z^{(n+1)} - Z^{(n)}\|_\infty$$
$$\leq \lim_{p\to\infty} \sum_{n\geq q} \|Z^{(n+1)} - Z^{(n)}\|_\infty$$
$$= 0,$$

hence the sequence $\left(Z^{(n)}\right)_{n\geq 0}$ is Cauchy in $\mathcal{C}_0([0,1])$ for the $\|\cdot\|_\infty$ norm. Since $\mathcal{C}_0([0,1])$ is a complete space for the $\|\cdot\|_\infty$ norm, this implies that, with probability one, the sequence $(Z^{(n)})_{n\geq 0}$ admits a limit in $\mathcal{C}_0([0,1])$.

j) 1. By construction we have $Z_0^{(n)} = 0$ for all $n \in \mathbb{N}$, hence $Z_0 = \lim_{n\to\infty} Z_0^{(n)} = 0$, almost surely.

2. The sample trajectories $t \mapsto Z_t$ are continuous, because the limit Z belongs to $\mathcal{C}_0([0,1])$ with probability 1.

3. The result of Question (e) shows that for any fixed $m \geq 1$, the sequences

$$Z_{t_1} - Z_{t_0}, Z_{t_2} - Z_{t_1}, \ldots, Z_{t_m} - Z_{t_{m-1}}$$

and

$$B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \ldots, B_{t_m} - B_{t_{m-1}}$$

have same distribution when the $t_k's$ are dyadic rationals of the form $t_k = i_n/2^n$, $k = 0, 1, \ldots, n$. This property extends to any sequence $t_0, t_1, \ldots, t_m$ of real numbers by approximation of each $t_k > 0$ by a sequence $(i_n)_{n\in\mathbb{N}}$ such that $t_k = \lim_{n\to\infty} i_n/2^n$ and taking the limit as n tends to infinity.

4. By a similar argument as in the above point 3, one can show that for any $0 \leq s < t$, $Z_t - Z_s$ has the Gaussian distribution $\mathcal{N}(0, t-s)$.

Problem 4.21

a) We have

$$\mathbb{E}\left[Q_T^{(n)}\right] = \sum_{k=1}^n \mathbb{E}\left[(B_{kT/n} - B_{(k-1)T/n})^2\right]$$
$$= \sum_{k=1}^n \left(k\frac{T}{n} - (k-1)\frac{T}{n}\right)$$
$$= T, \qquad n \geq 1.$$

b) We have

$$
\begin{aligned}
\mathbb{E}\left[(Q_T^{(n)})^2\right] &= \mathbb{E}\left[\left(\sum_{k=1}^{n}(B_{kT/n}-B_{(k-1)T/n})^2\right)^2\right] \\
&= \mathbb{E}\left[\sum_{k,l=1}^{n}(B_{kT/n}-B_{(k-1)T/n})^2(B_{lT/n}-B_{(l-1)T/n})^2\right] \\
&= \sum_{k=1}^{n}\mathbb{E}\left[(B_{kT/n}-B_{(k-1)T/n})^4\right] \\
&\quad +2\sum_{1\le k<l\le n}\mathbb{E}\left[(B_{kT/n}-B_{(k-1)T/n})^2\right]\mathbb{E}\left[(B_{lT/n}-B_{(l-1)T/n})^2\right] \\
&= 3\sum_{k=1}^{n}(kT/n-(k-1)T/n)^2 \\
&\quad +2\sum_{1\le k<l\le n}(kT/n-(k-1)T/n)(lT/n-(l-1)T/n) \\
&= 3\frac{T^2}{n}+\frac{n(n-1)T^2}{n^2} \\
&= T^2+\frac{2T^2}{n}, \qquad n\ge 1,
\end{aligned}
$$

hence

$$
\mathrm{Var}\left[Q_T^{(n)}\right] = \mathbb{E}\left[\left(Q_T^{(n)}\right)^2\right] - \left(\mathbb{E}\left[Q_T^{(n)}\right]\right)^2 = \frac{2T^2}{n}, \qquad n\ge 1.
$$

c) We have

$$
\begin{aligned}
\|Q_T^{(n)}-T\|_{L^2(\Omega)}^2 &= \mathbb{E}\left[(Q_T^{(n)}-\mathbb{E}[Q_T^{(n)}])^2\right] \\
&= \mathrm{Var}\left[Q_T^{(n)}\right] \\
&= \frac{n(n+2)T^2}{n^2}-T^2 \\
&= \frac{2T^2}{n},
\end{aligned}
$$

hence

$$
\lim_{n\to\infty}\|Q_T^{(n)}-T\|_{L^2(\Omega)}^2 = \lim_{n\to\infty}\frac{2T^2}{n}=0,
$$

showing that

$$
\lim_{n\to\infty}Q_T^{(n)}=T
$$

in $L^2(\Omega)$.

d) We have

$$\sum_{k=1}^n (B_{kT/n} - B_{(k-1)T/n}) B_{(k-1)T/n} = \frac{1}{2} \sum_{k=1}^n B_{kT/n}^2 - B_{(k-1)T/n}^2$$
$$-\frac{1}{2} \sum_{k=1}^n (B_{kT/n} - B_{(k-1)T/n})(B_{kT/n} - B_{(k-1)T/n})$$
$$= \frac{1}{2}((B_T)^2 - (B_0)^2)$$
$$-\frac{1}{2} \sum_{k=1}^n (B_{kT/n} - B_{(k-1)T/n})(B_{kT/n} - B_{(k-1)T/n})$$
$$= \frac{1}{2}\big((B_T)^2 - Q_T^{(n)}\big),$$

which converges to $((B_T)^2 - T)/2$ in $L^2(\Omega)$ as n tends to infinity, hence

$$\int_0^T B_t dB_t = \lim_{n\to\infty} \sum_{k=1}^n (B_{kT/n} - B_{(k-1)T/n}) B_{(k-1)T/n}$$
$$= \frac{(B_T)^2 - T}{2}.$$

e) We have

$$\mathbb{E}\left[\widetilde{Q}_T^{(n)}\right] = \sum_{k=1}^n \mathbb{E}\left[(B_{(k-1/2)T/n} - B_{(k-1)T/n})^2\right]$$
$$= \sum_{k=1}^n ((k-1/2)T/n - (k-1)T/n)$$
$$= \frac{T}{2}, \qquad n \geq 1.$$

Next, we have

$$\mathbb{E}\left[\big(\widetilde{Q}_T^{(n)}\big)^2\right] = \mathbb{E}\left[\left(\sum_{k=1}^n \big(B_{(k-1/2)T/n} - B_{(k-1)T/n}\big)^2\right)^2\right]$$
$$= \mathbb{E}\left[\sum_{k,l=1}^n (B_{(k-1/2)T/n} - B_{(k-1)T/n})^2 (B_{lT/n} - B_{(l-1)T/n})^2\right]$$
$$= \sum_{k=1}^n \mathbb{E}\left[(B_{(k-1/2)T/n} - B_{(k-1)T/n})^4\right]$$
$$+2 \sum_{1\leq k<l\leq n} \mathbb{E}\left[(B_{(k-1/2)T/n} - B_{(k-1)T/n})^2\right] \mathbb{E}\left[(B_{(l-1/2)T/n} - B_{(l-1)T/n})^2\right]$$

$$= 3\sum_{k=1}^{n}((k-1/2)T/n-(k-1)T/n)^2$$
$$+2\sum_{1\leq k<l\leq n}((k-1/2)T/n-(k-1)T/n)((l-1/2)T/n-(l-1)T/n)$$
$$= 3\frac{T^2}{4n}+\frac{n(n-1)T^2}{4n^2}$$
$$= \frac{n(n+2)T^2}{4n^2}, \qquad n\geq 1.$$

Finally, we find

$$\begin{aligned}\|\widetilde{Q}_T^{(n)}-T/2\|^2_{L^2(\Omega)} &= \mathbb{E}\left[(\widetilde{Q}_T^{(n)}-\mathbb{E}[\widetilde{Q}_T^{(n)}])^2\right]\\ &= \mathrm{Var}\left[\widetilde{Q}_T^{(n)}\right]\\ &= \frac{n(n+2)T^2}{4n^2}-\frac{T^2}{4}\\ &= \frac{T^2}{2n},\end{aligned}$$

hence

$$\lim_{n\to\infty}\|\widetilde{Q}_T^{(n)}-T/2\|^2_{L^2(\Omega)} = \lim_{n\to\infty}\frac{T^2}{2n} = 0,$$

showing that

$$\lim_{n\to\infty}\widetilde{Q}_T^{(n)} = \frac{T}{2}$$

in $L^2(\Omega)$.

f) We have

$$\sum_{k=1}^{n}\left(B_{kT/n}-B_{(k-1)T/n}\right)B_{(k-1/2)T/n}$$
$$= \sum_{k=1}^{n}\left(B_{kT/n}-B_{(k-1/2)T/n}\right)B_{(k-1/2)T/n}$$
$$+\sum_{k=1}^{n}\left(B_{(k-1/2)T/n}-B_{(k-1)T/n}\right)B_{(k-1/2)T/n}$$
$$= \frac{1}{2}\sum_{k=1}^{n}B^2_{kT/n}-B^2_{(k-1/2)T/n}$$
$$-\frac{1}{2}\sum_{k=1}^{n}\left(B_{kT/n}-B_{(k-1/2)T/n}\right)\left(B_{kT/n}-B_{(k-1/2)T/n}\right)$$
$$+\frac{1}{2}\sum_{k=1}^{n}B^2_{(k-1/2)T/n}-B^2_{(k-1)T/n}$$

$$+\frac{1}{2}\sum_{k=1}^{n}\big(B_{(k-1/2)T/n}-B_{(k-1)T/n}\big)\big(B_{(k-1/2)T/n}-B_{(k-1)T/n}\big)$$
$$=\frac{1}{2}(B_T)^2-\frac{1}{2}\sum_{k=1}^{n}\big(B_{kT/n}-B_{(k-1/2)T/n}\big)\big(B_{kT/n}-B_{(k-1/2)T/n}\big)$$
$$+\frac{1}{2}\sum_{k=1}^{n}\big(B_{(k-1/2)T/n}-B_{(k-1)T/n}\big)\big(B_{(k-1/2)T/n}-B_{(k-1)T/n}\big),$$

which converges to $((B_T)^2-T+T)/2=(B_T)^2/2$ in $L^2(\Omega)$ as n tends to infinity, hence

$$\int_0^T B_t\circ dB_t=\lim_{n\to\infty}\sum_{k=1}^{n}(B_{kT/n}-B_{(k-1)T/n})B_{(k-1/2)T/n}=\frac{(B_T)^2}{2},$$

see Section 2.4 of Mikosch (1998) for further details on the *Stratonovich integral*.

g) We have

$$\begin{aligned}\mathbb{E}\big[\widetilde{Q}_T^{(n)}\big]&=\sum_{k=1}^{n}\mathbb{E}\Big[\big(B_{(k-\alpha)T/n}-B_{(k-1)T/n}\big)^2\Big]\\&=\sum_{k=1}^{n}((k-\alpha)T/n-(k-1)T/n)\\&=(1-\alpha)\frac{T}{2},\qquad n\geq 1.\end{aligned}$$

Next, we have

$$\mathbb{E}\Big[\big(\widetilde{Q}_T^{(n)}\big)^2\Big]=\mathbb{E}\left[\left(\sum_{k=1}^{n}(B_{(k-\alpha)T/n}-B_{(k-1)T/n})^2\right)^2\right]$$
$$=\mathbb{E}\left[\sum_{k,l=1}^{n}(B_{(k-\alpha)T/n}-B_{(k-1)T/n})^2(B_{lT/n}-B_{(l-1)T/n})^2\right]$$
$$=\sum_{k=1}^{n}\mathbb{E}\big[(B_{(k-\alpha)T/n}-B_{(k-1)T/n})^4\big]$$
$$+2\sum_{1\leq k<l\leq n}\mathbb{E}\big[(B_{(k-\alpha)T/n}-B_{(k-1)T/n})^2\big]\,\mathbb{E}\big[(B_{(l-\alpha)T/n}-B_{(l-1)T/n})^2\big]$$
$$=3\sum_{k=1}^{n}((k-\alpha)T/n-(k-1)T/n)^2$$
$$+2\sum_{1\leq k<l\leq n}((k-\alpha)T/n-(k-1)T/n)((l-\alpha)T/n-(l-1)T/n)$$

$$
\begin{aligned}
&= 3(1-\alpha)^2\frac{T^2}{n} + (1-\alpha)^2\frac{n(n-1)T^2}{n^2} \\
&= (1-\alpha)^2\frac{n(n+2)T^2}{n^2}, \qquad n \geq 1.
\end{aligned}
$$

Finally we find

$$
\begin{aligned}
\|\widetilde{Q}_T^{(n)} - (1-\alpha)T/2\|_{L^2(\Omega)}^2 &= \mathbb{E}\left[\left(\widetilde{Q}_T^{(n)} - \mathbb{E}\left[\widetilde{Q}_T^{(n)}\right]\right)^2\right] \\
&= \mathrm{Var}\left[\widetilde{Q}_T^{(n)}\right] \\
&= (1-\alpha)^2\frac{n(n+2)T^2}{n^2} - (1-\alpha)^2T^2 \\
&= 2(1-\alpha)^2\frac{T^2}{n},
\end{aligned}
$$

hence

$$
\lim_{n\to\infty} \|\widetilde{Q}_T^{(n)} - (1-\alpha)T\|_{L^2(\Omega)}^2 = (1-\alpha)^2 \lim_{n\to\infty} \frac{T^2}{n} = 0.
$$

Next, we have

$$
\begin{aligned}
&\sum_{k=1}^n (B_{kT/n} - B_{(k-1)T/n})B_{(k-\alpha)T/n} \\
&= \sum_{k=1}^n (B_{kT/n} - B_{(k-\alpha)T/n})B_{(k-\alpha)T/n} + \sum_{k=1}^n (B_{(k-\alpha)T/n} - B_{(k-1)T/n})B_{(k-\alpha)T/n} \\
&= \frac{1}{2}\sum_{k=1}^n B_{kT/n}^2 - B_{(k-\alpha)T/n}^2 - \frac{1}{2}\sum_{k=1}^n (B_{kT/n} - B_{(k-\alpha)T/n})(B_{kT/n} - B_{(k-\alpha)T/n}) \\
&+ \frac{1}{2}\sum_{k=1}^n B_{(k-\alpha)T/n}^2 - B_{(k-1)T/n}^2 \\
&+ \frac{1}{2}\sum_{k=1}^n (B_{(k-\alpha)T/n} - B_{(k-1)T/n})(B_{(k-\alpha)T/n} - B_{(k-1)T/n}) \\
&= \frac{1}{2}(B_T)^2 - \frac{1}{2}\sum_{k=1}^n (B_{kT/n} - B_{(k-\alpha)T/n})(B_{kT/n} - B_{(k-\alpha)T/n}) \\
&+ \frac{1}{2}\sum_{k=1}^n (B_{(k-\alpha)T/n} - B_{(k-1)T/n})(B_{(k-\alpha)T/n} - B_{(k-1)T/n}),
\end{aligned}
$$

which converges to

$$
\frac{(B_T)^2 - \alpha T + (1-\alpha)T}{2} = \frac{(B_T)^2 + (1-2\alpha)T}{2}
$$

in $L^2(\Omega)$ as n tends to infinity, hence

$$\int_0^T B_t \circ d^\alpha B_t = \lim_{n\to\infty} \sum_{k=1}^n (B_{kT/n} - B_{(k-1)T/n}) B_{(k-\alpha)T/n}$$
$$= \frac{(B_T)^2 + (1-2\alpha)T}{2}.$$

In particular we find

$$\int_0^T B_t \circ d^0 B_t = \lim_{n\to\infty} \sum_{k=1}^n (B_{kT/n} - B_{(k-1)T/n}) B_{kT/n} = \frac{(B_T)^2 + T}{2},$$

and we note that

$$\int_0^T B_t \circ dB_t = \frac{1}{2}\left(\int_0^T B_t dB_t + \int_0^T B_t \circ d^1 B_t \right).$$

h) We have

$$\lim_{n\to\infty} \sum_{k=1}^n (k-\alpha) \frac{T}{n} \left(k\frac{T}{n} - (k-1)\frac{T}{n} \right)$$
$$= \lim_{n\to\infty} \frac{T}{n} \sum_{k=1}^n (k-\alpha) \frac{T}{n}$$
$$= \lim_{n\to\infty} \frac{T}{n} \sum_{k=1}^n k \frac{T}{n} - \alpha \lim_{n\to\infty} \frac{T}{n} \sum_{k=1}^n \frac{T}{n}$$
$$= T^2 \lim_{n\to\infty} \frac{n(n+1)}{2n^2} - \alpha \lim_{n\to\infty} \frac{T^2}{n}$$
$$= \frac{T^2}{2},$$

which does not depend on $\alpha \in [0,1]<$ hence the stochastic phenomenon of the previous questions does not occur when approximating the deterministic integral $\int_0^T t dt = T^2/2$ by Riemann sums.

In quantitative finance we choose to use the Itô integral (which corresponds to the choice $\alpha = 1$) because it is suitable for the modeling of market returns as

$$\frac{dS_t}{S_t} \simeq \frac{S_{t+\Delta t} - S_t}{S_t} = \mu \Delta t + \sigma \Delta B_t = \mu \Delta t + (B_{t+\Delta t} - B_t)\sigma$$

or

$$dS_t \simeq S_{t+\Delta t} - S_t = \mu S_t \Delta t + \sigma S_t \Delta B_t, = \mu S_t \Delta t + \sigma S_t (B_{t+\Delta t} - B_t),$$

based on the value S_t at the the *left endpoint* of the discretized time interval $[t, t+\Delta t]$.

Chapter 5

Exercise 5.1 For all $x \in \mathbb{R}$, we have

$$\begin{aligned}
\mathbb{P}(S_T \leq x) &= \mathbb{P}\big(S_0 \mathrm{e}^{\sigma B_T + (\mu - \sigma^2/2)T} \leq x\big) \\
&= \mathbb{P}\left(\sigma B_T + \left(\mu - \frac{\sigma^2}{2}\right) T \leq \log \frac{x}{S_0}\right) \\
&= \mathbb{P}\left(B_T \leq \frac{1}{\sigma}\left(\log \frac{x}{S_0} - \left(\mu - \frac{\sigma^2}{2}\right) T\right)\right) \\
&= \int_{-\infty}^{(\log(x/S_0) - (\mu - \sigma^2/2)T)/\sigma} \mathrm{e}^{-y^2/(2T)} \frac{dy}{\sqrt{2\pi T}} \\
&= \int_{-\infty}^{(\log(x/S_0) - (\mu - \sigma^2/2)T)/(\sigma\sqrt{T})} \mathrm{e}^{-z^2/2} \frac{dz}{\sqrt{2\pi}} \\
&= \Phi\left(\frac{1}{\sigma\sqrt{T}}\left(\log \frac{x}{S_0} - \left(\mu - \frac{\sigma^2}{2}\right) T\right)\right),
\end{aligned}$$

where

$$\Phi(x) := \int_{-\infty}^{x} \mathrm{e}^{-y^2/2} \frac{dy}{\sqrt{2\pi T}}, \qquad x \in \mathbb{R},$$

denotes the standard Gaussian cumulative distribution function. After differentiation with respect to x, we find the lognormal probability density function

$$\begin{aligned}
f(x) &= \frac{\mathrm{d}\mathbb{P}(S_T \leq x)}{dx} \\
&= \frac{\partial}{\partial x} \int_{-\infty}^{(\log(x/S_0) - (\mu - \sigma^2/2)T)/\sigma} \mathrm{e}^{-y^2/(2T)} \frac{dy}{\sqrt{2\pi T}} \\
&= \frac{\partial}{\partial x} \Phi\left(\frac{1}{\sigma\sqrt{T}}\left(\log \frac{x}{S_0} - \left(\mu - \frac{\sigma^2}{2}\right) T\right)\right) \\
&= \frac{1}{x\sigma\sqrt{T}} \varphi\left(\frac{1}{\sigma\sqrt{T}}\left(\log \frac{x}{S_0} - \left(\mu - \frac{\sigma^2}{2}\right) T\right)\right) \\
&= \frac{1}{x\sigma\sqrt{2\pi T}} \mathrm{e}^{-(-(\mu - \sigma^2/2)T + \log(x/S_0))^2/(2\sigma^2 T)}, \qquad x > 0,
\end{aligned}$$

where

$$\varphi(y) = \Phi'(y) := \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-y^2/2}, \qquad y \in \mathbb{R},$$

denotes the standard Gaussian probability density function.

Exercise 5.2

a) We have

$$d\log S_t = \frac{1}{S_t}dS_t - \frac{1}{2S_t^2}(dS_t)^2 = rdt + \sigma dB_t - \frac{\sigma^2}{2}dt, \quad t \geq 0.$$

b) We have $f(t) = f(0)\mathrm{e}^{ct}$ (continuous-time interest rate compounding), and

$$S_t = S_0\mathrm{e}^{\sigma B_t - \sigma^2 t/2 + rt}, \qquad t \geq 0,$$

(geometric Brownian motion).

c) Those quantities can be directly computed from the expression of S_t as a function of the $\mathcal{N}(0,t)$ random variable B_t. Namely, we have

$$\begin{aligned}\mathbb{E}[S_t] &= \mathbb{E}\left[S_0\mathrm{e}^{\sigma B_t - \sigma^2 t/2 + rt}\right] \\ &= S_0\mathrm{e}^{-\sigma^2 t/2 + rt}\,\mathbb{E}\left[\mathrm{e}^{\sigma B_t}\right] \\ &= S_0\mathrm{e}^{rt},\end{aligned}$$

where we used the Gaussian moment generating function (MGF) formula, *i.e.*

$$\mathbb{E}\left[\mathrm{e}^{\sigma B_t}\right] = \mathrm{e}^{\sigma^2 t/2}$$

for the normal random variable $B_t \simeq \mathcal{N}(0,t)$, $t > 0$. Similarly, we have

$$\begin{aligned}\mathbb{E}[S_t^2] &= \mathbb{E}\left[S_0^2\mathrm{e}^{2\sigma B_t - \sigma^2 t + 2rt}\right] \\ &= S_0^2\mathrm{e}^{-\sigma^2 t + 2rt}\,\mathbb{E}\left[\mathrm{e}^{2\sigma B_t}\right] \\ &= S_0^2\mathrm{e}^{\sigma^2 t + 2rt}, \qquad t \geq 0.\end{aligned}$$

d) We note that from the stochastic differential equation

$$S_t = S_0 + r\int_0^t S_s ds + \sigma\int_0^t S_s dB_s,$$

the function $u(t) := \mathbb{E}[S_t]$ satisfies the Ordinary Differential Equation (ODE) $u'(t) = ru(t)$ with $u(0) = S_0$ and solution $u(t) = \mathbb{E}[S_t] = S_0\mathrm{e}^{rt}$. On the other hand, by the Itô formula we have

$$dS_t^2 = 2S_t dS_t + (dS_t)^2 = 2rS_t^2 dt + \sigma^2 S_t^2 dt + 2\sigma S_t^2 dB_t,$$

hence letting $v(t) = \mathbb{E}\left[S_t^2\right]$ and taking expectations on both sides of

$$S_t^2 = S_0^2 + 2r\int_0^t S_u^2 du + \sigma^2\int_0^t S_u^2 du + 2\sigma\int_0^t S_u^2 dB_u,$$

we find

$$\begin{aligned}
v(t) &= \mathbb{E}\left[S_t^2\right] \\
&= S_0^2 + (2r+\sigma^2)\,\mathbb{E}\left[\int_0^t S_u^2 du\right] + 2\sigma\,\mathbb{E}\left[\int_0^t S_u^2 dB_u\right] \\
&= S_0^2 + (2r+\sigma^2)\int_0^t \mathbb{E}\left[S_u^2\right] du \\
&= S_0^2 + (2r+\sigma^2)\int_0^t v(u) du,
\end{aligned}$$

hence $v(t) := \mathbb{E}\left[S_t^2\right]$ satisfies the ordinary differential equation

$$v'(t) = (\sigma^2 + 2r)v(t),$$

with $v(0) = S_0^2$ and solution

$$v(t) = \mathbb{E}\left[S_t^2\right] = S_0^2 \mathrm{e}^{(\sigma^2+2r)t},$$

which recovers

$$\begin{aligned}
\mathrm{Var}[S_t] &= \mathbb{E}\left[S_t^2\right] - (\mathbb{E}[S_t])^2 \\
&= v(t) - u^2(t) \\
&= S_0^2 \mathrm{e}^{(\sigma^2+2r)t} - S_0^2 \mathrm{e}^{2rt} \\
&= S_0^2 \mathrm{e}^{2rt}(\mathrm{e}^{\sigma^2 t} - 1), \qquad t \geq 0.
\end{aligned}$$

Exercise 5.3 Using the bivariate Itô formula (4.26), we find

$$\begin{aligned}
df(S_t, Y_t) &= \frac{\partial f}{\partial x}(S_t, Y_t)dS_t + \frac{\partial f}{\partial y}(S_t, Y_t)dY_t \\
&\quad + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(S_t, Y_t)(dS_t)^2 + \frac{1}{2}\frac{\partial^2 f}{\partial y^2}(S_t, Y_t)(dY_t)^2 + \frac{\partial^2 f}{\partial x \partial y}(S_t, Y_t)dS_t \bullet dY_t \\
&= \frac{\partial f}{\partial x}(S_t, Y_t)(rS_t dt + \sigma S_t dB_t) + \frac{\partial f}{\partial y}(S_t, Y_t)(\mu Y_t dt + \eta Y_t dW_t) \\
&\quad + \frac{\sigma^2 S_t^2}{2}\frac{\partial^2 f}{\partial x^2}(S_t, Y_t)dt + \frac{\eta^2 Y_t^2}{2}\frac{\partial^2 f}{\partial y^2}(S_t, Y_t)dt + \rho\sigma\eta S_t Y_t \frac{\partial^2 f}{\partial x \partial y}(S_t, Y_t)dt.
\end{aligned}$$

Exercise 5.4 Taking expectations on both sides of (5.24) shows that

$$\mathbb{E}[S_T] = C(S_0, r, T) + \mathbb{E}\left[\int_0^T \zeta_{t,T} dB_t\right] = C(S_0, r, T),$$

hence

$$\begin{aligned}
C(S_0, r, T) &= \mathbb{E}[S_T] \\
&= \mathbb{E}[S_0 \mathrm{e}^{rT + \sigma B_T - \sigma^2 T/2}]
\end{aligned}$$

$$
\begin{aligned}
&= S_0 \mathrm{e}^{rT-\sigma^2T/2}\, \mathbb{E}[\mathrm{e}^{\sigma B_T}] \\
&= S_0 \mathrm{e}^{rT-\sigma^2T/2+\sigma^2T/2} \\
&= S_0 \mathrm{e}^{rT},
\end{aligned}
$$

where we used the moment generating function

$$\mathbb{E}[\mathrm{e}^{\sigma B_T}] = \mathrm{e}^{\sigma^2 T/2}$$

of the Gaussian random variable $B_T \simeq \mathcal{N}(0,T)$. On the other hand, the discounted asset price $X_t := \mathrm{e}^{-rt}S_t$ satisfies $dX_t = \sigma X_t dB_t$, which shows that

$$X_T = X_0 + \sigma \int_0^T X_t dB_t.$$

Multiplying both sides by e^{rT} shows that

$$S_T = \mathrm{e}^{rT} S_0 + \sigma \int_0^T \mathrm{e}^{rT} X_t dB_t = \mathrm{e}^{rT} S_0 + \sigma \int_0^T \mathrm{e}^{(T-t)r} S_t dB_t,$$

which recovers the relation $C(S_0, r, T) = S_0 \mathrm{e}^{rT}$, and shows that $\zeta_{t,T} = \sigma \mathrm{e}^{(T-t)r} S_t$, $t \in [0,T]$.

Exercise 5.5

a) We have $S_t = f(X_t)$, $t \geq 0$, where $f(x) = S_0 \mathrm{e}^x$ and $(X_t)_{t\in\mathbb{R}_+}$ is the Itô process given by

$$X_t := \int_0^t \sigma_s dB_s + \int_0^t u_s ds, \qquad t \geq 0,$$

or in differential form

$$dX_t := \sigma_t dB_t + u_t dt, \qquad t \geq 0,$$

hence

$$
\begin{aligned}
dS_t &= df(X_t) \\
&= f'(X_t) dX_t + \frac{1}{2} f''(X_t)(dX_t)^2 \\
&= u_t f'(X_t) dt + \sigma_t f'(X_t) dB_t + \frac{1}{2} \sigma_t^2 f''(X_t) dt \\
&= S_0 u_t \mathrm{e}^{X_t} dt + S_0 \sigma_t \mathrm{e}^{X_t} dB_t + \frac{1}{2} S_0 \sigma_t^2 \mathrm{e}^{X_t} dt \\
&= u_t S_t dt + \sigma_t S_t dB_t + \frac{1}{2} \sigma_t^2 S_t dt.
\end{aligned}
$$

b) The process $(S_t)_{t\in\mathbb{R}_+}$ satisfies the stochastic differential equation

$$dS_t = \left(u_t + \frac{1}{2}\sigma_t^2\right) S_t dt + \sigma_t S_t dB_t.$$

Exercise 5.6

a) We have $\mathbb{E}[S_t] = 1$ because the expected value of the Itô stochastic integral is zero by Relation (4.17) in Proposition 4.21. Regarding the variance, using the Itô isometry (4.16), we have

$$\begin{aligned}
\mathrm{Var}[S_t] &= \sigma^2\, \mathbb{E}\left[\left(\int_0^t \mathrm{e}^{\sigma B_s - \sigma^2 s/2} dB_s\right)^2\right] \\
&= \sigma^2\, \mathbb{E}\left[\int_0^t \left(\mathrm{e}^{\sigma B_s - \sigma^2 s/2}\right)^2 ds\right] \\
&= \sigma^2 \int_0^t \mathbb{E}\left[\left(\mathrm{e}^{\sigma B_s - \sigma^2 s/2}\right)^2\right] ds \\
&= \sigma^2 \int_0^t \mathbb{E}\left[\mathrm{e}^{2\sigma B_s - \sigma^2 s}\right] ds \\
&= \sigma^2 \int_0^t \mathrm{e}^{-\sigma^2 s}\, \mathbb{E}\left[\mathrm{e}^{2\sigma B_s}\right] ds \\
&= \sigma^2 \int_0^t \mathrm{e}^{-\sigma^2 s} \mathrm{e}^{2\sigma^2 s} ds \\
&= \sigma^2 \int_0^t \mathrm{e}^{\sigma^2 s} ds \\
&= \mathrm{e}^{\sigma^2 t} - 1.
\end{aligned}$$

b) Taking $f(x) = \log x$, we have

$$\begin{aligned}
d\log(S_t) &= df(S_t) \\
&= f'(S_t) dS_t + \frac{1}{2} f''(S_t)(dS_t)^2 \\
&= \sigma f'(S_t)\mathrm{e}^{\sigma B_t - \sigma^2 t/2} dB_t + \frac{1}{2}\sigma^2 f''(S_t)\mathrm{e}^{2\sigma B_t - \sigma^2 t} dt \\
&= \frac{\sigma}{S_t}\mathrm{e}^{\sigma B_t - \sigma^2 t/2} dB_t - \frac{\sigma^2}{2S_t^2}\mathrm{e}^{2\sigma B_t - \sigma^2 t} dt.
\end{aligned} \tag{S.5.20}$$

c) We check that letting $Z_t := e^{\sigma B_t - \sigma^2 t/2}$, $t \geq 0$, we have

$$\log Z_t = \sigma B_t - \sigma^2 t/2, \quad \text{and} \quad d\log Z_t = \sigma dB_t - \frac{\sigma^2}{2} dt.$$

On the other hand, we also have

$$\sigma dB_t - \frac{\sigma^2}{2} dt = \frac{\sigma}{Z_t}\mathrm{e}^{\sigma B_t - \sigma^2 t/2} dB_t - \frac{\sigma^2}{2Z_t^2}\mathrm{e}^{2\sigma B_t - \sigma^2 t} dt,$$

showing by (S.5.20) that the equation

$$d\log Z_t = \frac{\sigma}{Z_t}\mathrm{e}^{\sigma B_t-\sigma^2 t/2}dB_t - \frac{\sigma^2}{2Z_t^2}\mathrm{e}^{2\sigma B_t-\sigma^2 t}dt$$

is satisfied. By uniqueness of solutions to (S.5.20) we obtain

$$\log S_t = \log Z_t = \sigma B_t - \sigma^2 t/2,$$

and we conclude that $S_t = Z_t = e^{\sigma B_t-\sigma^2 t/2}$, $t \geq 0$.

Exercise 5.7

a) Leveraging with a factor $\beta : 1$ means that when the fund value is F_t, the amount $\xi_t S_t = \beta F_t$ is actually invested on the risky asset priced S_t. In this case, the fund value F_t at time $t \geq 0$ decomposes into the portfolio

$$F_t = \xi_t S_t + \eta_t A_t = \beta\frac{F_t}{S_t}S_t - (\beta-1)\frac{F_t}{A_t}A_t, \qquad t \geq 0,$$

with $\xi_t = \beta F_t/S_t$ and $\eta_t = -(\beta-1)F_t/A_t$, $t \geq 0$.

b) We have

$$\begin{aligned}
dF_t &= \xi_t dS_t + \eta_t dA_t \\
&= \beta\frac{F_t}{S_t}dS_t - (\beta-1)\frac{F_t}{A_t}dA_t \\
&= \beta\frac{F_t}{S_t}dS_t - (\beta-1)rF_t dt \\
&= \beta F_t(rdt + \sigma dB_t) - (\beta-1)rF_t dt \\
&= rF_t dt + \beta\sigma F_t dB_t, \qquad t \geq 0.
\end{aligned}$$

The above equation shows that the volatility $\beta\sigma$ of the fund is β times the volatility of the index. On the other hand, the risk-free rate r remains the same.

c) By Proposition 5.15 we have

$$\begin{aligned}
F_t &= F_0\mathrm{e}^{\beta\sigma B_t+rt-\beta^2\sigma^2 t/2} \\
&= F_0\left(\mathrm{e}^{\sigma B_t+rt/\beta-\beta\sigma^2 t/2}\right)^{\beta} \\
&= F_0\left(\mathrm{e}^{\sigma B_t+rt-\sigma^2 t/2-(1-1/\beta)rt-(\beta-1)\sigma^2 t/2}\right)^{\beta} \\
&= F_0\left(\mathrm{e}^{\sigma B_t+rt-\sigma^2 t/2}\right)^{\beta}\mathrm{e}^{-(\beta-1)rt-\beta(\beta-1)\sigma^2 t/2} \\
&= \left(S_0\mathrm{e}^{\sigma B_t+rt-\sigma^2 t/2}\right)^{\beta}\mathrm{e}^{-(\beta-1)rt-\beta(\beta-1)\sigma^2 t/2} \\
&= S_t^{\beta}\mathrm{e}^{-(\beta-1)rt-\beta(\beta-1)\sigma^2 t/2}, \qquad t \geq 0.
\end{aligned}$$

Exercise 5.8

a) For $t \in [0, T]$ and $i = 1, 2$ we have

$$\begin{aligned}\mathbb{E}\left[S_t^{(i)}\right] &= \mathbb{E}\left[S_0^{(i)} \mathrm{e}^{\mu t+\sigma_i W_t^{(i)}-\sigma_i^2 t/2}\right] \\ &= S_0^{(i)} \mathrm{e}^{\mu t-\sigma_i^2 t/2}\, \mathbb{E}\left[\mathrm{e}^{\sigma_i W_t^{(i)}}\right] \\ &= S_0^{(i)} \mathrm{e}^{\mu t-\sigma_i^2 t/2+\sigma_i^2 t/2} \\ &= S_0^{(i)} \mathrm{e}^{\mu t}.\end{aligned}$$

b) For all $t \in [0, T]$ and $i = 1, 2$, we have

$$\begin{aligned}\mathbb{E}\left[\big(S_t^{(i)}\big)^2\right] &= \mathbb{E}\left[\big(S_0^{(i)}\big)^2 \mathrm{e}^{2\mu t+2\sigma_i W_t^{(i)}-\sigma_i^2 t}\right] \\ &= \big(S_0^{(i)}\big)^2 \mathrm{e}^{2\mu t-\sigma_i^2 t}\, \mathbb{E}\left[\mathrm{e}^{2\sigma_i W_t^{(i)}}\right] \\ &= \big(S_0^{(i)}\big)^2 \mathrm{e}^{2\mu t-\sigma_i^2 t+2\sigma_i^2 t} \\ &= \big(S_0^{(i)}\big)^2 \mathrm{e}^{2\mu t+\sigma_i^2 t},\end{aligned}$$

hence

$$\begin{aligned}\mathrm{Var}\left[S_t^{(i)}\right] &= \mathbb{E}\left[\big(S_t^{(i)}\big)^2\right] - \big(\mathbb{E}\left[S_t^{(i)}\right]\big) \\ &= \big(S_0^{(i)}\big)^2 \mathrm{e}^{2\mu t+\sigma_i^2 t} - \big(S_0^{(i)}\big)^2 \mathrm{e}^{2\mu t} \\ &= \big(S_0^{(i)}\big)^2 \mathrm{e}^{2\mu t}\big(\mathrm{e}^{\sigma_i^2 t}-1\big), \quad t \in [0, T], \quad i = 1, 2.\end{aligned}$$

c) We have

$$\mathrm{Var}\left[S_t^{(2)} - S_t^{(1)}\right] = \mathrm{Var}\left[S_t^{(1)}\right] + \mathrm{Var}\left[S_t^{(2)}\right] - 2\,\mathrm{Cov}\big(S_t^{(1)}, S_t^{(2)}\big)$$

with

$$\begin{aligned}\mathbb{E}\left[S_t^{(1)} S_t^{(2)}\right] &= \mathbb{E}\left[S_0^{(1)} S_0^{(2)} \mathrm{e}^{2\mu t+\sigma_1 W_t^{(1)}-\sigma_1^2 t/2+\sigma_2 W_t^{(2)}-\sigma_2^2 t/2}\right] \\ &= S_0^{(1)} S_0^{(2)} \mathrm{e}^{2\mu t-\sigma_1^2 t/2-\sigma_2^2 t/2}\, \mathbb{E}\left[\mathrm{e}^{\sigma_1 W_t^{(1)}+\sigma_2 W_t^{(2)}}\right] \\ &= S_0^{(1)} S_0^{(2)} \mathrm{e}^{2\mu t-\sigma_1^2 t/2-\sigma_2^2 t/2} \exp\left(\frac{1}{2}\,\mathbb{E}\left[\big(\sigma_1 W_t^{(1)} + \sigma_2 W_t^{(2)}\big)^2\right]\right),\end{aligned}$$

with

$$\begin{aligned}\mathbb{E}\left[\big(\sigma_1 W_t^{(1)} + \sigma_2 W_t^{(2)}\big)^2\right] &= \mathbb{E}\left[\big(\sigma_1 W_t^{(1)}\big)^2\right] + 2\,\mathbb{E}\left[\sigma_1 W_t^{(1)} \sigma_2 W_t^{(2)}\right] + \mathbb{E}\left[\big(\sigma_2 W_t^{(2)}\big)^2\right] \\ &= \sigma_1^2 t + 2\rho\sigma_1\sigma_2 t + \sigma_2^2 t,\end{aligned}$$

hence

$$\mathbb{E}\left[S_t^{(1)} S_t^{(2)}\right] = S_0^{(1)} S_0^{(2)} \mathrm{e}^{2\mu t+\rho\sigma_1\sigma_2 t},$$

and

$$\mathrm{Cov}\left(S_t^{(1)}, S_t^{(2)}\right) = \mathbb{E}\left[S_t^{(1)} S_t^{(2)}\right] - \mathbb{E}\left[S_t^{(1)}\right] \mathbb{E}\left[S_t^{(2)}\right] = S_0^{(1)} S_0^{(2)} \mathrm{e}^{2\mu t}(\mathrm{e}^{\rho\sigma_1\sigma_2 t} - 1),$$

and therefore

$$\begin{aligned}
&\mathrm{Var}\left[S_t^{(2)} - S_t^{(1)}\right] \\
&= \left(S_0^{(1)}\right)^2 \mathrm{e}^{2\mu t}(\mathrm{e}^{\sigma_1^2 t} - 1) + \left(S_0^{(2)}\right)^2 \mathrm{e}^{2\mu t}(\mathrm{e}^{\sigma_2^2 t} - 1) - 2 S_0^{(1)} S_0^{(2)} \mathrm{e}^{2\mu t}(\mathrm{e}^{\rho\sigma_1\sigma_2 t} - 1) \\
&= \mathrm{e}^{2\mu t}\left(\left(S_0^{(1)}\right)^2 \mathrm{e}^{\sigma_1^2 t} + \left(S_0^{(2)}\right)^2 \mathrm{e}^{\sigma_2^2 t} - 2 S_0^{(1)} S_0^{(2)} \mathrm{e}^{\rho\sigma_1\sigma_2 t} - \left(S_0^{(2)} - S_0^{(1)}\right)^2\right).
\end{aligned}$$

Exercise 5.9 Letting $X_t := f(t)\mathrm{e}^{\sigma B_t - \sigma^2 t/2}$ and noting the relation

$$d\mathrm{e}^{\sigma B_t - \sigma^2 t/2} = \sigma f(t) \mathrm{e}^{\sigma B_t - \sigma^2 t/2} dB_t, \qquad t \geq 0,$$

see Proposition 5.15 and Relation (5.22) with $\mu = 0$, we have

$$\begin{aligned}
dX_t &= \mathrm{e}^{\sigma B_t - \sigma^2 t/2} f'(t) dt + f(t) d\mathrm{e}^{\sigma B_t - \sigma^2 t/2} \\
&= \mathrm{e}^{\sigma B_t - \sigma^2 t/2} f'(t) dt + \sigma f(t) \mathrm{e}^{\sigma B_t - \sigma^2 t/2} dB_t \\
&= \frac{f'(t)}{f(t)} X_t dt + \sigma X_t dB_t \\
&= h(t) X_t dt + \sigma X_t dB_t,
\end{aligned}$$

hence

$$\frac{d}{dt} \log f(t) = \frac{f'(t)}{f(t)} = h(t),$$

which shows that

$$\log f(t) = \log f(0) + \int_0^t h(s) ds,$$

and

$$\begin{aligned}
X_t &= f(t)\mathrm{e}^{\sigma B_t - \sigma^2 t/2} \\
&= f(0) \exp\left(\int_0^t h(s) ds + \sigma B_t - \frac{\sigma^2}{2} t\right) \\
&= X_0 \exp\left(\int_0^t h(s) ds + \sigma B_t - \frac{\sigma^2}{2} t\right), \qquad t \geq 0.
\end{aligned}$$

Exercise 5.10

a) We have

$$\begin{aligned}
S_t &= \mathrm{e}^{X_t} \\
&= \mathrm{e}^{X_0} + \int_0^t u_s \mathrm{e}^{X_s} dB_s + \int_0^t v_s \mathrm{e}^{X_s} ds + \frac{1}{2} \int_0^t u_s^2 \mathrm{e}^{X_s} ds \\
&= \mathrm{e}^{X_0} + \sigma \int_0^t \mathrm{e}^{X_s} dB_s + \nu \int_0^t \mathrm{e}^{X_s} ds + \frac{\sigma^2}{2} \int_0^t \mathrm{e}^{X_s} ds
\end{aligned}$$

$$= S_0 + \sigma \int_0^t S_s dB_s + \nu \int_0^t S_s ds + \frac{\sigma^2}{2} \int_0^t S_s ds.$$

b) Let $r > 0$. The process $(S_t)_{t\in\mathbb{R}_+}$ satisfies the stochastic differential equation

$$dS_t = rS_t dt + \sigma S_t dB_t$$

when $r = \nu + \sigma^2/2$.

c) We have

$$\mathrm{Var}[X_t] = \mathrm{Var}[(B_T - B_t)\sigma] = \sigma^2 \,\mathrm{Var}[B_T - B_t] = (T-t)\sigma^2, \quad t \in [0,T].$$

d) Let the process $(S_t)_{t\in\mathbb{R}_+}$ be defined by $S_t = S_0 \mathrm{e}^{\sigma B_t + \nu t}$, $t \geq 0$. Using the time splitting decomposition

$$S_T = S_t \frac{S_T}{S_t} = S_t \mathrm{e}^{(B_T - B_t)\sigma + \nu\tau},$$

we have

$$\begin{aligned}
\mathbb{P}(S_T > K \mid S_t = x) &= \mathbb{P}(S_t \mathrm{e}^{(B_T-B_t)\sigma + (T-t)\nu} > K \mid S_t = x) \\
&= \mathbb{P}(x\mathrm{e}^{(B_T-B_t)\sigma + (T-t)\nu} > K) \\
&= \mathbb{P}(\mathrm{e}^{(B_T-B_t)\sigma} > K\mathrm{e}^{-(T-t)\nu}/x) \\
&= \mathbb{P}\left(\frac{B_T - B_t}{\sqrt{T-t}} > \frac{1}{\sigma\sqrt{T-t}} \log\left(K\mathrm{e}^{-(T-t)\nu}/x\right)\right) \\
&= 1 - \Phi\left(\frac{\log\left(K\mathrm{e}^{-(T-t)\nu}/x\right)}{\sigma\sqrt{\tau}}\right) \\
&= \Phi\left(-\frac{\log\left(K\mathrm{e}^{-(T-t)\nu}/x\right)}{\sigma\sqrt{\tau}}\right) \\
&= \Phi\left(\frac{\log(x/K) + \nu\tau}{\sigma\sqrt{\tau}}\right),
\end{aligned}$$

where $\tau = T - t$.

Problem 5.11 (Exercise 4.19 continued).

a) The option payoff is $(B_T - K)^+$ at maturity.
b) We can ignore what happens between two crossings as every crossing resets the portfolio to its state right before the previous crossing. Based on this, It is clear that every of the four possible scenarios will lead to a portfolio value $(B_T - K)^+$ at maturity:

    i) If $B_0 < 1$ and $B_T < 1$ we issue the option for free and finish with an empty portfolio and zero payoff.

ii) If $B_0 < 1$ and $B_T > 1$ we issue the option for free and finish with one AUD and one SGD to refund, which yields the payoff $B_T - 1 = (B_T - 1)^+$.

iii) If $B_0 > 1$ and $B_T < 1$ we purchase one AUD and borrow one SGD at the start, however the AUD will be sold and the SGD refunded before maturity, resulting into an empty portfolio and zero payoff.

iv) If $B_0 > 1$ and $B_T > 1$ we purchase one AUD and borrow one SGD right before maturity, which yields the payoff $B_T - 1 = (B_T - 1)^+$.

Therefore we are hedging the option in all cases. Note that $\mathbb{P}(B_T = K) = 0$ so the case $B_T = 1$ can be ignored with probability one.

c) Since the portfolio strategy is to hold AU\$1 when $B_t > K$ and to and borrow SG\$1 when $B_t > K$, we let

$$\xi_t := \mathbb{1}_{(K,\infty)}(B_t) \quad \text{and} \quad \eta_t := -\mathbb{1}_{(K,\infty)}(B_t), \qquad t \in [0,T],$$

which is called a *stop-loss/start-gain* strategy.

d) Noting that $\int_0^t \eta_s dA_s = 0$ because $A_t = A_0$ is constant, $t \in [0,T]$, we find by the Itô-Tanaka formula (4.48) that

$$\begin{aligned}\int_0^t \eta_s dA_s + \int_0^t \xi_s dB_s &= \int_0^T \mathbb{1}_{[K,\infty)}(B_t) dB_t \\ &= (B_T - K)^+ - (B_0 - K)^+ - \frac{1}{2}\mathcal{L}^K_{[0,T]}.\end{aligned}$$

e) Question (d) shows that

$$(B_T - K)^+ = (B_0 - K)^+ + \int_0^t \eta_s dA_s + \int_0^t \xi_s dB_s + \frac{1}{2}\mathcal{L}^K_{[0,T]},$$

i.e. the initial premium $(B_0 - K)^+$ plus the sum of portfolio profits and losses is not sufficient to cover the terminal payoff $(B_T - K)^+$, and that we fall short of this by the positive amount $\frac{1}{2}\mathcal{L}^K_{[0,T]} > 0$. Therefore the portfolio allocation $(\xi_t, \eta_t)_{t\in[0,T]}$ is *not* self-financing.

Additional comments:
The stop-loss/start-gain strategy described here is difficult to implement in practice because it would require infinitely many transactions when Brownian motion crosses the level K, as illustrated in Figure S.16.

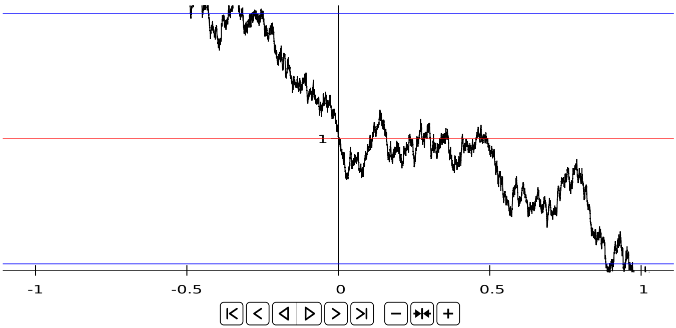


Fig. S.16: Brownian crossings of level 1.*

The arbitrage-free price of the option can in fact be computed as the expected discounted option payoff

$$
\begin{aligned}
\pi_t &= \mathrm{e}^{-(T-t)r}\,\mathbb{E}^*[(B_T-K)^+ \mid \mathcal{F}_t] \\
&= \mathrm{e}^{-(T-t)r}\,\mathbb{E}^*[(B_T-B_t+x-K)^+ \mid \mathcal{F}_t]_{x=B_t} \\
&= \mathrm{e}^{-(T-t)r}\,\mathbb{E}^*[(B_T-B_t+x-K)^+]_{x=B_t} \\
&= \mathrm{e}^{-(T-t)r}\int_{-\infty}^{\infty}(y+B_t-K)^+\mathrm{e}^{-y^2/(2(T-t))}\frac{dy}{\sqrt{2\pi(T-t)}} \\
&= \mathrm{e}^{-(T-t)r}\int_{K-B_t}^{\infty}(y+B_t-K)\mathrm{e}^{-y^2/(2(T-t))}\frac{dy}{\sqrt{2\pi(T-t)}} \\
&= \mathrm{e}^{-(T-t)r}\int_{K-B_t}^{\infty} y\mathrm{e}^{-y^2/(2(T-t))}\frac{dy}{\sqrt{2\pi(T-t)}} \\
&\quad +(B_t-K)\mathrm{e}^{-(T-t)r}\int_{K-B_t}^{\infty}\mathrm{e}^{-y^2/(2(T-t))}\frac{dy}{\sqrt{2\pi(T-t)}} \\
&= \mathrm{e}^{-(T-t)r}\int_{(K-B_t)/\sqrt{T-t}}^{\infty} y\mathrm{e}^{-y^2/2}\frac{dy}{\sqrt{2\pi}} \\
&\quad +(B_t-K)\mathrm{e}^{-(T-t)r}\int_{(K-B_t)/\sqrt{T-t}}^{\infty}\mathrm{e}^{-y^2/2}\frac{dy}{\sqrt{2\pi}} \\
&= \frac{\mathrm{e}^{-(T-t)r}}{\sqrt{2\pi}}\left[-\mathrm{e}^{-y^2/2}\right]_{(K-B_t)/\sqrt{T-t}}^{\infty} \\
&\quad +(B_t-K)\mathrm{e}^{-(T-t)r}\Phi\left(\frac{B_t-K}{\sqrt{T-t}}\right) \\
&= \frac{\mathrm{e}^{-(T-t)r}}{\sqrt{2\pi}}\mathrm{e}^{-(K-B_t)^2/(2(T-t))}
\end{aligned}
$$

$$
\begin{aligned}
&+(B_t-K)\mathrm{e}^{-(T-t)r}\Phi\left(\frac{B_t-K}{\sqrt{T-t}}\right)\\
&=: g(t,B_t),
\end{aligned}
$$

where the function

$$
g(t,x) := \frac{\mathrm{e}^{-(T-t)r}}{\sqrt{2\pi}}\mathrm{e}^{-(K-x)^2/(2(T-t))}+(x-K)\mathrm{e}^{-(T-t)r}\Phi\left(\frac{x-K}{\sqrt{T-t}}\right), \quad t\in[0,T),
$$

solves the Black-Scholes heat equation

$$
\frac{\partial g}{\partial t}(t,x)+r\frac{\partial g}{\partial x}(t,x)+\frac{1}{2}\frac{\partial^2 g}{\partial^2 x}(t,x)=0
$$

with terminal condition $g(T,x)=(x-K)^+$. The Delta gives the amount to be invested in AUD at time t and is given by

$$
\begin{aligned}
\xi_t &= \frac{\partial g}{\partial x}(t,B_t)\\
&=(K-B_t)\frac{\mathrm{e}^{-(T-t)r}}{\sqrt{2\pi}(T-t)}\mathrm{e}^{-(K-B_t)^2/(2(T-t))}\\
&+(B_t-K)\frac{\mathrm{e}^{-(T-t)r}}{\sqrt{2\pi(T-t)}}\mathrm{e}^{-(B_t-K)^2/(2(T-t))}+\mathrm{e}^{-(T-t)r}\Phi\left(\frac{B_t-K}{\sqrt{T-t}}\right)\\
&=\mathrm{e}^{-(T-t)r}\Phi\left(\frac{B_t-K}{\sqrt{(T-t)}}\right)\\
&=: h(t,B_t),
\end{aligned}
$$

with

$$
h(t,x):=\mathrm{e}^{-(T-t)r}\Phi\left(\frac{x-K}{\sqrt{T-t}}\right), \qquad t\in[0,T),
$$

and $h(T,x)=\mathbb{1}_{[K,\infty)}(x)$.

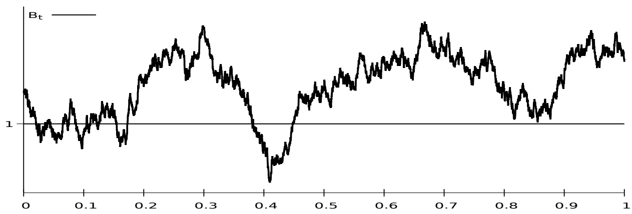


Fig. S.17: Brownian path started at $B_0>1$.

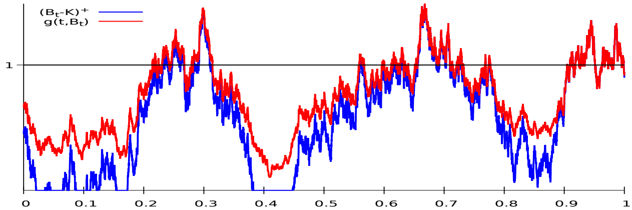


Fig. S.18: Risk-neutral pricing of the FX option by $\pi_t(B_t) = g(t, B_t)$ *vs* stop-loss/start-gain pricing.

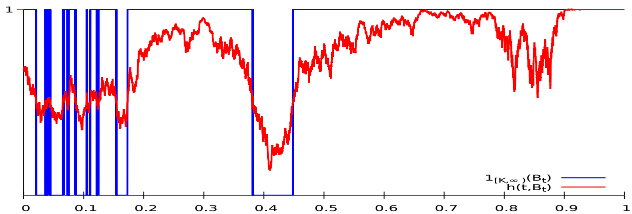


Fig. S.19: Delta hedging of the FX option by $\xi_t = h(t, B_t)$ *vs* the stop-loss/start-gain strategy.

The "one or nothing" stop-loss/start-gain strategy is not self-financing because in practice there is an impossibility to buy/sell the AUD at exactly SGD1.00 to the existence of an *order book* that generates a gap between bid/ask prices as in the sample of Figure S.20 with $383.16964 < 384.07141$.

Order Book (XBT/USD)

Buying Volume	Buying Price	Selling Price	Selling Volume
0.868	$383.16964	$384.07141	6.000
1.731	$382.50010	$384.07142	3.600
1.942	$382.50000	$384.93987	1.010
0.020	$382.01074	$384.95984	1.000
9.175	$381.77000	$384.95986	1.000
13.079	$380.79000	$384.95988	1.010
2.220	$380.48640	$385.04889	20.352
22.736	$379.80742	$385.99981	21.723
0.115	$379.61141	$386.81846	22.145
21.804	$378.98301	$386.97140	0.020
8.320	$378.95708	$387.70357	19.811

Fig. S.20: Bitcoin XBT/USD order book.

The existence of the order book will force buying and selling within a certain *range* $[K-\varepsilon, K+\varepsilon]$, typically resulting into selling lower than $K = 1.00$ and buying higher than $K = 1.00$. This potentially results into a *trading loss* that can be proportional to the time

$$\ell\left(\left\{t \in [0,T] \ : \ K-\varepsilon < B_t < K+\varepsilon\right\}\right)$$

spent by the exchange rate $(B_t)_{t\in[0,T]}$ within the range $[K-\varepsilon, K+\varepsilon]$.

The Itô-Tanaka formula (4.48)

$$(B_T - K)^+ = (B_0 - K)^+ + \int_0^T \mathbb{1}_{[K,\infty)}(B_t)dB_t + \frac{1}{2}\mathcal{L}^K_{[0,T]},$$

precisely shows that the trading loss equals half the *local time* $\mathcal{L}^K_{[0,T]}$ spent by $(B_t)_{t\in[0,T]}$ at the level K. When ε is small we have

$$\frac{1}{2}\mathcal{L}^K_{[0,T]} \simeq \frac{1}{4\varepsilon}\ell(\{t \in [0,T] \ : \ K-\varepsilon < B_t < K+\varepsilon\}),$$

therefore the trading loss is proportional to the time spent by Brownian motion $(B_t)_{t\in\mathbb{R}_+}$ within the interval $(K-\varepsilon, K+\varepsilon)$, with proportionality coefficient $1/(4\varepsilon)$.

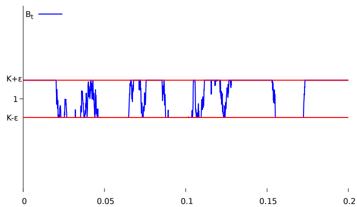


Fig. S.21: Time spent by Brownian motion within the range $(K-\varepsilon, K+\varepsilon)$.

More generally, one could show that there is no self-financing (buy and hold) portfolio that can remain constant over time intervals, and that the self-financing portfolio has to be constantly re-adjusted in time as illustrated in Figure S.19. This invalidates the stop-loss/start-gain strategy as a self-financing portfolio strategy.

Chapter 6

Exercise 6.1

a) By the Itô formula, we have

$$dV_t = dg(t, B_t) = \frac{\partial g}{\partial t}(t, B_t)dt + \frac{\partial g}{\partial x}(t, B_t)dB_t + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(t, B_t)dt. \quad \text{(S.6.21)}$$

Consider a hedging portfolio with value $V_t = \eta_t A_t + \xi_t B_t$, satisfying the self-financing condition

$$dV_t = \eta_t dA_t + \xi_t dB_t = \xi_t dB_t, \quad t \geq 0. \quad \text{(S.6.22)}$$

By respective identification of the terms in dB_t and dt in (S.6.21) and (S.6.22) we get

$$\begin{cases} 0 = \dfrac{\partial g}{\partial t}(t, B_t)dt + \dfrac{1}{2}\dfrac{\partial^2 g}{\partial x^2}(t, B_t)dt, \\ \\ \xi_t dB_t = \dfrac{\partial g}{\partial x}(t, B_t)dB_t, \end{cases}$$

hence

$$\begin{cases} 0 = \dfrac{\partial g}{\partial t}(t, B_t) + \dfrac{1}{2}\dfrac{\partial^2 g}{\partial x^2}(t, B_t), \\ \\ \xi_t = \dfrac{\partial g}{\partial x}(t, B_t), \end{cases}$$

and

$$\begin{cases} 0 = \dfrac{\partial g}{\partial t}(t, B_t) + \dfrac{1}{2}\dfrac{\partial^2 g}{\partial x^2}(t, B_t), \\ \\ \xi_t = \dfrac{\partial g}{\partial x}(t, B_t), \end{cases}$$

hence the function $g(t, x)$ satisfies the heat equation

$$0 = \frac{\partial g}{\partial t}(t, x) + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(t, x), \quad x > 0, \quad \text{(S.6.23)}$$

with terminal condition $g(T, x) = x^2$, and ξ_t is given by the partial derivative

$$\xi_t = \frac{\partial g}{\partial x}(t, B_t), \qquad t \geq 0.$$

b) In order to solve (S.6.23) we substitute a solution of the form $g(t, x) = x^2 + f(t)$ in to the partial differential equation, which yields $1 + f'(t) = 0$ with the terminal condition $f(T) = 0$. Therefore we have $f(T-t) = T-t$, and

$$g(t, x) = x^2 + f(t) = x^2 + T - t, \qquad 0 \leq t \leq T.$$

c) By (6.3), we have

$$\xi_t = \xi_t(B_t) = \frac{\partial g}{\partial x}(t, B_t) = 2B_t, \qquad 0 \le t \le T,$$

which recovers the value of ξ_t found page 163 in the power option example. We also have

$$\eta_t A_t = \eta_t A_0 = g(t, B_t) - \xi_t B_t = T - t - B_t^2, \qquad 0 \le t \le T.$$

Exercise 6.2 By the Itô formula, we have

$$\begin{aligned} dV_t &= dg(t, S_t) \qquad\qquad (S.6.24) \\ &= \frac{\partial g}{\partial t}(t, S_t)dt + \beta(\alpha - S_t)\frac{\partial g}{\partial x}(t, S_t)dt + \frac{1}{2}\sigma^2 S_t \frac{\partial^2 g}{\partial x^2}(t, S_t)dt + \sigma\sqrt{S_t}\frac{\partial g}{\partial x}(t, S_t)dB_t. \end{aligned}$$

By respective identification of the terms in dB_t and dt in (6.36) and (S.6.24) we get

$$\begin{cases} rg(t, S_t)dt + \beta(\alpha - S_t)\xi_t dt - r\xi_t S_t dt \\ \qquad = \dfrac{\partial g}{\partial t}(t, S_t)dt + \beta(\alpha - S_t)\dfrac{\partial g}{\partial x}(t, S_t)dt + \dfrac{1}{2}\sigma^2 S_t \dfrac{\partial^2 g}{\partial x^2}(t, S_t)dt, \\ \sigma\xi_t\sqrt{S_t}dB_t = \sigma\sqrt{S_t}\dfrac{\partial g}{\partial x}(t, S_t)dB_t, \end{cases}$$

hence

$$\begin{cases} rg(t, S_t) + \beta(\alpha - S_t)\xi_t - r\xi_t S_t = \dfrac{\partial g}{\partial t}(t, S_t) + \beta(\alpha - S_t)\dfrac{\partial g}{\partial x}(t, S_t) + \dfrac{1}{2}\sigma^2 S_t \dfrac{\partial^2 g}{\partial x^2}(t, S_t), \\ \xi_t = \dfrac{\partial g}{\partial x}(t, S_t), \end{cases}$$

and

$$\begin{cases} rg(t, S_t) - r\xi_t S_t = \dfrac{\partial g}{\partial t}(t, S_t) + \dfrac{1}{2}\sigma^2 S_t \dfrac{\partial^2 g}{\partial x^2}(t, S_t), \\ \xi_t = \dfrac{\partial g}{\partial x}(t, S_t), \end{cases}$$

hence the function $g(t, x)$ satisfies the PDE

$$rg(t, x) = \frac{\partial g}{\partial t}(t, x) + rx\frac{\partial g}{\partial x}(t, x) + \frac{1}{2}\sigma^2 x \frac{\partial^2 g}{\partial x^2}(t, x), \quad x > 0,$$

and ξ_t is given by the partial derivative

$$\xi_t = \frac{\partial g}{\partial x}(t, S_t), \qquad t \ge 0.$$

Exercise 6.3

a) Let $V_t := \xi_t S_t + \eta_t A_t$ denote the hedging portfolio value at time $t \in [0, T]$. Since the dividend yield δS_t per share is continuously reinvested in the portfolio, the portfolio change dV_t decomposes as

$$\begin{aligned} dV_t &= \underbrace{\eta_t dA_t + \xi_t dS_t}_{\text{trading profit and loss}} + \underbrace{\delta \xi_t S_t dt}_{\text{dividend payout}} \\ &= r\eta_t A_t dt + \xi_t((\mu - \delta) S_t dt + \sigma S_t dB_t) + \delta \xi_t S_t dt \\ &= r\eta_t A_t dt + \xi_t(\mu S_t dt + \sigma S_t dB_t) \\ &= r V_t dt + (\mu - r)\xi_t S_t dt + \sigma \xi_t S_t dB_t, \qquad t \geq 0. \end{aligned}$$

b) By Itô's formula we have

$$\begin{aligned} dg(t, S_t) &= \frac{\partial g}{\partial t}(t, S_t) dt + (\mu - \delta) S_t \frac{\partial g}{\partial x}(t, S_t) dt \\ &\quad + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 g}{\partial x^2}(t, S_t) dt + \sigma S_t \frac{\partial g}{\partial x}(t, S_t) dB_t, \end{aligned}$$

hence by identification of the terms in dB_t and dt in the expressions of dV_t and $dg(t, S_t)$, we get

$$\xi_t = \frac{\partial g}{\partial x}(t, S_t),$$

and we derive the Black-Scholes PDE with dividend

$$rg(t, x) = \frac{\partial g}{\partial t}(t, x) + (r - \delta) x \frac{\partial g}{\partial x}(t, x) + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 g}{\partial x^2}(t, x). \tag{S.6.25}$$

c) In order to solve (S.6.25) we note that, letting $f(t, x) := \mathrm{e}^{(T-t)\delta} g(t, x)$ and substituting $g(t, x) = \mathrm{e}^{-(T-t)\delta} f(t, x)$ into the PDE (S.6.25), we have

$$rf(t, x) = \delta f(t, x) + \frac{\partial f}{\partial t}(t, x) + (r - \delta) x \frac{\partial f}{\partial x}(t, x) + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 f}{\partial x^2}(t, x),$$

hence $f(t, x) := \mathrm{e}^{(T-t)\delta} g(t, x)$, satisfies the standard Black-Scholes PDE with interest rate $r - \delta$, *i.e.* we have

$$(r - \delta) f(t, x) = \frac{\partial f}{\partial t}(t, x) + (r - \delta) x \frac{\partial f}{\partial x}(t, x) + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 f}{\partial x^2}(t, x),$$

with same terminal condition $f(T, x) = g(T, x) = (x - K)^+$, hence we have

$$\begin{aligned} f(t, x) &= \mathrm{Bl}(x, K, \sigma, r - \delta, T - t) \\ &= x\Phi\big(d_+^\delta(T - t)\big) - K\mathrm{e}^{-(r-\delta)(T-t)}\Phi\big(d_-^\delta(T - t)\big), \end{aligned}$$

where

$$d_\pm^\delta(T - t) := \frac{\log(x/K) + (r - \delta \pm \sigma^2/2)(T - t)}{\sigma\sqrt{T - t}}.$$

Consequently, the pricing function of the European call option with dividend rate δ is

$$\begin{aligned} g(t,x) &= \mathrm{e}^{-(T-t)\delta} f(t,x) \\ &= \mathrm{e}^{-(T-t)\delta} \mathrm{Bl}(x, K, \sigma, r-\delta, T-t) \\ &= x\mathrm{e}^{-(T-t)\delta} \Phi\big(d_+^\delta(T-t)\big) - K\mathrm{e}^{-(T-t)r} \Phi\big(d_-^\delta(T-t)\big), \quad 0 \le t \le T. \end{aligned}$$

We also have

$$\begin{aligned} g(t,x) &= \mathrm{Bl}\big(x\mathrm{e}^{-(T-t)\delta}, K, \sigma, r, T-t\big) \\ &= \mathrm{e}^{-(T-t)\delta} \mathrm{Bl}\big(x, K\mathrm{e}^{(T-t)\delta}, \sigma, r, T-t\big), \qquad 0 \le t \le T. \end{aligned}$$

d) As in Proposition 6.4, we have

$$\xi_t = \frac{\partial g}{\partial x}(t, S_t) = \mathrm{e}^{-(T-t)\delta} \Phi\big(d_+^\delta(T-t)\big), \qquad 0 \le t < T.$$

Exercise 6.4

a) We check that $g_{\mathrm{c}}(t,0) = 0$, as when $x = 0$ we have $d_+(T-t) = d_-(T-t) = -\infty$ for all $t \in [0,T)$. On the other hand, we have

$$\lim_{t \nearrow T} d_+(T-t) = \lim_{t \nearrow T} d_-(T-t) = \begin{cases} +\infty, \ x > K, \\ 0, \qquad x = K, \\ -\infty, \ x < K, \end{cases}$$

which allows us to recover the boundary condition

$$\begin{aligned} g_{\mathrm{c}}(T,x) &= \lim_{t \nearrow T} g_{\mathrm{c}}(t,x) \\ &= \left\{ \begin{array}{ll} x\Phi(+\infty) - K\Phi(+\infty) = x - K, & x > K \\ \dfrac{x}{2} - \dfrac{K}{2} = 0, & x = K \\ x\Phi(-\infty) - K\Phi(-\infty) = 0, & x < K \end{array} \right\} = (x-K)^+ \end{aligned}$$

at $t = T$. Regarding the Delta of the European call option, we find

$$\lim_{t \nearrow T} \Phi(d_+(T-t)) = \left\{ \begin{array}{ll} \Phi(+\infty) = 1, & x > K \\ \Phi(0) = \dfrac{1}{2}, & x = K \\ \Phi(-\infty) = 0, & x < K \end{array} \right\}$$

see Figure 6.5. Similarly, we can check that

$$\lim_{T\to\infty} d_-(T-t) = \begin{cases} +\infty, & r > \dfrac{\sigma^2}{2}, \\ \\ 0, & r = \dfrac{\sigma^2}{2}, \\ \\ -\infty, & r < \dfrac{\sigma^2}{2}, \end{cases}$$

and $\lim_{T\to\infty} d_+(T-t) = +\infty$, hence

$$\begin{aligned} &\lim_{T\to\infty} \mathrm{Bl}(x, K, \sigma, r, T-t) \\ &= x \lim_{T\to\infty} \Phi(d_+(T-t)) - \lim_{T\to\infty} \left(\mathrm{e}^{-(T-t)r} \Phi(d_-(T-t))\right) \\ &= x, \qquad t \geq 0. \end{aligned}$$

b) We check that $g_{\mathrm{p}}(t,0) = K\mathrm{e}^{-(T-t)r}$ and $g_{\mathrm{p}}(t,\infty) = 0$ as when $x = 0$ we have $d_+(T-t) = d_-(T-t) = -\infty$ and as x tends to infinity we have $d_+(T-t) = d_-(T-t) = +\infty$ for all $t \in [0,T)$. On the other hand, we have

$$g_{\mathrm{p}}(T,x) = \left\{ \begin{array}{ll} K\Phi(+\infty) - x\Phi(+\infty) = K - x, & x < K \\ \\ \dfrac{K}{2} - \dfrac{x}{2} = 0, & x = K \\ \\ K\Phi(-\infty) - x\Phi(-\infty) = 0, & x > K \end{array} \right\} = (K-x)^+$$

at $t = T$. Regarding the Delta of the European put option, we find

$$-\lim_{t\nearrow T} \Phi(-d_+(T-t)) = \left\{ \begin{array}{ll} \Phi(-\infty) = 0, & x > K \\ \\ -\Phi(0) = -\dfrac{1}{2}, & x = K \\ \\ -\Phi(+\infty) = -1, & x < K \end{array} \right\}$$

see Figure 6.11. Similarly, we can check that

$$\lim_{T\to\infty} d_-(T-t) = \begin{cases} +\infty, & r > \dfrac{\sigma^2}{2}, \\ 0, & r = \dfrac{\sigma^2}{2}, \\ -\infty, & r < \dfrac{\sigma^2}{2}, \end{cases}$$

and $\lim_{T\to\infty} d_+(T-t) = +\infty$, hence

$$\lim_{T\to\infty} \mathrm{Bl}_{\mathrm{p}}(x, K, \sigma, r, T-t) = 0, \qquad t \geq 0.$$

Exercise 6.5 (Exercise 3.14 continued).

a) Substituting $g(x,t) = x^2 f(t)$ in (6.37), we find $f'(t) = -(r+\sigma^2)f(t)$, hence

$$f(t) = f(0)\mathrm{e}^{-(r+\sigma^2)t} = f(T)\mathrm{e}^{(r+\sigma^2)(T-t)},$$

hence $g(x,t) = f(T)x^2\mathrm{e}^{(r+\sigma^2)(T-t)} = x^2\mathrm{e}^{(r+\sigma^2)(T-t)}$ due to the terminal condition $g(x,T) = x^2$.

b) We have $\xi_t = \dfrac{\partial g}{\partial x} g(S_t, t) = 2S_t\mathrm{e}^{(r+\sigma^2)(T-t)}$, and

$$\begin{aligned} \eta_t &= \frac{1}{A_t}\left(g(S_t,t) - \xi_t S_t\right) \\ &= \frac{1}{A_0 e^{rt}}\left(S_t^2\mathrm{e}^{(r+\sigma^2)(T-t)} - 2S_t^2\mathrm{e}^{(r+\sigma^2)(T-t)}\right) \\ &= -\frac{S_t^2}{A_0}\mathrm{e}^{(T-2t)r+(T-t)\sigma^2}, \qquad t \in [0,T]. \end{aligned}$$

Exercise 6.6

a) Counting approximately 46 days to maturity, we have

$$\begin{aligned} d_-(T-t) &= \frac{(r-\sigma^2/2)(T-t) + \log(S_t/K)}{\sigma\sqrt{T-t}} \\ &= \frac{(0.04377 - (0.9)^2/2)(46/365) + \log(17.2/36.08)}{0.9\sqrt{46/365}} \\ &= -2.461179058, \end{aligned}$$

and

$$d_+(T-t) = d_-(T-t) + 0.9\sqrt{46/365} = -2.14167602.$$

From the standard Gaussian cumulative distribution table we get

$$\Phi(d_+(T-t)) = \Phi(-2.14) = 0.0161098$$

and

$$\Phi(d_-(T-t)) = \Phi(-2.46) = 0.00692406,$$

hence

$$\begin{aligned} f(t,S_t) &= S_t\Phi(d_+(T-t)) - K\mathrm{e}^{-(T-t)r}\Phi(d_-(T-t)) \\ &= 17.2 \times 0.0161098 - 36.08 \times \mathrm{e}^{-0.04377\times 46/365} \times 0.00692406 \\ &= \text{HK\$ } 0.028642744. \end{aligned}$$

For comparison, running the corresponding Black-Scholes R script of Figure 6.20 yields

$$\mathrm{BSCall}(17.2, 36.08, 0.04377, 46/365, 0.9) = 0.02864235.$$

b) We have

$$\eta_t = \frac{\partial f}{\partial x}(t,S_t) = \Phi(d_+(T-t)) = \Phi(-2.14) = 0.0161098, \qquad \text{(S.6.26)}$$

hence one should only hold a fractional quantity equal to 16.10 units in the risky asset in order to hedge 1000 such call options when $\sigma = 0.90$.

c) From the curve it turns out that when $f(t,S_t) = 10 \times 0.023 = \text{HK\$ } 0.23$, the volatility σ is approximately equal to $\sigma = 122\%$.

This approximate value of implied volatility can be found under the column "Implied Volatility (IV.)" on this set of market data from the Hong Kong Stock Exchange:

Updated: 6 November 2008

Basic Data

DW Code	Issuer	UL	Call /Put	DW Type	Listing (D-M-Y)	Maturity (D-M-Y)	Strike	Entitle-ment Ratio^
01897	FB	00066	Call	Standard	18-12-2007	23-12-2008	36.08	10

Market Data

Total Issue Size	O/S (%)	Delta (%)	IV. (%)	Day High ($)	Day Low ($)	Closing Price # ($)	T/O ('000)	UL Price ($)
138,000,000	16.43	0.780	125.375	0.000	0.000	0.023	0	17.200

Fig. S.22: Market data for the warrant #01897 on the MTR Corporation.

Remark: a typical value for the volatility in standard market conditions would be around 20%. The observed volatility value $\sigma = 1.22$ per year is actually quite high.

Exercise 6.7

a) We find $h(x) = x - K$.
b) Letting $g(t,x)$, the PDE rewrites as

$$(x - \alpha(t))r = -\alpha'(t) + rx,$$

hence $\alpha(t) = \alpha(0)\mathrm{e}^{rt}$ and $g(t,x) = x - \alpha(0)\mathrm{e}^{rt}$. The final condition

$$g(T,x) = h(x) = x - K$$

yields $\alpha(0) = K\mathrm{e}^{-rT}$ and $g(t,x) = x - K\mathrm{e}^{-(T-t)r}$.
c) We have

$$\xi_t = \frac{\partial g}{\partial x}(t, S_t) = 1,$$

hence

$$\eta_t = \frac{V_t - \xi_t S_t}{A_t} = \frac{g(t,S_t) - S_t}{A_t} = \frac{S_t - K\mathrm{e}^{-(T-t)r} - S_t}{A_t} = -K\mathrm{e}^{-rT}.$$

Note that we could also have directly used the identification

$$V_t = g(S_t,t) = S_t - K\mathrm{e}^{-(T-t)r} = S_t - K\mathrm{e}^{-rT}A_t = \xi_t S_t + \eta_t A_t,$$

which immediately yields $\xi_t = 1$ and $\eta_t = -K\mathrm{e}^{-rT}$.
d) It suffices to take $K = 0$, which shows that $g(t,x) = x$, $\xi_t = 1$ and $\eta_t = 0$.

Exercise 6.8

a) We develop two approaches.

(i) By financial intuition. We need to replicate a fixed amount of \$1 at maturity T, *without risk*. For this there is no need to invest in the stock. Simply invest $g(t,S_t) := \mathrm{e}^{-(T-t)r}$ at time $t \in [0,T]$ and at maturity T you will have $g(T,S_T) = \mathrm{e}^{(T-t)r}g(t,S_t) = \$1$.

(ii) By analysis and the Black-Scholes PDE. Given the hint, we try plugging a solution of the form $g(t,x) = f(t)$, *not depending on the variable* x, into the Black-Scholes PDE (6.38). Given that here we have

$$\frac{\partial g}{\partial x}(t,x) = 0, \qquad \frac{\partial^2 g}{\partial x^2}(t,x) = 0, \qquad \text{and} \qquad \frac{\partial g}{\partial t}(t,x) = f'(t),$$

we find that the Black-Scholes PDE reduces to $rf(t) = f'(t)$ with the terminal condition $f(T) = g(T,x) = 1$. This equation has for solution $f(t) = \mathrm{e}^{-(T-t)r}$ and this is also the unique solution $g(t,x) = f(t) = \mathrm{e}^{-(T-t)r}$ of the Black-Scholes PDE (6.38) with terminal condition $g(T,x) = 1$.

b) We develop two approaches.

(i) By financial intuition. Since the terminal payoff \$1 is risk-free we do not need to invest in the risky asset, hence we should keep $\xi_t = 0$. Our portfolio value at time t becomes

$$V_t = g(t, S_t) = \mathrm{e}^{-(T-t)r} = \xi_t S_t + \eta_t A_t = \eta_t A_t$$

with $A_t = \mathrm{e}^{rt}$, so that we find $\eta_t = \mathrm{e}^{-rT}$, $t \in [0,T]$. This portfolio strategy remains constant over time, hence it is clearly self-financing.

(ii) By analysis. The Black-Scholes theory of Proposition 6.1 tells us that

$$\xi_t = \frac{\partial g}{\partial x}(t,x) = 0,$$

and

$$\eta_t = \frac{V_t - \xi_t S_t}{A_t} = \frac{V_t}{A_t} = \frac{\mathrm{e}^{-(T-t)r}}{\mathrm{e}^{rt}} = \mathrm{e}^{-rT}.$$

Exercise 6.9 Log contracts.

a) Substituting the function $g(x,t) := f(t) + \log x$ in the PDE (6.39), we have

$$0 = f'(t) + r - \frac{\sigma^2}{2},$$

hence

$$f(t) = f(0) - \left(r - \frac{\sigma^2}{2}\right) t,$$

with $f(0) = \left(r - \frac{\sigma^2}{2}\right) T$ in order to match the terminal condition $g(x,T) := \log x$, hence we have

$$g(x,t) = \left(r - \frac{\sigma^2}{2}\right)(T-t) + \log x, \qquad x > 0.$$

b) Substituting the function

$$h(x,t) := u(t) g(x,t) = u(t)\left(\left(r - \frac{\sigma^2}{2}\right)(T-t) + \log x\right)$$

in the PDE (6.39), we find $u'(t) = ru(t)$, hence $u(t) = u(0)\mathrm{e}^{rt} = \mathrm{e}^{-(T-t)r}$, with $u(T) = 1$, and we conclude to

$$h(x,t) = u(t)g(x,t) = \mathrm{e}^{-(T-t)r}\left(\left(r - \frac{\sigma^2}{2}\right)(T-t) + \log x\right),$$

$x > 0$, $t \in [0,T]$.

c) We have

$$\xi_t = \frac{\partial h}{\partial x}(t, S_t) = \frac{\mathrm{e}^{-(T-t)r}}{S_t}, \qquad 0 \le t \le T,$$

and

$$\begin{aligned}\eta_t &= \frac{1}{A_t}\big(h(t,S_t) - \xi_t S_t\big) \\ &= \frac{\mathrm{e}^{-rT}}{A_0}\left(\left(r - \frac{\sigma^2}{2}\right)(T-t) + \log x - 1\right),\end{aligned}$$

$0 \le t \le T$.

Exercise 6.10 Binary options.

a) From Proposition 6.1, the function $C_d(t,x)$ solves the Black-Scholes PDE

$$\begin{cases} rC_d(t,x) = \dfrac{\partial C_d}{\partial t}(t,x) + rx\dfrac{\partial C_d}{\partial x}(t,x) + \dfrac{1}{2}\sigma^2 x^2 \dfrac{\partial^2 C_d}{\partial x^2}(t,x), \\ C_d(T,x) = \mathbb{1}_{[K,\infty)}(x). \end{cases}$$

b) We can check by direct differentiation that the Black-Scholes PDE is satisfied by the function $C_d(t,x)$, together with the terminal condition $C_d(T,x) = \mathbb{1}_{[K,\infty)}(x)$ as t tends to T.

Exercise 6.11

a) By (4.35) we have

$$S_t = S_0 \mathrm{e}^{\alpha t} + \sigma \int_0^t \mathrm{e}^{(t-s)\alpha} dB_s.$$

b) By the self-financing condition (5.8) we have

$$\begin{aligned} dV_t &= \eta_t dA_t + \xi_t dS_t \\ &= r\eta_t A_t dt + \alpha \xi_t S_t dt + \sigma \xi_t dB_t \\ &= rV_t dt + (\alpha - r)\xi_t S_t dt + \sigma \xi_t dB_t, \end{aligned} \tag{S.6.27}$$

$t \ge 0$. Rewriting (6.41) under the form of an Itô process

$$S_t = S_0 + \int_0^t v_s ds + \int_0^t u_s dB_s, \quad t \ge 0,$$

with

$$u_t = \sigma, \quad \text{and} \quad v_t = \alpha S_t, \qquad t \geq 0,$$

the application of Itô's formula Theorem 4.24 to $V_t = C(t, S_t)$ shows that

$$\begin{aligned} dC(t, S_t) &= v_t \frac{\partial C}{\partial x}(t, S_t)dt + u_t \frac{\partial C}{\partial x}(t, S_t)dB_t \\ &\quad + \frac{\partial C}{\partial t}(t, S_t)dt + \frac{1}{2}|u_t|^2 \frac{\partial^2 C}{\partial x^2}(t, S_t)dt \\ &= \frac{\partial C}{\partial t}(t, S_t)dt + \alpha S_t \frac{\partial C}{\partial x}(t, S_t)dt + \frac{1}{2}\sigma^2 \frac{\partial^2 C}{\partial x^2}(t, S_t)dt + \sigma \frac{\partial C}{\partial x}(t, S_t)dB_t. \end{aligned} \tag{S.6.28}$$

Identifying the terms in dB_t and dt in (S.6.27) and (S.6.28) above, we get

$$\begin{cases} rC(t, S_t) = \dfrac{\partial C}{\partial t}(t, S_t) + rS_t \dfrac{\partial C}{\partial x}(t, S_t) + \dfrac{\sigma^2}{2} \dfrac{\partial^2 C}{\partial x^2}(t, S_t), \\ \xi_t = \dfrac{\partial C}{\partial x}(t, S_t), \end{cases}$$

hence the function $C(t, x)$ satisfies the usual Black-Scholes PDE

$$rC(t, x) = \frac{\partial C}{\partial t}(t, x) + rx \frac{\partial C}{\partial x}(t, x) + \frac{1}{2}\sigma^2 \frac{\partial^2 C}{\partial x^2}(t, x), \quad x > 0, \quad 0 \leq t \leq T, \tag{S.6.29}$$

with the terminal condition $C(T, x) = \mathrm{e}^x$, $x \geq 0$.

c) Based on (6.42), we compute

$$\begin{cases} \dfrac{\partial C}{\partial t}(t, x) = \left(r + xh'(t) + \dfrac{\sigma^2}{2r} h(t)h'(t) \right) C(t, x), \\ \dfrac{\partial C}{\partial x}(t, x) = h(t)C(t, x) \\ \dfrac{\partial^2 C}{\partial x^2}(t, x) = (h(t))^2 C(t, x), \end{cases}$$

hence the substitution of (6.42) into the Black-Scholes PDE (S.6.29) yields the ordinary differential equation

$$xh'(t) + \frac{\sigma^2}{2r} h'(t)h(t) + rxh(t) + \frac{\sigma^2}{2}(h(t))^2 = 0, \quad x > 0, \quad 0 \leq t \leq T,$$

which reduces to the ordinary differential equation $h'(t) + rh(t) = 0$ with terminal condition $h(T) = 1$ and solution $h(t) = \mathrm{e}^{(T-t)r}$, $t \in [0, T]$, which yields

$$C(t, x) = \exp\left(-(T-t)r + x\mathrm{e}^{(T-t)r} + \frac{\sigma^2}{4r}(\mathrm{e}^{2(T-t)r} - 1) \right).$$

d) We have

$$\xi_t = \frac{\partial C}{\partial x}(t, S_t) = \exp\left(S_t \mathrm{e}^{(T-t)r} + \frac{\sigma^2}{4r}(\mathrm{e}^{2(T-t)r} - 1) \right).$$

Exercise 6.12

a) Noting that $\varphi(x) = \Phi'(x) = (2\pi)^{-1/2}\mathrm{e}^{-x^2/2}$, we have the

$$\begin{aligned}
\frac{\partial h}{\partial d}(S,d) &= S\varphi\big(d + \sigma\sqrt{T}\big) - K\mathrm{e}^{-rT}\varphi(d) \\
&= \frac{S}{\sqrt{2\pi}}\mathrm{e}^{-\left(d+\sigma\sqrt{T}\right)^2/2} - \frac{K}{\sqrt{2\pi}}\mathrm{e}^{-rT}\mathrm{e}^{-d^2/2} \\
&= \frac{S}{\sqrt{2\pi}}\mathrm{e}^{-d^2/2-\sigma\sqrt{T}d-\sigma^2T/2} - \frac{K}{\sqrt{2\pi}}\mathrm{e}^{-rT}\mathrm{e}^{-d^2/2},
\end{aligned}$$

hence the vanishing of $\dfrac{\partial h}{\partial d}(S, d_*(S))$ at $d = d_*(S)$ yields

$$\frac{S}{\sqrt{2\pi}}\mathrm{e}^{-d_*^2(S)/2-\sigma\sqrt{T}d_*(S)-\sigma^2T/2} - \frac{K}{\sqrt{2\pi}}\mathrm{e}^{-rT}\mathrm{e}^{-d_*^2(S)/2} = 0,$$

i.e. $d_*(S) = \dfrac{\log(S/K) + rT - \sigma^2T/2}{\sigma\sqrt{T}}$. We can also check that

$$\begin{aligned}
&\frac{\partial^2 h}{\partial d^2}(S, d_*(S)) = \frac{\partial}{\partial d}\left(\frac{S}{\sqrt{2\pi}}\mathrm{e}^{-\left(d_*(S)+\sigma\sqrt{T}\right)^2/2} - \frac{K}{\sqrt{2\pi}}\mathrm{e}^{-rT}\mathrm{e}^{-d_*^2(S)/2} \right) \\
&= -\big(d_*(S) + \sigma\sqrt{T}\big)\frac{S}{\sqrt{2\pi}}\mathrm{e}^{-\left(d_*(S)+\sigma\sqrt{T}\right)^2/2} + \frac{K}{\sqrt{2\pi}}d_*(S)\mathrm{e}^{-rT}\mathrm{e}^{-d_*^2(S)/2} \\
&= -\big(d_*(S) + \sigma\sqrt{T}\big)\frac{K}{\sqrt{2\pi}}\mathrm{e}^{-rT}\mathrm{e}^{-d_*^2(S)/2} + \frac{K}{\sqrt{2\pi}}d_*(S)\mathrm{e}^{-rT}\mathrm{e}^{-d_*^2(S)/2} \\
&= -\sigma\sqrt{T}\frac{K}{\sqrt{2\pi}}\mathrm{e}^{-rT}\mathrm{e}^{-d_*^2(S)/2} < 0,
\end{aligned}$$

hence the function $d \longmapsto h(S,d) := S\Phi\big(d + \sigma\sqrt{T}\big) - K\mathrm{e}^{-rT}\Phi(d)$ admits a *maximum* at $d = d_*(S)$, and

$$\begin{aligned}
&h(S, d_*(S)) = S\Phi\big(d_*(S) + \sigma\sqrt{T}\big) - K\mathrm{e}^{-rT}\Phi(d_*(S)) \\
&= S\Phi\left(\frac{\log(S/K) + (r+\sigma^2/2)T}{\sigma\sqrt{T}}\right) - K\mathrm{e}^{-rT}\Phi\left(\frac{\log(S/K) + (r-\sigma^2/2)T}{\sigma\sqrt{T}}\right)
\end{aligned}$$

is the Black-Scholes call option price.

b) Since $\frac{\partial h}{\partial d}(S, d_*(S)) = 0$, we find

$$\Delta = \frac{d}{dS}h(S, d_*(S)) = \frac{\partial h}{\partial S}(S, d_*(S)) + d_*'(S)\frac{\partial h}{\partial d}(S, d_*(S))$$

$$= \Phi\big(d_*(S) + \sigma\sqrt{T}\big) = \Phi\left(\frac{\log(S/K) + rT + \sigma^2 T/2}{\sigma\sqrt{T}}\right).$$

Exercise 6.13

a) When $\sigma > 0$ we have

$$\begin{aligned}\frac{\partial g_{\mathrm{c}}}{\partial \sigma} &= x\Phi'\big(d_+(T-t)\big)\frac{\partial}{\partial \sigma}d_+(T-t) - K\mathrm{e}^{-(T-t)r}\Phi'\big(d_-(T-t)\big)\frac{\partial}{\partial \sigma}d_-(T-t) \\ &= x\Phi'\big(d_+(T-t)\big)\frac{\partial}{\partial \sigma}d_+(T-t) \\ &\quad -K\mathrm{e}^{-(T-t)r}\Phi'\big(d_+(T-t)\big)\mathrm{e}^{(T-t)r+\log(x/K)}\frac{\partial}{\partial \sigma}d_-(T-t) \\ &= x\Phi'\big(d_+(T-t)\big)\frac{\partial}{\partial \sigma}\big(d_+(T-t) - d_-(T-t)\big) \\ &= x\Phi'\big(d_+(T-t)\big)\frac{\partial}{\partial \sigma}\big(\sigma\sqrt{T-t}\big) \\ &= x\sqrt{T-t}\Phi'\big(d_+(T-t)\big),\end{aligned}$$

where we used the fact that

$$\begin{aligned}\Phi'\big(d_-(T-t)\big) &= \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-(d_-(T-t))^2/2} \\ &= \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-(d_-(T-t))^2/2+(T-t)r+\log(x/K)} \\ &= \Phi'\big(d_+(T-t)\big)\mathrm{e}^{(T-t)r+\log(x/K)}.\end{aligned}$$

Relation (6.43) can be obtained from the equalities

$$\begin{aligned}&\big(d_+(T-t)\big)^2 - \big(d_-(T-t)\big)^2 \\ &= \big(d_+(T-t) + d_-(T-t)\big)\big(d_+(T-t) - d_-(T-t)\big) \\ &= 2r(T-t) + 2\log\frac{x}{K}.\end{aligned}$$

Due to the call-put parity relation (6.23), the Black-Scholes call and put Vega are identical, *i.e.*

$$\frac{\partial g_{\mathrm{p}}}{\partial \sigma} = \frac{\partial g_{\mathrm{c}}}{\partial \sigma} = x\sqrt{T-t}\Phi'\big(d_+(T-t)\big).$$

The Black-Scholes European call and put prices are increasing functions of the volatility parameter $\sigma > 0$.

b) We have

$$\frac{\partial g_{\mathrm{c}}}{\partial r} = x\Phi'(d_+(T-t))\frac{\partial}{\partial r}d_+(T-t) - K\mathrm{e}^{-(T-t)r}\Phi'(d_-(T-t))\frac{\partial}{\partial r}d_-(T-t)$$

$$
\begin{aligned}
&+(T-t)K\mathrm{e}^{-(T-t)r}\Phi(d_-(T-t)) \\
&= x\Phi'(d_+(T-t))\frac{\partial}{\partial r}d_+(T-t) \\
&\quad -K\mathrm{e}^{-(T-t)r}\Phi'(d_+(T-t))\mathrm{e}^{(T-t)r+\log(x/K)}\frac{\partial}{\partial r}d_-(T-t) \\
&\quad +(T-t)K\mathrm{e}^{-(T-t)r}\Phi(d_-(T-t)) \\
&= x\Phi'(d_+(T-t))\frac{\partial}{\partial r}\big(d_+(T-t)-d_-(T-t)\big)+(T-t)K\mathrm{e}^{-(T-t)r}\Phi(d_-(T-t)) \\
&= x\Phi'(d_+(T-t))\frac{\partial}{\partial r}\big(\sigma\sqrt{T-t}\big)+(T-t)K\mathrm{e}^{-(T-t)r}\Phi(d_-(T-t)) \\
&= (T-t)K\mathrm{e}^{-(T-t)r}\Phi(d_-(T-t)),
\end{aligned}
$$

where we used the fact that

$$
\begin{aligned}
\Phi'(d_-(T-t)) &= \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-(d_-(T-t))^2/2} \\
&= \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-(d_-(T-t))^2/2+(T-t)r+\log(x/K)} \\
&= \Phi'(d_+(T-t))\mathrm{e}^{(T-t)r+\log(x/K)}.
\end{aligned}
$$

The same relationship is used to simplify the formulas of the Black-Scholes Delta and Vega. We note that the Black-Scholes European call price is an increasing function of the interest rate parameter r.

Regarding put option prices $g_{\mathrm{p}}(t,x)$, the call-put parity relation (6.23) yields

$$
\begin{aligned}
\frac{\partial g_{\mathrm{p}}}{\partial r} &= \frac{\partial}{\partial r}\big(g_{\mathrm{c}}-(x-K\mathrm{e}^{-r(T-t)})\big) \\
&= (T-t)K\mathrm{e}^{-(T-t)r}\Phi(d_-(T-t))-(T-t)K\mathrm{e}^{-r(T-t)} \\
&= (T-t)K\mathrm{e}^{-(T-t)r}(\Phi(d_-(T-t))-1) \\
&= -(T-t)K\mathrm{e}^{-(T-t)r}\Phi(-d_-(T-t)),
\end{aligned}
$$

therefore the Black-Scholes European call price is a decreasing function of the interest rate parameter r.

Exercise 6.14

a) Given that

$$
p^* = \frac{r_N-a_N}{b_N-a_N} = \frac{1}{2} \quad \text{and} \quad q^* = \frac{b_N-r_N}{b_N-a_N} = \frac{1}{2},
$$

Relation (3.14) reads

$$\begin{aligned}\widetilde{v}(t,x) &= \frac{1}{2}\widetilde{v}\big(t+T/N, x(1+rT/N)(1-\sigma\sqrt{T/N})\big)\\&\quad+\frac{1}{2}\widetilde{v}\big(t+T/N, x(1+rT/N)(1+\sigma\sqrt{T/N})\big).\end{aligned}$$

After letting $\Delta T := T/N$ and applying Taylor's formula at the second order we obtain

$$\begin{aligned}0 &= \frac{1}{2}\big(\widetilde{v}\big(t+\Delta T, x\big(1+r\Delta T-\sigma\sqrt{\Delta T}\big)\big)-\widetilde{v}(t,x)\big)\\&\quad+\frac{1}{2}\big(\widetilde{v}\big(t+\Delta T, x\big(1+r\Delta T+\sigma\sqrt{\Delta T}\big)\big)-\widetilde{v}(t,x)\big)+o\big(\Delta T\big)\\&= \frac{1}{2}\left(\Delta T\frac{\partial\widetilde{v}}{\partial t}(t,x)+x\big(r\Delta T-\sigma\sqrt{\Delta T}\big)\frac{\partial\widetilde{v}}{\partial x}(t,x)\right.\\&\quad\left.+\frac{x^2}{2}\big(r\Delta T-\sigma\sqrt{\Delta T}\big)^2\frac{\partial^2\widetilde{v}}{\partial x^2}(t,x)+o(\Delta T)\right)\\&\quad+\frac{1}{2}\left(\Delta T\frac{\partial\widetilde{v}}{\partial t}(t,x)+x\big(r\Delta T+\sigma\sqrt{\Delta T}\big)\frac{\partial\widetilde{v}}{\partial x}(t,x)\right.\\&\quad\left.+\frac{x^2}{2}\big(r\Delta T+\sigma\sqrt{\Delta T}\big)^2\frac{\partial^2\widetilde{v}}{\partial x^2}(t,x)+o(\Delta T)\right)+o\big(\Delta T\big)\\&= \Delta T\frac{\partial\widetilde{v}}{\partial t}(t,x)+rx\Delta T\frac{\partial\widetilde{v}}{\partial x}(t,x)+\frac{x^2}{2}\big(\sigma\sqrt{\Delta T}\big)^2\frac{\partial^2\widetilde{v}}{\partial x^2}(t,x)+o\big(\Delta T\big),\end{aligned}$$

which shows that

$$\frac{\partial\widetilde{v}}{\partial t}(t,x)+rx\frac{\partial\widetilde{v}}{\partial x}(t,x)+x^2\frac{\sigma^2}{2}\frac{\partial^2\widetilde{v}}{\partial x^2}(t,x) = -\frac{o\big(\Delta T\big)}{\Delta T},$$

hence as N tends to infinity (or as ΔT tends to 0) we find*

$$0 = \frac{\partial\widetilde{v}}{\partial t}(t,x)+rx\frac{\partial\widetilde{v}}{\partial x}(t,x)+\frac{\sigma^2}{2}x^2\frac{\partial^2\widetilde{v}}{\partial x^2}(t,x),$$

showing that the function $v(t,x) := \mathrm{e}^{(T-t)r}\widetilde{v}(t,x)$ solves the classical Black-Scholes PDE

$$rv(t,x) = \frac{\partial v}{\partial t}(t,x)+rx\frac{\partial v}{\partial x}(t,x)+\frac{\sigma^2}{2}x^2\frac{\partial^2 v}{\partial x^2}(t,x).$$

b) Similarly, we have

$$\xi_t^{(1)}(x) = \frac{v\left(t,(1+b_N)x\right)-v\left(t,(1+a_N)x\right)}{x(b_N-a_N)}$$

* The notation $o(\Delta T)$ is used to represent any function of ΔT such that $\lim_{\Delta T\to 0} o(\Delta T)/\Delta T = 0$.

$$= \frac{v\left(t,(1+r/N)(1+\sigma\sqrt{T/N})x\right) - v\left(t,(1+r/N)(1-\sigma\sqrt{T/N})x\right)}{2x(1+r/N)\sigma\sqrt{T/N}}$$
$$\to \frac{\partial v}{\partial x}(t,x),$$

as N tends to infinity.

Problem 6.15

a) When the risk-free rate is $r = 0$ the two possible returns are $(5-4)/4 = 25\%$ and $(2-4)/4 = -50\%$. Under the risk-neutral probability measure given by $\mathbb{P}^*(S_1 = 5) = (4-2)/(5-2) = 2/3$ and $\mathbb{P}^*(S_1 = 2) = (5-4)/(5-2) = 1/3$ the expected return is $2 \times 25\%/3 - 50\%/3 = 0\%$. In general the expected return can be shown to be equal to the risk-free rate r.
b) The two possible returns become $(3\times5-4-2\times4)/4 = 75\%$ and $(3\times2-4-2\times4)/4 = -150\%$. Under the risk-neutral probability measure given by $\mathbb{P}^*(S_1 = 5) = (4-2)/(5-2) = 2/3$ and $\mathbb{P}^*(S_1 = 2) = (5-4)/(5-2) = 1/3$ the expected return is $2\times75\%/3-150\%/3 = 0\%$. Similarly to Question (a), the expected return can be shown to be equal to the risk-free rate r when $r \neq 0$.
c) We decompose the amount F_t invested in one unit of the fund as

$$F_t = \underbrace{\beta F_t}_{\text{Purchased/sold}} - \underbrace{(\beta-1)F_t,}_{\text{Borrowed/saved}}$$

meaning that we invest the amount βF_t in the risky asset S_t, and borrow/save the amount $-(\beta-1)F_t$ from/on the saving account.
d) We have

$$F_t = \xi_t S_t + \eta_t A_t = \beta\frac{F_t}{S_t}S_t - (\beta-1)\frac{F_t}{A_t}A_t, \qquad t \geq 0,$$

with $\xi_t = \beta F_t/S_t$ and $\eta_t = -(\beta-1)F_t/A_t$, $t \geq 0$.
e) We have

$$\begin{aligned} dF_t &= \xi_t dS_t + \eta_t dA_t \\ &= \beta\frac{F_t}{S_t}dS_t - (\beta-1)\frac{F_t}{A_t}dA_t \\ &= \beta\frac{F_t}{S_t}dS_t - (\beta-1)rF_t dt \qquad\qquad (\text{S.6.30}) \\ &= \beta F_t(rdt + \sigma dB_t) - (\beta-1)rF_t dt \\ &= rF_t dt + \beta\sigma F_t dB_t, \qquad t \geq 0. \end{aligned}$$

By (S.6.30), the return of the fund F_t is β times the return of the risky asset S_t, up to the cost of borrowing $(\beta-1)r$ per unit of time.

f) The discounted fund value $(e^{-rt}F_t)_{t\in\mathbb{R}_+}$ is a martingale under the risk-neutral probability measure $\mathbb{P}^*$ as we have

$$d(e^{-rt}F_t) = \beta\sigma e^{-rt}F_t dB_t, \qquad t \geq 0.$$

g) We have

$$F_t = F_0 e^{\beta\sigma B_t + rt - \beta^2\sigma^2 t/2}$$

and

$$S_t^\beta = \left(S_0 e^{\sigma B_t + rt - \sigma^2 t/2}\right)^\beta = F_0 e^{\beta\sigma B_t + \beta rt - \beta\sigma^2 t/2},$$

hence

$$F_t = S_t^\beta e^{-(\beta-1)rt - \beta(\beta-1)\sigma^2 t/2}, \qquad t \geq 0.$$

Note that when $\beta = 0$ we have $F_t = e^{rt}$, *i.e.* in this case the fund F_t coincides with the money market account.

h) We have

$$\begin{aligned}
&e^{-(T-t)r}\,\mathbb{E}^*\left[(F_T-K)^+ \mid \mathcal{F}_t\right] \\
&= F_t \Phi\left(\frac{\log(F_t/K) + (r+\beta^2\sigma^2/2)(T-t)}{|\beta|\sigma\sqrt{T-t}}\right) \\
&\quad - K e^{-(T-t)r}\Phi\left(\frac{\log(F_t/K) + (r-\beta^2\sigma^2/2)(T-t)}{|\beta|\sigma\sqrt{T-t}}\right),
\end{aligned}$$

$t \in [0, T)$.

i) We have

$$\begin{aligned}
&\Phi\left(\frac{\log(F_t/K) + (r+\beta^2\sigma^2/2)(T-t)}{|\beta|\sigma\sqrt{T-t}}\right) \\
&= \Phi\left(\frac{\log(S_t^\beta e^{-(\beta-1)rt-\beta(\beta-1)\sigma^2 t/2}/K) + (r+\beta^2\sigma^2/2)(T-t)}{|\beta|\sigma\sqrt{T-t}}\right) \\
&= \Phi\left(\frac{\log(S_t^\beta/K) - (\beta-1)rt - \beta(\beta-1)\sigma^2 t/2 + (r+\beta^2\sigma^2/2)(T-t)}{|\beta|\sigma\sqrt{T-t}}\right) \\
&= \Phi\left(\frac{\log(S_t^\beta/(K e^{(\beta-1)rT-(T/2-t)(\beta-1)\beta\sigma^2})) + (T-t)\beta r + (T-t)\beta\sigma^2/2}{|\beta|\sigma\sqrt{T-t}}\right) \\
&= \Phi\left(\frac{\log(S_t/K_\beta(t)) + (r+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}\right), \qquad 0 \leq t < T,
\end{aligned}$$

if $\beta > 0$, with $K_\beta(t) := K^{1/\beta} e^{(\beta-1)(rT/\beta - (T/2-t)\sigma^2)}$.

j) When $\beta < 0$ we find that the Delta of the call option on F_T with strike price K is

$$\begin{aligned}
&\Phi\left(\frac{\log(F_t/K)+(r+\beta^2\sigma^2/2)(T-t)}{|\beta|\sigma\sqrt{T-t}}\right)\\
&=\Phi\left(\frac{\log(S_t^\beta/K_\beta)+(T-t)\beta r+(T-t)\beta\sigma^2/2}{|\beta|\sigma\sqrt{T-t}}\right)\\
&=\Phi\left(-\frac{\log(S_t/K_\beta(t))+(r+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}\right), \qquad 0\le t<T,
\end{aligned}$$

which coincides, up to a negative sign, with the Delta of the *put* option on S_T with strike price $K_\beta(t) := K^{1/\beta}\mathrm{e}^{(\beta-1)(rT/\beta-(T/2-t)\sigma^2)}$.

Chapter 7

Exercise 7.1 (Exercise 6.1 continued). Since $r=0$ we have $\mathbb{P}=\mathbb{P}^*$ and

$$\begin{aligned}
g(t,B_t) &= \mathbb{E}^*\left[B_T^2 \mid \mathcal{F}_t\right]\\
&= \mathbb{E}^*\left[(B_T-B_t+B_t)^2 \mid \mathcal{F}_t\right]\\
&= \mathbb{E}^*\left[(B_T-B_t+x)^2\right]_{x=B_t}\\
&= \mathbb{E}^*\left[(B_T-B_t)^2+2x(B_T-B_t)+x^2\right]_{x=B_t}\\
&= \mathbb{E}^*\left[(B_T-B_t)^2\right]+2x\,\mathbb{E}^*[B_T-B_t]+B_t^2\\
&= B_t^2+T-t, \qquad 0\le t\le T,
\end{aligned}$$

hence ξ_t is given by the partial derivative

$$\xi_t=\frac{\partial g}{\partial x}(t,B_t)=2B_t, \qquad 0\le t\le T,$$

with

$$\begin{aligned}
\eta_t &= \frac{g(t,B_t)-\xi_tB_t}{A_0}\\
&= \frac{B_t^2+(T-t)-2B_t^2}{A_0}\\
&= \frac{(T-t)-B_t^2}{A_0}, \quad 0\le t\le T.
\end{aligned}$$

Exercise 7.2 Since $B_T \simeq \mathcal{N}(0,T)$, we have

$$\begin{aligned}
\mathbb{E}[\phi(S_T)] &= \mathbb{E}\left[\phi\big(S_0\mathrm{e}^{\sigma B_T+(r-\sigma^2/2)T}\big)\right]\\
&= \frac{1}{\sqrt{2\pi T}}\int_{-\infty}^{\infty}\phi(S_0\mathrm{e}^{\sigma y+(r-\sigma^2/2)T})\mathrm{e}^{-y^2/(2T)}dy
\end{aligned}$$

$$= \frac{1}{\sqrt{2\pi\sigma^2 T}} \int_{-\infty}^{\infty} \phi(x) e^{-((\sigma^2/2-r)T+\log x)^2/(2\sigma^2 T)} \frac{dx}{x}$$
$$= \int_{-\infty}^{\infty} \phi(x) g(x) dx,$$

under the change of variable

$$x = S_0 e^{\sigma y + (r-\sigma^2/2)T}, \quad \text{with } dx = \sigma S_0 e^{\sigma y+(r-\sigma^2/2)T} dy = \sigma x dy,$$

i.e.

$$y = \frac{(\sigma^2/2 - r)T + \log(x/S_0)}{\sigma} \quad \text{and} \quad dy = \frac{dx}{\sigma x},$$

where

$$g(x) := \frac{1}{x\sqrt{2\pi\sigma^2 T}} e^{-((\sigma^2/2-r)T+\log(x/S_0))^2/(2\sigma^2 T)}$$

is the *lognormal* probability density function with location parameter $(r - \sigma^2/2)T + \log S_0$ and scale parameter $\sigma\sqrt{T}$.

Exercise 7.3

a) By the Itô formula, we have

$$\begin{aligned} dS_t^p &= pS_t^{p-1} dS_t + \frac{p(p-1)}{2} S_t^{p-2} dS_t \bullet dS_t \\ &= pS_t^{p-1}(rS_t dt + \sigma S_t dB_t) + \frac{p(p-1)}{2} S_t^{p-2}(rS_t dt + \sigma S_t dB_t) \bullet (rS_t dt + \sigma S_t dB_t) \\ &= prS_t^p dt + \sigma p S_t^p dB_t + \sigma^2 \frac{p(p-1)}{2} S_t^p dt \\ &= \left(pr + \sigma^2 \frac{p(p-1)}{2} \right) S_t^p dt + \sigma p S_t^p dB_t. \end{aligned}$$

b) By the Girsanov Theorem 7.3, letting

$$\nu := \frac{1}{p\sigma} \left((p-1)r + \sigma^2 \frac{p(p-1)}{2} \right),$$

the drifted process

$$\widehat{B}_t := B_t + \nu t, \qquad 0 \le t \le T,$$

is a standard (centered) Brownian motion under the probability measure $\mathbb{Q}$ defined by

$$d\mathbb{Q}(\omega) = \exp\left(-\nu B_T - \frac{\nu^2}{2} T \right) d\mathbb{P}(\omega).$$

Therefore, the differential of $(S_t^p)_{t\in\mathbb{R}_+}$ can be written as

$$\begin{aligned}
dS_t^p &= \left(pr + \sigma^2 \frac{p(p-1)}{2}\right) S_t^p dt + \sigma p S_t^p dB_t \\
&= (r + p\sigma\nu)\, S_t^p dt + \sigma p S_t^p dB_t \\
&= r S_t^p dt + \sigma p S_t^p (dB_t + \nu dt) \\
&= r S_t^p dt + \sigma p S_t^p d\widehat{B}_t,
\end{aligned}$$

hence the discounted process $\widetilde{S}_t := \mathrm{e}^{-rt} S_t^p$ satisfies $d\widetilde{S}_t = \sigma p \widetilde{S}_t d\widehat{B}_t$, and $(\widetilde{S}_t)_{t\in\mathbb{R}_+}$ is a martingale under the probability measure $\mathbb{Q}$.

Exercise 7.4 We have

$$\begin{aligned}
\mathbb{E}^*[\phi(pS_{T_1} + qS_{T_2})] &\leq \mathbb{E}^*[p\phi(S_{T_1}) + q\phi(S_{T_2})] && \text{since } \phi \text{ is convex,} \\
&= p\,\mathbb{E}^*[\phi(S_{T_1})] + q\,\mathbb{E}^*[\phi(S_{T_2})] \\
&= p\,\mathbb{E}^*[\phi(\mathbb{E}^*[S_{T_2} \mid \mathcal{F}_{T_1}])] + q\,\mathbb{E}^*[\phi(S_{T_2})] && \text{because } (S_t)_{t\in\mathbb{R}_+} \text{ is a martingale,} \\
&\leq p\,\mathbb{E}^*[\mathbb{E}^*[\phi(S_{T_2}) \mid \mathcal{F}_{T_1}]] + q\,\mathbb{E}^*[\phi(S_{T_2})] && \text{by Jensen's inequality,} \\
&= p\,\mathbb{E}^*[\phi(S_{T_2})] + q\,\mathbb{E}^*[\phi(S_{T_2})] && \text{by the tower property,} \\
&= \mathbb{E}^*[\phi(S_{T_2})], && \text{because } p + q = 1,
\end{aligned}$$

see Exercise 13.7 for an extension to arbitrary summations.

Remark: This kind of technique can provide an upper price estimate from Black-Scholes when the actual option price is difficult to compute: here the closed-form computation would involve a double integration of the form

$$\begin{aligned}
&\mathbb{E}^*[\phi(pS_{T_1} + qS_{T_2})] = \mathbb{E}^*\left[\phi\left(pS_0 \mathrm{e}^{\sigma B_{T_1} - \sigma^2 T_1/2} + qS_0 \mathrm{e}^{\sigma B_{T_2} - \sigma^2 T_2/2}\right)\right] \\
&\quad = \mathbb{E}^*\left[\phi\left(S_0 \mathrm{e}^{\sigma B_{T_1} - \sigma^2 T_1/2}\left(p + q\mathrm{e}^{(B_{T_2} - B_{T_1})\sigma - (T_2 - T_1)\sigma^2/2}\right)\right)\right] \\
&\quad = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi\left(S_0 \mathrm{e}^{\sigma x - \sigma^2 T_1/2}\left(p + q\mathrm{e}^{\sigma y - (T_2 - T_1)\sigma^2/2}\right)\right) \\
&\qquad\qquad \times \mathrm{e}^{-x^2/(2T_1) - y^2(2(T_2 - T_1))} \frac{dxdy}{\sqrt{T_1(T_2 - T_1)}} \\
&\quad = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(S_0 \mathrm{e}^{\sigma x - \sigma^2 T_1/2}\left(p + q\mathrm{e}^{\sigma y - (T_2 - T_1)\sigma^2/2}\right) - K\right)^+ \\
&\qquad\qquad \times \mathrm{e}^{-x^2/(2T_1) - y^2(2(T_2 - T_1))} \frac{dxdy}{\sqrt{T_1(T_2 - T_1)}} \\
&\quad = \frac{1}{2\pi} \int_{\{(x,y)\in\mathbb{R}^2 \,:\, S_0 \mathrm{e}^{\sigma x}(p + q\mathrm{e}^{\sigma y - (T_2 - T_1)\sigma^2/2}) \geq K\mathrm{e}^{\sigma^2 T_1/2}\}} \\
&\qquad (S_0 \mathrm{e}^{\sigma x - \sigma^2 T_1/2}(p + q\mathrm{e}^{\sigma y - (T_2 - T_1)\sigma^2/2}) - K) \\
&\qquad\qquad \times \mathrm{e}^{-x^2/(2T_1) - y^2(2(T_2 - T_1))} \frac{dxdy}{\sqrt{T_1(T_2 - T_1)}} \\
&\quad = \cdots
\end{aligned}$$

Exercise 7.5

a) The European *call* option price $C(K) := \mathrm{e}^{-rT}\,\mathbb{E}^*[(S_T - K)^+]$ decreases with the strike price K, because the option payoff $(S_T - K)^+$ decreases and the expectation operator preserves the ordering of random variables.
b) The European *put* option price $C(K) := \mathrm{e}^{-rT}\,\mathbb{E}^*[(K - S_T)^+]$ increases with the strike price K, because the option payoff $(K - S_T)^+$ increases and the expectation operator preserves the ordering of random variables.

Exercise 7.6

a) Using Jensen's inequality and the martingale property of the discounted asset price process $(\mathrm{e}^{-rt}S_t)_{t\in\mathbb{R}_+}$ under the risk-neutral probability measure $\mathbb{P}^*$, we have

$$\begin{aligned}\mathrm{e}^{-(T-t)r}\,\mathbb{E}^*[(S_T - K)^+ \mid \mathcal{F}_t] &\geq \mathrm{e}^{-(T-t)r}\big(\mathbb{E}^*[S_T - K \mid \mathcal{F}_t]\big)^+\\ &= \mathrm{e}^{-(T-t)r}\big(\mathrm{e}^{(T-t)r}S_t - K\big)^+\\ &= \big(S_t - K\mathrm{e}^{-(T-t)r}\big)^+, \quad 0 \leq t \leq T.\end{aligned}$$

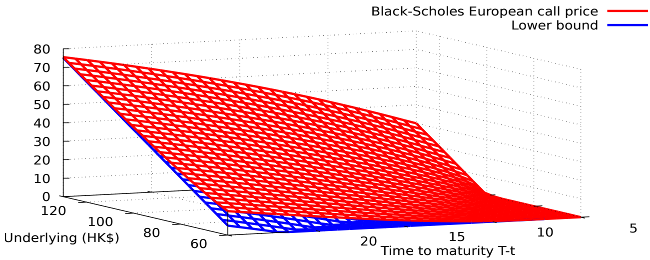


Fig. S.23: Lower bound *vs* Black-Scholes call price.

In terms of the *break-even* price defined as

$$\mathrm{BEP}_t := K + \mathrm{e}^{-(T-t)r}\,\mathbb{E}^*[(S_T - K)^+ \mid \mathcal{F}_t],$$

we obtain the bound

$$\mathrm{BEP}_t \geq K + \big(S_t - K\mathrm{e}^{-(T-t)r}\big)^+.$$

b) Similarly, by Jensen's inequality and the martingale property, we find

$$\begin{aligned}\mathrm{e}^{-(T-t)r}\,\mathbb{E}^*[(K - S_T)^+ \mid \mathcal{F}_t] &\geq \mathrm{e}^{-(T-t)r}\big(\mathbb{E}^*[K - S_T \mid \mathcal{F}_t]\big)^+\\ &= \mathrm{e}^{-(T-t)r}\big(K - \mathrm{e}^{(T-t)r}S_t\big)^+\end{aligned}$$

$$= \left(K\mathrm{e}^{-(T-t)r} - S_t\right)^+, \quad 0 \le t \le T.$$

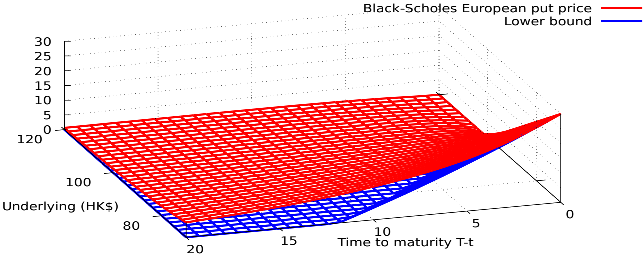


Fig. S.24: Lower bound *vs* Black-Scholes put option price.

We may also use the fact that a convex function of the martingale $(\mathrm{e}^{rt}S_t)_{t\in\mathbb{R}_+}$ under the risk-neutral probability measure $\mathbb{P}^*$ is a *sub*martingale, showing that

$$\begin{aligned}\mathrm{e}^{rt}\,\mathbb{E}^*[(\mathrm{e}^{-rT}K - \mathrm{e}^{-rT}S_T)^+ \mid \mathcal{F}_t] &\ge \mathrm{e}^{rt}(\mathrm{e}^{-rT}K - \mathrm{e}^{-rt}S_t)^+ \\ &= \left(K\mathrm{e}^{-(T-t)r} - S_t\right)^+, \quad 0 \le t \le T.\end{aligned}$$

In terms of the *break-even* price defined as

$$\mathrm{BEP}_t := K - \mathrm{e}^{-(T-t)r}\,\mathbb{E}^*[(S_T - K)^+ \mid \mathcal{F}_t],$$

we obtain the bound

$$\mathrm{BEP}_t \le K - \left(K\mathrm{e}^{-(T-t)r} - S_t\right)^+.$$

Exercise 7.7

a) (i) The bull spread option can be realized by purchasing one European call option with strike price K_1 and by short selling (or issuing) one European call option with strike price K_2, because the bull spread payoff function can be written as

$$x \longmapsto (x - K_1)^+ - (x - K_2)^+.$$

see https://optioncreator.com/st3ce7z.

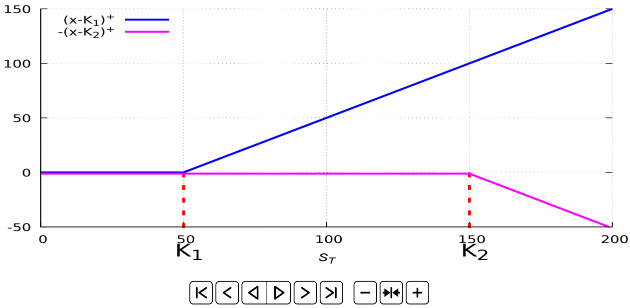


Fig. S.25: Bull spread option as a combination of call and put options.*

(ii) The bear spread option can be realized by purchasing one European put option with strike price K_2 and by short selling (or issuing) one European put option with strike price K_1, because the bear spread payoff function can be written as

$$x \longmapsto -(K_1 - x)^+ + (K_2 - x)^+,$$

see https://optioncreator.com/stmomsb.

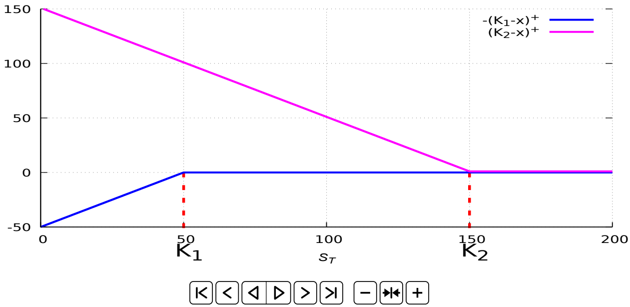


Fig. S.26: Bear spread option as a combination of call and put options.†

b) (i) The bull spread option can be priced at time $t \in [0, T)$ using the Black-Scholes formula as

* The animation works in Acrobat Reader on the entire pdf file.

† The animation works in Acrobat Reader on the entire pdf file.

$$\mathrm{Bl}(S_t, K_1, \sigma, r, T-t) - \mathrm{Bl}(S_t, K_2, \sigma, r, T-t).$$

(ii) The bear spread option can be priced at time $t \in [0, T)$ using the Black-Scholes formula as

$$\mathrm{Bl}(S_t, K_2, \sigma, r, T-t) - \mathrm{Bl}(S_t, K_1, \sigma, r, T-t).$$

Exercise 7.8

a) The payoff of the long box spread option is given in terms of K_1 and K_2 as

$$(x-K_1)^+ - (K_1-x)^+ - (x-K_2)^+ + (K_2-x)^+ = x - K_1 - (x-K_2) = K_2 - K_1.$$

b) By standard absence of arbitrage, the long box spread option payoff is priced $(K_2 - K_1)/(1+r)^{N-k}$ at times $k = 0, 1, \ldots, N$.

c) From Table S.6 below, we check that the strike prices suitable for a long box spread option on the Hang Seng Index (HSI) are $K_1 = 25,000$ and $K_2 = 25,200$.

Basic Data											Market Data							
DW Code	Issuer	UL	Call/ Put	DW Type	Listing (D-M-Y)	Maturity (D-M-Y)	Strike Currency	Strike	Entitle-ment Ratio^	Total Issue Size	O/S (%)	Delta (%)	IV. (%)	Trading Currency	Day High	Day Low	Closing Price	T/O ('000)
17334	HT	HSI	Put	Standard	12-11-2020	28-05-2021	-	24600	10000	400,000,000	5.11	(0.001)	33.268	HKD	0.034	0.025	0.034	314
17535	UB	HSI	Put	Standard	13-11-2020	28-05-2021	-	24600	10000	300,000,000	22.07	(0.001)	29.507	HKD	0.025	0.017	0.023	132
17589	CS	HSI	Put	Standard	13-11-2020	28-05-2021	-	24900	9500	425,000,000	8.61	(0.001)	30.838	HKD	0.033	0.028	0.033	80
18242	UB	HSI	Put	Standard	19-11-2020	28-05-2021	-	25000	9500	300,000,000	18.51	(0.001)	29.028	HKD	0.030	0.023	0.029	265
18606	SG	HSI	Put	Standard	23-11-2020	29-06-2021	-	25088	10000	300,000,000	8.01	(0.002)	30.968	HKD	0.054	0.042	0.053	459
19399	HT	HSI	Put	Standard	02-12-2020	29-06-2021	-	25200	10000	400,000,000	0.06	(0.002)	32.190	HKD	0.000	0.000	0.061	0
19485	BI	HSI	Put	Standard	03-12-2020	29-06-2021	-	25200	10000	150,000,000	21.41	(0.002)	28.154	HKD	0.044	0.037	0.044	59
22857	VT	HSI	Put	Standard	27-02-2020	29-06-2021	-	25000	8000	80,000,000	22.45	(0.002)	30.905	HKD	0.065	0.043	0.064	1,165
26601	BI	HSI	Call	Standard	28-12-2020	29-06-2021	-	25200	11000	150,000,000	0.00	0.018	25.347	HKD	0.390	0.360	0.370	84
27489	BP	HSI	Call	Standard	18-09-2020	29-06-2021	-	25000	7500	80,000,000	2.95	0.009	28.392	HKD	0.590	0.540	0.540	6
28231	HS	HSI	Call	Standard	30-09-2020	29-06-2021	-	25118	7500	200,000,000	0.00	0.012	24.897	HKD	0.000	0.000	0.570	0

Table S.6: Call and put options on the Hang Seng Index (HSI).

d) Based on the data provided, we note that the long box spread can be realized in two ways.

i) Using the put option issued by BI (BOCI Asia Ltd.) at 0.044.

In this case, the box spread option represents a short position priced

$$\underbrace{0.540}_{\text{Long call}} \times 7,500 \underbrace{-0.064}_{\text{Short put}} \times 8,000 \underbrace{-0.370}_{\text{Short call}} \times 11,000 \underbrace{+0.044}_{\text{Long put}} \times 10,000 = -92$$

index points, or $-92 \times \$50 = -\$4,600$ on 02 March 2021.

Note that according to Table 7.2, option prices are quoted in index points (to be multiplied by the relevant option/warrant entitlement ratio), and every index point is worth \$50.

ii) Using the put option issued by HT (Haitong Securities) at 0.061.

In this case, the box spread option represents a long position priced

$$\underbrace{0.540}_{\text{Long call}} \times 7,500 \underbrace{-0.044}_{\text{Short put}} \times 8,000 \underbrace{-0.370}_{\text{Short call}} \times 11,000 \underbrace{+0.061}_{\text{Long put}} \times 10,000 = +78$$

index points, or $78 \times \$50 = \$3,900$ on 02 March 2021.

e) As the option built in di) represents a short position paying \$4,600 today with an additional $\$50 \times (K_2 - K_1) = 200 = \$10,000$ payoff at maturity on June 29, I would definitely enter this position.

As for the option built in dii) it is less profitable because it costs \$3,900, however it is still profitable taking into account the \$10,000 payoff at maturity on June 29.

By the way, the put option at 0.061 has zero turnover (T/O).

In the early 2019 Robinhood incident, a member of the Reddit community /r/WallStreetBets realized a loss of more than \$57,000 on \$5,000 principal by attempting a box spread. This was due to the use of American call and put options that may be exercised by their holders at any time, instead of European options with fixed maturity time N. Robinhood subsequently announced that investors on the platform would no longer be able to open box spreads, a policy that remains in place as of early 2021

Remark. Searching for arbitrage opportunities via the existence of profitable long box spreads is a way to test the efficiency of the market (Billingsley and Chance (1985)). The data used for this test in Table 7.1 was in fact modified market data. The original 02 March 2021 data is displayed in Table S.7, and shows that the call option with strike price $K_2 = 25,200$ was actually not available for trading (N/A) at that time, with 0% outstanding quantity and zero turnover (T/O).

Basic Data															Market Data				
DW Code	Issuer	UL	Call/ Put	DW Type	Listing (D-M-Y)	Maturity (D-M-Y)	Strike Currency	Strike	Entitle-ment Ratio^	Total Issue Size	O/S (%)	Delta (%)	IV. (%)	Trading Currency	Day High	Day Low	Closing Price	T/O ('000)
18606	SG	HSI	Put	Standard	23-11-2020	29-06-2021	-	25088	10000	300,000,000	8.01	(0.002)	30.968	HKD	0.054	0.042	0.053	459
19399	HT	HSI	Put	Standard	02-12-2020	29-06-2021	-	25200	10000	400,000,000	0.06	(0.002)	32.190	HKD	0.000	0.000	0.061	0
19485	BI	HSI	Put	Standard	03-12-2020	29-06-2021	-	25200	10000	150,000,000	21.41	(0.002)	28.154	HKD	0.044	0.037	0.044	59
22857	VT	HSI	Put	Standard	27-02-2020	29-06-2021	-	25000	8000	80,000,000	22.45	(0.002)	30.905	HKD	0.065	0.043	0.064	1,165
26601	BI	HSI	Call	Standard	28-12-2020	29-06-2021	-	25200	11000	150,000,000	0.00	-	-	HKD	0.000	0.000	N/A	0
27489	BP	HSI	Call	Standard	18-09-2020	29-06-2021	-	25000	7500	80,000,000	2.95	-	-	HKD	0.590	0.590	0.540	6
28231	HS	HSI	Call	Standard	30-09-2020	29-06-2021	-	25118	7500	200,000,000	0.00	0.012	24.897	HKD	0.000	0.000	0.570	0

Table S.7: Original call/put options on the Hang Seng Index (HSI) as of 02/03/2021.

Exercise 7.9

a) The payoff function can be written as

$$\begin{aligned}(x-K_1)^+ &+ (x-K_2)^+ - 2(x-(K_1+K_2)/2)^+ \\ &= (x-50)^+ + (x-150)^+ - 2(x-100)^+,\end{aligned} \tag{S.7.31}$$

see also https://optioncreator.com/stnurzg.

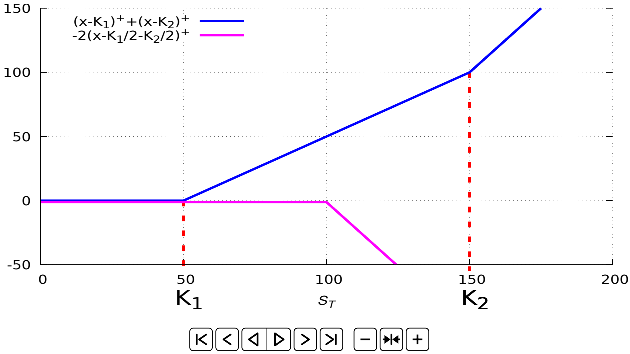


Fig. S.27: Butterfly option as a combination of call options.*

Hence the butterfly option can be realized by:

1. purchasing one *call option* with strike price $K_1 = 50$, and
2. purchasing one *call option* with strike price $K_2 = 150$, and
3. issuing (or selling) two *call options* with strike price $(K_1+K_2)/2 = 100$.

b) From (S.7.31), the long call butterfly option can be priced at time $t \in [0,T)$ using the Black-Scholes call formula as

$$\mathrm{Bl}(S_t, K_1, \sigma, r, T-t) + \mathrm{Bl}(S_t, K_2, \sigma, r, T-t) - 2\mathrm{Bl}(S_t, (K_1+K_2)/2, \sigma, r, T-t).$$

c) For example, in the discrete-time Cox-Ross-Rubinstein (Cox et al. (1979)) model, denoting by $\phi(x)$ the payoff function, the self-financing replicating portfolio strategy $(\xi_t(S_{t-1}))_{t=1,2,\ldots,N}$ hedging the contingent claim with payoff $C = \phi(S_N)$ is given as in Proposition 3.10 by

$$\xi_t(x) = \frac{\mathbb{E}^*\left[\phi\Big(x(1+b)\prod_{j=t+1}^N(1+R_j)\Big) - \phi\Big(x(1+a)\prod_{j=t+1}^N(1+R_j)\Big)\right]}{(b-a)(1+r)^{N-t}S_{t-1}}$$

* The animation works in Acrobat Reader.

with $x = S_{t-1}$. Therefore, $\xi_t(x)$ will be positive (holding) when $x = S_{t-1}$ is sufficiently below $(K_1+K_2)/2$, and $\xi_t(x)$ will be negative (short selling) when $x = S_{t-1}$ is sufficiently above $(K_1+K_2)/2$.

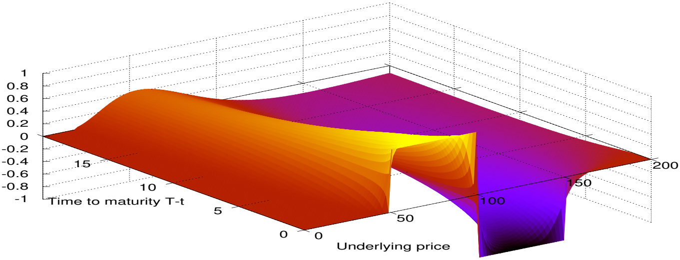


Fig. S.28: Delta of a butterfly option with strike prices $K_1 = 50$ and $K_2 = 150$.

Exercise 7.10

a) We have

$$\begin{aligned} C_t &= \mathrm{e}^{-(T-t)r}\,\mathbb{E}^*[S_T - K \mid \mathcal{F}_t] \\ &= \mathrm{e}^{-(T-t)r}\,\mathbb{E}^*[S_T \mid \mathcal{F}_t] - K\mathrm{e}^{-(T-t)r} \\ &= \mathrm{e}^{rt}\,\mathbb{E}^*\left[\mathrm{e}^{-rT}S_T \,\middle|\, \mathcal{F}_t\right] - K\mathrm{e}^{-(T-t)r} \\ &= \mathrm{e}^{rt}\mathrm{e}^{-rt}S_t - K\mathrm{e}^{-(T-t)r} \\ &= S_t - K\mathrm{e}^{-(T-t)r}. \end{aligned}$$

We can check that the function $g(x,t) = x - K\mathrm{e}^{-(T-t)r}$ satisfies the Black-Scholes PDE

$$rg(x,t) = \frac{\partial g}{\partial t}(x,t) + rx\frac{\partial g}{\partial x}(x,t) + \frac{\sigma^2}{2}x^2\frac{\partial^2 g}{\partial x^2}(x,t)$$

with terminal condition $g(x,T) = x - K$, since $\partial g(x,t)/\partial t = -rK\mathrm{e}^{-(T-t)r}$ and $\partial g(x,t)/\partial x = 1$.

b) We simply take $\xi_t = 1$ and $\eta_t = -K\mathrm{e}^{-rT}$ in order to have

$$C_t = \xi_t S_t + \eta_t \mathrm{e}^{rt} = S_t - K\mathrm{e}^{-(T-t)r}, \qquad 0 \le t \le T.$$

Note again that this hedging strategy is *constant* over time, and the relation $\xi_t = \partial g(S_t,t)/\partial x$ for the option Delta, cf. (S.6.26), is satisfied.

Exercise 7.11 Option pricing with dividends (Exercise 6.3 continued).

a) Let $\mathbb{P}^*$ denote the probability measure under which the process $(\widehat{B}_t)_{t\in\mathbb{R}_+}$ defined by

$$d\widehat{B}_t = \frac{\mu - r}{\sigma} dt + dB_t$$

is a standard Brownian motion. Under absence of arbitrage the asset price process $(S_t)_{t\in\mathbb{R}_+}$ has the dynamics

$$\begin{aligned} dS_t &= (\mu - \delta) S_t dt + \sigma S_t dB_t \\ &= (r - \delta) S_t dt + \sigma S_t d\widehat{B}_t, \end{aligned}$$

and the discounted asset price process $(\widetilde{S}_t)_{t\in\mathbb{R}_+} = (\mathrm{e}^{-rt} S_t)_{t\in\mathbb{R}_+}$ satisfies

$$d\widetilde{S}_t = -\delta \widetilde{S}_t dt + \sigma \widetilde{S}_t d\widehat{B}_t.$$

Assuming that the dividend yield δS_t per share is continuously reinvested in the portfolio, the self-financing portfolio condition

$$\begin{aligned} dV_t &= \underbrace{\eta_t dA_t + \xi_t dS_t}_{\text{Trading profit and loss}} + \underbrace{\delta \xi_t S_t dt}_{\text{Dividend payout}} \\ &= r\eta_t A_t dt + \xi_t ((r - \delta) S_t dt + \sigma S_t d\widehat{B}_t) + \delta \xi_t S_t dt \\ &= r\eta_t A_t dt + \xi_t (r S_t dt + \sigma S_t d\widehat{B}_t) \\ &= r V_t dt + \sigma \xi_t S_t d\widehat{B}_t, \qquad t \geq 0. \end{aligned}$$

In other words, no arbitrage is induced by the dividend payout. This yields

$$\begin{aligned} d\widetilde{V}_t &= d\left(\mathrm{e}^{-rt} V_t\right) \\ &= -r\mathrm{e}^{-rt} V_t dt + \mathrm{e}^{-rt} dV_t \\ &= \sigma \xi_t \mathrm{e}^{-rt} S_t d\widehat{B}_t \\ &= \sigma \xi_t \widetilde{S}_t d\widehat{B}_t \\ &= \xi_t (d\widetilde{S}_t + \delta \widetilde{S}_t dt), \qquad t \geq 0. \end{aligned}$$

Therefore, we have

$$\begin{aligned} \widetilde{V}_t - \widetilde{V}_0 &= \int_0^t d\widetilde{V}_u \\ &= \sigma \int_0^t \xi_u \widetilde{S}_u d\widehat{B}_u \\ &= \int_0^t \xi_u d\widetilde{S}_u + \delta \int_0^t \widetilde{S}_u du, \qquad t \geq 0. \end{aligned}$$

Here, the asset price process $(\mathrm{e}^{\delta t} S_t)_{t\in\mathbb{R}_+}$ with added dividend yield satisfies the equation

$$d\big(\mathrm{e}^{\delta t} S_t\big) = r\mathrm{e}^{\delta t} S_t dt + \sigma \mathrm{e}^{\delta t} S_t d\widehat{B}_t,$$

and after discount, the process $(\mathrm{e}^{-rt}\mathrm{e}^{\delta t} S_t)_{t\in\mathbb{R}_+} = (\mathrm{e}^{-(r-\delta)t} S_t)_{t\in\mathbb{R}_+}$ is a martingale under $\mathbb{P}^*$.

b) We have

$$\widetilde{V}_t = \widetilde{V}_0 + \sigma \int_0^t \xi_u \widetilde{S}_u d\widehat{B}_u, \qquad t \geq 0,$$

which is a martingale under $\mathbb{P}^*$ from Proposition 7.1, hence

$$\begin{aligned}\widetilde{V}_t &= \mathbb{E}^*\big[\widetilde{V}_T \bigm| \mathcal{F}_t\big] \\ &= \mathrm{e}^{-rT}\mathbb{E}^*[V_T \mid \mathcal{F}_t] \\ &= \mathrm{e}^{-rT}\mathbb{E}^*[C \mid \mathcal{F}_t],\end{aligned}$$

which implies

$$V_t = \mathrm{e}^{rt}\widetilde{V}_t = \mathrm{e}^{-(T-t)r}\mathbb{E}^*[C \mid \mathcal{F}_t], \qquad 0 \leq t \leq T.$$

c) After discounting the payoff $(S_T - K)^+$ at the continuously compounded interest rate r, we obtain

$$\begin{aligned}V_t &= \mathrm{e}^{-(T-t)r}\mathbb{E}^*\big[(S_T - K)^+ \mid \mathcal{F}_t\big] \\ &= \mathrm{e}^{-(T-t)r}\mathbb{E}^*\Big[\big(S_0\mathrm{e}^{\sigma\widehat{B}_T + (r-\delta-\sigma^2/2)T} - K\big)^+ \mid \mathcal{F}_t\Big] \\ &= \mathrm{e}^{-(T-t)\delta}\Big(\mathrm{e}^{-(T-t)(r-\delta)}\mathbb{E}^*\Big[\big(S_0\mathrm{e}^{\sigma\widehat{B}_T + (r-\delta-\sigma^2/2)T} - K\big)^+ \mid \mathcal{F}_t\Big]\Big) \\ &= \mathrm{e}^{-(T-t)\delta}\mathrm{Bl}(x, K, \sigma, r-\delta, T-t) \\ &= \mathrm{e}^{-(T-t)\delta}\Big(S_t\Phi\big(d_+^\delta(T-t)\big) - K\mathrm{e}^{-(T-t)(r-\delta)}\Phi\big(d_-^\delta(T-t)\big)\Big) \\ &= \mathrm{e}^{-(T-t)\delta}S_t\Phi\big(d_+^\delta(T-t)\big) - K\mathrm{e}^{-(T-t)r}\Phi\big(d_-^\delta(T-t)\big), \qquad 0 \leq t < T,\end{aligned}$$

where

$$d_+^\delta(T-t) := \frac{\log(S_t/K) + (r - \delta + \sigma^2/2)(T-t)}{|\sigma|\sqrt{T-t}}$$

and

$$d_-^\delta(T-t) := \frac{\log(S_t/K) + (r - \delta - \sigma^2/2)(T-t)}{|\sigma|\sqrt{T-t}}.$$

We also have

$$\begin{aligned}g(t,x) &= \mathrm{Bl}\big(x\mathrm{e}^{-(T-t)\delta}, K, \sigma, r, T-t\big) \\ &= \mathrm{e}^{-(T-t)\delta}\mathrm{Bl}\big(x, K\mathrm{e}^{(T-t)\delta}, \sigma, r, T-t\big), \qquad 0 \leq t \leq T.\end{aligned}$$

d) In view of the pricing formula

$$V_t = \mathrm{e}^{-(T-t)\delta}S_t\Phi\big(d_+^\delta(T-t)\big) - K\mathrm{e}^{-(T-t)r}\Phi\big(d_-^\delta(T-t)\big),$$

the Delta of the option is identified as

$$\xi_t = \mathrm{e}^{-(T-t)\delta}\Phi\big(d_+^\delta(T-t)\big), \qquad 0 \leq t < T,$$

which recovers the result of Exercise 6.3-(d).

Exercise 7.12 We start by pricing the "inner" at-the-money option with payoff $(S_{T_2} - S_{T_1})^+$ and strike price $K = S_{T_1}$ at time T_1 as

$$
\begin{aligned}
& \mathrm{e}^{-(T_2-T_1)r}\, \mathbb{E}^*\left[(S_{T_2} - S_{T_1})^+ \mid \mathcal{F}_{T_1}\right] \\
& = S_{T_1} \Phi\left(\frac{(r+\sigma^2/2)(T_2-T_1) + \log(S_{T_1}/S_{T_1})}{\sigma\sqrt{T_2-T_1}}\right) \\
& \quad - S_{T_1} \mathrm{e}^{-(T_2-T_1)r} \Phi\left(\frac{(r-\sigma^2/2)(T_2-T_1) + \log(S_{T_1}/S_{T_1})}{\sigma\sqrt{T_2-T_1}}\right) \\
& = S_{T_1} \Phi\left(\frac{r+\sigma^2/2}{\sigma}\sqrt{T_2-T_1}\right) - S_{T_1} \mathrm{e}^{-(T_2-T_1)r} \Phi\left(\frac{r-\sigma^2/2}{\sigma}\sqrt{T_2-T_1}\right),
\end{aligned}
$$

where we applied (7.24) with $T = T_2$, $t = T_1$, and $K = S_{T_1}$. As a consequence, the forward start option can be priced as

$$
\begin{aligned}
& \mathrm{e}^{-(T_1-t)r}\, \mathbb{E}^*\left[\mathrm{e}^{-(T_2-T_1)r}\, \mathbb{E}^*\left[(S_{T_2} - S_{T_1})^+ \mid \mathcal{F}_{T_1}\right] \Big| \mathcal{F}_t\right] \\
& = \mathrm{e}^{-(T_1-t)r} \\
& \quad \times \mathbb{E}^*\left[S_{T_1} \Phi\left(\frac{r+\sigma^2/2}{\sigma}\sqrt{T_2-T_1}\right) - S_{T_1} \mathrm{e}^{-(T_2-T_1)r} \Phi\left(\frac{r-\sigma^2/2}{\sigma}\sqrt{T_2-T_1}\right) \ \Big| \ \mathcal{F}_t\right] \\
& = \mathrm{e}^{-(T_1-t)r} \\
& \quad \times \left(\Phi\left(\frac{r+\sigma^2/2}{\sigma}\sqrt{T_2-T_1}\right) - \mathrm{e}^{-(T_2-T_1)r} \Phi\left(\frac{r-\sigma^2/2}{\sigma}\sqrt{T_2-T_1}\right)\right) \mathbb{E}^*[S_{T_1} \mid \mathcal{F}_t] \\
& = S_t \left(\Phi\left(\frac{r+\sigma^2/2}{\sigma}\sqrt{T_2-T_1}\right) - \mathrm{e}^{-(T_2-T_1)r} \Phi\left(\frac{r-\sigma^2/2}{\sigma}\sqrt{T_2-T_1}\right)\right),
\end{aligned}
$$

$0 \leq t \leq T_1$.

Exercise 7.13 From the Black-Scholes formula we have

$$
\begin{aligned}
\mathrm{e}^{-(T_k-T_{k-1})r}\, \mathbb{E}^*\left[\left(\frac{S_{T_k}}{S_{T_{k-1}}} - K\right)^+ \Big| \mathcal{F}_{T_{k-1}}\right] &= \Phi\left(\frac{(r+\sigma^2/2)(T_k-T_{k-1}) - \log K}{\sigma\sqrt{T_k-T_{k-1}}}\right) \\
& \quad - K \mathrm{e}^{-(T_k-T_{k-1})r} \Phi\left(\frac{(r-\sigma^2/2)(T_k-T_{k-1}) - \log K}{\sigma\sqrt{T_k-T_{k-1}}}\right).
\end{aligned}
$$

As a consequence, for $t = T_0 = 0$ we have

$$
\mathbb{E}^*\left[\left(\frac{S_{T_k}}{S_{T_{k-1}}} - K\right)^+ \Big| \mathcal{F}_t\right] = \mathbb{E}^*\left[\mathbb{E}^*\left[\left(\frac{S_{T_k}}{S_{T_{k-1}}} - K\right)^+ \Big| \mathcal{F}_{T_{k-1}}\right]\right]
$$

$$= \mathbb{E}^* \left[\mathrm{e}^{(T_k - T_{k-1})r} \Phi \left(\frac{(r+\sigma^2/2)(T_k - T_{k-1}) - \log K}{\sigma \sqrt{T_k - T_{k-1}}} \right) \right.$$
$$\left. -K\Phi \left(\frac{(r-\sigma^2/2)(T_k - T_{k-1}) - \log K}{\sigma \sqrt{T_k - T_{k-1}}} \right) \right]$$
$$= \mathrm{e}^{(T_k - T_{k-1})r} \Phi \left(\frac{(r+\sigma^2/2)(T_k - T_{k-1}) - \log K}{\sigma \sqrt{T_k - T_{k-1}}} \right)$$
$$- K\Phi \left(\frac{(r-\sigma^2/2)(T_k - T_{k-1}) - \log K}{\sigma \sqrt{T_k - T_{k-1}}} \right).$$

Hence, the cliquet option can be priced at time $t = T_0 = 0$ as

$$\sum_{k=1}^n \mathrm{e}^{-rT_k} \mathbb{E}^* \left[\left(\frac{S_{T_k}}{S_{T_{k-1}}} - K \right)^+ \right]$$
$$= \sum_{k=1}^n \left(\mathrm{e}^{-rT_{k-1}} \Phi \left(\frac{(r+\sigma^2/2)(T_k - T_{k-1}) - \log K}{\sigma \sqrt{T_k - T_{k-1}}} \right) \right.$$
$$\left. -K\mathrm{e}^{-rT_k} \Phi \left(\frac{(r-\sigma^2/2)(T_k - T_{k-1}) - \log K}{\sigma \sqrt{T_k - T_{k-1}}} \right) \right).$$

Exercise 7.14 (Exercise 6.9 continued). We have

$$\begin{aligned} C(t, S_t) &= \mathrm{e}^{-(T-t)r} \mathbb{E}^* \left[\log S_T \mid \mathcal{F}_t \right] \\ &= \mathrm{e}^{-(T-t)r} \mathbb{E}^* \left[\log S_t + (\widehat{B}_T - \widehat{B}_t)\sigma + \left(r - \frac{\sigma^2}{2} \right)(T-t) \mid \mathcal{F}_t \right] \\ &= \mathrm{e}^{-(T-t)r} \log S_t + \mathrm{e}^{-(T-t)r} \left(r - \frac{\sigma^2}{2} \right)(T-t), \qquad 0 \leq t \leq T. \end{aligned}$$

Exercise 7.15 (Exercise 6.5 continued).

a) By (5.20), for all $t \in [0, T]$, we have

$$\begin{aligned} C(t, S_t) &= \mathrm{e}^{-(T-t)r} \mathbb{E}[S_T^2 \mid \mathcal{F}_t] \\ &= \mathrm{e}^{-(T-t)r} \mathbb{E}\left[S_t^2 \frac{S_T^2}{S_t^2} \,\middle|\, \mathcal{F}_t \right] \\ &= \mathrm{e}^{-(T-t)r} S_t^2 \, \mathbb{E}\left[\frac{S_T^2}{S_t^2} \,\middle|\, \mathcal{F}_t \right] \\ &= \mathrm{e}^{-(T-t)r} S_t^2 \, \mathbb{E}\left[\frac{S_T^2}{S_t^2} \right] \\ &= \mathrm{e}^{-(T-t)r} S_t^2 \, \mathbb{E}\left[\mathrm{e}^{2(B_T - B_t)\sigma - (T-t)\sigma^2 + 2(T-t)r} \right] \end{aligned}$$

$$= \mathrm{e}^{-(T-t)r} S_t^2 \mathrm{e}^{-(T-t)\sigma^2+2(T-t)r} \,\mathbb{E}\left[\mathrm{e}^{2(B_T-B_t)\sigma}\right]$$
$$= S_t^2 \mathrm{e}^{(r+\sigma^2)(T-t)},$$

where we used the Gaussian moment generating function (MGF) formula, *i.e.*

$$\mathbb{E}\left[\mathrm{e}^{2(B_T-B_t)\sigma}\right] = \mathrm{e}^{2(T-t)\sigma^2}$$

for the normal random variable $B_T - B_t \simeq \mathcal{N}(0, T-t)$, $0 \le t < T$.

b) For all $t \in [0, T]$, we have

$$\xi_t = \frac{\partial C}{\partial x}(t, x)_{|x=S_t} = 2S_t \mathrm{e}^{(r+\sigma^2)(T-t)},$$

i.e.

$$\xi_t S_t = 2S_t^2 \mathrm{e}^{(r+\sigma^2)(T-t)} = 2C(t, S_t),$$

and

$$\eta_t = \frac{C(t,S_t) - \xi_t S_t}{A_t} = \frac{\mathrm{e}^{-rt}}{A_0}\big(S_t^2 \mathrm{e}^{(r+\sigma^2)(T-t)} - 2S_t^2 \mathrm{e}^{(r+\sigma^2)(T-t)}\big)$$
$$= -\frac{S_t^2}{A_0} \mathrm{e}^{\sigma^2(T-t)+(T-2t)r},$$

i.e.

$$\eta_t A_t = -S_t^2 \frac{A_t}{A_0} \mathrm{e}^{\sigma^2(T-t)+(T-2t)r} = -S_t^2 \mathrm{e}^{\sigma^2(T-t)+(T-t)r} = -C(t, S_t).$$

As for the self-financing condition, we have

$$\begin{aligned} dC(t,S_t) &= d\big(S_t^2 \mathrm{e}^{(r+\sigma^2)(T-t)}\big) \\ &= -(r+\sigma^2)\mathrm{e}^{(r+\sigma^2)(T-t)} S_t^2 dt + \mathrm{e}^{(r+\sigma^2)(T-t)} d\big(S_t^2\big) \\ &= -(r+\sigma^2)\mathrm{e}^{(r+\sigma^2)(T-t)} S_t^2 dt + \mathrm{e}^{(r+\sigma^2)(T-t)} \big(2S_t dS_t + \sigma^2 S_t^2 dt\big) \\ &= -r\mathrm{e}^{(r+\sigma^2)(T-t)} S_t^2 dt + 2S_t \mathrm{e}^{(r+\sigma^2)(T-t)} dS_t, \end{aligned}$$

and

$$\begin{aligned} \xi_t dS_t + \eta_t dA_t &= 2S_t \mathrm{e}^{(r+\sigma^2)(T-t)} dS_t - r\frac{S_t^2}{A_0} \mathrm{e}^{\sigma^2(T-t)+(T-2t)r} A_t dt \\ &= 2S_t \mathrm{e}^{(r+\sigma^2)(T-t)} dS_t - rS_t^2 \mathrm{e}^{\sigma^2(T-t)+(T-t)r} dt, \end{aligned}$$

which recovers $dC(t, S_t) = \xi_t dS_t + \eta_t dA_t$, *i.e.* the portfolio strategy is self-financing.

Exercise 7.16 (Exercise 6.11 continued).

a) The discounted process $X_t := \mathrm{e}^{-rt} S_t$ satisfies

$$dX_t = (\alpha - r) X_t dt + \sigma \mathrm{e}^{-rs} dB_s,$$

which is a martingale when $\alpha = r$ by Proposition 7.1, as in this case it becomes a stochastic integral with respect to a standard Brownian motion. This fact can be recovered by directly computing the conditional expectation $\mathbb{E}[X_t \mid \mathcal{F}_s]$ and showing it is equal to X_s. By (4.35), see Exercise 6.11, we have

$$S_t = S_0 \mathrm{e}^{\alpha t} + \sigma \int_0^t \mathrm{e}^{(t-s)\alpha} dB_s,$$

hence

$$X_t = S_0 + \sigma \int_0^t \mathrm{e}^{-rs} dB_s, \qquad t \geq 0,$$

and

$$\begin{aligned}
\mathbb{E}[X_t \mid \mathcal{F}_s] &= \mathbb{E}\left[S_0 + \sigma \int_0^t \mathrm{e}^{-ru} dB_u \,\Big|\, \mathcal{F}_s \right] \\
&= \mathbb{E}[S_0] + \sigma \, \mathbb{E}\left[\int_0^t \mathrm{e}^{-ru} dB_u \,\Big|\, \mathcal{F}_s \right] \\
&= S_0 + \sigma \, \mathbb{E}\left[\int_0^s \mathrm{e}^{-ru} dB_u \,\Big|\, \mathcal{F}_s \right] + \sigma \, \mathbb{E}\left[\int_s^t \mathrm{e}^{-ru} dB_u \,\Big|\, \mathcal{F}_s \right] \\
&= S_0 + \sigma \int_0^s \mathrm{e}^{-ru} dB_u + \sigma \, \mathbb{E}\left[\int_s^t \mathrm{e}^{-ru} dB_u \right] \\
&= S_0 + \sigma \int_0^s \mathrm{e}^{-ru} dB_u \\
&= X_s, \qquad 0 \leq s \leq t.
\end{aligned}$$

b) We rewrite the stochastic differential equation satisfied by $(S_t)_{t \in \mathbb{R}_+}$ as

$$dS_t = \alpha S_t dt + \sigma dB_t = r S_t dt + \sigma d\widehat{B}_t,$$

where

$$d\widehat{B}_t := \frac{\alpha - r}{\sigma} S_t dt + dB_t,$$

which allows us to rewrite (4.35), by taking $\alpha := -r$ therein, as

$$S_t = \mathrm{e}^{rt} \left(S_0 + \sigma \int_0^t \mathrm{e}^{-rs} d\widehat{B}_s \right) = S_0 \mathrm{e}^{rt} + \sigma \int_0^t \mathrm{e}^{(t-s)r} d\widehat{B}_s. \tag{S.7.32}$$

Taking

$$\psi_t := \frac{\alpha - r}{\sigma} S_t, \qquad 0 \leq t \leq T,$$

in the Girsanov Theorem 7.3, the process $(\widehat{B}_t)_{t \in \mathbb{R}_+}$ is a standard Brownian motion under the probability measure $\mathbb{P}_\alpha$ defined by

$$\frac{d\mathbb{P}_\alpha}{d\mathbb{P}} := \exp\left(-\int_0^T \psi_t dB_t - \frac{1}{2}\int_0^T \psi_t^2 dt\right)$$
$$= \exp\left(-\frac{\alpha - r}{\sigma}\int_0^T S_t dB_t - \frac{1}{2}\left(\frac{\alpha - r}{\sigma}\right)^2 \int_0^T S_t^2 dt\right),$$

and $(X_t)_{t\in\mathbb{R}_+}$ is a martingale under $\mathbb{P}_\alpha$.

c) Using (S.7.32) under the risk-neutral probability measure $\mathbb{P}^*$, we have

$$\begin{aligned}
&C(t, S_t) = \mathrm{e}^{-(T-t)r}\,\mathbb{E}_\alpha[\exp(S_T) \mid \mathcal{F}_t] \\
&= \mathrm{e}^{-(T-t)r}\,\mathbb{E}_\alpha\left[\exp\left(\mathrm{e}^{rT} S_0 + \sigma\int_0^T \mathrm{e}^{(T-u)r} d\widehat{B}_u\right) \,\Big|\, \mathcal{F}_t\right] \\
&= \mathrm{e}^{-(T-t)r}\,\mathbb{E}_\alpha\left[\exp\left(\mathrm{e}^{rT} S_0 + \sigma\int_0^t \mathrm{e}^{(T-u)r} d\widehat{B}_u + \sigma\int_t^T \mathrm{e}^{(T-u)r} d\widehat{B}_u\right) \,\Big|\, \mathcal{F}_t\right] \\
&= \exp\left(-(T-t)r + \mathrm{e}^{(T-t)r} S_t\right)\mathbb{E}_\alpha\left[\exp\left(\sigma\int_t^T \mathrm{e}^{(T-u)r} d\widehat{B}_u\right) \,\Big|\, \mathcal{F}_t\right] \\
&= \exp\left(-(T-t)r + \mathrm{e}^{(T-t)r} S_t\right)\mathbb{E}_\alpha\left[\exp\left(\sigma\int_t^T \mathrm{e}^{(T-u)r} d\widehat{B}_u\right)\right] \\
&= \exp\left(-(T-t)r + \mathrm{e}^{(T-t)r} S_t\right)\exp\left(\frac{\sigma^2}{2}\int_t^T \mathrm{e}^{2(T-u)r} du\right) \\
&= \exp\left(-(T-t)r + \mathrm{e}^{(T-t)r} S_t + \frac{\sigma^2}{4r}(\mathrm{e}^{2(T-t)r} - 1)\right), \qquad 0 \le t \le T.
\end{aligned}$$

d) We have

$$\xi_t = \frac{\partial C}{\partial x}(t, S_t) = \exp\left(S_t \mathrm{e}^{(T-t)r} + \frac{\sigma^2}{4r}(\mathrm{e}^{2(T-t)r} - 1)\right)$$

and

$$\begin{aligned}
\eta_t &= \frac{C(t, S_t) - \xi_t S_t}{A_t} \\
&= \frac{\mathrm{e}^{-(T-t)r}}{A_t}\exp\left(S_t \mathrm{e}^{(T-t)r} + \frac{\sigma^2}{4r}(\mathrm{e}^{2(T-t)r} - 1)\right) \\
&\quad - \frac{S_t}{A_t}\exp\left(S_t \mathrm{e}^{(T-t)r} + \frac{\sigma^2}{4r}(\mathrm{e}^{2(T-t)r} - 1)\right).
\end{aligned}$$

e) We have

$$\begin{aligned}
dC(t, S_t) &= r\mathrm{e}^{-(T-t)r}\exp\left(S_t \mathrm{e}^{(T-t)r} + \frac{\sigma^2}{4r}(\mathrm{e}^{2(T-t)r} - 1)\right)dt \\
&\quad - rS_t \exp\left(S_t \mathrm{e}^{(T-t)r} + \frac{\sigma^2}{4r}(\mathrm{e}^{2(T-t)r} - 1)\right)dt \\
&\quad - \frac{\sigma^2}{2}\mathrm{e}^{(T-t)r}\exp\left(S_t \mathrm{e}^{(T-t)r} + \frac{\sigma^2}{4r}(\mathrm{e}^{2(T-t)r} - 1)\right)dt
\end{aligned}$$

$$
\begin{aligned}
&+\exp\left(S_t \mathrm{e}^{(T-t)r} + \frac{\sigma^2}{4r}(\mathrm{e}^{2(T-t)r}-1)\right)dS_t \\
&+\frac{1}{2}\mathrm{e}^{(T-t)r}\exp\left(S_t \mathrm{e}^{(T-t)r} + \frac{\sigma^2}{4r}(\mathrm{e}^{2(T-t)r}-1)\right)\sigma^2 dt \\
&= r\mathrm{e}^{-(T-t)r}\exp\left(S_t \mathrm{e}^{(T-t)r} + \frac{\sigma^2}{4r}(\mathrm{e}^{2(T-t)r}-1)\right)dt \\
&-rS_t\exp\left(S_t \mathrm{e}^{(T-t)r} + \frac{\sigma^2}{4r}(\mathrm{e}^{2(T-t)r}-1)\right)dt + \xi_t dS_t.
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
&\xi_t dS_t + \eta_t dA_t = \xi_t dS_t \\
&+r\mathrm{e}^{-(T-t)r}\exp\left(S_t \mathrm{e}^{(T-t)r} + \frac{\sigma^2}{4r}(\mathrm{e}^{2(T-t)r}-1)\right)dt \\
&-rS_t\exp\left(S_t \mathrm{e}^{(T-t)r} + \frac{\sigma^2}{4r}(\mathrm{e}^{2(T-t)r}-1)\right)dt,
\end{aligned}
$$

showing that

$$
dC(t,S_t) = \xi_t dS_t + \eta_t dA_t,
$$

and confirming that the strategy $(\xi_t,\eta_t)_{t\in\mathbb{R}_+}$ is self-financing.

Exercise 7.17

a) Using (S.7.32) under the risk-neutral probability measure $\mathbb{P}^*$, we have

$$
\begin{aligned}
&C(t,S_t) = \mathrm{e}^{-(T-t)r}\,\mathbb{E}_\alpha[S_T^2 \mid \mathcal{F}_t] \\
&= \mathrm{e}^{-(T-t)r}\,\mathbb{E}_\alpha\left[\left(\mathrm{e}^{rT}S_0 + \sigma\int_0^T \mathrm{e}^{(T-u)r}d\widehat{B}_u\right)^2 \;\Big|\; \mathcal{F}_t\right] \\
&= \mathrm{e}^{-(T-t)r}\,\mathbb{E}_\alpha\left[\left(\mathrm{e}^{rT}S_0 + \sigma\int_0^t \mathrm{e}^{(T-u)r}d\widehat{B}_u + \sigma\int_t^T \mathrm{e}^{(T-u)r}d\widehat{B}_u\right)^2 \;\Big|\; \mathcal{F}_t\right] \\
&= \mathrm{e}^{-(T-t)r}\,\mathbb{E}_\alpha\left[\left(\mathrm{e}^{rT}S_0 + \sigma\int_0^t \mathrm{e}^{(T-u)r}d\widehat{B}_u\right)^2 \;\Big|\; \mathcal{F}_t\right] \\
&\quad +2\sigma\mathrm{e}^{-(T-t)r}\left(\mathrm{e}^{rT}S_0 + \sigma\int_0^t \mathrm{e}^{(T-u)r}d\widehat{B}_u\right)\mathbb{E}_\alpha\left[\int_t^T \mathrm{e}^{(T-u)r}d\widehat{B}_u \;\Big|\; \mathcal{F}_t\right] \\
&\quad +\sigma^2\mathrm{e}^{-(T-t)r}\,\mathbb{E}_\alpha\left[\left(\int_t^T \mathrm{e}^{(T-u)r}d\widehat{B}_u\right)^2 \;\Big|\; \mathcal{F}_t\right] \\
&= \mathrm{e}^{-(T-t)r}\left(\mathrm{e}^{rT}S_0 + \sigma\int_0^t \mathrm{e}^{(T-u)r}d\widehat{B}_u\right)^2 + \sigma^2\mathrm{e}^{-(T-t)r}\,\mathbb{E}_\alpha\left[\left(\int_t^T \mathrm{e}^{(T-u)r}d\widehat{B}_u\right)^2\right] \\
&= \mathrm{e}^{-(T-t)r}\left(\mathrm{e}^{rT}S_0 + \sigma\int_0^t \mathrm{e}^{(T-u)r}d\widehat{B}_u\right)^2 + \sigma^2\mathrm{e}^{-(T-t)r}\int_t^T \mathrm{e}^{2(T-u)r}du
\end{aligned}
$$

$$= e^{(T-t)r}S_t^2 + \frac{\sigma^2}{2r}\big(e^{(T-t)r} - e^{-(T-t)r}\big)$$
$$= e^{(T-t)r}S_t^2 + \sigma^2\frac{\sinh((T-t)r)}{r}, \qquad 0 \le t \le T.$$

b) We find

$$\xi_t = \frac{\partial C}{\partial x}(t, S_t) = 2e^{(T-t)r}S_t, \qquad 0 \le t \le T.$$

Exercise 7.18 (Exercise 5.8 continued, see Proposition 4.1 in Carmona and Durrleman (2003)). Letting $\alpha := \mathbb{E}^*[S_T] = e^{rt}\big(S_0^{(2)} - S_0^{(1)}\big)$ and

$$\eta^2 := \mathrm{Var}^*\left[S_t^{(2)} - S_t^{(1)}\right]$$
$$= e^{2rt}\Big(\big(S_0^{(1)}\big)^2 e^{\sigma_1^2 t} + \big(S_0^{(2)}\big)^2 e^{\sigma_2^2 t} - 2S_0^{(1)}S_0^{(2)}e^{\rho\sigma_1\sigma_2 t} - \big(S_0^{(2)} - S_0^{(1)}\big)^2\Big),$$

we approximate

$$e^{-rt}\,\mathbb{E}^*[(S_T - K)^+] \simeq \frac{e^{-rt}}{\sqrt{2\pi\eta^2}}\int_{-\infty}^{\infty}(x-K)^+ e^{-(x-\alpha)^2/(2\eta^2)}dx$$
$$= \frac{e^{-rt}}{\sqrt{2\pi\eta^2}}\int_K^{\infty}(x-K)e^{-(x-\alpha)^2/(2\eta^2)}dx$$
$$= \frac{e^{-rt}}{\sqrt{2\pi\eta^2}}\int_K^{\infty}x e^{-(x-\alpha)^2/(2\eta^2)}dx - \frac{Ke^{-rt}}{\sqrt{2\pi\eta^2}}\int_K^{\infty}e^{-(x-\alpha)^2/(2\eta^2)}dx$$
$$= \frac{\eta e^{-rt}}{\sqrt{2\pi}}\int_{(K-\alpha)/\eta}^{\infty}(x+\alpha)e^{-x^2/2}dx - \frac{Ke^{-rt}}{\sqrt{2\pi}}\int_{(K-\alpha)/\eta}^{\infty}e^{-x^2/2}dx$$
$$= -\frac{\eta e^{-rt}}{\sqrt{2\pi}}\left[e^{-x^2/2}\right]_{(K-\alpha)/\eta}^{\infty} - (K-\alpha)e^{-rt}\Phi\left(-\frac{K-\alpha}{\eta}\right)$$
$$= \frac{\eta e^{-rt}}{\sqrt{2\pi}}e^{-(K-\alpha)^2/(2\eta^2)} - (K-\alpha)e^{-rt}\Phi\left(-\frac{K-\alpha}{\eta}\right).$$

Remark: We note that the expected value $\mathbb{E}^*[\phi(S_T - K)]$ can be exactly computed from

$$\mathbb{E}^*[\phi(S_T)] = \mathbb{E}^*\left[\phi\big(S_T^{(2)} - S_T^{(1)}\big)\right] = \int_0^{\infty}\int_0^{\infty}\phi(x-y)\varphi_1(x)\varphi_2(y)dxdy,$$

where

$$\varphi_i(x) = \frac{1}{x\sigma_i\sqrt{2\pi T}}\exp\left(-\frac{(-(r-\sigma_i^2/2)T + \log(x/S_0))^2}{2\sigma_i^2 T}\right)$$

is the lognormal probability density function of $S_T^{(i)}$, $i = 1, 2$. In particular, we have

$$\mathbb{E}^*[\phi(S_T)] = \mathbb{E}^*\left[\phi\left(S_T^{(2)} - S_T^{(1)}\right)\right] = \int_0^\infty \phi(z)\varphi(z)dz,$$

where

$$\begin{aligned}\varphi(z) &= \int_0^\infty \varphi_1(z+y)\varphi_2(y)dydy \\ &= \int_0^\infty \mathrm{e}^{-(-(r-\sigma_1^2/2)T+\log((z+y)/S_0))^2/(2\sigma_1^2T)-(-(r-\sigma_2^2/2)T+\log(y/S_0))^2/(2\sigma_2^2T)} \\ &\qquad \times \frac{dy}{2\pi T\sigma_1\sigma_2(z+y)y}\end{aligned}$$

is the probability density function of S_T.

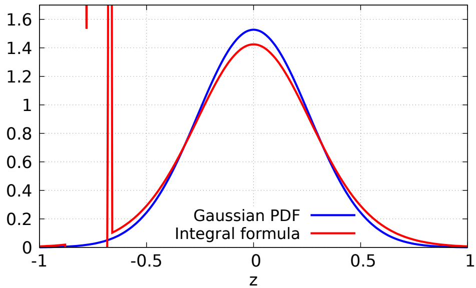


Fig. S.29: Gaussian approximation of spread probability density function.

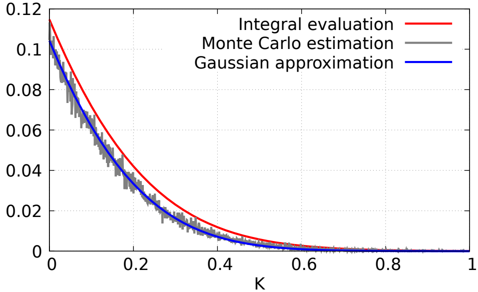


Fig. S.30: Gaussian approximation of spread option prices.

Exercise 7.19 (Exercise 6.2 continued). If C is a contingent claim payoff of the form $C = \phi(S_T)$ such that $(\xi_t, \eta_t)_{t\in[0,T]}$ hedges the claim payoff C, the arbitrage-free price of the claim payoff C at time $t \in [0, T]$ is given by

$$\pi_t(X) = V_t = \mathrm{e}^{-r(T-t)}\, \mathbb{E}^*[\phi(S_T) \mid \mathcal{F}_t], \qquad 0 \le t \le T,$$

where $\mathbb{E}^*$ denotes expectation under the risk-neutral measure $\mathbb{P}^*$. Hence, from the noncentral Chi square probability density function

$$\begin{aligned} & f_{T-t}(x) \\ & = \frac{2\beta}{\sigma^2\big(1-\mathrm{e}^{-\beta(T-t)}\big)} \exp\left(-\frac{2\beta\big(x + r_t\mathrm{e}^{-\beta(T-t)}\big)}{\sigma^2\big(1-\mathrm{e}^{-\beta(T-t)}\big)}\right) \left(\frac{x}{r_t\mathrm{e}^{-\beta(T-t)}}\right)^{\alpha\beta/\sigma^2-1/2} \\ & \quad \times I_{2\alpha\beta/\sigma^2-1}\left(\frac{4\beta\sqrt{r_t x \mathrm{e}^{-\beta(T-t)}}}{\sigma^2\big(1-\mathrm{e}^{-\beta(T-t)}\big)}\right), \end{aligned}$$

of S_T given S_t, $x > 0$, we find

$$\begin{aligned} & g(t, S_t) = \mathrm{e}^{-r(T-t)}\, \mathbb{E}^*[\phi(S_T) \mid \mathcal{F}_t] \\ & = \frac{2\beta\mathrm{e}^{-r(T-t)}}{\sigma^2\big(1-\mathrm{e}^{-\beta(T-t)}\big)} \int_0^\infty \phi(x) \left(\frac{x}{S_t\mathrm{e}^{-\beta(T-t)}}\right)^{\alpha\beta/\sigma^2-1/2} \mathrm{e}^{-2\frac{\beta(x+S_t\mathrm{e}^{-\beta(T-t)})}{\sigma^2(1-\mathrm{e}^{-\beta(T-t)})}} \\ & \qquad \times I_{2\alpha\beta/\sigma^2-1}\left(\frac{4\beta\sqrt{x S_t \mathrm{e}^{-\beta(T-t)}}}{\sigma^2\big(1-\mathrm{e}^{-\beta(T-t)}\big)}\right) dx \end{aligned}$$

$0 \le t \le T$, under the Feller condition $2\alpha\beta \ge \sigma^2$.

Exercise 7.20

a) We have

$$\frac{\partial f}{\partial t}(t,x) = (r - \sigma^2/2) f(t,x), \qquad \frac{\partial f}{\partial x}(t,x) = \sigma f(t,x),$$

and

$$\frac{\partial^2 f}{\partial x^2}(t,x) = \sigma^2 f(t,x),$$

hence

$$\begin{aligned} dS_t &= df(t, B_t) \\ &= \frac{\partial f}{\partial t}(t, B_t)dt + \frac{\partial f}{\partial x}(t, B_t)dB_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t, B_t)dt \\ &= \left(r - \frac{1}{2}\sigma^2\right) f(t, B_t)dt + \sigma f(t, B_t)dB_t + \frac{1}{2}\sigma^2 f(t, B_t)dt \\ &= r f(t, B_t)dt + \sigma f(t, B_t)dB_t \\ &= rS_t dt + \sigma S_t dB_t. \end{aligned}$$

b) We have

$$\begin{aligned}\mathbb{E}\left[\mathrm{e}^{\sigma B_T} \,\middle|\, \mathcal{F}_t\right] &= \mathbb{E}\left[\mathrm{e}^{(B_T-B_t+B_t)\sigma} \,\middle|\, \mathcal{F}_t\right] \\ &= \mathrm{e}^{\sigma B_t}\, \mathbb{E}\left[\mathrm{e}^{(B_T-B_t)\sigma} \,\middle|\, \mathcal{F}_t\right] \\ &= \mathrm{e}^{\sigma B_t}\, \mathbb{E}\left[\mathrm{e}^{(B_T-B_t)\sigma}\right] \\ &= \mathrm{e}^{\sigma B_t+\sigma^2(T-t)/2}.\end{aligned}$$

c) We have

$$\begin{aligned}\mathbb{E}[S_T \mid \mathcal{F}_t] &= \mathbb{E}\left[\mathrm{e}^{\sigma B_T+rT-\sigma^2T/2} \,\middle|\, \mathcal{F}_t\right] \\ &= \mathrm{e}^{rT-\sigma^2T/2}\, \mathbb{E}\left[\mathrm{e}^{\sigma B_T} \,\middle|\, \mathcal{F}_t\right] \\ &= \mathrm{e}^{rT-\sigma^2T/2}\mathrm{e}^{\sigma B_t+\sigma^2(T-t)/2} \\ &= \mathrm{e}^{rT+\sigma B_t-\sigma^2t/2} \\ &= \mathrm{e}^{(T-t)r+\sigma B_t+rt-\sigma^2t/2} \\ &= \mathrm{e}^{(T-t)r}S_t, \qquad 0 \le t \le T.\end{aligned}$$

d) We have

$$\begin{aligned}V_t &= \mathrm{e}^{-(T-t)r}\, \mathbb{E}[C \mid \mathcal{F}_t] \\ &= \mathrm{e}^{-(T-t)r}\, \mathbb{E}[S_T - K \mid \mathcal{F}_t] \\ &= \mathrm{e}^{-(T-t)r}\, \mathbb{E}[S_T \mid \mathcal{F}_t] - \mathrm{e}^{-(T-t)r}\, \mathbb{E}[K \mid \mathcal{F}_t] \\ &= S_t - \mathrm{e}^{-(T-t)r}K, \qquad 0 \le t \le T.\end{aligned}$$

e) We take $\xi_t = 1$ and $\eta_t = -K\mathrm{e}^{-rT}/A_0$, $t \in [0,T]$.

f) We find

$$V_T = \mathbb{E}[C \mid \mathcal{F}_T] = C.$$

Exercise 7.21 Binary options. (Exercise 6.10 continued).

a) By definition of the indicator (or step) functions $\mathbb{1}_{[K,\infty)}$ and $\mathbb{1}_{[0,K]}$ we have

$$\mathbb{1}_{[K,\infty)}(x) = \begin{cases} 1 & \text{if } x \ge K, \\ \\ 0 & \text{if } x < K, \end{cases} \qquad \textit{resp.} \quad \mathbb{1}_{[0,K]}(x) = \begin{cases} 1 & \text{if } x \le K, \\ \\ 0 & \text{if } x > K, \end{cases}$$

which shows the claimed result by the definition of C_b and P_b.

b) We have

$$\begin{aligned}\pi_t(C_b) &= \mathrm{e}^{-(T-t)r}\, \mathbb{E}[C_b \mid \mathcal{F}_t] \\ &= \mathrm{e}^{-(T-t)r}\, \mathbb{E}\left[\mathbb{1}_{[K,\infty)}(S_T) \,\middle|\, S_t\right] \\ &= \mathrm{e}^{-(T-t)r}\mathbb{P}(S_T \ge K \mid S_t)\end{aligned}$$

$$= C_b(t, S_t).$$

c) We have $\pi_t(C_b) = C_b(t, S_t)$, where

$$\begin{aligned} C_b(t,x) &= \mathrm{e}^{-(T-t)r}\mathbb{P}(S_T > K \mid S_t = x) \\ &= \mathrm{e}^{-(T-t)r}\Phi\left(\frac{(r-\sigma^2/2)(T-t)+\log(S_t/K)}{\sigma\sqrt{T-t}}\right) \\ &= \mathrm{e}^{-(T-t)r}\Phi\left(d_-(T-t)\right), \end{aligned}$$

with

$$d_-(T-t) = \frac{(r-\sigma^2/2)(T-t)+\log(S_t/K)}{\sigma\sqrt{T-t}}.$$

d) The price of this modified contract with payoff

$$C_\alpha = \mathbb{1}_{[K,\infty)}(S_T) + \alpha \mathbb{1}_{[0,K)}(S_T)$$

is given by

$$\begin{aligned} &\pi_t(C_\alpha) = \mathrm{e}^{-(T-t)r}\,\mathbb{E}\left[\mathbb{1}_{[K,\infty)}(S_T) + \alpha\mathbb{1}_{[0,K)}(S_T)\,\middle|\,S_t\right] \\ &= \mathrm{e}^{-(T-t)r}\mathbb{P}(S_T \geq K \mid S_t) + \alpha\mathrm{e}^{-(T-t)r}\mathbb{P}(S_T \leq K \mid S_t) \\ &= \mathrm{e}^{-(T-t)r}\mathbb{P}(S_T \geq K \mid S_t) + \alpha\mathrm{e}^{-(T-t)r}(1-\mathbb{P}(S_T \geq K \mid S_t)) \\ &= \alpha\mathrm{e}^{-(T-t)r}\mathrm{e}^{-(T-t)r} + (1-\alpha)\mathbb{P}(S_T \geq K \mid S_t) \\ &= \alpha\mathrm{e}^{-(T-t)r} + (1-\alpha)\mathrm{e}^{-(T-t)r}\Phi\left(\frac{(T-t)r-(T-t)\sigma^2/2+\log(S_t/K)}{\sigma\sqrt{T-t}}\right). \end{aligned}$$

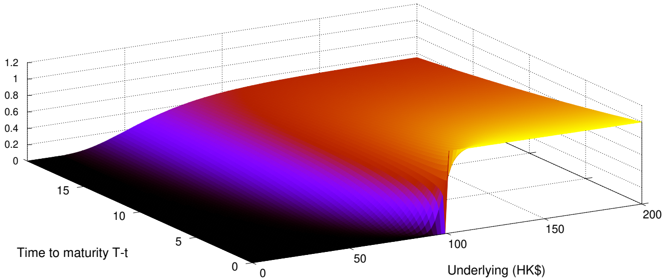


Fig. S.31: Price of a binary call option.

e) We note that

$$\mathbb{1}_{[K,\infty)}(S_T) + \mathbb{1}_{[0,K]}(S_T) = \mathbb{1}_{[0,\infty)}(S_T),$$

almost surely since $\mathbb{P}(S_T = K) = 0$, hence

$$\begin{aligned}
\pi_t(C_b) + \pi_t(P_b) &= \mathrm{e}^{-(T-t)r}\,\mathbb{E}[C_b \mid \mathcal{F}_t] + \mathrm{e}^{-(T-t)r}\,\mathbb{E}[P_b \mid \mathcal{F}_t] \\
&= \mathrm{e}^{-(T-t)r}\,\mathbb{E}[C_b + P_b \mid \mathcal{F}_t] \\
&= \mathrm{e}^{-(T-t)r}\,\mathbb{E}\left[\mathbb{1}_{[K,\infty)}(S_T) + \mathbb{1}_{[0,K]}(S_T) \,\middle|\, \mathcal{F}_t\right] \\
&= \mathrm{e}^{-(T-t)r}\,\mathbb{E}\left[\mathbb{1}_{[0,\infty)}(S_T) \,\middle|\, \mathcal{F}_t\right] \\
&= \mathrm{e}^{-(T-t)r}\,\mathbb{E}[1 \mid \mathcal{F}_t] \\
&= \mathrm{e}^{-(T-t)r}, \qquad 0 \leq t \leq T.
\end{aligned}$$

f) We have

$$\begin{aligned}
\pi_t(P_b) &= \mathrm{e}^{-(T-t)r} - \pi_t(C_b) \\
&= \mathrm{e}^{-(T-t)r} - \mathrm{e}^{-(T-t)r}\Phi\left(\frac{(T-t)r - (T-t)\sigma^2/2 + \log(x/K)}{\sigma\sqrt{T-t}}\right) \\
&= \mathrm{e}^{-(T-t)r}(1 - \Phi(d_-(T-t))) \\
&= \mathrm{e}^{-(T-t)r}\Phi(-d_-(T-t)).
\end{aligned}$$

g) We have

$$\begin{aligned}
\xi_t &= \frac{\partial C_b}{\partial x}(t, S_t) \\
&= \mathrm{e}^{-(T-t)r}\frac{\partial}{\partial x}\Phi\left(\frac{(T-t)r - (T-t)\sigma^2/2 + \log(x/K)}{\sigma\sqrt{T-t}}\right)_{x=S_t} \\
&= \mathrm{e}^{-(T-t)r}\frac{1}{\sigma S_t\sqrt{2(T-t)\pi}}\mathrm{e}^{-(d_-(T-t))^2/2} \\
&> 0.
\end{aligned}$$

The Black-Scholes hedging strategy of such a call option does not involve short selling because $\xi_t > 0$ at all times t, cf. Figure S.32 which represents the risky investment in the hedging portfolio of a binary call option.

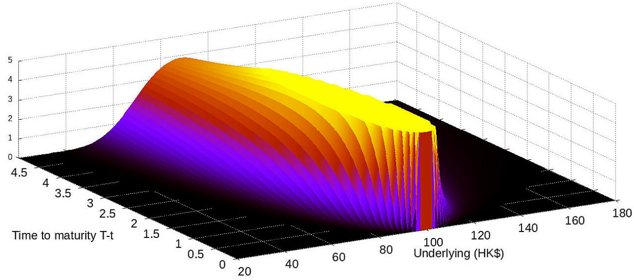


Fig. S.32: Risky hedging portfolio value for a binary call option.

Figure S.33 presents the risk-free hedging portfolio value for a binary call option.

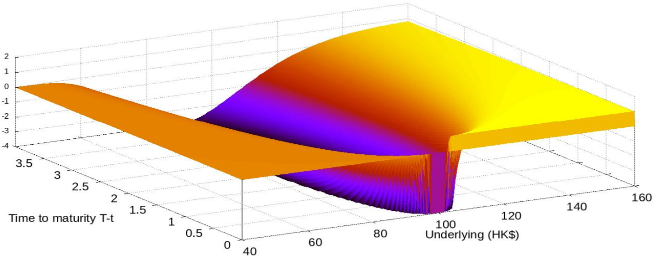


Fig. S.33: Risk-free hedging portfolio value for a binary call option.

h) Here, we have

$$\begin{aligned}
\xi_t &= \frac{\partial P_b}{\partial x}(t, S_t) \\
&= \mathrm{e}^{-(T-t)r} \frac{\partial}{\partial x} \Phi \left(- \frac{(T-t)r - (T-t)\sigma^2/2 + \log(x/K)}{\sigma\sqrt{T-t}} \right)_{x=S_t} \\
&= -\mathrm{e}^{-(T-t)r} \frac{1}{\sigma\sqrt{2(T-t)\pi} S_t} \mathrm{e}^{-(d_-(T-t))^2/2} \\
&< 0.
\end{aligned}$$

The Black-Scholes hedging strategy of such a put option does involve short selling because $\xi_t < 0$ for all t.

Exercise 7.22 Applying Itô's formula to $\big(\mathrm{e}^{-rt}\phi(S_t)\big)_{t\in\mathbb{R}_+}$ and using the fact that the expectation of the stochastic integral with respect to $(B_t)_{t\in\mathbb{R}_+}$ is zero, cf. Relation (4.17), we have

$$
\begin{aligned}
C(x,T) &= \mathbb{E}\left[\mathrm{e}^{-rT}\phi(S_T)\,\big|\,S_0=x\right] \qquad \text{(S.7.33)}\\
&= \mathbb{E}\left[\phi(x) - r\int_0^T \mathrm{e}^{-rt}\phi(S_t)dt + r\int_0^T \mathrm{e}^{-rt}S_t\phi'(S_t)dt\right.\\
&\qquad \left. +\sigma\int_0^T \mathrm{e}^{-rt}S_t\phi'(S_t)dB_t + \frac{1}{2}\int_0^T \mathrm{e}^{-rt}\phi''(S_t)\sigma^2(S_t)dt \,\Bigg|\, S_0=x\right]\\
&= \phi(x) - r\,\mathbb{E}\left[\int_0^T \mathrm{e}^{-rs}\phi(S_t)dt \,\Bigg|\, S_0=x\right] + r\,\mathbb{E}\left[\int_0^T \mathrm{e}^{-rt}S_t\phi'(S_t)dt \,\Bigg|\, S_0=x\right]\\
&\quad +\frac{1}{2}\,\mathbb{E}\left[\int_0^T \mathrm{e}^{-rt}\phi''(S_t)\sigma^2(S_t)dt \,\Bigg|\, S_0=x\right]\\
&= \phi(x) - \int_0^T r\mathrm{e}^{-rt}\,\mathbb{E}\left[\phi(S_t)\,\big|\,S_0=x\right]dt + r\int_0^T \mathrm{e}^{-rt}\,\mathbb{E}\left[S_t\phi'(S_t)\,\big|\,S_0=x\right]dt\\
&\quad +\frac{1}{2}\int_0^T \mathrm{e}^{-rt}\,\mathbb{E}\left[\phi''(S_t)\sigma^2(S_t)\,\big|\,S_0=x\right]dt,
\end{aligned}
$$

hence, by differentiation with respect to T we find

$$
\begin{aligned}
\mathrm{Theta}_T &= \frac{\partial}{\partial T}\big(\mathrm{e}^{-rT}\,\mathbb{E}[\phi(S_T)\mid S_0=x]\big)\\
&= -r\mathrm{e}^{-rT}\,\mathbb{E}\left[\phi(S_T)\big|\,S_0=x\right] + r\mathrm{e}^{-rT}\,\mathbb{E}\left[S_T\phi'(S_T)\,\big|\,S_0=x\right]\\
&\quad +\frac{1}{2}\mathrm{e}^{-rT}\,\mathbb{E}\left[\phi''(S_T)\sigma^2(S_T)\,\big|\,S_0=x\right].
\end{aligned}
$$

Problem 7.23 Chooser options.

a) We take conditional expectations in the equality

$$(S_T-K)^+ - (K-S_T)^+ = S_T - K$$

to find

$$
\begin{aligned}
&C(t,S_t,K,T) - P(t,S_t,K,T)\\
&= \mathrm{e}^{-(T-t)r}\,\mathbb{E}^*[(S_T-K)^+ \mid \mathcal{F}_t] - \mathrm{e}^{-(T-t)r}\,\mathbb{E}^*[(K-S_T)^+ \mid \mathcal{F}_t]\\
&= \mathrm{e}^{-(T-t)r}\,\mathbb{E}^*[S_T-K \mid \mathcal{F}_t]\\
&= \mathrm{e}^{-(T-t)r}\,\mathbb{E}^*[S_T \mid \mathcal{F}_t] - K\mathrm{e}^{-(T-t)r}\\
&= S_t - K\mathrm{e}^{-(T-t)r}, \qquad 0\le t\le T.
\end{aligned}
$$

b) The price this contract at time $t\in[0,T]$ can be written as

$$
\begin{aligned}
&\mathrm{e}^{-(T-t)r}\,\mathbb{E}^*\left[P(T,S_T,K,U)\mid\mathcal{F}_t\right]\\
&\qquad = \mathrm{e}^{-(T-t)r}\,\mathbb{E}^*\left[\mathrm{e}^{-(U-T)r}\,\mathbb{E}^*\left[(K-S_U)^+\mid\mathcal{F}_T\right]\mid\mathcal{F}_t\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \mathrm{e}^{-(U-t)r}\, \mathbb{E}^*\left[(K-S_U)^+ \mid \mathcal{F}_t\right] \\
&= P(t,S_t,K,U).
\end{aligned}
$$

c) From the call-put parity (7.47) the payoff of this contract can be written as

$$
\begin{aligned}
&\max(P(T,S_T,K,U),C(T,S_T,K,U)) \\
&= \max(P(T,S_T,K,U),P(T,S_T,K,U)+S_T-K\mathrm{e}^{-(U-T)r}) \\
&= P(T,S_T,K,U)+\max(S_T-K\mathrm{e}^{-(U-T)r},0).
\end{aligned}
$$

d) The contract of Question (c) is priced at any time $t\in[0,T]$ as

$$
\begin{aligned}
&\mathrm{e}^{-(T-t)r}\, \mathbb{E}^*\left[\max(P(T,S_T,K,U),C(T,S_T,K,U)) \mid \mathcal{F}_t\right] \\
&= \mathrm{e}^{-(T-t)r}\, \mathbb{E}^*\left[P(T,S_T,K,U) \mid \mathcal{F}_t\right] \\
&\quad +\mathrm{e}^{-(T-t)r}\, \mathbb{E}^*\left[\max(S_T-K\mathrm{e}^{-(U-T)r},0) \mid \mathcal{F}_t\right] \\
&= \mathrm{e}^{-(T-t)r}\, \mathbb{E}^*[\mathrm{e}^{-(U-T)r}\, \mathbb{E}^*[(K-S_U)^+ \mid \mathcal{F}_T] \mid \mathcal{F}_t] \\
&\quad +\mathrm{e}^{-(T-t)r}\, \mathbb{E}^*\left[\max\left(S_T-K\mathrm{e}^{-(U-T)r},0\right) \Big| \mathcal{F}_t\right] \\
&= \mathrm{e}^{-(U-t)r}\, \mathbb{E}^*[(K-S_U)^+ \mid \mathcal{F}_t] \\
&\quad +\mathrm{e}^{-(T-t)r}\, \mathbb{E}^*\left[\max\left(S_T-K\mathrm{e}^{-(U-T)r},0\right) \Big| \mathcal{F}_t\right] \\
&= P(t,S_t,K,U)+C\big(t,S_t,K\mathrm{e}^{-(U-T)r},T\big). \qquad \text{(S.7.34)}
\end{aligned}
$$

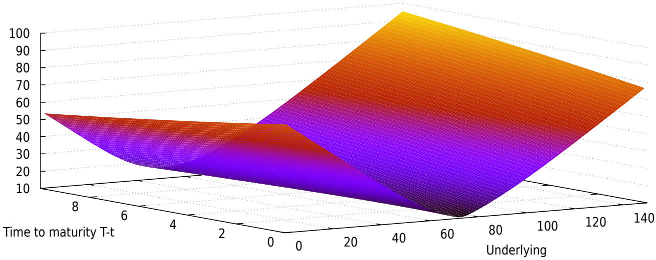


Fig. S.34: Black-Scholes price of the maximum chooser option.

e) By (S.7.34) and Relation (6.3) in Proposition 6.1 we have

$$
\begin{aligned}
\xi_t &= \frac{\partial C}{\partial x}\big(t,S_t,K\mathrm{e}^{-(U-T)r},T\big)+\frac{\partial P}{\partial x}(t,S_t,K,U) \\
&= \Phi\left(\frac{\log(\mathrm{e}^{(U-T)r}S_t/K)+(r+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}\right) \\
&\quad -\Phi\left(-\frac{\log(S_t/K)+(r+\sigma^2/2)(U-t)}{\sigma\sqrt{U-t}}\right)
\end{aligned}
$$

$$= \Phi\left(\frac{\log(S_t/K) + (U-t)r + (T-t)\sigma^2/2}{\sigma\sqrt{T-t}}\right)$$
$$-\Phi\left(-\frac{\log(S_t/K) + (r+\sigma^2/2)(U-t)}{\sigma\sqrt{U-t}}\right).$$

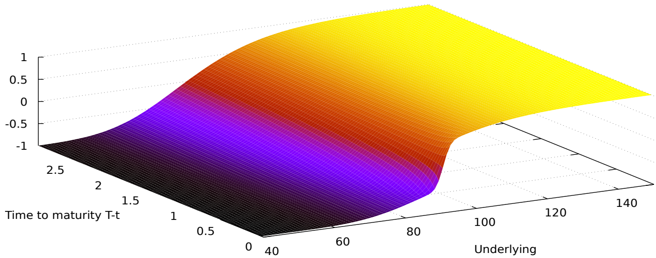


Fig. S.35: Delta of the maximum chooser option.

f) From the call-put parity (7.47) the payoff of this contract can be written as

$$\begin{aligned}
&\min(P(T,S_T,K,U), C(T,S_T,K,U)) \\
&= \min\left(C(T,S_T,K,U) - S_T + K\mathrm{e}^{-(U-T)r}, C(T,S_T,K,U)\right) \\
&= C(T,S_T,K,U) + \min(-S_T + K\mathrm{e}^{-(U-T)r}, 0) \\
&= C(T,S_T,K,U) - \max(S_T - K\mathrm{e}^{-(U-T)r}, 0).
\end{aligned}$$

g) The contract of Question (f) is priced at any time $t \in [0,T]$ as

$$\begin{aligned}
&\mathrm{e}^{-(T-t)r}\,\mathbb{E}^*\left[\min\left(P(T,S_T,K,U), C(T,S_T,K,U)\right) \,\middle|\, \mathcal{F}_t\right] \\
&= \mathrm{e}^{-(T-t)r}\,\mathbb{E}^*[C(T,S_T,K,U) \mid \mathcal{F}_t] \\
&\quad -\mathrm{e}^{-(T-t)r}\,\mathbb{E}^*\left[\max\left(S_T - K\mathrm{e}^{-(U-T)r}, 0\right) \,\middle|\, \mathcal{F}_t\right] \\
&= \mathrm{e}^{-(T-t)r}\,\mathbb{E}^*[\mathrm{e}^{-(U-T)r}\,\mathbb{E}^*[(S_U - K)^+ \mid \mathcal{F}_T] \mid \mathcal{F}_t] \\
&\quad -\mathrm{e}^{-(T-t)r}\,\mathbb{E}^*\left[\max\left(S_T - K\mathrm{e}^{-(U-T)r}, 0\right) \,\middle|\, \mathcal{F}_t\right] \\
&= \mathrm{e}^{-(U-t)r}\,\mathbb{E}^*\left[(S_U - K)^+ \mid \mathcal{F}_t\right] \\
&\quad -\mathrm{e}^{-(T-t)r}\,\mathbb{E}^*\left[\max\left(S_T - K\mathrm{e}^{-(U-T)r}, 0\right) \,\middle|\, \mathcal{F}_t\right] \\
&= C(t,S_t,K,U) - C\big(t,S_t,K\mathrm{e}^{-(U-T)r},T\big). \qquad \text{(S.7.35)}
\end{aligned}$$

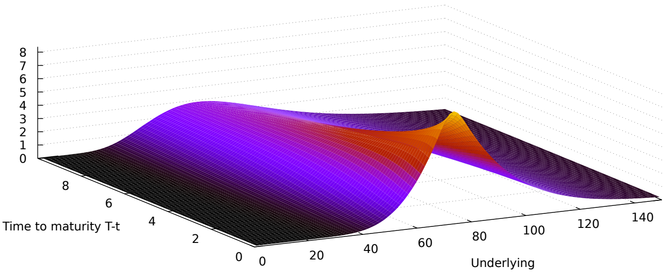


Fig. S.36: Black-Scholes price of the minimum chooser option.

h) By (S.7.35) and Relation (6.3) in Proposition 6.1 we have

$$\begin{aligned}\xi_t &= \frac{\partial C}{\partial x}(t,S_t,K,U) - \frac{\partial C}{\partial x}\big(t,S_t,K\mathrm{e}^{-(U-T)r},T\big)\\ &= \Phi\left(\frac{\log(S_t/K)+(r+\sigma^2/2)(U-t)}{\sigma\sqrt{U-t}}\right)\\ &\quad -\Phi\left(\frac{\log(\mathrm{e}^{(U-T)r}S_t/K)+(r+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}\right)\\ &= \Phi\left(\frac{\log(S_t/K)+(r+\sigma^2/2)(U-t)}{\sigma\sqrt{U-t}}\right)\\ &\quad -\Phi\left(\frac{\log(S_t/K)+(U-t)r+(T-t)\sigma^2/2}{\sigma\sqrt{T-t}}\right).\end{aligned}$$

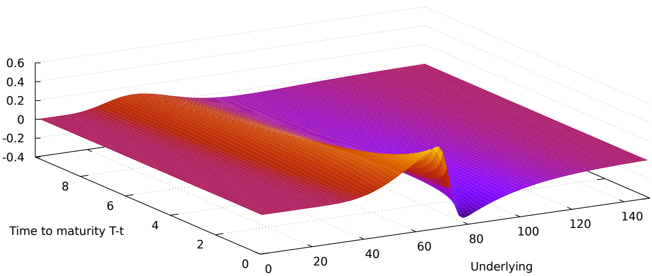


Fig. S.37: Delta of the minimum chooser option.

i) Such a contract is priced as the sum of a European call and a European put option with maturity U, and is priced at time $t \in [0,T]$ as $P(t,S_t,K,U)+C(t,S_t,K,U)$. Its hedging strategy is the sum of the hedging strategies of

Questions (e) and (h), *i.e.*

$$\begin{aligned}\xi_t &= \Phi\left(\frac{\log(S_t/K)+(r+\sigma^2/2)(U-t)}{\sigma\sqrt{T-t}}\right)\\ &\quad -\Phi\left(-\frac{\log(S_t/K)+(r+\sigma^2/2)(U-t)}{\sigma\sqrt{T-t}}\right)\\ &= 2\Phi\left(\frac{\log(S_t/K)+(r+\sigma^2/2)(U-t)}{\sigma\sqrt{T-t}}\right)-1.\end{aligned}$$

j) When $U=T$, the contracts of Questions (c), (f) and (i) have the respective payoffs

- $\max((S_T-K)^+,(K-S_T)^+)=|S_T-K|$,
- $\min((S_T-K)^+,(K-S_T)^+)=0$, and
- $(S_T-K)^+ + (K-S_T)^+ = |S_T-K|$,

where $|S_T-K|$ is known as the payoff of a *straddle option*.

Problem 7.24

a) The self-financing condition reads

$$\begin{aligned}dV_t &= \eta_t dA_t + \xi_t dS_t\\ &= r\eta_t A_t dt + \mu\xi_t S_t dt + \sigma\xi_t S_t dB_t\\ &= rV_t dt + (\mu-r)\xi_t S_t dt + \sigma\xi_t S_t dB_t,\end{aligned}$$

hence

$$\begin{aligned}V_T &= V_0 + \int_0^T (rV_t + (\mu-r)\xi_t S_t)dt + \sigma\int_0^T \xi_t S_t dB_t\\ &= V_t + \int_t^T (rV_s + (\mu-r)\xi_s S_s)ds + \sigma\int_t^T \xi_s S_s dB_s.\end{aligned}$$

b) The portfolio value V_t rewrites as

$$\begin{aligned}V_t &= V_T - \int_t^T \left(rV_s + \frac{\mu-r}{\sigma}\pi_s\right)ds - \int_t^T \pi_s dB_s\\ &= V_T - r\int_t^T V_s ds - \int_t^T \pi_s d\widehat{B}_s.\end{aligned}$$

c) We have

$$V_t = V_T - r\int_t^T V_s ds - \int_t^T \pi_s d\widehat{B}_s,$$

hence

$$dV_t = rV_t dt + \pi_t d\widehat{B}_t,$$

and after discounting we find

$$\begin{aligned} d\widetilde{V}_t &= -r\mathrm{e}^{-rt}V_t dt + \mathrm{e}^{-rt}dV_t \\ &= -r\mathrm{e}^{-rt}V_t dt + \mathrm{e}^{-rt}(rV_t dt + \pi_t d\widehat{B}_t) \\ &= \mathrm{e}^{-rt}\pi_t d\widehat{B}_t, \end{aligned}$$

which shows that

$$\widetilde{V}_T = V_0 + \int_0^T \mathrm{e}^{-rt}\pi_t d\widehat{B}_t,$$

after integration in $t \in [0, T]$.

d) We have

$$\begin{aligned} dV_t &= du(t, S_t) \\ &= \frac{\partial u}{\partial t}(t, S_t)dt + \mu S_t \frac{\partial u}{\partial x}(t, S_t)dt + \sigma S_t \frac{\partial u}{\partial x}(t, S_t)dB_t \\ &\quad + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 u}{\partial x^2}(t, S_t)dt. \end{aligned} \tag{S.7.36}$$

e) By matching the Itô formula (S.7.36) term by term to the BSDE (7.50) we find that $V_t = u(t, S_t)$ satisfies the PDE

$$\frac{\partial u}{\partial t}(t, x) + \mu \frac{\partial u}{\partial x}(t, x) + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 u}{\partial x^2}(t, x) + f\left(t, x, u(t, x), \sigma x \frac{\partial u}{\partial x}(t, x)\right) = 0.$$

f) In this case we have

$$\frac{\partial u}{\partial t}(t, x) + \mu x \frac{\partial u}{\partial x}(t, x) + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 u}{\partial x^2}(t, x) - ru(t, x) - (\mu - r)x \frac{\partial u}{\partial x}(t, x) = 0,$$

which recovers the Black-Scholes PDE

$$ru(t, x) = \frac{\partial u}{\partial t}(t, x) + rx \frac{\partial u}{\partial x}(t, x) + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 u}{\partial x^2}(t, x).$$

g) In the Black-Scholes model the Delta of the European call option is given by

$$\xi_t = \Phi\left(\frac{(r + \sigma^2/2)(T - t) + \log(S_t/K)}{\sigma\sqrt{T - t}}\right),$$

hence

$$\pi_t = \sigma \xi_t S_t = \sigma S_t \Phi\left(\frac{(r + \sigma^2/2)(T - t) + \log(S_t/K)}{\sigma\sqrt{T - t}}\right), \qquad 0 \le t \le T.$$

h) Replacing the self-financing condition with

$$\begin{aligned} dV_t &= \eta_t dA_t + \xi_t dS_t - \gamma S_t (\xi_t)^- dt \\ &= r\eta_t A_t dt + \mu \xi_t S_t dt + \sigma \xi_t S_t dB_t - \gamma S_t (\xi_t)^- dt \\ &= rV_t dt + (\mu - r)\xi_t S_t dt - \gamma S_t (\xi_t)^- dt + \sigma \xi_t S_t dB_t, \end{aligned}$$

we get the BSDE

$$V_t = V_T - \int_t^T \big(rV_s + (\mu - r)\pi_s + \gamma (\pi_s)^- \big) ds - \int_t^T \pi_s dB_s.$$

i) In this case we have

$$f(t,x,u,z) = -ru - \frac{\mu - r}{\sigma} z - \gamma z^-$$

and the BSDE reads

$$dV_t = ru(t,S_t)dt + (\mu - r)\xi_t S_t dt - \gamma S_t (\xi_t)^- dt + \sigma \xi_t S_t dB_t.$$

j) We find the nonlinear PDE

$$\frac{\partial u}{\partial t}(t,x) + rx \frac{\partial u}{\partial x}(t,x) + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 u}{\partial x^2}(t,x) - \gamma \sigma x \left(\frac{\partial u}{\partial x}(t,x) \right)^- = ru(t,x), \tag{S.7.37}$$

with the terminal condition $u(T,x) = g(x)$.

k) The self-financing condition reads

$$\begin{aligned} dV_t &= r \mathbb{1}_{\{\eta_t > 0\}} A_t \eta_t dt + R \mathbb{1}_{\{\eta_t < 0\}} A_t \eta_t dt + \xi_t dS_t \\ &= r A_t \eta_t dt + (R - r) \mathbb{1}_{\{\eta_t < 0\}} A_t \eta_t dt + \xi_t dS_t \\ &= rV_t dt - rS_t \xi_t dt - (R - r)(\eta_t A_t)^- dt + \xi_t dS_t \\ &= rV_t dt + (\mu - r) S_t \xi_t dt - (R - r)(V_t - \xi_t S_t)^- dt + \sigma \xi_t S_t dB_t, \end{aligned}$$

which yields the BSDE

$$V_t = V_T - \int_t^T \big(rV_s + (\mu - r)\pi_s - (R - r)(V_s - \xi_s S_s)^- \big) ds - \int_t^T \pi_s dB_s,$$

hence we have

$$f(t,x,u,z) = -ru - \frac{(\mu - r)}{\sigma} z + (R - r) \left(u - \frac{z}{\sigma} \right)^-$$

and the nonlinear PDE

$$\frac{\partial u}{\partial t}(t,x) + \mu \frac{\partial u}{\partial x}(t,x) + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 u}{\partial x^2}(t,x) + f\left(t, x, u(t,x), \sigma x \frac{\partial u}{\partial x}(t,x) \right) = 0$$

rewrites as

$$\frac{\partial u}{\partial t}(t,x)+r\frac{\partial u}{\partial x}(t,x)+\frac{1}{2}\sigma^2x^2\frac{\partial^2 u}{\partial x^2}(t,x) = ru(t,x)+(r-R)\left(u(t,x)-x\frac{\partial u}{\partial x}(t,x)\right)^-.$$

l) The sum of profits and losses of the portfolio $(\xi_t, \eta_t)_{t\in\mathbb{R}_+}$ is

$$\begin{aligned} V_0 + \int_0^T \eta_t dA_t + \int_0^T \xi_t dS_t &= V_0 + \int_0^T dV_t + \int_0^T dU_t \\ &= V_T + U_T - U_0 \\ &> V_T = C, \end{aligned}$$

hence the corresponding portfolio strategy superhedging the claim payoff $V_T = C$.

Exercise 7.25 Girsanov Theorem. For all $n \geq 1$, let

$$\psi_t^{(n)} := \mathbb{1}_{\{\psi_t\in[-n,n]\}}\psi_t, \qquad 0 \leq t \leq T.$$

Since $(\psi_t^{(n)})_{t\in[0,T]}$ is a bounded process it satisfies the Novikov integrability condition (7.11), hence for all $n \geq 1$ and random variable $F \in L^1(\Omega)$ we have

$$\mathbb{E}[F] = \mathbb{E}\left[F\left(B_\cdot + \int_0^\cdot \psi_s^{(n)} ds\right)\exp\left(-\int_0^T \psi_s^{(n)} dB_s - \frac{1}{2}\int_0^T (\psi_s^{(n)})^2 ds\right)\right],$$

which yields

$$\begin{aligned} \mathbb{E}[F] &= \lim_{n\to\infty}\mathbb{E}\left[F\left(B_\cdot + \int_0^\cdot \psi_s^{(n)} ds\right)\exp\left(-\int_0^T \psi_s^{(n)} dB_s - \frac{1}{2}\int_0^T (\psi_s^{(n)})^2 ds\right)\right] \\ &\geq \mathbb{E}\left[\liminf_{n\to\infty} F\left(B_\cdot + \int_0^\cdot \psi_s^{(n)} ds\right)\exp\left(-\int_0^T \psi_s^{(n)} dB_s - \frac{1}{2}\int_0^T (\psi_s^{(n)})^2 ds\right)\right] \\ &= \mathbb{E}\left[F\left(B_\cdot + \int_0^\cdot \psi_s ds\right)\exp\left(-\int_0^T \psi_s dB_s - \frac{1}{2}\int_0^T (\psi_s)^2 ds\right)\right], \end{aligned}$$

where we applied Fatou's Lemma.*

Problem 7.26

a) We have

$$\begin{aligned} &\frac{\operatorname{Cov}(dS_t/S_t, dM_t/M_t)}{\operatorname{Var}[dM_t/M_t]} \\ &= \frac{\operatorname{Cov}((r+\alpha)dt + \beta(dM_t/M_t - rdt) + \sigma_S dW_t, \mu dt + \sigma_M dB_t)}{\operatorname{Var}[\mu dt + \sigma_M dB_t]} \end{aligned}$$

* $\mathbb{E}[\lim_{n\to\infty} F_n] \leq \lim_{n\to\infty}\mathbb{E}[F_n]$ for any sequence $(F_n)_{n\in\mathbb{N}}$ of *nonnegative* random variables, provided that the limits exist, see MH4100 Real Analysis II.

$$
\begin{aligned}
&= \frac{\mathrm{Cov}\left((r+\alpha)dt + \beta(\mu dt + \sigma_M dB_t - rdt) + \sigma_S dW_t, \mu dt + \sigma_M dB_t\right)}{\mathrm{Var}[\mu dt + \sigma_M dB_t]} \\
&= \frac{\mathrm{Cov}\left(\beta\sigma_M dB_t + \sigma_S dW_t, \sigma_M dB_t\right)}{\mathrm{Var}[\sigma_M dB_t]} \\
&= \frac{\mathrm{Cov}\left(\beta\sigma_M dB_t, \sigma_M dB_t\right) + \mathrm{Cov}\left(\sigma_S dW_t, \sigma_M dB_t\right)}{\mathrm{Var}[\sigma_M dB_t]} \\
&= \frac{\mathrm{Cov}\left(\beta\sigma_M dB_t, \sigma_M dB_t\right)}{\mathrm{Var}[\sigma_M dB_t]} \\
&= \beta\frac{\mathrm{Cov}\left(\sigma_M dB_t, \sigma_M dB_t\right)}{\mathrm{Var}[\sigma_M dB_t]} \\
&= \beta.
\end{aligned}
$$

b) We have

$$
\begin{aligned}
dS_t &= (r+\alpha)S_t dt + \beta\left(\frac{dM_t}{M_t} - r\right)S_t dt + \sigma_S S_t dW_t \\
&= (r+\alpha)S_t dt + \beta S_t\left(\mu dt + \sigma_M dB_t - rdt\right) + \sigma_S S_t dW_t \\
&= (r+\alpha+\beta(\mu-r))S_t dt + S_t\big(\beta\sigma_M dB_t + \sigma_S dW_t\big) \\
&= (r+\alpha+\beta(\mu-r))S_t dt + S_t\sqrt{\beta^2\sigma_M^2 + \sigma_S^2}\,\frac{\beta\sigma_M dB_t + \sigma_S dW_t}{\sqrt{\beta^2\sigma_M^2 + \sigma_S^2}}.
\end{aligned}
$$

Now, we have

$$
\begin{aligned}
\left(\frac{\beta\sigma_M dB_t + \sigma_S dW_t}{\sqrt{\beta^2\sigma_M^2 + \sigma_S^2}}\right)^2 &= \frac{\big(\beta\sigma_M dB_t\big)^2 + \beta\sigma_M\sigma_S dB_t \bullet dW_t + \big(\sigma_S dW_t\big)^2}{\beta^2\sigma_M^2 + \sigma_S^2} \\
&= \frac{\beta^2\sigma_M^2(dB_t)^2 + \sigma_S^2(dW_t)^2}{\beta^2\sigma_M^2 + \sigma_S^2} \\
&= \frac{\beta^2\sigma_M^2 dt + \sigma_S^2 dt}{\beta^2\sigma_M^2 + \sigma_S^2} \\
&= dt.
\end{aligned}
$$

By the characterization of Brownian motion as the only continuous martingale whose quadratic variation is dt, it follows that the process $(Z_t)_{t\in\mathbb{R}_+}$ defined by

$$
dZ_t = \frac{\beta\sigma_M dB_t + \sigma_S dW_t}{\sqrt{\beta^2\sigma_M^2 + \sigma_S^2}}
$$

is a standard Brownian motion, see *e.g.* Theorem 7.36 page 203 of Klebaner (2005). Hence, we have

$$
dS_t = (r+\alpha+\beta(\mu-r))S_t dt + S_t\big(\beta\sigma_M dB_t + \sigma_S dW_t\big)
$$

$$= (r + \alpha + \beta(\mu - r))S_t dt + S_t\sqrt{\beta^2\sigma_M^2 + \sigma_S^2}dZ_t.$$

In what follows we assume that β is allowed to depend locally on the state of the benchmark market index on M_t, as $\beta(M_t)$, $t \in \mathbb{R}_+$.

c) We take

$$dB_t^* = dB_t + \frac{\mu - r}{\sigma_M}dt \tag{S.7.38}$$

and

$$dW_t^* = dW_t + \frac{\alpha}{\sigma_S}dt \tag{S.7.39}$$

in order to have

$$\begin{cases} \dfrac{dM_t}{M_t} = \mu dt + \sigma_M dB_t = rdt + \sigma_M dB_t^*, \\ \\ \dfrac{dS_t}{S_t} = (r+\alpha)dt + \beta(M_t) \times \left(\dfrac{dM_t}{M_t} - rdt\right) + \sigma_S dW_t \\ \qquad = (r+\alpha)dt + \sigma_M\beta(M_t)dB_t^* + \sigma_S dW_t \\ \qquad = rdt + \sigma_M\beta(M_t)dB_t^* + \sigma_S dW_t^*. \end{cases} \tag{S.7.40}$$

d) By the Girsanov theorem, $(B_t^*)_{t\in[0,T]}$ is a standard Brownian motion under the probability measure $\mathbb{P}_B^*$ defined by its Radon-Nikodym density

$$\frac{d\mathbb{P}_B^*}{d\mathbb{P}} = \exp\left(-\frac{\mu - r}{\sigma_M}B_T - \frac{(\mu - r)^2}{2\sigma_M^2}T\right),$$

and $(W_t^*)_{t\in[0,T]}$ is a standard Brownian motion under the probability measure $\mathbb{P}_W^*$ defined by its Radon-Nikodym density

$$\frac{d\mathbb{P}_W^*}{d\mathbb{P}} = \exp\left(-\frac{\alpha}{\sigma_S}W_T - \frac{\alpha^2}{2\sigma_S^2}T\right).$$

We conclude that $(B_t^*)_{t\in[0,T]}$ and $(W_t^*)_{t\in[0,T]}$ are independent standard Brownian motions under the probability measure $\mathbb{P}^*$ defined by its Radon-Nikodym density

$$\frac{d\mathbb{P}^*}{d\mathbb{P}} = \frac{d\mathbb{P}_B^*}{d\mathbb{P}} \times \frac{d\mathbb{P}_W^*}{d\mathbb{P}} = \exp\left(-\frac{\mu - r}{\sigma_M}B_T - \frac{\alpha}{\sigma_S}W_T - \frac{(\mu - r)^2}{2\sigma_M^2}T - \frac{\alpha^2}{2\sigma_S^2}T\right).$$

Indeed, for any sequence $t_0 = 0 < t_1 < \cdots < t_{n-1} < t_n = T$ we have

$$\mathbb{E}^*\left[f\left(B_{t_1}^* - B_{t_0}^*, \ldots, B_{t_n}^* - B_{t_{n-1}}^*\right)\right] = \mathbb{E}\left[\frac{d\mathbb{P}^*}{d\mathbb{P}}f\left(B_{t_1}^* - B_{t_0}^*, \ldots, B_{t_n}^* - B_{t_{n-1}}^*\right)\right]$$
$$= \mathbb{E}\Big[f\big(B_{t_1}^* - B_{t_0}^*, \ldots, B_{t_n}^* - B_{t_{n-1}}^*\big)$$

$$
\begin{aligned}
&\times \exp\left(-\frac{\mu-r}{\sigma_M}B_T-\frac{(\mu-r)^2}{2\sigma_M^2}T-\frac{\alpha}{\sigma_S}W_T-\frac{\alpha^2}{2\sigma_S^2}T\right)\Bigg]\\
&=\mathbb{E}\left[f\big(B^*_{t_1}-B^*_{t_0},\ldots,B^*_{t_n}-B^*_{t_{n-1}}\big)\exp\left(-\frac{\mu-r}{\sigma_M}B_T-\frac{(\mu-r)^2}{2\sigma_M^2}T\right)\right]\\
&\times \mathbb{E}\left[\exp\left(-\frac{\alpha}{\sigma_S}W_T-\frac{\alpha^2}{2\sigma_S^2}T\right)\right]\\
&=\mathbb{E}\left[f\big(B^*_{t_1}-B^*_{t_0},\ldots,B^*_{t_n}-B^*_{t_{n-1}}\big)\exp\left(-\frac{\mu-r}{\sigma_M}B_T-\frac{(\mu-r)^2}{2\sigma_M^2}T\right)\right]\\
&=\mathbb{E}\left[f\big(B_{t_1}-B_{t_0},\ldots,B_{t_n}-B_{t_{n-1}}\big)\right],
\end{aligned}
$$

and similarly for $(W^*_t)_{t\in[0,T]}$.

e) By (S.7.40), the discounted price processes

$$
(\widetilde{S}_t)_{t\in\mathbb{R}_+}:=(\mathrm{e}^{-rt}S_t)_{t\in\mathbb{R}_+}\quad\text{and}\quad(\widetilde{M}_t)_{t\in\mathbb{R}_+}:=(\mathrm{e}^{-rt}M_t)_{t\in\mathbb{R}_+}
$$

satisfy

$$
\begin{cases}
d\widetilde{M}_t=\sigma_M\widetilde{M}_t dB^*_t,\\
\\
d\widetilde{S}_t=\sigma_M\beta(M_t)\widetilde{S}_t dB^*_t+\sigma_S\widetilde{S}_t dW^*_t,
\end{cases}
$$

hence by the Girsanov theorem of Question (d) the discounted two-dimensional process $\big(\widetilde{S}_t,\widetilde{M}_t\big)_{t\in\mathbb{R}_+}$ is a martingale under the probability measure $\mathbb{P}^*$, showing that $\mathbb{P}^*$ is a risk-neutral probability measure. Therefore, by Theorem 6.8 the market made of S_t and M_t is without arbitrage opportunities due to the existence of a risk-neutral probability measure $\mathbb{P}^*$.

f) The self-financing condition for the portfolio strategy $(\xi_t,\zeta_t,\eta_t)_{t\in[0,T]}$ reads

$$
\eta_{t+dt}A_{t+dt}+\xi_{t+dt}S_{t+dt}+\zeta_{t+dt}M_{t+dt},=\eta_tA_{t+dt}+\xi_tS_{t+dt}+\zeta_tM_{t+dt}
$$

which yields

$$
A_{t+dt}d\eta_t+S_{t+dt}d\xi_t+M_{t+dt}\zeta_t=0,
$$

i.e.

$$
dA_t\bullet d\eta_t+dS_t\bullet d\xi_t+dM_t\bullet d\zeta_t+A_td\eta_t+S_td\xi_t+M_td\zeta_t=0,
$$

hence

$$
\begin{aligned}
dV_t&=\eta_tdA_t+\xi_tdS_t+\zeta_tdM_t\\
&\quad+A_td\eta_t+d\eta_t\bullet dA_t+S_td\xi_t+d\xi_t\bullet dS_t+M_td\zeta_t+d\zeta_t\bullet dM_t\\
&=\eta_tdA_t+\xi_tdS_t+\zeta_tdM_t\\
&=0.
\end{aligned}
$$

g) By the self-financing condition we have

$$\begin{aligned}
dV_t &= df(t, S_t, M_t) \\
&= \xi_t dS_t + \zeta_t dM_t + \eta_t dA_t \\
&= \xi_t(rS_t dt + \sigma_M \beta(M_t) S_t dB_t^* + \sigma_S S_t dW_t^*) + \zeta_t(rM_t dt + \sigma_M M_t dB_t^*) + r\eta_t A_t dt \\
&= r\xi_t S_t dt + \sigma_M \beta(M_t)\xi_t S_t dB_t^* + \sigma_S \xi_t S_t dW_t^* + r\zeta_t M_t dt + \sigma_M \zeta_t M_t dB_t^* + r\eta_t A_t dt \\
&= r\xi_t S_t dt + r\eta_t A_t dt + \sigma_S \xi_t S_t dW_t^* + r\zeta_t M_t dt + (\sigma_M \beta(M_t)\xi_t S_t + \sigma_M \zeta_t M_t) dB_t^* \\
&= rV_t dt + \sigma_S \xi_t S_t dW_t^* + (\sigma_M \beta(M_t)\xi_t S_t + \sigma_M \zeta_t M_t) dB_t^*. \qquad \text{(S.7.41)}
\end{aligned}$$

On the other hand, by the Itô formula for two state variables, we have

$$\begin{aligned}
df(t, S_t, M_t) &= \frac{\partial f}{\partial t}(t, S_t, M_t) dt + \frac{\partial f}{\partial x}(t, S_t, M_t) dS_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t, S_t, M_t)(dS_t)^2 \\
&\quad + \frac{\partial f}{\partial y}(t, S_t, M_t) dM_t + \frac{1}{2}\frac{\partial^2 f}{\partial y^2}(t, S_t, M_t)(dM_t)^2 + \frac{\partial^2 f}{\partial x \partial y}(t, S_t, M_t) dS_t \bullet dM_t \\
&= \frac{\partial f}{\partial t}(t, S_t, M_t) dt + \frac{\partial f}{\partial x}(t, S_t, M_t)(rS_t dt + \sigma_M \beta(M_t) S_t dB_t^* + \sigma_S S_t dW_t^*) \\
&\quad + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t, S_t, M_t)(\sigma_M^2 \beta^2(M_t) S_t^2 + \sigma_S^2 S_t^2) dt \\
&\quad + \frac{\partial f}{\partial y}(t, S_t, M_t)(rM_t dt + \sigma_M M_t dB_t^*) + \frac{1}{2}\frac{\partial^2 f}{\partial y^2}(t, S_t, M_t)\sigma_M^2 M_t^2 dt \\
&\quad + \sigma_M^2 S_t M_t \beta(M_t) \frac{\partial^2 f}{\partial x \partial y}(t, S_t, M_t) dt \\
&= \frac{\partial f}{\partial t}(t, S_t, M_t) dt + rS_t \frac{\partial f}{\partial x}(t, S_t, M_t) dt + \sigma_M \beta(M_t) S_t \frac{\partial f}{\partial x}(t, S_t, M_t) dB_t^* \\
&\quad + \sigma_S S_t \frac{\partial f}{\partial x}(t, S_t, M_t) dW_t^* + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t, S_t, M_t)(\sigma_M^2 \beta^2(M_t) S_t^2 dt + \sigma_S^2 S_t^2 dt) \\
&\quad + rM_t \frac{\partial f}{\partial y}(t, S_t, M_t) dt + \sigma_M M_t \frac{\partial f}{\partial y}(t, S_t, M_t) dB_t^* \\
&\quad + \frac{1}{2}\frac{\partial^2 f}{\partial y^2}(t, S_t, M_t)\sigma_M^2 M_t^2 dt + \sigma_M^2 S_t M_t \beta(M_t) \frac{\partial^2 f}{\partial x \partial y}(t, S_t, M_t) dt \\
&= \frac{\partial f}{\partial t}(t, S_t, M_t) dt + rM_t \frac{\partial f}{\partial y}(t, S_t, M_t) dt + rS_t \frac{\partial f}{\partial x}(t, S_t, M_t) dt \\
&\quad + \frac{1}{2}\sigma_M^2 M_t^2 \frac{\partial^2 f}{\partial y^2}(t, S_t, M_t) dt + \frac{1}{2}\sigma_M^2 \beta^2(M_t) S_t^2 \frac{\partial^2 f}{\partial x^2}(t, S_t, M_t) dt \\
&\quad + \frac{1}{2}\sigma_S^2 S_t^2 \frac{\partial^2 f}{\partial x^2}(t, S_t, M_t) dt) + \sigma_M^2 S_t M_t \beta(M_t) \frac{\partial^2 f}{\partial x \partial y}(t, S_t, M_t) dt \\
&\quad + \left(\sigma_M \beta(M_t) S_t \frac{\partial f}{\partial x}(t, S_t, M_t) + \sigma_M M_t \frac{\partial f}{\partial y}(t, S_t, M_t) \right) dB_t^* \qquad \text{(S.7.42)} \\
&+ \sigma_S S_t \frac{\partial f}{\partial x}(t, S_t, M_t) dW_t^*.
\end{aligned}$$

By identification of the terms in dt in (S.7.41) and (S.7.42), we find

$$rf(t,S_t,M_t) = \frac{\partial f}{\partial t}(t,S_t,M_t) + rS_t\frac{\partial f}{\partial x}(t,S_t,M_t)$$
$$+\frac{1}{2}(\sigma_S^2+\sigma_M^2\beta^2(M_t))S_t^2\frac{\partial^2 f}{\partial x^2}(t,S_t,M_t)$$
$$+rM_t\frac{\partial f}{\partial y}(t,S_t,M_t)+\frac{1}{2}\sigma_M^2M_t^2\frac{\partial^2 f}{\partial y^2}(t,S_t,M_t)+\sigma_M^2S_tM_t\beta(M_t)\frac{\partial^2 f}{\partial x\partial y}(t,S_t,M_t)dt,$$

which yields the PDE

$$\begin{aligned} & rf(t,x,y) \\ & = \frac{\partial f}{\partial t}(t,x,y)+rx\frac{\partial f}{\partial x}(t,x,y)+\frac{1}{2}x^2(\sigma_S^2+\sigma_M^2\beta^2(y))\frac{\partial^2 f}{\partial x^2}(t,x,y) \\ & \quad +ry\frac{\partial f}{\partial y}(t,x,y)+\frac{1}{2}\sigma_M^2y^2\frac{\partial^2 f}{\partial y^2}(t,x,y)+\sigma_M^2xy\beta(y)\frac{\partial^2 f}{\partial x\partial y}(t,x,y), \end{aligned} \tag{S.7.43}$$

with the terminal condition

$$f(T,x,y)=h(x,y), \qquad x,y>0.$$

h) By identification of terms in dB_t^* and dW_t^* in (S.7.41) and (S.7.42), we find

$$\xi_t = \frac{\partial f}{\partial x}(t,S_t,M_t)$$

and

$$\sigma_M\beta(M_t)S_t\frac{\partial f}{\partial x}(t,S_t,M_t)+\sigma_MM_t\frac{\partial f}{\partial y}(t,S_t,M_t)=\sigma_M\beta(M_t)\xi_tS_t+\sigma_M\zeta_tM_t,$$

hence

$$\zeta_t = \frac{\partial f}{\partial y}(t,S_t,M_t),$$

and by the relation $V_t=\xi_tS_t+\zeta_tM_t+\eta_tA_t$ we find

$$\begin{aligned}\eta_t &= \frac{V_t-\xi_tS_t-\zeta_tM_t}{A_t} \\ &= \frac{f(t,S_t,M_t)-S_t\frac{\partial f}{\partial x}(t,S_t,M_t)-M_t\frac{\partial f}{\partial y}(t,S_t,M_t)}{A_0\mathrm{e}^{rt}}, \qquad 0\le t\le T.\end{aligned}$$

i) When the option payoff depends only on S_T we can look for a solution of (S.7.43) of the form $f(t,x)$, in which case (S.7.43) simplifies to

$$rf(t,x,y)=\frac{\partial f}{\partial t}(t,x,y)+rx\frac{\partial f}{\partial x}(t,x,y)+\frac{1}{2}x^2(\sigma_S^2+\sigma_M^2\beta^2(y))\frac{\partial^2 f}{\partial x^2}(t,x,y), \tag{S.7.44}$$

When $\beta(M_t) = \beta$ is a constant, (S.7.44) becomes the Black-Scholes PDE with squared volatility parameter

$$\sigma^2 := \sigma_S^2 + \sigma_M^2 \beta^2.$$

When the option is the European call option with strike price K on S_T, its solution is given by the Black-Scholes function

$$\boxed{\begin{aligned} f(t,x) &= \mathrm{Bl}(x, K, \sigma, r, T-t) \\ &= x\Phi\big(d_+(T-t)\big) - K\mathrm{e}^{-(T-t)r}\Phi\big(d_-(T-t)\big), \end{aligned} \\ \text{with} \\ \begin{aligned} d_+(T-t) &:= \frac{\log(x/K) + (r + (\sigma_S^2 + \sigma_M^2\beta^2)/2)(T-t)}{|\sigma|\sqrt{T-t}}, \\ d_-(T-t) &:= \frac{\log(x/K) + (r - (\sigma_S^2 + \sigma_M^2\beta^2)/2)(T-t)}{|\sigma|\sqrt{T-t}}, \end{aligned}}$$

and

$$\xi_t = \frac{\partial f}{\partial x}(t, S_t, M_t) = \Phi\big(d_+(T-t)\big),$$

with

$$\eta_t = -\frac{K}{A_t}\mathrm{e}^{-(T-t)r}\Phi\big(d_-(T-t)\big) = -\frac{K}{A_0}\mathrm{e}^{-Tr}\Phi\big(d_-(T-t)\big), \quad 0 \le t < T.$$

j) Similarly to Question (i), when the option is the European put option with strike price K on S_T, its solution is given by the Black-Scholes put price function

$$\boxed{f(t,x) = K\mathrm{e}^{-(T-t)r}\Phi\big(-d_-(T-t)\big) - x\Phi\big(-d_+(T-t)\big),}$$

with

$$\zeta_t = \frac{\partial f}{\partial y}(t, S_t, M_t) = -\Phi\big(-d_+(T-t)\big), \qquad 0 \le t < T.$$

and

$$\eta_t = \frac{K}{A_0}\mathrm{e}^{-Tr}\Phi\big(-d_-(T-t)\big), \qquad 0 \le t < T.$$

Remark. By the answer to Question (b) we have

$$dS_t = (r + \alpha + \beta(\mu - r))S_t dt + S_t\sqrt{\beta^2\sigma_M^2 + \sigma_S^2} dZ_t$$

where $(Z_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion, hence the answers to Questions (i) and j can be recovered from the pricing relation

$$f(t, S_t) = \mathrm{e}^{-(T-t)r}\,\mathbb{E}^*\big[\phi(S_T) \mid \mathcal{F}_t\big], \qquad 0 \leq t \leq T.$$

Problem 7.27

1) a) It suffices to let $\tau_n := T$, $n \geq 1$. Then, the sequence $(\tau_n)_{n\geq 1}$ clearly satisfies Conditions $(v)-(vi)$, and the process $(M_{\tau_n\wedge t})_{t\in[0,T]} = (M_t)_{t\in[0,T]}$ is a (true) martingale under $\mathbb{P}$.

   b) Applying

      i) the local martingale property to a suitable sequence $(\tau_n)_{n\geq 1}$ of stopping times, and

      ii) Fatou's lemma to the non-negative sequence $(M_{\tau_n\wedge t})_{n\geq 1}$,

   we have

$$\begin{aligned}
\mathbb{E}[M_t \mid \mathcal{F}_s] &= \mathbb{E}\big[\lim_{n\to\infty} M_{\tau_n\wedge t} \,\big|\, \mathcal{F}_s\big] \\
&\leq \liminf_{n\to\infty} \mathbb{E}[M_{\tau_n\wedge t} \mid \mathcal{F}_s] \\
&= \liminf_{n\to\infty} M_{\tau_n\wedge s} \\
&= \lim_{n\to\infty} M_{\tau_n\wedge s} \\
&= M_s, \qquad 0 \leq s \leq t \leq T,
\end{aligned}$$

   which shows that $(M_t)_{t\in[0,T]}$ is a supermartingale.

   c) Since $(M_t)_{t\in[0,T]}$ is a supermartingale by Question (1b), for any $t \in [0,T]$ we have $\mathbb{E}[M_T \mid \mathcal{F}_t] - M_t \leq 0$ a.s., and there exists $t \in [0,T]$ such that $\mathbb{E}[\mathbb{E}[M_T \mid \mathcal{F}_t] - M_t] < 0$, otherwise we would have $\mathbb{E}[M_T \mid \mathcal{F}_t] - M_t = 0$ *a.s.* for all $t \in [0,T]$, and $(M_t)_{t\in[0,T]}$ would be a martingale by the tower property.* Therefore, using again the tower property, we find

$$\mathbb{E}[M_T - M_0] = \mathbb{E}[\mathbb{E}[M_T \mid \mathcal{F}_t] - M_0] \leq \mathbb{E}[\mathbb{E}[M_T \mid \mathcal{F}_t] - M_t] < 0.$$

   d) We have

$$\begin{aligned}
C(0, M_0) - P(0, M_0) &= \mathrm{e}^{-rT}\,\mathbb{E}[(\mathrm{e}^{rT} M_T - K)^+ - (K - \mathrm{e}^{rT} M_T)^+] \\
&= \mathbb{E}[(M_T - \mathrm{e}^{-rT} K)^+ - (\mathrm{e}^{-rT} K - M_T)^+] \\
&= \mathbb{E}[M_T - \mathrm{e}^{-rT} K] \\
&< \mathbb{E}[M_0] - \mathrm{e}^{-rT} K,
\end{aligned}$$

   showing that the call-put parity relation $C(0, M_0) - P(0, M_0) = \mathbb{E}[M_0] - \mathrm{e}^{-rT} K$ is not satisfied.

2) a) The stochastic differential equation can be rewritten as $dS_t = \sigma(t, S_t)dB_t$ where $\sigma(t,x) = x/\sqrt{T-t}$, $t \in [0, T-\varepsilon]$, satisfies the global Lipschitz condition

* If $M_t = \mathbb{E}[M_T \mid \mathcal{F}_t]$ for all $t \in [0,t]$ then $M_s = \mathbb{E}[M_T \mid \mathcal{F}_s] = \mathbb{E}[\mathbb{E}[M_T \mid \mathcal{F}_t] \mid \mathcal{F}_s] = \mathbb{E}[M_T \mid \mathcal{F}_s]$, $0 \leq s \leq t \leq T$.

$$|\sigma(t,x)-\sigma(t,y)| = \left|\frac{x-y}{\sqrt{T-t}}\right| \leq \frac{|x-y|}{T-\varepsilon}, \qquad x,y\in\mathbb{R}.$$

Hence by *e.g.* Theorem V-7 in Protter (2004) this stochastic differential equation admits unique (strong) solution such that

$$S_t = S_0 + \int_0^t \frac{S_s}{T-s} dB_s, \qquad 0 \leq t \leq T-\varepsilon.$$

Next, we have

$$\begin{aligned} S_t &= S_0 \exp\left(\int_0^t \frac{dB_s}{\sqrt{T-s}} - \frac{1}{2}\int_0^t \frac{ds}{T-s}\right) \\ &= S_0 \exp\left(\int_0^t \frac{dB_s}{\sqrt{T-s}} - \frac{1}{2}\log\frac{T}{T-t}\right) \\ &= S_0\sqrt{1-\frac{t}{T}} \exp\left(\int_0^t \frac{dB_s}{\sqrt{T-s}}\right), \qquad 0 \leq t \leq T-\varepsilon. \end{aligned}$$

b) We have $S_T = 0$, as can be checked from the graphs of Question (d) below.
c) Consider the stopping times

$$\tau_n := \left(\left(1-\frac{1}{n}\right)T\right) \wedge \inf\{t\in[0,T] \ : \ |S_t| \geq n\}, \qquad n \geq 1.$$

for all $n \geq 1$ the stopped process $(S_{\tau_n\wedge t})_{t\in[0,T]}$ is given by

$$S_{\tau_n\wedge t} = S_0 + \int_0^{\tau_n\wedge t} \frac{S_u}{\sqrt{T-u}} dB_u = S_0 + \int_0^t \mathbb{1}_{[0,\tau_n]}(u)\frac{S_u}{\sqrt{T-u}} dB_u,$$

$0 \leq t \leq T$, and the process $(\mathbb{1}_{[0,\tau_n]}(u)S_u/\sqrt{T-u})_{0\leq u\leq\tau_n\wedge t}$ is square integrable as

$$\mathbb{E}\left[\int_0^T \mathbb{1}_{[0,\tau_n]}(u)\frac{S_u^2}{T-u}du\right] \leq \mathbb{E}\left[\int_0^{(1-1/n)T} \mathbb{1}_{[0,\tau_n]}(u)\frac{n^2}{T-u}du\right],$$

hence by Proposition 8.1 the stopped process $(S_{\tau_n\wedge t})_{t\in[0,T]}$ is a (true) martingale under $\mathbb{P}$ for all $n \geq 1$, and therefore $(S_t)_{t\in[0,T]}$ is a local martingale on $[0,T]$. Finally, we note that since $0 = \mathbb{E}[S_T] \neq S_0$, the process $(S_t)_{t\in[0,T]}$ is not a martingale.
d) The following code solves the stochastic differential equation $dS_t = S_t dB_t/\sqrt{1-t}$ by the Euler scheme.

```
N=10000; t <- 0:N; dt <- 1.0/N;
dB <- rnorm(N,mean=0,sd=sqrt(dt));S <- rep(0,N+1);S[1]=1.0
for (k in 2:(N-1)){S[k]=S[k-1]+S[k-1]*dB[k]/sqrt(1-k*dt)}
plot(t*dt, S, xlab = "t", ylab = "", type = "l", ylim = c(0,1.05*max(S)), col = "blue",
      xaxs = "i", yaxs = "i",cex.axis=1.6,cex.lab=1.8)
```

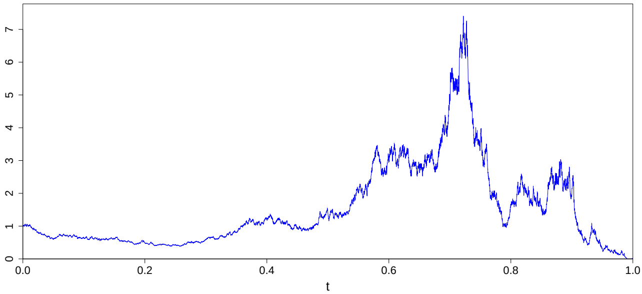


Fig. S.38: Sample path of $dS_t = S_t dB_t / \sqrt{1-t}$.

3) a) The following code solves the stochastic differential equation $dS_t = S_t^2 dB_t$ by the Euler scheme.

```
N=10000; t <- 0:N; dt <- 1.0/N;
dB <- rnorm(N+1,mean=0,sd=sqrt(dt));S <- rep(0,N+1);S[1]=1.0
for (k in 2:(N+1)){S[k]=S[k-1]+S[k-1]^2*dB[k]}
plot(t*dt, S, xlab = "t", ylab = "", type = "l", ylim = c(1.05*min(S),1.05*max(S)), col
      = "blue", xaxs = "i", yaxs = "i",cex.axis=1.6,cex.lab=1.8)
```

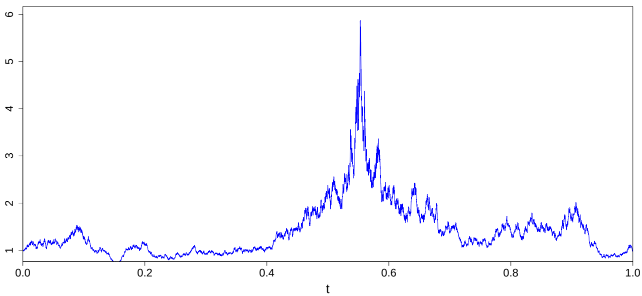


Fig. S.39: Sample path of $dS_t = S_t^2 dB_t$.

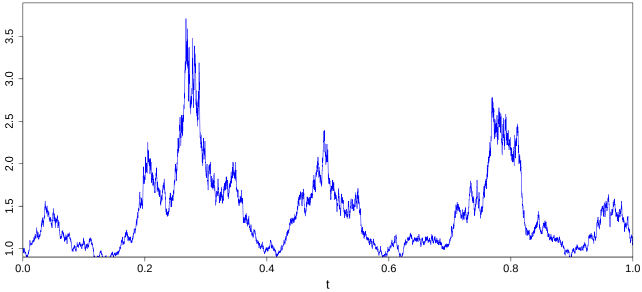


Fig. S.40: Sample path of $dS_t = S_t^2 dB_t$.

b) With the change of variable $y = 1/x$ and $dy = -dx/x^2$, we have

$$\begin{aligned}
\mathbb{E}[S_T] &= \int_0^\infty x\varphi_T(x)dx \\
&= \int_0^\infty \frac{S_0}{x^2\sqrt{2\pi T}} \left(\exp\left(-\frac{(1/x - 1/S_0)^2}{2T} \right) - \exp\left(-\frac{(1/x + 1/S_0)^2}{2T} \right) \right) dx \\
&= \int_0^\infty \frac{S_0}{\sqrt{2\pi T}} \exp\left(-\frac{(y - 1/S_0)^2}{2T} \right) dy - \int_0^\infty \frac{S_0}{\sqrt{2\pi T}} \exp\left(-\frac{(y + 1/S_0)^2}{2T} \right) dy \\
&= \int_{-1/S_0}^\infty \frac{S_0}{\sqrt{2\pi T}} \exp\left(-\frac{y^2}{2t} \right) dy - \int_{1/S_0}^\infty \frac{S_0}{\sqrt{2\pi T}} \exp\left(-\frac{y^2}{2T} \right) dy \\
&= \int_{-1/(S_0\sqrt{T})}^\infty \frac{S_0}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2} \right) dy - \int_{1/(S_0\sqrt{T})}^\infty \frac{S_0}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2} \right) dy \\
&= S_0 \Phi(1/(S_0\sqrt{T})) - S_0 \Phi(-1/(S_0\sqrt{T})) \\
&= S_0(1 - 2\Phi(-1/(S_0\sqrt{T}))).
\end{aligned}$$

c) We have

$$\begin{aligned}
\mathbb{E}[S_T] &= 2S_0 \left(\Phi\left(\frac{1}{S_0\sqrt{T}} \right) - \frac{1}{2} \right) \\
&= 2S_0 \left(\int_{-\infty}^{1/(S_0\sqrt{T})} \mathrm{e}^{-x^2/2} \frac{dx}{\sqrt{2\pi}} - \int_{-\infty}^0 \mathrm{e}^{-x^2/2} \frac{dx}{\sqrt{2\pi}} \right) \\
&= 2S_0 \int_0^{1/(S_0\sqrt{T})} \mathrm{e}^{-x^2/2} \frac{dx}{\sqrt{2\pi}} \\
&= 2 \int_0^{1/\sqrt{T}} \mathrm{e}^{-(y/S_0)^2/2} \frac{dy}{\sqrt{2\pi}},
\end{aligned}$$

hence by the *dominated convergence theorem* we have

$$\begin{aligned}
\lim_{S_0\to\infty}\mathbb{E}[S_T] &= \lim_{S_0\to\infty} 2\int_0^{1/\sqrt{T}} \mathrm{e}^{-(y/S_0)^2/2}\frac{dy}{\sqrt{2\pi}}\\
&= 2\int_0^{1/\sqrt{T}} \lim_{S_0\to\infty}\mathrm{e}^{-(y/S_0)^2/2}\frac{dy}{\sqrt{2\pi}}\\
&= 2\int_0^{1/\sqrt{T}} 1\frac{dy}{\sqrt{2\pi}}\\
&= \sqrt{\frac{2}{\pi T}}.
\end{aligned}$$

d) With the change of variable $y = 1/x$ and $dy = -dx/x^2$, we have

$$\begin{aligned}
&\mathbb{E}[(S_T-K)^+] = \int_0^\infty (x-K)^+\varphi_T(x)dx\\
&= S_0\int_K^\infty \frac{x-K}{x^3\sqrt{2\pi T}}\left(\exp\left(-\frac{(1/x-1/S_0)^2}{2T}\right)-\exp\left(-\frac{(1/x+1/S_0)^2}{2T}\right)\right)dx\\
&= \int_0^{1/K}\frac{S_0}{\sqrt{2\pi T}}\left(\exp\left(-\frac{(y-1/S_0)^2}{2T}\right)-\exp\left(-\frac{(y+1/S_0)^2}{2T}\right)\right)dy\\
&\quad -KS_0\int_0^{1/K}\frac{y}{\sqrt{2\pi T}}\left(\exp\left(-\frac{(y-1/S_0)^2}{2T}\right)-\exp\left(-\frac{(y+1/S_0)^2}{2T}\right)\right)dy\\
&= \int_{-1/S_0}^{1/K-1/S_0}\frac{S_0}{\sqrt{2\pi T}}\exp\left(-\frac{y^2}{2T}\right)dy-\int_{1/S_0}^{1/K+1/S_0}\frac{S_0}{\sqrt{2\pi T}}\exp\left(-\frac{y^2}{2T}\right)dy\\
&\quad -KS_0\int_{-1/S_0}^{1/K-1/S_0}\frac{y+1/S_0}{\sqrt{2\pi T}}\exp\left(-\frac{y^2}{2T}\right)dy\\
&\quad +KS_0\int_{1/S_0}^{1/K+1/S_0}\frac{y-1/S_0}{\sqrt{2\pi T}}\exp\left(-\frac{y^2}{2T}\right)dy\\
&= S_0\Phi\left(\frac{1}{K\sqrt{T}}-\frac{1}{S_0\sqrt{T}}\right)-S_0\Phi\left(-\frac{1}{S_0\sqrt{T}}\right)\\
&\quad -S_0\Phi\left(\frac{1}{K\sqrt{T}}+\frac{1}{S_0\sqrt{T}}\right)+S_0\Phi\left(\frac{1}{S_0\sqrt{T}}\right)\\
&\quad +KS_0\sqrt{T}\varphi\left(\frac{1}{K\sqrt{T}}-\frac{1}{S_0\sqrt{T}}\right)-KS_0\sqrt{T}\varphi\left(\frac{1}{K\sqrt{T}}+\frac{1}{S_0\sqrt{T}}\right)\\
&\quad -K\Phi\left(\frac{1}{K\sqrt{T}}-\frac{1}{S_0\sqrt{T}}\right)+K\Phi\left(-\frac{1}{S_0\sqrt{T}}\right)\\
&\quad -K\Phi\left(\frac{1}{K\sqrt{T}}+\frac{1}{S_0\sqrt{T}}\right)+K\Phi\left(\frac{1}{S_0\sqrt{T}}\right)\\
&= S_0\Phi\left(\frac{1}{K\sqrt{T}}-\frac{1}{S_0\sqrt{T}}\right)-S_0\Phi\left(-\frac{1}{S_0\sqrt{T}}\right)\\
&\quad -S_0\Phi\left(\frac{1}{K\sqrt{T}}+\frac{1}{S_0\sqrt{T}}\right)+S_0\Phi\left(\frac{1}{S_0\sqrt{T}}\right)
\end{aligned}$$

$$
\begin{aligned}
&+KS_0\sqrt{T}\varphi\left(\frac{1}{K\sqrt{T}}-\frac{1}{S_0\sqrt{T}}\right)-KS_0\sqrt{T}\varphi\left(\frac{1}{K\sqrt{T}}+\frac{1}{S_0\sqrt{T}}\right)\\
&-K\Phi\left(\frac{1}{K\sqrt{T}}-\frac{1}{S_0\sqrt{T}}\right)-K\Phi\left(\frac{1}{K\sqrt{T}}+\frac{1}{S_0\sqrt{T}}\right)+1\\
={}&S_0\Phi\left(\frac{1}{K\sqrt{T}}-\frac{1}{S_0\sqrt{T}}\right)-S_0\Phi\left(-\frac{1}{S_0\sqrt{T}}\right)\\
&-S_0\Phi\left(\frac{1}{K\sqrt{T}}+\frac{1}{S_0\sqrt{T}}\right)+S_0\Phi\left(\frac{1}{S_0\sqrt{T}}\right)\\
&+KS_0\sqrt{T}\varphi\left(\frac{1}{K\sqrt{T}}-\frac{1}{S_0\sqrt{T}}\right)-KS_0\sqrt{T}\varphi\left(\frac{1}{K\sqrt{T}}+\frac{1}{S_0\sqrt{T}}\right)\\
&+K\Phi\left(\frac{1}{S_0\sqrt{T}}-\frac{1}{K\sqrt{T}}\right)-K\Phi\left(\frac{1}{K\sqrt{T}}+\frac{1}{S_0\sqrt{T}}\right),
\end{aligned}
$$

where $\varphi(z) := \mathrm{e}^{-z^2}/\sqrt{2\pi}$ is the standard normal probability density function, see Relation (2.1.2) in Jacquier (2017).

e) We have

$$
\begin{aligned}
\mathbb{E}[(S_T-K)^+] &\leq \mathbb{E}[S_T]\\
&= 2\int_0^{1/\sqrt{T}} \mathrm{e}^{-(y/S_0)^2/2}\frac{dy}{\sqrt{2\pi}}\\
&\leq 2\int_0^{1/\sqrt{T}} \frac{dy}{\sqrt{2\pi}}\\
&\leq \sqrt{\frac{2}{\pi T}}.
\end{aligned}
$$

Fig. S.41: “Infogrames” stock price curve.

Problem 7.28

a) Relation (7.54) can be checked to hold first on the event A_α, and then on its complement A_α^c. Taking the $\mathbb{Q}$-expectation on both sides of (7.54) yields

$$\mathbb{E}_{\mathbb{Q}}\left[\left(\frac{d\mathbb{P}}{d\mathbb{Q}}-\alpha\right)(2\mathbb{1}_{A_\alpha}-1)\right] \geq \mathbb{E}_{\mathbb{Q}}\left[\left(\frac{d\mathbb{P}}{d\mathbb{Q}}-\alpha\right)(2\mathbb{1}_{A}-1)\right],$$

i.e.

$$\mathbb{E}_{\mathbb{Q}}\left[\frac{d\mathbb{P}}{d\mathbb{Q}}(2\mathbb{1}_{A_\alpha}-1)\right]-\alpha\,\mathbb{E}_{\mathbb{Q}}[2\mathbb{1}_{A_\alpha}-1] \geq \mathbb{E}_{\mathbb{Q}}\left[\frac{d\mathbb{P}}{d\mathbb{Q}}(2\mathbb{1}_{A}-1)\right]-\alpha\,\mathbb{E}_{\mathbb{Q}}[2\mathbb{1}_{A}-1],$$

i.e.

$$2\mathbb{P}(A_\alpha)-1-\alpha(2\mathbb{Q}(A_\alpha)-1) \geq 2\mathbb{P}(A)-1-\alpha(2\mathbb{Q}(A)-1),$$

which shows that

$$\mathbb{P}(A_\alpha)-\mathbb{P}(A) \geq \alpha(\mathbb{Q}(A_\alpha)-\mathbb{Q}(A)),$$

allowing us to conclude to $\mathbb{P}(A_\alpha)-\mathbb{P}(A) \geq 0$ since $\alpha \geq 0$.

b) We check that $d\mathbb{Q}^*/d\mathbb{P}^* \geq 0$ since $C \geq 0$, and

$$\begin{aligned}
\mathbb{Q}^*(\Omega) &= \int_\Omega d\mathbb{Q}^* \\
&= \int_\Omega \frac{d\mathbb{Q}^*}{d\mathbb{P}^*}d\mathbb{P}^* \\
&= \int_\Omega \frac{C}{\mathbb{E}_{\mathbb{P}^*}[C]}d\mathbb{P}^* \\
&= \mathbb{E}_{\mathbb{P}^*}\left[\frac{C}{\mathbb{E}_{\mathbb{P}^*}[C]}\right] \\
&= \frac{\mathbb{E}_{\mathbb{P}^*}[C]}{\mathbb{E}_{\mathbb{P}^*}[C]} \\
&= 1.
\end{aligned}$$

In the next questions we consider a nonnegative contingent claim payoff $C \geq 0$ with maturity $T > 0$, priced $\mathrm{e}^{-rT}\,\mathbb{E}_{\mathbb{P}^*}[C]$ at time 0 under the risk-neutral measure $\mathbb{P}^*$.

Budget constraint. We assume that no more than a certain fraction $\beta \in (0,1]$ of the claim price $\mathrm{e}^{-rT}\,\mathbb{E}_{\mathbb{P}^*}[C]$ is available to construct the initial hedging portfolio V_0 at time 0.

Since a self-financing portfolio process $(V_t)_{t\in\mathbb{R}_+}$ started at $V_0 = \beta\mathrm{e}^{-rT}\,\mathbb{E}_{\mathbb{P}^*}[C]$ may not able to hedge the claim C when $\beta < 1$, we will attempt to maximize the probability $\mathbb{P}(V_T \geq C)$ of successful hedging.

For this, given A an event we consider the portfolio process $(V_t^A)_{t\in[0,T]}$ hedging the claim $C\mathbb{1}_A$, priced $V_0^A = \mathrm{e}^{-rT}\,\mathbb{E}_{\mathbb{P}^*}[C\mathbb{1}_A]$ at time 0, and such that $V_T^A = C\mathbb{1}_A$ at maturity T.

c) Using the probability measure $\mathbb{Q}^*$, we rewrite the condition (7.56) as

$$\mathbb{Q}^*(A) = \mathbb{E}_{\mathbb{Q}^*}[\mathbb{1}_A] = \mathbb{E}_{\mathbb{P}^*}\left[\mathbb{1}_A \frac{d\mathbb{Q}^*}{d\mathbb{P}^*}\right] = \frac{\mathbb{E}_{\mathbb{P}^*}[C\mathbb{1}_A]}{\mathbb{E}_{\mathbb{P}^*}[C]} \leq \beta,$$

i.e.

$$\mathbb{Q}^*(A) \leq \mathbb{Q}^*(A_\alpha) = \beta.$$

By the Neyman-Pearson Lemma, for any event A, the inequality $\mathbb{Q}^*(A) \leq \mathbb{Q}^*(A_\alpha) = \beta$ implies $\mathbb{P}(A) \leq \mathbb{P}(A_\alpha)$, which shows that the event $A = A_\alpha$ realizes the maximum under the required condition.

d) The obvious inequality is

$$\mathbb{P}(A_\alpha) \leq \mathbb{P}(C\mathbb{1}_{A_\alpha} \geq C) = \mathbb{P}\big(V_T^{A_\alpha} \geq C\big).$$

In the other direction, we note that the event $B_\alpha := \{C\mathbb{1}_{A_\alpha} \geq C\} = \big\{V_T^{A_\alpha} \geq C\big\}$ satisfies (7.56), as

$$\begin{aligned}
\mathrm{e}^{-rT}\,\mathbb{E}_{\mathbb{P}^*}\left[V_T^{B_\alpha}\right] &= \mathrm{e}^{-rT}\,\mathbb{E}_{\mathbb{P}^*}\left[C\mathbb{1}_{B_\alpha}\right] \\
&= \mathrm{e}^{-rT}\,\mathbb{E}_{\mathbb{P}^*}\left[C\mathbb{1}_{\{C\mathbb{1}_{A_\alpha}\geq C\}}\right] \\
&= \mathrm{e}^{-rT}\,\mathbb{E}_{\mathbb{P}^*}\left[C\mathbb{1}_{A_\alpha}\right] \\
&= \beta\mathrm{e}^{-rT}\,\mathbb{E}_{\mathbb{P}^*}[C],
\end{aligned}$$

where the last equality $\mathbb{E}_{\mathbb{P}^*}\left[C\mathbb{1}_{A_\alpha}\right] = \beta\,\mathbb{E}_{\mathbb{P}^*}[C]$ follows from $\mathbb{Q}^*(A_\alpha) = \beta$ and the definition (7.55) of $\mathbb{Q}^*$.

Therefore, by the result of Question (c) we have

$$\mathbb{P}(C\mathbb{1}_{A_\alpha} \geq C) = \mathbb{P}\big(V_T^{A_\alpha} \geq C\big) = \mathbb{P}(B_\alpha) \leq \mathbb{P}(A_\alpha).$$

e) This hedging strategy starts from the initial amount

$$V_0^{A_\alpha} = \mathrm{e}^{-rT}\,\mathbb{E}_{\mathbb{P}^*}[C\mathbb{1}_{A_\alpha}] = \beta\mathrm{e}^{-rT}\,\mathbb{E}_{\mathbb{P}^*}[C],$$

and it satisfies

$$\mathbb{P}\big(V_T^{A_\alpha} \geq C\big) = \mathbb{P}(C\mathbb{1}_{A_\alpha} \geq C) = \mathbb{P}(A_\alpha)$$

which is the maximum hedging probability under the constraint (7.56).

f) We have

$$S_t = S_0\mathrm{e}^{\sigma B_t + rt - \sigma^2 t/2} = S_0\mathrm{e}^{\sigma B_t} = S_0\mathrm{e}^{B_t}, \qquad t \geq 0.$$

g) We have

$$\begin{aligned}
\mathbb{P}\big(V_T^{A_\alpha} \geq C\big) &= \mathbb{P}(A_\alpha) \\
&= \mathbb{P}\left(\frac{d\mathbb{P}}{d\mathbb{Q}^*} > \alpha\right) \\
&= \mathbb{P}\left(\frac{d\mathbb{P}}{d\mathbb{P}^*} > \alpha \frac{d\mathbb{Q}^*}{d\mathbb{P}^*}\right) \\
&= \mathbb{P}\left(\frac{d\mathbb{P}}{d\mathbb{P}^*} > \alpha \frac{C}{\mathbb{E}_{\mathbb{P}^*}[C]}\right) \\
&= \mathbb{P}\left(\alpha C < \mathbb{E}_{\mathbb{P}^*}[C]\right) \\
&= \mathbb{P}\big((S_T - K)^+ < \mathbb{E}_{\mathbb{P}^*}[C]/\alpha\big) \\
&= \mathbb{P}\big(S_T - K < \mathbb{E}_{\mathbb{P}^*}[C]/\alpha\big) \\
&= \mathbb{P}\big(S_T < K + \mathbb{E}_{\mathbb{P}^*}[C]/\alpha\big) \\
&= \mathbb{P}\big(S_0 \mathrm{e}^{B_T} < K + \mathbb{E}_{\mathbb{P}^*}[C]/\alpha\big) \\
&= \mathbb{P}\left(B_T < \log(K + \mathbb{E}_{\mathbb{P}^*}[C]/\alpha)\right) \\
&= \Phi\left(\frac{\log(K + \mathbb{E}_{\mathbb{P}^*}[C]/\alpha)}{\sqrt{T}}\right).
\end{aligned}$$

h) We have

$$\alpha = \frac{\mathbb{E}_{\mathbb{P}^*}[C]}{\exp\left(\sqrt{T}\Phi^{-1}\big(\mathbb{P}\big(V_T^{A_\alpha} \geq C\big)\big)\right) - K}.$$

i) We have

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}^*}[C\mathbb{1}_{A_\alpha}] &= \mathbb{E}_{\mathbb{P}^*}[(S_T - K)^+ \mathbb{1}_{A_\alpha}] \\
&= \mathbb{E}_{\mathbb{P}^*}\left[(S_T - K)^+ \mathbb{1}_{\{B_T < \log(K + \mathbb{E}_{\mathbb{P}^*}[C]/\alpha)\}}\right] \\
&= \mathbb{E}_{\mathbb{P}^*}\left[\big(\mathrm{e}^{B_T} - K\big)^+ \mathbb{1}_{\{B_T < \log(K + \mathbb{E}_{\mathbb{P}^*}[C]/\alpha)\}}\right] \\
&= \int_{(\log K)/\sqrt{T}}^{\log(K + \mathbb{E}_{\mathbb{P}^*}[C]/\alpha)/\sqrt{T}} (\mathrm{e}^x - K)\mathrm{e}^{-x^2/2} \frac{dx}{\sqrt{2\pi}} \\
&= \int_{(\log K)/\sqrt{T}}^{\log(K + \mathbb{E}_{\mathbb{P}^*}[C]/\alpha)/\sqrt{T}} \mathrm{e}^x \mathrm{e}^{-x^2/2} \frac{dx}{\sqrt{2\pi}} \\
&\quad - K \int_{(\log K)/\sqrt{T}}^{\log(K + \mathbb{E}_{\mathbb{P}^*}[C]/\alpha)/\sqrt{T}} \mathrm{e}^{-x^2/2} \frac{dx}{\sqrt{2\pi}} \\
&= \int_{(\log K)/\sqrt{T}}^{\log(K + \mathbb{E}_{\mathbb{P}^*}[C]/\alpha)/\sqrt{T}} \mathrm{e}^{1/2 - (x-1)^2/2} \frac{dx}{\sqrt{2\pi}} \\
&\quad - K\big(\Phi(\log(K + \mathbb{E}_{\mathbb{P}^*}[C]/\alpha)) - \Phi(\log K)\big) \\
&= \int_{(-1+\log K)/\sqrt{T}}^{-1/\sqrt{T} + \log(K + \mathbb{E}_{\mathbb{P}^*}[C]/\alpha)/\sqrt{T}} \mathrm{e}^{1/2 - x^2/2} \frac{dx}{\sqrt{2\pi}} \\
&\quad - K\big(\Phi(\log(K + \mathbb{E}_{\mathbb{P}^*}[C]/\alpha)) - \Phi(\log K)\big)
\end{aligned}$$

$$= \sqrt{e}\left(\Phi\left(\frac{-1+\log(K+\mathbb{E}_{\mathbb{P}^*}[C]/\alpha)}{\sqrt{T}}\right) - \Phi\left(\frac{-1+\log K}{\sqrt{T}}\right)\right)$$
$$-K\left(\Phi\left(\frac{\log(K+\mathbb{E}_{\mathbb{P}^*}[C]/\alpha)}{\sqrt{T}}\right) - \Phi\left(\frac{\log K}{\sqrt{T}}\right)\right),$$

hence

$$\begin{aligned}
&\mathrm{e}^{-rT}\,\mathbb{E}_{\mathbb{P}^*}[C\mathbb{1}_{A_\alpha}] \\
&= \mathrm{e}^{-rT+1/2}\left(\Phi\left(\frac{-1+\log(K+\mathbb{E}_{\mathbb{P}^*}[C]/\alpha)}{\sqrt{T}}\right) - \Phi\left(\frac{-1+\log K}{\sqrt{T}}\right)\right) \\
&\quad -K\mathrm{e}^{-rT}\left(\Phi\left(\frac{\log(K+\mathbb{E}_{\mathbb{P}^*}[C]/\alpha)}{\sqrt{T}}\right) - \Phi\left(\frac{\log K}{\sqrt{T}}\right)\right).
\end{aligned}$$

j) i) We have $d_+(T) = 1$, $d_-(T) = 0$, hence by the Black-Scholes formula we find

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}^*}[(S_T-K)^+] &= \sqrt{e}S_0\Phi(d_+(T)) - K\Phi(d_-(T)) \\
&= 1.64872 \times 0.84134 - 1/2 \\
&= 0.88713,
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{e}^{-rT}\,\mathbb{E}_{\mathbb{P}^*}[(S_T-K)^+] &= S_0\Phi(d_+(T)) - K\mathrm{e}^{-Tr}\Phi(d_-(T)) \\
&= 0.84134 - 0.60653/2 \\
&= 0.53807.
\end{aligned}$$

ii) By the result of Question (h), we have

$$\begin{aligned}
\alpha &= \frac{\mathbb{E}_{\mathbb{P}^*}[C]}{\exp\left(\Phi^{-1}\big(\mathbb{P}\big(V_T^{A_\alpha} \geq C\big)\big)\right) - K} \\
&= \frac{0.88714}{\exp\left(\Phi^{-1}(0.9)\right) - 1} \\
&= \frac{0.88714}{\mathrm{e}^{1.28} - 1} = 0.34165.
\end{aligned}$$

iii) By the result of Question (i), we find

$$\begin{aligned}
\mathrm{e}^{-rT}\,\mathbb{E}_{\mathbb{P}^*}[C\mathbb{1}_{A_\alpha}] &= \big(\Phi(-1+\log(K+\mathbb{E}_{\mathbb{P}^*}[C]/\alpha)) - \Phi(-1+\log K)\big) \\
&\quad -K\mathrm{e}^{-1/2}\big(\Phi(\log(K+\mathbb{E}_{\mathbb{P}^*}[C]/\alpha)) - \Phi(\log K)\big) \\
&= \big(\Phi(-1+\log(1+\mathbb{E}_{\mathbb{P}^*}[C]/\alpha)) - \Phi(-1)\big) \\
&\quad -\mathrm{e}^{-1/2}\big(\Phi(\log(1+\mathbb{E}_{\mathbb{P}^*}[C]/\alpha)) - \Phi(0)\big) \\
&= \big(\Phi(-1+\log(1+0.88713/0.34165)) - \Phi(-1)\big) \\
&\quad -\mathrm{e}^{-1/2}\big(\Phi(\log(3.596604712)) - \Phi(0)\big)
\end{aligned}$$

$$
\begin{aligned}
&= \big(\Phi(0.27999) - \Phi(-1)\big) \\
&\quad -0.60653 \times \big(\Phi(1.279990265) - \Phi(0)\big) \\
&= (0.61026 - 0.158655) - 0.60653 \times (0.899726 - 0.5) \\
&= 0.20915.
\end{aligned}
$$

In addition, we find

$$
\beta = \frac{\mathrm{e}^{-rT}\,\mathbb{E}_{\mathbb{P}^*}[C\mathbb{1}_{A_\alpha}]}{\mathrm{e}^{-rT}\,\mathbb{E}_{\mathbb{P}^*}[C]} = \frac{0.20915}{0.53807} = 37.53\%.
$$

Chapter 8

Exercise 8.1 We need to compute the average

$$
\frac{1}{T}\,\mathbb{E}\left[\int_0^T v_t dt\right] = \frac{1}{T}\int_0^T \mathbb{E}[v_t]dt = \frac{1}{T}\int_0^T u(t)dt,
$$

where $u(t) := \mathbb{E}[v_t]$. Taking expectation on both sides of the equation

$$
v_t = v_0 - \lambda\int_0^t (v_s - m)ds + \eta\int_0^t \sqrt{v_s}dB_s,
$$

we find

$$
\begin{aligned}
u(t) &= \mathbb{E}[v_t] \\
&= \mathbb{E}\left[v_0 - \lambda\int_0^t (v_s - m)ds + \eta\int_0^t \sqrt{v_s}dB_s\right] \\
&= v_0 - \lambda\,\mathbb{E}\left[\int_0^t (v_s - m)ds\right] \\
&= v_0 - \lambda\int_0^t (\mathbb{E}[v_s] - m)ds \\
&= v_0 - \lambda\int_0^t (u(s) - m)ds, \qquad t \geq 0,
\end{aligned}
$$

hence by differentiation with respect to $t \in \mathbb{R}$ we find the ordinary differential equation

$$
u'(t) = \lambda m - \lambda u(t),
$$

cf. *e.g.* Exercise 4.18-(b). This equation can be rewritten as

$$
(\mathrm{e}^{\lambda t}u(t))' = \lambda\mathrm{e}^{\lambda t}u(t) + \mathrm{e}^{\lambda t}u'(t) = \lambda m\mathrm{e}^{\lambda t},
$$

which can be integrated as

$$
\begin{aligned}
e^{\lambda t}u(t) &= \left(u(0)+\lambda m\int_0^t e^{\lambda s}ds\right)\\
&= \mathbb{E}[v_0]+m(e^{\lambda t}-1)\\
&= me^{\lambda t}+\mathbb{E}[v_0]-m \qquad t\in\mathbb{R}_+,
\end{aligned}
$$

from which we conclude that

$$
u(t)=m+(\mathbb{E}[v_0]-m)e^{-\lambda t}, \qquad t\in\mathbb{R}_+,
$$

and

$$
\begin{aligned}
\frac{1}{T}\mathbb{E}\left[\int_0^T v_t dt\right] &= \frac{1}{T}\int_0^T u(t)dt\\
&= \frac{1}{T}\int_0^T \big(m+(\mathbb{E}[v_0]-m)e^{-\lambda t}\big)dt\\
&= m+\frac{\mathbb{E}[v_0]-m}{T}\int_0^T e^{-\lambda t}dt\\
&= m+\big(\,\mathbb{E}[v_0]-m\big)\frac{1-e^{-\lambda T}}{\lambda T}.
\end{aligned}
$$

Exercise 8.2

a) By *e.g.* Exercise 4.18-(b), we have

$$
\mathbb{E}[v_t]=\mathbb{E}[v_0]e^{-\lambda t}+m(1-e^{-\lambda t}), \qquad t\in\mathbb{R}_+,
$$

hence

$$
\begin{aligned}
\mathrm{VS}_T &= \frac{1}{T}\mathbb{E}\left[\int_0^T \frac{1}{S_t^2}(dS_t)^2\right]\\
&= \frac{1}{T}\mathbb{E}\left[\int_0^T \frac{1}{S_t^2}\Big((r-\alpha v_t)S_tdt+S_t\sqrt{\beta+v_t}dB_t^{(1)}\Big)^2\right]\\
&= \frac{1}{T}\mathbb{E}\left[\int_0^T (\beta+v_t)dt\right]\\
&= \beta+\frac{1}{T}\int_0^T \mathbb{E}[v_t]dt,
\end{aligned}
$$

which yields

$$
\begin{aligned}
\mathrm{VS}_T &= \beta+\frac{1}{T}\int_0^T(\mathbb{E}[v_0]e^{-\lambda t}+m(1-e^{-\lambda t}))dt\\
&= \beta+\frac{1}{T}\int_0^T(\mathbb{E}[v_0]e^{-\lambda t}+m(1-e^{-\lambda t}))dt\\
&= \beta+m+\frac{1}{T}(\mathbb{E}[v_0]-m)\int_0^T e^{-\lambda t}dt
\end{aligned}
$$

$$= \beta + m + (\mathbb{E}[v_0] - m)\frac{e^{\lambda T} - 1}{\lambda T}.$$

Note that if the process $(v_t)_{t \in \mathbb{R}_+}$ is started in the gamma stationary distribution then we have $\mathbb{E}[v_0] = \mathbb{E}[v_t] = m$, $t \in \mathbb{R}_+$, and the variance swap rate $\mathrm{VS}_T = \beta + m$ becomes independent of the time T.

b) The stochastic differential equation $d\sigma_t = \alpha \sigma_t dB_t^{(2)}$ is solved as

$$\sigma_t = \sigma_0 e^{\alpha B_t^{(2)} - \alpha^2 t/2}, \qquad t \in \mathbb{R}_+,$$

hence we have

$$\begin{aligned}
\mathrm{VS}_T &= \frac{1}{T}\mathbb{E}\left[\int_0^T \frac{1}{S_t^2}(dS_t)^2\right] \\
&= \frac{1}{T}\mathbb{E}\left[\int_0^T \frac{1}{S_t^2}\big(\sigma_t S_t dB_t^{(1)}\big)^2\right] \\
&= \frac{1}{T}\mathbb{E}\left[\int_0^T \sigma_t^2 dt\right] \\
&= \frac{\sigma_0^2}{T}\int_0^T \mathbb{E}\left[e^{2\alpha B_t^{(2)} - \alpha^2 t}\right] dt \\
&= \frac{\sigma_0^2}{T}\int_0^T e^{-\alpha^2 t}\,\mathbb{E}\left[e^{2\alpha B_t^{(2)}}\right] dt \\
&= \frac{\sigma_0^2}{T}\int_0^T e^{-\alpha^2 t + 2\alpha^2 t} dt \\
&= \frac{\sigma_0^2}{T}\int_0^T e^{\alpha^2 t} dt \\
&= \frac{\sigma_0^2}{\alpha^2 T}(e^{\alpha^2 T} - 1).
\end{aligned}$$

Exercise 8.3

a) Taking $x = R_{0,T}^2$ and $x_0 = \mathbb{E}\left[R_{0,T}^2\right]$, we have

$$R_{0,T} \approx \sqrt{\mathbb{E}\left[R_{0,T}^2\right]} + \frac{R_{0,T}^2 - \mathbb{E}\left[R_{0,T}^2\right]}{2\sqrt{\mathbb{E}\left[R_{0,T}^2\right]}} - \frac{\left(R_{0,T}^2 - \mathbb{E}\left[R_{0,T}^2\right]\right)^2}{8\left(\mathbb{E}\left[R_{0,T}^2\right]\right)^{3/2}}, \quad \text{(S.8.45)}$$

provided that $R_{0,T}^2$ is sufficiently close to $\mathbb{E}\left[R_{0,T}^2\right]$.

b) Taking expectations on both sides of (S.8.45), we find

$$\mathbb{E}^*[R_{0,T}] \approx \sqrt{\mathbb{E}\left[R_{0,T}^2\right]} + \frac{\mathbb{E}\left[R_{0,T}^2\right] - \mathbb{E}\left[R_{0,T}^2\right]}{2\sqrt{\mathbb{E}\left[R_{0,T}^2\right]}} - \frac{\mathbb{E}\left[\left(R_{0,T}^2 - \mathbb{E}[R_{0,T}^2]\right)^2\right]}{8\left(\mathbb{E}\left[R_{0,T}^2\right]\right)^{3/2}}$$

$$= \sqrt{\mathbb{E}\left[R_{0,T}^2\right]} - \frac{\mathbb{E}\left[\left(R_{0,T}^2 - \mathbb{E}\left[R_{0,T}^2\right]\right)^2\right]}{8\big(\mathbb{E}\left[R_{0,T}^2\right]\big)^{3/2}}$$

$$= \sqrt{\mathbb{E}\left[R_{0,T}^2\right]} - \frac{\mathrm{Var}\left[R_{0,T}^2\right]}{8\big(\mathbb{E}\left[R_{0,T}^2\right]\big)^{3/2}},$$

provided that $R_{0,T}^2$ is sufficiently close to $\mathbb{E}\left[R_{0,T}^2\right]$.

Exercise 8.4 We have

$$\begin{aligned}
\mathbb{E}\left[\sum_{n=1}^{N_T}\left(\log\frac{S_{T_k}}{S_{T_{k-1}}}\right)^2\right] &= \mathbb{E}\left[\int_0^T\left(\log\frac{S_t}{S_{t^-}}\right)^2 dN_t\right] \\
&= \mathbb{E}\left[\int_0^T \left(Z_{N_{t^-}}\right)^2 dN_t\right] \\
&= \lambda\int_0^T \mathbb{E}\left[\left(Z_{N_{t^-}}\right)^2\right] dt \\
&= \lambda\int_0^T (\eta^2+\delta^2) dt \\
&= \lambda(\eta^2+\delta^2)T.
\end{aligned}$$

Exercise 8.5

a) We have $S_t = S_0 \mathrm{e}^{\sigma B_t - \sigma^2 t/2 + rt}$, $t \in \mathbb{R}_+$.

b) Letting $\widetilde{S}_t := \mathrm{e}^{-rt} S_t$, $t \in \mathbb{R}_+$, we have $\widetilde{S}_T = S_0 \mathrm{e}^{\sigma B_T - \sigma^2 T/2}$ and $d\widetilde{S}_t = \sigma \widetilde{S}_t dB_t$, hence

$$\widetilde{S}_T = S_0 + \sigma\int_0^T \widetilde{S}_t dB_t,$$

and

$$\begin{aligned}
&2\,\mathbb{E}^*\left[\frac{\mathrm{e}^{-rT}S_T}{S_0}\log\frac{\mathrm{e}^{-rT}S_T}{S_0}\right] = 2\,\mathbb{E}^*\left[\frac{\widetilde{S}_T}{S_0}\log\frac{\widetilde{S}_T}{S_0}\right] \\
&= 2\,\mathbb{E}^*\left[\left(1+\sigma\int_0^T\frac{\widetilde{S}_t}{S_0}dB_t\right)\left(\sigma B_T - \frac{\sigma^2 T}{2}\right)\right] \\
&= 2\,\mathbb{E}^*\left[\sigma B_T - \frac{\sigma^2 T}{2}\right] + 2\sigma^2\,\mathbb{E}^*\left[B_T\int_0^T\frac{\widetilde{S}_t}{S_0}dB_t\right] - \sigma^2 T\,\mathbb{E}^*\left[\int_0^T\frac{\widetilde{S}_t}{S_0}dB_t\right] \\
&= -\sigma^2 T + 2\sigma^2\,\mathbb{E}^*\left[\int_0^T dB_t\int_0^T\frac{\widetilde{S}_t}{S_0}dB_t\right] \\
&= -\sigma^2 T + 2\sigma^2\,\mathbb{E}^*\left[\int_0^T\frac{\widetilde{S}_t}{S_0}dt\right]
\end{aligned}$$

$$
\begin{aligned}
&= -\sigma^2 T + 2\sigma^2 \int_0^T \mathbb{E}^*\left[\frac{\widetilde{S}_t}{S_0}\right] dt \\
&= -\sigma^2 T + 2\sigma^2 \int_0^T dt \\
&= -\sigma^2 T + 2\sigma^2 T \\
&= \sigma^2 T.
\end{aligned}
$$

Alternatively, we could also write

$$
\begin{aligned}
&2\,\mathbb{E}^*\left[\frac{\mathrm{e}^{-rT} S_T}{S_0} \log \frac{\mathrm{e}^{-rT} S_T}{S_0}\right] = 2\,\mathbb{E}^*\left[\frac{\widetilde{S}_T}{S_0} \log \frac{\widetilde{S}_T}{S_0}\right] \\
&= 2\,\mathbb{E}^*\left[\mathrm{e}^{\sigma B_T - \sigma^2 T/2} \log \mathrm{e}^{\sigma B_T - \sigma^2 T/2}\right] \\
&= 2\,\mathbb{E}^*\left[\mathrm{e}^{\sigma B_T - \sigma^2 T/2} \left(\sigma B_T - \frac{\sigma^2 T}{2}\right)\right] \\
&= 2\sigma \mathrm{e}^{-\sigma^2 T/2}\,\mathbb{E}^*\left[B_T \mathrm{e}^{\sigma B_T}\right] - \sigma^2 T\,\mathbb{E}^*\left[\mathrm{e}^{\sigma B_T - \sigma^2 T/2}\right] \\
&= 2\sigma \mathrm{e}^{-\sigma^2 T/2} \frac{\partial}{\partial \sigma}\,\mathbb{E}^*\left[\mathrm{e}^{\sigma B_T}\right] - \sigma^2 T \\
&= 2\sigma \mathrm{e}^{-\sigma^2 T/2} \frac{\partial}{\partial \sigma} \mathrm{e}^{\sigma^2 T/2} - \sigma^2 T \\
&= 2\sigma^2 T \mathrm{e}^{-\sigma^2 T/2} \mathrm{e}^{\sigma^2 T/2} - \sigma^2 T \\
&= \sigma^2 T.
\end{aligned}
$$

Exercise 8.6

a) By the Itô formula, we have

$$
\log \frac{S_T}{S_0} = \log S_T - \log S_0 = \int_0^T \frac{dS_t}{S_t} - \frac{1}{2} \int_0^T \frac{\sigma_t^2}{S_t^2} dt.
$$

b) By (8.47) we have

$$
\begin{aligned}
\mathbb{E}^*\left[\int_0^T \sigma_t^2 dt \,\middle|\, \mathcal{F}_t\right] &= 2\,\mathbb{E}^*\left[\int_0^T \frac{dS_t}{S_t} \,\middle|\, \mathcal{F}_t\right] - 2\,\mathbb{E}^*\left[\log \frac{S_T}{S_0} \,\middle|\, \mathcal{F}_t\right] \\
&= 2 \int_0^t \frac{dS_u}{S_u} + 2r(T-t) - 2\,\mathbb{E}^*\left[\log \frac{S_T}{S_0} \,\middle|\, \mathcal{F}_t\right].
\end{aligned}
$$

c) At time $t \in [0, T]$ we check that

$$
\begin{aligned}
&L_t + \mathrm{e}^{-(T-t)r} \frac{2}{S_t} S_t + 2\mathrm{e}^{-rT} \left(\int_0^t \frac{dS_u}{S_u} + (T-t)r - 1\right) A_t \\
&\qquad = L_t + 2r(T-t)\mathrm{e}^{-(T-t)r} + 2\mathrm{e}^{-(T-t)r} \int_0^t \frac{dS_u}{S_u}
\end{aligned}
$$

$$= V_t.$$

d) By (8.48) we have

$$\begin{aligned}
dV_t &= d\left(L_t + 2r(T-t)\mathrm{e}^{-(T-t)r} + 2\mathrm{e}^{-(T-t)r}\int_0^t \frac{dS_u}{S_u}\right) \\
&= dL_t - 2r\mathrm{e}^{-(T-t)r}dt + 2r^2(T-t)\mathrm{e}^{-(T-t)r}dt \\
&\quad +2r\mathrm{e}^{-(T-t)r}\int_0^t \frac{dS_u}{S_u}dt + 2\mathrm{e}^{-(T-t)r}\frac{dS_t}{S_t} \\
&= dL_t + \mathrm{e}^{-(T-t)r}\frac{2}{S_t}dS_t + 2\mathrm{e}^{-rT}\left(\int_0^t \frac{dS_u}{S_u} + (T-t)r - 1\right)dA_t,
\end{aligned}$$

with $dA_t = r\mathrm{e}^{rt}dt$, hence the portfolio is self-financing.

Exercise 8.7 By second differentiation of the moment generating function (8.9), we find the two expressions

$$\mathbb{E}^*\left[R_{0,T}^4\right] = 4\,\mathbb{E}^*\left[\left(\log\frac{S_T}{F_0}\right)^2 + 2\log\frac{S_T}{F_0}\right] = 4\,\mathbb{E}^*\left[\left(\log\frac{S_T}{F_0}\right)^2\right] - 4\,\mathbb{E}^*\left[R_{0,T}^2\right],$$

and

$$\begin{aligned}
\mathbb{E}^*\left[R_{0,T}^4\right] &= 4\,\mathbb{E}^*\left[\frac{S_T}{F_0}\left(\left(\log\frac{S_T}{F_0}\right)^2 - 2\log\frac{S_T}{F_0}\right)\right] \\
&= 4\,\mathbb{E}^*\left[\frac{S_T}{F_0}\left(\log\frac{S_T}{F_0}\right)^2\right] - 4\,\mathbb{E}^*\left[R_{0,T}^2\right].
\end{aligned}$$

Chapter 9

Exercise 9.1

a) We have $\displaystyle\frac{\partial C}{\partial x}(T-t,x,K) = \frac{\partial f}{\partial z}\left(T-t,\frac{x}{K}\right)$ and

$$\begin{aligned}
\frac{\partial C}{\partial K}(T-t,x,K) &= \frac{\partial}{\partial K}\left(Kf\left(T-t,\frac{x}{K}\right)\right) \\
&= f\left(T-t,\frac{x}{K}\right) - \frac{x}{K}\frac{\partial f}{\partial z}\left(T-t,\frac{x}{K}\right) \\
&= \frac{1}{K}C(T-t,x,K) - \frac{x}{K}\frac{\partial C}{\partial x}(T-t,x,K),
\end{aligned}$$

hence

$$\frac{\partial C}{\partial x}(T-t,x,K) = \frac{1}{x}C(T-t,x,K) - \frac{K}{x}\frac{\partial C}{\partial K}(T-t,x,K).$$

b) We have $\displaystyle\frac{\partial^2 C}{\partial x^2}(T-t,x,K) = \frac{1}{K}\frac{\partial^2 f}{\partial z^2}\left(T-t,\frac{x}{K}\right)$ and

$$\begin{aligned}
&\frac{\partial^2 C}{\partial K^2}(T-t,x,K)\\
&= -\frac{x}{K^2}\frac{\partial f}{\partial z}\left(T-t,\frac{x}{K}\right) + \frac{x}{K^2}\frac{\partial f}{\partial z}\left(T-t,\frac{x}{K}\right) + \frac{x^2}{K^3}\frac{\partial f}{\partial z}\left(T-t,\frac{x}{K}\right)\\
&= \frac{x^2}{K^3}\frac{\partial^2 f}{\partial z^2}\left(T-t,\frac{x}{K}\right)\\
&= \frac{x^2}{K^2}\frac{\partial^2 C}{\partial x^2}(T-t,x,K),
\end{aligned}$$

hence

$$\frac{\partial^2 C}{\partial x^2}(T-t,x,K) = \frac{K^2}{x^2}\frac{\partial^2 C}{\partial K^2}(T-t,x,K).$$

c) Noting that

$$\frac{\partial C}{\partial t}(T-t,x,K) = -\frac{\partial C}{\partial T}(T-t,x,K),$$

we can rewrite the Black-Scholes PDE as

$$\begin{aligned}
rC(T-t,x,K) = &-\frac{\partial C}{\partial T}(T-t,x,K)\\
&+rx\left(\frac{1}{x}C(T-t,x,K) - \frac{K}{x}\frac{\partial C}{\partial K}(T-t,x,K)\right)\\
&+\frac{\sigma^2 x^2}{2}\frac{K^2}{x^2}\frac{\partial^2 C}{\partial K^2}(T-t,x,K),
\end{aligned}$$

i.e.

$$\frac{\partial C}{\partial T}(T-t,x,K) = -rK\frac{\partial C}{\partial K}(T-t,x,K) + \frac{\sigma^2 x^2}{2}\frac{K^2}{x^2}\frac{\partial^2 C}{\partial K^2}(T-t,x,K).$$

Remarks:

1. Using the Black-Scholes Greek Gamma expression

$$\begin{aligned}
\frac{\partial^2 C}{\partial x^2}(T-t,x,K) &= \frac{1}{\sigma x\sqrt{T-t}}\Phi'(d_+(T-t))\\
&= \frac{1}{\sigma x\sqrt{2\pi(T-t)}}\mathrm{e}^{-(d_+(T-t))^2/2},
\end{aligned}$$

we can recover the lognormal probability density function $\varphi_T(y)$ of geometric Brownian motion S_T as follows:

$$\begin{aligned}\varphi_T(K) &= \mathrm{e}^{(T-t)r}\frac{\partial^2 C}{\partial K^2}(T-t,x,K)\\ &= \mathrm{e}^{(T-t)r}\frac{x^2}{K^2}\frac{\partial^2 C}{\partial x^2}(T-t,x,K)\\ &= \frac{\mathrm{e}^{(T-t)r}x}{\sigma K^2\sqrt{2\pi(T-t)}}\mathrm{e}^{-(d_+(T-t))^2/2}\\ &= \frac{1}{\sigma K\sqrt{2\pi(T-t)}}\mathrm{e}^{-(d_-(T-t))^2/2}\\ &= \frac{1}{\sigma K\sqrt{2\pi(T-t)}}\exp\left(-\frac{\big((r-\sigma^2/2)(T-t)+\log(x/K)\big)^2}{2(T-t)\sigma^2}\right),\end{aligned}$$

knowing that

$$\begin{aligned}-\frac{1}{2}\big(d_-(T-t)\big)^2 &= -\frac{1}{2}\left(\frac{\log(x/K)+(r-\sigma^2/2)(T-t)}{|\sigma|\sqrt{T-t}}\right)^2\\ &= -\frac{1}{2}\left(\frac{\log(x/K)+(r+\sigma^2/2)(T-t)}{|\sigma|\sqrt{T-t}}\right)^2+(T-t)r+\log\frac{x}{K}\\ &= -\frac{1}{2}\big(d_+(T-t)\big)^2+(T-t)r+\log\frac{x}{K},\end{aligned}$$

which can be obtained from the relation

$$\begin{aligned}&\big(d_+(T-t)\big)^2-\big(d_-(T-t)\big)^2\\ &= \big(\big(d_+(T-t)+d_-(T-t)\big)\big(\big(d_+(T-t)-d_-(T-t)\big)\\ &= 2r(T-t)+2\log\frac{x}{K}.\end{aligned}$$

2. Using the expressions of the Black-Scholes Greeks Delta and Theta we can also recover

$$\begin{aligned}&2\frac{\dfrac{\partial C}{\partial T}(T-t,x,K)+rK\dfrac{\partial C}{\partial K}(T-t,x,K)}{K^2\dfrac{\partial^2 C}{\partial K^2}(T-t,x,K)}\\ &= 2\frac{-\dfrac{\partial C}{\partial t}(T-t,x,K)+rK\left(\dfrac{1}{K}C(T-t,x,K)-\dfrac{x}{K}\dfrac{\partial C}{\partial x}(T-t,x,K)\right)}{x^2\dfrac{\partial^2 C}{\partial x^2}(T-t,x,K)}\end{aligned}$$

$$
\begin{aligned}
&= 2\frac{x\sigma\Phi'(d_+(T-t))/(2\sqrt{T-t}) + rK\mathrm{e}^{-(T-t)r}\Phi(d_-(T-t))}{x^2\Phi'(d_+(T-t))/(x\sigma\sqrt{T-t})} \\
&\quad +2\frac{rC(T-t,x,K) - rx\Phi(d_+(T-t))}{x^2\Phi'(d_+(T-t))/(x\sigma\sqrt{T-t})} \\
&= \sigma^2.
\end{aligned}
$$

Exercise 9.2 We have

$$
\begin{aligned}
\frac{\partial C}{\partial K}(S_0,K,T) &= -(K-S_0)\frac{\mathrm{e}^{-(K-S_0)^2/(2T)}}{\sqrt{2\pi T}} \\
&\quad -\Phi\left(-\frac{K-S_0}{\sqrt{T}}\right) + \frac{K-S_0}{\sqrt{T}}\varphi\left(-\frac{K-S_0}{\sqrt{T}}\right) \\
&= -\Phi\left(-\frac{K-S_0}{\sqrt{T}}\right),
\end{aligned}
$$

and

$$
\frac{\partial C^2}{\partial K^2}(S_0,K,T) = \frac{1}{\sqrt{2\pi T}}\mathrm{e}^{-(K-S_0)^2/(2T)},
$$

which is the Gaussian probability density function of $S_T = S_0 + B_T$. We also have

$$
\begin{aligned}
\frac{\partial C}{\partial T}(S_0,K,T) &= \frac{1}{2\sqrt{2\pi T}}\mathrm{e}^{-(K-S_0)^2/(2T)} - \frac{(K-S_0)^2}{2T^2}\sqrt{\frac{T}{2\pi}}\mathrm{e}^{-(K-S_0)^2/(2T)} \\
&\quad + \frac{(K-S_0)^2}{2T^{3/2}}\varphi\left(-\frac{K-S_0}{\sqrt{T}}\right). \\
&= \frac{1}{2\sqrt{2\pi T}}\mathrm{e}^{-(K-S_0)^2/(2T)} \\
&= \frac{1}{2}\frac{\partial C^2}{\partial K^2}(S_0,K,T),
\end{aligned}
$$

hence

$$
|\sigma(t,y)| = \sqrt{\frac{2\frac{\partial C^M}{\partial t}(t,y) + 2ry\frac{\partial C^M}{\partial y}(t,y)}{y^2\frac{\partial^2 C^M}{\partial y^2}(t,y)}} = \sqrt{\frac{2\frac{\partial C^M}{\partial t}(t,y)}{y^2\frac{\partial^2 C^M}{\partial y^2}(t,y)}} = \sqrt{\frac{1}{y^2}} = \frac{1}{|y|},
$$

and the equation satisfied by $(S_t)_{t\in\mathbb{R}_+}$ is

$$
dS_t = S_t\sigma(t,S_t)dB_t = \frac{S_t}{|S_t|}dB_t = \mathrm{sign}\,(S_t)dB_t = dW_t,
$$

where $dW_t := \text{sign}\,(S_t)dB_t$ is also a standard Brownian motion by the Lévy characterization theorem, $\sigma(t,y) = 1/y$, and $S_t = S_0 + B_t$. Indeed, as in Quiz 2 of FE8815, the price of the call option in the Bachelier model is given by

$$\begin{aligned}
C(S_0,K,T) &= \mathbb{E}[(S_T-K)^+] \\
&= \mathbb{E}[(S_0+B_T-K)^+] \\
&= \int_{K-S_0}^{\infty}(x+S_0-K)\mathrm{e}^{-x^2/(2T)}\frac{dx}{\sqrt{2\pi T}} \\
&= \int_{K-S_0}^{\infty} x\mathrm{e}^{-x^2/(2T)}\frac{dx}{\sqrt{2\pi T}} - (K-S_0)\int_{K-S_0}^{\infty}\mathrm{e}^{-x^2/(2T)}\frac{dx}{\sqrt{2\pi T}} \\
&= \sqrt{\frac{T}{2\pi}}\left[-\mathrm{e}^{-x^2/(2T)}\right]_{K-S_0}^{\infty} - (K-S_0)\int_{(K-S_0)/\sqrt{T}}^{\infty}\mathrm{e}^{-y^2/2}\frac{dy}{\sqrt{2\pi}} \\
&= \sqrt{\frac{T}{2\pi}}\mathrm{e}^{-(K-S_0)^2/(2T)} - (K-S_0)\Phi\left(-\frac{K-S_0}{\sqrt{T}}\right).
\end{aligned}$$

Exercise 9.3

a) We have

$$\frac{\partial M_C}{\partial K}(K,S,r,\tau) = \frac{\partial C}{\partial K}(K,S,\sigma_{\text{imp}}(K),r,\tau) + \sigma'_{\text{imp}}(K)\frac{\partial C}{\partial \sigma}(K,S,\sigma_{\text{imp}}(K),r,\tau).$$

b) We have

$$\frac{\partial C}{\partial K}(K,S,\sigma_{\text{imp}}(K),r,\tau) + \sigma'_{\text{imp}}(K)\frac{\partial C}{\partial \sigma}(K,S,\sigma_{\text{imp}}(K),r,\tau) \leq 0,$$

which shows that

$$\sigma'_{\text{imp}}(K) \leq -\frac{\dfrac{\partial C}{\partial K}(K,S,\sigma_{\text{imp}}(K),r,\tau)}{\dfrac{\partial C}{\partial \sigma}(K,S,\sigma_{\text{imp}}(K),r,\tau)}$$

c) We have

$$\frac{\partial P}{\partial K}(K,S,\sigma_{\text{imp}}(K),r,\tau) + \sigma'_{\text{imp}}(K)\frac{\partial P}{\partial \sigma}(K,S,\sigma_{\text{imp}}(K),r,\tau) \geq 0,$$

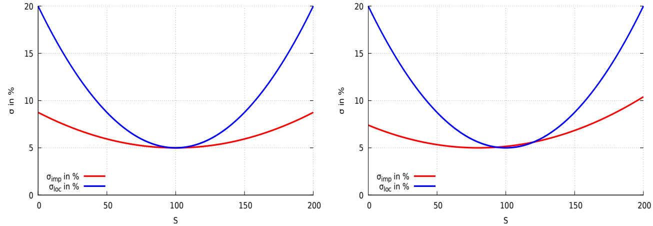
which shows that

$$\sigma'_{\text{imp}}(K) \geq -\frac{\dfrac{\partial P}{\partial K}(K,S,\sigma_{\text{imp}}(K),r,\tau)}{\dfrac{\partial P}{\partial \sigma}(K,S,\sigma_{\text{imp}}(K),r,\tau)}$$

Exercise 9.4

a) We have

$$\sigma_{\rm imp}(K,S) \simeq \sigma_{\rm loc}\left(\frac{K+S}{2}\right)$$
$$= \sigma_0 + \beta\left(\frac{K+S}{2} - S_0\right)^2$$
$$= \sigma_0 + \frac{\beta}{4}(K - (2S_0 - S))^2.$$



(a) At the money $K = S_0$.

(b) Out of the money $K > S_0$.

Fig. S.42: Implied *vs* local volatility.

b) We find

$$\frac{\partial}{\partial S}\big(\mathrm{Bl}\big(S,K,T,\sigma_{\rm imp}(K,S),r\big)\big) = \frac{\partial \mathrm{Bl}}{\partial x}\big(x,K,T,\sigma_{\rm imp}(K,S),r\big)_{x=S}$$
$$+\frac{\partial \sigma_{\rm imp}}{\partial S}\frac{\partial \mathrm{Bl}}{\partial \sigma}\big(x,K,T,\sigma,r\big)_{\sigma=\sigma_{\rm imp}(K,S)}$$
$$= \Delta + \nu\frac{\beta}{2}(K - (2S_0 - S)),$$

where

$$\Delta = \frac{\partial \mathrm{Bl}}{\partial x}\big(x,K,T,\sigma_{\rm imp}(K,S),r\big)_{x=S}$$

is the Black-Scholes Delta and

$$\nu = \frac{\partial \mathrm{Bl}}{\partial \sigma}\big(S,K,T,\sigma,r\big)_{\sigma=\sigma_{\rm imp}(K,S)}$$

is the Black-Scholes Vega, cf. §2.2 of Hagan et al. (2002).

Exercise 9.5 We take $t = 0$ for simplicity. We start by showing that for every $\lambda > 0$, we have

$$\frac{1}{\lambda}\mathbb{E}\left[\exp\left(\lambda\int_0^\tau\sigma_t^2dt\right)-1\right]=\frac{2\mathrm{e}^{r\tau}}{S_0^{p_\lambda}}\left(\int_0^{F_0}P(\tau,K)\frac{dK}{K^{2-p_\lambda}}+\int_{F_0}^\infty C(\tau,K)\frac{dK}{K^{2-p_\lambda}}\right).$$

By Lemma 8.2, we have

$$\frac{1}{\lambda}\mathbb{E}\left[\exp\left(\lambda\int_0^\tau\sigma_t^2dt\right)-1\right]=\frac{1}{\lambda}\mathbb{E}\left[\left(\frac{S_\tau}{F_0}\right)^{p_\lambda}-1\right],\quad\lambda>0.$$

Using Relation (9.18), *i.e.*

$$\varphi_\tau(K)=\mathrm{e}^{r\tau}\frac{\partial^2C}{\partial y^2}^M(\tau,y)=\mathrm{e}^{r\tau}\frac{\partial^2P}{\partial y^2}^M(\tau,y),$$

we have

$$\begin{aligned}
&\frac{1}{\lambda}\mathbb{E}\left[\exp\left(\lambda\int_0^\tau\sigma_t^2dt\right)-1\right]=\frac{1}{\lambda}\mathbb{E}\left[\left(\frac{S_\tau}{F_0}\right)^{p_\lambda}-1\right]\\
&=\frac{1}{\lambda F_0^{p_\lambda}}\mathbb{E}\left[S_\tau^{p_\lambda}-F_0^{p_\lambda}\right]\\
&=\frac{1}{\lambda F_0^{p_\lambda}}\left(\int_0^\infty K^{p_\lambda}\varphi_\tau(K)dK-F_0^{p_\lambda}\right)\\
&=\frac{1}{\lambda F_0^{p_\lambda}}\left(\int_0^{F_0}K^{p_\lambda}\varphi_\tau(K)dK+\int_{F_0}^\infty K^{p_\lambda}\varphi_\tau(K)dK-F_0^{p_\lambda}\right)\\
&=\frac{1}{\lambda S_0^{p_\lambda}}\left(\mathrm{e}^{r\tau}\int_0^{F_0}K^{p_\lambda}\frac{\partial^2P}{\partial K^2}(\tau,K)dK+\mathrm{e}^{r\tau}\int_{F_0}^\infty K^{p_\lambda}\frac{\partial^2C}{\partial K^2}(\tau,K)dK-S_0^{p_\lambda}\right).
\end{aligned}$$

Next, integrating by parts over the intervals $[0,F_0]$ and $[F_0,\infty)$ and using the boundary conditions

$$P(\tau,0)=C(\tau,\infty)=0,\qquad\frac{\partial P}{\partial K}(\tau,0)=\frac{\partial C}{\partial K}(\tau,\infty)=0,$$

with the relation

$$\frac{\partial P}{\partial K}(\tau,K)-\frac{\partial C}{\partial K}(\tau,K)-\mathrm{e}^{-r\tau}=0$$

and the call-put parity

$$P(\tau,F_0)-C(\tau,F_0)=S_0-F_0\mathrm{e}^{r\tau}=0$$

as boundary conditions, we find

$$\begin{aligned}
&\frac{1}{\lambda}\mathbb{E}\left[\exp\left(\lambda\int_0^\tau\sigma_t^2dt\right)-1\right]\\
&=\frac{1}{\lambda S_0^{p_\lambda}}\left(\mathrm{e}^{r\tau}S_0^{p_\lambda}\frac{\partial P}{\partial K}(\tau,F_0)-p_\lambda\mathrm{e}^{r\tau}\int_0^{F_0}K^{p_\lambda-1}\frac{\partial P}{\partial K}(\tau,K)dK\right.
\end{aligned}$$

$$
\begin{aligned}
&-\mathrm{e}^{r\tau} S_0^{p_\lambda} \frac{\partial C}{\partial K}(\tau, F_0) - p_\lambda \mathrm{e}^{r\tau} \int_{F_0}^\infty K^{p_\lambda - 1} \frac{\partial C}{\partial K}(\tau, K) dK - S_0^{p_\lambda} \Big) \\
&= -p_\lambda \frac{\mathrm{e}^{r\tau}}{\lambda S_0^{p_\lambda}} \left(\int_0^{F_0} K^{p_\lambda - 1} \frac{\partial P}{\partial K}(\tau, K) dK + \int_{F_0}^\infty K^{p_\lambda - 1} \frac{\partial C}{\partial K}(\tau, K) dK \right) \\
&= \frac{p_\lambda \mathrm{e}^{r\tau}}{\lambda S_0^{p_\lambda}} \Big(S_0^{p_\lambda - 1} P(\tau, F_0) + (p_\lambda - 1) \int_0^{F_0} K^{p_\lambda - 2} P(\tau, K) dK \\
&\quad - S_0^{p_\lambda - 1} C(\tau, F_0) + (p_\lambda - 1) \int_{F_0}^\infty K^{p_\lambda - 2} C(\tau, K) dK \Big) \\
&= \frac{p_\lambda (p_\lambda - 1)}{\lambda S_0^{p_\lambda}} \mathrm{e}^{r\tau} \left(\int_0^{F_0} K^{p_\lambda - 2} P(\tau, K) dK + \int_{F_0}^\infty K^{p_\lambda - 2} C(\tau, K) dK \right) \\
&= \frac{2\mathrm{e}^{r\tau}}{S_0^{p_\lambda}} \left(\int_0^{F_0} P(\tau, K) \frac{dK}{K^{2-p_\lambda}} + \int_{F_0}^\infty C(\tau, K) \frac{dK}{K^{2-p_\lambda}} \right) \\
&= \frac{2\mathrm{e}^{r\tau}}{S_0^{p_\lambda}} \left(\int_0^{F_0} P(\tau, K) \frac{dK}{K^{2-p_\lambda}} + \int_{F_0}^\infty C(\tau, K) \frac{dK}{K^{2-p_\lambda}} \right).
\end{aligned}
$$

Finally, taking

$$
p_\lambda := p_\lambda^- = 1/2 - \sqrt{1/4 + 2\lambda}
$$

and letting λ tend to zero, we find

$$
\begin{aligned}
\mathbb{E}\left[\int_0^\tau \sigma_t^2 dt \right] &= \lim_{\lambda \to 0} \frac{1}{\lambda} \mathbb{E}\left[\exp\left(\lambda \int_0^\tau \sigma_t^2 dt \right) - 1 \right] \\
&= \lim_{\lambda \to 0} \frac{2\mathrm{e}^{r\tau}}{S_0^{p_\lambda}} \left(\int_0^{F_0} P(\tau, K) \frac{dK}{K^{2-p_\lambda}} + \int_{F_0}^\infty C(\tau, K) \frac{dK}{K^{2-p_\lambda}} \right) \\
&= 2\mathrm{e}^{r\tau} \left(\int_0^{F_0} P(\tau, K) \frac{dK}{K^2} + \int_{F_0}^\infty C(\tau, K) \frac{dK}{K^2} \right).
\end{aligned}
$$

Exercise 9.6 (Exercise 8.7 continued). Taking $\phi(x) = (\log(x/F_0))^2$ with $y = F_0$, we have

$$
\phi'(x) = \frac{2}{x} \log \frac{x}{F_0} \quad \text{and} \quad \phi''(x) = \frac{2}{x^2} \left(1 - \log \frac{x}{F_0} \right),
$$

hence

$$
\begin{aligned}
\left(\log \frac{S_T}{F_0} \right)^2 &= \phi(F_0) + (S_T - F_0)\phi'(F_0) \\
&\quad + \int_0^{F_0} (z - S_T)^+ \phi''(z) dz + \int_{F_0}^\infty (S_T - z)^+ \phi''(z) dz \\
&= 2 \int_0^{F_0} (K - S_T)^+ \left(1 - \log \frac{K}{F_0} \right) \frac{dK}{K^2} \\
&\quad + 2 \int_{F_0}^\infty (S_T - K)^+ \left(1 - \log \frac{K}{F_0} \right) \frac{dK}{K^2}.
\end{aligned}
$$

Therefore, we have

$$\begin{aligned}\mathbb{E}^*\left[\left(\log\frac{S_T}{F_0}\right)^2\right] &= 2\int_0^{F_0}\mathbb{E}^*[(K-S_T)^+]\frac{dK}{K^2}+2\int_{F_0}^{\infty}\mathbb{E}^*[(S_T-K)^+]\frac{dK}{K^2}\\ &\quad -2\int_0^{F_0}\mathbb{E}^*[(K-S_T)^+]\log\frac{K}{F_0}\frac{dK}{K^2}-2\int_{F_0}^{\infty}\mathbb{E}^*[(S_T-K)^+]\log\frac{K}{F_0}\frac{dK}{K^2}\\ &= \mathbb{E}^*\left[R_{0,T}^2\right]-2\mathrm{e}^{rT}\int_0^{F_0}P(T,K)\log\frac{K}{F_0}\frac{dK}{K^2}-2\mathrm{e}^{rT}\int_{F_0}^{\infty}C(T,K)\log\frac{K}{F_0}\frac{dK}{K^2},\end{aligned}$$

and

$$\mathbb{E}^*\left[R_{0,T}^4\right]=8\mathrm{e}^{rT}\int_0^{F_0}P(T,K)\left(\log\frac{F_0}{K}\right)\frac{dK}{K^2}-8\mathrm{e}^{rT}\int_{F_0}^{\infty}C(T,K)\left(\log\frac{K}{F_0}\right)\frac{dK}{K^2}. \tag{S.9.46}$$

Alternatively, taking $\phi(x)=(x/F_0)(\log(x/F_0))^2$ with $y=F_0$, we have

$$\phi'(x)=\frac{1}{F_0}\left(\log\frac{x}{F_0}\right)^2+\frac{2}{F_0}\log\frac{x}{F_0}$$

and

$$\phi''(x)=\frac{2}{xF_0}\log\frac{x}{F_0}+\frac{2}{xF_0}=\frac{2}{xF_0}\left(1+\log\frac{x}{F_0}\right),$$

hence

$$\begin{aligned}\left(\log\frac{S_T}{F_0}\right)^2 &= \int_0^{F_0}(K-S_T)^+\frac{2}{KF_0}\left(1+\log\frac{K}{F_0}\right)dK\\ &\quad +\int_{F_0}^{\infty}(S_T-K)^+\frac{2}{KF_0}\left(1+\log\frac{K}{F_0}\right)dK.\end{aligned}$$

Therefore, we have

$$\begin{aligned}\mathbb{E}^*\left[\left(\log\frac{S_T}{F_0}\right)^2\right] &= \int_0^{F_0}\mathbb{E}[(K-S_T)^+]\frac{2}{KF_0}\left(1+\log\frac{K}{F_0}\right)dK\\ &\quad +\int_{F_0}^{\infty}\mathbb{E}[(S_T-K)^+]\frac{2}{KF_0}\left(1+\log\frac{K}{F_0}\right)dK.,\end{aligned}$$

and

$$\begin{aligned}\mathbb{E}^*\left[R_{0,T}^4\right] &= \frac{8}{F_0}\mathrm{e}^{rT}\int_0^{F_0}P(T,K)\left(1+\log\frac{K}{F_0}\right)\frac{dK}{K}\\ &\quad +\frac{8}{F_0}\mathrm{e}^{rT}\int_{F_0}^{\infty}P(T,K)\left(1+\log\frac{K}{F_0}\right)\frac{dK}{K}-4\,\mathbb{E}^*\left[R_{0,T}^2\right].\end{aligned}$$

Exercise 9.7

a) We have

$$\begin{aligned}\int_0^\infty \frac{\mathrm{e}^{-\nu x}-\mathrm{e}^{-\mu x}}{x^{\rho+1}}dx &= \int_0^\infty \frac{\mathrm{e}^{-\nu x}-\mathrm{e}^{-\mu x}}{x^{\rho+1}}dx \\ &= -\frac{1}{\rho}\left[\frac{\mathrm{e}^{-\nu x}-\mathrm{e}^{-\mu x}}{x^{\rho}}\right]_0^\infty + \frac{1}{\rho}\int_0^\infty \frac{-\nu\mathrm{e}^{-\nu x}+\mu\mathrm{e}^{-\mu x}}{x^{\rho}}dx \\ &= -\frac{\nu}{\rho}\int_0^\infty \mathrm{e}^{-\nu x}x^{-\rho}dx + \frac{\mu}{\rho}\int_0^\infty \mathrm{e}^{-\mu x}x^{-\rho}dx \\ &= \frac{\mu^\rho-\nu^\rho}{\rho}\Gamma(1-\rho).\end{aligned}$$

b) Taking $\nu=0$ and $\mu=R_{t,T}$, we find

$$\begin{aligned}\mathbb{E}^*[R_{t,T}] &= \frac{\rho}{\Gamma(1-\rho)}\mathbb{E}^*\left[\int_0^\infty \left(1-\mathrm{e}^{-\lambda R_{t,T}^2}\right)\frac{d\lambda}{\lambda^{\rho+1}}\right] \\ &= \frac{\rho}{\Gamma(1-\rho)}\int_0^\infty \left(1-\mathbb{E}^*\left[\mathrm{e}^{-\lambda R_{t,T}^2}\right]\right)\frac{d\lambda}{\lambda^{\rho+1}},\end{aligned}$$

see § 3.1 in Friz and Gatheral (2005) with $\rho=1/2$.

c) Letting $p_\lambda^\pm := 1/2 \pm \sqrt{1/4-2\lambda}$, we have

$$\begin{aligned}\mathbb{E}^*[R_{t,T}] &= \frac{\rho}{\Gamma(1-\rho)}\int_0^\infty \left(1-\mathbb{E}^*\left[\mathrm{e}^{-\lambda R_{t,T}^2}\right]\right)\frac{d\lambda}{\lambda^{\rho+1}} \\ &= \frac{\rho}{\Gamma(1-\rho)}\int_0^\infty \left(1-\mathrm{e}^{-rp_\lambda^\pm T}\,\mathbb{E}^*\left[\left(\frac{S_T}{S_0}\right)^{p_\lambda^\pm}\right]\right)\frac{d\lambda}{\lambda^{\rho+1}} \\ &= \frac{\rho}{\Gamma(1-\rho)}\int_0^\infty \mathbb{E}^*\left[1-\left(\frac{S_T}{F_0}\right)^{p_\lambda^\pm}\right]\frac{d\lambda}{\lambda^{\rho+1}} \\ &= \frac{\rho}{\Gamma(1-\rho)}\int_0^{1/8} \mathbb{E}^*\left[1-\left(\frac{S_T}{F_0}\right)^{p_\lambda^\pm}\right]\frac{d\lambda}{\lambda^{\rho+1}} \\ &\quad +\frac{\rho}{\Gamma(1-\rho)}\int_{1/8}^\infty \mathbb{E}^*\left[1-\sqrt{\frac{S_T}{F_0}}\exp\left(\pm\frac{i}{2}\sqrt{8\lambda-1}\log\frac{S_T}{F_0}\right)\right]\frac{d\lambda}{\lambda^{\rho+1}} \\ &= \frac{8^\rho}{\rho+1}+\frac{\rho}{\Gamma(1-\rho)}\int_0^{1/8} \mathbb{E}^*\left[1-\left(\frac{S_T}{F_0}\right)^{p_\lambda^\pm}\right]\frac{d\lambda}{\lambda^{\rho+1}} \\ &\quad -\frac{\rho}{\Gamma(1-\rho)}\int_{1/8}^\infty \mathbb{E}^*\left[\sqrt{\frac{S_T}{F_0}}\cos\left(\frac{1}{2}\sqrt{8\lambda-1}\log\frac{S_T}{F_0}\right)\right]\frac{d\lambda}{\lambda^{\rho+1}} \\ &= \frac{\rho}{\Gamma(1-\rho)}\int_0^{1/8} \mathbb{E}^*[\phi_\lambda(S_T)]\frac{d\lambda}{\lambda^{\rho+1}} + \frac{\rho}{\Gamma(1-\rho)}\mathbb{E}^*\left[\psi(S_T)\right],\end{aligned}$$

where

$$\phi_\lambda(x) = 1 - \left(\frac{x}{F_0}\right)^{p_\lambda^\pm},$$

we have

$$\phi_\lambda'(x) = -p_\lambda^\pm \frac{x^{p_\lambda^\pm - 1}}{F_0^{p_\lambda^\pm}} \quad \text{and} \quad \phi_\lambda''(x) = -p_\lambda^\pm (p_\lambda^\pm - 1)\frac{x^{p_\lambda^\pm - 2}}{F_0^{p_\lambda^\pm}} = 2\lambda \frac{x^{p_\lambda^\pm - 2}}{F_0^{p_\lambda^\pm}},$$

hence with $y := F_0$ we have $\phi_\lambda(y) = 0$ and

$$\begin{aligned}
&\mathbb{E}^*[\phi_\lambda(S_T)] \\
&= \mathbb{E}^*\left[(S_T - F_0)\frac{p_\lambda^\pm}{F_0} - 2\lambda \int_0^{F_0} (K - S_T)^+ \frac{K^{p_\lambda^\pm - 2}}{F_0^{p_\lambda^\pm}} dK - 2\lambda \int_{F_0}^\infty (S_T - K)^+ \frac{K^{p_\lambda^\pm - 2}}{F_0^{p_\lambda^\pm}} dK\right] \\
&= 2\lambda \int_0^{F_0} \mathbb{E}^*[(K - S_T)^+] \frac{K^{p_\lambda^\pm - 2}}{F_0^{p_\lambda^\pm}} dK + \int_{F_0}^\infty \mathbb{E}^*[(S_T - K)^+] \frac{K^{p_\lambda^\pm - 2}}{F_0^{p_\lambda^\pm}} dK \\
&= 2\lambda \frac{\mathrm{e}^{rT}}{F_0^{p_\lambda^\pm}} \left(\int_0^{F_0} P(T,K) K^{p_\lambda^\pm - 2} dK + \int_{F_0}^\infty C(T,K) K^{p_\lambda^\pm - 2} dK\right).
\end{aligned}$$

Taking now

$$\psi(x) := \int_{1/8}^\infty \left(1 - \sqrt{\frac{x}{F_0}} \cos\left(\frac{1}{2}\sqrt{8\lambda - 1} \log \frac{x}{F_0}\right)\right) \frac{d\lambda}{\lambda^{\rho+1}},$$

we have

$$\begin{aligned}
\psi'(x) = &-\frac{1}{2\sqrt{F_0 x}} \int_{1/8}^\infty \cos\left(\frac{1}{2}\sqrt{8\lambda - 1}\log\frac{x}{F_0}\right)\frac{d\lambda}{\lambda^{\rho+1}} \\
&+ \frac{1}{2\sqrt{F_0 x}} \int_{1/8}^\infty \sin\left(\frac{1}{2}\sqrt{8\lambda - 1}\log\frac{x}{F_0}\right)\sqrt{8\lambda - 1}\frac{d\lambda}{\lambda^{\rho+1}},
\end{aligned}$$

which converges provided that $\rho > 1/2$, while $\psi''(x)$ cannot be written as a converging integral but can be estimated numerically from $\psi'(x)$. Hence, we have

$$\begin{aligned}
&\mathbb{E}^*[R_{t,T}] \\
&= \frac{\rho \mathrm{e}^{rT}}{\Gamma(1-\rho)} \\
&\times \left(\int_0^{1/8} \frac{2}{F_0^{\pm\sqrt{1/4 - 2\lambda}}} \left(\int_0^{F_0} P(T,K) K^{p_\lambda^\pm - 2} dK + \int_{F_0}^\infty C(T,K) K^{p_\lambda^\pm - 2} dK\right) \frac{d\lambda}{\lambda^p}\right. \\
&\left. + \int_0^{F_0} P(T,K)\psi''(K) dK + \int_{F_0}^\infty C(T,K)\psi''(K) dK\right).
\end{aligned}$$

```
library(quantmod)
today <- as.Date(Sys.Date(), format="%Y-%m-%d"); getSymbols("^SPX", src = "yahoo")
lastBusDay=last(row.names(as.data.frame(Ad(SPX))))
S0 = as.vector(tail(Ad(SPX),1)); T = 30/365;r=0.02;F0 = S0*exp(r*T)
maturity<- as.Date("2021-07-07", format="%Y-%m-%d") # Choose a maturity in 30 days
SPX.OPTS <- getOptionChain("^SPX", maturity)
Call <- as.data.frame(SPX.OPTS$calls); Put <- as.data.frame(SPX.OPTS$puts)
Call_OTM <- Call[Call$Strike>F0,];Call_OTM$dif = c(min(Call_OTM$Strike)-F0,
      diff(Call_OTM$Strike))
Put_OTM <-Put[Put$Strike<F0,];Put_OTM$dif = c(diff(Put_OTM$Strike),
      F0-max(Put_OTM$Strike))
```

```
pl <- function(lambda){return( 1/2+sqrt(1/4-2*lambda ))}; rho=0.9
g1 <- function(x){ f1 <- function(lambda){ - cos ( 0.5*sqrt(lambda*8-1)*log
      (x/F0))/lambda^(rho+1)/sqrt(x*F0)/2}; return(f1)}
g2 <- function(x){ f2 <- function(lambda){ sin ( 0.5*sqrt(lambda*8-1)*log
      (x/F0))/lambda^(rho+1)/sqrt(lambda*8-1)/sqrt(x*F0)/2}; return(f2)}
g3 <- function (x) { integrate(g1(x), lower=0.125, upper=Inf,stop.on.error = FALSE)$value}
g4 <- function (x) { if (x>F0) {integrate(g2(x), lower=0.125, upper=1000000,stop.on.error =
      FALSE)$value} else {integrate(g2(x), lower=0.125, upper=100000,stop.on.error =
      FALSE)$value}}
eps=1;psi2nd <- function(x){(g3(x+eps)+g4(x+eps)-g3(x)-g4(x))/eps}
f <- function(lambda){ return (2*(sum(Put_OTM$Last*Put_OTM$Strike**(pl(lambda)-2)
      *Put_OTM$dif)) +sum(Call_OTM$Last *Call_OTM$Strike**(pl(lambda)-2)
      *Call_OTM$dif)/F0**pl(lambda)/lambda**rho)}
(sum(Put_OTM$Last*as.numeric(lapply(Put_OTM$Strike,psi2nd)) *Put_OTM$dif)
      +sum(Call_OTM$Last*as.numeric(lapply(Call_OTM$Strike,psi2nd))
      *Call_OTM$dif)+integrate(Vectorize(f), lower=0,
      upper=0.125)$value)*rho*exp(r*T)/gamma(1-rho)
```

Chapter 10

Exercise 10.1

a) By differentiating (10.2) with respect to T, we find

$$\begin{aligned}\varphi_{\tau_a}(T) &= \frac{\partial}{\partial T}\mathbb{P}(\tau_a < T) \\ &= 2\frac{\partial}{\partial T}\mathbb{P}(W_T > a) \\ &= \frac{2}{\sqrt{2\pi T}}\frac{\partial}{\partial T}\int_a^\infty \mathrm{e}^{-x^2/(2T)}dx \\ &= \frac{2}{\sqrt{2\pi}}\frac{\partial}{\partial T}\int_{a/\sqrt{T}}^\infty \mathrm{e}^{-y^2/2}dy \\ &= \frac{a}{\sqrt{2\pi T^3}}\mathrm{e}^{-a^2/(2T)}, \qquad T > 0. \qquad \text{(S.10.47)}\end{aligned}$$

b) By differentiating (10.13) with respect to T, we find

$$\begin{aligned}
\varphi_{\tilde{\tau}_a}(T) &= \frac{\partial}{\partial T}\mathbb{P}(\tilde{\tau}_a < T) \\
&= -\frac{\partial}{\partial T}\mathbb{P}(\tilde{\tau}_a \geq T) \\
&= -\frac{\partial}{\partial T}\mathbb{P}\big(\widehat{X}_0^T \leq a\big) \\
&= -\frac{\partial}{\partial T}\Phi\left(\frac{a-\mu T}{\sqrt{T}}\right) + \mathrm{e}^{2\mu a}\frac{\partial}{\partial T}\Phi\left(\frac{-a-\mu T}{\sqrt{T}}\right) \\
&= \left(\frac{a}{2\sqrt{2\pi T^3}} + \frac{\mu}{\sqrt{2\pi T}}\right)\mathrm{e}^{-(a-\mu T)^2/(2T)} \\
&\quad + \left(\frac{a}{2\sqrt{2\pi T^3}} - \frac{\mu}{\sqrt{2\pi T}}\right)\mathrm{e}^{2\mu a-(a+\mu T)^2/(2T)} \\
&= \frac{a}{2\sqrt{2\pi T^3}}\mathrm{e}^{-(a-\mu T)^2/(2T)}, \qquad T > 0.
\end{aligned}$$

c) By differentiating (10.15) with respect to T, for $x > S_0$ we find

$$\begin{aligned}
\varphi_{\hat{\tau}_x}(T) &= \frac{\partial}{\partial T}\mathbb{P}(\hat{\tau}_a < T) \\
&= -\frac{\partial}{\partial T}\mathbb{P}(\hat{\tau}_x \geq T) \\
&= -\frac{\partial}{\partial T}\mathbb{P}\big(M_0^T \leq x\big) \\
&= -\frac{\partial}{\partial T}\Phi\left(\frac{-(r-\sigma^2/2)T + \log(x/S_0)}{\sigma\sqrt{T}}\right) \\
&\quad + \left(\frac{S_0}{x}\right)^{1-2r/\sigma^2}\frac{\partial}{\partial T}\Phi\left(\frac{-(r-\sigma^2/2)T - \log(x/S_0)}{\sigma\sqrt{T}}\right) \\
&= \frac{\log(x/S_0)}{\sigma\sqrt{2\pi T^3}}\exp\left(-\frac{1}{2\sigma^2 T}((r-\sigma^2/2)T - \log(x/S_0))^2\right), \quad T > 0,
\end{aligned}$$

which can also be recovered from (S.10.47) by taking $a := \log(S_0/x)/\sigma$ and $\mu := r/\sigma - \sigma/2$. Similarly, when $0 < x < S_0$ we can differentiate (10.18) in Corollary 10.8 to find

$$\begin{aligned}
\varphi_{\hat{\tau}_x}(T) &= \frac{\partial}{\partial T}\mathbb{P}(\hat{\tau}_x < T) \\
&= \frac{\partial}{\partial T}\mathbb{P}\big(m_0^T \leq x\big) \\
&= \frac{\partial}{\partial T}\Phi\left(\frac{-(r-\sigma^2/2)T + \log(x/S_0)}{\sigma\sqrt{T}}\right) \\
&\quad + \left(\frac{S_0}{x}\right)^{1-2r/\sigma^2}\frac{\partial}{\partial T}\Phi\left(\frac{(r-\sigma^2/2)T + \log(x/S_0)}{\sigma\sqrt{T}}\right)
\end{aligned}$$

$$= \frac{\log(S_0/x)}{\sigma\sqrt{2\pi T^3}} \exp\left(-\frac{1}{2\sigma^2 T}((r-\sigma^2/2)T - \log(x/S_0))^2\right), \quad T > 0,$$

which yields

$$\varphi_{\hat{\tau}_x}(T) = \frac{|\log(S_0/x)|}{\sigma\sqrt{2\pi T^3}} \exp\left(-\frac{1}{2\sigma^2 T}((r-\sigma^2/2)T - \log(x/S_0))^2\right), \quad T > 0,$$

for all $x > 0$.

Exercise 10.2

a) We use Relation (10.14) and the integration by parts identity

$$\int_0^\infty v'(z)u(z)dz = u(+\infty)v(+\infty) - u(0)v(0) - \int_0^\infty v(z)u'(z)dz$$

with

$$u(y) = \Phi\left(\frac{-y-\mu T/\sigma}{\sqrt{T}}\right) \quad \text{and} \quad v'(y) = \frac{2\mu}{\sigma} y e^{2\mu y/\sigma}$$

which satisfy

$$u'(y) = -\frac{1}{\sqrt{2\pi T}} e^{-(y+\mu T/\sigma)^2/(2T)} \quad \text{and} \quad v(y) = ye^{2\mu y/\sigma} - \frac{\sigma}{2\mu} e^{2\mu y/\sigma},$$

we have

$$\begin{aligned}
&\mathbb{E}\left[\max_{t\in[0,T]} \widetilde{W}_t\right] = \sigma\,\mathbb{E}\left[\max_{t\in[0,T]} (W_t + \mu t/\sigma)\right] \\
&= \sigma\sqrt{\frac{2}{\pi T}} \int_0^\infty y e^{-(y-\mu T/\sigma)^2/(2T)} dy - 2\mu \int_0^\infty y e^{2\mu y/\sigma} \Phi\left(\frac{-y-\mu T/\sigma}{\sqrt{T}}\right) dy \\
&= \sigma\sqrt{\frac{2}{\pi T}} \int_0^\infty y e^{-(y-\mu T/\sigma)^2/(2T)} dy - \sigma \int_0^\infty v'(y)u(y)dy \\
&= \sigma\sqrt{\frac{2}{\pi T}} \int_0^\infty y e^{-(y-\mu T/\sigma)^2/(2T)} dy - \sigma u(+\infty)v(+\infty) + \sigma u(0)v(0) + \sigma \int_0^\infty u'(y)v(y)dy \\
&= \sigma\sqrt{\frac{2}{\pi T}} \int_0^\infty y e^{-(y-\mu T/\sigma)^2/(2T)} dy - \frac{\sigma^2}{2\mu}\Phi\left(-\frac{\mu\sqrt{T}}{\sigma}\right) \\
&\quad -\frac{\sigma}{\sqrt{2\pi T}} \int_0^\infty y e^{2\mu y/\sigma-(y+\mu T/\sigma)^2/(2T)} dy + \frac{\sigma^2}{2\mu\sqrt{2\pi T}} \int_0^\infty e^{2\mu y/\sigma-(y+\mu T/\sigma)^2/(2T)} dy \\
&= \sigma\sqrt{\frac{2}{\pi T}} \int_0^\infty y e^{-(y-\mu T/\sigma)^2/(2T)} dy - \frac{\sigma}{\sqrt{2\pi T}} \int_0^\infty y e^{-(y-\mu T/\sigma)^2/(2T)} dy \\
&\quad -\frac{\sigma^2}{2\mu}\Phi\left(-\frac{\mu\sqrt{T}}{\sigma}\right) + \frac{\sigma^2}{2\mu\sqrt{2\pi T}} \int_0^\infty e^{-(y-\mu T/\sigma)^2/(2T)} dy
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\sigma}{\sqrt{2\pi T}} \int_0^\infty y \mathrm{e}^{-(y-\mu T/\sigma)^2/(2T)} dy - \frac{\sigma^2}{2\mu} \Phi\left(-\frac{\mu\sqrt{T}}{\sigma}\right) \\
&\quad + \frac{\sigma^2}{2\mu\sqrt{2\pi T}} \int_0^\infty \mathrm{e}^{-(y-\mu T/\sigma)^2/(2T)} dy \\
&= \frac{\sigma}{\sqrt{2\pi T}} \int_{-\mu T/\sigma}^\infty \left(y + \frac{\mu T}{\sigma}\right) \mathrm{e}^{-y^2/(2T)} dy - \frac{\sigma^2}{2\mu} \Phi\left(-\frac{\mu\sqrt{T}}{\sigma}\right) \\
&\quad + \frac{\sigma^2}{2\mu\sqrt{2\pi T}} \int_{-\mu T/\sigma}^\infty \mathrm{e}^{-y^2/(2T)} dy \\
&= \frac{\sigma}{\sqrt{2\pi T}} \int_{-\mu T/\sigma}^\infty y \mathrm{e}^{-y^2/(2T)} dy + \frac{\mu T + \sigma^2/(2\mu)}{\sqrt{2\pi T}} \int_{-\mu T/\sigma}^\infty \mathrm{e}^{-y^2/(2T)} dy - \frac{\sigma^2}{2\mu} \Phi\left(-\frac{\mu\sqrt{T}}{\sigma}\right) \\
&= \frac{\sigma}{\sqrt{2\pi T}} \left[-T\mathrm{e}^{-y^2/(2T)}\right]_{-\mu T/\sigma}^\infty + \mu T \Phi\left(\frac{\mu\sqrt{T}}{\sigma}\right) + \frac{\sigma^2}{2\mu} \Phi\left(\frac{\mu\sqrt{T}}{\sigma}\right) - \frac{\sigma^2}{2\mu} \Phi\left(-\frac{\mu\sqrt{T}}{\sigma}\right) \\
&= \sigma\sqrt{\frac{T}{2\pi}} \mathrm{e}^{-\mu^2 T/2} + \left(\mu T + \frac{\sigma^2}{2\mu}\right) \Phi\left(\frac{\mu\sqrt{T}}{\sigma}\right) - \frac{\sigma^2}{2\mu} \Phi\left(-\frac{\mu\sqrt{T}}{\sigma}\right).
\end{aligned}
$$

As σ tends to zero, we find

$$
\mathbb{E}\left[\max_{t\in[0,T]} \widetilde{W}_t\right] = \begin{cases} \mu T \Phi(+\infty) = \mu T & \text{if } \mu \geq 0, \\ \\ \mu T \Phi(-\infty) = 0 & \text{if } \mu \leq 0. \end{cases}
$$

We also have

$$
\begin{aligned}
\mathbb{E}\left[\max_{t\in[0,T]} \widetilde{W}_t\right] &= \sigma\sqrt{\frac{T}{2\pi}} \mathrm{e}^{-\mu^2 T/2} + \mu T \Phi\left(\frac{\mu\sqrt{T}}{\sigma}\right) + \frac{\sigma^2}{2\mu} \left(\Phi\left(\frac{\mu\sqrt{T}}{\sigma}\right) - \Phi\left(-\frac{\mu\sqrt{T}}{\sigma}\right)\right) \\
&= \sigma\sqrt{\frac{T}{2\pi}} \mathrm{e}^{-\mu^2 T/2} + \mu T \Phi\left(\frac{\mu\sqrt{T}}{\sigma}\right) + \frac{\sigma^2}{2\mu\sqrt{2\pi}} \int_{-\mu\sqrt{T}/\sigma}^{\mu\sqrt{T}/\sigma} \mathrm{e}^{-y^2/2} dy.
\end{aligned}
$$

Hence, as μ tends to zero we find

$$
\mathbb{E}\left[\max_{t\in[0,T]} \widetilde{W}_t\right] = \sigma\sqrt{\frac{T}{2\pi}} \mathrm{e}^{-\mu^2 T/2} + \mu T \Phi\left(\frac{\mu\sqrt{T}}{\sigma}\right) + \sigma\sqrt{\frac{T}{2\pi}} + o(\mu), \qquad [\mu \to 0],
$$

and for $\mu = 0$ and $\sigma = 1$ we recover the average maximum of standard Brownian motion

$$
\mathbb{E}\left[\max_{t\in[0,T]} W_t\right] = \sqrt{\frac{2T}{\pi}},
$$

which represents two times the expected maximum

$$\begin{aligned}\mathbb{E}[\max(W_T,0)] &= \frac{1}{\sqrt{2\pi T}}\int_{-\infty}^{\infty}\max(y,0)\mathrm{e}^{-y^2/(2T)}dy\\ &= \frac{1}{\sqrt{2\pi T}}\int_0^{\infty} y\mathrm{e}^{-y^2/(2T)}dy\\ &= \frac{1}{\sqrt{2\pi T}}\left[-T\mathrm{e}^{-y^2/(2T)}\right]_0^{\infty}\\ &= \sqrt{\frac{T}{2\pi}}.\end{aligned}$$

b) By part (a) the identity in distribution $(-W_t)_{t\in\mathbb{R}_+} \approx (W_t)_{t\in\mathbb{R}_+}$, we have

$$\begin{aligned}&\mathbb{E}\left[\min_{t\in[0,T]}\widetilde{W}_t\right] = \sigma\,\mathbb{E}\left[\min_{t\in[0,T]}(W_t+\mu t/\sigma)\right]\\ &= -\sigma\,\mathbb{E}\left[\max_{t\in[0,T]}(-W_t-\mu t/\sigma)\right]\\ &= -\sigma\,\mathbb{E}\left[\max_{t\in[0,T]}(W_t-\mu t/\sigma)\right]\\ &= -\sigma\sqrt{\frac{T}{2\pi}}\mathrm{e}^{-\mu^2T/2} + \left(\mu T+\frac{\sigma^2}{2\mu}\right)\Phi\left(\frac{-\mu\sqrt{T}}{\sigma}\right) - \frac{\sigma^2}{2\mu}\Phi\left(\frac{\mu\sqrt{T}}{\sigma}\right).\end{aligned}$$

In particular, as σ tends to zero, we find

$$\mathbb{E}\left[\min_{t\in[0,T]}\widetilde{W}_t\right] = \begin{cases}\mu T\Phi(+\infty)=\mu T & \text{if } \mu\le 0,\\[2ex] \mu T\Phi(-\infty)=0 & \text{if } \mu\ge 0.\end{cases}$$

Exercise 10.3

a) We have $S_t = S_0\mathrm{e}^{\sigma W_t}$, $t\in\mathbb{R}_+$.
b) We have

$$\mathbb{E}[S_T] = S_0\,\mathbb{E}\left[\mathrm{e}^{\sigma W_T}\right] = S_0\mathrm{e}^{\sigma^2T/2}.$$

c) We have

$$\mathbb{P}\left(\max_{t\in[0,T]}W_t\ge a\right) = 2\int_a^{\infty}\mathrm{e}^{-x^2/(2T)}\frac{dx}{\sqrt{2\pi T}}, \qquad a\ge 0,$$

and

$$P\left(\max_{t\in[0,T]}W_t\le a\right) = 2\int_0^{a}\mathrm{e}^{-x^2/(2T)}\frac{dx}{\sqrt{2\pi T}}, \qquad a\ge 0,$$

hence the probability density function φ of $\max_{t\in[0,T]}W_t$ is given by

$$\varphi(a) = \sqrt{\frac{2}{\pi T}}\mathrm{e}^{-a^2/(2T)}\mathbb{1}_{[0,\infty)}(a), \qquad a\in\mathbb{R}.$$

d) We have

$$\begin{aligned}
\mathbb{E}\left[M_0^T\right] &= S_0\,\mathbb{E}\left[\exp\left(\sigma \max_{t\in[0,T]} W_t\right)\right] = S_0\int_0^\infty \mathrm{e}^{\sigma x}\varphi(x)dx\\
&= \frac{2S_0}{\sqrt{2\pi T}}\int_0^\infty \mathrm{e}^{\sigma x - x^2/(2T)}dx = \frac{2S_0}{\sqrt{2\pi\sigma^2 T}}\int_0^\infty \mathrm{e}^{-(x-\sigma T)^2/(2T)+\sigma^2 T/2}dx\\
&= \frac{2S_0}{\sqrt{2\pi T}}\mathrm{e}^{\sigma^2 T/2}\int_{-\sigma T}^\infty \mathrm{e}^{-x^2/(2T)}dx = \frac{2S_0}{\sqrt{2\pi}}\mathrm{e}^{\sigma^2 T/2}\int_{-\sigma\sqrt{T}}^\infty \mathrm{e}^{-x^2/2}dx\\
&= 2S_0\mathrm{e}^{\sigma^2 T/2}\int_{-\infty}^{\sigma\sqrt{T}} \mathrm{e}^{-x^2/2}dx\\
&= 2S_0\mathrm{e}^{\sigma^2 T/2}\Phi\big(\sigma\sqrt{T}\big)\\
&= 2\,\mathbb{E}[S_T]\Phi\big(\sigma\sqrt{T}\big).
\end{aligned}$$

Remarks:

(i) From the inequality

$$\begin{aligned}
0 &\leq \mathbb{E}[(W_T-\sigma T)^+]\\
&= \frac{1}{\sqrt{2\pi T}}\int_{-\infty}^\infty (x-\sigma T)^+\mathrm{e}^{-x^2/(2T)}dx\\
&= -\frac{1}{\sqrt{2\pi T}}\int_{\sigma T}^\infty (x-\sigma T)\mathrm{e}^{-x^2/(2T)}dx\\
&= \frac{1}{\sqrt{2\pi T}}\int_{\sigma T}^\infty x\mathrm{e}^{-x^2/(2T)}dx - \frac{\sigma T}{\sqrt{2\pi T}}\int_{\sigma T}^\infty \mathrm{e}^{-x^2/(2T)}dx\\
&= \sqrt{\frac{T}{2\pi}}\int_{\sigma\sqrt{T}}^\infty x\mathrm{e}^{-x^2/2}dx - \frac{\sigma T}{\sqrt{2\pi}}\int_{\sigma\sqrt{T}}^\infty \mathrm{e}^{-x^2/2}dx\\
&= \sqrt{\frac{T}{2\pi}}\left[\mathrm{e}^{-x^2/2}\right]_{\sigma\sqrt{T}}^\infty - \sigma T\Phi\big(-\sigma\sqrt{T}\big)\\
&= \sqrt{\frac{T}{2\pi}}\mathrm{e}^{-\sigma^2 T/2} - \sigma T\big(1-\Phi\big(\sigma\sqrt{T}\big)\big),
\end{aligned}$$

we get

$$\Phi\big(\sigma\sqrt{T}\big) \geq 1 - \frac{\mathrm{e}^{-\sigma^2 T/2}}{\sigma\sqrt{2\pi T}},$$

hence

$$\begin{aligned}
\mathbb{E}\left[M_0^T\right] &= 2S_0\mathrm{e}^{\sigma^2 T/2}\Phi\big(\sigma\sqrt{T}\big)\\
&\geq 2S_0\mathrm{e}^{\sigma^2 T/2}\left(1-\frac{\mathrm{e}^{-\sigma^2 T/2}}{\sigma\sqrt{2\pi T}}\right)\\
&= 2\,\mathbb{E}[S_T]\left(1-\frac{\mathrm{e}^{-\sigma^2 T/2}}{\sigma\sqrt{2\pi T}}\right)
\end{aligned}$$

$$= 2S_0 \left(\mathrm{e}^{\sigma^2 T/2} - \frac{1}{\sigma\sqrt{2\pi T}} \right).$$

(ii) We observe that the ratio between the expected gains by selling at the maximum and selling at time T is given by $2\Phi\big(\sigma\sqrt{T}\big)$, which cannot be greater than 2.

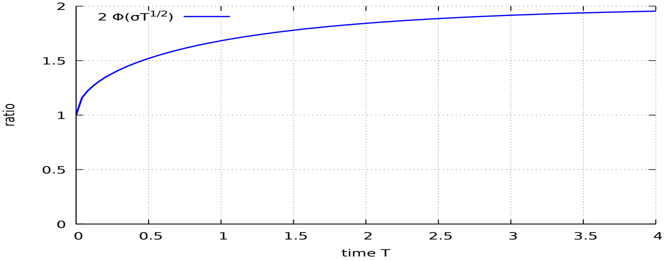


Fig. S.43: Average return by selling at the maximum *vs* selling at maturity.

e) By a symmetry argument, we have

$$\begin{aligned}
\mathbb{P}\left(\min_{t\in[0,T]} W_t \leq a\right) &= \mathbb{P}\left(-\max_{t\in[0,T]}(-W_t) \leq a\right) \\
&= \mathbb{P}\left(-\max_{t\in[0,T]} W_t \leq a\right) \\
&= \mathbb{P}\left(\max_{t\in[0,T]} W_t \geq -a\right) \\
&= 2\int_{-a}^{\infty} \mathrm{e}^{-x^2/(2T)} \frac{dx}{\sqrt{2\pi T}}, \qquad a < 0,
\end{aligned}$$

i.e. the probability density function φ of $\min_{t\in[0,T]} W_t$ is given by

$$\varphi(a) = \sqrt{\frac{2}{\pi T}} \mathrm{e}^{-a^2/(2T)} \mathbb{1}_{(-\infty,0]}(a), \qquad a \in \mathbb{R}.$$

f) We have

$$\begin{aligned}
\mathbb{E}\left[m_0^T\right] &= S_0\, \mathbb{E}\left[\exp\left(\sigma \min_{t\in[0,T]} W_t\right)\right] \\
&= S_0 \int_{-\infty}^{0} \mathrm{e}^{\sigma x} \varphi(x) dx \\
&= \frac{2S_0}{\sqrt{2\pi T}} \int_{-\infty}^{0} \mathrm{e}^{\sigma x - x^2/(2T)} dx
\end{aligned}$$

$$
\begin{aligned}
&= \frac{2S_0}{\sqrt{2\pi T}} \int_{-\infty}^{0} \mathrm{e}^{-(x-\sigma T)^2/(2T)+\sigma^2 T/2} dx \\
&= \frac{2S_0}{\sqrt{2\pi T}} \mathrm{e}^{\sigma^2 T/2} \int_{-\infty}^{-\sigma T} \mathrm{e}^{-x^2/(2T)} dx \\
&= \frac{2S_0}{\sqrt{2\pi}} \mathrm{e}^{\sigma^2 T/2} \int_{-\infty}^{-\sigma\sqrt{T}} \mathrm{e}^{-x^2/2} dx \\
&= 2S_0 \mathrm{e}^{\sigma^2 T/2} \Phi\big(-\sigma\sqrt{T}\big) \\
&= 2\,\mathbb{E}[S_T]\Phi\big(-\sigma\sqrt{T}\big).
\end{aligned}
$$

Remarks:

(i) From the inequality

$$
\begin{aligned}
0 &\le \mathbb{E}[(-\sigma T - W_T)^+] \\
&= \frac{1}{\sqrt{2\pi T}} \int_{-\infty}^{\infty} (-\sigma T - x)^+ \mathrm{e}^{-x^2/(2T)} dx \\
&= -\frac{1}{\sqrt{2\pi T}} \int_{-\infty}^{-\sigma T} (\sigma T + x) \mathrm{e}^{-x^2/(2T)} dx \\
&= -\frac{\sigma T}{\sqrt{2\pi T}} \int_{-\infty}^{-\sigma T} \mathrm{e}^{-x^2/(2T)} dx - \frac{1}{\sqrt{2\pi T}} \int_{-\infty}^{-\sigma T} x \mathrm{e}^{-x^2/(2T)} dx \\
&= -\frac{\sigma T}{\sqrt{2\pi}} \int_{-\infty}^{-\sigma\sqrt{T}} \mathrm{e}^{-x^2/2} dx - \sqrt{\frac{T}{2\pi}} \int_{-\infty}^{-\sigma\sqrt{T}} x \mathrm{e}^{-x^2/2} dx \\
&= -\sigma T \Phi\big(-\sigma\sqrt{T}\big) + \sqrt{\frac{T}{2\pi}} \left[\mathrm{e}^{-x^2/2}\right]_{-\infty}^{-\sigma\sqrt{T}} \\
&= -\sigma T \Phi\big(-\sigma\sqrt{T}\big) + \sqrt{\frac{T}{2\pi}} \mathrm{e}^{-\sigma^2 T/2},
\end{aligned}
$$

we get

$$
\mathrm{e}^{\sigma^2 T/2} \Phi\big(-\sigma\sqrt{T}\big) \le \frac{1}{\sigma\sqrt{2\pi T}}, \quad \text{hence} \quad \mathbb{E}\left[m_0^T\right] \le \frac{2S_0}{\sigma\sqrt{2\pi T}}.
$$

(ii) The ratio between the expected gains by maturity T *vs* selling at the minimum is given by $2\Phi\big(-\sigma\sqrt{T}\big)$, which is at most 1 and tends to 0 as σ and T tend to infinity.

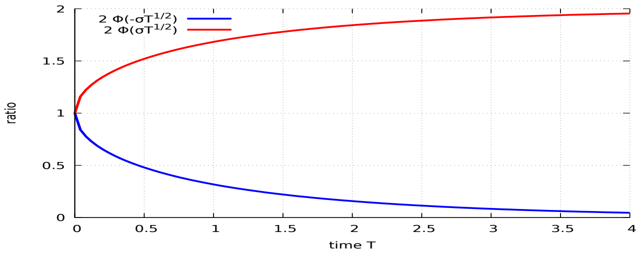


Fig. S.44: Average returns by selling at the minimum *vs* selling at maturity.

(iii) Given that $\mathbb{E}\left[M_0^T\right] = 2\,\mathbb{E}[S_T]\Phi\big(\sigma\sqrt{T}\big)$, we find the bound

$$2\,\mathbb{E}[S_T]\Phi\big(-\sigma\sqrt{T}\big) \le \mathbb{E}[S_T] \le 2\,\mathbb{E}[S_T]\Phi\big(\sigma\sqrt{T}\big),$$

with equality if $\sigma = 0$ or $T = 0$. We also have

$$\begin{aligned} 2\,\mathbb{E}[S_T] - \mathbb{E}\left[M_0^T\right] &= 2\mathrm{e}^{\sigma^2 T/2}\big(1 - \Phi\big(\sigma\sqrt{T}\big)\big) \\ &= 2\mathrm{e}^{\sigma^2 T/2}\Phi\big(-\sigma\sqrt{T}\big) \\ &= \mathbb{E}\left[m_0^T\right], \end{aligned}$$

hence we have

$$\mathbb{E}\left[m_0^T\right] + \mathbb{E}\left[M_0^T\right] = 2\,\mathbb{E}[S_T], \quad \text{or} \quad \mathbb{E}[S_T] - \mathbb{E}\left[m_0^T\right] = \mathbb{E}\left[M_0^T\right] - \mathbb{E}[S_T],$$

and

$$2\,\mathbb{E}[S_T] - \frac{2S_0}{\sigma\sqrt{2\pi T}} \le \mathbb{E}\left[M_0^T\right] \le 2\,\mathbb{E}[S_T].$$

Exercise 10.4 (Exercise 10.3 continued).

a) Regarding call option prices we have, assuming $K \ge S_0$,

$$\begin{aligned} &\mathbb{E}\left[\big(M_0^T - K\big)^+\right] = S_0\,\mathbb{E}\left[\left(\exp\left(\sigma \max_{t\in[0,T]} W_t\right) - K\right)^+\right] \\ &= \int_0^\infty (S_0\mathrm{e}^{\sigma x} - K)^+ \varphi(x)dx \\ &= \frac{2}{\sqrt{2\pi T}}\int_0^\infty \big(S_0\mathrm{e}^{\sigma x} - K\big)^+ \mathrm{e}^{-x^2/(2T)}dx \\ &= \frac{2}{\sqrt{2\pi T}}\int_{\sigma^{-1}\log(K/S_0)}^\infty \big(S_0\mathrm{e}^{\sigma x} - K\big)\mathrm{e}^{-x^2/(2T)}dx \\ &= \frac{2S_0}{\sqrt{2\pi T}}\int_{\sigma^{-1}\log(K/S_0)}^\infty \mathrm{e}^{\sigma x - x^2/(2T)}dx \end{aligned}$$

$$
\begin{aligned}
&-\frac{2K}{\sqrt{2\pi T}}\int_{\sigma^{-1}\log(K/S_0)}^\infty \mathrm{e}^{-x^2/(2T)}dx\\
&= \frac{2S_0}{\sqrt{2\pi T}}\int_{\sigma^{-1}\log(K/S_0)}^\infty \mathrm{e}^{-(x-\sigma T)^2/(2T)+\sigma^2 T/2}dx\\
&\quad -\frac{2K}{\sqrt{2\pi T}}\int_{\sigma^{-1}\log(K/S_0)}^\infty \mathrm{e}^{-x^2/(2T)}dx\\
&= \frac{2S_0}{\sqrt{2\pi T}}\mathrm{e}^{\sigma^2 T/2}\int_{-\sigma T+\sigma^{-1}\log(K/S_0)}^\infty \mathrm{e}^{-x^2/(2T)}dx\\
&\quad -\frac{2K}{\sqrt{2\pi T}}\int_{\sigma^{-1}\log(K/S_0)}^\infty \mathrm{e}^{-x^2/(2T)}dx\\
&= 2S_0\mathrm{e}^{\sigma^2 T/2}\Phi\big(\sigma\sqrt{T}+\sigma^{-1}\log(S_0/K)/\sqrt{T}\big)\\
&\quad -2K\Phi\big(\sigma^{-1}\log(S_0/K)/\sqrt{T}\big).
\end{aligned}
$$

When $K \leq S_0$, by "completion of the square" and use of the Gaussian cumulative distribution function $\Phi(\cdot)$, we find

$$
\begin{aligned}
\mathbb{E}\left[\left(\max_{t\in[0,T]} S_t - K\right)^+\right] &= \mathbb{E}\left[\max_{t\in[0,T]} S_t - K\right]\\
&= \mathbb{E}\left[\max_{t\in[0,T]} S_t\right] - \mathbb{E}[K]\\
&= \mathbb{E}\left[\max_{t\in[0,T]} S_t\right] - K\\
&= S_0\,\mathbb{E}\left[\exp\left(\sigma \max_{t\in[0,T]} W_t\right)\right] - K\\
&= S_0\int_0^\infty \mathrm{e}^{\sigma x}\varphi(x)dx - K\\
&= \frac{2S_0}{\sqrt{2\pi T}}\int_0^\infty \mathrm{e}^{\sigma x - x^2/(2T)}dx\\
&= \frac{2S_0}{\sqrt{2\pi T}}\int_0^\infty \mathrm{e}^{-(x-\sigma T)^2/(2T)+\sigma^2 T/2}dx - K\\
&= \frac{2S_0}{\sqrt{2\pi T}}\mathrm{e}^{\sigma^2 T/2}\int_{-\sigma T}^\infty \mathrm{e}^{-x^2/(2T)}dx - K\\
&= 2S_0\mathrm{e}^{\sigma^2 T/2}\Phi\big(\sigma\sqrt{T}\big) - K\\
&= 2S_0\mathrm{e}^{\sigma^2 T/2}\left(1-\Phi\big(-\sigma\sqrt{T}\big)\right) - K\\
&= 2\,\mathbb{E}[S_T]\Phi\big(\sigma\sqrt{T}\big) - K,
\end{aligned}
$$

hence

$$
\mathrm{e}^{-\sigma^2 T/2}\,\mathbb{E}\left[\left(M_0^T - K\right)^+\right] = 2S_0\Phi\big(\sigma\sqrt{T}\big) - K\mathrm{e}^{-\sigma^2 T/2}.
$$

Recall that when $r = \sigma^2/2$ the price of the finite expiration American call option price is the Black-Scholes price with maturity T, with

$$
\begin{aligned}
&\mathrm{Bl}_{\mathrm{Call}}(S_0, K, \sigma, r, T)\\
&= S_0 \Phi\big(\sigma\sqrt{T} + \sigma^{-1}\log(S_0/K)/\sqrt{T}\big) - K\mathrm{e}^{-\sigma^2 T/2}\Phi\big(\sigma^{-1}\log(S_0/K)/\sqrt{T}\big)\\
&\leq \begin{cases} 2S_0\Phi\big(\sigma\sqrt{T} + \sigma^{-1}\log(S_0/K)/\sqrt{T}\big) - 2K\mathrm{e}^{-\sigma^2 T/2}\Phi\big(\sigma^{-1}\log(S_0/K)/\sqrt{T}\big) \\ \qquad \text{if } K \geq S_0, \\ \\ 2S_0\Phi\big(\sigma\sqrt{T}\big) - K\mathrm{e}^{-\sigma^2 T/2} \\ \qquad \text{if } K \leq S_0. \end{cases}\\
&= \begin{cases} 2 \times \mathrm{Bl}_{\mathrm{Call}}(S_0, K, \sigma, r, T) & \text{if } K \geq S_0, \\ \\ 2S_0\Phi\big(\sigma\sqrt{T}\big) - K\mathrm{e}^{-\sigma^2 T/2} & \text{if } K \leq S_0, \end{cases}\\
&= \max\left(2 \times \mathrm{Bl}_{\mathrm{Call}}(S_0, K, \sigma, r, T), 2S_0\Phi\big(\sigma\sqrt{T}\big) - K\mathrm{e}^{-\sigma^2 T/2}\right).
\end{aligned}
$$

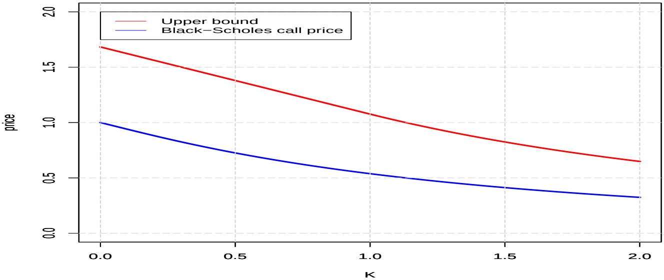


Fig. S.45: Black-Scholes call price upper bound with $S_0 = 1$.

b) Regarding put option prices we have, assuming $S_0 \geq K$,

$$
\begin{aligned}
\mathbb{E}\left[\big(K - m_0^T\big)^+\right] &= S_0\,\mathbb{E}\left[\left(K - \exp\left(\sigma \min_{t\in[0,T]} W_t\right)\right)^+\right]\\
&= \int_0^\infty (K - S_0\mathrm{e}^{\sigma x})^+ \varphi(x)dx\\
&= \frac{2}{\sqrt{2\pi T}}\int_{-\infty}^0 \big(K - S_0\mathrm{e}^{\sigma x}\big)^+ \mathrm{e}^{-x^2/(2T)}dx\\
&= \frac{2}{\sqrt{2\pi T}}\int_{-\infty}^{\sigma^{-1}\log(K/S_0)} \big(K - S_0\mathrm{e}^{\sigma x}\big)\mathrm{e}^{-x^2/(2T)}dx
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{2K}{\sqrt{2\pi T}} \int_{-\infty}^{\sigma^{-1}\log(K/S_0)} \mathrm{e}^{-x^2/(2T)} dx \\
&\quad - \frac{2S_0}{\sqrt{2\pi T}} \int_{-\infty}^{\sigma^{-1}\log(K/S_0)} \mathrm{e}^{\sigma x - x^2/(2T)} dx \\
&= \frac{2K}{\sqrt{2\pi T}} \int_{-\infty}^{\sigma^{-1}\log(K/S_0)} \mathrm{e}^{-x^2/(2T)} dx \\
&\quad - \frac{2S_0}{\sqrt{2\pi T}} \int_{-\infty}^{\sigma^{-1}\log(K/S_0)} \mathrm{e}^{-(x-\sigma T)^2/(2T)+\sigma^2 T/2} dx \\
&= \frac{2K}{\sqrt{2\pi T}} \int_{-\infty}^{\sigma^{-1}\log(K/S_0)} \mathrm{e}^{-x^2/(2T)} dx \\
&\quad - \frac{2S_0}{\sqrt{2\pi T}} \mathrm{e}^{\sigma^2 T/2} \int_{-\infty}^{-\sigma T+\sigma^{-1}\log(K/S_0)} \mathrm{e}^{-x^2/(2T)} dx \\
&= 2K\Phi(-\sigma^{-1}\log\left(S_0/K)/\sqrt{T}\right) \\
&\quad -2S_0\mathrm{e}^{\sigma^2 T/2}\Phi\big(-\sigma\sqrt{T}-\sigma^{-1}\log(S_0/K)/\sqrt{T}\big),
\end{aligned}
$$

with

$$
\mathrm{e}^{-\sigma^2 T/2}\,\mathbb{E}\left[\left(K-m_0^T\right)^+\right] = K\mathrm{e}^{-\sigma^2 T/2} - 2S_0\Phi\big(-\sigma\sqrt{T}\big)
$$

if $S_0 \le K$. Therefore we deduce the bounds

$$
\begin{aligned}
&\mathrm{Bl}_{\mathrm{Put}}(S_0, K, \sigma, r, T) \\
&= K\mathrm{e}^{-\sigma^2 T/2}\Phi\big(-\sigma^{-1}\log(S_0/K)/\sqrt{T}\big) - S_0\Phi\big(-\sigma\sqrt{T}-\sigma^{-1}\log(S_0/K)/\sqrt{T}\big) \\
&\le \text{American put option price} \\
&\le \begin{cases} 2K\mathrm{e}^{-\sigma^2 T/2}\Phi\big(-\sigma^{-1}\log(S_0/K)/\sqrt{T}\big) - 2S_0\Phi\big(-\sigma\sqrt{T}-\sigma^{-1}\log(S_0/K)/\sqrt{T}\big) \\ \qquad \text{if } S_0 \ge K, \\ K\mathrm{e}^{-\sigma^2 T/2} - 2S_0\Phi\big(-\sigma\sqrt{T}\big) \qquad \text{if } S_0 \le K, \end{cases} \\
&= \begin{cases} 2 \times \mathrm{Bl}_{\mathrm{Put}}(S_0, K, \sigma, r, T) & \text{if } S_0 \ge K, \\ K\mathrm{e}^{-\sigma^2 T/2} - 2S_0\Phi\big(-\sigma\sqrt{T}\big) & \text{if } S_0 \le K, \end{cases} \\
&= \max\left(2 \times \mathrm{Bl}_{\mathrm{Put}}(S_0, K, \sigma, r, T), K\mathrm{e}^{-\sigma^2 T/2} - 2S_0\Phi\big(-\sigma\sqrt{T}\big)\right)
\end{aligned}
$$

for the finite expiration American put option price when $r = \sigma^2/2$.

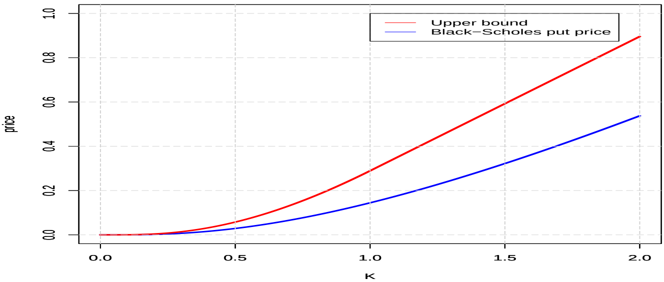


Fig. S.46: Black-Scholes put price upper bound with $S_0 = 1$.

Exercise 10.5 (Exercise 10.4 continued).

a) Using the expression

$$\varphi_{\check{X}_0^T}(x) = \sqrt{\frac{2}{\pi T}} \mathrm{e}^{-(x-\mu T)^2/(2T)} + 2\mu \mathrm{e}^{2\mu x} \Phi\left(\frac{x+\mu T}{\sqrt{T}}\right), \quad x \leq 0.$$

of the probability density function of the minimum

$$\check{X}_0^T := \min_{t\in[0,T]} \widetilde{W}_t = \min_{t\in[0,T]} (W_t + \mu t)$$

of drifted Brownian motion $\widetilde{W}_t = W_t + \mu t$ over $t \in [0, T]$ given in Proposition 10.7, we find

$$\begin{aligned}
\mathbb{E}\left[\min_{t\in[0,T]} S_t\right] &= S_0 \int_{-\infty}^0 \mathrm{e}^{\sigma x} \varphi_{\check{X}_0^T}(x) dx \\
&= S_0 \int_{-\infty}^0 \mathrm{e}^{\sigma x} \sqrt{\frac{2}{\pi T}} \mathrm{e}^{-(x-\mu T)^2/(2T)} dx \\
&\quad +2\mu S_0 \int_{-\infty}^0 \mathrm{e}^{\sigma x} \mathrm{e}^{2\mu x} \Phi\left(\frac{x+\mu T}{\sqrt{T}}\right) dx \\
&= 2S_0 \mathrm{e}^{\sigma^2 T/2 - \mu\sigma T} \Phi\big((\mu-\sigma)\sqrt{T}\big) + \frac{2\mu S_0}{2\mu-\sigma} \Phi\big(-\mu\sqrt{T}\big) \\
&\quad -\frac{2\mu S_0}{2\mu-\sigma} \mathrm{e}^{\sigma^2 T/2 - \mu\sigma T} \Phi\big((\mu-\sigma)\sqrt{T}\big),
\end{aligned}$$

with $\mu := r/\sigma - \sigma/2$, which yields

$$\mathbb{E}\left[\min_{t\in[0,T]} S_t\right] = S_0\left(1-\frac{\sigma^2}{2r}\right)\Phi\left(\frac{r-\sigma^2/2}{\sigma}\sqrt{T}\right)$$
$$+S_0\left(1+\frac{\sigma^2}{2r}\right)\Phi\left(-\frac{r+\sigma^2/2}{\sigma}\sqrt{T}\right).$$

See Exercise 12.1-(b) for the computation of $\mathbb{E}\left[\min_{t\in[0,1]} S_t\right]$ when $r=0$.

b) When $S_0 \leq K$, we have

$$\begin{aligned}
\mathbb{E}\left[\left(K-\min_{t\in[0,T]} S_t\right)^+\right] &= \mathbb{E}\left[K-\min_{t\in[0,T]} S_t\right] \\
&= K - \mathbb{E}\left[\min_{t\in[0,T]} S_t\right] \\
&= K - S_0\left(1-\frac{\sigma^2}{2r}\right)\Phi\left(\frac{r-\sigma^2/2}{\sigma}\sqrt{T}\right) \\
&\quad -S_0\mathrm{e}^{rT}\left(1+\frac{\sigma^2}{2r}\right)\Phi\left(-\frac{r+\sigma^2/2}{\sigma}\sqrt{T}\right).
\end{aligned}$$

Next, when $S_0 \geq K$ we have, using the probability density function $\varphi_{\check{X}_0^T}(x)$,

$$\begin{aligned}
&\mathbb{E}\left[\left(K-\min_{t\in[0,T]} S_t\right)^+\right] = \mathbb{E}\left[\left(K-S_0\min_{t\in[0,T]} \mathrm{e}^{\sigma \check{X}_0^T}\right)^+\right] \\
&= \int_{-\infty}^0 \left(K-S_0\mathrm{e}^{\sigma x}\right)^+ \varphi_{\check{X}_0^T}(x)dx \\
&= S_0\sqrt{\frac{2}{\pi T}}\int_{-\infty}^0 \left(K-S_0\mathrm{e}^{\sigma x}\right)^+ \mathrm{e}^{-(x-\mu T)^2/(2T)}dx \\
&\quad +2\mu S_0\int_{-\infty}^0 \left(K-S_0\mathrm{e}^{\sigma x}\right)^+ \mathrm{e}^{2\mu x}\Phi\left(\frac{x+\mu T}{\sqrt{T}}\right)dx \\
&= K\Phi\left(-\frac{(r-\sigma^2/2)T+\log(S_0/K)}{\sigma\sqrt{T}}\right) \\
&\quad +K\left(\frac{S_0}{K}\right)^{1-2r/\sigma^2}\Phi\left(\frac{(r-\sigma^2/2)T+\log(K/S_0)}{\sigma\sqrt{T}}\right) \\
&\quad -S_0\left(1-\frac{\sigma^2}{2r}\right)\left(\frac{S_0}{K}\right)^{-2r/\sigma^2}\Phi\left(\frac{(r-\sigma^2/2)T+\log(K/S_0)}{\sigma\sqrt{T}}\right) \\
&\quad -S_0\mathrm{e}^{rT}\left(1+\frac{\sigma^2}{2r}\right)\Phi\left(-\frac{(r+\sigma^2/2)T+\log(S_0/K)}{\sigma\sqrt{T}}\right).
\end{aligned}$$

In Figure S.47, using a finite expiration American put option pricer from the R fOptions package, we plot the graph of American put option price *vs* (S.10.48)-(S.10.49), together with the European put option price, according to the following R code.

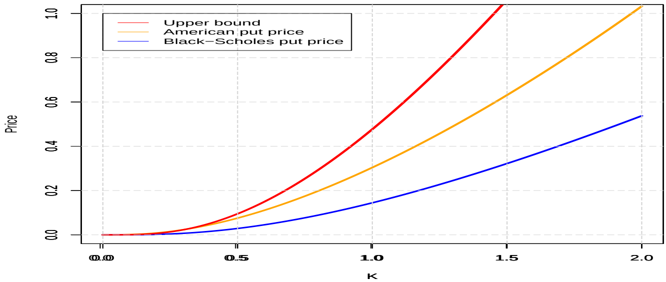


Fig. S.47: "Optimal exercise" put price upper bound with $S_0 = 1$.

```r
d1 <- function(S,K,r,T,sig) {return((log(S/K)+(r+sig^2/2)*T)/(sig*sqrt(T)))}
d2 <- function(S,K,r,T,sig) {return(d1(S, K, r, T, sig) - sig * sqrt(T))}
BSPut <- function(S, K, r, T, sig){return(K*exp(-r*T) * pnorm(-d2(S, K, r, T, sig)) -
        S*pnorm(-d1(S, K, r, T, sig)))}
Optimal_Put_Option <- function(S,K,r,T,sig){
if (r==0) {if (S>=K) {return(K*pnorm(d1(K,S,0,T,sig))-S*(1+sig*sig*T/2
        +log(S/K))*pnorm(-d1(S,K,0,T,sig))
        +S*sig*sqrt(T/(2*pi))*exp(-d1(S,K,0,T,sig)*d1(S,K,0,T,sig)/(2*sig*sig*T)))}
else {return(K-2*S*(1+sig*sig*T/4)*pnorm(-sig*sqrt(T)/2)
        +S*sig*sqrt(T/(2*pi))*exp(-sig*sig*T/8))}}
else {if (S>=K) {return(K*pnorm(-d2(S,K,r,T,sig))
        +K*(S/K)**(1-2*r/sig/sig)*pnorm(d2(K,S,r,T,sig))
        -2*S*exp(r*T)*pnorm(-d1(S,K,r,T,sig))
        -S*(1-sig*sig/2/r)*(S/K)**(-2*r/sig/sig)*pnorm(d1(K,S,r,T,sig))
        +S*exp(r*T)*(1-sig*sig/2/r)*pnorm(-d1(S,K,r,T,sig)))}
else {return(K-S*(1-sig*sig/2/r)*pnorm((r-sig*sig/2)*sqrt(T)/sig)
        -S*(1+sig*sig/2/r)*pnorm(-(r+sig*sig/2)*sqrt(T)/sig))}}}
r=0.5;sig=1;S=1;T=1
library(fOptions)
curve(BAWAmericanApproxOption("p",S,x,T,r,b=0,sig,title = NULL, description =
        NULL)@price, from=0.01, to=2 , xlab="K", lwd = 3, ylim=c(0,1),ylab="",col="orange")
par(new=TRUE)
curve(BSPut(S,x,r,T,sig), from=0, to=2 , xlab="K", lwd = 3, ylim=c(0,1),
        ylab="Price",col="blue")
par(new=TRUE)
curve(Optimal_Put_Option(S,x,r,T,sig), from=0, to=2 , xlab="K", lwd = 3,
        ylim=c(0,1),ylab="",col="red")
grid (lty = 5)
legend(.0,1.0,legend=c("Upper bound","American put price","Black-Scholes put
        price"),col=c("red","orange", "blue"), lty=1:1, cex=1.)
```

c) When $r = 0$ and $S_0 \leq K$, we find

$$\mathbb{E}\left[\left(K - \min_{t\in[0,T]} S_t\right)^+\right] = K - 2S_0\left(1 + \frac{\sigma^2 T}{4}\right)\Phi\left(-\frac{\sigma\sqrt{T}}{2}\right) + \sigma S_0\sqrt{\frac{T}{2\pi}}\mathrm{e}^{-\sigma^2 T/8}. \tag{S.10.48}$$

Next, when $r = 0$ and $S_0 \geq K$, we find

$$\mathbb{E}\left[\left(K - \min_{t\in[0,T]} S_t\right)^+\right] = K\Phi\left(\frac{\sigma^2 T/2 + \log(K/S_0)}{\sigma\sqrt{T}}\right) \tag{S.10.49}$$

$$-S_0\left(1+\log\frac{S_0}{K}+\frac{\sigma^2T}{2}\right)\Phi\left(-\frac{\sigma^2T/2+\log(S_0/K)}{\sigma\sqrt{T}}\right)$$
$$+S_0\sigma\sqrt{\frac{T}{2\pi}}\mathrm{e}^{-(\sigma^2T/2+\log(S_0/K))^2/(2\sigma^2T)}.$$

From the above R code we can check that when $r \simeq 0$ the price of the finite expiration American put option coincides with the price of the standard European put option, as noted in Proposition 15.9.

Exercise 10.6

a) We have

$$\begin{aligned}P(\tau_a \geq t) &= P(X_t > a)\\ &= \int_a^\infty \varphi_{X_t}(x)dx\\ &= \sqrt{\frac{2}{\pi t}}\int_y^\infty \mathrm{e}^{-x^2/(2t)}dx, \qquad y > 0.\end{aligned}$$

b) We have

$$\begin{aligned}\varphi_{\tau_a}(t) &= \frac{d}{dt}P(\tau_a \leq t)\\ &= \frac{d}{dt}\int_a^\infty \varphi_{X_t}(x)dx\\ &= -\frac{1}{2}\sqrt{\frac{2}{\pi}}t^{-3/2}\int_a^\infty \mathrm{e}^{-x^2/(2t)}dx + \frac{1}{2}\sqrt{\frac{2}{\pi}}t^{-3/2}\int_a^\infty \frac{x^2}{t}\mathrm{e}^{-x^2/(2t)}dx\\ &= \frac{1}{2}\sqrt{\frac{2}{\pi}}t^{-3/2}\left(-\int_a^\infty \mathrm{e}^{-x^2/(2t)}dx + a\mathrm{e}^{-a^2/(2t)} + \int_a^\infty \mathrm{e}^{-x^2/(2t)}dx\right)\\ &= \frac{a}{\sqrt{2\pi t^3}}\mathrm{e}^{-a^2/(2t)}, \qquad t > 0.\end{aligned}$$

c) We have

$$\begin{aligned}\mathbb{E}\left[(\tau_a)^{-2}\right] &= \int_0^\infty t^{-2}\varphi_{\tau_a}(t)dt\\ &= \frac{a}{\sqrt{2\pi}}\int_0^\infty t^{-7/2}\mathrm{e}^{-a^2/(2t)}dt\\ &= \frac{2a}{\sqrt{2\pi}}\int_0^\infty x^4\mathrm{e}^{-a^2x^2/2}dx\\ &= \frac{3}{a^4},\end{aligned}$$

by the change of variable $x = t^{-1/2}$, *i.e.* $x^2 = 1/t$, $t = x^{-2}$, and $dt = -2x^{-3}dx$.

Remark: We have

$$\mathbb{E}[\tau_a] = \frac{a}{\sqrt{2\pi}} \int_0^\infty t^{-1/2} \mathrm{e}^{-a^2/(2t)} dt = +\infty.$$

Exercise 10.7 Starting from the probability density function

$$\varphi_{\widehat{X}_0^T, \widetilde{W}_T}(a, b) = \mathbb{1}_{\{a \geq \max(b,0)\}} \frac{1}{T} \sqrt{\frac{2}{\pi T}} (2a - b) \mathrm{e}^{\mu b - (2a-b)^2/(2T) - \mu^2 T/2}$$

of the drifted Brownian motion $\widetilde{W}_T := W_T + \mu T$ and its maximum $\widehat{X}_0^T = \max_{t \in [0,T]} \widetilde{W}_t$, we take $\mu := r/\sigma - \sigma/2$ and let the functions f and g be defined as

$$f(x, y) := \frac{1}{\sigma} \log \frac{x}{S_0} \quad \text{and} \quad g(x, y) := \frac{1}{\sigma} \log \frac{y}{S_0},$$

with the Jacobian

$$|J(x, y)| = \begin{vmatrix} \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} \\ \dfrac{\partial g}{\partial x} & \dfrac{\partial g}{\partial y} \end{vmatrix} = \begin{vmatrix} \dfrac{1}{\sigma x} & 0 \\ 0 & \dfrac{1}{\sigma y} \end{vmatrix} = \frac{1}{\sigma^2 xy},$$

$x, y > 0$, which yields the joint density

$$\begin{aligned} &\varphi_{M_0^T, S_T}(x, y) = |J(x, y)| \varphi_{\widehat{X}_0^T, \widetilde{W}_T}(f(x, y), g(x, y)) \\ &= \frac{1}{\sigma^3 Txy} \mathbb{1}_{\{x \geq \max(y, S_0)\}} \sqrt{\frac{2}{\pi T}} \left(\log \frac{x^2}{S_0 y} \right) \exp\left(\frac{\mu}{\sigma} \log \frac{y}{S_0} - \frac{1}{2\sigma^2 T} \left(\log \frac{x^2}{S_0 y} \right)^2 - \mu^2 \frac{T}{2} \right) \end{aligned}$$

of S_T and its maximum $M_0^T = \max_{t \in [0,T]} S_t$ over $t \in [0, T]$, $x, y > 0$.

Chapter 11

Exercise 11.1 Barrier options.

a) By (11.26) and (12.15) we find

$$\begin{aligned} \xi_t = \frac{\partial g}{\partial y}(t, S_t) &= \Phi\left(\delta_+^{T-t}\left(\frac{S_t}{K} \right) \right) - \Phi\left(\delta_+^{T-t}\left(\frac{S_t}{B} \right) \right) \\ &+ \frac{K}{B} \mathrm{e}^{-(T-t)r} \left(1 - \frac{2r}{\sigma^2} \right) \left(\frac{S_t}{B} \right)^{-2r/\sigma^2} \left(\Phi\left(\delta_-^{T-t}\left(\frac{B^2}{KS_t} \right) \right) - \Phi\left(\delta_-^{T-t}\left(\frac{B}{S_t} \right) \right) \right) \end{aligned}$$

$$+\frac{2r}{\sigma^2}\left(\frac{S_t}{B}\right)^{-1-2r/\sigma^2}\left(\Phi\left(\delta_+^{T-t}\left(\frac{B^2}{KS_t}\right)\right)-\Phi\left(\delta_+^{T-t}\left(\frac{B}{S_t}\right)\right)\right)$$
$$-\frac{2}{\sigma\sqrt{2\pi(T-t)}}\left(1-\frac{K}{B}\right)\exp\left(-\frac{1}{2}\left(\delta_+^{T-t}\left(\frac{S_t}{B}\right)\right)^2\right),$$

$0 < S_t \leq B$, $0 \leq t \leq T$, cf. also Exercise 7.1-(ix) of Shreve (2004) and Figure 11.11 above. At maturity for $t = T$ we find $\xi_T = \mathbb{1}_{[K,B]}(S_T)$.

b) We find

$$\mathbb{P}(Y_T \leq a \text{ and } W_T \geq b) = \mathbb{P}(W_T \leq 2a - b), \qquad a < b < 0,$$

hence

$$f_{Y_T,W_T}(a,b) = \frac{d\mathbb{P}(Y_T \leq a \text{ and } W_T \leq b)}{dadb} = -\frac{d\mathbb{P}(Y_T \leq a \text{ and } W_T \geq b)}{dadb},$$

$a, b \in \mathbb{R}$, satisfies

$$f_{Y_T,W_T}(a,b) = \sqrt{\frac{2}{\pi T}}\mathbb{1}_{(-\infty,\min(0,b)]}(a)\frac{(b-2a)}{T}\mathrm{e}^{-(2a-b)^2/(2T)}$$

$$= \begin{cases} \sqrt{\dfrac{2}{\pi T}}\dfrac{(b-2a)}{T}\mathrm{e}^{-(2a-b)^2/(2T)}, & a < \min(0,b), \\ 0, & a > \min(0,b). \end{cases}$$

c) We find

$$f_{\check{Y}_T,\widehat{W}_T}(a,b) = \mathbb{1}_{(-\infty,\min(0,b)]}(a)\frac{1}{T}\sqrt{\frac{2}{\pi T}}(b-2a)\mathrm{e}^{-\mu^2 T/2+\mu b-(2a-b)^2/(2T)}$$

$$= \begin{cases} \dfrac{1}{T}\sqrt{\dfrac{2}{\pi T}}(b-2a)\mathrm{e}^{-\mu^2 T/2+\mu b-(2a-b)^2/(2T)}, & a < \min(0,b), \\ 0, & a > \min(0,b). \end{cases}$$

d) The function $g(t,x)$ is given by the Relations (11.10) and (11.11) above.

Exercise 11.2

a) By Corollary 10.8, the probability density function of the minimum

$$m_0^{\Delta\tau} = \min_{s\in[0,\Delta\tau]} S_{\tau+s}$$

with $S_\tau = B$ is given by

$$\begin{aligned}
\varphi_{m_0^{\Delta\tau}}(x) &= \frac{1}{\sigma x\sqrt{2\pi\Delta\tau}} \exp\left(-\frac{(-(r-\sigma^2/2)\Delta\tau + \log(x/B))^2}{2\sigma^2\Delta\tau}\right) \\
&+ \frac{1}{\sigma x\sqrt{2\pi\Delta\tau}} \left(\frac{B}{x}\right)^{1-2r/\sigma^2} \exp\left(-\frac{((r-\sigma^2/2)\Delta\tau + \log(x/B))^2}{2\sigma^2\Delta\tau}\right) \\
&+ \frac{1}{x}\left(\frac{2r}{\sigma^2}-1\right)\left(\frac{B}{x}\right)^{1-2r/\sigma^2} \Phi\left(\frac{(r-\sigma^2/2)\Delta\tau + \log(x/B)}{\sigma\sqrt{\Delta\tau}}\right),
\end{aligned}$$

$0 < x \leq B$, see also Proposition 10.7 for the probability density function of the minimum of the drifted Brownian motion $\widetilde{W}_t = W_t + \mu t$ over $t \in [0, T]$. Hence, we have

$$\begin{aligned}
&\mathbb{E}\left[\left(\min_{s\in[0,\Delta\tau]} S_{\tau+s} - K\right)^+ \,\Big|\, \mathcal{F}_\tau\right] = \int_0^B (x-K)^+ \varphi_{m_0^{\Delta\tau}}(x)dx \\
&= B\left(1+\frac{\sigma^2}{2r}\right)\mathrm{e}^{r\Delta\tau}\Phi\left(-\left(\frac{r}{\sigma}+\frac{\sigma}{2}\right)\sqrt{\Delta\tau}\right) + K\Phi\left(-\frac{\log(B/K)+(r-\sigma^2/2)\Delta\tau}{\sigma\sqrt{\Delta\tau}}\right) \\
&\quad - B\left(1+\frac{\sigma^2}{2r}\right)\mathrm{e}^{r\Delta\tau}\Phi\left(-\frac{\log(B/K)+(r+\sigma^2/2)\Delta\tau}{\sigma\sqrt{\Delta\tau}}\right) \\
&\quad + B\left(1-\frac{\sigma^2}{2r}\right)\Phi\left(\left(\frac{r}{\sigma}-\frac{\sigma}{2}\right)\sqrt{\Delta\tau}\right) \\
&\quad + B\frac{\sigma^2}{2r}\left(\frac{K}{B}\right)^{2r/\sigma^2}\Phi\left(-\frac{\log(B/K)-(r-\sigma^2/2)\Delta\tau}{\sigma\sqrt{\Delta\tau}}\right) - K,
\end{aligned}$$

with $r > 0$.

b) When $r = 0$, we find

$$\begin{aligned}
&\mathbb{E}\left[\left(\min_{s\in[0,\Delta\tau]} S_{\tau+s} - K\right)^+ \,\Big|\, \mathcal{F}_\tau\right] = B\left(2+\frac{\sigma^2}{2}\Delta\tau\right)\Phi\left(-\frac{\sigma}{2}\sqrt{\Delta\tau}\right) \\
&\quad - B\left(1+\frac{\sigma^2}{2}\Delta\tau + \log\frac{B}{K}\right)\Phi\left(-\frac{\log(B/K)+\frac{\sigma^2}{2}\Delta\tau}{\sigma\sqrt{\Delta\tau}}\right) + K\Phi\left(-\frac{\log(B/K)-\frac{\sigma^2}{2}\Delta\tau}{\sigma\sqrt{\Delta\tau}}.\right) \\
&\quad - \frac{1}{\sqrt{2\pi}}\sigma B\sqrt{\Delta\tau}\left(\mathrm{e}^{-\sigma^2\Delta\tau/8} - \mathrm{e}^{-d_+^2/2}\right) - K.
\end{aligned}$$

c) By the solution of Exercise 10.1-(c), the probability density function of τ_B is given by

$$\varphi_{\tau_B}(x) = \frac{|\log(S_0/B)|}{\sigma\sqrt{2\pi x^3}} \exp\left(-\frac{1}{2\sigma^2 x}((r-\sigma^2/2)x - \log(B/S_0))^2\right), \quad x > 0.$$

d) We have

$$
\begin{aligned}
& \mathrm{e}^{-\Delta\tau}\,\mathbb{E}\left[\mathrm{e}^{-\tau}\mathbb{1}_{[0,T]}(\tau)\Big(\min_{t\in[\tau,\tau+\Delta\tau]} S_t - K\Big)^+\right] \\
&= \mathrm{e}^{-\Delta\tau}\,\mathbb{E}\left[\mathrm{e}^{-r\tau}\mathbb{1}_{[0,T]}(\tau)\,\mathbb{E}\left[\Big(\min_{t\in[\tau,\tau+\Delta\tau]} S_t - K\Big)^+ \;\Big|\; \mathcal{F}_\tau\right]\right] \\
&= \mathrm{e}^{-\Delta\tau}\,\mathbb{E}\left[\mathrm{e}^{-r\tau}\mathbb{1}_{[0,T]}(\tau)\right]\mathbb{E}\left[\Big(\min_{t\in[\tau,\tau+\Delta\tau]} S_t - K\Big)^+ \;\Big|\; \mathcal{F}_\tau\right],
\end{aligned}
$$

where $\mathbb{E}\left[\left(\min_{t\in[\tau,\tau+\Delta\tau]} S_t - K\right)^+ \;\Big|\; \mathcal{F}_\tau\right]$ is given by Questions (a)-(b), and

$$
\begin{aligned}
&\mathbb{E}\left[\mathrm{e}^{-r\tau}\mathbb{1}_{[0,T]}(\tau)\right] = \int_0^T \mathrm{e}^{-rx}\varphi_{\tau_B}(x)dx \\
&= \left|\log\frac{S_0}{B}\right| \int_0^T \mathrm{e}^{-rx\tau}\frac{1}{\sigma\sqrt{2\pi x^3}}\exp\left(-\frac{1}{2\sigma^2 x}((r-\sigma^2/2)x - \log(B/S_0))^2\right)dx \\
&= \left|\log\frac{S_0}{B}\right| \exp\left(\frac{\log(B/S_0)\big(r-\sigma^2/2-\sqrt{(r-\sigma^2/2)^2+2r\sigma^2}\big)}{\sigma^2}\right) \\
&\quad \times \int_0^T \frac{1}{\sigma\sqrt{2\pi x^3}}\exp\left(-\frac{1}{2\sigma^2 x}\Big(x\sqrt{(r-\sigma^2/2)^2+2r\sigma^2} - \log(B/S_0)\Big)^2\right)dx \\
&= \exp\left(\frac{\log(B/S_0)\big(r-\sigma^2/2-\sqrt{(r-\sigma^2/2)^2+2r\sigma^2}\big)}{\sigma^2}\right) \\
&\quad \times \int_0^T \frac{1}{\sigma\sqrt{2\pi x^3}}\exp\left(-\frac{1}{2\sigma^2 x}\Big(x\sqrt{(r-\sigma^2/2)^2+2r\sigma^2} - \log(B/S_0)\Big)^2\right)dx \\
&= \left(\frac{B}{S_0}\right)^{\left(r-\sigma^2/2-\sqrt{(r-\sigma^2/2)^2+2r\sigma^2}\right)/\sigma^2}\Phi\left(\frac{\log(B/S_0) - T\sqrt{(r-\sigma^2/2)^2+2r\sigma^2}}{\sigma\sqrt{T}}\right) \\
&\quad + \left(\frac{B}{S_0}\right)^{\left(r-\sigma^2/2+\sqrt{(r-\sigma^2/2)^2+2r\sigma^2}\right)/\sigma^2}\Phi\left(\frac{\log(B/S_0) + T\sqrt{(r-\sigma^2/2)^2+2r\sigma^2}}{\sigma\sqrt{T}}\right),
\end{aligned}
$$

where the last identity follows from Proposition 10.4 and the relation

$$
\begin{aligned}
\Phi\left(\frac{a-\mu T}{\sqrt{T}}\right) - \mathrm{e}^{2\mu a}\Phi\left(\frac{-a-\mu T}{\sqrt{T}}\right) &= \mathbb{P}\big(\widehat{X}_0^T \le a\big) \\
&= \mathbb{P}(\tilde{\tau}_a \le T) \\
&= \int_0^T \varphi_{\tilde{\tau}_a}(x)dx \\
&= \frac{1}{2}\int_0^T \frac{a}{\sqrt{2\pi x^3}}\mathrm{e}^{-(a-\mu x)^2/(2x)}dx, \quad T>0.
\end{aligned}
$$

with $a := \log(B/S_0)/\sigma$, for a Brownian motion with drift $\mu = \sqrt{(r - \sigma^2/2)^2 + 2r\sigma^2}/\sigma$, see Exercise 10.1-(b).

Exercise 11.3 Barrier forward contracts.

a) Up-and-in barrier long forward contract. We have

$$\begin{aligned}
\mathrm{e}^{-(T-t)r}\,\mathbb{E}[C \mid \mathcal{F}_t] &= \mathrm{e}^{-(T-t)r}\,\mathbb{E}\left[(S_T - K)\,\mathbb{1}_{\left\{\max\limits_{0\le u\le T} S_u > B\right\}} \,\Big|\, \mathcal{F}_t\right] \\
&= \mathbb{1}_{\left\{\max\limits_{0\le u\le t} S_u > B\right\}}\big(S_t - K\mathrm{e}^{-(T-t)r}\big) + \mathbb{1}_{\left\{\max\limits_{0\le u\le t} S_u \le B\right\}}\phi(t, S_t),
\end{aligned} \tag{S.11.50}$$

where the function

$$\begin{aligned}
\phi(t,x) := {} & x\Phi\left(\delta_+^{T-t}(x/B)\right) - K\mathrm{e}^{-(T-t)r}\Phi\left(\delta_-^{T-t}(x/B)\right) \\
& + B(B/x)^{2r/\sigma^2}\Phi\left(-\delta_+^{T-t}(B/x)\right) \\
& - K\mathrm{e}^{-(T-t)r}(B/x)^{-1+2r/\sigma^2}\Phi\left(-\delta_-^{T-t}(B/x)\right)
\end{aligned}$$

solves the Black-Scholes PDE with the terminal condition

$$\phi(T,x) = \left(x - K + \left(\frac{B}{x}\right)^{2r/\sigma^2}\left(B - x\frac{K}{B}\right)\right)\mathbb{1}_{[B,\infty)}(x),$$

as in the proof of Proposition 11.3. Note that only the values of $\phi(t,x)$ with $x \in [0,B]$ are used for pricing.

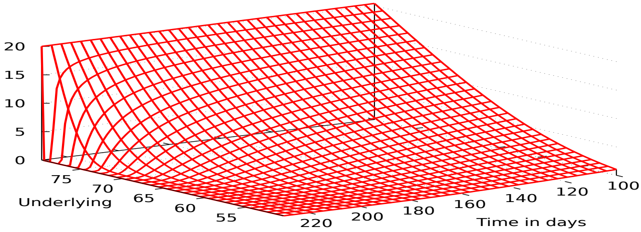


Fig. S.48: Price of the up-and-in long forward contract with $K = 60 < B = 80$.

As for the hedging strategy, we find

$$\begin{aligned}
\xi_t &= \frac{\partial \phi}{\partial x}(t, S_t) = \Phi\left(\delta_+^{T-t}(x/B)\right) + \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-(\delta_+^{T-t}(x/B))^2/2} \\
&\quad - \frac{1}{x\sqrt{2\pi}} K \mathrm{e}^{-(T-t)r-(\delta_-^{T-t}(x/B))^2/2} - \frac{2r}{\sigma^2}(B/x)^{1+2r/\sigma^2} \Phi\left(-\delta_+^{T-t}(B/x)\right) \\
&\quad + \frac{1}{\sqrt{2\pi}}(B/x)^{1+2r/\sigma^2} \mathrm{e}^{-(\delta_+^{T-t}(B/x))^2/2} \\
&\quad - \frac{K(1-2r/\sigma^2)}{B} \mathrm{e}^{-(T-t)r}(B/x)^{2r/\sigma^2} \Phi\left(-\delta_-^{T-t}(B/x)\right) \\
&\quad - \frac{K}{B\sqrt{2\pi}}(B/x)^{2r/\sigma^2} \mathrm{e}^{-(T-t)r-(\delta_-^{T-t}(B/x))^2/2} \\
&= \Phi\left(\delta_+^{T-t}(x/B)\right) - \frac{2r}{\sigma^2}(B/x)^{1+2r/\sigma^2} \Phi\left(-\delta_+^{T-t}(B/x)\right) \\
&\quad + \frac{1}{\sqrt{2\pi}}(1-K/B)\left(\mathrm{e}^{-(\delta_+^{T-t}(x/B))^2/2} + \frac{B}{x} \mathrm{e}^{-(T-t)r-(\delta_-^{T-t}(x/B))^2/2}\right) \\
&\quad - \frac{K}{B}(1-2r/\sigma^2) \mathrm{e}^{-(T-t)r}(B/x)^{2r/\sigma^2} \Phi\left(-\delta_-^{T-t}(B/x)\right),
\end{aligned}$$

since by (12.22) we have

$$\mathrm{e}^{-(\delta_-^{T-t}(B/x))^2/2} = \mathrm{e}^{r(T-t)}(x/B)^{2r/\sigma^2} \mathrm{e}^{-(\delta_+^{T-t}(x/B))^2/2}$$

and

$$\mathrm{e}^{-(\delta_-^{T-t}(x/B))^2/2} = \mathrm{e}^{r(T-t)}(B/x)^{2r/\sigma^2} \mathrm{e}^{-(\delta_+^{T-t}(B/x))^2/2}.$$

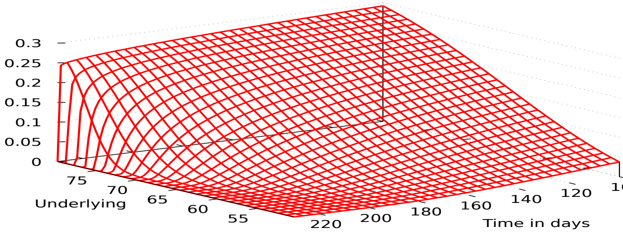


Fig. S.49: Delta of the up-and-in long forward contract with $K = 60 < B = 80$.

b) Up-and-out barrier long forward contract. We have

$$\begin{aligned} \mathrm{e}^{-(T-t)r}\, \mathbb{E}[C \mid \mathcal{F}_t] &= \mathrm{e}^{-(T-t)r}\, \mathbb{E}\left[(S_T - K)\, \mathbb{1}_{\left\{ \max\limits_{0\le u\le T} S_u < B \right\}} \,\Big|\, \mathcal{F}_t \right] \\ &= \mathbb{1}_{\left\{ \max\limits_{0\le u\le t} S_u \le B \right\}} \phi(t, S_t), \end{aligned} \qquad \text{(S.11.51)}$$

where the function

$$\begin{aligned} \phi(t,x) := {} & x\Phi\left(-\delta_+^{T-t}(x/B)\right) - K\mathrm{e}^{-(T-t)r}\Phi\left(-\delta_-^{T-t}(x/B)\right) \\ & -B(B/x)^{2r/\sigma^2}\Phi\left(-\delta_+^{T-t}(B/x)\right) \\ & +K\mathrm{e}^{-(T-t)r}(B/x)^{-1+2r/\sigma^2}\Phi\left(-\delta_-^{T-t}(B/x)\right) \end{aligned}$$

solves the Black-Scholes PDE with the terminal condition

$$\phi(T,x) = (x-K)\mathbb{1}_{[0,B]}(x) - \left(\frac{B}{x}\right)^{2r/\sigma^2}\left(B - x\frac{K}{B}\right)\mathbb{1}_{[B,\infty)}(x).$$

Note that only the values of $\phi(t,x)$ with $x \in [B,\infty)$ are used for pricing.

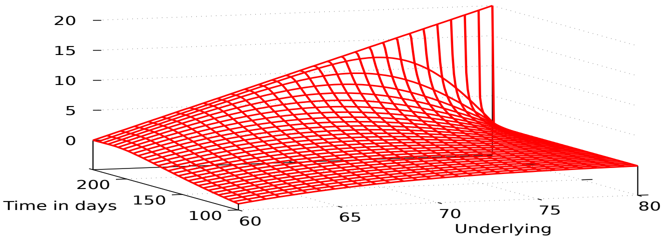


Fig. S.50: Price of the up-and-out long forward contract with $K = 60 < B = 80$.

As for the hedging strategy, we find

$$\begin{aligned}
\xi_t &= \frac{\partial \phi}{\partial x}(t, S_t) = \Phi\left(-\delta_+^{T-t}(x/B)\right) - \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-(\delta_+^{T-t}(x/B))^2/2} \\
&\quad + \frac{1}{x\sqrt{2\pi}} K \mathrm{e}^{-(T-t)r-(\delta_-^{T-t}(x/B))^2/2} + \frac{2r}{\sigma^2}(B/x)^{1+2r/\sigma^2}\Phi\left(-\delta_+^{T-t}(B/x)\right) \\
&\quad - \frac{1}{\sqrt{2\pi}}(B/x)^{1+2r/\sigma^2}\mathrm{e}^{-(\delta_+^{T-t}(B/x))^2/2} \\
&\quad + \frac{K(1-2r/\sigma^2)}{B}\mathrm{e}^{-(T-t)r}(B/x)^{2r/\sigma^2}\Phi\left(-\delta_-^{T-t}(B/x)\right) \\
&\quad + \frac{K}{B\sqrt{2\pi}}(B/x)^{2r/\sigma^2}\mathrm{e}^{-(T-t)r-(\delta_-^{T-t}(B/x))^2/2} \\
&= \Phi\left(-\delta_+^{T-t}(x/B)\right) + \frac{2r}{\sigma^2}(B/x)^{1+2r/\sigma^2}\Phi\left(-\delta_+^{T-t}(B/x)\right) \\
&\quad - \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-(\delta_+^{T-t}(x/B))^2/2} - \frac{1}{\sqrt{2\pi}}\frac{B}{x}\mathrm{e}^{-(T-t)r-(\delta_-^{T-t}(x/B))^2/2} \\
&\quad + \frac{K}{B\sqrt{2\pi}}\mathrm{e}^{-(\delta_+^{T-t}(x/B))^2/2} + \frac{1}{\sqrt{2\pi}}\frac{K}{x}\mathrm{e}^{-(T-t)r-(\delta_-^{T-t}(x/B))^2/2} \\
&\quad + \frac{K}{B}(1-2r/\sigma^2)\mathrm{e}^{-(T-t)r}(B/x)^{2r/\sigma^2}\Phi\left(-\delta_-^{T-t}(B/x)\right) \\
&= \Phi\left(-\delta_+^{T-t}(x/B)\right) + \frac{2r}{\sigma^2}(B/x)^{1+2r/\sigma^2}\Phi\left(-\delta_+^{T-t}(B/x)\right) \\
&\quad - \frac{1}{\sqrt{2\pi}}(1-K/B)\left(\mathrm{e}^{-(\delta_+^{T-t}(x/B))^2/2} + \frac{B}{x}\mathrm{e}^{-(T-t)r-(\delta_-^{T-t}(x/B))^2/2}\right) \\
&\quad + \frac{K}{B}\left(1-\frac{2r}{\sigma^2}\right)\mathrm{e}^{-(T-t)r}\left(\frac{B}{x}\right)^{2r/\sigma^2}\Phi\left(-\delta_-^{T-t}\left(\frac{B}{x}\right)\right),
\end{aligned}$$

by (12.22).

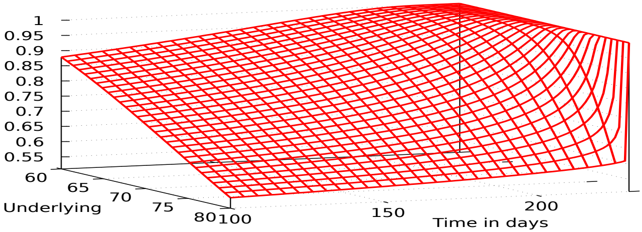


Fig. S.51: Delta of the up-and-out long forward contract price with $K = 60 < B = 80$.

c) Down-and-in barrier long forward contract. We have

$$\mathrm{e}^{-(T-t)r}\,\mathbb{E}[C \mid \mathcal{F}_t] = \mathrm{e}^{-(T-t)r}\,\mathbb{E}\left[(S_T - K)\mathbb{1}_{\left\{\min\limits_{0\le u\le T} S_u < B\right\}} \,\Big|\, \mathcal{F}_t\right]$$
$$= \mathbb{1}_{\left\{\min\limits_{0\le u\le t} S_u < B\right\}}(S_t - K\mathrm{e}^{-(T-t)r}) + \mathbb{1}_{\left\{\min\limits_{0\le u\le t} S_u \ge B\right\}}\phi(t, S_t) \tag{S.11.52}$$

where the function

$$\begin{aligned}\phi(t,x) := {} & x\Phi\left(-\delta_+^{T-t}(x/B)\right) - K\mathrm{e}^{-(T-t)r}\Phi\left(-\delta_-^{T-t}(x/B)\right)\\ & +B(B/x)^{2r/\sigma^2}\Phi\left(\delta_+^{T-t}(B/x)\right)\\ & -K\mathrm{e}^{-(T-t)r}(B/x)^{-1+2r/\sigma^2}\Phi\left(\delta_-^{T-t}(B/x)\right)\end{aligned}$$

solves the Black-Scholes PDE with the terminal condition

$$\phi(T,x) = \left(x - K + \left(\frac{B}{x}\right)^{2r/\sigma^2}\left(B - x\frac{K}{B}\right)\right)\mathbb{1}_{[0,B]}(x).$$

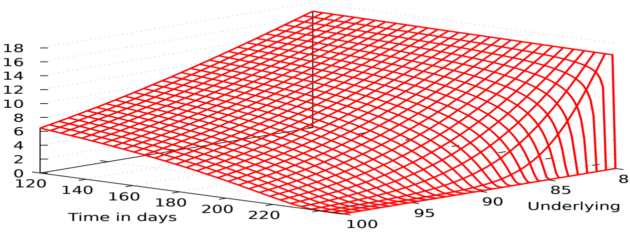


Fig. S.52: Price of the down-and-in long forward contract with $K = 60 < B = 80$.

As for the hedging strategy, we find

$$\begin{aligned}\xi_t &= \frac{\partial \phi}{\partial x}(t, S_t)\\ &= \Phi\left(-\delta_+^{T-t}(x/B)\right) + \frac{2r}{\sigma^2}(B/x)^{1+2r/\sigma^2}\Phi\left(\delta_+^{T-t}(B/x)\right)\\ &\quad -\frac{1}{\sqrt{2\pi}}(1-K/B)\left(\mathrm{e}^{-(\delta_+^{T-t}(x/B))^2/2} + \frac{B}{x}\mathrm{e}^{-(T-t)r-(\delta_-^{T-t}(x/B))^2/2}\right)\\ &\quad +\frac{K}{B}(1-2r/\sigma^2)\mathrm{e}^{-(T-t)r}(B/x)^{2r/\sigma^2}\Phi\left(\delta_-^{T-t}(B/x)\right).\end{aligned}$$

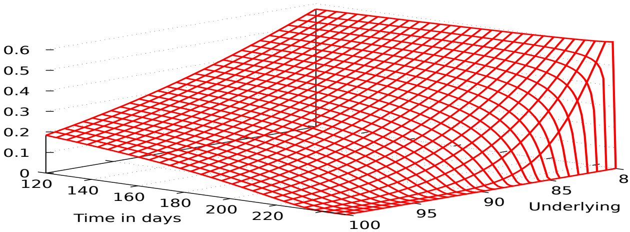


Fig. S.53: Delta of the down-and-in long forward contract with $K = 60 < B = 80$.

d) Down-and-out barrier long forward contract. We have

$$\mathrm{e}^{-(T-t)r}\,\mathbb{E}[C \mid \mathcal{F}_t] = \mathrm{e}^{-(T-t)r}\,\mathbb{E}\left[(S_T - K)\mathbb{1}_{\left\{\min\limits_{0\le u\le T} S_u > B\right\}} \,\Big|\, \mathcal{F}_t\right]$$

$$= \mathbb{1}_{\left\{ \min\limits_{0 \le u \le t} S_u \ge B \right\}} \phi(t, S_t) \tag{S.11.53}$$

where the function

$$\begin{aligned}\phi(t,x) := x\Phi\left(\delta_+^{T-t}\left(\frac{x}{B}\right)\right) - K\mathrm{e}^{-(T-t)r}\Phi\left(\delta_-^{T-t}\left(\frac{x}{B}\right)\right) \\ -B(B/x)^{2r/\sigma^2}\Phi\left(\delta_+^{T-t}\left(\frac{B}{x}\right)\right) \\ +K\mathrm{e}^{-(T-t)r}(B/x)^{-1+2r/\sigma^2}\Phi\left(\delta_-^{T-t}\left(\frac{B}{x}\right)\right)\Bigg)\end{aligned}$$

solves the Black-Scholes PDE with the terminal condition

$$\phi(T,x) = (x-K)\mathbb{1}_{[B,\infty)}(x) - \left(B - x\frac{K}{B}\right)\left(\frac{B}{x}\right)^{2r/\sigma^2} \mathbb{1}_{[0,B]}(x).$$

Note that $\phi(t,x)$ above coincides with the price of (11.11) of the standard down-and-out barrier call option in the case $K < B$, cf. Exercise 11.1-(d).

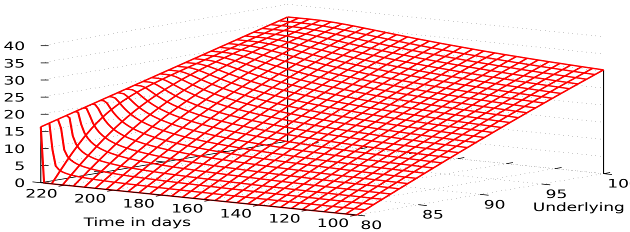


Fig. S.54: Price of the down-and-out long forward contract with $K = 60 < B = 80$.

As for the hedging strategy, we find

$$\begin{aligned}\xi_t &= \frac{\partial\phi}{\partial x}(t, S_t) \\ &= \Phi\left(\delta_+^{T-t}\left(\frac{x}{B}\right)\right) - \frac{2r}{\sigma^2}(B/x)^{1+2r/\sigma^2}\Phi\left(\delta_+^{T-t}(B/x)\right) \\ &\quad + \frac{1}{\sqrt{2\pi}}\left(1 - \frac{K}{B}\right)\left(\mathrm{e}^{-(\delta_+^{T-t}(x/B))^2/2} + \frac{B}{x}\mathrm{e}^{-(T-t)r-(\delta_-^{T-t}(x/B))^2/2}\right) \\ &\quad - \frac{K}{B}\left(1 - \frac{2r}{\sigma^2}\right)\mathrm{e}^{-(T-t)r}\left(\frac{B}{x}\right)^{2r/\sigma^2}\Phi\left(\delta_-^{T-t}\left(\frac{B}{x}\right)\right).\end{aligned}$$

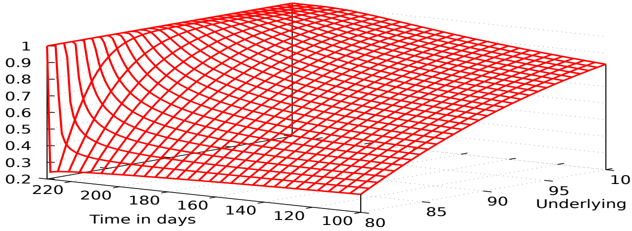


Fig. S.55: Delta of the down-and-out long forward contract with $K = 60 < B = 80$.

e) Up-and-in barrier short forward contract. The price of the up-and-in barrier short forward contract is identical to (S.11.50) with a negative sign.

f) Up-and-out barrier short forward contract. The price of the up-and-out barrier short forward contract is identical to (S.11.51) with a negative sign. Note that $\phi(t, x)$ coincides with the price of (11.8) of the standard up-and-out barrier put option in the case $B < K$.

g) Down-and-in barrier short forward contract. The price of the down-and-in barrier short forward contract is identical to (S.11.52) with a negative sign.

h) Down-and-out barrier short forward contract. The price of the down-and-out barrier short forward contract is identical to (S.11.53) with a negative sign.

Exercise 11.4 When $B < K$, we find

$$\begin{aligned}
&\mathrm{Vega}_{\text{down-and-out-call}} \\
&= S_t\sqrt{\frac{T-t}{2\pi}}\mathrm{e}^{-(\delta_+^{T-t}(S_t/K))^2/2} \\
&- \frac{4r}{\sigma^3}\left(\frac{S_t}{B}\right)^{1-2r/\sigma^2}\left(\frac{B^2}{S_t}\Phi\left(\delta_+^{T-t}\left(\frac{B^2}{KS_t}\right)\right) - K\mathrm{e}^{-(T-t)r}\Phi\left(\delta_-^{T-t}\left(\frac{B^2}{KS_t}\right)\right)\right)\log\frac{S_t}{B} \\
&- \sqrt{\frac{T-t}{2\pi}}\frac{B^2}{S_t}\left(\frac{S_t}{B}\right)^{1-2r/\sigma^2}\mathrm{e}^{-(\delta_+^{T-t}(B^2/K/S_t))^2/2}.
\end{aligned}$$

When $B > K$, we find

$$\begin{aligned}
&\mathrm{Vega}_{\text{down-and-out-call}} \\
&= \frac{S_t}{\sqrt{2\pi}}\mathrm{e}^{-(\delta_+^{T-t}(S_t/K))^2/2}\left(\left(\frac{K}{B}-1\right)\left(\frac{\delta_-^{T-t}(S_t/B)}{\sigma}+\sqrt{T-t}\right)+\sqrt{T-t}\right)
\end{aligned}$$

$$
-\frac{4r}{\sigma^3}\left(\frac{S_t}{B}\right)^{1-2r/\sigma^2}\left(\frac{B^2}{S_t}\Phi\left(\delta_+^{T-t}\left(\frac{B}{S_t}\right)\right)-K\mathrm{e}^{-(T-t)r}\Phi\left(\delta_-^{T-t}\left(\frac{B}{S_t}\right)\right)\right)\log\frac{S_t}{B}
$$
$$
-\frac{1}{\sqrt{2\pi}}\frac{B^2}{S_t}\mathrm{e}^{-(\delta_+^{T-t}(S_t/B))^2/2}\left(\left(\frac{K}{B}-1\right)\left(\frac{\delta_-^{T-t}(B/S_t)}{\sigma}+\sqrt{T-t}\right)+\sqrt{T-t}\right).
$$

The corresponding formulas for the down-and-in call option can be obtained from the parity relation (11.2) and the value $S_t\sqrt{\frac{T-t}{2\pi}}\mathrm{e}^{-(\delta_+^{T-t}(S_t/K))^2/2}$ of the Black-Scholes Vega, see Table 6.1.

Exercise 11.5 We have

$$
\begin{aligned}
&\mathbb{E}^*[C]=\mathbb{E}^*\left[\mathbb{1}_{\{S_T\geq K\}}\mathbb{1}_{\{M_0^T\leq B\}}\right]\\
&=\mathbb{E}^*\left[\mathbb{1}_{\{S_0\mathrm{e}^{\sigma\widehat{W}_T}\geq K\}}\mathbb{1}_{\{S_0\mathrm{e}^{\sigma\widehat{X}_0^T}\leq B\}}\right]\\
&=\int_{-\infty}^{\infty}\int_{y\vee 0}^{\infty}\mathbb{1}_{\{S_0\mathrm{e}^{\sigma y}\geq K\}}\mathbb{1}_{\{S_0\mathrm{e}^{\sigma x}\leq B\}}d\mathbb{P}(\widehat{X}_0^T\leq x,\widehat{W}_T\leq y)\\
&=\int_{-\infty}^{\infty}\int_{y\vee 0}^{\infty}\mathbb{1}_{\{S_0\mathrm{e}^{\sigma y}\geq K\}}\mathbb{1}_{\{S_0\mathrm{e}^{\sigma x}\leq B\}}f_{\widehat{X}_T,\widehat{W}_T}(x,y)dxdy\\
&=\frac{1}{T}\sqrt{\frac{2}{\pi T}}\int_{-\infty}^{\sigma^{-1}\log(B/S_0)}\\
&\quad\int_{y\vee 0}^{\infty}\mathbb{1}_{\{S_0\mathrm{e}^{\sigma y}\geq K\}}\mathbb{1}_{\{S_0\mathrm{e}^{\sigma x}\leq B\}}(2x-y)\mathrm{e}^{-\mu^2T/2+\mu y-(2x-y)^2/(2T)}dxdy\\
&=\frac{1}{T}\sqrt{\frac{2}{\pi T}}\int_{\sigma^{-1}\log(K/S_0)}^{\sigma^{-1}\log(B/S_0)}\int_{y\vee 0}^{\sigma^{-1}\log(B/S_0)}(2x-y)\mathrm{e}^{-\mu^2T/2+\mu y-(2x-y)^2/(2T)}dxdy,
\end{aligned}
$$

if $B\geq S_0$ (otherwise the option price is 0), with $\mu=r/\sigma-\sigma/2$ and $y\vee 0=\max(y,0)$. Next, letting $a=y\vee 0$ and $b=\sigma^{-1}\log(B/S_0)$, we have

$$
\int_a^b(2x-y)\mathrm{e}^{2x(y-x)/T}dx=\frac{T}{2}(1-\mathrm{e}^{2b(y-b)/T}),
$$

hence, letting $c=\sigma^{-1}\log(K/S_0)$, we have

$$
\begin{aligned}
\mathbb{E}^*[C]&=\mathrm{e}^{-\mu^2T/2}\frac{1}{\sqrt{2\pi T}}\int_c^b\mathrm{e}^{\mu y-y^2/(2T)}(1-\mathrm{e}^{2b(y-b)/T})dy\\
&=\mathrm{e}^{-\mu^2T/2}\frac{1}{\sqrt{2\pi T}}\int_c^b\mathrm{e}^{\mu y-y^2/(2T)}dy\\
&\quad-\mathrm{e}^{-\mu^2T/2-2b^2/T}\frac{1}{\sqrt{2\pi T}}\int_c^b\mathrm{e}^{y(\mu+2b/T)-y^2/(2T)}dy.
\end{aligned}
$$

Using the relation

$$
\frac{1}{\sqrt{2\pi T}}\int_c^b\mathrm{e}^{\gamma y-y^2/(2T)}dy=\mathrm{e}^{\gamma^2T/2}\left(\Phi\left(\frac{-c+\gamma T}{\sqrt{T}}\right)-\Phi\left(\frac{-b+\gamma T}{\sqrt{T}}\right)\right),
$$

we find

$$\begin{aligned}
&\mathbb{E}^*[C] = \mathbb{E}^*\left[(S_T-K)^+ \mathbb{1}_{\{M_0^T\le B\}}\right] \\
&= \Phi\left(\frac{-c+\mu T}{\sqrt{T}}\right) - \Phi\left(\frac{-b+\mu T}{\sqrt{T}}\right) \\
&\quad - \mathrm{e}^{-\mu^2T/2-2b^2/T+(\mu+2b/T)^2T/2}\left(\Phi\left(\frac{-c+(\mu+2b/T)T}{\sqrt{T}}\right) - \Phi\left(\frac{-b+(\mu+2b/T)T}{\sqrt{T}}\right)\right) \\
&= \Phi\left(\delta_-^T\left(\frac{S_0}{K}\right)\right) - \Phi\left(\delta_-^T\left(\frac{S_0}{B}\right)\right) \\
&\quad - \mathrm{e}^{-\mu^2T/2-2b^2T+(\mu+2b/T)^2T/2}\left(\Phi\left(\delta_-\left(\frac{B^2}{KS_0}\right)\right) - \Phi\left(\delta_-\left(\frac{B}{S_0}\right)\right)\right),
\end{aligned}$$

$0 \le x \le B$. Given the relation

$$-\frac{\mu^2T}{2} - 2\frac{b^2}{T} + \frac{T}{2}\left(\mu+\frac{2b}{T}\right)^2 = \left(-1+\frac{2r}{\sigma^2}\right)\log\frac{B}{S_0},$$

we get

$$\begin{aligned}
\mathrm{e}^{-rT}\,\mathbb{E}^*[C] &= \mathrm{e}^{-rT}\,\mathbb{E}^*\left[\mathbb{1}_{\{S_T\ge K\}}\mathbb{1}_{\{M_0^T\le B\}}\right] \\
&= \mathrm{e}^{-rT}\left(\Phi\left(\delta_-^T\left(\frac{S_0}{K}\right)\right) - \Phi\left(\delta_-^T\left(\frac{S_0}{B}\right)\right)\right. \\
&\quad \left. - \left(\frac{S_0}{B}\right)^{1-2r/\sigma^2}\left(\Phi\left(\delta_-^T\left(\frac{B^2}{KS_0}\right)\right) - \Phi\left(\delta_-^T\left(\frac{B}{S_0}\right)\right)\right)\right).
\end{aligned}$$

Exercise 11.6

a) For $x = B$ and $t \in [0,T]$ we check that

$$\begin{aligned}
g(t,B) = B &\left(\Phi\left(\delta_+^{T-t}\left(\frac{B}{K}\right)\right) - \Phi\left(\delta_+^{T-t}(1)\right)\right) \\
&-\mathrm{e}^{-(T-t)r}K\left(\Phi\left(\delta_-^{T-t}\left(\frac{B}{K}\right)\right) - \Phi\left(\delta_-^{T-t}(1)\right)\right) \\
&-B\left(\Phi\left(\delta_+^{T-t}\left(\frac{B}{K}\right)\right) - \Phi\left(\delta_+^{T-t}(1)\right)\right) \\
&+\mathrm{e}^{-(T-t)r}K\left(\Phi\left(\delta_-^{T-t}\left(\frac{B}{K}\right)\right) - \Phi\left(\delta_-^{T-t}(1)\right)\right) \\
= 0&,
\end{aligned}$$

and the function $g(t,x)$ is extended to $x > B$ by letting

$$g(t, x) = 0, \qquad x > B.$$

b) For $x = K$ and $t = T$, we find

$$\delta^0_{\pm}(s) = -\infty \times \mathbb{1}_{\{s<1\}} + \infty \times \mathbb{1}_{\{s>1\}} = \begin{cases} +\infty & \text{if } s > 1, \\ 0 & \text{if } s = 1, \\ -\infty & \text{if } s < 1, \end{cases}$$

hence when $\underline{x < K < B}$ we have

$$\begin{aligned} g(T, x) &= x\left(\Phi(-\infty) - \Phi(-\infty)\right) \\ &\quad -K\left(\Phi(-\infty) - \Phi(-\infty)\right) \\ &\quad -B\left(\frac{B}{x}\right)^{2r/\sigma^2}\left(\Phi(+\infty) - \Phi(+\infty)\right) \\ &\quad +K\left(\frac{B}{K}\right)^{2r/\sigma^2}\left(\Phi(+\infty) - \Phi(+\infty)\right) \\ &= 0, \end{aligned}$$

c) when $\underline{K < x < B}$, we get

$$\begin{aligned} g(T, x) &= x\left(\Phi(+\infty) - \Phi(-\infty)\right) \\ &\quad -K\left(\Phi(+\infty) - \Phi(-\infty)\right) \\ &\quad -B\left(\frac{B}{x}\right)^{2r/\sigma^2}\left(\Phi(+\infty) - \Phi(+\infty)\right) \\ &\quad +K\left(\frac{B}{K}\right)^{2r/\sigma^2}\left(\Phi(+\infty) - \Phi(+\infty)\right) \\ &= x - K. \end{aligned}$$

Finally, for $x > B$ we obtain

$$\begin{aligned} g(T, K) &= x\left(\Phi(+\infty) - \Phi(+\infty)\right) \\ &\quad -K\left(\Phi(+\infty) - \Phi(+\infty)\right) \\ &\quad -B\left(\frac{B}{x}\right)^{2r/\sigma^2}\left(\Phi(-\infty) - \Phi(-\infty)\right) \\ &\quad +K\left(\frac{B}{K}\right)^{2r/\sigma^2}\left(\Phi(-\infty) - \Phi(-\infty)\right) \\ &= 0. \end{aligned}$$

Exercise 11.7

a) The price at time $t \in [0,T]$ of the European knock-out call option is given by

$$\mathrm{EKOC}_t = \mathrm{e}^{-(T-t)r}\,\mathbb{E}^*\left[(S_T-K)^+\mathbb{1}_{\{S_T\le B\}}\,\middle|\,\mathcal{F}_t\right], \qquad 0\le t\le T.$$

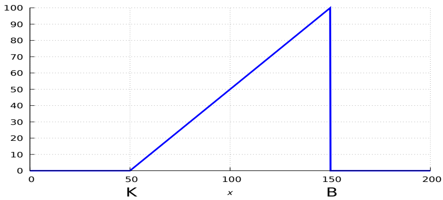


Fig. S.56: Payoff function of the European knock-out call option.

Using the relation

$$S_T = S_t \mathrm{e}^{(T-t)r+(\widehat{B}_T-\widehat{B}_t)\sigma-(T-t)\sigma^2/2}, \qquad 0\le t\le T,$$

we have

$$\begin{aligned}
&\mathrm{e}^{-(T-t)r}\,\mathbb{E}^*\left[(S_T-K)^+\mathbb{1}_{\{S_T\le B\}}\,\middle|\,\mathcal{F}_t\right]\\
&= \mathrm{e}^{-(T-t)r}\,\mathbb{E}^*\Big[\big(x\mathrm{e}^{(T-t)r+\left(\widehat{B}_T-\widehat{B}_t\right)\sigma-(T-t)\sigma^2/2}-K\big)^+\\
&\qquad \times \mathbb{1}_{\{x\mathrm{e}^{(T-t)r+(\widehat{B}_T-\widehat{B}_t)\sigma-(T-t)\sigma^2/2}\le B\}}\Big]_{x=S_t}\\
&= \mathrm{e}^{-(T-t)r}\,\mathbb{E}^*\left[(\mathrm{e}^{m(x)+X}-K)^+\mathbb{1}_{\{\mathrm{e}^{m(x)+X}\le B\}}\right]_{x=S_t}, \qquad 0\le t\le T,
\end{aligned}$$

where

$$m(x) := (T-t)r - \frac{\sigma^2}{2}(T-t) + \log x$$

and

$$X := \big(\widehat{B}_T-\widehat{B}_t\big)\sigma \simeq \mathcal{N}(0,(T-t)\sigma^2)$$

under $\mathbb{P}^*$. Next, as in Lemma 7.7 we note that if X is a centered Gaussian random variable with variance $v^2>0$ and $B\ge K$, for any $m\in\mathbb{R}$ we have

$$\begin{aligned}
&\mathbb{E}\left[(\mathrm{e}^{m+X}-K)^+\mathbb{1}_{\{\mathrm{e}^{m+X}\le B\}}\right]\\
&= \frac{1}{\sqrt{2\pi v^2}}\int_{-\infty}^{\infty}(\mathrm{e}^{m+x}-K)^+\mathbb{1}_{\{\mathrm{e}^{m+x}\le B\}}\mathrm{e}^{-x^2/(2v^2)}dx\\
&= \frac{1}{\sqrt{2\pi v^2}}\int_{-m+\log K}^{-m+\log B}(\mathrm{e}^{m+x}-K)\mathrm{e}^{-x^2/(2v^2)}dx
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\mathrm{e}^{m}}{\sqrt{2\pi v^2}} \int_{-m+\log K}^{-m+\log B} \mathrm{e}^{x-x^2/(2v^2)} dx - \frac{K}{\sqrt{2\pi v^2}} \int_{-m+\log K}^{-m+\log B} \mathrm{e}^{-x^2/(2v^2)} dx \\
&= \frac{\mathrm{e}^{m+v^2/2}}{\sqrt{2\pi v^2}} \int_{-m+\log K}^{-m+\log B} \mathrm{e}^{-(v^2-x)^2/(2v^2)} dx - \frac{K}{\sqrt{2\pi}} \int_{(-m+\log K)/v}^{(-m+\log B)/v} \mathrm{e}^{-x^2/2} dx \\
&= \frac{\mathrm{e}^{m+v^2/2}}{\sqrt{2\pi v^2}} \int_{-v^2-m+\log K}^{-v^2-m+\log B} \mathrm{e}^{-y^2/(2v^2)} dy \\
&\quad -K\big(\Phi((m-\log K)/v) - \Phi((m-\log B)/v)\big) \\
&= \mathrm{e}^{m+v^2/2}\big(\Phi(v+(m-\log K)/v) - \Phi(v+(m-\log B)/v)\big) \\
&\quad -K\big(\Phi((m-\log K)/v) - \Phi((m-\log B)/v)\big).
\end{aligned}
$$

Hence, the price of the European knock-out call option is given, if $B \geq K$, by

$$
\begin{aligned}
&\mathrm{EKOC}_t = \mathrm{e}^{-(T-t)r}\, \mathbb{E}^*\left[(S_T-K)^+ \mathbb{1}_{\{S_T \leq B\}} \,\middle|\, \mathcal{F}_t\right] \\
&= \mathrm{e}^{-(T-t)r} \mathrm{e}^{m(S_t)+\sigma^2(T-t)/2} \left(\Phi\left(v + \frac{m(S_t)-\log K}{v}\right) - \Phi\left(v + \frac{m(S_t)-\log K}{v}\right)\right) \\
&\quad - K\mathrm{e}^{-(T-t)r} \left(\Phi\left(\frac{m(S_t)-\log K}{v}\right) - \Phi\left(\frac{m(S_t)-\log B}{v}\right)\right) \\
&= S_t \Phi\left(\frac{(T-t)r + (T-t)\sigma^2/2 + \log(S_t/K)}{\sigma\sqrt{T-t}}\right) \\
&\quad - S_t \Phi\left(\frac{(T-t)r + (T-t)\sigma^2/2 + \log(S_t/B)}{\sigma\sqrt{T-t}}\right) \\
&\quad - K\mathrm{e}^{-(T-t)r} \Phi\left(\frac{(T-t)r - (T-t)\sigma^2/2 + \log(S_t/K)}{\sigma\sqrt{T-t}}\right) \\
&\quad + K\mathrm{e}^{-(T-t)r} \Phi\left(\frac{(T-t)r - (T-t)\sigma^2/2 + \log(S_t/B)}{\sigma\sqrt{T-t}}\right),
\end{aligned}
$$

$0 \leq t \leq T$, with $\mathrm{EKOC}_t = 0$ if $B \leq K$.

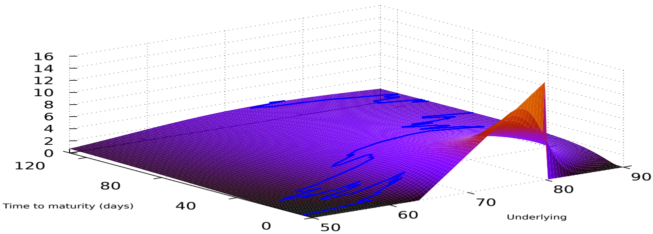


Fig. S.57: Price map of the European knock-out call option.

b) By computations similar to part (a) we find that, if $B \leq K$,

$$\begin{aligned}\mathrm{EKIP}_t = & K\mathrm{e}^{-(T-t)r}\Phi\left(-\frac{(T-t)r-(T-t)\sigma^2/2+\log(S_t/B)}{\sigma\sqrt{T-t}}\right) \\ & -S_t\Phi\left(-\frac{(T-t)r+(T-t)\sigma^2/2+\log(S_t/B)}{\sigma\sqrt{T-t}}\right).\end{aligned}$$

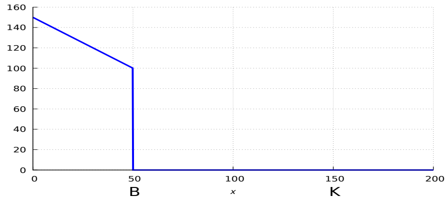


Fig. S.58: Payoff function of the European knock-in put option.

When $B \geq K$, we find the Black-Scholes put option price

$$\begin{aligned}\mathrm{EKIP}_t & = \mathrm{e}^{-(T-t)r}\,\mathbb{E}^*\left[(K-S_T)^+\mathbb{1}_{\{S_T\leq B\}}\,\middle|\,\mathcal{F}_t\right] \\ & = \mathrm{e}^{-(T-t)r}\,\mathbb{E}^*\left[(K-S_T)^+\,\middle|\,\mathcal{F}_t\right] \\ & = K\mathrm{e}^{-(T-t)r}\Phi\left(-\frac{(T-t)r-(T-t)\sigma^2/2+\log(S_t/K)}{\sigma\sqrt{T-t}}\right) \\ & \quad - S_t\Phi\left(-\frac{(T-t)r+(T-t)\sigma^2/2+\log(S_t/K)}{\sigma\sqrt{T-t}}\right),\end{aligned}$$

$0 \leq t \leq T$.

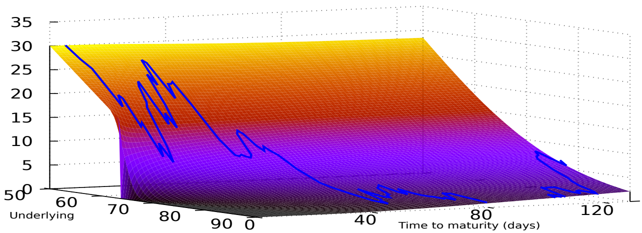


Fig. S.59: Price map of the European knock-in put option.

c) Using the in-out parity relation

$$\mathrm{EKOC}_t + \mathrm{EKIC}_t = \mathrm{e}^{-(T-t)r}\,\mathbb{E}^*[(S_T-K)^+ \mid \mathcal{F}_t]$$

$$= S_t \Phi\left(\frac{(T-t)r + (T-t)\sigma^2/2 + \log(S_t/K)}{\sigma\sqrt{T-t}}\right)$$
$$-K\mathrm{e}^{-(T-t)r}\Phi\left(\frac{(T-t)r - (T-t)\sigma^2/2 + \log(S_t/K)}{\sigma\sqrt{T-t}}\right),$$

which is the price of the European call put option with strike price K, the price at time $t \in [0, T]$ of the European knock-in call option is given, if $B \geq K$, as

$$\begin{aligned}
\mathrm{EKIC}_t &= \mathrm{e}^{-(T-t)r}\,\mathbb{E}^*\left[(S_T - K)^+ \mathbb{1}_{\{S_T \geq B\}} \,\middle|\, \mathcal{F}_t\right] \\
&= S_t \Phi\left(\frac{(T-t)r + (T-t)\sigma^2/2 + \log(S_t/B)}{\sigma\sqrt{T-t}}\right) \\
&\quad -K\mathrm{e}^{-(T-t)r}\Phi\left(\frac{(T-t)r - (T-t)\sigma^2/2 + \log(S_t/B)}{\sigma\sqrt{T-t}}\right),
\end{aligned}$$

$0 \leq t \leq T$.

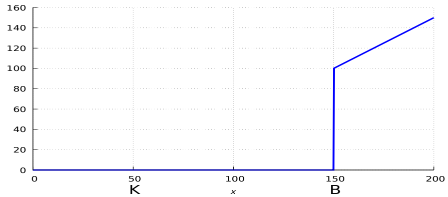


Fig. S.60: Payoff function of the European knock-in call option.

When $B \leq K$, we find the Black-Scholes call option price

$$\begin{aligned}
\mathrm{EKIC}_t &= \mathrm{e}^{-(T-t)r}\,\mathbb{E}^*\left[(S_T - K)^+ \mathbb{1}_{\{S_T \geq B\}} \,\middle|\, \mathcal{F}_t\right] \\
&= \mathrm{e}^{-(T-t)r}\,\mathbb{E}^*\left[(S_T - K)^+ \,\middle|\, \mathcal{F}_t\right] \\
&= S_t \Phi\left(\frac{(T-t)r + (T-t)\sigma^2/2 + \log(S_t/K)}{\sigma\sqrt{T-t}}\right) \\
&\quad -K\mathrm{e}^{-(T-t)r}\Phi\left(\frac{(T-t)r - (T-t)\sigma^2/2 + \log(S_t/K)}{\sigma\sqrt{T-t}}\right).
\end{aligned}$$

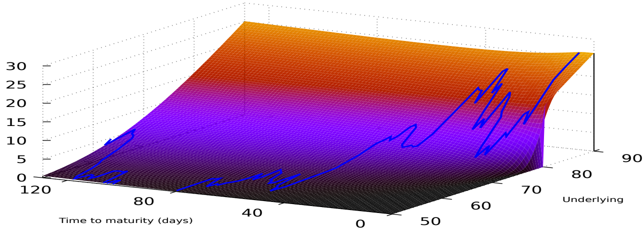


Fig. S.61: Price map of the European knock-in call option.

d) Using the in-out parity relation

$$\mathrm{EKOP}_t + \mathrm{EKIP}_t = \mathrm{e}^{-(T-t)r}\, \mathbb{E}^*[(K - S_T)^+ \mid \mathcal{F}_t],$$

which is the price of the European put option with strike price K, we find that the price at time $t \in [0, T]$ of the European knock-in put option is given, if $B \leq K$, as

$$\begin{aligned}
\mathrm{EKOP}_t &= \mathrm{e}^{-(T-t)r}\, \mathbb{E}^*\left[(K - S_T)^+ \mathbb{1}_{\{S_T \geq B\}} \,\middle|\, \mathcal{F}_t\right] \\
&= \mathrm{e}^{-(T-t)r}\, \mathbb{E}^*[(K - S_T)^+ \mid \mathcal{F}_t] - \mathrm{EKIP}_t \\
&= K\mathrm{e}^{-(T-t)r}\Phi\left(-\frac{(T-t)r - (T-t)\sigma^2/2 + \log(S_t/K)}{\sigma\sqrt{T-t}}\right) \\
&\quad -S_t\Phi\left(-\frac{(T-t)r + (T-t)\sigma^2/2 + \log(S_t/K)}{\sigma\sqrt{T-t}}\right) \\
&\quad +S_t\Phi\left(-\frac{(T-t)r + (T-t)\sigma^2/2 + \log(S_t/B)}{\sigma\sqrt{T-t}}\right) \\
&\quad -K\mathrm{e}^{-(T-t)r}\Phi\left(-\frac{(T-t)r - (T-t)\sigma^2/2 + \log(S_t/B)}{\sigma\sqrt{T-t}}\right).
\end{aligned}$$

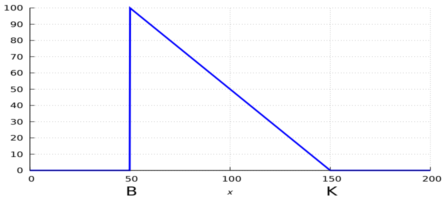


Fig. S.62: Payoff function of the European knock-out put option.

When $B \geq K$, we have

$$\mathrm{EKIP}_t = \mathrm{e}^{-(T-t)r}\, \mathbb{E}^*\left[(K - S_T)^+ \mathbb{1}_{\{S_T \geq B\}} \,\middle|\, \mathcal{F}_t \right] = 0.$$

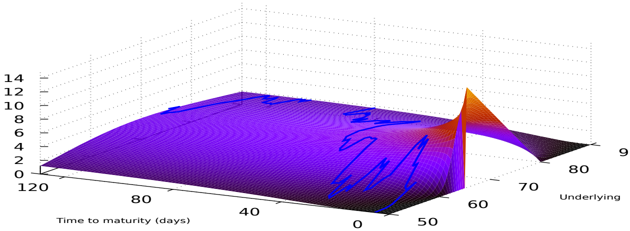


Fig. S.63: Price map of the European knock-out put option.

In addition, by the results of Questions (d) and (c) we can verify the call-put parity relation

$$\begin{aligned}
&\mathrm{EKIC}_t - \mathrm{EKIP}_t = \mathrm{e}^{-(T-t)r}\, \mathbb{E}^*\left[(S_T - K) \mathbb{1}_{\{S_T \geq B\}} \,\middle|\, \mathcal{F}_t \right] \\
&= S_t \Phi\left(\frac{(T-t)r + (T-t)\sigma^2/2 + \log(S_t/B)}{\sigma\sqrt{T-t}} \right) \\
&\quad - K\mathrm{e}^{-(T-t)r} \Phi\left(\frac{(T-t)r - (T-t)\sigma^2/2 + \log(S_t/B)}{\sigma\sqrt{T-t}} \right).
\end{aligned}$$

Chapter 12

Exercise 12.1

a) This probability density function is given by

$$x \longmapsto \sqrt{\frac{2}{\pi T}} \mathrm{e}^{-(x-\sigma T/2)^2/(2T)} - \sigma \mathrm{e}^{\sigma x} \Phi\left(\frac{-x - \sigma T/2}{\sqrt{T}} \right), \qquad x \geq 0.$$

b) We have

$$\begin{aligned}
&\mathbb{E}\left[\min_{t\in[0,T]} S_t \right] = S_0\, \mathbb{E}\left[\min_{t\in[0,T]} \mathrm{e}^{\sigma B_t - \sigma^2 t/2} \right] \\
&= S_0\, \mathbb{E}\left[\mathrm{e}^{-\sigma \max_{t\in[0,T]}(B_t + \sigma t/2)} \right] \\
&= S_0 \int_0^\infty \mathrm{e}^{-\sigma x} \left(\sqrt{\frac{2}{\pi T}} \mathrm{e}^{-(x-\sigma T/2)^2/(2T)} - \sigma \mathrm{e}^{\sigma x} \Phi\left(\frac{-x - \sigma T/2}{\sqrt{T}} \right) \right) dx
\end{aligned}$$

$$
\begin{aligned}
&= \frac{2S_0}{\sqrt{2\pi T}} \int_0^\infty \mathrm{e}^{-(x+\sigma T/2)^2/(2T)} dx - S_0\sigma \int_0^\infty \Phi\left(\frac{-x-\sigma T/2}{\sqrt{T}}\right) dx \\
&= \frac{2S_0}{\sqrt{2\pi T}} \int_{\sigma T/2}^\infty \mathrm{e}^{-x^2/(2T)} dx - \frac{\sigma S_0}{\sqrt{2\pi T}} \int_0^\infty x\mathrm{e}^{-(x+\sigma T/2)^2/(2T)} dx \\
&= \frac{2S_0}{\sqrt{2\pi T}} \int_{\sigma T/2}^\infty \mathrm{e}^{-x^2/(2T)} dx - \frac{\sigma S_0}{\sqrt{2\pi T}} \int_{\sigma T/2}^\infty (x-\sigma T/2)\mathrm{e}^{-x^2/(2T)} dx \\
&= 2S_0(1+\sigma^2 T/4)\Phi(-\sigma\sqrt{T}/2) - S_0\sigma\sqrt{\frac{T}{2\pi}}\mathrm{e}^{-\sigma^2 T/8}. \qquad \text{(S.12.54)}
\end{aligned}
$$

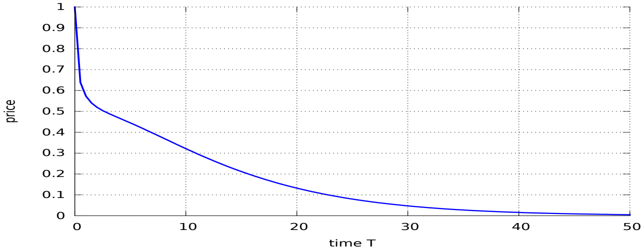


Fig. S.64: Expected minimum of geometric Brownian motion over $[0, T]$.

c) We have

$$
\begin{aligned}
\mathbb{E}\left[\left(K - \min_{t\in[0,T]} S_t\right)^+\right] &= \mathbb{E}\left[K - \min_{t\in[0,T]} S_t\right] \\
&= K - S_0\left(2\left(1+\frac{\sigma^2 T}{4}\right)\Phi\left(-\frac{\sigma\sqrt{T}}{2}\right) - \sigma\sqrt{\frac{T}{2\pi}}\mathrm{e}^{-\sigma^2 T/8}\right).
\end{aligned}
$$

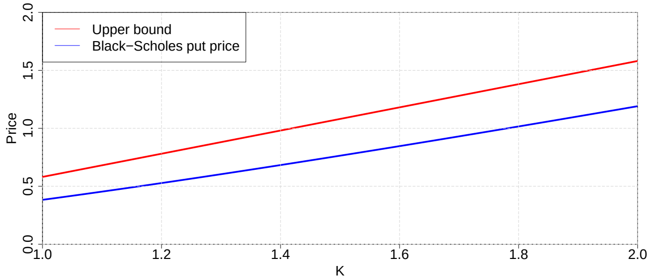


Fig. S.65: Black-Scholes put price upper bound with $S_0 = 1$.

The derivative with respect to time is given by

$$\frac{\partial}{\partial T}\mathbb{E}\left[\min_{t\in[0,T]} S_t\right] = S_0\frac{\sigma^2}{2}\Phi\big(-\sigma\sqrt{T}/2\big) - 2S_0\left(1+\frac{\sigma^2 T}{4}\right)\frac{\sigma}{4\sqrt{2\pi T}}\mathrm{e}^{-\sigma^2 T/8}$$
$$-\frac{\sigma S_0}{\sqrt{8\pi T}}\mathrm{e}^{-\sigma^2 T/8} + \frac{S_0\sigma^3}{8}\sqrt{\frac{T}{2\pi}}\mathrm{e}^{-\sigma^2 T/8}$$
$$= \frac{S_0\sigma^2}{2}\Phi\left(-\sigma\frac{\sqrt{T}}{2}\right) - \frac{S_0\sigma}{\sqrt{2\pi T}}\mathrm{e}^{-\sigma^2 T/8}\left(1+\frac{3\sigma^2 T}{4}\right).$$

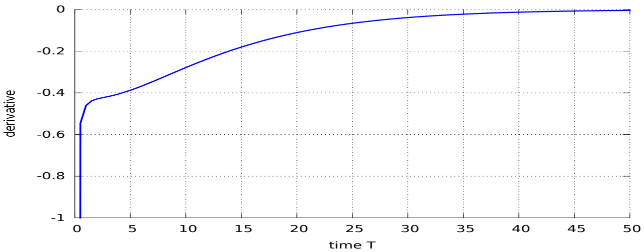


Fig. S.66: Time derivative of the expected minimum of geometric Brownian motion.

On the other hand, when $r > 0$ we have

$$\mathbb{E}^*\left[m_0^T \mid \mathcal{F}_t\right] = m_0^t\Phi\left(\delta_-^{T-t}\left(\frac{S_t}{m_0^t}\right)\right) - S_t\frac{\sigma^2}{2r}\left(\frac{m_0^t}{S_t}\right)^{2r/\sigma^2}\Phi\left(\delta_-^{T-t}\left(\frac{m_0^t}{S_t}\right)\right)$$
$$+S_t\mathrm{e}^{(T-t)r}\left(1+\frac{\sigma^2}{2r}\right)\Phi\left(-\delta_+^{T-t}\left(\frac{S_t}{m_0^t}\right)\right).$$

When r tends to 0, this minimum tends to

$$m_0^t\Phi\left(\frac{\log(S_t/m_0^t)-\sigma^2T/2}{\sigma\sqrt{T}}\right) + S_t\Phi\left(-\frac{\log(S_t/m_0^t)+\sigma^2T/2}{\sigma\sqrt{T}}\right)$$
$$+\sigma^2 S_t\lim_{r\to 0}\frac{1}{2r}\left(\mathrm{e}^{(T-t)r}\Phi\left(-\delta_+^{T-t}\left(\frac{S_t}{m_0^t}\right)\right) - \left(\frac{m_0^t}{S_t}\right)^{2r/\sigma^2}\Phi\left(\delta_-^{T-t}\left(\frac{m_0^t}{S_t}\right)\right)\right),$$

where

$$\lim_{r\to 0}\frac{1}{2r}\left(\mathrm{e}^{(T-t)r}\Phi\left(-\delta_+^{T-t}\left(\frac{S_t}{m_0^t}\right)\right) - \left(\frac{m_0^t}{S_t}\right)^{2r/\sigma^2}\Phi\left(\delta_-^{T-t}\left(\frac{m_0^t}{S_t}\right)\right)\right)$$
$$= \lim_{r\to 0}\frac{1}{2r}\left((1+(T-t)r)\Phi\left(-\frac{\log(S_t/m_0^t)+\sigma^2T/2+rT}{\sigma\sqrt{T}}\right)\right.$$

$$
\begin{aligned}
& - \left(1+\frac{2r}{\sigma^2}\log\frac{m_0^t}{S_t}\right)\Phi\left(\frac{\log(m_0^t/S_t)-\sigma^2T/2+rT}{\sigma\sqrt{T}}\right)\Bigg)\\
&= \frac{1}{2}\left(T-t+\frac{2}{\sigma^2}\log\frac{m_0^t}{S_t}\right)\Phi\left(-\frac{\log(S_t/m_0^t)+\sigma^2T/2}{\sigma\sqrt{T}}\right)\\
&\quad + \lim_{r\to 0}\frac{1}{r\sqrt{8\pi}}\left(\int_{-\infty}^{-(\log(S_t/m_0^t)+\sigma^2T/2+rT)/(\sigma\sqrt{T})}\mathrm{e}^{-y^2/2}dy\right.\\
&\quad \left. - \int_{-\infty}^{(-\log(S_t/m_0^t)-\sigma^2T/2+rT)/(\sigma\sqrt{T})}\mathrm{e}^{-y^2/2}dy\right)\\
&= \frac{1}{2}\left(T-t+\frac{2}{\sigma^2}\log\frac{m_0^t}{S_t}\right)\Phi\left(-\frac{\log(S_t/m_0^t)+\sigma^2T/2}{\sigma\sqrt{T}}\right)\\
&\quad - \lim_{r\to 0}\frac{1}{r\sqrt{8\pi}}\int_{(-\log(S_t/m_0^t)-\sigma^2T/2-rT)/(\sigma\sqrt{T})}^{(-\log(S_t/m_0^t)-\sigma^2T/2+rT)/(\sigma\sqrt{T})}\mathrm{e}^{-y^2/2}dy\\
&= \frac{1}{2}\left(T-t+\frac{2}{\sigma^2}\log\frac{m_0^t}{S_t}\right)\Phi\left(-\frac{\log(S_t/m_0^t)+\sigma^2T/2}{\sigma\sqrt{T}}\right)\\
&\quad -\frac{\sqrt{T}}{\sigma\sqrt{2\pi}}\mathrm{e}^{-((\log(S_t/m_0^t)+\sigma^2T/2)/(\sigma\sqrt{T}))^2/2},
\end{aligned}
$$

hence

$$
\begin{aligned}
\mathbb{E}^*\left[m_0^T \mid \mathcal{F}_t\right] &= m_0^t\Phi\left(\frac{\log(S_t/m_0^t)-\sigma^2T/2}{\sigma\sqrt{T}}\right)+S_t\Phi\left(-\frac{\log(S_t/m_0^t)+\sigma^2T/2}{\sigma\sqrt{T}}\right)\\
&\quad +\frac{S_t}{2}\left((T-t)\sigma^2+2\log\frac{m_0^t}{S_t}\right)\Phi\left(-\frac{\log(S_t/m_0^t)+\sigma^2T/2}{\sigma\sqrt{T}}\right)\\
&\quad -\sigma S_t\sqrt{\frac{T}{2\pi}}\mathrm{e}^{-((\log(S_t/m_0^t)+\sigma^2T/2)/(\sigma\sqrt{T}))^2/2}.
\end{aligned}
$$

In particular, when T tends to infinity we find that

$$
\lim_{T\to\infty}\frac{\mathbb{E}^*\left[m_0^T \mid \mathcal{F}_t\right]}{\mathbb{E}^*[S_T \mid \mathcal{F}_t]} = 0, \qquad r \geq 0.
$$

When $t=0$ we have $S_0 = m_0^0$, and we recover

$$
\mathbb{E}^*\left[m_0^T\right] = 2\left(S_0+\frac{\sigma^2T}{4}\right)\Phi\left(-\sigma\frac{\sqrt{T}}{2}\right)-\sigma S_0\sqrt{\frac{T}{2\pi}}\mathrm{e}^{-\sigma^2T/8}.
$$

Exercise 12.2

a) By (S.12.54), we have

$$
\mathbb{E}\left[\max_{t\in[0,T]} S_t\right] = \mathbb{E}\left[\mathrm{e}^{\sigma\max_{t\in[0,T]}(B_t-\sigma t/2)}\right]
$$

$$
\begin{aligned}
&= S_0 \, \mathbb{E}\left[\mathrm{e}^{-(-\sigma)\max_{t\in[0,T]}(B_t-(-\sigma)t/2)} \right] \\
&= 2S_0(1+\sigma^2T/4)\Phi(\sigma\sqrt{T}/2) + S_0\sigma\sqrt{\frac{T}{2\pi}}\mathrm{e}^{-\sigma^2T/8}.
\end{aligned}
$$

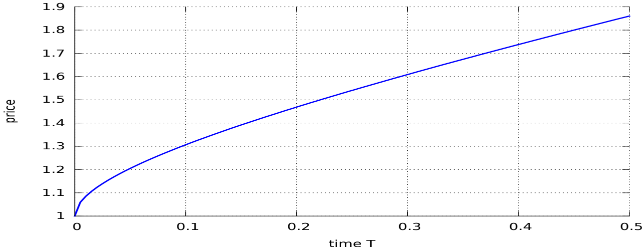


Fig. S.67: Expected maximum of geometric Brownian motion over $[0, T]$.

b) We have

$$
\begin{aligned}
\mathbb{E}\left[\left(S_0 \max_{t\in[0,T]} \mathrm{e}^{\sigma B_t - \sigma^2 t/2} - K \right)^+ \right] &= \mathbb{E}\left[S_0 \max_{t\in[0,T]} \mathrm{e}^{\sigma B_t - \sigma^2 t/2} \right] - K \\
&= 2S_0\left(1+\frac{\sigma^2T}{4}\right)\Phi\left(\sigma\frac{\sqrt{T}}{2}\right) + S_0\sigma\sqrt{\frac{T}{2\pi}}\mathrm{e}^{-\sigma^2T/8} - K.
\end{aligned}
$$

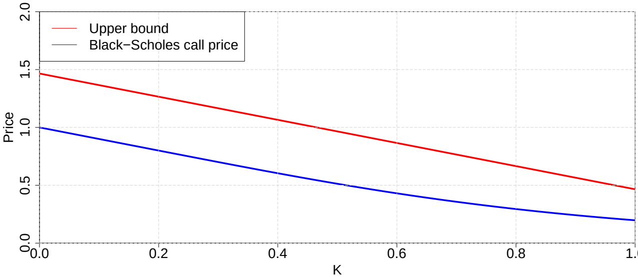


Fig. S.68: Black-Scholes call price upper bound with $S_0 = 1$.

The derivative with respect to time is given by

$$
\frac{\partial}{\partial T}\mathbb{E}\left[\max_{t\in[0,T]} S_t \right] = \frac{S_0\sigma^2}{2}\Phi\left(\sigma\frac{\sqrt{T}}{2}\right) + \frac{S_0\sigma}{\sqrt{2\pi T}}\mathrm{e}^{-\sigma^2T/8}\left(1+\frac{3\sigma^2T}{4}\right).
$$

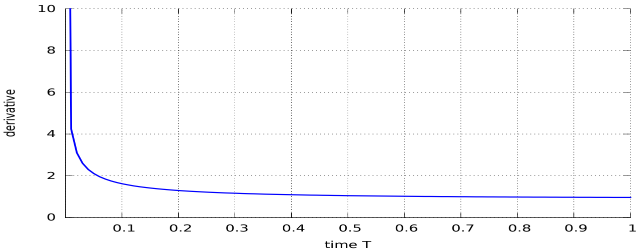


Fig. S.69: Time derivative of the expected maximum of geometric Brownian motion.

Note that when $r > 0$ we have

$$\mathbb{E}^*\left[M_0^T \mid \mathcal{F}_t\right] = M_0^t \Phi\left(-\delta_-^{T-t}\left(\frac{S_t}{M_0^t}\right)\right) + S_t \mathrm{e}^{(T-t)r}\left(1+\frac{\sigma^2}{2r}\right)\Phi\left(\delta_+^{T-t}\left(\frac{S_t}{M_0^t}\right)\right)$$
$$-S_t\frac{\sigma^2}{2r}\left(\frac{M_0^t}{S_t}\right)^{2r/\sigma^2}\Phi\left(-\delta_-^{T-t}\left(\frac{M_0^t}{S_t}\right)\right).$$

When r tends to 0, this maximum tends to

$$M_0^t\Phi\left(-\frac{\log(S_t/M_0^t)-\sigma^2T/2}{\sigma\sqrt{T}}\right) + S_t\Phi\left(\frac{\log(S_t/M_0^t)+\sigma^2T/2}{\sigma\sqrt{T}}\right)$$
$$+\sigma^2 S_t \lim_{r\to 0}\frac{1}{2r}\left(\mathrm{e}^{(T-t)r}\Phi\left(\delta_+^{T-t}\left(\frac{S_t}{M_0^t}\right)\right) - \left(\frac{M_0^t}{S_t}\right)^{2r/\sigma^2}\Phi\left(-\delta_-^{T-t}\left(\frac{M_0^t}{S_t}\right)\right)\right),$$

where

$$\begin{aligned}
&\lim_{r\to 0}\frac{1}{2r}\left(\mathrm{e}^{(T-t)r}\Phi\left(\delta_+^{T-t}\left(\frac{S_t}{M_0^t}\right)\right) - \left(\frac{M_0^t}{S_t}\right)^{2r/\sigma^2}\Phi\left(-\delta_-^{T-t}\left(\frac{M_0^t}{S_t}\right)\right)\right)\\
&\quad = \lim_{r\to 0}\frac{1}{2r}\left((1+(T-t)r)\Phi\left(\frac{\log\left(\frac{S_t}{M_0^t}\right)+\frac{\sigma^2}{2}T+rT}{\sigma\sqrt{T}}\right)\right.\\
&\qquad \left. - \left(1+\frac{2r}{\sigma^2}\log\frac{M_0^t}{S_t}\right)\Phi\left(\frac{\log(S_t/M_0^t)+\sigma^2T/2-rT}{\sigma\sqrt{T}}\right)\right)\\
&\quad = \frac{1}{2}\left(T-t+\frac{2}{\sigma^2}\log\frac{M_0^t}{S_t}\right)\Phi\left(\frac{\log(S_t/M_0^t)+\sigma^2T/2}{\sigma\sqrt{T}}\right)\\
&\qquad + \lim_{r\to 0}\frac{1}{r\sqrt{8\pi}}\left(\int_{-\infty}^{(\log(S_t/M_0^t)+\sigma^2T/2+rT)/(\sigma\sqrt{T})}\mathrm{e}^{-y^2/2}dy\right.\\
&\qquad \left. -\int_{-\infty}^{(\log(S_t/M_0^t)+\sigma^2T/2-rT)/(\sigma\sqrt{T})}\mathrm{e}^{-y^2/2}dy\right)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\left(T-t+\frac{2}{\sigma^2}\log\frac{M_0^t}{S_t}\right)\Phi\left(\frac{\log(S_t/M_0^t)+\sigma^2T/2}{\sigma\sqrt{T}}\right)\\
&\quad+\lim_{r\to 0}\frac{1}{r\sqrt{8\pi}}\int_{(\log(S_t/M_0^t)+\sigma^2T/2-rT)/(\sigma\sqrt{T})}^{(\log(S_t/M_0^t)+\sigma^2T/2+rT)/(\sigma\sqrt{T})}\mathrm{e}^{-y^2/2}dy\\
&= \frac{1}{2}\left(T-t+\frac{2}{\sigma^2}\log\frac{M_0^t}{S_t}\right)\Phi\left(\frac{\log(S_t/M_0^t)+\sigma^2T/2}{\sigma\sqrt{T}}\right)\\
&\quad+\frac{\sqrt{T}}{\sigma\sqrt{2\pi}}\mathrm{e}^{-((\log(S_t/M_0^t)+\sigma^2T/2)/(\sigma\sqrt{T}))^2/2},
\end{aligned}
$$

hence

$$
\begin{aligned}
\mathbb{E}^*\left[M_0^T \mid \mathcal{F}_t\right] &= M_0^t\Phi\left(-\frac{\log(S_t/M_0^t)-\sigma^2T/2}{\sigma\sqrt{T}}\right)+S_t\Phi\left(\frac{\log(S_t/M_0^t)+\sigma^2T/2}{\sigma\sqrt{T}}\right)\\
&\quad+\frac{S_t}{2}\left((T-t)\sigma^2+2\log\frac{M_0^t}{S_t}\right)\Phi\left(\frac{\log(S_t/M_0^t)+\sigma^2T/2}{\sigma\sqrt{T}}\right)\\
&\quad+\sigma S_t\sqrt{\frac{T}{2\pi}}\mathrm{e}^{-((\log(S_t/M_0^t)+\sigma^2T/2)/(\sigma\sqrt{T}))^2/2}.
\end{aligned}
$$

In particular, when T tends to infinity we find that

$$
\lim_{T\to\infty}\frac{\mathbb{E}^*\left[M_0^T \mid \mathcal{F}_t\right]}{\mathbb{E}^*[S_T \mid \mathcal{F}_t]} = \begin{cases} 1+\dfrac{\sigma^2}{2r} & \text{if } r>0,\\ \\ +\infty & \text{if } r=0. \end{cases}
$$

When $t=0$ we have $S_0=M_0^0$, and we recover

$$
\mathbb{E}^*\left[M_0^T\right] = 2\left(S_0+\frac{\sigma^2T}{4}\right)\Phi\left(\sigma\frac{\sqrt{T}}{2}\right)+\sigma S_0\sqrt{\frac{T}{2\pi}}\mathrm{e}^{-\sigma^2T/8}.
$$

Exercise 12.3

a) We have

$$
P\left(\min_{t\in[0,T]}B_t\leq a\right) = 2\int_{-\infty}^{a}\mathrm{e}^{-x^2/(2T)}\frac{dx}{\sqrt{2\pi T}}, \qquad a<0,
$$

i.e. the probability density function φ of $\sup_{t\in[0,T]} B_t$ is given by

$$
\varphi(a) = \sqrt{\frac{2}{\pi T}}\mathrm{e}^{-a^2/(2T)}\mathbb{1}_{(-\infty,0]}(a), \qquad a\in\mathbb{R}.
$$

b) We have

$$
\begin{aligned}
&\mathbb{E}\left[\min_{t\in[0,T]} S_t\right] = S_0\,\mathbb{E}\left[\exp\left(\sigma \min_{t\in[0,T]} B_t\right)\right] \\
&= \frac{2S_0}{\sqrt{2\pi T}}\int_{-\infty}^0 \mathrm{e}^{\sigma x - x^2/(2T)}dx = \frac{2S_0}{\sqrt{2\pi\sigma^2 T}}\int_{-\infty}^0 \mathrm{e}^{-(x-\sigma T)^2/(2T)+\sigma^2 T/2}dx \\
&= \frac{2S_0}{\sqrt{2\pi T}}\mathrm{e}^{\sigma^2 T/2}\int_{-\infty}^{-\sigma T} \mathrm{e}^{-x^2/(2T)}dx = \frac{2S_0}{\sqrt{2\pi}}\mathrm{e}^{\sigma^2 T/2}\int_{-\infty}^{-\sigma\sqrt{T}} \mathrm{e}^{-x^2/2}dx \\
&= 2S_0\mathrm{e}^{\sigma^2 T/2}\Phi\big(-\sigma\sqrt{T}\big) = 2\,\mathbb{E}[S_T]\big(1-\Phi\big(\sigma\sqrt{T}\big)\big),
\end{aligned}
$$

hence

$$
\begin{aligned}
\mathbb{E}\left[S_T - \min_{t\in[0,T]} S_t\right] &= \mathbb{E}[S_T] - \mathbb{E}\left[\min_{t\in[0,T]} S_t\right] = \mathbb{E}[S_T] - 2\,\mathbb{E}[S_T]\left(1-\Phi\left(\sigma\sqrt{T}\right)\right) \\
&= \mathbb{E}[S_T]\left(2\Phi\left(\sigma\sqrt{T}\right)-1\right) = 2S_0\mathrm{e}^{\sigma^2 T/2}\left(\Phi\left(\sigma\sqrt{T}\right)-\frac{1}{2}\right),
\end{aligned}
$$

and

$$
\mathrm{e}^{-\sigma^2 T/2}\,\mathbb{E}\left[S_T - \min_{t\in[0,T]} S_t\right] = S_0\left(2\Phi\left(\sigma\sqrt{T}\right)-1\right) = S_0\left(1-2\Phi\left(-\sigma\sqrt{T}\right)\right).
$$

Remark: We note that the price of the lookback option converges to S_0 as T goes to infinity.

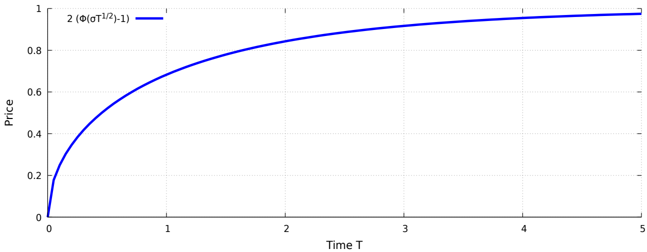


Fig. S.70: Lookback call option price as a function of T with $S_0 = 1$.

Exercise 12.4 We have

$$
\begin{aligned}
&\mathbb{E}^*\left[\mathrm{e}^{-r\tau}\mathbb{1}_{\{\tau\le T\}}\mathbb{1}_{\{M_0^\tau - S_\tau \ge K\}}\right] \\
&= \int_1^T\int_0^\infty\int_{K+x}^\infty \mathrm{e}^{-rt} f_{(\tau,S_\tau,M_\tau)}(t,x,y)dydxdt \\
&= \int_1^T\int_K^\infty\int_0^{y-K} \mathrm{e}^{-rt} f_{(\tau,S_\tau,M_\tau)}(t,x,y)dxdydt
\end{aligned}
$$

for $T \geq 1$, and $\mathbb{E}^*\left[\mathrm{e}^{-r\tau}\mathbb{1}_{\{\tau\leq T\}}\mathbb{1}_{\{M_0^\tau - S_\tau \geq K\}}\right] = 0$ if $T \in [0,1]$.

Exercise 12.5

a) i) The boundary condition (12.3a) is explained by the fact that

$$\begin{aligned} f(t,0,y) &= \mathrm{e}^{-(T-t)r}\,\mathbb{E}^*\left[M_0^T - S_T \,\middle|\, S_t = 0,\ M_0^t = y\right] \\ &= \mathrm{e}^{-(T-t)r}\,\mathbb{E}^*\left[M_0^t - S_T \,\middle|\, S_t = 0,\ M_0^t = y\right] \\ &= \mathrm{e}^{-(T-t)r}\,\mathbb{E}^*\left[M_0^t \,\middle|\, M_0^t = y\right] - \mathrm{e}^{-(T-t)r}\,\mathbb{E}^*[S_T \mid S_t = 0] \\ &= y\mathrm{e}^{-(T-t)r}, \end{aligned}$$

since $\mathbb{E}^*[S_T \mid S_t = 0] = 0$ as $S_t = 0$ implies $S_T = 0$ from the relation

$$S_T = S_t \mathrm{e}^{\sigma(B_T - B_t) + (\mu - \sigma^2/2)(T-t)}, \qquad 0 \leq t \leq T.$$

ii) The boundary condition (12.3b), *i.e.*

$$\frac{\partial f}{\partial y}(t,x,y)_{y=x} = 0, \qquad 0 \leq x \leq y,$$

is illustrated in the following Figure S.71, see also Figure 12.3.

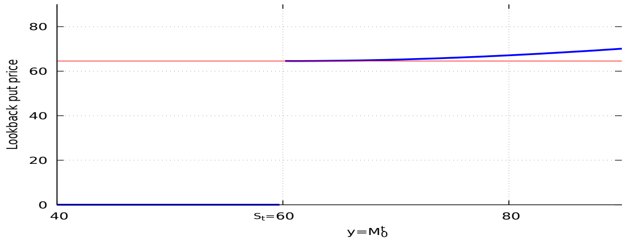


Fig. S.71: Graph of the lookback put option price (2D) with $S_t = 60$.

iii) Condition (12.3c) follows from the fact that

$$f(T,x,y) = \mathbb{E}^*\left[M_0^T - S_T \,\middle|\, S_T = x,\ M_0^T = y\right] = y - x.$$

b) i) The boundary condition (12.14a) is explained by the fact that

$$\begin{aligned} f(t,x,0) &= \mathrm{e}^{-(T-t)r}\,\mathbb{E}^*\left[S_T - m_0^T \,\middle|\, S_t = x,\ m_0^t = 0\right] \\ &= \mathrm{e}^{-(T-t)r}\,\mathbb{E}^*\left[S_T \,\middle|\, S_t = x,\ m_0^t = 0\right] \\ &= \mathrm{e}^{-(T-t)r}\,\mathbb{E}^*[S_T \mid S_t = x] \end{aligned}$$

$$= \mathrm{e}^{-(T-t)r}x, \qquad x > 0.$$

ii) Condition (12.14b) follows from the fact that

$$f(T,x,y) = \mathbb{E}^*\left[S_T - m_0^T \,\middle|\, S_T = x,\ m_0^T = y\right] = x - y.$$

We have

$$f(t,x,x) = xC(T-t),$$

with

$$\begin{aligned} C(\tau) &= 1 - \mathrm{e}^{-r\tau}\Phi\big(\delta_-^\tau(1)\big) \\ &\quad - \left(1 + \frac{\sigma^2}{2r}\right)\Phi\big(-\delta_+^\tau(1)\big) + \mathrm{e}^{-r\tau}\frac{\sigma^2}{2r}\Phi\big(\delta_-^\tau(1)\big), \quad \tau > 0, \end{aligned}$$

hence

$$\frac{\partial f}{\partial x}(t,x,x) = C(T-t), \qquad 0 \le t \le T,$$

while we also have

$$\frac{\partial f}{\partial y}(t,x,y)_{y=x} = 0, \qquad 0 \le x \le y,$$

see also Figure 12.8.

Chapter 13

Exercise 13.1 We have

$$\begin{aligned} \mathbb{E}\left[\int_\tau^T S_t dt\right] &= \int_\tau^T \mathbb{E}[S_t]dt \\ &= S_0\int_\tau^T \mathbb{E}[\mathrm{e}^{\sigma B_t + rt - \sigma^2 t/2}]dt \\ &= S_0\int_\tau^T \mathrm{e}^{rt-\sigma^2 t/2}\,\mathbb{E}[\mathrm{e}^{\sigma B_t}]dt \\ &= S_0\int_\tau^T \mathrm{e}^{rt}dt \\ &= S_0\frac{\mathrm{e}^{rT} - \mathrm{e}^{r\tau}}{r}, \qquad 0 \le \tau \le T, \end{aligned}$$

and

$$\mathbb{E}\left[\left(\int_\tau^T S_t dt\right)^2\right] = \mathbb{E}\left[\int_\tau^T S_t dt \int_\tau^T S_u du\right]$$

$$
\begin{aligned}
&= \mathbb{E}\left[\int_\tau^T \int_\tau^T S_u S_t dt du\right] \\
&= 2\int_\tau^T \int_\tau^u \mathbb{E}[S_u S_t] dt du \\
&= 2S_0^2 \int_\tau^T \int_\tau^u \mathbb{E}[\mathrm{e}^{\sigma B_u + ru - \sigma^2 u/2} \mathrm{e}^{\sigma B_t + rt - \sigma^2 t/2}] dt du \\
&= 2S_0^2 \int_\tau^T \int_\tau^u \mathrm{e}^{ru - \sigma^2 u/2 + rt - \sigma^2 t/2}\, \mathbb{E}[\mathrm{e}^{\sigma B_u + \sigma B_t}] dt du \\
&= 2S_0^2 \int_\tau^T \mathrm{e}^{(r-\sigma^2/2)u} \int_\tau^u \mathrm{e}^{(r-\sigma^2/2)t}\, \mathbb{E}[\mathrm{e}^{\sigma B_u + \sigma B_t}] dt du \\
&= 2S_0^2 \int_\tau^T \mathrm{e}^{(r-\sigma^2/2)u} \int_\tau^u \mathrm{e}^{(r-\sigma^2/2)t}\, \mathbb{E}[\mathrm{e}^{2\sigma B_t + \sigma(B_u - B_t)}] dt du \\
&= 2S_0^2 \int_\tau^T \mathrm{e}^{(r-\sigma^2/2)u} \int_\tau^u \mathrm{e}^{(r-\sigma^2/2)t}\, \mathbb{E}[\mathrm{e}^{2\sigma B_t}]\, \mathbb{E}[\mathrm{e}^{\sigma(B_u - B_t)}] dt du \\
&= 2S_0^2 \int_\tau^T \mathrm{e}^{(r-\sigma^2/2)u} \int_\tau^u \mathrm{e}^{(r-\sigma^2/2)t} \mathrm{e}^{2\sigma^2 t} \mathrm{e}^{\sigma^2(u-t)/2} dt du \\
&= 2S_0^2 \int_\tau^T \mathrm{e}^{ru} \int_\tau^u \mathrm{e}^{rt + \sigma^2 t} dt du \\
&= \frac{2S_0^2}{\sigma^2 + r} \int_\tau^T \left(\mathrm{e}^{(2r+\sigma^2)u} - \mathrm{e}^{ru} \mathrm{e}^{(r+\sigma^2)\tau}\right) du \\
&= 2S_0^2 \frac{r\mathrm{e}^{(\sigma^2+2r)T} - (\sigma^2 + 2r)\mathrm{e}^{rT + (\sigma^2+r)\tau} + (\sigma^2 + r)\mathrm{e}^{(\sigma^2+2r)\tau}}{(\sigma^2 + r)(\sigma^2 + 2r)r}, \quad 0 \le \tau \le T.
\end{aligned}
$$

Exercise 13.2

a) The integral $\int_0^T r_s ds$ has a centered Gaussian distribution with variance

$$
\begin{aligned}
\mathbb{E}\left[\left(\int_0^T r_s ds\right)^2\right] &= \sigma^2\, \mathbb{E}\left[\int_0^T \int_0^T B_s B_t ds dt\right] \\
&= \sigma^2 \int_0^T \int_0^T \mathbb{E}[B_s B_t] ds dt \\
&= \sigma^2 \int_0^T \int_0^T \min(s,t) ds dt \\
&= 2\sigma^2 \int_0^T \int_0^t s ds dt \\
&= \sigma^2 \int_0^T t^2 dt \\
&= T^3 \frac{\sigma^2}{3}.
\end{aligned}
$$

b) Since the integral $\int_0^T r_s ds$ is a random variable with probability density

$$
\varphi(x) = \frac{1}{\sqrt{2\pi T^3/3}} \mathrm{e}^{-3x^2/(2\pi T^3)},
$$

we have

$$
\begin{aligned}
& \mathrm{e}^{-rT}\,\mathbb{E}\left[\left(\int_0^T r_u du-\kappa\right)^+\right] = \mathrm{e}^{-rT}\int_{-\infty}^{\infty}(x-\kappa)^+\varphi(x)dx \\
& = \frac{\mathrm{e}^{-rT}}{\sqrt{2\pi\sigma^2T^3/3}}\int_\kappa^\infty (x-\kappa)\mathrm{e}^{-3x^2/(2\sigma^2T^3)}dx \\
& = \frac{\mathrm{e}^{-rT}}{\sqrt{2\pi}}\int_{\kappa/\sqrt{\sigma^2T^3/3}}^\infty (x\sqrt{\sigma^2T^3/3}-\kappa)\mathrm{e}^{-x^2/2}dx \\
& = \frac{\mathrm{e}^{-rT}\sqrt{\sigma^2T^3/3}}{\sqrt{2\pi}}\int_{\kappa/\sqrt{\sigma^2T^3/3}}^\infty x\mathrm{e}^{-x^2/2}dx - \kappa\frac{\mathrm{e}^{-rT}}{\sqrt{2\pi}}\int_{\kappa/\sqrt{\sigma^2T^3/3}}^\infty \mathrm{e}^{-x^2/2}dx \\
& = -\frac{\mathrm{e}^{-rT}\sqrt{\sigma^2T^3/3}}{\sqrt{2\pi}}\left[\mathrm{e}^{-x^2/2}\right]_{\kappa/\sqrt{\sigma^2T^3/3}}^\infty - \kappa\frac{\mathrm{e}^{-rT}}{\sqrt{2\pi}}(1-\Phi(\kappa/\sqrt{\sigma^2T^3/3})) \\
& = \frac{\mathrm{e}^{-rT}\sqrt{\sigma^2T^3/3}}{\sqrt{2\pi}}\mathrm{e}^{-3\kappa^2/(2\sigma^2T^3)} - \kappa\frac{\mathrm{e}^{-rT}}{\sqrt{2\pi}}(1-\Phi(\kappa/\sqrt{\sigma^2T^3/3})) \\
& = \mathrm{e}^{-rT}\sqrt{\frac{\sigma^2T^3}{6\pi}}\mathrm{e}^{-3\kappa^2/(2\sigma^2T^3)} - \kappa\frac{\mathrm{e}^{-rT}}{\sqrt{2\pi}}\Phi\left(-\kappa\sqrt{\frac{3}{\sigma^2T^3}}\right).
\end{aligned}
$$

Exercise 13.3 We have

$$
\begin{aligned}
& \mathrm{e}^{-(T-t)r}\,\mathbb{E}\left[\left(\frac{1}{T}\int_0^T S_u du-\kappa\right)^+ \;\Big|\; \mathcal{F}_t\right] = \mathrm{e}^{-(T-t)r}\,\mathbb{E}\left[\frac{1}{T}\int_0^T S_u du-\kappa \;\Big|\; \mathcal{F}_t\right] \\
& = \mathrm{e}^{-(T-t)r}\,\mathbb{E}\left[\frac{1}{T}\int_0^T S_u du \;\Big|\; \mathcal{F}_t\right] - \kappa\mathrm{e}^{-(T-t)r} \\
& = \mathrm{e}^{-(T-t)r}\frac{1}{T}\,\mathbb{E}\left[\int_0^t S_u du \;\Big|\; \mathcal{F}_t\right] + \mathrm{e}^{-(T-t)r}\frac{1}{T}\,\mathbb{E}\left[\int_t^T S_u du \;\Big|\; \mathcal{F}_t\right] - \kappa\mathrm{e}^{-(T-t)r} \\
& = \mathrm{e}^{-(T-t)r}\frac{1}{T}\int_0^t S_u du + \mathrm{e}^{-(T-t)r}\frac{1}{T}\,\mathbb{E}\left[\int_t^T S_u du \;\Big|\; \mathcal{F}_t\right] - \kappa\mathrm{e}^{-(T-t)r} \\
& = \mathrm{e}^{-(T-t)r}\frac{1}{T}\int_0^t S_u du + \mathrm{e}^{-(T-t)r}\frac{1}{T}\int_t^T \mathbb{E}[S_u \mid \mathcal{F}_t]du - \kappa\mathrm{e}^{-(T-t)r} \\
& = \mathrm{e}^{-(T-t)r}\frac{1}{T}\int_0^t S_u du + \mathrm{e}^{-(T-t)r}\frac{1}{T}\int_t^T S_t\mathrm{e}^{(u-t)r}du - \kappa\mathrm{e}^{-(T-t)r} \\
& = \mathrm{e}^{-(T-t)r}\frac{1}{T}\int_0^t S_u du + \mathrm{e}^{-(T-t)r}\frac{S_t}{T}\int_0^{T-t} \mathrm{e}^{ru}du - \kappa\mathrm{e}^{-(T-t)r} \\
& = \mathrm{e}^{-(T-t)r}\frac{1}{T}\int_0^t S_u du + \mathrm{e}^{-(T-t)r}\frac{S_t}{rT}(\mathrm{e}^{(T-t)r}-1) - \kappa\mathrm{e}^{-(T-t)r} \\
& = \mathrm{e}^{-(T-t)r}\frac{1}{T}\int_0^t S_u du + S_t\frac{1-\mathrm{e}^{-(T-t)r}}{rT} - \kappa\mathrm{e}^{-(T-t)r},
\end{aligned}
$$

$t \in [0,T]$, cf. Geman and Yor (1993) page 361. We check that the function $f(t,x,y) = \mathrm{e}^{-(T-t)r}(y/T-\kappa) + x(1-\mathrm{e}^{-(T-t)r})/(rT)$ satisfies the PDE

$$rf(t,x,y) = \frac{\partial f}{\partial t}(t,x,y) + x\frac{\partial f}{\partial y}(t,x,y) + rx\frac{\partial f}{\partial x}(t,x,y) + \frac{1}{2}x^2\sigma^2\frac{\partial^2 f}{\partial x^2}(t,x,y),$$

$t, x > 0$, and the boundary conditions $f(t,0,y) = \mathrm{e}^{-(T-t)r}(y/T - \kappa)$, $0 \le t \le T$, $y \in \mathbb{R}_+$, and $f(T,x,y) = y/T - \kappa$, $x, y \in \mathbb{R}_+$. However, the condition $\lim_{y\to-\infty} f(t,x,y) = 0$ is not satisfied because we need to take $y > 0$ in the above calculation.

Exercise 13.4

a) We have

$$\begin{aligned}
&\mathrm{e}^{-(T-t)r}\,\mathbb{E}^*\left[\frac{1}{T}\int_0^T S_u du - K \,\middle|\, \mathcal{F}_t\right] \\
&= \mathrm{e}^{-(T-t)r}\,\mathbb{E}^*\left[\frac{1}{T}\int_0^T S_u du \,\middle|\, \mathcal{F}_t\right] - K\mathrm{e}^{-(T-t)r} \\
&= \frac{\mathrm{e}^{-(T-t)r}}{T}\,\mathbb{E}^*\left[\int_0^t S_u du \,\middle|\, \mathcal{F}_t\right] + \frac{\mathrm{e}^{-(T-t)r}}{T}\,\mathbb{E}^*\left[\int_t^T S_u du \,\middle|\, \mathcal{F}_t\right] - K\mathrm{e}^{-(T-t)r} \\
&= \frac{\mathrm{e}^{-(T-t)r}}{T}\int_0^t S_u du + \frac{\mathrm{e}^{-(T-t)r}}{T}\int_t^T \mathbb{E}^*[S_u \mid \mathcal{F}_t] du - K\mathrm{e}^{-(T-t)r} \\
&= \frac{\mathrm{e}^{-(T-t)r}}{T}\int_0^t S_u du + \frac{\mathrm{e}^{-(T-t)r}}{T}\int_t^T \mathrm{e}^{(u-t)r} S_t du - K\mathrm{e}^{-(T-t)r} \\
&= \frac{\mathrm{e}^{-(T-t)r}}{T}\int_0^t S_u du + S_t\frac{\mathrm{e}^{-rT}}{T}\int_t^T \mathrm{e}^{ru} du - K\mathrm{e}^{-(T-t)r} \\
&= \frac{\mathrm{e}^{-(T-t)r}}{T}\int_0^t S_u du + \frac{S_t}{rT}\left(1 - \mathrm{e}^{-(T-t)r}\right) - K\mathrm{e}^{-(T-t)r}.
\end{aligned}$$

b) Using the relation

$$(x-K)^+ - (K-x)^+ = x - K, \qquad K, x \in \mathbb{R},$$

we have

$$\begin{aligned}
C(t,K) - P(t,K) &= \mathrm{e}^{-(T-t)r}\,\mathbb{E}^*\left[\left(\frac{1}{T}\int_0^T S_u du - K\right)^+ \,\middle|\, \mathcal{F}_t\right] \\
&\quad -\mathrm{e}^{-(T-t)r}\,\mathbb{E}^*\left[\left(K - \frac{1}{T}\int_0^T S_u du\right)^+ \,\middle|\, \mathcal{F}_t\right] \\
&= \mathrm{e}^{-(T-t)r}\,\mathbb{E}^*\left[\left(\frac{1}{T}\int_0^T S_u du - K\right)^+ - \left(K - \frac{1}{T}\int_0^T S_u du\right)^+ \,\middle|\, \mathcal{F}_t\right] \\
&= \mathrm{e}^{-(T-t)r}\,\mathbb{E}^*\left[\frac{1}{T}\int_0^T S_u du - K \,\middle|\, \mathcal{F}_t\right]
\end{aligned}$$

$$= \frac{e^{-(T-t)r}}{T} \int_0^t S_u du + \frac{S_t}{rT}\left(1 - e^{-(T-t)r}\right) - K e^{-(T-t)r}. \qquad \text{(S.13.55)}$$

c) Any self-financing portfolio strategy $(\xi_t)_{t\in\mathbb{R}_+}$ with price process $(V_t)_{t\in\mathbb{R}_+}$ has to satisfy the equation

$$\begin{aligned} dV_t &= \eta_t dA_t + \xi_t dS_t \\ &= r\eta_t A_t dt + \mu\xi_t S_t dt + \sigma\xi_t S_t dB_t \\ &= rV_t dt + (\mu - r)\xi_t S_t dt + \sigma\xi_t S_t dB_t, \qquad t \in \mathbb{R}_+. \end{aligned}$$

On the other hand, by part (a) we have

$$\begin{aligned} dV_t &= d\left(\frac{e^{-(T-t)r}}{T}\int_0^t S_s ds + \frac{S_t}{rT}(1 - e^{-(T-t)r}) - K e^{-(T-t)r}\right) \\ &= \frac{r}{T} e^{-(T-t)r} \int_0^t S_s ds dt + \frac{e^{-(T-t)r}}{T} S_t dt - \frac{S_t}{T} e^{-(T-t)r} dt \\ &\quad + \frac{1 - e^{-(T-t)r}}{rT} dS_t - rK e^{-(T-t)r} dt \\ &= \frac{r}{T} e^{-(T-t)r} \int_0^t S_s ds dt + \frac{1 - e^{-(T-t)r}}{rT} dS_t - rK e^{-(T-t)r} dt \\ &= rV_t dt + \frac{1 - e^{-(T-t)r}}{rT} dS_t - S_t(1 - e^{-(T-t)r}) dt \\ &= rV_t dt + \frac{1 - e^{-(T-t)r}}{rT} ((\mu - r)S_t dt + \sigma S_t dB_t), \end{aligned}$$

hence

$$\xi_t = \frac{1 - e^{-(T-t)r}}{rT}, \qquad t \in [0, T],$$

which can be recovered by differentiating the pricing function

$$\frac{e^{-(T-t)r}}{T} y + \frac{x}{rT}\left(1 - e^{-(T-t)r}\right) - K e^{-(T-t)r}$$

in (S.13.55) with respect to $x = S_t$, with $y = \int_0^t S_u du$. We also have

$$V_t = \frac{e^{-(T-t)r}}{T} \int_0^t S_s ds + \xi_t S_t - K e^{-(T-t)r}, \qquad t \in [0, T].$$

d) The following code yields \$7.906436 for the price of the long forward contract.

```
T=1;t=63/252;r=0.0209;K=80;dt=1/252;S=as.numeric(last(futures));
exp(-(T-t)*r)*sum(futures)*dt/T+S*(1-exp(-(T-t)*r))/(r*T)-K*exp(-(T-t)*r)
```

Exercise 13.5 The geometric mean price G satisfies

$$
\begin{aligned}
G &= \exp\left(\frac{1}{T}\int_0^T \log S_u du\right) = \exp\left(\frac{1}{T}\int_0^t \log S_u du + \frac{1}{T}\int_t^T \log S_u du\right) \\
&= \exp\left(\frac{1}{T}\int_0^t \log S_u du + \frac{T-t}{T}\log S_t + \frac{1}{T}\int_t^T \log\frac{S_u}{S_t} du\right) \\
&= \exp\left(\frac{1}{T}\int_0^t \log S_u du + \frac{T-t}{T}\log S_t \right.\\
&\quad \left. + \frac{1}{T}\int_t^T (r(u-t) + (B_u - B_t)\sigma - (u-t)\sigma^2/2) du\right) \\
&= \exp\left(\frac{1}{T}\int_0^t \log S_u du + \frac{T-t}{T}\log S_t \right.\\
&\quad \left. + \frac{1}{T}\int_0^{T-t} (ru - \sigma^2 u/2) du + \frac{\sigma}{T}\int_t^T (B_u - B_t) du\right) \\
&= (S_t)^{(T-t)/T}\exp\left(\frac{1}{T}\int_0^t \log S_u du + \frac{(T-t)^2}{2T}(r-\sigma^2/2) + \frac{\sigma}{T}\int_t^T (B_u - B_t) du\right)
\end{aligned}
$$

where $\int_t^T B_u du$ is centered Gaussian with conditional variance

$$
\begin{aligned}
\mathbb{E}\left[\left(\int_t^T B_u du\right)^2 \,\Big|\, \mathcal{F}_t\right] &= \mathbb{E}\left[\left(\int_t^T (B_u - B_t) du\right)^2 \,\Big|\, \mathcal{F}_t\right] \\
&= \mathbb{E}\left[\left(\int_t^T (B_u - B_t) du\right)^2\right] \\
&= \mathbb{E}\left[\left(\int_0^{T-t} (B_u - B_t) du\right)^2\right] = \int_0^{T-t}\int_0^{T-t} \mathbb{E}[B_s B_u]\, ds du \\
&= 2\int_0^{T-t}\int_0^u s ds du = \int_0^{T-t} u^2 du = \frac{(T-t)^3}{3}.
\end{aligned}
$$

Hence, letting

$$
m := \frac{1}{T}\int_0^t \log S_u du + \frac{T-t}{T}\log S_t + \frac{(T-t)^2}{2T}(r-\sigma^2/2), \quad X := \frac{\sigma}{T}\int_t^T B_u du,
$$

and $v^2 = (T-t)\sigma^2/3$, we find

$$
\begin{aligned}
&\mathrm{e}^{-(T-t)r}\,\mathbb{E}^*\left[\left(\exp\left(\frac{1}{T}\int_0^T \log S_u du\right) - K\right)^+ \,\Big|\, \mathcal{F}_t\right] \\
&= (S_t)^{(T-t)/T}\mathrm{e}^{-(T-t)r}\exp\left(\frac{1}{T}\int_0^t \log S_u du + \frac{(T-t)^2}{4T}(2r-\sigma^2) + \frac{\sigma^2}{6}(T-t)\right)
\end{aligned}
$$

$$\times\Phi\left(\frac{(T-t)\sigma^2/3+\frac{1}{T}\int_0^t\log S_u du+\log\frac{S_t^{(T-t)/T}}{K}+\frac{(T-t)^2}{2T}(r-\sigma^2/2)}{\sigma\sqrt{(T-t)/3}}\right)$$
$$-K\mathrm{e}^{-(T-t)r}\Phi\left(\frac{\frac{1}{T}\int_0^t\log S_u du+\log\frac{S_t^{(T-t)/T}}{K}+\frac{(T-t)^2}{2T}(r-\sigma^2/2)}{\sigma\sqrt{(T-t)/3}}\right),$$

$0\le t\le T$. In case $t=0$, we get

$$\begin{aligned}
&\mathrm{e}^{-rT}\,\mathbb{E}^*\left[\left(\exp\left(\frac{1}{T}\int_0^T\log S_u du\right)-K\right)^+\right]\\
&=S_0\mathrm{e}^{-T(r+\sigma^2/6)/2}\Phi\left(\frac{\log(S_0/K)+T(r+\sigma^2/6)/2}{\sigma\sqrt{T/3}}\right)\\
&\quad -K\mathrm{e}^{-rT}\Phi\left(\frac{\log(S_0/K)+T(r-\sigma^2/2)/2}{\sigma\sqrt{T/3}}\right).
\end{aligned}$$

Exercise 13.6 Under the above condition we have, taking $t\in[\tau,T]$,

$$\begin{aligned}
&\mathrm{e}^{-(T-t)r}\,\mathbb{E}^*\left[\left(\frac{1}{T-\tau}\int_\tau^T r_s ds-K\right)^+\,\Big|\,\mathcal{F}_t\right]\\
&=\mathrm{e}^{-(T-t)r}\,\mathbb{E}^*\left[\left(\Lambda_t+\frac{1}{T-\tau}\int_t^T r_s ds-K\right)^+\,\Big|\,\mathcal{F}_t\right]\\
&=\mathrm{e}^{-(T-t)r}\,\mathbb{E}^*\left[\Lambda_t+\frac{1}{T-\tau}\int_t^T r_s ds-K\,\Big|\,\mathcal{F}_t\right]\\
&=\mathrm{e}^{-(T-t)r}(\Lambda_t-K)+\frac{\mathrm{e}^{-(T-t)r}}{T-\tau}\,\mathbb{E}^*\left[\int_t^T r_s ds\,\Big|\,\mathcal{F}_t\right]\\
&=\mathrm{e}^{-(T-t)r}(\Lambda_t-K)+\frac{\mathrm{e}^{-(T-t)r}}{T-\tau}\int_t^T\mathbb{E}^*[r_s\mid\mathcal{F}_t]ds,\qquad t\in[\tau,T],
\end{aligned}$$

where

$$\mathbb{E}^*[r_s\mid\mathcal{F}_t]=v_t\mathrm{e}^{-(s-t)\lambda}+m(1-\mathrm{e}^{-(s-t)\lambda}),\qquad 0\le s\le t,$$

hence

$$\begin{aligned}
&\mathbb{E}^*\left[\left(\frac{1}{T-\tau}\int_\tau^T r_s ds-K\right)^+\,\Big|\,\mathcal{F}_t\right]=\mathbb{E}^*\left[\left(\Lambda_t+\frac{1}{T-\tau}\int_t^T r_s ds-K\right)^+\,\Big|\,\mathcal{F}_t\right]\\
&=\Lambda_t-K+\frac{1}{T-\tau}\int_t^T\mathbb{E}^*[r_s\mid\mathcal{F}_t]ds\\
&=\Lambda_t-K+\frac{1}{T-\tau}\int_t^T\big(r_t\mathrm{e}^{-(s-t)\lambda}+m(1-\mathrm{e}^{-(s-t)\lambda})\big)ds
\end{aligned}$$

$$
\begin{aligned}
&= \Lambda_t - K + \frac{1}{T-\tau}(r_t - m)\int_0^{T-t} \mathrm{e}^{-\lambda s} ds + m(T-t)\frac{\mathrm{e}^{-(T-t)r}}{T-\tau} \\
&= \Lambda_t - K + (r_t - m)\frac{1}{T-\tau}\int_0^{T-t} \mathrm{e}^{-\lambda s} ds + m\frac{T-t}{T-\tau} \\
&= \Lambda_t - K + \frac{1-\mathrm{e}^{-(T-t)\lambda}}{(T-\tau)\lambda}(r_t - m) + m\frac{T-t}{T-\tau}.
\end{aligned}
$$

Exercise 13.7 This question extends Exercise 7.4 to $n \geq 3$. If $(S_t)_{t\in\mathbb{R}_+}$ is a martingale then for any convex payoff function ϕ we can write

$$
\begin{aligned}
&\mathbb{E}^*\left[\phi\left(\frac{S_{T_1}+\cdots+S_{T_n}}{n}\right)\right] \leq \mathbb{E}^*\left[\frac{\phi(S_{T_1})+\cdots+\phi(S_{T_n})}{n}\right] && \text{since } \phi \text{ is convex,} \\
&= \frac{\mathbb{E}^*[\phi(S_{T_1})]+\cdots+\mathbb{E}^*[\phi(S_{T_n})]}{n} \\
&= \frac{\mathbb{E}^*[\phi(\mathbb{E}^*[S_{T_n} \mid \mathcal{F}_{T_1}])]+\cdots+\mathbb{E}^*[\phi(\mathbb{E}^*[S_{T_n} \mid \mathcal{F}_{T_n}])]}{n} && \text{because } (S_t)_{t\in\mathbb{R}_+} \text{ is a martingale,} \\
&\leq \frac{\mathbb{E}^*[\mathbb{E}^*[\phi(S_{T_n}) \mid \mathcal{F}_{T_1}]]+\cdots+\mathbb{E}^*[\mathbb{E}^*[\phi(S_{T_n}) \mid \mathcal{F}_{T_n}]]}{n} && \text{by Jensen's inequality,} \\
&= \frac{\mathbb{E}^*[\phi(S_{T_n})]+\cdots+\mathbb{E}^*[\phi(S_{T_n})]}{n} && \text{by the tower property,} \\
&= \mathbb{E}^*[\phi(S_{T_n})].
\end{aligned}
$$

On the other hand, if $(S_t)_{t\in\mathbb{R}_+}$ is only a *sub*martingale then the above argument still applies to a convex non-decreasing payoff function ϕ such as $\phi(x) = (x-K)^+$.

Exercise 13.8 Taking $t \in [\tau, T]$, under the condition

$$
\Lambda_t := \frac{1}{T-\tau}\int_\tau^t S_s ds \geq K,
$$

we have

$$
\begin{aligned}
&\mathrm{e}^{-(T-t)r}\,\mathbb{E}^*\left[\left(\frac{1}{T-\tau}\int_\tau^T S_s ds - K\right)^+ \,\middle|\, \mathcal{F}_t\right] \\
&= \mathrm{e}^{-(T-t)r}\,\mathbb{E}^*\left[\left(\Lambda_t + \frac{1}{T-\tau}\int_t^T S_s ds - K\right)^+ \,\middle|\, \mathcal{F}_t\right] \\
&= \mathrm{e}^{-(T-t)r}\,\mathbb{E}^*\left[\Lambda_t + \frac{1}{T-\tau}\int_t^T S_s ds - K \,\middle|\, \mathcal{F}_t\right] \\
&= \mathrm{e}^{-(T-t)r}(\Lambda_t - K) + \frac{\mathrm{e}^{-(T-t)r}}{T-\tau}\,\mathbb{E}^*\left[\int_t^T S_s ds \,\middle|\, \mathcal{F}_t\right] \\
&= \mathrm{e}^{-(T-t)r}(\Lambda_t - K) + \frac{\mathrm{e}^{-(T-t)r}}{T-\tau}\int_t^T \mathbb{E}^*[S_s \mid \mathcal{F}_t] ds
\end{aligned}
$$

$$
\begin{aligned}
&= \mathrm{e}^{-(T-t)r}(\Lambda_t - K) + S_t \frac{\mathrm{e}^{-(T-t)r}}{T-\tau} \int_t^T \mathrm{e}^{(s-t)r} ds \\
&= \mathrm{e}^{-(T-t)r}(\Lambda_t - K) + S_t \frac{\mathrm{e}^{-(T-t)r}}{T-\tau} \int_0^{T-t} \mathrm{e}^{rs} ds \\
&= \mathrm{e}^{-(T-t)r}(\Lambda_t - K) + S_t \frac{\mathrm{e}^{-(T-t)r}}{(T-\tau)r} (\mathrm{e}^{(T-t)r} - 1) \\
&= \mathrm{e}^{-(T-t)r}(\Lambda_t - K) + S_t \frac{1-\mathrm{e}^{-(T-t)r}}{(T-\tau)r}, \qquad t \in [\tau, T].
\end{aligned}
$$

Exercise 13.9 The Asian option price can be written as

$$
\begin{aligned}
\mathrm{e}^{-r(T-t)}\, \mathbb{E}^* \left[\left(\frac{1}{T} \int_0^T S_u du - K \right)^+ \;\middle|\; \mathcal{F}_t \right] &= S_t \widehat{\mathbb{E}} \left[(U_T)^+ \mid U_t \right] \\
&= S_t h(t, U_t) = S_t g(t, Z_t),
\end{aligned}
$$

which shows that

$$
g(t, Z_t) = h(t, U_t),
$$

and it remains to use the relation

$$
U_t = \frac{1-\mathrm{e}^{-(T-t)r}}{rT} + \mathrm{e}^{-(T-t)r} Z_t, \qquad t \in [0, T].
$$

Exercise 13.10

i) By change of variable. We note that $\widetilde{Z}_t = \mathrm{e}^{-(T-t)r} Z_t$, where

$$
Z_t := \frac{1}{S_t} \left(\frac{1}{T} \int_0^t S_u du - K \right) = \frac{1}{S_t} \left(\frac{\Lambda_t}{T} - K \right), \qquad 0 \le t \le T,
$$

and the pricing function $g(t, Z_t)$ satisfies the Rogers-Shi PDE

$$
\frac{\partial g}{\partial t}(t,z) + \left(\frac{1}{T} - rz \right) \frac{\partial g}{\partial z}(t,z) + \frac{1}{2} \sigma^2 z^2 \frac{\partial^2 g}{\partial z^2}(t,z) = 0.
$$

Letting $\widetilde{z} := \mathrm{e}^{-(T-t)r} z$ and $\widetilde{g}(t, \widetilde{z}) := g\big(t, \mathrm{e}^{(T-t)r} \widetilde{z}\big) = g(t,z) = \widetilde{g}\big(t, \mathrm{e}^{-(T-t)r} z\big)$, we note that

$$\begin{cases} \dfrac{\partial g}{\partial t}(t,z) &= \dfrac{\partial}{\partial t}\widetilde{g}\big(t, \mathrm{e}^{-(T-t)r}z\big) \\ &= \dfrac{\partial \widetilde{g}}{\partial t}\big(t, \mathrm{e}^{-(T-t)r}z\big) + r\mathrm{e}^{-(T-t)r}z\dfrac{\partial \widetilde{g}}{\partial x}\big(t, \mathrm{e}^{-(T-t)r}z\big) \\ &= \dfrac{\partial \widetilde{g}}{\partial t}(t,\widetilde{z}) + r\widetilde{z}\dfrac{\partial \widetilde{g}}{\partial x}(t,\widetilde{z}), \\ \dfrac{\partial g}{\partial z}(t,z) &= \mathrm{e}^{-(T-t)r}\dfrac{\partial \widetilde{g}}{\partial \widetilde{z}}\big(t, \mathrm{e}^{-(T-t)r}z\big) = \mathrm{e}^{-(T-t)r}\dfrac{\partial \widetilde{g}}{\partial \widetilde{z}}(t,\widetilde{z}), \\ \dfrac{\partial^2 g}{\partial z^2}(t,z) &= \mathrm{e}^{-2(T-t)r}\dfrac{\partial^2 \widetilde{g}}{\partial \widetilde{z}^2}\big(t, \mathrm{e}^{-(T-t)r}z\big) = \mathrm{e}^{-2(T-t)r}\dfrac{\partial^2 \widetilde{g}}{\partial \widetilde{z}^2}(t,\widetilde{z}), \end{cases}$$

hence

$$\begin{aligned} 0 &= \frac{\partial g}{\partial t}(t,z) + \left(\frac{1}{T} - rz\right)\frac{\partial g}{\partial z}(t,z) + \frac{1}{2}\sigma^2 z^2 \frac{\partial^2 g}{\partial z^2}(t,z) \\ &= \frac{\partial \widetilde{g}}{\partial t}(t,\widetilde{z}) + r\widetilde{z}\frac{\partial \widetilde{g}}{\partial x}(t,\widetilde{z}) + \left(\frac{1}{T} - rz\right)\mathrm{e}^{-(T-t)r}\frac{\partial \widetilde{g}}{\partial \widetilde{z}}(t,\widetilde{z}) \\ &\quad + \frac{1}{2}\sigma^2 z^2 \mathrm{e}^{-2(T-t)r}\frac{\partial^2 \widetilde{g}}{\partial \widetilde{z}^2}(t,\widetilde{z}) \\ &= \frac{\partial \widetilde{g}}{\partial t}(t,\widetilde{z}) + \frac{1}{T}\mathrm{e}^{-(T-t)r}\frac{\partial \widetilde{g}}{\partial \widetilde{z}}(t,\widetilde{z}) + \frac{1}{2}\sigma^2\widetilde{z}^2\frac{\partial^2 \widetilde{g}}{\partial \widetilde{z}^2}(t,\widetilde{z}), \end{aligned}$$

and the (simpler) PDE

$$\frac{\partial \widetilde{g}}{\partial t}(t,\widetilde{z}) + \frac{1}{T}\mathrm{e}^{-(T-t)r}\frac{\partial \widetilde{g}}{\partial \widetilde{z}}(t,\widetilde{z}) + \frac{1}{2}\sigma^2\widetilde{z}^2\frac{\partial^2 \widetilde{g}}{\partial \widetilde{z}^2}(t,\widetilde{z}) = 0.$$

ii) Using the Itô formula. Given that

$$\begin{aligned} d\widetilde{Z}_t &= d(\mathrm{e}^{-(T-t)r}Z_t) \\ &= r\mathrm{e}^{-(T-t)r}Z_t dt + \mathrm{e}^{-(T-t)r}dZ_t \\ &= r\widetilde{Z}_t dt + \mathrm{e}^{-(T-t)r}dZ_t, \end{aligned}$$

and

$$dS_t = rS_t dt + \sigma S_t dB_t,$$

under the risk-neutral probability measure $\mathbb{P}^*$, an application of Itô's formula to the discounted portfolio price leads to

$$\begin{aligned} & d\big(\mathrm{e}^{-rt}S_t\widetilde{g}(t,\widetilde{Z}_t)\big) \\ &= \mathrm{e}^{-rt}\big(-r\widetilde{g}(t,\widetilde{Z}_t)dt + \widetilde{g}(t,\widetilde{Z}_t)dS_t + S_t d\widetilde{g}(t,\widetilde{Z}_t) + dS_t \bullet d\widetilde{g}(t,\widetilde{Z}_t)\big) \\ &= \mathrm{e}^{-rt}\left(-rS_t\widetilde{g}(t,\widetilde{Z}_t)dt + \widetilde{g}(t,\widetilde{Z}_t)dS_t + S_t\frac{\partial \widetilde{g}}{\partial t}(t,\widetilde{Z}_t)dt + S_t\frac{\partial \widetilde{g}}{\partial z}(t,\widetilde{Z}_t)d\widetilde{Z}_t\right) \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}\mathrm{e}^{-rt}\left(S_t\frac{\partial^2\widetilde{g}}{\partial z^2}(t,\widetilde{Z}_t)(d\widetilde{Z}_t)^2+dS_t\bullet d\widetilde{g}(t,\widetilde{Z}_t)\right)\\
&=\mathrm{e}^{-rt}\left(-rS_t\widetilde{g}(t,\widetilde{Z}_t)dt+\widetilde{g}(t,\widetilde{Z}_t)dS_t+S_t\frac{\partial\widetilde{g}}{\partial t}(t,\widetilde{Z}_t)dt\right.\\
&\quad\left.+r\widetilde{Z}_tS_t\frac{\partial\widetilde{g}}{\partial z}(t,\widetilde{Z}_t)dt+S_t\mathrm{e}^{-(T-t)r}\frac{\partial\widetilde{g}}{\partial z}(t,\widetilde{Z}_t)dZ_t\right)\\
&\quad+\frac{1}{2}\mathrm{e}^{-rt}\left(S_t\frac{\partial^2\widetilde{g}}{\partial z^2}(t,\widetilde{Z}_t)(d\widetilde{Z}_t)+dS_t\bullet d\widetilde{g}(t,\widetilde{Z}_t)\right)\\
&=\mathrm{e}^{-rt}\left(-rS_t\widetilde{g}(t,\widetilde{Z}_t)dt+rS_t\widetilde{g}(t,\widetilde{Z}_t)dt+\sigma S_t\widetilde{g}(t,\widetilde{Z}_t)dB_t+r\widetilde{Z}_tS_t\frac{\partial\widetilde{g}}{\partial z}(t,\widetilde{Z}_t)dt\right)\\
&\quad+\mathrm{e}^{-rt}\left(\mathrm{e}^{-(T-t)r}S_tZ_t\left(-r+\sigma^2\right)\frac{\partial\widetilde{g}}{\partial z}(t,\widetilde{Z}_t)dt\right.\\
&\quad\left.+\frac{1}{T}\mathrm{e}^{-(T-t)r}S_t\frac{\partial\widetilde{g}}{\partial z}(t,\widetilde{Z}_t)dt-\sigma\mathrm{e}^{-(T-t)r}S_tZ_t\frac{\partial\widetilde{g}}{\partial z}(t,\widetilde{Z}_t)dB_t\right)\\
&\quad+\mathrm{e}^{-rt}\left(\frac{1}{2}\sigma^2\widetilde{Z}_t^2S_t\frac{\partial^2\widetilde{g}}{\partial z^2}(t,\widetilde{Z}_t)dt-\sigma^2S_t\widetilde{Z}_t\frac{\partial\widetilde{g}}{\partial z}(t,\widetilde{Z}_t)dt\right)\\
&=\mathrm{e}^{-rt}S_t\left(\frac{\partial\widetilde{g}}{\partial t}(t,\widetilde{Z}_t)+\frac{1}{T}\mathrm{e}^{-(T-t)r}\frac{\partial\widetilde{g}}{\partial z}(t,\widetilde{Z}_t)+\frac{1}{2}\sigma^2\widetilde{Z}_t^2\frac{\partial^2\widetilde{g}}{\partial z^2}(t,\widetilde{Z}_t)\right)dt\\
&\quad+S_t\mathrm{e}^{-rt}\left(\sigma\widetilde{g}(t,\widetilde{Z}_t)-\sigma\widetilde{Z}_t\frac{\partial\widetilde{g}}{\partial z}(t,\widetilde{Z}_t)\right)dB_t.
\end{aligned}
$$

Since the discounted portfolio price process is a martingale under the risk-neutral probability measure $\mathbb{P}^*$, the sum of components in dt should vanish in the above expression, which yields

$$
\frac{\partial\widetilde{g}}{\partial t}(t,\widetilde{Z}_t)+\frac{1}{T}\mathrm{e}^{-(T-t)r}\frac{\partial\widetilde{g}}{\partial z}(t,\widetilde{Z}_t)+\frac{1}{2}\sigma^2\widetilde{Z}_t^2\frac{\partial^2\widetilde{g}}{\partial z^2}(t,\widetilde{Z}_t)=0,
$$

and the PDE

$$
\frac{\partial\widetilde{g}}{\partial t}(t,\widetilde{z})+\frac{1}{T}\mathrm{e}^{-(T-t)r}\frac{\partial\widetilde{g}}{\partial\widetilde{z}}(t,\widetilde{z})+\frac{1}{2}\sigma^2\widetilde{z}^2\frac{\partial^2\widetilde{g}}{\partial\widetilde{z}^2}(t,\widetilde{z})=0,
$$

under the terminal condition $\widetilde{g}(T,\widetilde{z})=\widetilde{z}^+$, $\widetilde{z}\in\mathbb{R}$.

Exercise 13.11

a) When $\Lambda_t/T\geq K$ we have

$$
f(t,S_t,\Lambda_t)=\mathrm{e}^{-(T-t)r}\left(\frac{\Lambda_t}{T}-K\right)+S_t\frac{1-\mathrm{e}^{-(T-t)r}}{rT},
$$

see Exercise 13.8.

b) When $\Lambda_t/T\geq K$ we have

$$\xi_t = \frac{1 - \mathrm{e}^{-(T-t)r}}{rT} \quad \text{and} \quad \eta_t A_t = \mathrm{e}^{(T-t)r}\left(\frac{\Lambda_t}{T} - K\right), \qquad 0 \le t \le T.$$

c) At maturity we have $f(T, S_T, \Lambda_T) = (\Lambda_T/T - K)^+$, hence $\xi_T = 0$ and

$$\eta_T A_T = A_T \frac{\mathrm{e}^{-rT}}{A_0}\left(\frac{\Lambda_T}{T} - K\right)\mathbb{1}_{\{\Lambda_T > KT\}} = \left(\frac{\Lambda_T}{T} - K\right)^+.$$

d) By Proposition 13.12 we have

$$\xi_t = \frac{1}{S_t}\left(f(t, S_t, \Lambda_t) - \left(\frac{\Lambda_t}{T} - K\right)\frac{\partial g}{\partial z}\left(t, \frac{1}{S_t}\left(\frac{\Lambda_t}{T} - K\right)\right)\right)$$

where the function $g(t,z)$ satisfies $f(t,x,y) = xg(t, (y/T - K)/x))$ and

$$g(t,z) = z\mathrm{e}^{-(T-t)r} + \frac{1 - \mathrm{e}^{-(T-t)r}}{rT}, \qquad z > 0,$$

and solves the PDE

$$\frac{\partial g}{\partial t}(t,z) + \left(\frac{1}{T} - rz\right)\frac{\partial g}{\partial z}(t,z) + \frac{1}{2}\sigma^2 z^2 \frac{\partial^2 g}{\partial z^2}(t,z) = 0,$$

under the terminal condition $g(T,z) = z^+$, hence letting

$$h(t,z) := \mathrm{e}^{(T-t)r}\frac{\partial g}{\partial z}(t,z),$$

we have

$$\mathrm{e}^{(T-t)r}\frac{\partial g}{\partial t}(t,z) + \mathrm{e}^{(T-t)r}\left(\frac{1}{T} - rz\right)\frac{\partial g}{\partial z}(t,z) + \frac{1}{2}\sigma^2 z^2 \frac{\partial^2 g}{\partial z^2}(t,z) = 0,$$

with $h(t,z) = 1$, $z > 0$, hence

$$\mathrm{e}^{(T-t)r}\frac{\partial^2 g}{\partial t \partial z}(t,z) - r\mathrm{e}^{(T-t)r}\frac{\partial g}{\partial z}(t,z) + \mathrm{e}^{(T-t)r}\left(\frac{1}{T} - rz\right)\frac{\partial^2 g}{\partial z^2}(t,z)$$
$$+\sigma^2 z\mathrm{e}^{(T-t)r}\frac{\partial^2 g}{\partial z^2}(t,z) + \frac{1}{2}\mathrm{e}^{(T-t)r}\sigma^2 z^2 \frac{\partial^3 g}{\partial z^3}(t,z) = 0,$$

or

$$\frac{\partial h}{\partial t}(t,z) + \left(\frac{1}{T} + (\sigma^2 - r)z\right)\frac{\partial h}{\partial z}(t,z) + \frac{1}{2}\sigma^2 z^2 \frac{\partial^2 h}{\partial z^2}(t,z) = 0,$$

with the terminal condition $h(T,z) = \mathbb{1}_{\{z>0\}}$. On the other hand, we have

$$\eta_t = \frac{1}{A_t}\left(f(t, S_t, \Lambda_t) - \xi_t S_t\right)$$

$$
\begin{aligned}
&= \frac{1}{A_t}\left(\frac{\Lambda_t}{T} - K\right)\frac{\partial g}{\partial z}\left(t, \frac{1}{S_t}\left(\frac{\Lambda_t}{T} - K\right)\right) \\
&= \frac{\mathrm{e}^{-(T-t)r}}{A_t}\left(\frac{\Lambda_t}{T} - K\right) h\left(t, \frac{1}{S_t}\left(\frac{\Lambda_t}{T} - K\right)\right).
\end{aligned}
$$

Exercise 13.12 Asian options with dividends. When reinvesting dividends, the portfolio self-financing condition reads

$$
\begin{aligned}
dV_t = {} & \underbrace{\eta_t dA_t + \xi_t dS_t}_{\text{Trading profit and loss}} + \underbrace{\delta \xi_t S_t dt}_{\text{Dividend payout}} \\
&= r\eta_t A_t dt + \xi_t((\mu - \delta)S_t dt + \sigma S_t dB_t) + \delta \xi_t S_t dt \\
&= r\eta_t A_t dt + \xi_t(\mu S_t dt + \sigma S_t dB_t) \\
&= rV_t dt + (\mu - r)\xi_t S_t dt + \sigma \xi_t S_t dB_t, \qquad t \in \mathbb{R}_+.
\end{aligned}
$$

On the other hand, by Itô's formula we have

$$
\begin{aligned}
& dg_\delta(t, S_t, \Lambda_t) \\
&= \frac{\partial g_\delta}{\partial t}(t, S_t, \Lambda_t)dt + \frac{\partial g_\delta}{\partial y}(t, S_t, \Lambda_t)d\Lambda_t + (\mu - \delta)S_t\frac{\partial g_\delta}{\partial x}(t, S_t, \Lambda_t)dt \\
&\quad + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 g_\delta}{\partial x^2}(t, S_t, \Lambda_t)dt + \sigma S_t \frac{\partial g_\delta}{\partial x}(t, S_t, \Lambda_t)dB_t \\
&= \frac{\partial g_\delta}{\partial t}(t, S_t, \Lambda_t)dt + S_t\frac{\partial g_\delta}{\partial y}(t, S_t, \Lambda_t)dt + (\mu - \delta)S_t\frac{\partial g_\delta}{\partial x}(t, S_t, \Lambda_t)dt \\
&\quad + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 g_\delta}{\partial x^2}(t, S_t, \Lambda_t)dt + \sigma S_t \frac{\partial g_\delta}{\partial x}(t, S_t, \Lambda_t)dB_t,
\end{aligned}
$$

hence by identification of the terms in dB_t and dt in the expressions of dV_t and $dg_\delta(t, S_t)$, we get

$$
\xi_t = \frac{\partial g_\delta}{\partial x}(t, S_t, \Lambda_t),
$$

and we derive the Black-Scholes PDE with dividend

$$
\begin{aligned}
rg_\delta(t, x, y) = {} & \frac{\partial g_\delta}{\partial t}(t, x, y) + y\frac{\partial g_\delta}{\partial y}(t, x, y) \\
& + (r - \delta)x\frac{\partial g_\delta}{\partial x}(t, x, y) + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 g_\delta}{\partial x^2}(t, x, y).
\end{aligned}
\tag{S.13.56}
$$

Defining $f(t, x, y) := \mathrm{e}^{(T-t)\delta} g_\delta(t, x, y)$ and substituting

$$
g_\delta(t, x, y) = \mathrm{e}^{(T-t)\delta} f(t, x, y)
$$

in (S.13.56) yields the equation

$$rf(t,x,y) = \delta f(t,x,y) + y\frac{\partial f}{\partial y}(t,x,y) + \frac{\partial f}{\partial t}(t,x,y)$$
$$+(r-\delta)x\frac{\partial f}{\partial x}(t,x,y) + \frac{1}{2}\sigma^2x^2\frac{\partial^2 f}{\partial x^2}(t,x,y),$$

i.e.

$$(r-\delta)f(t,x,y) = \frac{\partial f}{\partial t}(t,x,y) + y\frac{\partial f}{\partial y}(t,x,y)$$
$$+(r-\delta)x\frac{\partial f}{\partial x}(t,x,y) + \frac{1}{2}\sigma^2x^2\frac{\partial^2 f}{\partial x^2}(t,x,y),$$

whose solution $f(t,x,y)$ is the Asian option pricing function with modified interest rate $r-\delta$ and no dividends, under the terminal condition

$$f(T,x,y) = g_\delta(T,x,y) = \left(\frac{y}{T} - K\right)^+.$$

Therefore the Asian option price $g_\delta(t,S_t,\Lambda_t)$ with dividend rate δ can be recovered from the relation

$$g_\delta(t,x,y) = \mathrm{e}^{(T-t)\delta} f(t,x,y), \qquad t\in[0,T],\ x,y>0.$$

Note that we can also define

$$h(t,x,y) := g_\delta\big(t, x\mathrm{e}^{-\delta(T-t)}, y\big)$$

and substituting

$$g_\delta(t,x,y) = h\big(t, x\mathrm{e}^{\delta(T-t)}, y\big)$$

in (S.13.56) yields the equation

$$rh(t,x,y) = y\frac{\partial h}{\partial y}(t,x,y) + \frac{\partial h}{\partial t}(t,x,y)$$
$$+rx\frac{\partial h}{\partial x}(t,x,y) + \frac{1}{2}\sigma^2x^2\frac{\partial^2 h}{\partial x^2}(t,x,y),$$

whose solution $h(t,x,y)$ is the Asian option pricing function with interest rate r and no dividends, under the terminal condition

$$h(T,x,y) = g_\delta(T,x,y) = \left(\frac{y}{T} - K\right)^+.$$

Finally, the Asian option price $g_\delta(t,S_t,\Lambda_t)$ with dividend rate δ can be also recovered from the relation

$$g_\delta(t,x,y) = h\big(t, x\mathrm{e}^{-(T-t)\delta}, y\big), \qquad t\in[0,T],\ x,y>0.$$

Chapter 14

Exercise 14.1

a) The process $((2-B_t)^+)_{t\in\mathbb{R}_+}$ is a convex function $x \longmapsto (2-x)^+$ of the Brownian martingale $(B_t)_{t\in\mathbb{R}_+}$, hence it is a *sub*martingale by Proposition 14.4-(a).

b) Taking $\sigma := 1$ and $\mu := \sigma^2/2 > 0$, the process e^{B_t} can be written as

$$\mathrm{e}^{B_t} = \mathrm{e}^{\sigma B_t - \sigma^2 t/2 + \mu t} = \mathrm{e}^{\mu t}\mathrm{e}^{\sigma B_t - \sigma^2 t/2}, \qquad t \in \mathbb{R}_+,$$

hence it is a *sub*martingale as the driftless geometric Brownian motion $\mathrm{e}^{\sigma B_t - \sigma^2 t/2}$ is a martingale.

c) When $t > 0$, the question "is $\nu > t$?" cannot be answered at time t without waiting to know the value of B_{2t} at time $2t > t$. Therefore ν is *not* a stopping time.

d) For any $t \in \mathbb{R}_+$, the question "is $\tau > t$?" can be answered based on the observation of the paths of $(B_s)_{0\le s\le t}$ and of the (deterministic) curve $(\mathrm{e}^{s/2} + \alpha s \mathrm{e}^{s/2})_{0\le s\le t}$ up to the time t. Therefore τ is a stopping time.

e) Since τ is a stopping time and $\left(\mathrm{e}^{B_t - t/2}\right)_{t\in\mathbb{R}_+}$ is a martingale, the *Stopping Time Theorem* 14.7 shows that $\left(\mathrm{e}^{B_{t\wedge\tau} - (t\wedge\tau)/2}\right)_{t\in\mathbb{R}_+}$ is also a martingale and, in particular, its expected value*

$$\mathbb{E}\left[\mathrm{e}^{B_{t\wedge\tau} - (t\wedge\tau)/2}\right] = \mathbb{E}\left[\mathrm{e}^{B_{0\wedge\tau} - (0\wedge\tau)/2}\right] = \mathbb{E}\left[\mathrm{e}^{B_0 - 0/2}\right] = 1$$

is constantly equal to 1 for all $t \ge 0$. This shows that

$$\begin{aligned}\mathbb{E}\left[\mathrm{e}^{B_\tau - \tau/2}\right] &= \mathbb{E}\left[\lim_{t\to\infty} \mathrm{e}^{B_{t\wedge\tau} - (t\wedge\tau)/2}\right] \\ &= \lim_{t\to\infty} \mathbb{E}\left[\mathrm{e}^{B_{t\wedge\tau} - (t\wedge\tau)/2}\right] \\ &= 1.\end{aligned}$$

Next, we note that $\mathrm{e}^{B_\tau} = (\alpha + \beta\tau)\mathrm{e}^{\tau/2}$ at time τ, hence $\alpha + \beta\tau = \mathrm{e}^{B_\tau - \tau/2}$ and

$$\alpha + \beta\,\mathbb{E}[\tau] = \mathbb{E}[\alpha + \beta\tau] = \mathbb{E}\left[\mathrm{e}^{B_\tau - \tau/2}\right] = 1,$$

i.e. $\mathbb{E}[\tau] = (1-\alpha)/\beta$.

Remark: This argument also recovers $\mathbb{E}[\tau] = 0$ when $\alpha = 1$, however it fails when ($\alpha > 1$ and $\beta > 0$) and when ($\alpha < 1$ and $\beta < 0$), because τ is not *a.s.* finite ($\mathbb{P}(\tau < \infty) < 1$) in those cases.

Exercise 14.2 Stopping times.

* We let $t \wedge \tau := \min(t, \tau)$.

a) When $0 \leq t < 1$ the question "is $\nu > t$?" cannot be answered at time t without waiting to know the value of B_1 at time 1. Therefore ν is *not* a stopping time.

b) For any $t \in \mathbb{R}_+$, the question "is $\tau > t$?" can be answered based on the observation of the paths of $(B_s)_{0\leq s\leq t}$ and of the (deterministic) curve $\left(\alpha \mathrm{e}^{-s/2}\right)_{0\leq s\leq t}$ up to the time t. Therefore τ is a stopping time.

Since τ is a stopping time and $(B_t)_{t\in\mathbb{R}_+}$ is a martingale, the *Stopping Time Theorem* 14.7 shows that $\left(\mathrm{e}^{B_{t\wedge\tau}-(t\wedge\tau)/2}\right)_{t\in\mathbb{R}_+}$ is also a martingale and in particular its expected value

$$\mathbb{E}\left[\mathrm{e}^{B_{t\wedge\tau}-(t\wedge\tau)/2}\right] = \mathbb{E}\left[\mathrm{e}^{B_{0\wedge\tau}-(0\wedge\tau)/2}\right] = \mathbb{E}\left[\mathrm{e}^{B_0-0/2}\right] = 1$$

is constantly equal to 1 for all t. This shows that

$$\mathbb{E}\left[\mathrm{e}^{B_\tau-\tau/2}\right] = \mathbb{E}\left[\lim_{t\to\infty} \mathrm{e}^{B_{\tau\wedge t}-(\tau\wedge t)/2}\right] = \lim_{t\to\infty} \mathbb{E}\left[\mathrm{e}^{B_{\tau\wedge t}-(\tau\wedge t)/2}\right] = 1.$$

Next, we note that we have $\mathrm{e}^{B_\tau} = \alpha \mathrm{e}^{-\tau/2}$ at time τ, hence

$$\alpha\, \mathbb{E}\left[\mathrm{e}^{-\tau}\right] = \mathbb{E}\left[\mathrm{e}^{B_\tau-\tau/2}\right] = 1, \quad i.e. \quad \mathbb{E}\left[\mathrm{e}^{-\tau}\right] = \frac{1}{\alpha} \leq 1.$$

Remark: This argument fails when $\alpha < 1$ because in that case τ is not *a.s.* finite.

c) For any $t \in \mathbb{R}_+$, the question "is $\tau > t$?" can be answered based on the observation of the paths of $(B_s)_{0\leq s\leq t}$ and of the (deterministic) curve $(1+\alpha s)_{0\leq s\leq t}$ up to the time t. Therefore τ is a stopping time.

Since τ is a stopping time and $(B_t)_{t\in\mathbb{R}_+}$ is a martingale, the *Stopping Time Theorem* 14.7 shows that $(B^2_{t\wedge\tau} - (t\wedge\tau))_{t\in\mathbb{R}_+}$ is also a martingale and in particular its expected value

$$\mathbb{E}[B^2_{t\wedge\tau} - (t\wedge\tau)] = \mathbb{E}[B^2_{0\wedge\tau} - (0\wedge\tau)] = \mathbb{E}[B^2_0 - 0] = 0$$

is constantly equal to 0 for all t. This shows that

$$\mathbb{E}[B^2_\tau - \tau] = \mathbb{E}\left[\lim_{t\to\infty}(B^2_{t\wedge\tau} - (t\wedge\tau))\right] = \lim_{t\to\infty}\mathbb{E}\left[(B^2_{t\wedge\tau} - (t\wedge\tau))\right] = 0.$$

Next, we note that $B^2_\tau = 1+\alpha\tau$ at time τ, hence

$$1+\alpha\,\mathbb{E}[\tau] = \mathbb{E}[1+\alpha\tau] = \mathbb{E}[B^2_\tau] - \mathbb{E}[\tau] = 0,$$

i.e.

$$\mathbb{E}[\tau] = \frac{1}{1-\alpha}.$$

Remark: This argument is valid whenever $\alpha \leq 1$ and yields $\mathbb{E}[\tau] = +\infty$ when $\alpha = 1$, however it fails when $\alpha > 1$ because in that case τ is not *a.s.*

finite.

Exercise 14.3

a) By the Stopping Time Theorem 14.7, for all $n \geq 0$ we have

$$\begin{aligned}
1 &= \mathbb{E}\left[e^{\sqrt{2r}B_{\tau_L \wedge n} - r(\tau_L \wedge n)}\right] \\
&= \mathbb{E}\left[e^{\sqrt{2r}B_{\tau_L \wedge n} - r(\tau_L \wedge n)} \mathbb{1}_{\{\tau_L < n\}}\right] + \mathbb{E}\left[e^{\sqrt{2r}B_{\tau_L \wedge n} - r(\tau_L \wedge n)} \mathbb{1}_{\{\tau_L \geq n\}}\right] \\
&= \mathbb{E}\left[e^{\sqrt{2r}B_{\tau_L} - r\tau_L} \mathbb{1}_{\{\tau_L < n\}}\right] + \mathbb{E}\left[e^{\sqrt{2r}B_n - rn} \mathbb{1}_{\{\tau_L \geq n\}}\right] \\
&= e^{L\sqrt{2r}} \mathbb{E}\left[e^{-r\tau_L} \mathbb{1}_{\{\tau_L < n\}}\right] + \mathbb{E}\left[e^{\sqrt{2r}B_n - rn} \mathbb{1}_{\{\tau_L \geq n\}}\right].
\end{aligned}$$

The first term above converges to

$$e^{L\sqrt{2r}} \mathbb{E}\left[e^{-r\tau_L} \mathbb{1}_{\{\tau_L < \infty\}}\right] = e^{L\sqrt{2r}} \mathbb{E}\left[e^{-r\tau_L}\right]$$

as n tends to infinity, by dominated or monotone convergence and the fact that $\underline{r > 0}$. The second term can be bounded as

$$0 \leq \mathbb{E}\left[e^{\sqrt{2r}B_n - rn} \mathbb{1}_{\{\tau_L \geq n\}}\right] \leq e^{-rn} \mathbb{E}\left[e^{L\sqrt{2r}} \mathbb{1}_{\{\tau_L \geq n\}}\right] \leq e^{-rn} e^{L\sqrt{2r}},$$

which tends to 0 as n tends to infinity because $\underline{r > 0}$. Therefore we have

$$1 = \lim_{n \to \infty} \mathbb{E}\left[e^{\sqrt{2r}B_{\tau_L \wedge n} - r(\tau_L \wedge n)}\right] = e^{L\sqrt{2r}} \mathbb{E}\left[e^{-r\tau_L}\right],$$

which yields $\mathbb{E}\left[e^{-r\tau_L}\right] = e^{-L\sqrt{2r}}$ for any $\underline{r \geq 0}$. When $r < 0$ we could in fact show that $\mathbb{E}\left[e^{-r\tau_L}\right] = +\infty$.

b) In order to maximize the quantity

$$\begin{aligned}
\mathbb{E}\left[e^{-r\tau_L} B_{\tau_L}\right] &= \mathbb{E}\left[e^{-r\tau_L} B_{\tau_L} \mathbb{1}_{\{\tau_L < \infty\}}\right] \\
&= L\, \mathbb{E}\left[e^{-r\tau_L} \mathbb{1}_{\{\tau_L < \infty\}}\right] \\
&= L\, \mathbb{E}\left[e^{-r\tau_L}\right] \\
&= L e^{-L\sqrt{2r}},
\end{aligned}$$

we differentiate

$$\frac{\partial}{\partial L}(L e^{-L\sqrt{2r}}) = e^{-L\sqrt{2r}} - L\sqrt{2r} e^{-L\sqrt{2r}} = 0,$$

which yields the optimal level $L^* = 1/\sqrt{2r}$.

This shows that when the value of r is "large" the better strategy is to opt for a "small gain" at the level $L^* = 1/\sqrt{2r}$ rather than to wait for a longer time.

Exercise 14.4 See *e.g.* Theorem 6.16 page 161 of Klebaner (2005). By the Itô formula, we have

$$\begin{aligned}
X_t &= f(B_t) - \frac{1}{2}\int_0^t f''(B_s)ds \\
&= f(B_0) + \int_0^t f'(B_s)dB_s + \frac{1}{2}\int_0^t f''(B_s)ds - \frac{1}{2}\int_0^t f''(B_s)ds \\
&= f(B_0) + \int_0^t f'(B_s)dB_s,
\end{aligned}$$

hence the process $(X_t)_{t\in\mathbb{R}_+}$ is a *martingale*. By the Stopping Time Theorem 14.7 we have

$$\begin{aligned}
f(x) &= \mathbb{E}[X_0 \mid B_0 = x] \\
&= \mathbb{E}[X_{\tau\wedge t} \mid B_0 = x] \\
&= \mathbb{E}[f(B_{\tau\wedge t}) \mid B_0 = x] - \frac{1}{2}\mathbb{E}\left[\int_0^{\tau\wedge t} f''(B_s)ds \mid B_0 = x\right] \\
&= \mathbb{E}[f(B_{\tau\wedge t}) \mid B_0 = x] + \mathbb{E}[\tau\wedge t \mid B_0 = x],
\end{aligned}$$

since $f''(y) = -2$ for all $y \in \mathbb{R}$. We note that, by dominated convergence,

$$\begin{aligned}
\mathbb{E}\left[\tau \mid B_0 = x\right] &= \mathbb{E}\left[\lim_{t\to\infty}(\tau\wedge t) \mid B_0 = x\right] \\
&= \lim_{t\to\infty}\mathbb{E}[\tau\wedge t \mid B_0 = x] \\
&\leq |f(x)| + \max_{y\in[a,b]}|f(y)| \\
&< \infty,
\end{aligned}$$

hence $\mathbb{E}[\tau] < \infty$ and therefore $\mathbb{P}(\tau < \infty) = 1$, allowing us to write $\lim_{t\to\infty}(\tau\wedge t) = \tau < \infty$ with probability $\mathbb{P}(\tau < \infty) = 1$. Next, we have

$$\begin{aligned}
f(x) &= \lim_{t\to\infty}\mathbb{E}[f(B_{\tau\wedge t}) \mid B_0 = x] + \lim_{t\to\infty}\mathbb{E}[\tau\wedge t \mid B_0 = x] \\
&= \mathbb{E}\left[\lim_{t\to\infty} f(B_{\tau\wedge t}) \mid B_0 = x\right] + \mathbb{E}\left[\lim_{t\to\infty}(\tau\wedge t) \mid B_0 = x\right] \\
&= \mathbb{E}[f(B_\tau) \mid B_0 = x] + \mathbb{E}[\tau \mid B_0 = x] \\
&= \mathbb{E}[\tau \mid B_0 = x],
\end{aligned}$$

since $f''(x) = -2$ and $f(a) = f(b) = 0$ with $B_\tau \in \{a, b\}$.

Remarks.

i) The above exchanges between $\lim_{t\to\infty}$ and the expectation operator $\mathbb{E}[\,\cdot \mid B_0 = x]$ is justified by the *dominated convergence theorem*, since

$$|f(B_{\tau\wedge t})| \leq \max_{y\in[a,b]} |f(y)|, \qquad t \in \mathbb{R}_+.$$

ii) The function $f(x)$ can be determined by searching for a quadratic solution of the form $f(x) = \alpha+\beta x+\gamma x^2$, which shows that $f''(x) = 2\gamma = -2$ hence $\gamma = -1$, and

$$\begin{cases} f(a) = \alpha + \beta a - a^2 = 0, \\ \\ f(b) = \alpha + \beta b - b^2 = 0, \end{cases}$$

hence $\alpha = -ab$ and $\beta = a + b$. Therefore, we have

$$\mathbb{E}[\tau \mid B_0 = x] = f(x) = -ab + (a + b)x - x^2 = (x - a)(b - x).$$

Exercise 14.5 We use the Stopping Time Theorem 14.7 and the fact that $\left(e^{\sigma B_t - \sigma^2 t/2}\right)_{t\in\mathbb{R}_+}$ is a martingale for all $\sigma \in \mathbb{R}$. By the stopping time theorem, for all $n \geq 0$ we have

$$\begin{aligned} 1 &= \mathbb{E}\left[e^{\sigma B_{\tau\wedge n} - \sigma^2(\tau\wedge n)/2}\right] \\ &= \mathbb{E}\left[e^{\sigma B_{\tau\wedge n} - \sigma^2(\tau\wedge n)/2}\mathbb{1}_{\{\tau<n\}}\right] + \mathbb{E}\left[e^{\sigma B_{\tau\wedge n} - \sigma^2(\tau\wedge n)/2}\mathbb{1}_{\{\tau\geq n\}}\right] \\ &= \mathbb{E}\left[e^{\sigma B_\tau - \sigma^2\tau/2}\mathbb{1}_{\{\tau<n\}}\right] + \mathbb{E}\left[e^{\sigma B_n - \sigma^2 n/2}\mathbb{1}_{\{\tau\geq n\}}\right] \\ &= e^{\sigma\alpha}\,\mathbb{E}\left[e^{\sigma\beta\tau - \sigma^2\tau/2}\mathbb{1}_{\{\tau<n\}}\right] + \mathbb{E}\left[e^{\sigma B_n - \sigma^2 n/2}\mathbb{1}_{\{\tau\geq n\}}\right]. \end{aligned} \tag{S.14.57}$$

Under the condition $\sigma^2 \geq 2\sigma\beta$ we have $0 \leq e^{\sigma\beta\tau-\sigma^2\tau/2} \leq 1$, hence by dominated or monotone convergence we find

$$\lim_{n\to\infty} \mathbb{E}\left[e^{\sigma\beta\tau-\sigma^2\tau/2}\mathbb{1}_{\{\tau<n\}}\right] = \mathbb{E}\left[e^{\sigma\beta\tau-\sigma^2\tau/2}\lim_{n\to\infty}\mathbb{1}_{\{\tau<n\}}\right] = \mathbb{E}\left[e^{\sigma\beta\tau-\sigma^2\tau/2}\mathbb{1}_{\{\tau<\infty\}}\right],$$

and the first term in (S.14.57) converges to

$$e^{\sigma\alpha}\,\mathbb{E}\left[e^{-(\sigma^2/2-\sigma\beta)\tau}\mathbb{1}_{\{\tau<\infty\}}\right] = e^{\sigma\alpha}\,\mathbb{E}\left[e^{-(\sigma^2/2-\sigma\beta)\tau}\right]$$

as n tends to infinity. In what follows we use the solutions $\sigma_\pm = \beta \pm \sqrt{\beta^2 + 2r}$ of the equation $r = \sigma^2/2 - \sigma\beta$ with $\sigma_+ \geq 0$ and $\sigma_- \leq 0$, and we distinguish two cases.

a) If $\alpha \geq 0$, we have $B_n \leq \alpha + \beta n$, $n \leq \tau$, hence the second term above can be bounded as

$$\begin{aligned} 0 &\leq \mathbb{E}\left[e^{\sigma_+ B_n - n\sigma_+^2/2}\mathbb{1}_{\{\tau\geq n\}}\right] \\ &\leq e^{-n\sigma_+^2/2}\,\mathbb{E}\left[e^{\alpha\sigma_+ + \beta\sigma_+ n}\mathbb{1}_{\{\tau\geq n\}}\right] \\ &\leq e^{\alpha\sigma_+ - n\sigma_+^2/2 + \beta\sigma_+ n} \end{aligned}$$

$$= e^{\alpha\sigma_+ - rn},$$

which tends to 0 as n tends to infinity. Therefore, we have

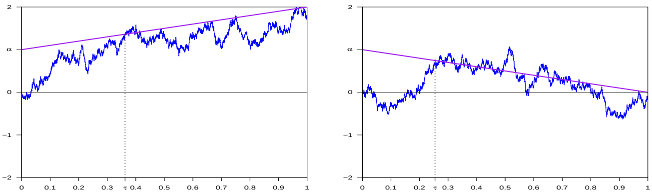
$$1 = \lim_{n\to\infty} \mathbb{E}\left[e^{\sigma_+ B_{\tau\wedge n} - \sigma_+^2(\tau\wedge n)/2}\right] = e^{\alpha\sigma_+} \mathbb{E}\left[e^{-(\sigma_+^2/2-\beta\sigma_+)\tau}\right],$$

which yields

$$\begin{aligned}\mathbb{E}\left[e^{-r\tau}\right] &= \mathbb{E}\left[e^{-(\sigma_+^2/2-\beta\sigma_+)\tau}\right] \\ &= e^{-\alpha\sigma_+} \\ &= e^{-\alpha\beta-\alpha\sqrt{\beta^2+2r}} \\ &= e^{-\alpha\beta-|\alpha|\sqrt{\beta^2+2r}},\end{aligned}$$

with

$$\mathbb{P}(\tau < +\infty) = \mathbb{E}[\mathbb{1}_{\{\tau<\infty\}}] = \lim_{r\to 0} \mathbb{E}\left[\mathbb{1}_{\{\tau<\infty\}} e^{-r\tau}\right] = \begin{cases} e^{-2\alpha\beta} & \text{if } \beta \geq 0, \\ 1 & \text{if } \beta \leq 0. \end{cases}$$



(a) $\alpha = 1$, $\beta = 0.5$

(b) $\alpha = 1$, $\beta = -0.5$

Fig. S.72: Hitting times of a straight line started at $\alpha < 0$.

b) If $\alpha \leq 0$, we have $B_n \geq \alpha + \beta n$, $n \leq \tau$, hence the second term above can be bounded as

$$\begin{aligned}0 &\leq \mathbb{E}\left[e^{\sigma_- B_n - n\sigma_-^2/2}\mathbb{1}_{\{\tau\geq n\}}\right] \\ &\leq e^{-n\sigma_-^2/2}\,\mathbb{E}\left[e^{\alpha\sigma_- + \beta n\sigma_-}\mathbb{1}_{\{\tau\geq n\}}\right] \leq e^{\alpha\sigma_- - n\sigma_-^2/2 + \beta n\sigma_-} \\ &= e^{\alpha\sigma_- - rn},\end{aligned}$$

which tends to 0 as n tends to infinity. Therefore, we have

$$1 = \lim_{n\to\infty} \mathbb{E}\left[e^{\sigma_- B_{\tau\wedge n} - \sigma_-^2(\tau\wedge n)/2}\right] = e^{\alpha\sigma_-} \mathbb{E}\left[e^{-(\sigma_-^2/2-\beta\sigma_-)\tau}\right],$$

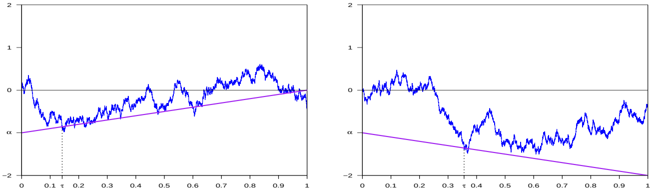
which yields

$$\mathbb{E}\left[e^{-r\tau}\right] = \mathbb{E}\left[e^{-(\sigma_-^2/2-\beta\sigma_-)\tau}\right] = e^{-\alpha\sigma_-}$$

$$
\begin{aligned}
&= \mathrm{e}^{-\alpha\beta+\alpha\sqrt{\beta^2+2r}} \\
&= \mathrm{e}^{-\alpha\beta-|\alpha|\sqrt{\beta^2+2r}},
\end{aligned}
$$

with

$$
\mathbb{P}(\tau < +\infty) = \mathbb{E}[\mathbb{1}_{\{\tau<\infty\}}] = \lim_{r\to 0} \mathbb{E}\left[\mathbb{1}_{\{\tau<\infty\}}\mathrm{e}^{-r\tau}\right] = \begin{cases} 1 & \text{if } \beta \geq 0, \\ \mathrm{e}^{-2\alpha\beta} & \text{if } \beta \leq 0. \end{cases}
$$



(a) $\alpha = -1$, $\beta = 0.5$

(b) $\alpha = -1$, $\beta = -0.5$

Fig. S.73: Hitting times of a straight line started at $\alpha < 0$.

Exercise 14.6

a) Letting $A_0 := 0$,

$$
A_{n+1} := A_n + \mathbb{E}[M_{n+1} - M_n \mid \mathcal{F}_n], \quad n \geq 0,
$$

and

$$
N_n := M_n - A_n, \qquad n \in \mathbb{N}, \tag{S.14.58}
$$

we have

(i) for all $n \in \mathbb{N}$,

$$
\begin{aligned}
\mathbb{E}[N_{n+1} \mid \mathcal{F}_n] &= \mathbb{E}[M_{n+1} - A_{n+1} \mid \mathcal{F}_n] \\
&= \mathbb{E}[M_{n+1} - A_n - \mathbb{E}[M_{n+1} - M_n \mid \mathcal{F}_n] \mid \mathcal{F}_n] \\
&= \mathbb{E}[M_{n+1} - A_n \mid \mathcal{F}_n] - \mathbb{E}[\mathbb{E}[M_{n+1} - M_n \mid \mathcal{F}_n] \mid \mathcal{F}_n] \\
&= \mathbb{E}[M_{n+1} - A_n \mid \mathcal{F}_n] - \mathbb{E}[M_{n+1} - M_n \mid \mathcal{F}_n] \\
&= -\mathbb{E}[A_n \mid \mathcal{F}_n] + \mathbb{E}[M_n \mid \mathcal{F}_n] \\
&= M_n - A_n \\
&= N_n,
\end{aligned}
$$

hence $(N_n)_{n\in\mathbb{N}}$ is a martingale with respect to $(\mathcal{F}_n)_{n\in\mathbb{N}}$.

(ii) We have

$$
A_{n+1} - A_n = \mathbb{E}[M_{n+1} - M_n \mid \mathcal{F}_n]
$$

$$\begin{aligned} &= \mathbb{E}[M_{n+1} \mid \mathcal{F}_n] - \mathbb{E}[M_n \mid \mathcal{F}_n] \\ &= \mathbb{E}[M_{n+1} \mid \mathcal{F}_n] - M_n \geq 0, \qquad n \in \mathbb{N}, \end{aligned}$$

since $(M_n)_{n\in\mathbb{N}}$ is a *sub*martingale.

(iii) By induction we have

$$A_n = A_{n-1} + \mathbb{E}[M_n - M_{n-1} \mid \mathcal{F}_{n-1}], \quad n \geq 1,$$

which is $\mathcal{F}_{n-1}$-measurable provided that A_n is $\mathcal{F}_{n-1}$-measurable, $n \geq 1$.

(iv) This property is obtained by construction in (S.14.58).

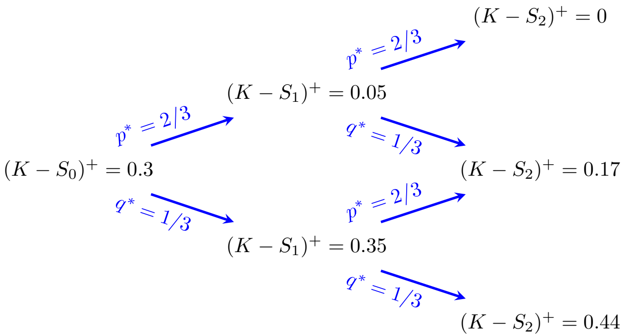
b) For all bounded stopping times σ and τ such that $\sigma \leq \tau$ *a.s.*, we have

$$\begin{aligned} \mathbb{E}[M_\sigma] &= \mathbb{E}[N_\sigma] + \mathbb{E}[A_\sigma] \\ &\leq \mathbb{E}[N_\sigma] + \mathbb{E}[A_\tau] \\ &= \mathbb{E}[N_\tau] + \mathbb{E}[A_\tau] \\ &= \mathbb{E}[M_\tau], \end{aligned}$$

by (14.7), since $(M_n)_{n\in\mathbb{N}}$ is a martingale and $(A_n)_{n\in\mathbb{N}}$ is non-decreasing.

Chapter 15

Exercise 15.1 The option payoffs at immediate exercise are given as follows:



On the other hand, the expected payoffs are given by:

$$\begin{tikzcd}[column sep=large]
 & (K-S_2)^+ = 0 \\
\mathbb{E}^*[(K-S_2)^+ \mid S_1 = 1.2] = 0.17/3 \arrow[ur] \arrow[dr] & \\
 & (K-S_2)^+ = 0.17 \\
\mathbb{E}^*[(K-S_2)^+ \mid S_1 = 0.9] = 0.26 \arrow[ur] \arrow[dr] & \\
 & (K-S_2)^+ = 0.44
\end{tikzcd}$$

Consequently, at time $t = 1$ we would exercise immediately if $S_1 = 0.9$, and wait if $S_1 = 1.2$. At time $t = 0$ with $S_0 = 1$ the initial value of the option is $(0.34/3 + 0.35)/3 = 1.39/9 \simeq 0.154 < 0.25$ so we would exercise immediately as well.

Exercise 15.2

a) Taking $f(x) := Cx^{-2r/\sigma^2}$, we have

$$\begin{aligned}
rxf'(x) + \frac{1}{2}\sigma^2 x^2 f''(x) &= -C\frac{2r^2}{\sigma^2}x^{-2r/\sigma^2} + Cr\left(1 + \frac{2r}{\sigma^2}\right)x^{-2r/\sigma^2} \\
&= Crx^{-2r/\sigma^2} \\
&= rf(x),
\end{aligned}$$

and the condition $\lim_{x\to\infty} f(x) = 0$ is satisfied since $r > 0$.

b) The conditions $f(L^*) = K - L^*$ and $f'(L^*) = -1$ read

$$\begin{cases}
C(L^*)^{-2r/\sigma^2} = K - L^*, \\[2ex]
-\dfrac{2r}{\sigma^2}C(L^*)^{-1-2r/\sigma^2} = -1,
\end{cases}$$

i.e.

$$\begin{cases}
C(L^*)^{-2r/\sigma^2} = K - L^* \\[2ex]
\dfrac{2r}{\sigma^2}(K - L^*) = L^*,
\end{cases}$$

hence

$$\begin{cases}
L^* = \dfrac{2rK}{2r + \sigma^2} \\[2ex]
C = \dfrac{K\sigma^2}{2r + \sigma^2}\left(\dfrac{2rK}{2r + \sigma^2}\right)^{2r/\sigma^2} = \dfrac{\sigma^2}{2r}\left(\dfrac{2rK}{2r + \sigma^2}\right)^{1+2r/\sigma^2}
\end{cases}.$$

Exercise 15.3

a) This an American put option with strike price considered at $S_0 \geq L^*$, hence by Propositions 15.2 and 15.4 the price of this option is

$$(K - L^*) \mathbb{E}^* \left[\mathrm{e}^{-\tau_{L^*}} \right] = (K - L^*) \left(\frac{S_0}{L^*} \right)^{-2r/\sigma^2}.$$

b) This an American put option with strike price K and immediately exercised at $S_0 \leq L^*$, hence by Propositions 15.2 and 15.4 the price of this option is $K - S_0$.

c) This is an American call option with strike price $2\widehat{K} - K$ exercised at the optimal level $L^* = \widehat{K}$, hence by Equation (15.23) the price of this option is

$$(L^* - (2\widehat{K} - K)) \mathbb{E}^* \left[\mathrm{e}^{-\tau_{L^*}} \right] = (\widehat{K} - (2\widehat{K} - K)) \mathbb{E}^* \left[\mathrm{e}^{-\tau_{L^*}} \right] = (K - \widehat{K}) \frac{S_0}{\widehat{K}}.$$

In conclusion, the pricing function is obtained by pasting together the pricing functions of American call and put options.

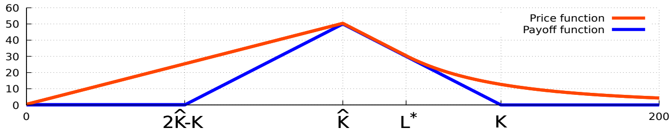


Fig. S.74: American butterfly payoff and price functions.

Exercise 15.4

a) Given the value

$$\frac{\partial}{\partial x} \mathrm{BS}_{\mathrm{p}}(x, T) = -\Phi(-d_+(x, T))$$

of the Delta of the Black-Scholes put option, see Proposition 6.7, the smooth fit condition states that at $x = S^*$, the left derivative of (15.31), which is

$$\frac{\partial}{\partial x} \mathrm{BS}_{\mathrm{p}}(x, T) + \alpha \frac{\partial}{\partial x} (x/S^*)^{-2r/\sigma^2} = -\Phi(-d_+(x, T)) + \alpha \frac{(S^*)^{2r/\sigma^2}}{x^{1+2r/\sigma^2}},$$

$x > S^*$, should match the right derivative of (15.32), which is -1, hence

$$-1 = -\Phi(-d_+(S^*, T)) - \frac{2r\alpha}{\sigma^2} (S^*)^{-1},$$

which yields

$$\alpha^* = \frac{\sigma^2 S^*}{2r}(1 - \Phi(-d_+(S^*, T))) = \frac{\sigma^2 S^*}{2r}\Phi(d_+(S^*, T)),$$

and

$$f(x,T) \simeq \begin{cases} \mathrm{BS_p}(x,T) + \dfrac{\sigma^2 (S^*)^{1+2r/\sigma^2}}{2r x^{2r/\sigma^2}}\Phi(d_+(S^*,T)), & x > S^*, \\ K - x, & x \le S^*. \end{cases}$$

Note that at maturity ($T = 0$ here) we have $d_+(S^*, 0) = -\infty$ since $S^* < K$, hence $\Phi(d_+(S^*, 0)) = 0$ and $f(x, 0) = K - x$ as expected.

b) Equating (15.31) to (15.32) at $x = S^*$ yields the equation

$$K - S^* = \mathrm{BS_p}(x,T) + \alpha^*,$$

i.e.

$$1 = \mathrm{e}^{-rT}\Phi(-d_-(S^*,T)) + \frac{S^*}{K}\left(1 + \frac{\sigma^2}{2r}\right)\Phi(d_+(S^*,T)),$$

which can be used to determine the value of S^*, and then the corresponding value of α. The proposed strategy is to exercise the put option as soon as the underlying asset price reaches the critical level S^*.

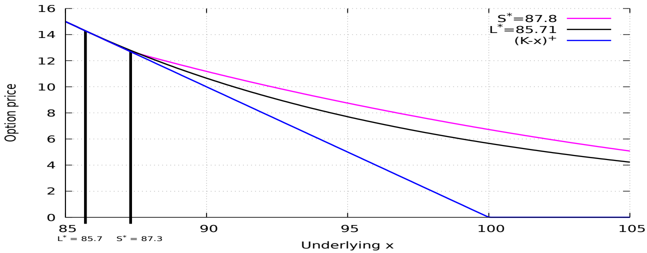


Fig. S.75: Perpetual *vs* finite expiration American put option price.

The plot in Figure S.75 yields a finite expiration critical price $S^* = 87.3$ which is expectedly higher than the perpetual critical price $L^* = 85.71$, with $K = 100$, $\sigma = 10\%$, and $r = 3\%$. The perpetual price, however, appears higher than the finite expiration price.

In Figure S.76 we plot the graph of a Barone-Adesi and Whaley (1987) approximation, together with the European put option price, using the R fOptions package. Note that this approximation is valid only for certain parameter ranges.

```r
r=0.1;sig=0.15;T=0.5;K=100;library(ragtop)
library(fOptions);payoff <- function(x){return(max(K-x,0))};vpayoff <- Vectorize(payoff)
par(new=TRUE)
curve(vpayoff, from=85, to=120, xlab="", lwd = 3, ylim=c(0,10),ylab="",col="red")
par(new=TRUE)
curve(blackscholes(callput=-1, x, K, r, T, sig, 0)$Price, from=85, to=120, xlab="", lwd = 3,
      ylim=c(0,10),ylab="",col="orange")
par(new=TRUE)
curve(BAWAmericanApproxOption("p",x,K,T,r,b=0,sig,title = NULL, description =
      NULL)@price, from=85, to=120 , xlab="Underlying asset price", lwd =
      3,ylim=c(0,10),ylab="",col="blue")
grid (lty = 5);legend(105,9.5,legend=c("Approximation","European payoff","Black-Scholes
      put"),col=c("blue","red","orange"),lty=1:1, cex=1.)
```

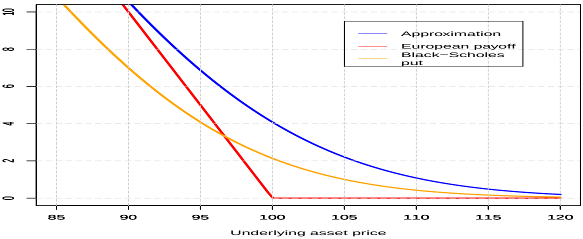


Fig. S.76: American put price approximation.

Exercise 15.5

a) We have

$$\tau_\epsilon = \begin{cases} \epsilon & \text{if } Z = 1, \\ \\ +\infty & \text{if } Z = 0. \end{cases}$$

b) First, we note that

$$\mathcal{F}_t = \begin{cases} \{\emptyset, \Omega\} & \text{if } t = 0, \\ \\ \{\emptyset, \Omega, \{Z = 0\}, \{Z = 1\}\} & \text{if } t > 0. \end{cases}$$

Next, we have

$$\{\tau_\epsilon > 0\} = \{Z = 0\},$$

hence

$$\{\tau_\epsilon > 0\} \notin \mathcal{F}_0 = \{\emptyset, \Omega\},$$

and therefore τ_0 *is not* an $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$-stopping time.

c) i) For $t = 0$ we have $\{\tau_\epsilon > 0\} = \{Z = 0\} \cup \{Z = 1\} = \Omega$, hence

$$\{\tau_\epsilon > 0\} \in \mathcal{F}_0 = \{\emptyset, \Omega\}.$$

ii) For $0 < t < \epsilon$ we have $\{\tau_\epsilon > t\} = \Omega$, hence

$$\{\tau_\epsilon > t\} \in \mathcal{F}_t = \{\emptyset, \Omega, \{Z = 0\}, \{Z = 1\}\}.$$

iii) For $t > \epsilon$ we have $\{\tau_\epsilon > t\} = \{Z = 0\}$, hence

$$\{\tau_\epsilon > t\} \in \mathcal{F}_t = \{\emptyset, \Omega, \{Z = 0\}, \{Z = 1\}\}.$$

Therefore τ_ϵ *is* an $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$-stopping time when $\epsilon > 0$.

Note that here the filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ is not right-continuous, as

$$\{\emptyset, \Omega\} = \mathcal{F}_0 \neq \mathcal{F}_{0^+} := \bigcap_{t>0} \mathcal{F}_t = \{\emptyset, \Omega, \{Z = 0\}, \{Z = 1\}\}.$$

Exercise 15.6

a) This intrinsic payoff is $\kappa - S_0$.
b) We note that the process $(Z_t)_{t\in\mathbb{R}_+}$ defined as

$$\begin{aligned} Z_t &:= \left(\frac{S_t}{S_0}\right)^\lambda \mathrm{e}^{-(r-\delta)\lambda t + \lambda\sigma^2 t/2 - \lambda^2\sigma^2 t/2} \\ &= \left(\mathrm{e}^{(r-\delta)t + \sigma\widehat{B}_t - \sigma^2 t/2}\right)^\lambda \mathrm{e}^{-(r-\delta)\lambda t + \lambda\sigma^2 t/2 - \lambda^2\sigma^2 t/2} \\ &= \mathrm{e}^{\lambda\sigma\widehat{B}_t - \lambda^2\sigma^2 t/2}, \qquad t \geq 0, \end{aligned}$$

is a geometric Brownian motion without drift, hence a martingale, under the risk-neutral probability measure $\mathbb{P}^*$.

c) The parameter λ should satisfy the equation

$$r = (r - \delta)\lambda - \frac{\sigma^2}{2}\lambda(1 - \lambda),$$

i.e.

$$\lambda^2\sigma^2/2 + \lambda(r - \delta - \sigma^2/2) - r = 0.$$

This equation admits two solutions

$$\lambda_\pm = \frac{-(r - \delta - \sigma^2/2) \pm \sqrt{(r - \delta - \sigma^2/2)^2 + 4r\sigma^2/2}}{\sigma^2},$$

d) We have

$$\begin{aligned} 0 \leq Z_t^{(\lambda_-)} &= \left(\frac{S_t}{S_0}\right)^{\lambda_-} \mathrm{e}^{-rt} \\ &\leq \left(\frac{S_t}{S_0}\right)^{\lambda_-} \end{aligned}$$

$$\leq \left(\frac{L}{S_0}\right)^{\lambda_-}, \qquad 0 \leq t < \tau_L,$$

since $\lambda_- < 0$ and $S_t > L$ for $t \in [0, \tau_L)$.

e) By the Stopping Time Theorem 14.7 we have

$$\mathbb{E}^*[Z_{\tau_L}] = \mathbb{E}^*[Z_0] = 1,$$

which rewrites as

$$\mathbb{E}^*\left[\left(\frac{S_{\tau_L}}{S_0}\right)^{\lambda} \mathrm{e}^{-((r-\delta)\lambda - \lambda\sigma^2/2 + \lambda^2\sigma^2/2)\tau_L}\right] = 1,$$

or, given the relation $S_{\tau_L} = L$,

$$\left(\frac{L}{S_0}\right)^{\lambda} \mathbb{E}^*\left[\mathrm{e}^{-((r-\delta)\lambda - \lambda\sigma^2/2 + \lambda^2\sigma^2/2)\tau_L}\right] = 1,$$

i.e.

$$\mathbb{E}^*\left[\mathrm{e}^{-r\tau_L}\right] = \left(\frac{S_0}{L}\right)^{\lambda},$$

provided that we choose λ such that

$$-((r-\delta)\lambda - \lambda\sigma^2/2 + \lambda^2\sigma^2/2) = -r, \tag{S.15.59}$$

i.e.

$$\lambda = \frac{-(r-\delta-\sigma^2/2) \pm \sqrt{(r-\delta-\sigma^2/2)^2 + 4r\sigma^2/2}}{\sigma^2},$$

and we choose the negative solution

$$\lambda := \frac{-(r-\delta-\sigma^2/2) - \sqrt{(r-\delta-\sigma^2/2)^2 + 4r\sigma^2/2}}{\sigma^2}$$

since $S_0/L = x/L > 1$ and the expectation $\mathbb{E}^*\left[\mathrm{e}^{-r\tau_L}\right] < 1$ is lower than 1 as $r \geq 0$.

f) This follows from (15.9) and the fact that $r > 0$. Using the fact that $S_{\tau_L} = L < K$ when $\tau_L < \infty$, we find

$$\begin{aligned}
\mathbb{E}^*\left[\mathrm{e}^{-r\tau_L}(K - S_{\tau_L})^+ \,\Big|\, S_0 = x\right] &= \mathbb{E}^*\left[\mathrm{e}^{-r\tau_L}(K - S_{\tau_L})^+ \mathbb{1}_{\{\tau_L < x\}} \,\Big|\, S_0 = x\right] \\
&= \mathbb{E}^*\left[\mathrm{e}^{-r\tau_L}(K - L) \mathbb{1}_{\{\tau_L < x\}} \,\Big|\, S_0 = x\right] \\
&= (K - L)\,\mathbb{E}^*\left[\mathrm{e}^{-r\tau_L} \mathbb{1}_{\{\tau_L < x\}} \,\Big|\, S_0 = x\right] \\
&= (K - L)\,\mathbb{E}^*\left[\mathrm{e}^{-r\tau_L} \,\Big|\, S_0 = x\right].
\end{aligned}$$

Next, noting that $\tau_L = 0$ if $S_0 \leq L$, for all $L \in (0, K)$ we have

$$\begin{aligned}
&\mathbb{E}^* \left[\mathrm{e}^{-r\tau_L} (K - S_{\tau_L})^+ \mid S_0 = x \right] \\
&= \begin{cases} K - x, & 0 < x \leq L, \\ \\ \mathbb{E}\left[\mathrm{e}^{-r\tau_L} (K - L)^+ \mid S_0 = x \right], & x \geq L. \end{cases} \\
&= \begin{cases} K - x, & 0 < x \leq L, \\ \\ (K - L)\,\mathbb{E}\left[\mathrm{e}^{-r\tau_L} \mid S_0 = x \right], & x \geq L. \end{cases} \\
&= \begin{cases} K - x, & 0 < x \leq L, \\ \\ (K - L)\left(\dfrac{x}{L}\right)^{\frac{-(r-a-\sigma^2/2)-\sqrt{(r-a-\sigma^2/2)^2+4r\sigma^2/2}}{\sigma^2}}, & x \geq L. \end{cases}
\end{aligned}$$

g) In order to compute L^* we observe that, geometrically, the slope of $x \longmapsto f_L(x) = (K - L)(x/L)^{\lambda_-}$ at $x = L^*$ is equal to -1, *i.e.*

$$f'_{L^*}(L^*) = \lambda_-(K - L^*)\frac{(L^*)^{\lambda_- - 1}}{(L^*)^{\lambda_-}} = -1,$$

hence

$$\lambda_-(K - L^*) = L^*, \quad \text{or} \quad L^* = \frac{\lambda_-}{\lambda_- - 1}K < K.$$

Equivalently we may recover the value of L^* from the optimality condition

$$\frac{\partial f_L(x)}{\partial L} = -\left(\frac{x}{L}\right)^{\lambda_-} - \lambda_- x(K - L)\left(\frac{x}{L}\right)^{\lambda_- + 1} = 0,$$

at $L = L^*$, hence

$$-\left(\frac{x}{L}\right)^{\lambda_-} - \lambda_-(K - L)x^{\lambda_-}L^{-\lambda_- - 1} = 0,$$

hence

$$L^* = \frac{\lambda_-}{1 - \lambda_-}K = \frac{1}{1 - 1/\lambda_-}K,$$

and

$$\sup_{L \in (0,K)} \mathbb{E}^* \left[\mathrm{e}^{-r\tau_L} (K - S_{\tau_L})^+ \mid S_0 = x \right] = -\frac{1}{\lambda_-}\left(\frac{K}{1 - 1/\lambda_-}\right)^{1-\lambda_-} x^{\lambda_-}.$$

h) For $x \geq L$ we have

$$f_{L^*}(x) = (K - L^*)\left(\frac{x}{L^*}\right)^{\lambda_-}$$

$$
\begin{aligned}
&= \left(K - \frac{\lambda_-}{\lambda_- - 1} K\right) \left(\frac{x}{\frac{\lambda_-}{\lambda_- - 1} K}\right)^{\lambda_-} \\
&= \left(-\frac{K}{\lambda_- - 1}\right) \left(\frac{x(\lambda_- - 1)}{\lambda_- K}\right)^{\lambda_-} \\
&= \left(-\frac{K}{\lambda_- - 1}\right) \left(\frac{x}{-\lambda_-}\right)^{\lambda_-} \left(\frac{\lambda_- - 1}{-K}\right)^{\lambda_-} \\
&= \left(\frac{x}{-\lambda_-}\right)^{\lambda_-} \left(\frac{\lambda_- - 1}{-K}\right)^{\lambda_- - 1} \\
&= \left(\frac{x}{K}\right)^{\lambda_-} \left(\frac{\lambda_- - 1}{\lambda_-}\right)^{\lambda_-} \frac{K}{1 - \lambda_-}. \qquad \text{(S.15.60)}
\end{aligned}
$$

i) Let us check that the relation

$$
f_{L^*}(x) \geq (K - x)^+ \qquad \text{(S.15.61)}
$$

holds. For all $x \leq K$ we have

$$
\begin{aligned}
f_{L^*}(x) - (K - x) &= \left(\frac{x}{K}\right)^{\lambda_-} \left(\frac{\lambda_- - 1}{\lambda_-}\right)^{\lambda_-} \frac{K}{1 - \lambda_-} + x - K \\
&= K \left(\left(\frac{x}{K}\right)^{\lambda_-} \left(\frac{\lambda_- - 1}{\lambda_-}\right)^{\lambda_-} \frac{1}{1 - \lambda_-} + \frac{x}{K} - 1 \right).
\end{aligned}
$$

Hence it suffices to take $K = 1$ and to show that for all

$$
L^* = \frac{\lambda_-}{\lambda_- - 1} \leq x \leq 1
$$

we have

$$
f_{L^*}(x) - (1 - x) = \frac{x^{\lambda_-}}{1 - \lambda_-} \left(\frac{\lambda_- - 1}{\lambda_-}\right)^{\lambda_-} + x - 1 \geq 0.
$$

Equality to 0 holds for $x = \lambda_- / (\lambda_- - 1)$. By differentiation of this relation we get

$$
\begin{aligned}
f'_{L^*}(x) - (1 - x)' &= \lambda_- x^{\lambda_- - 1} \left(\frac{\lambda_- - 1}{\lambda_-}\right)^{\lambda_-} \frac{1}{1 - \lambda_-} + 1 \\
&= x^{\lambda_- - 1} \left(\frac{\lambda_- - 1}{\lambda_-}\right)^{\lambda_- - 1} + 1 \\
&\geq 0,
\end{aligned}
$$

hence the function $f_{L^*}(x) - (1-x)$ is non-decreasing and the inequality holds throughout the interval $[\lambda_-/(\lambda_- - 1), K]$.

On the other hand, using (S.15.59) it can be checked by hand that f_{L^*} given by (S.15.60) satisfies the equality

$$(r-\delta)x f'_{L^*}(x) + \frac{1}{2}\sigma^2 x^2 f''_{L^*}(x) = r f_{L^*}(x) \tag{S.15.62}$$

for $x \geq L^* = \dfrac{\lambda_-}{\lambda_- - 1}K$. In case

$$0 \leq x \leq L^* = \frac{\lambda_-}{\lambda_- - 1}K < K,$$

we have

$$f_{L^*}(x) = K - x = (K-x)^+,$$

hence the relation

$$\left(r f_{L^*}(x) - (r-\delta)x f'_{L^*}(x) - \frac{1}{2}\sigma^2 x^2 f''_{L^*}(x) \right) \left(f_{L^*}(x) - (K-x)^+ \right) = 0$$

always holds. On the other hand, in that case we also have

$$(r-\delta)x f'_{L^*}(x) + \frac{1}{2}\sigma^2 x^2 f''_{L^*}(x) = -(r-\delta)x,$$

and to conclude we need to show that

$$(r-\delta)x f'_{L^*}(x) + \frac{1}{2}\sigma^2 x^2 f''_{L^*}(x) \leq r f_{L^*}(x) = r(K-x), \tag{S.15.63}$$

which is true if

$$\delta x \leq rK.$$

Indeed, by (S.15.59) we have

$$\begin{aligned}(r-\delta)\lambda_- &= r + \lambda_-(\lambda_- - 1)\sigma^2/2 \\ &\geq r,\end{aligned}$$

hence

$$\delta \frac{\lambda_-}{\lambda_- - 1} \leq r,$$

since $\lambda_- < 0$, which yields

$$\delta x \leq \delta L^* \leq \delta \frac{\lambda_-}{\lambda_- - 1} K \leq rK.$$

j) By Itô's formula and the relation

$$dS_t = (r-\delta)S_t dt + \sigma S_t d\widehat{B}_t$$

we have

$$\begin{aligned}
d\big(\widetilde{f}_{L^*}(S_t)\big) &= -r\mathrm{e}^{-rt} f_{L^*}(S_t)dt + \mathrm{e}^{-rt} df_{L^*}(S_t) \\
&= -r\mathrm{e}^{-rt} f_{L^*}(S_t)dt + \mathrm{e}^{-rt} f'_{L^*}(S_t)dS_t + \frac{\sigma^2}{2}\mathrm{e}^{-rt} S_t^2 f''_{L^*}(S_t) \\
&= \mathrm{e}^{-rt}\left(-rf_{L^*}(S_t) + (r-\delta)S_t f'_{L^*}(S_t) + \frac{\sigma^2}{2}S_t^2 f''_{L^*}(S_t)\right)dt \\
&\quad + \mathrm{e}^{-rt}\sigma S_t f'_{L^*}(S_t)d\widehat{B}_t,
\end{aligned}$$

and from Equations (S.15.62) and (S.15.63) we have

$$(r-\delta)xf'_{L^*}(x) + \frac{1}{2}\sigma^2 x^2 f''_{L^*}(x) \le rf_{L^*}(x),$$

hence

$$t \longmapsto \mathrm{e}^{-rt} f_{L^*}(S_t)$$

is a *super*martingale.

k) By the *super*martingale property of

$$t \longmapsto \mathrm{e}^{-rt} f_{L^*}(S_t),$$

for all stopping times τ we have

$$f_{L^*}(S_0) \ge \mathbb{E}^*\left[\mathrm{e}^{-r\tau} f_{L^*}(S_\tau) \;\middle|\; S_0\right] \ge \mathbb{E}^*\left[\mathrm{e}^{-r\tau}(K-S_\tau)^+ \;\middle|\; S_0\right],$$

by (S.15.61), hence

$$f_{L^*}(S_0) \ge \sup_{\tau \text{ stopping time}} \mathbb{E}^*\left[\mathrm{e}^{-r\tau}(K-S_\tau)^+ \;\middle|\; S_0\right].$$

l) The stopped process

$$t \longmapsto \mathrm{e}^{-rt\wedge\tau_{L^*}} f_{L^*}(S_{t\wedge\tau_{L^*}})$$

is a martingale since it has vanishing drift up to time τ_{L^*} by (S.15.62), and it is constant after time τ_{L^*}, hence by the Stopping Time Theorem 14.7 we find

$$\begin{aligned}
f_{L^*}(S_0) &= \mathbb{E}^*\left[\mathrm{e}^{-r\tau} f_{L^*}(S_{\tau_{L^*}}) \;\middle|\; S_0\right] \\
&= \mathbb{E}^*\left[\mathrm{e}^{-r\tau} f_{L^*}(L^*) \;\middle|\; S_0\right] \\
&= \mathbb{E}^*\left[\mathrm{e}^{-r\tau}(K-S_{\tau_{L^*}})^+ \;\middle|\; S_0\right] \\
&\le \sup_{\tau \text{ stopping time}} \mathbb{E}^*\left[\mathrm{e}^{-r\tau}(K-S_\tau)^+ \;\middle|\; S_0\right].
\end{aligned}$$

m) By combining the above results and conditioning at time t instead of time 0 we deduce that

$$f_{L^*}(S_t) = \mathbb{E}^*\left[e^{-r(\tau_{L^*}-t)}(K - S_{\tau_{L^*}})^+ \;\Big|\; S_t \right]$$
$$= \begin{cases} K - S_t, & 0 < S_t \leq \dfrac{\lambda_-}{\lambda_- - 1}K, \\[2ex] \left(\dfrac{\lambda_- - 1}{-K}\right)^{\lambda_- - 1}\left(\dfrac{-S_t}{\lambda_-}\right)^{\lambda_-}, & S_t \geq \dfrac{\lambda_-}{\lambda_- - 1}K, \end{cases}$$

for all $t \in \mathbb{R}_+$, where

$$\tau_{L^*} = \inf\{u \geq t \;:\; S_u \leq L\}.$$

We note that the perpetual put option price does not depend on the value of $t \geq 0$.

Exercise 15.7

a) We have

$$Z_t^{(\lambda)} = (S_t)^\lambda e^{-t((r-\delta)\lambda - \lambda(1-\lambda)\sigma^2/2)} = (S_0)^\lambda e^{\lambda\sigma\widehat{B}_t - \lambda^2\sigma^2 t/2},$$

which is a driftless geometric Brownian motion, and therefore a martingale under $\mathbb{P}^*$.

b) The condition is $r = (r-\delta)\lambda - \lambda(1-\lambda)\sigma^2/2$, with solutions

$$\lambda_- = \frac{\delta - r + \sigma^2/2 - \sqrt{(\delta - r + \sigma^2/2)^2 + 2r\sigma^2}}{\sigma^2} \leq 0,$$

$$\lambda_+ = \frac{\delta - r + \sigma^2/2 + \sqrt{(\delta - r + \sigma^2/2)^2 + 2r\sigma^2}}{\sigma^2} \geq 1.$$

c) Due to the inequality

$$0 \leq Z_t^{(\lambda_+)} = (S_t)^{\lambda_+} e^{-rt} \leq L^{\lambda_+},$$

which holds because $\lambda_+ > 0$ and $S_t \leq L$, $0 \leq t < \tau_L$, we note that since $\lim_{t\to\infty} Z_t^{(\lambda_+)} = 0$, we have

$$\begin{aligned} L^{\lambda_+}\, \mathbb{E}^*\left[e^{-r\tau_L}\right] &= \mathbb{E}^*\left[\left(S_{\tau_L}\right)^{\lambda_+} e^{-r\tau_L} \mathbb{1}_{\{\tau_L<\infty\}}\right] = \mathbb{E}^*\left[Z_{\tau_L}^{(\lambda_+)} \mathbb{1}_{\{\tau_L<\infty\}}\right] \\ &= \mathbb{E}^*\left[\lim_{t\to\infty} Z_{\tau_L\wedge t}^{(\lambda_+)}\right] = \lim_{t\to\infty} \mathbb{E}^*\left[Z_{\tau_L\wedge t}^{(\lambda_+)}\right] \\ &= \mathbb{E}^*\left[Z_0^{(\lambda_+)}\right] = (S_0)^{\lambda_+}. \end{aligned}$$

Therefore, we have

$$\begin{aligned}
\mathbb{E}^*\left[\mathrm{e}^{-r\tau_L}(S_{\tau_L}-K)^+ \;\middle|\; S_0=x\right] &= \mathbb{E}^*\left[(S_{\tau_L}-K)\mathrm{e}^{-r\tau_L}\mathbb{1}_{\{\tau_L<\infty\}} \;\middle|\; S_0=x\right] \\
&= (L-K)\,\mathbb{E}^*\left[\mathrm{e}^{-r\tau_L}\mathbb{1}_{\{\tau_L<\infty\}} \;\middle|\; S_0=x\right] \\
&= (L-K)\,\mathbb{E}^*\left[\mathrm{e}^{-r\tau_L} \;\middle|\; S_0=x\right] \\
&= (L-K)\left(\frac{x}{L}\right)^{\lambda_+},
\end{aligned}$$

when $S_0=x>L$. In order to maximize

$$\sup_{L\in(0,K)} \mathbb{E}^*\left[\mathrm{e}^{-r\tau_L}(K-S_{\tau_L})^+ \;\middle|\; S_0=x\right],$$

we differentiate $L\longmapsto (L-K)\,(x/L)^{\lambda_+}$ with respect to L, to find

$$\left(\frac{x}{L}\right)^{\lambda+} - \lambda_+(L-K)x^{\lambda_+}L^{-\lambda_+-1}=0,$$

hence

$$L^*_\delta = \frac{\lambda_+}{\lambda_+-1}K = \frac{K}{1-1/\lambda_+},$$

and

$$\sup_{L\in(0,K)} \mathbb{E}^*\left[\mathrm{e}^{-r\tau_L}(K-S_{\tau_L})^+ \;\middle|\; S_0=x\right] = \frac{1}{\lambda_+}\left(\frac{K}{1-1/\lambda_+}\right)^{1-\lambda_+} x^{\lambda_+}.$$

We note that as $\delta\searrow 0$ we have $\lambda_+\searrow 1$ and $L^*_\delta\nearrow\infty$, and since

$$\left(\frac{K}{1-1/\lambda_+}\right)^{1-\lambda_+} = \exp\left((\lambda_+-1)\log\frac{\lambda_+-1}{\lambda_+K}\right)\to 1,$$

we find that the perpetual American call option price without dividend ($\delta=0$) is $S_0=x$.

Exercise 15.8

a) By the definition (15.36) of $S_1(t)$ and $S_2(t)$ we have

$$\begin{aligned}
Z_t &= \mathrm{e}^{-rt}S_2(t)\left(\frac{S_1(t)}{S_2(t)}\right)^\alpha \\
&= \mathrm{e}^{-rt}S_1(t)^\alpha S_2(t)^{1-\alpha} \\
&= S_1(0)^\alpha S_2(0)^{1-\alpha}\mathrm{e}^{(\alpha\sigma_1+(1-\alpha)\sigma_2)W_t-\sigma_2^2t/2},
\end{aligned}$$

which is a martingale when

$$\sigma_2^2 = (\alpha\sigma_1+(1-\alpha)\sigma_2)^2,$$

i.e.

$$\alpha\sigma_1 + (1-\alpha)\sigma_2 = \pm\sigma_2,$$

which yields either $\alpha = 0$ or

$$\alpha = \frac{2\sigma_2}{\sigma_2 - \sigma_1} = 1 + \frac{\sigma_1 + \sigma_2}{\sigma_2 - \sigma_1} > 1,$$

since $0 \le \sigma_1 < \sigma_2$.

b) We have

$$\begin{aligned}\mathbb{E}\left[e^{-r\tau_L}(S_1(\tau_L) - S_2(\tau_L))^+\right] &= \mathbb{E}\left[e^{-r\tau_L}(LS_2(\tau_L) - S_2(\tau_L))^+\right] \\ &= (L-1)^+\,\mathbb{E}\left[e^{-r\tau_L}S_2(\tau_L)\right]. \quad (S.15.64)\end{aligned}$$

c) Since $\tau_L \wedge t$ is a bounded stopping time we can write

$$S_2(0)\left(\frac{S_1(0)}{S_2(0)}\right)^\alpha = \mathbb{E}\left[e^{-r(\tau_L\wedge t)}S_2(\tau_L\wedge t)\left(\frac{S_1(\tau_L\wedge t)}{S_2(\tau_L\wedge t)}\right)^\alpha\right] \quad (S.15.65)$$
$$= \mathbb{E}\left[e^{-r\tau_L}S_2(\tau_L)\left(\frac{S_1(\tau_L)}{S_2(\tau_L)}\right)^\alpha \mathbb{1}_{\{\tau_L<t\}}\right] + \mathbb{E}\left[e^{-rt}S_2(t)\left(\frac{S_1(t)}{S_2(t)}\right)^\alpha \mathbb{1}_{\{\tau_L>t\}}\right]$$

We have

$$e^{-rt}S_2(t)\left(\frac{S_1(t)}{S_2(t)}\right)^\alpha \mathbb{1}_{\{\tau_L>t\}} \le e^{-rt}S_2(t)L^\alpha\mathbb{1}_{\{\tau_L>t\}} \le e^{-rt}S_2(t)L^\alpha,$$

hence by a uniform integrability argument,

$$\lim_{t\to\infty}\mathbb{E}\left[e^{-rt}S_2(t)\left(\frac{S_1(t)}{S_2(t)}\right)^\alpha \mathbb{1}_{\{\tau_L>t\}}\right] = 0,$$

and letting t go to infinity in (S.15.65) shows that

$$S_2(0)\left(\frac{S_1(0)}{S_2(0)}\right)^\alpha = \mathbb{E}\left[e^{-r\tau_L}S_2(\tau_L)\left(\frac{S_1(\tau_L)}{S_2(\tau_L)}\right)^\alpha\right] = L^\alpha\,\mathbb{E}\left[e^{-r\tau_L}S_2(\tau_L)\right],$$

since $S_1(\tau_L)/S_2(\tau_L) = L/L = 1$. The conclusion

$$\mathbb{E}\left[e^{-r\tau_L}(S_1(\tau_L) - S_2(\tau_L))^+\right] = (L-1)^+L^{-\alpha}S_2(0)\left(\frac{S_1(0)}{S_2(0)}\right)^\alpha \quad (S.15.66)$$

then follows by an application of (S.15.64).

d) In order to maximize (S.15.66) as a function of L we consider the derivative

$$\frac{\partial}{\partial L}\left(\frac{L-1}{L^\alpha}\right) = \frac{1}{L^\alpha} - \alpha(L-1)L^{-\alpha-1} = 0,$$

which vanishes for

$$L^* = \frac{\alpha}{\alpha - 1},$$

and we substitute L in (S.15.66) with the value of L^*.

e) In addition to $r = \sigma_2^2/2$ it is sufficient to let $S_1(0) = \kappa$ and $\sigma_1 = 0$ which yields $\alpha = 2$, $L^* = 2$, and we find

$$\sup_{\tau \text{ stopping time}} \mathbb{E}\left[\mathrm{e}^{-r\tau}(\kappa - S_2(\tau))^+\right] = \frac{1}{S_2(0)}\left(\frac{\kappa}{2}\right)^2,$$

which coincides with the result of Proposition 15.4.

Exercise 15.9

a) It suffices to check the sign of the quantity

$$(\lambda - 1)(\lambda + 2r/\sigma^2), \tag{S.15.67}$$

in (15.38), which is positive when $\lambda \in (-\infty, -2r/\sigma^2] \cup [1, \infty)$, and negative when $-2r/\sigma^2 \leq \lambda \leq 1$.

b) The sign of (S.15.67) is positive when $\lambda \in (-\infty, 1] \cup [-2r/\sigma^2, \infty)$, and negative when $1 \leq \lambda \leq -2r/\sigma^2$.

c) By the Stopping Time Theorem 14.7, for any $n \geq 0$ we have

$$\begin{aligned}
x^\lambda &= \mathbb{E}^*\left[\mathrm{e}^{-r(\tau_L \wedge n)} Z^{(\lambda)}_{\tau_L \wedge n} \mid S_0 = x\right] \\
&= \mathbb{E}^*\left[Z^{(\lambda)}_{\tau_L} \mathbb{1}_{\{\tau_L < n\}} \mid S_0 = x\right] + \mathrm{e}^{-rn}\, \mathbb{E}^*\left[Z^{(\lambda)}_n \mathbb{1}_{\{\tau_L > n\}} \mid S_0 = x\right] \\
&\geq \mathbb{E}^*\left[\mathrm{e}^{-r\tau_L}(S_{\tau_L})^\lambda \mathbb{1}_{\{\tau_L < n\}} \mid S_0 = x\right] \\
&= L^\lambda\, \mathbb{E}^*\left[\mathrm{e}^{-r\tau_L} \mathbb{1}_{\{\tau_L < n\}} \mid S_0 = x\right].
\end{aligned}$$

By the results of Questions (a)-(b), the process $\left(Z^{(\lambda)}_t\right)_{t\in\mathbb{R}_+}$ is a martingale when $\lambda \in \{1, -2r\sigma^2/2\}$. Next, letting n to infinity, by monotone convergence we find

$$\mathbb{E}^*\left[\mathrm{e}^{-r\tau_L} \mathbb{1}_{\{\tau_L < \infty\}} \mid S_0 = x\right] \leq \left(\frac{x}{L}\right)^\lambda \leq \begin{cases} \left(\dfrac{x}{L}\right)^{\max(1,-2r/\sigma^2)}, & x \geq L, \\[2ex] \left(\dfrac{x}{L}\right)^{\min(1,-2r/\sigma^2)}, & 0 < x \leq L. \end{cases}$$

d) We note that $\mathbb{P}^*(\tau_L < \infty) = 1$ by (14.15), hence if $-\sigma^2/2 \leq r < 0$ we have

$$\begin{aligned}
&\mathbb{E}^*\left[\mathrm{e}^{-r\tau_L}(K - S_{\tau_L})^+ \mathbb{1}_{\{\tau_L < \infty\}} \mid S_0 = x\right] \\
&= (K - L)\, \mathbb{E}^*\left[\mathrm{e}^{-r\tau_L} \mid S_0 = x\right] \leq \begin{cases} (K - L)\left(\dfrac{x}{L}\right)^{-2r/\sigma^2}, & x \geq L, \\[2ex] (K - L)\dfrac{x}{L}, & 0 < x \leq L. \end{cases}
\end{aligned}$$

Similarly, if $r \le -\sigma^2/2$ we have

$$\mathbb{E}^*\left[\mathrm{e}^{-r\tau_L}(K-S_{\tau_L})^+\mathbb{1}_{\{\tau_L<\infty\}} \mid S_0=x\right] = (K-L)\,\mathbb{E}^*\left[\mathrm{e}^{-r\tau_L} \mid S_0=x\right] \le \begin{cases} (K-L)\dfrac{x}{L}, & x \ge L, \\ (K-L)\left(\dfrac{x}{L}\right)^{-2r/\sigma^2}, & 0<x\le L. \end{cases}$$

e) This follows by noting that $(K-L)(x/L) = (K/L-1)x$ increases to ∞ when L tends to zero.

f) If $-\sigma^2/2 \le r < 0$ we have

$$\begin{aligned} &\mathbb{E}^*\left[\mathrm{e}^{-r\tau_L}(S_{\tau_L}-K)^+\mathbb{1}_{\{\tau_L<\infty\}} \mid S_0=x\right] \\ &= (L-K)\,\mathbb{E}^*\left[\mathrm{e}^{-r\tau_L}\mathbb{1}_{\{\tau_L<\infty\}} \mid S_0=x\right] \\ &\le \begin{cases} (L-K)\left(\dfrac{x}{L}\right)^{-2r/\sigma^2}, & x \ge L, \\ (L-K)\dfrac{x}{L}, & 0<x\le L. \end{cases} \end{aligned}$$

If $r \le -\sigma^2/2$ we have

$$\begin{aligned} &\mathbb{E}^*\left[\mathrm{e}^{-r\tau_L}(S_{\tau_L}-K)^+\mathbb{1}_{\{\tau_L<\infty\}} \mid S_0=x\right] \\ &= (L-K)\,\mathbb{E}^*\left[\mathrm{e}^{-r\tau_L}\mathbb{1}_{\{\tau_L<\infty\}} \mid S_0=x\right] \\ &\le \begin{cases} (L-K)\dfrac{x}{L}, & x \ge L, \\ (L-K)\left(\dfrac{x}{L}\right)^{-2r/\sigma^2}, & 0<x\le L. \end{cases} \end{aligned}$$

g) This follows by noting that for fixed $x>0$, the quantity $(L-K)x/L = (1-K/L)x$ increases to x when L tends to infinity.

Exercise 15.10 Perpetual American binary options.

a) Similarly, for $x \ge K$, immediate exercise is the optimal strategy and we have $C_b^{\mathrm{Am}}(t,x) = 1$. When $x<K$ the optimal exercise level of the perpetual American binary call option is $L^* = K$ with the optimal exercise time τ_K, and by *e.g.* (4.4.22) page 135 we have

$$\begin{aligned} C_b^{\mathrm{Am}}(t,x) &= \sup_{\substack{\tau \ge t \\ \tau \text{ stopping time}}} \mathbb{E}^*\left[\mathrm{e}^{-(\tau-t)r}\mathbb{1}_{\{S_\tau \ge K\}} \,\middle|\, S_t=x\right] \\ &= \mathbb{E}^*\left[\mathrm{e}^{-(\tau_K-t)r} \,\middle|\, S_t=x\right] \\ &= \frac{x}{K}, \qquad x<K. \end{aligned}$$

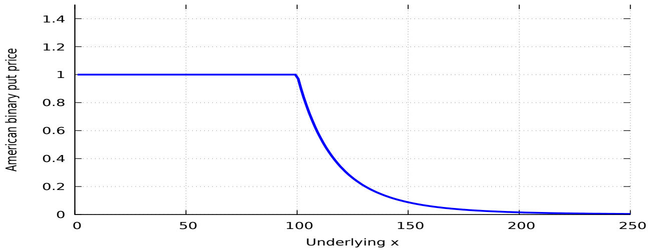


Fig. S.77: Perpetual American binary put price map with $K = 100$.

b) For $x \leq K$, immediate exercise is the optimal strategy and we have $P_b^{\mathrm{Am}}(t,x) = 1$. When $x > K$ the optimal exercise level of the perpetual American binary put option is $L^* = K$ with the optimal exercise time τ_K, and by *e.g.* (4.4.11) page 125 we have

$$\begin{aligned} P_b^{\mathrm{Am}}(t,x) &= \sup_{\substack{\tau \geq t \\ \tau \text{ stopping time}}} \mathbb{E}^* \left[\mathrm{e}^{-(\tau-t)r} \mathbb{1}_{\{S_\tau \leq K\}} \,\middle|\, S_t = x \right] \\ &= \mathbb{E}^* \left[\mathrm{e}^{-(\tau_K-t)r} \,\middle|\, S_t = x \right] \\ &= \left(\frac{x}{K} \right)^{-2r/\sigma^2}, \qquad x > K. \end{aligned}$$

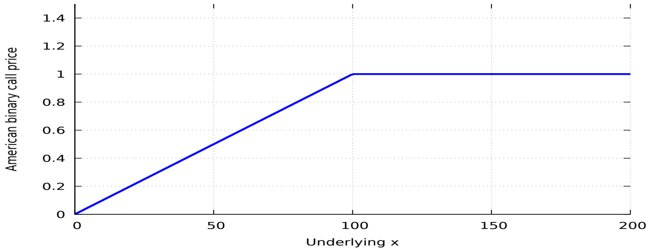


Fig. S.78: Perpetual American binary call price map with $K = 100$.

Exercise 15.11 Finite expiration American binary options.

a) The optimal strategy is as follows:

(i) if $S_t \geq K$, then exercise immediately.
(ii) if $S_t < K$, then wait.

b) The optimal strategy is as follows:

(i) if $S_t > K$, then wait.
(ii) if $S_t \leq K$, exercise immediately.

c) Based on the answers to Question (a) we set

$$C_d^{\mathrm{Am}}(t, T, K) = 1, \qquad 0 \leq t < T,$$

and

$$C_d^{\mathrm{Am}}(T, T, x) = 0, \qquad 0 \leq x < K.$$

d) Based on the answers to Question (b), we set

$$P_d^{\mathrm{Am}}(t, T, K) = 1, \qquad 0 \leq t < T,$$

and

$$P_d^{\mathrm{Am}}(T, T, x) = 0, \qquad x > K.$$

e) Starting from $S_t \leq K$, the maximum possible payoff is clearly reached as soon as S_t hits the level K before the expiration date T, hence the discounted optimal payoff of the option is $\mathrm{e}^{-r(\tau_K - t)} \mathbb{1}_{\{\tau_K < T\}}$.

f) From Relation (10.13), we find that the first hitting time τ_a of the level a by a μ-drifted Brownian motion $(W_u + \mu u)_{u \in \mathbb{R}_+}$ satisfies

$$\mathbb{P}(\tau_a \leq u) = \Phi\left(\frac{a - \mu u}{\sqrt{u}}\right) - \mathrm{e}^{2\mu a} \Phi\left(\frac{-a - \mu u}{\sqrt{u}}\right), \qquad u > 0,$$

and by differentiation with respect to u this yields the probability density function

$$f_{\tau_a}(u) = \frac{\partial}{\partial u} \mathbb{P}(\tau_a \leq u) = \frac{a}{\sqrt{2\pi u^3}} \mathrm{e}^{-(a-\mu u)^2/(2u)} \mathbb{1}_{[0,\infty)}(u)$$

of the first hitting time of level a by Brownian motion with drift μ. Given the relation

$$S_u = S_t \mathrm{e}^{\sigma W_{u-t} - (u-t)\sigma^2/2 + (u-t)r} = S_t \mathrm{e}^{(W_{u-t} + (u-t)\mu)\sigma}, \quad u \geq t,$$

with $\mu = r/\sigma - \sigma/2$, we find that $(S_u)_{u \in [t,\infty)}$ hits the level K at a time $\tau_K = t + \tau_a$, such that

$$S_{\tau_K} = S_t \mathrm{e}^{\sigma W_{\tau_a} - \sigma^2 \tau_a/2 + r\tau_a} = S_t \mathrm{e}^{(W_{\tau_a} + \mu\tau_a)\sigma} = K,$$

i.e.

$$a = W_{\tau_a} + \mu \tau_a = \frac{1}{\sigma} \log \frac{K}{S_t}.$$

Therefore, the probability density function of the first hitting time τ_K of level K after time t by $(S_u)_{u \in [t,\infty)}$ is given by

$$s \longmapsto \frac{a}{\sqrt{2\pi(s-t)^3}} \mathrm{e}^{-(a-(s-t)\mu)^2/(2(s-t))}, \qquad s > t,$$

with

$$\mu := \frac{1}{\sigma}\left(r - \frac{\sigma^2}{2}\right) \quad \text{and} \quad a := \frac{1}{\sigma}\log\frac{K}{x},$$

given that $S_t = x$. Hence, for $x \in (0, K)$ we have

$$\begin{aligned}
C_d^{\mathrm{Am}}(t,T,x) &= \mathbb{E}\left[\mathrm{e}^{-r(\tau_K - t)}\mathbb{1}_{\{\tau_K < T\}} \mid S_t = x\right] \\
&= \int_t^T \mathrm{e}^{-(s-t)r}\frac{a}{\sqrt{2\pi(s-t)^3}}\mathrm{e}^{-(a-(s-t)\mu)^2/(2(s-t))}ds \\
&= \int_0^{T-t} \mathrm{e}^{-rs}\frac{a}{\sqrt{2\pi s^3}}\mathrm{e}^{-(a-\mu s)^2/(2s)}ds \\
&= \int_0^{T-t} \frac{\log(K/x)}{\sigma\sqrt{2\pi s^3}}\exp\left(-rs - \frac{1}{2\sigma^2 s}\left(-\left(r-\frac{\sigma^2}{2}\right)s + \log\frac{K}{x}\right)^2\right)ds \\
&= \left(\frac{K}{x}\right)^{\left(r/\sigma^2 - 1/2\right) \pm \left(r/\sigma^2 + 1/2\right)} \\
&\quad \times \int_0^{T-t} \frac{\log(K/x)}{\sigma\sqrt{2\pi s^3}}\exp\left(-\frac{1}{2\sigma^2 s}\left(\pm\left(r+\frac{\sigma^2}{2}\right)s + \log\frac{K}{x}\right)^2\right)ds \\
&= \frac{1}{\sqrt{2\pi}}\frac{x}{K}\int_{y_-}^\infty \mathrm{e}^{-y^2/2}dy + \frac{1}{\sqrt{2\pi}}\left(\frac{K}{x}\right)^{2r/\sigma^2}\int_{y_+}^\infty \mathrm{e}^{-y^2/2}dy \\
&= \frac{x}{K}\Phi\left(\frac{(r+\sigma^2/2)(T-t) + \log(x/K)}{\sigma\sqrt{T-t}}\right) \\
&\quad + \left(\frac{x}{K}\right)^{-2r/\sigma^2}\Phi\left(\frac{-(r+\sigma^2/2)(T-t) + \log(x/K)}{\sigma\sqrt{T-t}}\right), \qquad 0 < x < K,
\end{aligned}$$

where

$$y_\pm = \frac{1}{\sigma\sqrt{T-t}}\left(\pm\left(r+\frac{\sigma^2}{2}\right)(T-t) + \log\frac{K}{x}\right),$$

and we used the decomposition

$$\log\frac{K}{x} = \frac{1}{2}\left(\left(r+\frac{\sigma^2}{2}\right)s + \log\frac{K}{x}\right) + \frac{1}{2}\left(-\left(r+\frac{\sigma^2}{2}\right)s + \log\frac{K}{x}\right).$$

We check that

$$C_d^{\mathrm{Am}}(T,T,K) = \Phi(0) + \Phi(0) = 1,$$

and

$$C_d^{\mathrm{Am}}(T,T,x) = \frac{x}{K}\Phi(-\infty) + \left(\frac{x}{K}\right)^{-2r/\sigma^2}\Phi(-\infty) = 0, \qquad x < K,$$

since $t = T$, which is consistent with the answers to Question (c).

In addition, as T tends to infinity we have

$$\begin{aligned}\lim_{T\to\infty} C_d^{\mathrm{Am}}(t,T,x) &= \frac{x}{K}\lim_{T\to\infty}\Phi\left(\frac{(r+\sigma^2/2)(T-t)+\log(x/K)}{\sigma\sqrt{T-t}}\right)\\ &\quad + \left(\frac{x}{K}\right)^{-2r/\sigma^2}\lim_{T\to\infty}\Phi\left(\frac{-(r+\sigma^2/2)(T-t)+\log(x/K)}{\sigma\sqrt{T-t}}\right)\\ &= \frac{x}{K}, \qquad 0<x<K,\end{aligned}$$

which is consistent with the answer to Question (a) of Exercise 15.10.

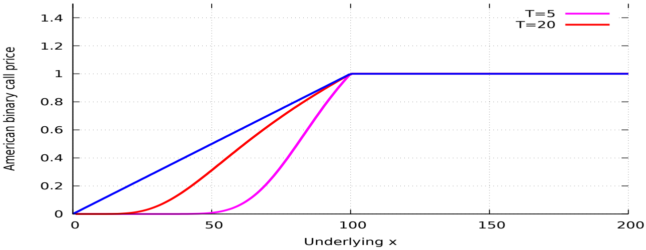


Fig. S.79: Finite expiration American binary call price map with $K = 100$.

g) Starting from $S_t \geq K$, the maximum possible payoff is clearly reached as soon as S_t hits the level K before the expiration date T, hence the discounted optimal payoff of the option is $\mathrm{e}^{-r(\tau_K-t)}\mathbb{1}_{\{\tau_K<T\}}$.

h) Using the notation and answer to Question (f), for $x > K$ we find

$$\begin{aligned}P_d^{\mathrm{Am}}(t,T,x) &= \mathbb{E}\left[\mathrm{e}^{-r(\tau_K-t)}\mathbb{1}_{\{\tau_K<T\}} \mid S_t = x\right]\\ &= \int_0^{T-t}\mathrm{e}^{-rs}\frac{a}{\sqrt{2\pi s^3}}\mathrm{e}^{-(a-\mu s)^2/(2s)}ds\\ &= \int_0^{T-t}\frac{\log(x/K)}{\sigma\sqrt{2\pi s^3}}\exp\left(-rs-\frac{1}{2\sigma^2 s}\left(\left(r-\frac{\sigma^2}{2}\right)s+\log\frac{x}{K}\right)^2\right)ds\\ &= \left(\frac{K}{x}\right)^{\left(\frac{r}{\sigma^2}-\frac{1}{2}\right)\pm\left(\frac{r}{\sigma^2}+\frac{1}{2}\right)}\\ &\quad\times\int_0^{T-t}\frac{\log(x/K)}{\sigma\sqrt{2\pi s^3}}\exp\left(-\frac{1}{2\sigma^2 s}\left(\mp\left(r+\frac{\sigma^2}{2}\right)s+\log\frac{x}{K}\right)^2\right)ds\\ &= \frac{1}{\sqrt{2\pi}}\frac{x}{K}\int_{y_-}^{\infty}\mathrm{e}^{-y^2/2}dy + \frac{1}{\sqrt{2\pi}}\left(\frac{x}{K}\right)^{2r/\sigma^2}\int_{y_+}^{\infty}\mathrm{e}^{-y^2/2}dy\end{aligned}$$

$$= \frac{x}{K}\Phi\left(\frac{-(r+\sigma^2/2)(T-t)-\log(x/K)}{\sigma\sqrt{T-t}}\right)$$
$$+\left(\frac{x}{K}\right)^{-2r/\sigma^2}\Phi\left(\frac{(r+\sigma^2/2)(T-t)-\log(x/K)}{\sigma\sqrt{T-t}}\right), \qquad x > K,$$

with

$$y_\pm = \frac{1}{\sigma\sqrt{T-t}}\left(\mp\left(r+\frac{\sigma^2}{2}\right)(T-t)+\log\frac{x}{K}\right).$$

We check that

$$P_d^{\mathrm{Am}}(T,T,K) = \Phi(0)+\Phi(0) = 1,$$

and

$$P_d^{\mathrm{Am}}(T,T,x) = \frac{x}{K}\Phi(-\infty) + \left(\frac{x}{K}\right)^{-2r/\sigma^2}\Phi(-\infty) = 0, \qquad 0 < x < K,$$

since $t = T$, which is consistent with the answers to Question (c).

In addition, as T tends to infinity we have

$$\lim_{T\to\infty} P_d^{\mathrm{Am}}(t,T,x) = \frac{x}{K}\lim_{T\to\infty}\Phi\left(\frac{-(r+\sigma^2/2)(T-t)-\log(x/K)}{\sigma\sqrt{T-t}}\right)$$
$$+\left(\frac{x}{K}\right)^{-2r/\sigma^2}\lim_{T\to\infty}\Phi\left(\frac{(r+\sigma^2/2)(T-t)-\log(x/K)}{\sigma\sqrt{T-t}}\right)$$
$$= \left(\frac{x}{K}\right)^{-2r/\sigma^2}, \qquad x > K,$$

which is consistent with the answer to Question (b) of Exercise 15.10

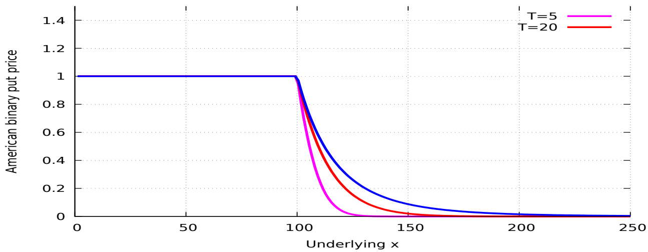


Fig. S.80: Finite expiration American binary put price map with $K = 100$.

i) The call-put parity does not hold for American binary options since for $x \in (0,K)$ we have

$$C_d^{\mathrm{Am}}(t,T,x)+P_d^{\mathrm{Am}}(t,T,x)=1+\frac{x}{K}\Phi\left(\frac{(r+\sigma^2/2)(T-t)+\log(x/K)}{\sigma\sqrt{T-t}}\right)$$
$$+\left(\frac{x}{K}\right)^{-2r/\sigma^2}\Phi\left(\frac{-(r+\sigma^2/2)(T-t)+\log(x/K)}{\sigma\sqrt{T-t}}\right),$$

while for $x>K$ we find

$$C_d^{\mathrm{Am}}(t,T,x)+P_d^{\mathrm{Am}}(t,T,x)=1+\frac{x}{K}\Phi\left(\frac{-(r+\sigma^2/2)(T-t)-\log(x/K)}{\sigma\sqrt{T-t}}\right)$$
$$+\left(\frac{x}{K}\right)^{-2r/\sigma^2}\Phi\left(\frac{(r+\sigma^2/2)(T-t)-\log(x/K)}{\sigma\sqrt{T-t}}\right).$$

Exercise 15.12 American forward Contracts.

a) For all (bounded) stopping times $\tau\in[t,T]$, since the discounted asset price process $\big(\widetilde{S}_u\big)_{u\in[t,\infty)}:=\big(\mathrm{e}^{-(u-t)r}S_u\big)_{u\in[t,\infty)}$ is a martingale and the stopped process $\big(\widetilde{S}_{\tau\wedge u}\big)_{u\in[t,\infty)}=\big(\mathrm{e}^{-(\tau\wedge u-t)r}S_{\tau\wedge u}\big)_{u\in[t,\infty)}$ is also a martingale by the Stopping Time Theorem 14.7, we have

$$\begin{aligned}
\mathbb{E}^*\left[\mathrm{e}^{-r(\tau-t)}(K-S_\tau)\;\middle|\;\mathcal{F}_t\right]&=K\,\mathbb{E}^*\left[\mathrm{e}^{-r(\tau-t)}\;\middle|\;\mathcal{F}_t\right]-\mathbb{E}^*\left[\mathrm{e}^{-r(\tau-t)}S_\tau\;\middle|\;\mathcal{F}_t\right]\\
&=K\,\mathbb{E}^*\left[\mathrm{e}^{-r(\tau-t)}\;\middle|\;\mathcal{F}_t\right]-\mathbb{E}^*\left[\widetilde{S}_{\tau\wedge T}\;\middle|\;\mathcal{F}_t\right]\\
&=K\,\mathbb{E}^*\left[\mathrm{e}^{-r(\tau-t)}\;\middle|\;\mathcal{F}_t\right]-\widetilde{S}_{\tau\wedge t}\\
&=K\,\mathbb{E}^*\left[\mathrm{e}^{-r(\tau-t)}\;\middle|\;\mathcal{F}_t\right]-\widetilde{S}_t\\
&=K\,\mathbb{E}^*\left[\mathrm{e}^{-r(\tau-t)}\;\middle|\;\mathcal{F}_t\right]-S_t,
\end{aligned}$$

and the above quantity is clearly maximized by taking $\tau=t$. Hence we have

$$f(t,S_t)=\sup_{\substack{t\le\tau\le T\\ \tau\ \text{stopping time}}}\mathbb{E}^*\left[\mathrm{e}^{-r(\tau-t)}(K-S_\tau)\;\middle|\;\mathcal{F}_t\right]=K-S_t,$$

and the optimal strategy is to exercise immediately (or to avoid purchasing the option) at time t and price K due to the effect of time value of money when $r>0$.

b) Similarly to the above, we have

$$\begin{aligned}
\mathbb{E}^*\left[\mathrm{e}^{-r(\tau-t)}(S_\tau-K)\;\middle|\;\mathcal{F}_t\right]&=\mathbb{E}^*\left[\mathrm{e}^{-r(\tau-t)}S_\tau\;\middle|\;\mathcal{F}_t\right]-K\,\mathbb{E}^*\left[\mathrm{e}^{-r(\tau-t)}\;\middle|\;\mathcal{F}_t\right]\\
&=S_t-K\,\mathbb{E}^*\left[\mathrm{e}^{-r(\tau-t)}\;\middle|\;\mathcal{F}_t\right],
\end{aligned}$$

since $\tau\in[t,T]$ is bounded and $(\mathrm{e}^{-rt}S_t)_{t\in\mathbb{R}_+}$ is a martingale. As the above quantity is clearly maximized by taking $\tau=T$, we have

$$f(t,S_t) = \sup_{\substack{t\le\tau\le T\\ \tau \text{ stopping time}}} \mathbb{E}^*\left[\mathrm{e}^{-r(\tau-t)}(S_\tau-K)\ \middle|\ \mathcal{F}_t\right] = S_t - \mathrm{e}^{-(T-t)r}K,$$

and the optimal strategy is to wait until the maturity time T in order to exercise at price K, due to the effect of time value of money when $r>0$.

c) Regarding the perpetual American long forward contract, since the discounted asset price process $\big(\widetilde{S}_u\big)_{u\in[t,\infty)} := \big(\mathrm{e}^{-(u-t)r}S_u\big)_{u\in[t,\infty)}$ is a martingale, by the Stopping Time Theorem 14.7, for all stopping times $\tau\ge t$ we have*,

$$\begin{aligned}
&\mathbb{E}^*\left[\mathrm{e}^{-r(\tau-t)}(S_\tau-K)\ \middle|\ \mathcal{F}_t\right] = \mathbb{E}^*\left[\mathrm{e}^{-r(\tau-t)}S_\tau\ \middle|\ \mathcal{F}_t\right] - K\,\mathbb{E}^*\left[\mathrm{e}^{-r(\tau-t)}\ \middle|\ \mathcal{F}_t\right]\\
&= \mathbb{E}^*\left[\widetilde{S}_\tau\ \middle|\ \mathcal{F}_t\right] - K\,\mathbb{E}^*\left[\mathrm{e}^{-r(\tau-t)}\ \middle|\ \mathcal{F}_t\right]\\
&= \widetilde{S}_t - K\,\mathbb{E}^*\left[\mathrm{e}^{-r(\tau-t)}\ \middle|\ \mathcal{F}_t\right]\\
&= S_t - K\,\mathbb{E}^*\left[\mathrm{e}^{-r(\tau-t)}\ \middle|\ \mathcal{F}_t\right]\\
&\le S_t, \qquad t\ge 0.
\end{aligned}$$

On the other hand, for all fixed $T>0$ we have

$$\begin{aligned}
\mathbb{E}^*\left[\mathrm{e}^{-r(T-t)}(S_T-K)\ \middle|\ \mathcal{F}_t\right] &= \mathrm{e}^{-r(T-t)}\,\mathbb{E}^*[S_T\mid\mathcal{F}_t] - \mathrm{e}^{-r(T-t)}\,\mathbb{E}^*[K\mid\ \mathcal{F}_t]\\
&= S_t - \mathrm{e}^{-r(T-t)}K, \qquad t\in[0,T],
\end{aligned}$$

hence

$$(S_t - \mathrm{e}^{-r(T-t)}K) \le \sup_{\substack{t\ge\tau\\ \tau \text{ stopping time}}} \mathbb{E}^*\left[\mathrm{e}^{-r(\tau-t)}(S_\tau-K)\ \middle|\ \mathcal{F}_t\right] \le S_t, \quad T\ge t,$$

and letting $T\to\infty$ we get

$$\begin{aligned}
S_t &= \lim_{T\to\infty}(S_t - \mathrm{e}^{-r(T-t)}K)\\
&\le \sup_{\substack{\tau\ge t\\ \tau \text{ stopping time}}} \mathbb{E}^*\left[\mathrm{e}^{-r(\tau-t)}(S_\tau-K)\ \middle|\ \mathcal{F}_t\right]\\
&\le S_t,
\end{aligned}$$

hence we have

$$f(t,S_t) = \sup_{\substack{\tau\ge t\\ \tau \text{ stopping time}}} \mathbb{E}^*\left[\mathrm{e}^{-r(\tau-t)}(S_\tau-K)\ \middle|\ \mathcal{F}_t\right] = S_t,$$

and the optimal strategy $\tau^* = +\infty$ is to wait indefinitely.

Regarding the perpetual American short forward contract, we have

* Using Fatou's Lemma.

$$\begin{aligned} f(t,S_t) &= \sup_{\substack{\tau\geq t\\ \tau \text{ stopping time}}} \mathbb{E}^*\left[\mathrm{e}^{-r(\tau-t)}(K-S_\tau) \;\middle|\; \mathcal{F}_t\right] \\ &\leq \sup_{\substack{\tau\geq t\\ \tau \text{ stopping time}}} \mathbb{E}^*\left[\mathrm{e}^{-r(\tau-t)}(K-S_\tau)^+ \;\middle|\; \mathcal{F}_t\right] \\ &= f_{L^*}(S_t), \end{aligned} \tag{S.15.68}$$

with

$$L^* = \frac{2r}{2r+\sigma^2}K < K$$

as defined in (15.12). On the other hand, for $\tau = \tau_{L^*}$ we have

$$(K - S_{\tau_{L^*}}) = (K - L^*) = (K - L^*)^+$$

since $0 < L^* = 2Kr/(2r+\sigma^2) < K$, hence

$$\begin{aligned} f_{L^*}(S_t) &= \mathbb{E}^*\left[\mathrm{e}^{-r(\tau_{L^*}-t)}(K-S_{\tau_{L^*}})^+ \;\middle|\; \mathcal{F}_t\right] \\ &= \mathbb{E}^*\left[\mathrm{e}^{-r(\tau_{L^*}-t)}(K-S_{\tau_{L^*}}) \;\middle|\; \mathcal{F}_t\right] \\ &\leq \sup_{\substack{\tau\geq t\\ \tau \text{ stopping time}}} \mathbb{E}^*\left[\mathrm{e}^{-r(\tau-t)}(K-S_\tau) \;\middle|\; \mathcal{F}_t\right] \\ &= f(t,S_t), \end{aligned}$$

which, together with (S.15.68), shows that

$$f(t,S_t) = f_{L^*}(S_t),$$

i.e. the perpetual American short forward contract has same price and exercise strategy as the perpetual American put option.

Exercise 15.13

a) We have

$$\begin{aligned} Y_t &= \mathrm{e}^{-rt}\big(S_0\mathrm{e}^{rt+\sigma\widehat{B}_t-\sigma^2t/2}\big)^{-2r/\sigma^2} \\ &= S_0^{-2r/\sigma^2}\mathrm{e}^{-rt-2r^2t/\sigma^2+2r\widehat{B}_t/\sigma+rt} \\ &= S_0^{-2r/\sigma^2}\mathrm{e}^{2r\widehat{B}_t/\sigma-(2r/\sigma)^2t/2}, \qquad t\geq 0, \end{aligned}$$

and

$$Z_t = \mathrm{e}^{-rt}S_t = S_0\mathrm{e}^{\sigma\widehat{B}_t-\sigma^2t/2}, \qquad t\geq 0,$$

which are both martingales under $\mathbb{P}^*$ because they are standard geometric Brownian motions with respective volatilities σ and $2r/\sigma$.

b) Since $(Y_t)_{t\in\mathbb{R}_+}$ and $(Z_t)_{t\in\mathbb{R}_+}$ are both martingales and τ_L is a stopping time, we have

$$\begin{aligned} S_0^{-2r/\sigma^2} &= \mathbb{E}^*[Y_0] \\ &= \mathbb{E}^*[Y_{\tau_L}] \\ &= \mathbb{E}^*\left[\mathrm{e}^{-r\tau_L} S_{\tau_L}^{-2r/\sigma^2}\right] \\ &= \mathbb{E}^*\left[\mathrm{e}^{-r\tau_L} L^{-2r/\sigma^2}\right] \\ &= L^{-2r/\sigma^2}\,\mathbb{E}^*\left[\mathrm{e}^{-r\tau_L}\right], \end{aligned}$$

hence

$$\mathbb{E}^*\left[\mathrm{e}^{-r\tau_L}\right] = \left(\frac{x}{L}\right)^{-2r/\sigma^2}$$

if $S_0 = x \geq L$ (note that in this case $Y_{\tau_L \wedge t}$ remains bounded by L^{-2r/σ^2}), and

$$S_0 = \mathbb{E}^*[Z_0] = \mathbb{E}^*[Z_{\tau_L}] = \mathbb{E}^*\left[\mathrm{e}^{-r\tau_L} S_{\tau_L}\right] = \mathbb{E}^*\left[\mathrm{e}^{-r\tau_L} L\right] = L\,\mathbb{E}^*\left[\mathrm{e}^{-r\tau_L}\right],$$

hence

$$\mathbb{E}^*\left[\mathrm{e}^{-r\tau_L}\right] = \frac{x}{L}$$

if $S_0 = x \leq L$. Note that in this case $Z_{\tau_L \wedge t}$ remains bounded by L.

c) We find

$$\begin{aligned} \mathbb{E}\left[\mathrm{e}^{-r\tau_L}(K - S_{\tau_L}) \mid S_0 = x\right] &= (K-L)\,\mathbb{E}^*\left[\mathrm{e}^{-r\tau_L} \mid S_0 = x\right] \\ &= \begin{cases} x\dfrac{K-L}{L}, & 0 < x \leq L, \\ (K-L)\left(\dfrac{x}{L}\right)^{-2r/\sigma^2}, & x \geq L. \end{cases} \end{aligned} \tag{S.15.69}$$

d) We check that when $L \geq x$, the maximum value of (S.15.69) is $K - x$, and that it is reached at $L^* := x$. On the other hand, when $L < x$, (S.15.69) can be maximized using $L^* := 2rK/(2r+\sigma^2)$ as in the perpetual American put option setting.

e) The stopping strategy $\tau_{L_x^*}$ would be suboptimal in comparison with the perpetual American put option stopping strategy, see Exercise 15.12-(c).

Exercise 15.14

a) The option payoff equals $(\kappa - S_t)^p$ if $S_t \leq L$.

b) We have

$$\begin{aligned} f_L(S_t) &= \mathbb{E}^*\left[\mathrm{e}^{-r(\tau_L - t)}((\kappa - S_{\tau_L})^+)^p \mid \mathcal{F}_t\right] \\ &= \mathbb{E}^*\left[\mathrm{e}^{-r(\tau_L - t)}((\kappa - L)^+)^p \mid \mathcal{F}_t\right] \\ &= (\kappa - L)^p\,\mathbb{E}^*\left[\mathrm{e}^{-r(\tau_L - t)} \mid \mathcal{F}_t\right]. \end{aligned}$$

c) We have

$$f_L(x) = \mathbb{E}^*\left[e^{-r(\tau_L - t)}(\kappa - S_{\tau_L})^+ \ \Big| \ \mathcal{F}_t = x \right]$$
$$= \begin{cases} (\kappa - x)^p, & 0 < x \le L, \\ \\ (\kappa - L)^p \left(\dfrac{L}{x}\right)^{2r/\sigma^2}, & x \ge L. \end{cases} \qquad \text{(S.15.70)}$$

d) By the differentiation $\dfrac{d}{dx}(\kappa - x)^p = -p(\kappa - x)^{p-1}$ we find

$$\frac{\partial f_L(x)}{\partial L} = \frac{2r}{\sigma^2 L}(\kappa - L)^p \left(\frac{L}{x}\right)^{2r/\sigma^2} - p(\kappa - L)^{p-1}\left(\frac{L}{x}\right)^{2r/\sigma^2},$$

hence the condition $\dfrac{\partial f'_{L^*}(x)}{\partial L}_{|x=L^*} = 0$ reads

$$\frac{2r}{\sigma^2 L^*}(\kappa - L^*) - p = 0, \quad \text{or} \quad L^* = \frac{2r}{2r + p\sigma^2}\kappa < \kappa.$$

e) By (S.15.70) the price can be computed as

$$f(t, S_t) = f_{L^*}(S_t) = \begin{cases} (\kappa - S_t)^p, & 0 < S_t \le L^*, \\ \\ \left(\dfrac{p\sigma^2\kappa}{2r + p\sigma^2}\right)^p \left(\dfrac{2r + p\sigma^2}{2r}\dfrac{S_t}{\kappa}\right)^{-2r/\sigma^2}, & S_t \ge L^*, \end{cases}$$

using (14.13) as in the proof of Proposition 15.4, since the process

$$u \longmapsto e^{-ru} f_{L^*}(S_u), \qquad u \ge t,$$

is a nonnegative *super*martingale.

Exercise 15.15

a) The option payoff is $\kappa - (S_t)^p$.
b) We have

$$\begin{aligned} f_L(S_t) &= \mathbb{E}^*\left[e^{-r(\tau_L - t)}(\kappa - (S_{\tau_L})^p) \ \Big| \ \mathcal{F}_t \right] \\ &= \mathbb{E}^*\left[e^{-r(\tau_L - t)}(\kappa - L^p) \ \Big| \ \mathcal{F}_t \right] \\ &= (\kappa - L^p)\mathbb{E}^*\left[e^{-r(\tau_L - t)} \ \Big| \ \mathcal{F}_t \right]. \end{aligned}$$

c) We have

$$f_L(x) = \mathbb{E}^*\left[e^{-r(\tau_L - t)}(\kappa - (S_{\tau_L})^p) \;\Big|\; S_t = x \right]$$
$$= \begin{cases} \kappa - x^p, & 0 < x \le L, \\ \\ (\kappa - L^p)\left(\dfrac{x}{L}\right)^{-2r/\sigma^2}, & x \ge L. \end{cases}$$

d) We have

$$f'_{L^*}(L^*) = -\frac{2r}{\sigma^2}(\kappa - (L^*)^p)\frac{(L^*)^{-2r/\sigma^2-1}}{(L^*)^{-2r/\sigma^2}} = -p(L^*)^{p-1},$$

i.e.

$$\frac{2r}{\sigma^2}(\kappa - (L^*)^p) = p(L^*)^p,$$

or

$$L^* = \left(\frac{2r\kappa}{2r + p\sigma^2}\right)^{1/p} < (\kappa)^{1/p}. \tag{S.15.71}$$

Remark: We may also compute L^* by maximizing $L \longmapsto f_L(x)$ for all fixed x. The derivative $\partial f_L(x)/\partial L$ can be computed as

$$\frac{\partial f_L(x)}{\partial L} = \frac{\partial}{\partial L}\left((\kappa - L^p)\left(\frac{L}{x}\right)^{2r/\sigma^2}\right)$$
$$= -pL^{p-1}\left(\frac{L}{x}\right)^{2r/\sigma^2} + \frac{2r}{\sigma^2}L^{-1}(\kappa - L^p)\left(\frac{L}{x}\right)^{2r/\sigma^2},$$

and equating $\partial f_L(x)/\partial L$ to 0 at $L = L^*$ yields

$$-p(L^*)^{p-1} + \frac{2r}{\sigma^2}(L^*)^{-1}(\kappa - (L^*)^p) = 0,$$

which recovers (S.15.71).

e) We have

$$f_{L^*}(S_t) = \begin{cases} \kappa - (S_t)^p, & 0 < S_t \le L^*, \\ \\ (\kappa - (L^*)^p)\dfrac{(S_t)^{-2r/\sigma^2}}{(L^*)^{-2r/\sigma^2}}, & S_t \ge L^* \end{cases}$$
$$= \begin{cases} \kappa - (S_t)^p, & 0 < S_t \le L^*, \\ \\ \dfrac{\sigma^2}{2r}p(S_t)^{-2r/\sigma^2}(L^*)^{p+2r/\sigma^2}, & S_t \ge L^*, \end{cases}$$

$$= \begin{cases} \kappa - (S_t)^p, & 0 < S_t \leq L^*, \\ \\ \dfrac{p\sigma^2\kappa}{2r+p\sigma^2} \left(\dfrac{2r+p\sigma^2}{2r} \dfrac{S_t^p}{\kappa} \right)^{-2r/(p\sigma^2)} < \kappa, & S_t \geq L^*, \end{cases}$$

however we cannot conclude as in Exercise 15.14-(e) since the process

$$u \longmapsto \mathrm{e}^{-ru} f_{L^*}(S_u), \qquad u \geq t,$$

does not remain nonnegative when $p > 1$, so that (14.13) cannot be applied as in the proof of Proposition 15.4.

Chapter 16

Exercise 16.1

a) We have

$$\begin{aligned} d\widehat{X}_t &= d\left(\frac{X_t}{N_t}\right) \\ &= \frac{X_0}{N_0} d\left(\mathrm{e}^{(\sigma-\eta)B_t - (\sigma^2-\eta^2)t/2}\right) \\ &= \frac{X_0}{N_0}(\sigma-\eta)\mathrm{e}^{(\sigma-\eta)B_t-(\sigma^2-\eta^2)t/2}dB_t \\ &\quad + \frac{X_0}{2N_0}(\sigma-\eta)^2\mathrm{e}^{(\sigma-\eta)B_t-(\sigma^2-\eta^2)t/2}dt \\ &\quad - \frac{X_0}{2N_0}(\sigma^2-\eta^2)\mathrm{e}^{(\sigma-\eta)B_t-(\sigma^2-\eta^2)t/2}dt \\ &= -\frac{X_t}{2N_t}(\sigma^2-\eta^2)dt + \frac{X_t}{N_t}(\sigma-\eta)dB_t + \frac{X_t}{2N_t}(\sigma-\eta)^2dt \\ &= -\frac{X_t}{N_t}\eta(\sigma-\eta)dt + \frac{X_t}{N_t}(\sigma-\eta)dB_t \\ &= \frac{X_t}{N_t}(\sigma-\eta)(dB_t - \eta dt) \\ &= (\sigma-\eta)\frac{X_t}{N_t}d\widehat{B}_t = (\sigma-\eta)\widehat{X}_t d\widehat{B}_t, \end{aligned}$$

where $d\widehat{B}_t = dB_t - \eta dt$ is a *standard Brownian motion* under $\widehat{\mathbb{P}}$.

b) By change of numéraire, we have

$$\mathbb{E}[(X_T - \lambda N_T)^+] = \widehat{\mathbb{E}}\left[\frac{N_0}{N_T}(X_T-\lambda N_T)^+\right] = N_0\widehat{\mathbb{E}}\big[(\widehat{X}_T - \lambda)^+\big].$$

Next, by the result of Question (a), $\widehat{X}_t$ is a driftless geometric Brownian motion with volatility $\sigma - \eta$ under $\widehat{\mathbb{P}}$, hence we have

$$\widehat{\mathbb{E}}[(\widehat{X}_T - \lambda)^+] = \widehat{X}_0 \Phi\left(\frac{\log(\widehat{X}_0/\lambda)}{\widehat{\sigma}\sqrt{T}} + \frac{\widehat{\sigma}\sqrt{T}}{2}\right) - \lambda\Phi\left(\frac{\log(\widehat{X}_0/\lambda)}{\widehat{\sigma}\sqrt{T}} - \frac{\widehat{\sigma}\sqrt{T}}{2}\right),$$

by the Black-Scholes formula with zero interest rate and volatility parameter $\widehat{\sigma} = \sigma - \eta$. By multiplication by N_0 and the relation $X_0 = N_0\widehat{X}_0$ we conclude to (16.36), *i.e.*

$$\begin{aligned}
\mathbb{E}\left[(X_T - \lambda N_T)^+\right] &= N_0\widehat{\mathbb{E}}\left[(\widehat{X}_T - \lambda)^+\right] \\
&= N_0\widehat{X}_0\Phi(d_+) - \lambda N_0\Phi(d^-) \\
&= X_0\Phi(d_+) - \lambda N_0\Phi(d^-).
\end{aligned}$$

c) We have $\widehat{\sigma} = \sigma - \eta$.

Exercise 16.2

a) By the Girsanov Theorem 16.7, the processes

$$d\widehat{B}_t^{(1)} = dB_t^{(1)} - \frac{1}{N_t} dN_t \bullet dB_t^{(1)} = dB_t^{(1)} - \frac{1}{S_t^{(2)}} dS_t^{(2)} \bullet dB_t^{(1)} = dB_t^{(1)} - \eta\rho dt,$$

and

$$d\widehat{B}_t^{(2)} = dB_t^{(2)} - \frac{1}{N_t} dN_t \bullet dB_t^{(2)} = dB_t^{(2)} - \frac{1}{S_t^{(2)}} dS_t^{(2)} \bullet dB_t^{(2)} = dB_t^{(2)} - \eta dt$$

are both standard Brownian motions (and martingales) under $\widehat{\mathbb{P}}_2$.

b) We have

$$\begin{aligned}
d\widehat{S}_t^{(1)} &= d\left(\frac{S_t^{(1)}}{S_t^{(2)}}\right) \\
&= \frac{S_0^{(1)}}{S_0^{(2)}} d\left(\mathrm{e}^{\sigma B_t^{(1)} - \eta B_t^{(2)} - (\sigma^2 - \eta^2)t/2}\right) \\
&= \frac{S_0^{(1)}}{S_0^{(2)}} \mathrm{e}^{\sigma B_t^{(1)} - \eta B_t^{(2)} - (\sigma^2 - \eta^2)t/2} \\
&\quad \times \left(\sigma dB_t^{(1)} + \frac{\sigma^2}{2} dt - \eta dB_t^{(2)} + \frac{\eta^2}{2} dt - \frac{\sigma^2 - \eta^2}{2} dt - \sigma\eta\rho dt\right) \\
&= \widehat{S}_t^{(1)}\left(\sigma d\widehat{B}_t^{(1)} - \eta d\widehat{B}_t^{(2)}\right).
\end{aligned}$$

c) We note that the driftless geometric Brownian motion $(\widehat{S}_t^{(1)})_{t\in\mathbb{R}}$ can be written as
$$d\widehat{S}_t^{(1)} = \widehat{\sigma}\widehat{S}_t^{(1)} d\widehat{W}_t,$$
where $(\widehat{W}_t)_{t\in\mathbb{R}}$ is a standard Brownian motion under $\widehat{\mathbb{P}}_2$. In order to determine $\widehat{\sigma}$ we note that
$$\begin{aligned}\widehat{\sigma}^2 dt &= \widehat{\sigma}^2 d\widehat{W}_t \bullet d\widehat{W}_t \\ &= \frac{d\widehat{S}_t^{(1)}}{\widehat{S}_t^{(1)}} \bullet \frac{d\widehat{S}_t^{(1)}}{\widehat{S}_t^{(1)}} \\ &= \left(\sigma dB_t^{(1)} - \eta d\widehat{B}_t^{(2)}\right) \cdot \left(\sigma dB_t^{(1)} - \eta d\widehat{B}_t^{(2)}\right) \\ &= (\sigma^2 + \eta^2 - 2\sigma\eta\rho)dt,\end{aligned}$$
hence $\widehat{\sigma}^2 = \sigma^2 + \eta^2 - 2\sigma\eta\rho$. We conclude by applying the change of numéraire formula
$$\mathrm{e}^{-rT}\,\mathbb{E}^*\left[\left(S_T^{(1)} - \lambda S_T^{(2)}\right)^+\right] = S_0^{(2)}\widehat{\mathbb{E}}\left[\left(\widehat{S}_T^{(1)} - \lambda\right)^+\right]$$
and the Black-Scholes formula to the the driftless geometric Brownian motion $(\widehat{S}_t^{(1)})_{t\in\mathbb{R}}$.

Exercise 16.3 We have $N_t = P(t,T)$ and from (17.25) and the relations $P(t,T) = F(t,r_t)$ and $P(t,S) = G(t,r_t)$ we find
$$\begin{cases} \dfrac{dP(t,S)}{P(t,S)} = r_t dt + \sigma(t,r_t)\dfrac{\partial}{\partial x}\log G(t,r_t)dW_t, \\ \\ \dfrac{dN_t}{N_t} = \dfrac{dP(t,T)}{P(t,T)} = r_t dt + \sigma(t,r_t)\dfrac{\partial}{\partial x}\log F(t,r_t)dW_t. \end{cases}$$
By the Girsanov Theorem (16.12) we also have
$$d\widehat{W}_t = dW_t - \frac{dN_t}{N_t} \bullet dW_t = dW_t - \sigma(t,r_t)\frac{\partial}{\partial x}\log F(t,r_t)dt,$$
hence
$$\frac{dP(t,S)}{P(t,S)} = r_t dt + \sigma^2(t,r_t)\frac{\partial}{\partial x}\log F(t,r_t)\frac{\partial}{\partial x}\log G(t,r_t)dt + \sigma(t,r_t)\frac{\partial}{\partial x}\log G(t,r_t)d\widehat{W}_t.$$
Using the relation $P(t,S) = G(t,r_t)$ we can also write
$$dP(t,S) = r_t P(t,S)dt + \sigma^2(t,r_t)\frac{\partial}{\partial x}\log F(t,r_t)\frac{\partial}{\partial x}G(t,r_t)dt + \sigma(t,r_t)\frac{\partial}{\partial x}G(t,r_t)d\widehat{W}_t.$$

Exercise 16.4 Forward contract. Taking $N_t := P(t,T)$, $t \in [0,T]$, we have

$$\begin{aligned}
\mathbb{E}^*\left[\exp\left(-\int_t^T r_s ds\right)(P(T,S)-K)\,\Big|\,\mathcal{F}_t\right] &= N_t \widehat{\mathbb{E}}\left[\frac{(P(T,S)-K)}{N_T}\,\Big|\,\mathcal{F}_t\right] \\
&= P(t,T)\widehat{\mathbb{E}}\left[\frac{(P(T,S)-K)}{P(T,T)}\,\Big|\,\mathcal{F}_t\right] \\
&= P(t,T)\widehat{\mathbb{E}}[P(T,S)-K \mid \mathcal{F}_t] \\
&= P(t,T)\widehat{\mathbb{E}}[P(T,S) \mid \mathcal{F}_t] - KP(t,T) \\
&= P(t,T)\widehat{\mathbb{E}}\left[\frac{P(T,S)}{P(T,T)} \mid \mathcal{F}_t\right] - KP(t,T) \\
&= P(t,T)\frac{P(t,S)}{P(t,T)} - KP(t,T) \\
&= P(t,S) - KP(t,T),
\end{aligned}$$

since

$$t \longmapsto \frac{P(t,T)}{N_t} = \frac{P(t,S)}{P(t,T)}$$

is a martingale under the forward measure $\widehat{\mathbb{P}}$. The corresponding (static) hedging strategy is given by buying one bond with maturity S and by short selling K units of the bond with maturity T.

Remark: The above result can also be obtained by a direct argument using the tower property of conditional expectations:

$$\begin{aligned}
&\mathbb{E}^*\left[\exp\left(-\int_t^T r_s ds\right)(P(T,S)-K)\,\Big|\,\mathcal{F}_t\right] \\
&= \mathbb{E}^*\left[\exp\left(-\int_t^T r_s ds\right)\left(\mathbb{E}^*\left[\exp\left(-\int_T^S r_s ds\right)\,\Big|\,\mathcal{F}_T\right] - K\right)\,\Big|\,\mathcal{F}_t\right] \\
&= \mathbb{E}^*\left[\exp\left(-\int_t^T r_s ds\right)\mathbb{E}^*\left[\exp\left(-\int_T^S r_s ds\right) - K\,\Big|\,\mathcal{F}_T\right]\,\Big|\,\mathcal{F}_t\right] \\
&= \mathbb{E}^*\left[\mathbb{E}^*\left[\exp\left(-\int_t^S r_s ds\right) - K\exp\left(-\int_t^T r_s ds\right)\,\Big|\,\mathcal{F}_T\right]\,\Big|\,\mathcal{F}_t\right] \\
&= \mathbb{E}^*\left[\exp\left(-\int_t^S r_s ds\right) - K\exp\left(-\int_t^T r_s ds\right)\,\Big|\,\mathcal{F}_t\right] \\
&= P(t,S) - KP(t,T), \qquad t \in [0,T].
\end{aligned}$$

Exercise 16.5

a) We choose $N_t := S_t$ as numéraire because this allows us to write the option payoff as $(S_T(S_T-K))^+ = N_T(S_T-K)^+$. In this case, the forward measure $\widehat{\mathbb{P}}$ satisfies

$$\frac{d\widehat{\mathbb{P}}}{d\mathbb{P}} = \mathrm{e}^{-rT} \frac{N_T}{N_0} = \mathrm{e}^{-rT} \frac{S_T}{S_0},$$

or

$$\frac{d\widehat{\mathbb{P}}_{|\mathcal{F}_t}}{d\mathbb{P}_{|\mathcal{F}_t}} = \mathrm{e}^{-(T-t)r} \frac{N_T}{N_t} = \mathrm{e}^{-(T-t)r} \frac{S_T}{S_t}, \qquad 0 \leq t \leq T.$$

b) By the change of numéraire formula of Proposition 16.5, the option price becomes

$$\begin{aligned} \mathrm{e}^{-(T-t)r}\, \mathbb{E}^* \left[(S_T(S_T-K))^+ \,\middle|\, \mathcal{F}_t \right] &= \mathbb{E}^* \left[\mathrm{e}^{-(T-t)r} N_T (S_T-K)^+ \,\middle|\, \mathcal{F}_t \right] \\ &= N_t \widehat{\mathbb{E}}\big[(S_T-K)^+ \,\big|\, \mathcal{F}_t \big] \\ &= S_t \widehat{\mathbb{E}}\big[(S_T-K)^+ \,\big|\, \mathcal{F}_t \big]. \qquad \text{(S.16.72)} \end{aligned}$$

c) In order to compute (S.16.72) it remains to determine the dynamics of $(S_t)_{t\in\mathbb{R}_+}$ under $\widehat{\mathbb{P}}$. Since $S_t = S_0 \mathrm{e}^{\sigma B_t + rt - \sigma^2 t/2}$, we have

$$\frac{d\widehat{\mathbb{P}}}{d\mathbb{P}} = \mathrm{e}^{-rT} \frac{S_T}{S_0} = \mathrm{e}^{\sigma B_T - \sigma^2 T/2},$$

hence by the Girsanov Theorem 7.3, $\widehat{B}_t := B_t - \sigma t$ is a standard Brownian motion under $\widehat{\mathbb{P}}$, with

$$\begin{aligned} S_T &= S_0 \mathrm{e}^{rT + \sigma B_T - \sigma^2 T/2} \\ &= S_0 \mathrm{e}^{(r+\sigma^2)T + \sigma \widehat{B}_T - \sigma^2 T/2} \\ &= S_t \mathrm{e}^{(r+\sigma^2)(T-t) + (\widehat{B}_T - \widehat{B}_t)\sigma - (T-t)\sigma^2/2}, \qquad 0 \leq t \leq T. \end{aligned}$$

d) According to the above, (S.16.72) becomes

$$\begin{aligned} &\mathrm{e}^{-(T-t)r}\, \mathbb{E}^* \left[\big(S_T(S_T-K) \big)^+ \,\middle|\, \mathcal{F}_t \right] = S_t \widehat{\mathbb{E}}\big[\big(S_T - K \big)^+ \,\big|\, \mathcal{F}_t \big] \\ &= S_t \widehat{\mathbb{E}}\Big[\big(S_0 \mathrm{e}^{rT + \sigma^2 T + \sigma \widehat{B}_T - \sigma^2 T/2} - K \big)^+ \,\Big|\, \mathcal{F}_t \Big] \\ &= S_t \widehat{\mathbb{E}}\Big[\big(S_t \mathrm{e}^{(r+\sigma^2)(T-t) + (\widehat{B}_T - \widehat{B}_t)\sigma - (T-t)\sigma^2/2} - K \big)^+ \,\Big|\, \mathcal{F}_t \Big] \\ &= S_t \mathrm{e}^{(T-t)(r+\sigma^2)} \mathrm{Bl}(S_t, K, r+\sigma^2, \sigma, T-t), \qquad 0 \leq t \leq T, \end{aligned}$$

since the Black-Scholes formula with interest rate $r + \sigma^2$ reads

$$\begin{aligned} &\mathrm{e}^{-(T-t)(r+\sigma^2)} \widehat{\mathbb{E}}\Big[\big(S_t \mathrm{e}^{(r+\sigma^2)(T-t) + (\widehat{B}_T - \widehat{B}_t)\sigma - (T-t)\sigma^2/2} - K \big)^+ \,\Big|\, \mathcal{F}_t \Big] \\ &\qquad = \mathrm{Bl}(S_t, K, r+\sigma^2, \sigma, T-t), \qquad 0 \leq t \leq T. \end{aligned}$$

Remarks:

i) The option price can be rewritten using other Black-Scholes parametrizations, such as for example

$$S_t \mathrm{Bl}\big(S_t \mathrm{e}^{(T-t)(r+\sigma^2)}, K, 0, \sigma, T-t\big),$$

or

$$S_t \mathrm{e}^{(T-t)(r+\sigma^2)} \mathrm{Bl}\big(S_t, K\mathrm{e}^{-(T-t)(r+\sigma^2)}, 0, \sigma, T-t\big),$$

however we prefer to choose the simplest possibility.

ii) Deflated (or forward) processes such as $S_t/N_1 = 1$ or

$$\frac{\mathrm{e}^{-(T-t)r}}{N_t} \mathbb{E}^*\left[(S_T(S_T-K))^+ \,\middle|\, \mathcal{F}_t\right] = \widehat{\mathbb{E}}\big[(S_T-K)^+ \,\big|\, \mathcal{F}_t\big], \quad 0 \le t \le T,$$

are martingales under the forward measure $\widehat{\mathbb{P}}$.

iii) This option can also be priced via an integral calculation instead of using change of numéraire, as follows:

$$\begin{aligned}
&\mathrm{e}^{-(T-t)r}\, \mathbb{E}^*[S_T(S_T-K)^+ \mid \mathcal{F}_t] \\
&= \mathrm{e}^{-(T-t)r}\, \mathbb{E}^*\Big[S_t \mathrm{e}^{(T-t)r+(B_T-B_t)\sigma-(T-t)\sigma^2/2} \\
&\quad \times (S_t \mathrm{e}^{(T-t)r+(B_T-B_t)\sigma-(T-t)\sigma^2/2} - K)^+ \,\Big|\, \mathcal{F}_t\Big] \\
&= S_t \mathrm{e}^{-(T-t)\sigma^2/2}\, \mathbb{E}^*\left[(S_t \mathrm{e}^{(T-t)r+2(B_T-B_t)\sigma-(T-t)\sigma^2/2} - K\mathrm{e}^{(B_T-B_t)\sigma})^+ \,\middle|\, \mathcal{F}_t\right] \\
&= S_t \mathrm{e}^{-(T-t)\sigma^2/2}\, \mathbb{E}^*\left[(x\mathrm{e}^{(T-t)r+2(B_T-B_t)\sigma-(T-t)\sigma^2/2} - K\mathrm{e}^{(B_T-B_t)\sigma})^+\right]_{x=S_t} \\
&= S_t \mathrm{e}^{-(T-t)\sigma^2/2}\, \mathbb{E}^*\left[(\mathrm{e}^{m(x)+2X} - K\mathrm{e}^{X})^+\right]_{x=S_t}, \qquad 0 \le t \le T,
\end{aligned}$$

where $X \simeq \mathcal{N}(0, v^2)$ with $v^2 = (T-t)\sigma^2$ and $m(x) = (T-t)r - (T-t)\sigma^2/2 + \log x$. Next, we note that

$$\begin{aligned}
&\mathbb{E}\left[\left(\mathrm{e}^{m+2X} - K\mathrm{e}^{X}\right)^+\right] = \frac{1}{\sqrt{2\pi v^2}} \int_{-\infty}^{\infty} \left(\mathrm{e}^{m+2x} - K\mathrm{e}^{x}\right)^+ \mathrm{e}^{-x^2/(2v^2)} dx \\
&= \frac{1}{\sqrt{2\pi v^2}} \int_{-m+\log K}^{\infty} (\mathrm{e}^{m+2x} - K\mathrm{e}^{x}) \mathrm{e}^{-x^2/(2v^2)} dx \\
&= \frac{\mathrm{e}^{m}}{\sqrt{2\pi v^2}} \int_{-m+\log K}^{\infty} \mathrm{e}^{2x-x^2/(2v^2)} dx - \frac{K}{\sqrt{2\pi v^2}} \int_{-m+\log K}^{\infty} \mathrm{e}^{x-x^2/(2v^2)} dx \\
&= \frac{\mathrm{e}^{m+2v^2}}{\sqrt{2\pi v^2}} \int_{-m+\log K}^{\infty} \mathrm{e}^{-(2v^2-x)^2/(2v^2)} dx - \frac{K\mathrm{e}^{v^2/2}}{\sqrt{2\pi}} \int_{-m+\log K}^{\infty} \mathrm{e}^{-(v^2-x)^2/(2v^2)} dx \\
&= \frac{\mathrm{e}^{m+2v^2}}{\sqrt{2\pi v^2}} \int_{-2v^2-m+\log K}^{\infty} \mathrm{e}^{-x^2/(2v^2)} dx - \frac{K\mathrm{e}^{v^2/2}}{\sqrt{2\pi v^2}} \int_{-v^2-m+\log K}^{\infty} \mathrm{e}^{-x^2/(2v^2)} dx \\
&= \frac{\mathrm{e}^{m+2v^2}}{\sqrt{2\pi}} \int_{(-2v^2-m+\log K)/v}^{\infty} \mathrm{e}^{-x^2/2} dx - \frac{K\mathrm{e}^{v^2/2}}{\sqrt{2\pi}} \int_{(-v^2-m+\log K)/v}^{\infty} \mathrm{e}^{-x^2/2} dx \\
&= \mathrm{e}^{m+2v^2} \Phi\left(2v + \frac{m-\log K}{v}\right) - K\mathrm{e}^{v^2/2} \Phi\left(v + \frac{m-\log K}{v}\right),
\end{aligned}$$

hence

$$
\begin{aligned}
&\mathrm{e}^{-(T-t)r}\,\mathbb{E}^*\left[S_T\big(S_T-K\big)^+\,\middle|\,\mathcal{F}_t\right]\\
&= S_t\mathrm{e}^{-(T-t)\sigma^2/2}\,\mathbb{E}^*\left[\big(\mathrm{e}^{m(x)+2X}-K\mathrm{e}^{X}\big)^+\right]_{x=S_t}\\
&= S_t^2\mathrm{e}^{(T-t)(r+\sigma^2)}\Phi\left(\frac{(T-t)(r+\sigma^2)+(T-t)\sigma^2/2+\log(S_t/K)}{\sigma\sqrt{T-t}}\right)\\
&\quad -KS_t\Phi\left(\frac{(T-t)(r+\sigma^2)-(T-t)\sigma^2/2+\log(S_t/K)}{\sigma\sqrt{T-t}}\right)\\
&= S_t\mathrm{e}^{(T-t)(r+\sigma^2)}\mathrm{Bl}(S_t,K,r+\sigma^2,\sigma,T-t), \qquad 0\le t\le T.
\end{aligned}
$$

Exercise 16.6

a) Knowing that $S_t = S_0\mathrm{e}^{\sigma W_t+rt-\sigma^2t/2}$, we check that the discounted numéraire process

$$
\begin{aligned}
\mathrm{e}^{-rt}N_t &:= S_t^n\mathrm{e}^{-(n-1)n\sigma^2t/2-nrt}\\
&= S_0^n\mathrm{e}^{n\sigma W_t+nrt-n\sigma^2t/2}\mathrm{e}^{-(n-1)n\sigma^2t/2-nrt}\\
&= S_0\mathrm{e}^{n\sigma W_t-(n\sigma)^2t/2}, \qquad 0\le t\le T,
\end{aligned}
$$

is a martingale under $\mathbb{P}^*$, and

$$
\frac{d\widehat{\mathbb{P}}}{d\mathbb{P}^*} = \mathrm{e}^{-rT}\frac{N_T}{N_0} = \frac{S_T^n}{S_0^n}\mathrm{e}^{-n\sigma^2T-nrT} = \mathrm{e}^{n\sigma W_T-(n\sigma)^2T/2}. \tag{S.16.73}
$$

b) From Equation (S.16.73) and the Girsanov theorem, the process $\widehat{W}_t := W_t-\sigma nt$ is a standard Brownian motion under $\widehat{\mathbb{P}}$, and we have

$$
\begin{aligned}
S_t &= S_0\mathrm{e}^{rt+\sigma W_t-\sigma^2t/2}\\
&= S_0\mathrm{e}^{rt+\sigma(\widehat{W}_t+n\sigma t)-\sigma^2t/2}\\
&= S_0\mathrm{e}^{(r+n\sigma^2)t+\sigma\widehat{W}_t+\sigma^2t/2}, \qquad t\ge 0.
\end{aligned}
$$

c) We have

$$
\begin{aligned}
&\mathrm{e}^{-(T-t)r}\,\mathbb{E}^*\left[S_T^n\mathbb{1}_{\{S_T\ge K\}}\,\middle|\,\mathcal{F}_t\right] = \mathbb{E}^*\left[\mathrm{e}^{-(T-t)r}S_T^n\mathbb{1}_{\{S_T\ge K\}}\,\middle|\,\mathcal{F}_t\right]\\
&= \mathrm{e}^{(n-1)n\sigma^2T/2+(n-1)rT}\,\mathbb{E}^*\left[N_T\mathbb{1}_{\{S_T\ge K\}}\,\middle|\,\mathcal{F}_t\right]\\
&= \mathrm{e}^{(n-1)n\sigma^2T/2+(n-1)rT}N_t\widehat{\mathbb{E}}\left[\mathbb{1}_{\{S_T\ge K\}}\,\middle|\,\mathcal{F}_t\right]\\
&= \mathrm{e}^{(n-1)n\sigma^2T/2+(n-1)rT}N_t\widehat{\mathbb{P}}(S_T\ge K\mid\mathcal{F}_t)\\
&= \mathrm{e}^{(n-1)n\sigma^2T/2+(n-1)rT}N_t\Phi\left(\frac{\log(S_t/K)+(r+(n-1/2)\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right)\\
&= S_t^n\mathrm{e}^{(n-1)n\sigma^2(T-t)/2+(n-1)r(T-t)}\Phi\left(\frac{\log(S_t/K)+(r+(n-1/2)\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right),
\end{aligned}
$$

$0 \leq t \leq T$, $n \geq 0$.

d) The power call option with payoff $(S_T^n - K^n)^+$ is priced at time $t \in [0, T]$ as

$$\begin{aligned}
&\mathrm{e}^{-(T-t)r}\,\mathbb{E}^*\left[\left(S_T^n - K^n\right)^+ \,\middle|\, \mathcal{F}_t\right] \\
&= S_t^n \mathrm{e}^{(n-1)n\sigma^2(T-t)/2+(n-1)r(T-t)}\Phi\left(\frac{\log(S_t/K)+(r+(n-1/2)\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right) \\
&\quad - K^n \mathrm{e}^{-(T-t)r}\Phi\left(\frac{\log(S_t/K)+(r-\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}\right), \qquad n \geq 1.
\end{aligned}$$

Exercise 16.7 Bond options.

a) Itô's formula yields

$$\begin{aligned}
d\left(\frac{P(t,S)}{P(t,T)}\right) &= \frac{P(t,S)}{P(t,T)}\big(\zeta^S(t) - \zeta^T(t)\big)(dW_t - \zeta^T(t)dt) \\
&= \frac{P(t,S)}{P(t,T)}\big(\zeta^S(t) - \zeta^T(t)\big)d\widehat{W}_t, \qquad \text{(S.16.74)}
\end{aligned}$$

where $(\widehat{W}_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion under $\widehat{\mathbb{P}}$ by the Girsanov Theorem 7.3.

b) From (S.16.74) or (19.7) we have

$$\frac{P(t,S)}{P(t,T)} = \frac{P(0,S)}{P(0,T)}\exp\left(\int_0^t \big(\zeta^S(s)-\zeta^T(s)\big)d\widehat{W}_s - \frac{1}{2}\int_0^t \left|\zeta^S(s)-\zeta^T(s)\right|^2 ds\right),$$

hence

$$\frac{P(u,S)}{P(u,T)} = \frac{P(t,S)}{P(t,T)}\exp\left(\int_t^u \big(\zeta^S(s)-\zeta^T(s)\big)d\widehat{W}_s - \frac{1}{2}\int_t^u \left|\zeta^S(s)-\zeta^T(s)\right|^2 ds\right),$$

$t \in [0, u]$, and for $u = T$ this yields

$$P(T,S) = \frac{P(t,S)}{P(t,T)}\exp\left(\int_t^T \big(\zeta^S(s)-\zeta^T(s)\big)d\widehat{W}_s - \frac{1}{2}\int_t^T \left|\zeta^S(s)-\zeta^T(s)\right|^2 ds\right),$$

since $P(T,T) = 1$. Let $\widehat{\mathbb{P}}$ denote the forward measure associated to the numéraire

$$N_t := P(t,T), \qquad 0 \leq t \leq T.$$

c) For all $S \geq T > 0$ we have

$$\begin{aligned}
&\mathbb{E}\left[\mathrm{e}^{-\int_t^T r_s ds}(P(T,S)-K)^+ \,\middle|\, \mathcal{F}_t\right] \\
&= P(t,T)\widehat{\mathbb{E}}\left[\left(\frac{P(t,S)}{P(t,T)}\exp\left(X - \frac{1}{2}\int_t^T \left|\zeta^S(s)-\zeta^T(s)\right|^2 ds\right) - K\right)^+ \,\middle|\, \mathcal{F}_t\right]
\end{aligned}$$

$$= P(t,T)\widehat{\mathbb{E}}\left[\left(e^{X+m(t,T,S)} - K\right)^+ \,\middle|\, \mathcal{F}_t\right],$$

where X is a centered Gaussian random variable with variance

$$v^2(t,T,S) = \int_t^T \left|\zeta^S(s) - \zeta^T(s)\right|^2 ds$$

given $\mathcal{F}_t$, and

$$m(t,T,S) = -\frac{1}{2}v^2(t,T,S) + \log\frac{P(t,S)}{P(t,T)}.$$

Recall that when X is a centered Gaussian random variable with variance v^2, the expectation of $(e^{m+X} - K)^+$ is given, as in the standard Black-Scholes formula, by

$$\mathbb{E}[(e^{m+X} - K)^+] = e^{m+v^2/2}\Phi(v + (m - \log K)/v) - K\Phi((m - \log K)/v),$$

where

$$\Phi(z) = \int_{-\infty}^z e^{-y^2/2}\frac{dy}{\sqrt{2\pi}}, \qquad z \in \mathbb{R},$$

denotes the Gaussian cumulative distribution function and for simplicity of notation we dropped the indices t, T, S in $m(t,T,S)$ and $v^2(t,T,S)$.

Consequently we have

$$\begin{aligned}&\mathbb{E}\left[e^{-\int_t^T r_s ds}(P(T,S) - K)^+ \,\middle|\, \mathcal{F}_t\right]\\&= P(t,S)\Phi\left(\frac{v}{2} + \frac{1}{v}\log\frac{P(t,S)}{KP(t,T)}\right) - KP(t,T)\Phi\left(-\frac{v}{2} + \frac{1}{v}\log\frac{P(t,S)}{KP(t,T)}\right).\end{aligned}$$

d) The self-financing hedging strategy that hedges the bond option is obtained by holding a (possibly fractional) quantity

$$\Phi\left(\frac{v}{2} + \frac{1}{v}\log\frac{P(t,S)}{KP(t,T)}\right)$$

of the bond with maturity S, and by shorting a quantity

$$K\Phi\left(-\frac{v}{2} + \frac{1}{v}\log\frac{P(t,S)}{KP(t,T)}\right)$$

of the bond with maturity T.

Exercise 16.8

a) The process

$$e^{-rt}S_2(t) = S_2(0)e^{\sigma_2 W_t + (\mu - r)t}$$

is a martingale if

$$r - \mu = \frac{1}{2}\sigma_2^2.$$

b) We note that

$$\begin{aligned}
\mathrm{e}^{-rt} X_t &= \mathrm{e}^{-rt} \mathrm{e}^{(r-\mu)t - \sigma_1^2 t/2} S_1(t) \\
&= \mathrm{e}^{-rt} \mathrm{e}^{(\sigma_2^2 - \sigma_1^2)t/2} S_1(t) \\
&= \mathrm{e}^{-\mu t - \sigma_1^2 t/2} S_1(t) \\
&= S_1(0) \mathrm{e}^{\mu t - \sigma_1^2 t/2} \mathrm{e}^{\sigma_1 W_t + \mu t} \\
&= S_1(0) \mathrm{e}^{\sigma_1 W_t - \sigma_1^2 t/2}
\end{aligned}$$

is a martingale, where

$$X_t = \mathrm{e}^{(r-\mu)t - \sigma_1^2 t/2} S_1(t) = \mathrm{e}^{(\sigma_2^2 - \sigma_1^2)t/2} S_1(t).$$

c) By (16.38) we have

$$\begin{aligned}
\widehat{X}(t) &= \frac{X_t}{N_t} \\
&= \mathrm{e}^{(\sigma_2^2 - \sigma_1^2)t/2} \frac{S_1(t)}{S_2(t)} \\
&= \frac{S_1(0)}{S_2(0)} \mathrm{e}^{(\sigma_2^2 - \sigma_1^2)t/2 + (\sigma_1 - \sigma_2) W_t} \\
&= \frac{S_1(0)}{S_2(0)} \mathrm{e}^{(\sigma_2^2 - \sigma_1^2)t/2 + (\sigma_1 - \sigma_2) \widehat{W}_t + \sigma_2(\sigma_1 - \sigma_2)t} \\
&= \frac{S_1(0)}{S_2(0)} \mathrm{e}^{(\sigma_1 - \sigma_2) \widehat{W}_t + \sigma_2 \sigma_1 t - (\sigma_2^2 + \sigma_1^2)t/2} \\
&= \frac{S_1(0)}{S_2(0)} \mathrm{e}^{(\sigma_1 - \sigma_2) \widehat{W}_t - (\sigma_1 - \sigma_2)^2 t/2},
\end{aligned}$$

where

$$\widehat{W}_t := W_t - \sigma_2 t$$

is a standard Brownian motion under the forward measure $\widehat{\mathbb{P}}$ defined by

$$\begin{aligned}
\frac{d\widehat{\mathbb{P}}}{d\mathbb{P}} &= \mathrm{e}^{-\int_0^T r_s ds} \frac{N_T}{N_0} \\
&= \mathrm{e}^{-rT} \frac{S_2(T)}{S_2(0)} \\
&= \mathrm{e}^{-rT} \mathrm{e}^{\sigma_2 W_T + \mu T} \\
&= \mathrm{e}^{\sigma_2 W_T + (\mu - r)T} \\
&= \mathrm{e}^{\sigma_2 W_T - \sigma_2^2 t/2}.
\end{aligned}$$

d) Given that $X_t = \mathrm{e}^{(\sigma_2^2-\sigma_1^2)t/2} S_1(t)$ and $\widehat{X}(t) = X_t/N_t = X_t/S_2(t)$, we have

$$\begin{aligned}
&\mathrm{e}^{-rT}\,\mathbb{E}[(S_1(T)-\kappa S_2(T))^+] = \mathrm{e}^{-rT}\,\mathbb{E}[(\mathrm{e}^{-(\sigma_2^2-\sigma_1^2)T/2}X_T-\kappa S_2(T))^+] \\
&= \mathrm{e}^{-rT}\mathrm{e}^{-(\sigma_2^2-\sigma_1^2)T/2}\,\mathbb{E}[(X_T-\kappa \mathrm{e}^{(\sigma_2^2-\sigma_1^2)T/2}S_2(T))^+] \\
&= S_2(0)\mathrm{e}^{-(\sigma_2^2-\sigma_1^2)T/2}\widehat{\mathbb{E}}[(\widehat{X}_T-\kappa \mathrm{e}^{(\sigma_2^2-\sigma_1^2)T/2})^+] \\
&= S_2(0)\mathrm{e}^{-(\sigma_2^2-\sigma_1^2)T/2}\widehat{\mathbb{E}}[(\widehat{X}_0\mathrm{e}^{(\sigma_1-\sigma_2)\widehat{W}_T-(\sigma_1-\sigma_2)^2T/2}-\kappa \mathrm{e}^{(\sigma_2^2-\sigma_1^2)T/2})^+] \\
&= S_2(0)\mathrm{e}^{-(\sigma_2^2-\sigma_1^2)T/2}\big(\widehat{X}_0\Phi^0_+(T,\widehat{X}_0)-\kappa \mathrm{e}^{(\sigma_2^2-\sigma_1^2)T/2}\Phi^0_-(T,\widehat{X}_0)\big) \\
&= S_2(0)\mathrm{e}^{-(\sigma_2^2-\sigma_1^2)T/2}\widehat{X}_0\Phi^0_+(T,\widehat{X}_0) \\
&\quad -\kappa S_2(0)\mathrm{e}^{-(\sigma_2^2-\sigma_1^2)T/2}\mathrm{e}^{(\sigma_2^2-\sigma_1^2)T/2}\Phi^0_-(T,\widehat{X}_0) \\
&= X_0\mathrm{e}^{-(\sigma_2^2-\sigma_1^2)T/2}\Phi^0_+(T,\widehat{X}_0)-\kappa S_2(0)\Phi^0_-(T,\widehat{X}_0) \\
&= S_1(0)\mathrm{e}^{-(\sigma_2^2-\sigma_1^2)T/2}\Phi^0_+(T,\widehat{X}_0)-\kappa S_2(0)\Phi^0_-(T,\widehat{X}_0),
\end{aligned}$$

where

$$\begin{aligned}
\Phi^0_+(T,x) &= \Phi\left(\frac{\log(x/\kappa)}{|\sigma_1-\sigma_2|\sqrt{T}}+\frac{(\sigma_1-\sigma_2)^2-(\sigma_2^2-\sigma_1^2)}{2|\sigma_1-\sigma_2|}\sqrt{T}\right) \\
&= \begin{cases} \Phi\left(\dfrac{\log(x/\kappa)}{|\sigma_1-\sigma_2|\sqrt{T}}+\sigma_1\sqrt{T}\right), & \sigma_1>\sigma_2, \\[2ex] \Phi\left(\dfrac{\log(x/\kappa)}{|\sigma_1-\sigma_2|\sqrt{T}}-\sigma_1\sqrt{T}\right), & \sigma_1<\sigma_2, \end{cases}
\end{aligned}$$

and

$$\begin{aligned}
\Phi^0_-(T,x) &= \Phi\left(\frac{\log(x/\kappa)}{|\sigma_1-\sigma_2|\sqrt{T}}-\frac{(\sigma_1-\sigma_2)^2+(\sigma_2^2-\sigma_1^2)}{2|\sigma_1-\sigma_2|}\sqrt{T}\right) \\
&= \begin{cases} \Phi\left(\dfrac{\log(x/\kappa)}{|\sigma_1-\sigma_2|\sqrt{T}}+\sigma_2\sqrt{T}\right), & \sigma_1>\sigma_2, \\[2ex] \Phi\left(\dfrac{\log(x/\kappa)}{|\sigma_1-\sigma_2|\sqrt{T}}-\sigma_2\sqrt{T}\right), & \sigma_1<\sigma_2, \end{cases}
\end{aligned}$$

if $\sigma_1 \neq \sigma_2$. In case $\sigma_1 = \sigma_2$, we find

$$\begin{aligned}
\mathrm{e}^{-rT}\,\mathbb{E}[(S_1(T)-\kappa S_2(T))^+] &= \mathrm{e}^{-rT}\,\mathbb{E}[S_1(T)(1-\kappa S_2(0)/S_1(0))^+] \\
&= (1-\kappa S_2(0)/S_1(0))^+\mathrm{e}^{-rT}\,\mathbb{E}[S_1(T)] \\
&= (S_1(0)-\kappa S_2(0))\mathbb{1}_{\{S_1(0)>\kappa S_2(0)\}}.
\end{aligned}$$

Exercise 16.9 We have

$$\mathrm{e}^{-(T-t)r}\,\mathbb{E}^*\left[\mathbb{1}_{\{R_T\geq\kappa\}}\,\middle|\,R_t\right] = \mathrm{e}^{-(T-t)r}\mathbb{P}^*\big(R_T\geq\kappa\,\big|\,R_t\big)$$

$$
\begin{aligned}
&= \mathrm{e}^{-(T-t)r}\mathbb{P}^*\left(R_t \mathrm{e}^{(W_T-W_t)\sigma+(r-r^{\mathrm{f}})(T-t)-(T-t)\sigma^2/2} \geq \kappa \,\middle|\, R_t\right)\\
&= \mathrm{e}^{-(T-t)r}\mathbb{P}^*\left(x\mathrm{e}^{(W_T-W_t)\sigma+(r-r^{\mathrm{f}})(T-t)-(T-t)\sigma^2/2} \geq \kappa\right)_{x=R_t}\\
&= \mathrm{e}^{-(T-t)r}\Phi\left(\frac{(r-r^{\mathrm{f}})(T-t)-(T-t)\sigma^2/2-\log(\kappa/R_t)}{\sigma\sqrt{T-t}}\right),
\end{aligned}
$$

after applying the hint provided, with

$$
\eta^2 := (T-t)\sigma^2 \quad \text{and} \quad \mu := (r-r^{\mathrm{f}})(T-t)-(T-t)\sigma^2/2.
$$

Remark: Binary options are often proposed *at the money*, *i.e.* $\kappa = R_t$, with a short time to maturity, for example the small value

$$
T-t \simeq 30 \text{ seconds} = 0.000000951 = 9.51 \times 10^{-7} \text{ year}^{-1},
$$

in which case we have

$$
\begin{aligned}
\mathrm{e}^{-(T-t)r}\,\mathbb{E}^*\left[\mathbb{1}_{\{R_T\geq\kappa\}} \,\middle|\, R_t\right] &= \mathrm{e}^{-(T-t)r}\Phi\left(\left(\frac{r-r^{\mathrm{f}}}{\sigma}-\frac{\sigma}{2}\right)\sqrt{T-t}\right)\\
&\simeq \Phi(0)\\
&= \frac{1}{2}.
\end{aligned}
$$

Taking for example $r-r^{\mathrm{f}} = 0.02 = 2\%$ and $\sigma = 0.3 = 30\%$, we have

$$
\left(\frac{r-r^{\mathrm{f}}}{\sigma}-\frac{\sigma}{2}\right)\sqrt{T-t} = \left(\frac{0.02}{0.3}-\frac{0.3}{2}\right)\sqrt{9.51\times 10^{-7}} = -0.000081279
$$

and

$$
\begin{aligned}
\mathrm{e}^{-(T-t)r}\,\mathbb{E}^*\left[\mathbb{1}_{\{R_T\geq\kappa\}} \,\middle|\, R_t\right] &= \mathrm{e}^{-(T-t)r}\Phi(-0.000081279)\\
&= \mathrm{e}^{-r\times 0.000000951}\times 0.499968\\
&= 0.49996801\\
&\simeq \frac{1}{2},
\end{aligned}
$$

with $r = 0.02 = 2\%$.

Exercise 16.10

a) It suffices to check that the definition of $(W_t^N)_{t\in\mathbb{R}_+}$ implies the correlation identity $dW_t^S \bullet dW_t^N = \rho dt$ by Itô's calculus.
b) We let

$$
\widehat{\sigma}_t = \sqrt{(\sigma_t^S)^2 - 2\rho\sigma_t^R\sigma_t^S + (\sigma_t^R)^2}
$$

and

$$dW_t^X = \frac{\sigma_t^S - \rho\sigma_t^N}{\widehat{\sigma}_t} dW_t^S - \sqrt{1-\rho^2}\frac{\sigma_t^N}{\widehat{\sigma}_t} dW_t, \qquad t \geq 0,$$

which defines a standard Brownian motion under $\mathbb{P}^*$ due to the definition of $\widehat{\sigma}_t$.

Exercise 16.11

a) We have $\widehat{\sigma} = \sqrt{(\sigma^S)^2 - 2\rho\sigma^R\sigma^S + (\sigma^R)^2}$.

b) Letting $\widetilde{X}_t = \mathrm{e}^{-rt} X_t = \mathrm{e}^{(a-r)t} S_t/R_t$, $t \geq 0$, we have

$$\begin{aligned}
&\mathbb{E}^*\left[\left(\frac{S_T}{R_T} - \kappa\right)^+ \middle| \mathcal{F}_t\right] = \mathrm{e}^{-aT}\,\mathbb{E}^*\left[\left(X_T - \mathrm{e}^{aT}\kappa\right)^+ \middle| \mathcal{F}_t\right] \\
&= \mathrm{e}^{-(a-r)T}\,\mathbb{E}^*\left[\left(\widetilde{X}_T - \mathrm{e}^{(a-r)T}\kappa\right)^+ \middle| \mathcal{F}_t\right] \\
&= \mathrm{e}^{-(a-r)T}\left(\widetilde{X}_t \Phi\left(\frac{(r-a+\widehat{\sigma}^2/2)(T-t)}{\widehat{\sigma}\sqrt{T-t}} + \frac{1}{\widehat{\sigma}\sqrt{T-t}}\log\frac{S_t}{\kappa R_t}\right)\right. \\
&\quad \left. -\kappa\mathrm{e}^{(a-r)T}\Phi\left(\frac{(r-a-\widehat{\sigma}^2/2)(T-t)}{\widehat{\sigma}\sqrt{T-t}} + \frac{1}{\widehat{\sigma}\sqrt{T-t}}\log\frac{S_t}{\kappa R_t}\right)\right) \\
&= \frac{S_t}{R_t}\mathrm{e}^{(r-a)(T-t)}\Phi\left(\frac{(r-a+\widehat{\sigma}^2/2)(T-t)}{\widehat{\sigma}\sqrt{T-t}} + \frac{1}{\widehat{\sigma}\sqrt{T-t}}\log\frac{S_t}{\kappa R_t}\right) \\
&\quad -\kappa\Phi\left(\frac{(r-a-\widehat{\sigma}^2/2)(T-t)}{\widehat{\sigma}\sqrt{T-t}} + \frac{1}{\widehat{\sigma}\sqrt{T-t}}\log\frac{S_t}{\kappa R_t}\right),
\end{aligned}$$

hence the price of the quanto option is

$$\begin{aligned}
&\mathrm{e}^{-(T-t)r}\,\mathbb{E}^*\left[\left(\frac{S_T}{R_T} - \kappa\right)^+ \middle| \mathcal{F}_t\right] \\
&= \frac{S_t}{R_t}\mathrm{e}^{-a(T-t)}\Phi\left(\frac{(r-a+\widehat{\sigma}^2/2)(T-t)}{\widehat{\sigma}\sqrt{T-t}} + \frac{1}{\widehat{\sigma}\sqrt{T-t}}\log\frac{S_t}{\kappa R_t}\right) \\
&\quad -\kappa\mathrm{e}^{-r(T-t)}\Phi\left(\frac{(r-a-\widehat{\sigma}^2/2)(T-t)}{\widehat{\sigma}\sqrt{T-t}} + \frac{1}{\widehat{\sigma}\sqrt{T-t}}\log\frac{S_t}{\kappa R_t}\right).
\end{aligned}$$

Chapter 17

Exercise 17.1

a) We have

$$dr_t = r_0 d\mathrm{e}^{-bt} + \frac{a}{b} d(1-\mathrm{e}^{-bt}) + \sigma d\left(\mathrm{e}^{-bt}\int_0^t \mathrm{e}^{bs} dB_s\right)$$

$$
\begin{aligned}
&= -br_0 e^{-bt} dt + a e^{-bt} dt + \sigma e^{-bt} d \int_0^t e^{bs} dB_s + \sigma \int_0^t e^{bs} dB_s d e^{-bt} \\
&= -br_0 e^{-bt} dt + a e^{-bt} dt + \sigma e^{-bt} e^{bt} dB_t - \sigma b \int_0^t e^{bs} dB_s e^{-bt} dt \\
&= -br_0 e^{-bt} dt + a e^{-bt} dt + \sigma dB_t - \sigma b \int_0^t e^{bs} dB_s e^{-bt} dt \\
&= -br_0 e^{-bt} dt + a e^{-bt} dt + \sigma dB_t - b \left(r_t - r_0 e^{-bt} - \frac{a}{b} \big(1 - e^{-bt}\big) \right) dt \\
&= (a - br_t) dt + \sigma dB_t,
\end{aligned}
$$

which shows that r_t solves (17.46).

b) We note that

$$
\begin{aligned}
r_t &= r_0 e^{-bt} + \frac{a}{b} \big(1 - e^{-bt}\big) + \sigma \int_0^t e^{-(t-u)b} dB_u \\
&= r_0 e^{-bs} e^{-(t-s)b} + \frac{a}{b} e^{-(t-s)b} \big(1 - e^{-bs}\big) \\
&\quad + \frac{a}{b} \big(1 - e^{-(t-s)b}\big) + \sigma e^{-(t-s)b} \int_0^s e^{-(s-u)b} dB_u + \sigma \int_s^t e^{-(t-u)b} dB_u \\
&= r_s e^{-(t-s)b} + \frac{a}{b} \big(1 - e^{-(t-s)b}\big) + \sigma \int_s^t e^{-(t-u)b} dB_u, \quad 0 \leq s \leq t.
\end{aligned}
$$

Hence, assuming that r_s has the $\mathcal{N}(a/b, \sigma^2/(2b))$ distribution, the distribution of r_t is Gaussian with mean

$$
\begin{aligned}
\mathbb{E}[r_t] &= e^{-(t-s)b} \, \mathbb{E}[r_s] + \frac{a}{b} \big(1 - e^{(t-s)b}\big) \\
&= \frac{a}{b} e^{-(t-s)b} + \frac{a}{b} \big(1 - e^{(t-s)b}\big) \\
&= \frac{a}{b},
\end{aligned}
$$

and variance

$$
\begin{aligned}
\mathrm{Var}[r_t] &= \mathrm{Var}\left[r_s e^{-(t-s)b} + \frac{a}{b} \big(1 - e^{(t-s)b}\big) + \sigma \int_s^t e^{-(t-u)b} dB_u \right] \\
&= \mathrm{Var}\left[r_s e^{-(t-s)b} + \sigma \int_s^t e^{-(t-u)b} dB_u \right] \\
&= \mathrm{Var}\left[r_s e^{-(t-s)b} \right] + \mathrm{Var}\left[\sigma \int_s^t e^{-(t-u)b} dB_u \right] \\
&= e^{-2(t-s)b} \, \mathrm{Var}\left[r_s \right] + \sigma^2 \, \mathrm{Var}\left[\int_s^t e^{-(t-u)b} dB_u \right] \\
&= e^{-2(t-s)b} \frac{\sigma^2}{2b} + \sigma^2 \int_s^t e^{-2(t-u)b} du \\
&= e^{-2(t-s)b} \frac{\sigma^2}{2b} + \sigma^2 \int_0^{t-s} e^{-2bu} du
\end{aligned}
$$

$$= \frac{\sigma^2}{2b}, \qquad t \geq 0.$$

Exercise 17.2

a) The zero-coupon bond price $P(t,T)$ in the Vasicek model is given by

$$\log P(t,T) = A(T-t) + r_t C(T-t), \qquad 0 \leq t \leq T,$$

where

$$C(T-t) := -\frac{1}{b}\big(1 - \mathrm{e}^{-(T-t)b}\big),$$

and

$$\begin{aligned} A(T-t) &:= \frac{4ab - 3\sigma^2}{4b^3} + \frac{\sigma^2 - 2ab}{2b^2}(T-t) \\ &\quad + \frac{\sigma^2 - ab}{b^3}\mathrm{e}^{-(T-t)b} - \frac{\sigma^2}{4b^3}\mathrm{e}^{-2(T-t)b} \quad 0 \leq t \leq T. \end{aligned} \tag{S.17.75}$$

Since $\lim_{T\to\infty} C(T-t)/(T-t) = 0$ and

$$\lim_{T\to\infty} A(T-t)/(T-t) = (\sigma^2 - 2ab)/(2b^2),$$

we find

$$r_\infty = -\lim_{T\to\infty} \frac{\log P(t,T)}{T-t} = -\frac{\sigma^2 - 2ab}{2b^2} = \frac{a}{b} - \frac{\sigma^2}{2b^2}.$$

b) We have

$$\begin{aligned} \log \frac{P(t,T)}{P(0,T)} &= \log P(t,T) - \log P(0,T) \\ &= A(T-t) - A(T) + r_t C(T-t) - r_0 C(T) \\ &= -t\frac{\sigma^2 - 2ab}{2b^2} - \mathrm{e}^{-(T-t)b}\left(\frac{\sigma^2 - ab}{b^3} + \frac{\sigma^2}{4b^3}\mathrm{e}^{-(T-t)b}\right) \\ &\quad + \mathrm{e}^{-bT}\left(\frac{\sigma^2}{4b^3} - \frac{\sigma^2 - ab}{b^3}\right) - \frac{r_t}{b}\big(1 - \mathrm{e}^{-(T-t)b}\big) + \frac{r_0}{b}\big(1 - \mathrm{e}^{-bT}\big), \end{aligned}$$

hence

$$\lim_{T\to\infty} \log \frac{P(t,T)}{P(0,T)} = \left(\frac{a}{b} - \frac{\sigma^2}{2b^2}\right)t - \frac{r_t - r_0}{b} = -\frac{r_t - r_0}{b} + r_\infty t,$$

and*

* The log function is continuous on $(0,\infty)$.

$$\lim_{T\to\infty} \log \frac{P(t,T)}{P(0,T)} = \mathrm{e}^{-(r_t-r_0)/b+t(a/b-\sigma^2/(2b^2))} = \mathrm{e}^{-(r_t-r_0)/b+r_\infty t},$$

which shows that the yield of the long-bond return is the asymptotic bond yield r_∞.

Remark: By Relations (17.32)-(17.34), the Vasicek bond price $P(t,T)$ can be rewritten in terms of the asymptotic bond yield r_∞ as

$$P(t,T) = \mathrm{e}^{-(T-t)r_\infty+(r_t-r_\infty)C(T-t)-\sigma^2C^2(T-t)/(4b)}, \qquad 0 \le t \le T,$$

see *e.g.* Relation (3.12) in Brody et al. (2018).

Exercise 17.3 An estimator of σ can be obtained from the orthogonality relation

$$\sum_{l=0}^{n-1}\left(\left(\tilde r_{t_{l+1}} - a\Delta t - (1-b\Delta t)\tilde r_{t_l}\right)^2 - \sigma^2\Delta t(\tilde r_{t_l})^{2\gamma}\right) = \sigma^2\sum_{l=0}^{n-1}(\tilde r_{t_l})^{2\gamma}\left((Z_l)^2-\Delta t\right)$$
$$\simeq 0,$$

which is due to the independence of t_{t_l} and Z_l, $l=0,\ldots,n-1$, and yields

$$\widehat{\sigma}^2 = \frac{\displaystyle\sum_{l=0}^{n-1}\left(\tilde r_{t_{l+1}} - a\Delta t - (1-b\Delta t)\tilde r_{t_l}\right)^2}{\displaystyle\Delta t\sum_{l=0}^{n-1}(\tilde r_{t_l})^{2\gamma}}.$$

Regarding the estimation of γ, we can combine the above relation with the second orthogonality relation

$$\begin{aligned}&\sum_{l=0}^{n-1}\left(\left(\tilde r_{t_{l+1}} - a\Delta t - (1-b\Delta t)\tilde r_{t_l}\right)^2 - \sigma^2\Delta t(\tilde r_{t_l})^{2\gamma}\right)\tilde r_{t_l}\\ &= \sigma^2\sum_{l=0}^{n-1}(\tilde r_{t_l})^{2\gamma+1}\left((Z_l)^2-\Delta t\right)\\ &\simeq 0,\end{aligned}$$

cf. § 2.2 of Faff and Gray (2006). One may also attempt to minimize the residual

$$\sum_{l=0}^{n-1}\left(\left(\tilde r_{t_{l+1}} - a\Delta t - (1-b\Delta t)\tilde r_{t_l}\right)^2 - \sigma^2\Delta t(\tilde r_{t_l})^{2\gamma}\right)^2$$

by equating the following derivatives to zero, as

$$
\begin{aligned}
&\frac{\partial}{\partial \sigma} \sum_{l=0}^{n-1} \left(\left(\tilde{r}_{t_{l+1}} - a\Delta t - (1 - b\Delta t)\tilde{r}_{t_l} \right)^2 - \sigma^2 \Delta t (\tilde{r}_{t_l})^{2\gamma} \right)^2 \\
&= -4\sigma \sum_{l=0}^{n-1} (\tilde{r}_{t_l})^{2\gamma} \left(\left(\tilde{r}_{t_{l+1}} - a\Delta t - (1 - b\Delta t)\tilde{r}_{t_l} \right)^2 - \sigma^2 \Delta t (\tilde{r}_{t_l})^{2\gamma} \right) \\
&= 0,
\end{aligned}
$$

hence

$$
\sum_{l=0}^{n-1} (\tilde{r}_{t_l})^{2\gamma} \left(\tilde{r}_{t_{l+1}} - a\Delta t - (1 - b\Delta t)\tilde{r}_{t_l} \right)^2 - \sigma^2 \Delta t \sum_{l=0}^{n-1} (\tilde{r}_{t_l})^{4\gamma} = 0,
$$

which yields

$$
\widehat{\sigma}^2 = \frac{\displaystyle\sum_{l=0}^{n-1} (\tilde{r}_{t_l})^{2\gamma} \left(\tilde{r}_{t_{l+1}} - a\Delta t - (1 - b\Delta t)\tilde{r}_{t_l} \right)^2}{\displaystyle\Delta t \sum_{l=0}^{n-1} (\tilde{r}_{t_l})^{4\gamma}}. \tag{S.17.76}
$$

We also have

$$
\begin{aligned}
&\frac{\partial}{\partial \gamma} \sum_{l=0}^{n-1} \left(\left(\tilde{r}_{t_{l+1}} - a\Delta t - (1 - b\Delta t)\tilde{r}_{t_l} \right)^2 - \sigma^2 \Delta t (\tilde{r}_{t_l})^{2\gamma} \right)^2 \\
&= -4\sigma^2 \Delta t \sum_{l=0}^{n-1} (\tilde{r}_{t_l})^{2\gamma} \left(\left(\tilde{r}_{t_{l+1}} - a\Delta t - (1 - b\Delta t)\tilde{r}_{t_l} \right)^2 - \sigma^2 \Delta t (\tilde{r}_{t_l})^{2\gamma} \right) \log \tilde{r}_{t_l} \\
&= 0,
\end{aligned}
$$

which yields

$$
\widehat{\sigma}^2 = \frac{\displaystyle\sum_{l=0}^{n-1} (\tilde{r}_{t_l})^{2\gamma} \left(\tilde{r}_{t_{l+1}} - a\Delta t - (1 - b\Delta t)\tilde{r}_{t_l} \right)^2 \log \tilde{r}_{t_l}}{\displaystyle\Delta t \sum_{l=0}^{n-1} (\tilde{r}_{t_l})^{4\gamma} \log \tilde{r}_{t_l}}, \tag{S.17.77}
$$

and shows that γ can be estimated by matching Relations (S.17.76) and (S.17.77), *i.e.*

$$\frac{\sum_{l=0}^{n-1} (\tilde{r}_{t_l})^{2\gamma} \big(\tilde{r}_{t_{l+1}} - a\Delta t - (1 - b\Delta t)\tilde{r}_{t_l}\big)^2}{\sum_{l=0}^{n-1} (\tilde{r}_{t_l})^{4\gamma}}$$
$$= \frac{\sum_{l=0}^{n-1} (\tilde{r}_{t_l})^{2\gamma} \big(\tilde{r}_{t_{l+1}} - a\Delta t - (1 - b\Delta t)\tilde{r}_{t_l}\big)^2 \log \tilde{r}_{t_l}}{\sum_{l=0}^{n-1} (\tilde{r}_{t_l})^{4\gamma} \log \tilde{r}_{t_l}}.$$

Remarks.

i) Estimators of a and b can be obtained by minimizing the residual

$$\sum_{l=0}^{n-1} \big(\tilde{r}_{t_{l+1}} - a\Delta t - (1 - b\Delta t)\tilde{r}_{t_l}\big)^2$$

as in the Vašíček (1977) model, *i.e.* from the equations

$$\sum_{l=0}^{n-1} \big(\tilde{r}_{t_{l+1}} - a\Delta t - (1 - b\Delta t)\tilde{r}_{t_l}\big) = 0$$

and

$$\sum_{l=0}^{n-1} \big(\tilde{r}_{t_{l+1}} - a\Delta t - (1 - b\Delta t)\tilde{r}_{t_l}\big)\, \tilde{r}_{t_l} = 0.$$

ii) Instead of using the (generalised) method of moments, parameter estimation for stochastic differential equations can be achieved by maximum likelihood estimation, see *e.g.* Lindström (2007) and references therein.

Exercise 17.4

a) We have $r_t = r_0 + at + \sigma B_t$, and

$$F(t, r_t) = F(t, r_0 + at + \sigma B_t),$$

hence by Proposition 17.2 the PDE satisfied by $F(t, x)$ is

$$-xF(t,x) + \frac{\partial F}{\partial t}(t,x) + a\frac{\partial F}{\partial x}(t,x) + \frac{1}{2}\sigma^2 \frac{\partial^2 F}{\partial x^2}(t,x) = 0, \qquad \text{(S.17.78)}$$

with terminal condition $F(T, x) = 1$.

b) Using the relation $r_t = r_0 + at + \sigma B_t$ and the fact that the stochastic integral $\int_t^T (T - s)dB_s$ is independent of $\mathcal{F}_t$, we have

$$
\begin{aligned}
F(t,r_t) &= \mathbb{E}^*\left[\exp\left(-\int_t^T r_s ds\right) \Big| \mathcal{F}_t\right] \\
&= \mathbb{E}^*\left[\exp\left(-r_0(T-t) - a\int_t^T s ds - \sigma\int_t^T B_s ds\right) \Big| \mathcal{F}_t\right] \\
&= \mathbb{E}^*\left[\mathrm{e}^{-r_0(T-t)-a(T^2-t^2)/2}\exp\left(-(T-t)\sigma B_t - \sigma\int_t^T (T-s)dB_s\right) \Big| \mathcal{F}_t\right] \\
&= \mathrm{e}^{-r_0(T-t)-a(T^2-t^2)/2-(T-t)\sigma B_t}\,\mathbb{E}^*\left[\exp\left(-\sigma\int_t^T (T-s)dB_s\right) \Big| \mathcal{F}_t\right] \\
&= \mathrm{e}^{-r_0(T-t)-a(T-t)(T+t)/2-(T-t)\sigma B_t}\,\mathbb{E}^*\left[\exp\left(-\sigma\int_t^T (T-s)dB_s\right)\right] \\
&= \exp\left(-(T-t)r_t - a(T-t)^2/2 + \frac{\sigma^2}{2}\int_t^T (T-s)^2 ds\right) \\
&= \exp\left(-(T-t)r_t - a(T-t)^2/2 + (T-t)^3\sigma^2/6\right),
\end{aligned}
$$

hence $F(t,x) = \exp\left(-(T-t)x - a(T-t)^2/2 + (T-t)^3\sigma^2/6\right)$.

Note that the PDE (S.17.78) can also be solved by looking for a solution of the form $F(t,x) = \mathrm{e}^{A(T-t)+xC(T-t)}$, in which case one would find $A(s) = -as^2/2 + \sigma^2 s^3/6$ and $C(s) = -s$.

c) We check that the function $F(t,x)$ of Question (b) satisfies the PDE (S.17.78) of Question (a), since $F(T,x) = 1$ and

$$
\begin{aligned}
-xF(t,x) + \left(x + a(T-t) - \frac{\sigma^2}{2}(T-t)^2\right)F(t,x) - a(T-t)F(t,x) \\
+\frac{\sigma^2}{2}(T-t)^2F(t,x) = 0.
\end{aligned}
$$

Exercise 17.5

a) We check from (17.51) and the differentiation rule $d\int_0^t f(u)du = f(t)dt$ that

$$
\begin{aligned}
dr_t &= \alpha\beta d\left(S_t\int_0^t \frac{1}{S_u}du\right) + r_0 dS_t \\
&= \alpha\beta S_t d\int_0^t \frac{1}{S_u}du + \alpha\beta\int_0^t \frac{1}{S_u}du dS_t + r_0 dS_t \\
&= \alpha\beta\frac{S_t}{S_t}dt + \alpha\beta\int_0^t \frac{S_t}{S_u}du\frac{dS_t}{S_t} + r_0 dS_t \\
&= \alpha\beta dt + (r_t - r_0 S_t)\frac{dS_t}{S_t} + r_0 dS_t \\
&= \alpha\beta dt + r_t\frac{dS_t}{S_t} \\
&= \alpha\beta dt + r_t(-\beta dt + \sigma dB_t)
\end{aligned}
$$

$$= \beta(\alpha - r_t)dt + \sigma dB_t, \qquad t \geq 0.$$

b) Taking $\mu(t,x) := \beta(\alpha - x)$ and $\sigma(t,x) = \sigma x$, by Itô's formula we have

$$\begin{aligned}
d\left(\mathrm{e}^{-\int_0^t r_s ds} P(t,T)\right) &= -r_t \mathrm{e}^{-\int_0^t r_s ds} P(t,T)dt + \mathrm{e}^{-\int_0^t r_s ds} dP(t,T) \\
&= -r_t \mathrm{e}^{-\int_0^t r_s ds} F(t,r_t)dt + \mathrm{e}^{-\int_0^t r_s ds} dF(t,r_t) \\
&= -r_t \mathrm{e}^{-\int_0^t r_s ds} F(t,r_t)dt + \mathrm{e}^{-\int_0^t r_s ds} \frac{\partial F}{\partial x}(t,r_t)(\mu(t,r_t)dt + \sigma(t,r_t)dB_t) \\
&\quad + \mathrm{e}^{-\int_0^t r_s ds}\left(\frac{1}{2}\sigma^2(t,r_t)\frac{\partial^2 F}{\partial x^2}(t,r_t) + \frac{\partial F}{\partial t}(t,r_t)\right)dt \\
&= \mathrm{e}^{-\int_0^t r_s ds}\sigma(t,r_t)\frac{\partial F}{\partial x}(t,r_t)dB_t \\
&+ \mathrm{e}^{-\int_0^t r_s ds}\left(-r_t F(t,r_t) + \mu(t,r_t)\frac{\partial F}{\partial x}(t,r_t) + \frac{1}{2}\sigma^2(t,r_t)\frac{\partial^2 F}{\partial x^2}(t,r_t) + \frac{\partial F}{\partial t}(t,r_t)\right)dt.
\end{aligned} \tag{S.17.79}$$

Given that $t \longmapsto \mathrm{e}^{-\int_0^t r_s ds} P(t,T)$ is a martingale, the above expression (S.17.79) should only contain terms in dB_t and all terms in dt should vanish inside (S.17.79). This leads us to the identities

$$\begin{cases}
r_t F(t,r_t) \\
= \mu(t,r_t)\dfrac{\partial F}{\partial x}(t,r_t) + \dfrac{1}{2}\sigma^2(t,r_t)\dfrac{\partial^2 F}{\partial x^2}(t,r_t) + \dfrac{\partial F}{\partial t}(t,r_t) & \text{(S.17.80a)} \\
\\
d\left(\mathrm{e}^{-\int_0^t r_s ds} P(t,T)\right) = \mathrm{e}^{-\int_0^t r_s ds}\sigma(t,r_t)\dfrac{\partial F}{\partial x}(t,r_t)dB_t, & \text{(S.17.80b)}
\end{cases}$$

and in particular to the bond pricing PDE

$$xF(t,x) = \beta(\alpha - x)\frac{\partial F}{\partial x}(t,x) + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 F}{\partial x^2}(t,x) + \frac{\partial F}{\partial t}(t,x).$$

Exercise 17.6

a) Applying the Itô formula

$$df(r_t) = f'(r_t)dr_t + \frac{1}{2}f''(r_t)(dr_t)^2$$

to the function $f(x) = x^{2-\gamma}$ with

$$f'(x) = (2-\gamma)x^{1-\gamma} \quad \text{and} \quad f''(x) = (2-\gamma)(1-\gamma)x^{-\gamma},$$

we have

$$
\begin{aligned}
dR_t &= dr_t^{2-\gamma} \\
&= df(r_t) \\
&= f'(r_t)dr_t + \frac{1}{2}f''(r_t)(dr_t)^2 \\
&= f'(r_t)\big((\beta r_t^{\gamma-1} + \alpha r_t)dt + \sigma r_t^{\gamma/2}dB_t\big)dr_t + \\
&\quad \frac{1}{2}f''(r_t)\big((\beta r_t^{\gamma-1} + \alpha r_t)dt + \sigma r_t^{\gamma/2}dB_t\big)^2 \\
&= f'(r_t)\big((\beta r_t^{\gamma-1} + \alpha r_t)dt + \sigma r_t^{\gamma/2}dB_t\big)dr_t + \frac{\sigma^2}{2}f''(r_t)r_t^{\gamma}dt \\
&= (2-\gamma)r_t^{1-\gamma}\big((\beta r_t^{\gamma-1} + \alpha r_t)dt + \sigma r_t^{\gamma/2}dB_t\big) + \frac{\sigma^2}{2}(2-\gamma)(1-\gamma)r_t^{\gamma}r_t^{-\gamma}dt \\
&= (2-\gamma)(\beta + \alpha r_t^{2-\gamma})dt + \frac{\sigma^2}{2}(2-\gamma)(1-\gamma)dt + \sigma(2-\gamma)r_t^{1-\gamma/2}dB_t \\
&= (2-\gamma)\left(\beta + \frac{\sigma^2}{2}(1-\gamma) + \alpha R_t\right)dt + (2-\gamma)\sigma\sqrt{R_t}dB_t.
\end{aligned}
$$

We conclude that the process $R_t = r_t^{2-\gamma}$ follows the CIR equation

$$
dR_t = b(a - R_t)dt + \eta\sqrt{R_t}dB_t
$$

with initial condition $R_0 = r_0^{2-\gamma}$ and coefficients

$$
b = (2-\gamma)\alpha, \quad a = \frac{1}{\alpha}\left(\beta + (1-\gamma)\frac{\sigma^2}{2}\right), \quad \text{and} \quad \eta = (2-\gamma)\sigma.
$$

b) By Itô's formula and the relation $P(t,T) = F(t,r_t)$, $t \in [0,T]$, we have

$$
\begin{aligned}
& d\left(\mathrm{e}^{-\int_0^t r_s ds}P(t,T)\right) = -r_t\mathrm{e}^{-\int_0^t r_s ds}P(t,T)dt + \mathrm{e}^{-\int_0^t r_s ds}dP(t,T) \\
&= -r_t\mathrm{e}^{-\int_0^t r_s ds}F(t,r_t)dt + \mathrm{e}^{-\int_0^t r_s ds}dF(t,r_t) \\
&= -r_t\mathrm{e}^{-\int_0^t r_s ds}F(t,r_t)dt + \mathrm{e}^{-\int_0^t r_s ds}\frac{\partial F}{\partial t}(t,r_t)dt + \mathrm{e}^{-\int_0^t r_s ds}\frac{\partial F}{\partial x}(t,r_t)dr_t \\
&\quad + \mathrm{e}^{-\int_0^t r_s ds}\frac{1}{2}\frac{\partial^2 F}{\partial x^2}(t,r_t)(dr_t)^2 \\
&= -r_t\mathrm{e}^{-\int_0^t r_s ds}F(t,r_t)dt + \mathrm{e}^{-\int_0^t r_s ds}\frac{\partial F}{\partial x}(t,r_t)((\beta r_t^{-(1-\gamma)} + \alpha r_t)dt + \sigma r_t^{\gamma/2}dB_t) \\
&\quad + \mathrm{e}^{-\int_0^t r_s ds}\left(\frac{\partial F}{\partial t}(t,r_t) + \frac{1}{2}\sigma^2 r_t^{\gamma}\frac{\partial^2 F}{\partial x^2}(t,r_t)\right)dt \\
&= \mathrm{e}^{-\int_0^t r_s ds}\sigma r_t^{\gamma/2}\frac{\partial F}{\partial x}(t,r_t)dB_t \\
&\quad + \mathrm{e}^{-\int_0^t r_s ds}\left(-r_tF(t,r_t) + (\beta r_t^{-(1-\gamma)} + \alpha r_t)\frac{\partial F}{\partial x}(t,r_t)\right.
\end{aligned}
$$

$$+\frac{1}{2}\sigma^2 r_t^\gamma \frac{\partial^2 F}{\partial x^2}(t, r_t) + \frac{\partial F}{\partial t}(t, r_t)\Big) dt. \quad (S.17.81)$$

Given that $t \longmapsto \mathrm{e}^{-\int_0^t r_s ds} P(t,T)$ is a martingale, the above expression (S.17.81) should only contain terms in dB_t and all terms in dt should vanish inside (S.17.81). This leads to the identities

$$\begin{cases} r_t F(t, r_t) = (\beta r_t^{-(1-\gamma)} + \alpha r_t)\dfrac{\partial F}{\partial x}(t, r_t) + \dfrac{1}{2}\sigma^2 r_t^\gamma \dfrac{\partial^2 F}{\partial x^2}(t, r_t) + \dfrac{\partial F}{\partial t}(t, r_t) \\ \\ d\left(\mathrm{e}^{-\int_0^t r_s ds} P(t,T)\right) = \sigma \mathrm{e}^{-\int_0^t r_s ds} r_t^{\gamma/2} \dfrac{\partial F}{\partial x}(t, r_t) dB_t, \end{cases}$$

and to the PDE

$$xF(t,x) = \frac{\partial F}{\partial t}(t,x) + (\beta x^{-(1-\gamma)} + \alpha x)\frac{\partial F}{\partial x}(t,x) + \frac{\sigma^2}{2} x^\gamma \frac{\partial^2 F}{\partial x^2}(t,x).$$

Exercise 17.7

a) The discounted bond prices process $\mathrm{e}^{-\int_0^t r_s ds} F(t, r_t)$ is a martingale, and we have

$$\begin{aligned} & d\left(\mathrm{e}^{-\int_0^t r_s ds} F(t, r_t)\right) \\ &= \mathrm{e}^{-\int_0^t r_s ds}\left(-r_t F(t,r_t)dt + \frac{\partial F}{\partial t}(t,r_t)dt + \frac{\partial F}{\partial x}(t,r_t)dr_t + \frac{\sigma^2}{2}\frac{\partial^2 F}{\partial x^2}(t,r_t)(dr_t)^2\right) \\ &= \mathrm{e}^{-\int_0^t r_s ds}\left(-r_t F(t,r_t)dt + \frac{\partial F}{\partial t}(t,r_t)dt + \frac{\partial F}{\partial x}(t,r_t)(-ar_t dt + \sigma\sqrt{r_t}dB_t)\right) \\ &\quad + r_t \mathrm{e}^{-\int_0^t r_s ds}\frac{\sigma^2}{2}\frac{\partial^2 F}{\partial x^2}(t,r_t)dt, \end{aligned}$$

hence $F(t,x)$ satisfies the *affine PDE*

$$-xF(t,x) + \frac{\partial F}{\partial t}(t,x) - ax\frac{\partial F}{\partial x}(t,x) + x\frac{\sigma^2}{2}\frac{\partial^2 F}{\partial x^2}(t,x) = 0. \quad (S.17.83)$$

b) Plugging $F(t,x) = \mathrm{e}^{A(T-t)+xC(T-t)}$ into the PDE (S.17.83) shows that

$$\mathrm{e}^{A(T-t)+xC(T-t)}\left(-x - A'(T-t) - xC'(T-t) - axC(T-t) + \frac{\sigma^2 x}{2}C^2(T-t)\right) \\ = 0.$$

Taking successively $x = 0$ and $x = 1$ in the above relation then yields the two equations

$$\begin{cases} A'(T-t) = 0, \\ \\ -1 - C'(T-t) - aC(T-t) + \dfrac{\sigma^2}{2} C^2(T-t) = 0. \end{cases}$$

<u>Remark</u>: The initial condition $A(0) = 0$ shows that $A(s) = 1$, and it can be shown from the condition $C(0) = 0$ that

$$C(T-t) = \frac{2(1 - \mathrm{e}^{\gamma(T-t)})}{2\gamma + (a+\gamma)(\mathrm{e}^{\gamma(T-t)} - 1)}, \qquad t \in [0, T],$$

with $\gamma = \sqrt{a^2 + 2\sigma^2}$, see *e.g.* Eq. (3.25) page 66 of Brigo and Mercurio (2006).

Exercise 17.8

a) The payoff of the convertible bond is given by $\max(\alpha S_\tau, P(\tau, T))$.
b) We have

$$\begin{aligned} \max(\alpha S_\tau, P(\tau, T)) &= P(\tau, T) \mathbb{1}_{\{\alpha S_\tau \leq P(\tau,T))\}} + \alpha S_\tau \mathbb{1}_{\{\alpha S_\tau > P(\tau,T))\}} \\ &= P(\tau, T) + (\alpha S_\tau - P(\tau, T)) \mathbb{1}_{\{\alpha S_\tau > P(\tau,T))\}} \\ &= P(\tau, T) + (\alpha S_\tau - P(\tau, T))^+ \\ &= P(\tau, T) + \alpha (S_\tau - P(\tau, T)/\alpha)^+, \end{aligned}$$

where the latter European call option payoff has the strike price $K := P(\tau, T)/\alpha$.
c) From the Markov property applied at time $t \in [0, \tau]$, we will write the corporate bond price as a function $C(t, S_t, r_t)$ of the underlying asset price and interest rate, hence we have

$$C(t, S_t, r_t) = \mathbb{E}^* \left[\mathrm{e}^{-\int_t^\tau r_s ds} \max(\alpha S_\tau, P(\tau, T)) \,\Big|\, \mathcal{F}_t \right].$$

The martingale property follows from the equalities

$$\begin{aligned} \mathrm{e}^{-\int_0^t r_s ds} C(t, S_t, r_t) &= \mathrm{e}^{-\int_0^t r_s ds} \mathbb{E}^* \left[\mathrm{e}^{-\int_t^\tau r_s ds} \max(\alpha S_\tau, P(\tau, T)) \,\Big|\, \mathcal{F}_t \right] \\ &= \mathbb{E}^* \left[\mathrm{e}^{-\int_0^\tau r_s ds} \max(\alpha S_\tau, P(\tau, T)) \,\Big|\, \mathcal{F}_t \right]. \end{aligned}$$

d) We have

$$\begin{aligned} & d\left(\mathrm{e}^{-\int_0^t r_s ds} C(t, S_t, r_t) \right) \\ &= -r_t \mathrm{e}^{-\int_0^t r_s ds} C(t, S_t, r_t) dt + \mathrm{e}^{-\int_0^t r_s ds} \frac{\partial C}{\partial t}(t, S_t, r_t) dt \\ &\quad + \mathrm{e}^{-\int_0^t r_s ds} \frac{\partial C}{\partial x}(t, S_t, r_t)(r S_t dt + \sigma S_t dB_t^{(1)}) \end{aligned}$$

$$
\begin{aligned}
&+\mathrm{e}^{-\int_0^t r_s ds}\frac{\partial C}{\partial y}(t,S_t,r_t)(\gamma(t,r_t)dt+\eta(t,S_t)dB_t^{(2)})\\
&+\mathrm{e}^{-\int_0^t r_s ds}\frac{\sigma^2}{2}S_t^2\frac{\partial^2 C}{\partial x^2}(t,S_t,r_t)dt+\mathrm{e}^{-\int_0^t r_s ds}\eta^2(t,r_t)\frac{1}{2}\frac{\partial^2 C}{\partial y^2}(t,S_t,r_t)dt\\
&+\rho\sigma S_t\eta(t,r_t)\mathrm{e}^{-\int_0^t r_s ds}\frac{\partial^2 C}{\partial x\partial y}(t,S_t,r_t)dt. \qquad \text{(S.17.84)}
\end{aligned}
$$

The martingale property of $\left(\mathrm{e}^{-\int_0^t r_s ds}C(t,S_t,r_t)\right)_{t\in\mathbb{R}_+}$ shows that the sum of terms in factor of dt vanishes in the above Relation (S.17.84), which yields the PDE

$$
\begin{aligned}
0=&-yC(t,x,y)+\frac{\partial C}{\partial t}(t,x,y)dt+ry\frac{\partial C}{\partial x}(t,x,y)+\gamma(t,y)\frac{\partial C}{\partial y}(t,x,y)\\
&+\frac{\sigma^2}{2}x^2\frac{\partial^2 C}{\partial x^2}(t,x,y)+\eta^2(t,y)\frac{1}{2}\frac{\partial^2 C}{\partial y^2}(t,x,y)+\rho\sigma x\eta(t,y)\frac{\partial^2 C}{\partial x\partial y}(t,x,y),
\end{aligned}
$$

with the terminal condition

$$
C(\tau,x,y)=\max(\alpha x,F(\tau,y)),\quad \text{where}\quad F(\tau,r_\tau)=P(\tau,T)
$$

is the bond pricing function.

e) The convertible bond can be priced as

$$
\begin{aligned}
&\mathbb{E}^*\left[\mathrm{e}^{-\int_t^\tau r_s ds}\max(\alpha S_\tau,P(\tau,T))\,\Big|\,\mathcal{F}_t\right]\\
&=\mathbb{E}^*\left[\mathrm{e}^{-\int_t^\tau r_s ds}P(\tau,T)\,\Big|\,\mathcal{F}_t\right]+\alpha\,\mathbb{E}^*\left[\mathrm{e}^{-\int_t^\tau r_s ds}(S_\tau-P(\tau,T)/\alpha)^+\,\Big|\,\mathcal{F}_t\right]\\
&=P(t,T)+\alpha P(t,T)\widehat{\mathbb{E}}\big[(S_\tau/P(\tau,T)-1/\alpha)^+\,\big|\,\mathcal{F}_t\big]. \qquad \text{(S.17.85)}
\end{aligned}
$$

f) By Proposition 16.8 we find

$$
dZ_t=(\sigma-\sigma_B(t))Z_t d\widehat{W}_t,
$$

where $(\widehat{W}_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion under the forward measure $\widehat{\mathbb{P}}$.

g) By modeling $(Z_t)_{t\in\mathbb{R}_+}$ as the geometric Brownian motion

$$
dZ_t=\sigma(t)Z_t d\widehat{W}_t,
$$

Relation (S.17.85) shows that the convertible bond is priced as

$$
P(t,T)+\alpha S_t\Phi(d_+)-P(t,T)\Phi(d_-),
$$

where

$$d_+ = \frac{1}{v(t,T)}\left(\log\frac{S_t}{P(t,T)} + \frac{v^2(t,\tau)}{2}\right),$$

$$d_- = \frac{1}{v(t,T)}\left(\log\frac{S_t}{P(t,T)} - \frac{v^2(t,\tau)}{2}\right),$$

and $v^2(t,T) = \int_t^T \sigma^2(s,T)ds$, $0 < t < T$.

Exercise 17.9 We have

$$\begin{aligned}\frac{\partial}{\partial r}P_c(0,n) &= \frac{\partial}{\partial r}\left(\frac{1}{(1+r)^n} + \frac{c}{r}\left(1 - \frac{1}{(1+r)^n}\right)\right)\\ &= -\frac{n}{(1+r)^{n+1}} - \frac{c}{r^2}\left(1 - \frac{1}{(1+r)^n}\right) + \frac{nc}{r(1+r)^{n+1}},\end{aligned}$$

hence

$$\begin{aligned}D_c(0,n) &= -\frac{1+r}{P_c(0,n)}\frac{\partial}{\partial r}P_c(0,n)\\ &= -\frac{-\dfrac{n}{(1+r)^n} - \dfrac{(1+r)c}{r^2}\left(1 - \dfrac{1}{(1+r)^n}\right) + \dfrac{nc}{r(1+r)^n}}{\dfrac{1}{(1+r)^n} + \dfrac{c}{r}\left(1 - \dfrac{1}{(1+r)^n}\right)}\\ &= -\frac{-nr - \dfrac{1+r}{r}\left(c((1+r)^n - 1)\right) + nc}{r + c\left((1+r)^n - 1\right)}\\ &= -\frac{1 + r - nr - \dfrac{1+r}{r}\left(r + c((1+r)^n - 1)\right) + nc}{r + c\left((1+r)^n - 1\right)}\\ &= \frac{1+r}{r} - \frac{1 + r + n(c-r)}{r + c\left((1+r)^n - 1\right)}\\ &= \frac{(1 - c/r)n}{1 + c\left((1+r)^n - 1\right)/r} + \frac{1+r}{r}\left(\frac{c\left((1+r)^n - 1\right)}{r + c\left((1+r)^n - 1\right)}\right),\end{aligned}$$

with $D_0(0,n) = n$. We note that

$$\lim_{n\to\infty} D_c(0,n) = \lim_{n\to\infty}\left(\frac{1+r}{r} - \frac{1 + r + n(c-r)}{r + c\left((1+r)^n - 1\right)}\right) = 1 + \frac{1}{r}.$$

When n becomes large, the duration (or relative sensitivity) of the bond price converges to $1+1/r$ whenever the (nonnegative) coupon amount c is nonzero, otherwise the bond duration of $P_c(0,n)$ is n. In particular, the presence of a nonzero coupon makes the duration (or relative sensitivity) of the bond price bounded as n increases, whereas the duration n of $P_0(0,n)$ goes to infinity as n increases.

As a consequence, the presence of the coupon tends to put an upper limit the risk and sensitivity of bond prices with respect to the market interest rate r as n becomes large, which can be used for bond *immunization*. Note that the duration $D_c(0,n)$ can also be written as the relative average

$$D_c(0,n) = \frac{1}{P_c(0,n)}\left(\frac{n}{(1+r)^n} + c\sum_{k=1}^n \frac{k}{(1+r)^k}\right)$$

of zero coupon bond durations, weighted by their respective zero-coupon prices.

Exercise 17.10

a) We have

$$P(t,T) = P(s,T)\exp\left(\int_s^t r_u du + \int_s^t \sigma_u^T dB_u - \frac{1}{2}\int_s^t |\sigma_u^T|^2 du\right),$$

$0 \leq s \leq t \leq T$.

b) We have

$$d\left(\mathrm{e}^{-\int_0^t r_s ds} P(t,T)\right) = \mathrm{e}^{-\int_0^t r_s ds}\sigma_t^T P(t,T) dB_t,$$

which gives a martingale after integration, from the properties of the Itô integral.

c) By the martingale property of the previous question we have

$$\begin{aligned}\mathbb{E}\left[\mathrm{e}^{-\int_0^T r_s ds}\,\Big|\,\mathcal{F}_t\right] &= \mathbb{E}\left[P(T,T)\mathrm{e}^{-\int_0^T r_s ds}\,\Big|\,\mathcal{F}_t\right] \\ &= P(t,T)\mathrm{e}^{-\int_0^t r_s ds}, \qquad 0 \leq t \leq T.\end{aligned}$$

d) By the previous question we have

$$\begin{aligned}P(t,T) &= \mathrm{e}^{\int_0^t r_s ds}\,\mathbb{E}\left[\mathrm{e}^{-\int_0^T r_s ds}\,\Big|\,\mathcal{F}_t\right] \\ &= \mathbb{E}\left[\mathrm{e}^{\int_0^t r_s ds}\mathrm{e}^{-\int_0^T r_s ds}\,\Big|\,\mathcal{F}_t\right] \\ &= \mathbb{E}\left[\mathrm{e}^{-\int_t^T r_s ds}\,\Big|\,\mathcal{F}_t\right], \qquad 0 \leq t \leq T,\end{aligned}$$

since $\mathrm{e}^{-\int_0^t r_s ds}$ is an $\mathcal{F}_t$-measurable random variable.

e) We have

$$\frac{P(t,S)}{P(t,T)} = \frac{P(s,S)}{P(s,T)}\exp\left(\int_s^t (\sigma_u^S - \sigma_u^T) dB_u - \frac{1}{2}\int_s^t (|\sigma_u^S|^2 - |\sigma_u^T|^2) du\right)$$

$$= \frac{P(s,S)}{P(s,T)} \exp\left(\int_s^t (\sigma_u^S - \sigma_u^T) dB_u^T - \frac{1}{2}\int_s^t (\sigma_u^S - \sigma_u^T)^2 du\right),$$

$0 \leq t \leq T$, hence letting $s = t$ and $t = T$ in the above expression, we find

$$P(T,S) = \frac{P(t,S)}{P(t,T)} \exp\left(\int_t^T \left(\sigma_s^S - \sigma_s^T\right) dB_s^T - \frac{1}{2}\int_t^T \left(\sigma_s^S - \sigma_s^T\right)^2 ds\right).$$

f) We have

$$\begin{aligned}
&P(t,T)\widehat{\mathbb{E}}\big[(P(T,S)-\kappa)^+\big] \\
&= P(t,T)\widehat{\mathbb{E}}\left[\left(\frac{P(t,S)}{P(t,T)}\mathrm{e}^{\int_t^T (\sigma_s^S-\sigma_s^T)dB_s^T - \int_t^T (\sigma_s^S-\sigma_s^T)^2 ds/2} - \kappa\right)^+\right] \\
&= P(t,T)\,\mathbb{E}^*\left[\left(\mathrm{e}^X - \kappa\right)^+ \,\middle|\, \mathcal{F}_t\right] \\
&= P(t,T)\mathrm{e}^{m(t)+v^2(t)/2}\Phi\left(\frac{v(t)}{2} + \frac{1}{v(t)}(m(t)+v^2(t)/2 - \log\kappa)\right) \\
&\quad -\kappa P(t,T)\Phi\left(-\frac{v(t)}{2} + \frac{1}{v(t)}(m(t)+v^2(t)/2-\log\kappa)\right),
\end{aligned}$$

where

$$m(t) := \log\frac{P(t,S)}{P(t,T)} - \frac{1}{2}\int_t^T \left(\sigma_s^S - \sigma_s^T\right)^2 ds,$$

and X is a centered Gaussian random variable with variance

$$v^2(t) := \int_t^T \left(\sigma_s^S - \sigma_s^T\right)^2 ds,$$

given $\mathcal{F}_t$. This yields

$$\begin{aligned}
&P(t,T)\widehat{\mathbb{E}}\big[(P(T,S)-\kappa)^+\big] \\
&= P(t,S)\Phi\left(\frac{v(t)}{2} + \frac{1}{v(t)}\log\frac{P(t,S)}{\kappa P(t,T)}\right) - \kappa P(t,T)\Phi\left(-\frac{v(t)}{2} + \frac{1}{v(t)}\log\frac{P(t,S)}{\kappa P(t,T)}\right).
\end{aligned}$$

Exercise 17.11 (Exercise 4.18 continued). From Proposition 17.2, the bond pricing PDE is

$$\begin{cases} \dfrac{\partial F}{\partial t}(t,x) = xF(t,x) - (\alpha - \beta x)\dfrac{\partial F}{\partial x}(t,x) - \dfrac{1}{2}\sigma^2 x^2 \dfrac{\partial^2 F}{\partial x^2}(t,x) \\[2ex] F(T,x) = 1. \end{cases}$$

We search for a solution of the form

$$F(t,x) = \mathrm{e}^{A(T-t) - xB(T-t)},$$

with $A(0) = B(0) = 0$, which implies

$$\begin{cases} A'(s) = 0 \\ \\ B'(s) + \beta B(s) + \dfrac{1}{2}\sigma^2 B^2(s) = 1, \end{cases}$$

hence in particular $A(s) = 0$, $s \in \mathbb{R}$, and $B(s)$ solves a Riccati equation, whose solution can be checked to be

$$B(s) = \frac{2(\mathrm{e}^{\gamma s} - 1)}{2\gamma + (\beta + \gamma)(\mathrm{e}^{\gamma s} - 1)},$$

with $\gamma = \sqrt{\beta^2 + 2\sigma^2}$.

Exercise 17.12

a) We have

$$\begin{cases} y_{0,1} = -\dfrac{1}{T_1} \log P(0, T_1) = 9.53\%, \\ y_{0,2} = -\dfrac{1}{T_2} \log P(0, T_2) = 9.1\%, \\ y_{1,2} = -\dfrac{1}{T_2 - T_1} \log \dfrac{P(0, T_2)}{P(T_1, T_2)} = 8.6\%, \end{cases}$$

with $T_1 = 1$ and $T_2 = 2$.

b) We have

$$P_c(1, 2) = (\$1 + \$0.1) \times P_0(1, 2) = (\$1 + \$0.1) \times \mathrm{e}^{-(T_2 - T_2)y_{1,2}} = \$1.00914,$$

and

$$\begin{aligned} P_c(0, 2) &= (\$1 + \$0.1) \times P_0(0, 2) + \$0.1 \times P_0(0, 1) \\ &= (\$1 + \$0.1) \times \mathrm{e}^{-(T_2 - T_2)y_{0,2}} + \$0.1 \times \mathrm{e}^{-(T_2 - T_2)y_{0,1}} \\ &= \$1.00831. \end{aligned}$$

Exercise 17.13

a) The discretization $r_{t_{k+1}} := r_{t_k} + (a - br_{t_k})\Delta t \pm \sigma\sqrt{\Delta t}$ does not lead to a binomial tree because r_{t_2} can be obtained in *four* different ways from r_{t_0}, as

$$r_{t_2} = r_{t_1}(1 - b\Delta t) + a\Delta t \pm \sigma\sqrt{\Delta t}$$

$$= \begin{cases} (r_{t_0}(1-b\Delta t)+a\Delta t+\sigma\sqrt{\Delta t})(1-b\Delta t)+a\Delta t+\sigma\sqrt{\Delta t} \\ (r_{t_0}(1-b\Delta t)+a\Delta t+\sigma\sqrt{\Delta t})(1-b\Delta t)+a\Delta t-\sigma\sqrt{\Delta t} \\ (r_{t_0}(1-b\Delta t)+a\Delta t-\sigma\sqrt{\Delta t})(1-b\Delta t)+a\Delta t+\sigma\sqrt{\Delta t} \\ (r_{t_0}(1-b\Delta t)+a\Delta t-\sigma\sqrt{\Delta t})(1-b\Delta t)+a\Delta t-\sigma\sqrt{\Delta t}. \end{cases}$$

b) By the Girsanov Theorem, the process $(r_t/\sigma)_{t\in[0,T]}$ with

$$\frac{dr_t}{\sigma} = \frac{a-br_t}{\sigma}dt + dB_t$$

is a standard Brownian motion under the probability measure $\mathbb{Q}$ with Radon-Nikodym density

$$\begin{aligned} \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} &= \exp\left(-\frac{1}{\sigma}\int_0^T (a-br_t)dB_t - \frac{1}{2\sigma^2}\int_0^T (a-br_t)^2 dt\right) \\ &\simeq \exp\left(-\frac{1}{\sigma^2}\int_0^T (a-br_t)(dr_t-(a-br_t)dt) - \frac{1}{2\sigma^2}\int_0^T (a-br_t)^2 dt\right) \\ &= \exp\left(-\frac{1}{\sigma^2}\int_0^T (a-br_t)dr_t + \frac{1}{2\sigma^2}\int_0^T (a-br_t)^2 dt\right). \end{aligned}$$

In other words, if we generate $(r_t/\sigma)_{t\in[0,T]}$ and the increments $\sigma^{-1}dr_t \simeq \pm\sqrt{\Delta t}$ as a standard Brownian motion under $\mathbb{Q}$, then, under the probability measure $\mathbb{P}$ such that

$$\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{Q}} = \exp\left(\frac{1}{\sigma^2}\int_0^T (a-br_t)dr_t - \frac{1}{2\sigma^2}\int_0^T (a-br_t)^2 dt\right),$$

the process

$$dB_t = \frac{dr_t}{\sigma} - \frac{a-br_t}{\sigma}dt$$

will be a standard Brownian motion under $\mathbb{P}$, and the samples

$$dr_t = (a-br_t)dt + \sigma dB_t$$

of $(r_t)_{t\in[0,T]}$ will be distributed as a Vasicek process under $\mathbb{P}$.

c) Approximating the standard Brownian increment $\sigma^{-1}dr_t$ under $\mathbb{Q}$ by $\pm\sqrt{\Delta t}$, we have

$$\begin{aligned} 2^{T/\Delta T}\prod_{0<t<T}\left(\frac{1}{2}\pm\frac{a-br_t}{2\sigma}\sqrt{\Delta t}\right) &= \prod_{0<t<T}\left(1\pm\frac{a-br_t}{\sigma}\sqrt{\Delta t}\right) \\ &= \exp\left(\log\prod_{0<t<T}\left(1\pm\frac{a-br_t}{\sigma}\sqrt{\Delta t}\right)\right) \end{aligned}$$

$$
\begin{aligned}
&= \exp\left(\sum_{0<t<T} \log\left(1 \pm \frac{a-br_t}{\sigma}\sqrt{\Delta t}\right)\right) \\
&\simeq \exp\left(\sum_{0<t<T} \frac{a-br_t}{\sigma}(\pm\sqrt{\Delta t}) - \frac{1}{2}\sum_{0<t<T} \frac{(a-br_t)^2}{\sigma^2}\Delta t\right) \\
&\simeq \exp\left(\frac{1}{\sigma^2}\int_0^T (a-br_t)dr_t - \frac{1}{2\sigma^2}\int_0^T (a-br_t)^2 dt\right) \\
&= \frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{Q}}.
\end{aligned}
$$

d) We check that

$$
\begin{aligned}
\mathbb{E}[\Delta r_{t_1} \mid r_{t_0}] &= (a-br_{t_0})\Delta t \\
&= p(r_{t_0})\sigma\sqrt{\Delta t} - (1-p(r_{t_0}))\sigma\sqrt{\Delta t} \\
&= \sigma p(r_{t_0})\sqrt{\Delta t} - \sigma q(r_{t_0})\sqrt{\Delta t},
\end{aligned}
$$

with

$$
p(r_{t_0}) = \frac{1}{2} + \frac{a-br_{t_0}}{2\sigma}\sqrt{\Delta t} \quad \text{and} \quad q(r_{t_0}) = \frac{1}{2} - \frac{a-br_{t_0}}{2\sigma}\sqrt{\Delta t}.
$$

Similarly, we have

$$
\begin{aligned}
\mathbb{E}[\Delta r_{t_2} \mid r_{t_1}] &= (a-br_{t_1})\Delta t \\
&= p(r_{t_1})\sigma\sqrt{\Delta t} - (1-p(r_{t_1}))\sigma\sqrt{\Delta t} \\
&= \sigma p(r_{t_1})\sqrt{\Delta t} - \sigma q(r_{t_1})\sqrt{\Delta t},
\end{aligned}
$$

with

$$
p(r_{t_1}) = \frac{1}{2} + \frac{a-br_{t_1}}{2\sigma}\sqrt{\Delta t}, \qquad q(r_{t_1}) = \frac{1}{2} - \frac{a-br_{t_1}}{2\sigma}\sqrt{\Delta t}.
$$

Exercise 17.14

a) We have

$$
P(1,2) = \mathbb{E}^*\left[\frac{100}{1+r_1}\right] = \frac{100}{2(1+r_1^u)} + \frac{100}{2(1+r_1^d)}.
$$

b) We have

$$
P(0,2) = \frac{100}{2(1+r_0)(1+r_1^u)} + \frac{100}{2(1+r_0)(1+r_1^d)}.
$$

c) We have $P(0,1) = 91.74 = 100/(1+r_0)$, hence

$$r_0 = \frac{100 - P(0,1)}{P(0,1)} = 100/91.74 - 1 = 0.090037061 \simeq 9\%.$$

d) We have

$$83.40 = P(0,2) = \frac{P(0,1)}{2(1+r_1^u)} + \frac{P(0,1)}{2(1+r_1^d)}$$

and $r_1^u/r_1^d = \mathrm{e}^{2\sigma\sqrt{\Delta t}}$, hence

$$83.40 = P(0,2) = \frac{P(0,1)}{2(1+r_1^d \mathrm{e}^{2\sigma\sqrt{\Delta t}})} + \frac{P(0,1)}{2(1+r_1^d)}$$

or

$$\mathrm{e}^{2\sigma\sqrt{\Delta t}}(r_1^d)^2 + 2r_1^d \mathrm{e}^{\sigma\sqrt{\Delta t}} \cosh\left(\sigma\sqrt{\Delta t}\right)\left(1 - \frac{P(0,1)}{2P(0,2)}\right) + 1 - \frac{P(0,1)}{P(0,2)} = 0,$$

and

$$\begin{aligned} r_1^d &= \mathrm{e}^{-\sigma\sqrt{\Delta t}} \left(\cosh\left(\sigma\sqrt{\Delta t}\right)\left(\frac{P(0,1)}{2P(0,2)} - 1\right) \right. \\ &\qquad \left. \pm \sqrt{\left(\frac{P(0,1)}{2P(0,2)} - 1\right)^2 \cosh^2\left(\sigma\sqrt{\Delta t}\right) + \frac{P(0,1)}{P(0,2)} - 1} \right) \\ &= 0.078684844 \simeq 7.87\%, \end{aligned}$$

and

$$\begin{aligned} r_1^u &= r_1^d \mathrm{e}^{2\sigma\sqrt{\Delta t}} \\ &= \mathrm{e}^{\sigma\sqrt{\Delta t}} \left(\cosh\left(\sigma\sqrt{\Delta t}\right)\left(\frac{P(0,1)}{2P(0,2)} - 1\right) \right. \\ &\qquad \left. \pm \sqrt{\left(\frac{P(0,1)}{2P(0,2)} - 1\right)^2 \cosh^2\left(\sigma\sqrt{\Delta t}\right) + \frac{P(0,1)}{P(0,2)} - 1} \right) \\ &= 0.122174525 \simeq 12.22\%. \end{aligned}$$

We also find

$$\mu = \frac{1}{\Delta t}\left(\sigma\sqrt{\Delta t} + \log\frac{r_1^d}{r_0}\right) = \frac{1}{\Delta t}\left(-\sigma\sqrt{\Delta t} + \log\frac{r_1^u}{r_0}\right) = 0.085229181 \simeq 8.52\%.$$

Exercise 17.15

a) When $n = 1$ the relation (17.55) shows that $\widetilde{f}(t,t,T_1) = f(t,t,T_1)$ with $F(t,x) = c_1 \mathrm{e}^{-(T_1-1)x}$ and $P(t,T_1) = c_1 \mathrm{e}^{f(t,t,T_1)}$, hence

$$D(t,T_1) := -\frac{1}{P(t,T_1)}\frac{\partial F}{\partial x}(t,f(t,t,T_1)) = T_1 - t, \qquad 0 \leq t \leq T_1.$$

b) In general, we have

$$\begin{aligned}
D(t,T_n) &= -\frac{1}{P(t,T_n)}\frac{\partial F}{\partial x}\big(t,\widetilde{f}(t,t,T_n)\big) \\
&= \frac{1}{P(t,T_n)}\sum_{k=1}^{n}(T_k-t)c_k \mathrm{e}^{-(T_k-t)\tilde{f}(t,t,T_n)} \\
&= \sum_{k=1}^{n}(T_k-t)w_k,
\end{aligned}$$

where

$$\begin{aligned}
w_k &:= \frac{c_k}{P(t,T_n)}\mathrm{e}^{-(T_k-t)\tilde{f}(t,t,T_n)} \\
&= \frac{c_k \mathrm{e}^{-(T_k-t)\tilde{f}(t,t,T_n)}}{\displaystyle\sum_{l=1}^{n} c_l \mathrm{e}^{-(T_l-t)f(t,t,T_l)}} \\
&= \frac{c_k \mathrm{e}^{-(T_k-t)\tilde{f}(t,t,T_n)}}{\displaystyle\sum_{l=1}^{n} c_l \mathrm{e}^{-(T_l-t)\tilde{f}(t,t,T_n)}}, \qquad k = 1,2,\ldots,n,
\end{aligned}$$

and the weights $w_1, w_2, \ldots, w_n$ satisfy

$$\sum_{k=1}^{n} w_k = 1.$$

c) We have

$$\begin{aligned}
&C(t,T_n) = \frac{1}{P(t,T_n)}\frac{\partial^2 F}{\partial x^2}\big(t,\widetilde{f}(t,t,T_n)\big) \\
&= \sum_{k=1}^{n}(T_k-t)^2 w_k \\
&= \sum_{k=1}^{n}(T_k-t-D(t,T_n))^2 w_k + 2D(t,T_n)\sum_{k=1}^{n}(T_k-t)w_k - (D(t,T_n))^2 \\
&= (D(t,T_n))^2 + \sum_{k=1}^{n}(T_k-t-D(t,T_n))^2 w_k \\
&= (D(t,T_n))^2 + (S(t,T_n))^2,
\end{aligned}$$

where

$$(S(t,T_n))^2 := \sum_{k=1}^n (T_k - t - D(t,T_n))^2 w_k.$$

d) We have

$$\begin{aligned}
D(t,T_n) &= \frac{1}{P(t,T_n)} \sum_{k=1}^n c_k B(T_k - t) \mathrm{e}^{A(T_k-t)+B(T_k-t)f_\alpha(t,t,T_n)} \\
&= \frac{1}{\mathrm{e}^{A(T_n-t)+B(T_n-t)f_\alpha(t,t,T_n)}} \sum_{k=1}^n c_k B(T_k - t) \mathrm{e}^{A(T_k-t)+B(T_k-t)f_\alpha(t,t,T_n)} \\
&= \sum_{k=1}^n c_k B(T_k - t) \mathrm{e}^{A(T_k-t)-A(T_n-t)+(B(T_k-t)-B(T_n-t))f_\alpha(t,t,T_n)}.
\end{aligned}$$

e) We have

$$\begin{aligned}
D(t,T_n) &= \frac{1}{b} \sum_{k=1}^n \left(1 - \mathrm{e}^{-(T_k-t)b}\right) c_k \mathrm{e}^{A(T_k-t)-A(T_n-t)+\left(\mathrm{e}^{-(T_n-t)b}-\mathrm{e}^{-(T_k-t)b}\right)f_\alpha(t,t,T_n)/b} \\
&= \frac{1}{b} \sum_{k=1}^n \left(1 - \mathrm{e}^{-(T_k-t)b}\right) c_k \mathrm{e}^{A(T_k-t)-A(T_n-t)} (P(t, t+\alpha(T_n-t))^{\frac{(\mathrm{e}^{-(T_n-t)b}-\mathrm{e}^{-(T_k-t)b})}{(T_n-t)\alpha b}}.
\end{aligned}$$

Chapter 18

Exercise 18.1

a) By partial differentiation with respect to T under the expectation $\widehat{\mathbb{E}}$, we have

$$\begin{aligned}
\frac{\partial P}{\partial T}(t,T) &= \frac{\partial}{\partial T} \mathbb{E}^* \left[\mathrm{e}^{-\int_t^T r_s ds} \;\middle|\; \mathcal{F}_t \right] \\
&= \mathbb{E}^* \left[-r_T \mathrm{e}^{-\int_t^T r_s ds} \;\middle|\; \mathcal{F}_t \right] \\
&= -P(t,T) \widehat{\mathbb{E}}[r_T \mid \mathcal{F}_t].
\end{aligned}$$

b) As a consequence of Question (a), we find

$$f(t,T) = -\frac{1}{P(t,T)} \frac{\partial P}{\partial T}(t,T) = \widehat{\mathbb{E}}[r_T \mid \mathcal{F}_t], \qquad 0 \le t \le T, \qquad \text{(S.18.86)}$$

see Relation (22) page 10 of Mamon (2004).

c) The martingale property of $(f(t,T))_{t\in[0,T]}$ under the forward measure $\widehat{\mathbb{E}}$ follows from Relation (S.18.86) and the tower property of conditional expectations as in *e.g.* (7.1) or (7.42).

Remark. In the Vasicek model, by (17.2) and (19.9) we have

$$\begin{aligned}
r_T &= r_0 \mathrm{e}^{-bT} + \frac{a}{b}(1-\mathrm{e}^{-bT}) + \sigma \int_0^T \mathrm{e}^{-(T-s)b} dW_s \\
&= r_0 \mathrm{e}^{-bT} + \frac{a}{b}(1-\mathrm{e}^{-bT}) + \sigma \int_0^T \mathrm{e}^{-(T-s)b} d\widehat{W}_s \\
&\quad - \frac{\sigma^2}{b} \int_0^T \mathrm{e}^{-(T-s)b}(1-\mathrm{e}^{-(T-s)b}) ds \\
&= r_0 \mathrm{e}^{-bT} + \frac{a}{b}(1-\mathrm{e}^{-bT}) + \sigma \int_0^T \mathrm{e}^{-(T-s)b} d\widehat{W}_s \\
&\quad - \frac{\sigma^2}{b} \int_0^T \mathrm{e}^{-(T-s)b} ds + \frac{\sigma^2}{b} \int_0^T \mathrm{e}^{-2(T-s)b} ds,
\end{aligned}$$

hence

$$\begin{aligned}
\widehat{\mathbb{E}}[r_T \mid \mathcal{F}_t] &= r_0 \mathrm{e}^{-bT} + \frac{a}{b}\big(1-\mathrm{e}^{-bT}\big) + \sigma \int_0^t \mathrm{e}^{-(T-s)b} d\widehat{W}_s \\
&\quad - \frac{\sigma^2}{b} \int_0^T \mathrm{e}^{-(T-s)b} ds + \frac{\sigma^2}{b} \int_0^T \mathrm{e}^{-2(T-s)b} ds \\
&= r_0 \mathrm{e}^{-bT} + \frac{a}{b}\big(1-\mathrm{e}^{-bT}\big) + \sigma \int_0^t \mathrm{e}^{-(T-s)b} dW_s \\
&\quad + \frac{\sigma^2}{b} \int_0^t \mathrm{e}^{-(T-s)b}\big(1-\mathrm{e}^{-(T-s)b}\big) ds \\
&\quad - \frac{\sigma^2}{b} \int_0^T \mathrm{e}^{-(T-s)b} ds + \frac{\sigma^2}{b} \int_0^T \mathrm{e}^{-2(T-s)b} ds \\
&= r_0 \mathrm{e}^{-bT} + \frac{a}{b}\big(1-\mathrm{e}^{-bT}\big) + \mathrm{e}^{-(T-t)b}\Big(r_t - r_0 \mathrm{e}^{-bt} - \frac{a}{b}\big(1-\mathrm{e}^{-bt}\big)\Big) \\
&\quad + \frac{\sigma^2}{b} \int_0^t \mathrm{e}^{-(T-s)b} ds - \frac{\sigma^2}{b} \int_0^t \mathrm{e}^{-2(T-s)b} ds \\
&\quad - \frac{\sigma^2}{b} \int_0^T \mathrm{e}^{-(T-s)b} ds + \frac{\sigma^2}{b} \int_0^T \mathrm{e}^{-2(T-s)b} ds \\
&= \frac{a}{b} + \mathrm{e}^{-(T-t)b}\left(r_t - \frac{a}{b}\right) - \frac{\sigma^2}{b} \int_t^T \mathrm{e}^{-(T-s)b} ds + \frac{\sigma^2}{b} \int_t^T \mathrm{e}^{-2(T-s)b} ds \\
&= \frac{a}{b} + \mathrm{e}^{-(T-t)b}\left(r_t - \frac{a}{b}\right) - \frac{\sigma^2}{b} \int_0^{T-t} \mathrm{e}^{-bs} ds + \frac{\sigma^2}{b} \int_0^{T-t} \mathrm{e}^{-2bs} ds \\
&= \frac{a}{b} + \mathrm{e}^{-(T-t)b}\left(r_t - \frac{a}{b}\right) - \frac{\sigma^2}{b^2}\big(1-\mathrm{e}^{-(T-t)b}\big) + \frac{\sigma^2}{b^2}\big(1-\mathrm{e}^{-2(T-t)b}\big) \\
&= \frac{a}{b} - \frac{\sigma^2}{2b^2} + \mathrm{e}^{-(T-t)b}\left(r_t - \frac{a}{b} + \frac{\sigma^2}{b^2}\right) - \frac{\sigma^2}{b^2} \mathrm{e}^{-2(T-t)b},
\end{aligned}$$

which recovers (18.31).

Exercise 18.2 We have

$$P(0,T_2) = \exp\left(-\int_0^{T_2} f(t,s)ds\right) = \mathrm{e}^{-r_1T_1 - r_2(T_2-T_1)}, \qquad t \in [0,T_2],$$

and

$$P(T_1,T_2) = \exp\left(-\int_{T_1}^{T_2} f(t,s)ds\right) = \mathrm{e}^{-r_2(T_2-T_1)}, \qquad t \in [0,T_2],$$

from which we deduce

$$r_2 = -\frac{1}{T_2-T_1}\log P(T_1,T_2),$$

and

$$\begin{aligned} r_1 &= -r_2\frac{T_2-T_1}{T_1} - \frac{1}{T_1}\log P(0,T_2) \\ &= \frac{1}{T_1}\log P(T_1,T_2) - \frac{1}{T_1}\log P(0,T_2) \\ &= -\frac{1}{T_1}\log\frac{P(0,T_2)}{P(T_1,T_2)}. \end{aligned}$$

Exercise 18.3 (Exercise 4.15 continued).

a) We check that $P(T,T) = \mathrm{e}^{X_T^T} = 1$.
b) We have

$$\begin{aligned} f(t,T,S) &= -\frac{1}{S-T}\left(X_t^S - X_t^T - \mu(S-T)\right) \\ &= \mu - \sigma\frac{1}{S-T}\left((S-t)\int_0^t \frac{1}{S-s}dB_s - (T-t)\int_0^t \frac{1}{T-s}dB_s\right) \\ &= \mu - \sigma\frac{1}{S-T}\int_0^t\left(\frac{S-t}{S-s} - \frac{T-t}{T-s}\right)dB_s \\ &= \mu - \sigma\frac{1}{S-T}\int_0^t \frac{(T-s)(S-t)-(T-t)(S-s)}{(S-s)(T-s)}dB_s \\ &= \mu + \frac{\sigma}{S-T}\int_0^t \frac{(s-t)(S-T)}{(S-s)(T-s)}dB_s. \end{aligned}$$

c) We have

$$f(t,T) = \mu - \sigma\int_0^t \frac{t-s}{(T-s)^2}dB_s.$$

d) We note that

$$\lim_{T\searrow t} f(t,T) = \mu - \sigma\int_0^t \frac{1}{t-s}dB_s$$

does not exist in $L^2(\Omega)$.

e) By Itô's calculus we have

$$\begin{aligned}\frac{dP(t,T)}{P(t,T)} &= \sigma dB_t + \frac{1}{2}\sigma^2 dt + \mu dt - \frac{X_t^T}{T-t}dt \\ &= \sigma dB_t + \frac{1}{2}\sigma^2 dt - \frac{\log P(t,T)}{T-t}dt, \qquad t \in [0,T].\end{aligned}$$

f) Letting

$$\begin{aligned} r_t^{\mathrm{T}} &:= \mu + \frac{1}{2}\sigma^2 - \frac{X_t^T}{T-t} \\ &= \mu + \frac{1}{2}\sigma^2 - \sigma \int_0^t \frac{dB_s}{T-s}, \end{aligned}$$

by Question (e) we find that

$$\frac{dP(t,T)}{P(t,T)} = \sigma dB_t + r_t^T dt, \qquad 0 \leq t \leq T.$$

g) The equation of Question (f) can be solved as

$$P(t,T) = P(0,T)\exp\left(\sigma B_t - \frac{\sigma^2 t}{2} + \int_0^t r_s^T ds\right), \qquad 0 \leq t \leq T,$$

hence the process

$$P(t,T)\exp\left(-\int_0^t r_s^T ds\right) = P(0,T)\exp\left(\sigma B_t - \frac{\sigma^2}{2}t\right), \quad 0 \leq t \leq T,$$

is a martingale under $\mathbb{P}^*$, with the relation

$$\begin{aligned} P(t,T)\exp\left(-\int_0^t r_s^T ds\right) &= \mathbb{E}^*\left[P(T,T)\exp\left(-\int_0^T r_s^T ds\right)\Big| \mathcal{F}_t\right] \\ &= \mathbb{E}^*\left[\exp\left(-\int_0^T r_s^T ds\right)\Big| \mathcal{F}_t\right], \qquad 0 \leq t \leq T, \end{aligned}$$

showing that

$$\begin{aligned} P(t,T) &= \exp\left(\int_0^t r_s^T ds\right)\mathbb{E}^*\left[\exp\left(-\int_0^T r_s^T ds\right)\Big| \mathcal{F}_t\right] \\ &= \mathbb{E}^*\left[\exp\left(\int_0^t r_s^T ds\right)\exp\left(-\int_0^T r_s^T ds\right)\Big| \mathcal{F}_t\right] \\ &= \mathbb{E}^*\left[\exp\left(-\int_t^T r_s^T ds\right)\Big| \mathcal{F}_t\right], \qquad 0 \leq t \leq T. \end{aligned}$$

h) By Question (g) we have

$$\mathbb{E}\left[\frac{d\mathbb{P}_T}{d\mathbb{P}}\,\Big|\,\mathcal{F}_t\right] = \frac{P(t,T)}{P(0,T)}\mathrm{e}^{-\int_0^t r_s^T ds} = \mathrm{e}^{\sigma B_t - \sigma^2 t/2}, \qquad 0 \le t \le T.$$

i) By the Girsanov Theorem, the process $\widehat{B}_t := B_t - \sigma t$ is a standard Brownian motion under $\mathbb{P}_T$.

j) We have

$$\begin{aligned}
\log P(T,S) &= -\mu(S-T) + \sigma \int_0^T \frac{S-T}{S-s} dB_s \\
&= -\mu(S-T) + \sigma \int_0^t \frac{S-T}{S-s} dB_s + \sigma \int_t^T \frac{S-T}{S-s} dB_s \\
&= \frac{S-T}{S-t} \log P(t,S) + \sigma \int_t^T \frac{S-T}{S-s} dB_s \\
&= \frac{S-T}{S-t} \log P(t,S) + \sigma \int_t^T \frac{S-T}{S-s} d\widehat{B}_s + \sigma^2 \int_t^T \frac{S-T}{S-s} ds \\
&= \frac{S-T}{S-t} \log P(t,S) + \sigma \int_t^T \frac{S-T}{S-s} d\widehat{B}_s + (S-T)\sigma^2 \log \frac{S-t}{S-T},
\end{aligned}$$

$0 < T < S$.

k) We have

$$\begin{aligned}
& P(t,T)\,\mathbb{E}_T\left[(P(T,S)-K)^+ \,\big|\, \mathcal{F}_t\right] \\
&= P(t,T)\,\mathbb{E}\left[\left(\mathrm{e}^X - K\right)^+ \mid \mathcal{F}_t\right] \\
&= P(t,T)\mathrm{e}^{m_t + v_t^2/2}\Phi\left(\frac{v_t}{2} + \frac{1}{v_t}(m_t + v_t^2/2 - \log\kappa)\right) \\
&\quad -\kappa P(t,T)\Phi\left(-\frac{v_t}{2} + \frac{1}{v_t}(m_t + v_t^2/2 - \log\kappa)\right) \\
&= P(t,T)\mathrm{e}^{m_t + v_t^2/2}\Phi\left(v_t + \frac{1}{v_t}(m_t - \log\kappa)\right) - \kappa P(t,T)\Phi\left(\frac{1}{v_t}(m_t - \log\kappa)\right),
\end{aligned}$$

with

$$m_t = \frac{S-T}{S-t}\log P(t,S) + (S-T)\sigma^2 \log\frac{S-t}{S-T}$$

and

$$\begin{aligned}
v_t^2 &= \sigma^2 \int_t^T \frac{(S-T)^2}{(S-s)^2} ds \\
&= (S-T)^2\sigma^2 \left(\frac{1}{S-T} - \frac{1}{S-t}\right) \\
&= (S-T)\sigma^2 \frac{(T-t)}{(S-t)},
\end{aligned}$$

hence

$$\begin{aligned}
&P(t,T)\,\mathbb{E}_T\left[(P(T,S)-K)^+ \,\middle|\, \mathcal{F}_t\right] \\
&= P(t,T)(P(t,S))^{(S-T)(S-t)}\left(\frac{S-t}{S-T}\right)^{(S-T)\sigma^2} \mathrm{e}^{v_t^2/2} \\
&\quad \times\Phi\left(v_t+\frac{1}{v_t}\log\left(\frac{(P(t,S))^{(S-T)(S-t)}}{\kappa}\left(\frac{S-t}{S-T}\right)^{(S-T)\sigma^2}\right)\right) \\
&-\kappa P(t,T)\Phi\left(\frac{1}{v_t}\log\left(\frac{(P(t,S))^{(S-T)(S-t)}}{\kappa}\left(\frac{S-t}{S-T}\right)^{(S-T)\sigma^2}\right)\right).
\end{aligned}$$

Exercise 18.4

a) In the Vašíček (1977) model, by (17.32) we have

$$\begin{aligned}
P(t,T) &= \mathbb{E}\left[\exp\left(-\int_t^T r_s ds\right)\right] \\
&= \mathbb{E}\left[\exp\left(-\int_t^T h(s)ds-\int_t^T X_s ds\right)\right] \\
&= \exp\left(-\int_t^T h(s)ds\right)\mathbb{E}\left[\exp\left(-\int_t^T X_s ds\right)\right] \\
&= \exp\left(-\int_t^T h(s)ds + A(T-t)+X_tC(T-t)\right), \quad 0\le t\le T,
\end{aligned}$$

hence, since $X_0=0$ we find $P(0,T)=\exp\left(-\int_0^T h(s)ds + A(T)\right)$.

b) By the identification

$$\begin{aligned}
P(t,T) &= \exp\left(-\int_t^T h(s)ds + A(T-t)+X_tC(T-t)\right) \\
&= \exp\left(-\int_t^T f(t,s)ds\right),
\end{aligned}$$

we find

$$\int_t^T h(s)ds = \int_t^T f(t,s)ds + A(T-t)+X_tC(T-t),$$

and by differentiation with respect ot T this yields

$$h(T)=f(t,T)+A'(T-t)+X_tC'(T-t), \qquad 0\le t\le T,$$

where

$$A(T-t) = \frac{4ab - 3\sigma^2}{4b^3} + \frac{\sigma^2 - 2ab}{2b^2}(T-t) + \frac{\sigma^2 - ab}{b^3}\mathrm{e}^{-b(T-t)} - \frac{\sigma^2}{4b^3}\mathrm{e}^{-2b(T-t)}.$$

Given an initial market data curve $f^M(0,T)$, the matching $f^M(0,T) = f(0,T)$ can be achieved at time $t = 0$ by letting

$$h(T) := f^M(0,T) + A'(T) = f^M(0,T) + \frac{\sigma^2 - 2ab}{2b^2} - \frac{\sigma^2 - ab}{b^2}\mathrm{e}^{-bT} + \frac{\sigma^2}{2b^2}\mathrm{e}^{-2bT},$$

$T > 0$. Note however that in general, at time $t \in (0,T]$ we will have

$$h(T) = f(t,T) + A'(T-t) + X_t C'(T-t) = f^M(0,T) + A'(T),$$

and the relation

$$f(t,T) = f^M(0,T) + A'(T) - A'(T-t) - X_t C'(T-t), \quad t \in [0,T],$$

will allow us to match market data at time $t = 0$ only, *i.e.* for the initial curve. In any case, model calibration is to be done at time $t = 0$.

Exercise 18.5 (Exercise 4.12 continued, see also Proposition 4.1 in Carmona and Durrleman (2003)). Letting $\sigma := \sqrt{\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2}$, we have

$$dS_t = rS_t dt + \sigma dW_t,$$

where $(W_t)_{t\in[0,T]}$, is a standard Brownian motions under $\mathbb{P}^*$, hence

$$S_t = S_0\mathrm{e}^{rt} + \sigma \int_0^t \mathrm{e}^{(t-s)r} dW_s, \quad t \geq 0,$$

The spread S_T has a Gaussian distribution with mean $\alpha := \mathbb{E}^*[S_T] = S_0\mathrm{e}^{rT}$ and variance

$$\begin{aligned}
\eta^2 &:= \mathrm{Var}^*[S_T] \\
&= \mathrm{Var}\left[\sigma \int_0^T \mathrm{e}^{(T-s)r} dB_s\right] \\
&= \sigma^2 \int_0^T \left(\mathrm{e}^{(T-s)r}\right)^2 ds \\
&= \frac{\sigma^2}{2r}\left(\mathrm{e}^{2rT} - 1\right),
\end{aligned}$$

and probability density function

$$\varphi(x) = \frac{\sqrt{r/\pi}}{\sigma\sqrt{\mathrm{e}^{2rT} - 1}} \exp\left(-\frac{\left(S_0\mathrm{e}^{rT} - x\right)^2}{\sigma^2\left(\mathrm{e}^{2rT} - 1\right)/r}\right), \quad x \in \mathbb{R}.$$

Hence, we have

$$
\begin{aligned}
\mathrm{e}^{-rt}\,\mathbb{E}^*[(S_T-K)^+] &= \frac{\mathrm{e}^{-rt}}{\sqrt{2\pi\eta^2}}\int_{-\infty}^{\infty}(x-K)^+\mathrm{e}^{-(x-\alpha)^2/(2\eta^2)}dx \\
&= \frac{\mathrm{e}^{-rt}}{\sqrt{2\pi\eta^2}}\int_{K}^{\infty}(x-K)\mathrm{e}^{-(x-\alpha)^2/(2\eta^2)}dx \\
&= \frac{\mathrm{e}^{-rt}}{\sqrt{2\pi\eta^2}}\int_{K}^{\infty}x\mathrm{e}^{-(x-\alpha)^2/(2\eta^2)}dx - \frac{K\mathrm{e}^{-rt}}{\sqrt{2\pi\eta^2}}\int_{K}^{\infty}\mathrm{e}^{-(x-\alpha)^2/(2\eta^2)}dx \\
&= \frac{\eta\mathrm{e}^{-rt}}{\sqrt{2\pi}}\int_{(K-\alpha)/\eta}^{\infty}(x+\alpha)\mathrm{e}^{-x^2/2}dx - \frac{K\mathrm{e}^{-rt}}{\sqrt{2\pi}}\int_{(K-\alpha)/\eta}^{\infty}\mathrm{e}^{-x^2/2}dx \\
&= -\frac{\eta\mathrm{e}^{-rt}}{\sqrt{2\pi}}\left[\mathrm{e}^{-x^2/2}\right]_{(K-\alpha)/\eta}^{\infty} - (K-\alpha)\mathrm{e}^{-rt}\Phi\left(-\frac{K-\alpha}{\eta}\right) \\
&= \frac{\eta\mathrm{e}^{-rt}}{\sqrt{2\pi}}\mathrm{e}^{-(K-\alpha)^2/(2\eta^2)} - (K-\alpha)\mathrm{e}^{-rt}\Phi\left(-\frac{K-\alpha}{\eta}\right).
\end{aligned}
$$

Exercise 18.6 From the definition

$$
L(t,t,T) = \frac{1}{T-t}\left(\frac{1}{P(t,T)}-1\right),
$$

we have

$$
P(t,T) = \frac{1}{1+(T-t)L(t,t,T)},
$$

and similarly

$$
P(t,S) = \frac{1}{1+(S-t)L(t,t,S)}.
$$

Hence we get

$$
\begin{aligned}
L(t,T,S) &= \frac{1}{S-T}\left(\frac{P(t,T)}{P(t,S)}-1\right) \\
&= \frac{1}{S-T}\left(\frac{1+(S-t)L(t,t,S)}{1+(T-t)L(t,t,T)}-1\right) \\
&= \frac{1}{S-T}\left(\frac{(S-t)L(t,t,S)-(T-t)L(t,t,T)}{1+(T-t)L(t,t,T)}\right).
\end{aligned}
$$

When $T=$ one year and $L(0,0,T)=2\%$, $L(0,0,2T)=2.5\%$ we find

$$
L(t,T,S) = \frac{1}{T}\left(\frac{2TL(0,0,2T)-TL(0,0,T)}{1+TL(0,0,T)}\right) = \frac{2\times 0.025-0.02}{1+0.02} = 2.94\%,
$$

so that we would prefer a spot rate at $L(T,T,2T)=2\%$ over a forward contract with rate $L(0,T,2T)=2.94\%$.

Exercise 18.7 (Exercise 17.4 continued).

a) By Definition 18.1, we have

$$\begin{aligned}
f(t,T,S) &= \frac{1}{S-T}(\log P(t,T) - \log P(t,S)) \\
&= \frac{1}{S-T}\left(\left(-(T-t)r_t + \frac{\sigma^2}{6}(T-t)^3\right) - \left(-(S-t)r_t + \frac{\sigma^2}{6}(S-t)^3\right)\right) \\
&= r_t + \frac{1}{S-T}\frac{\sigma^2}{6}\big((T-t)^3 - (S-t)^3\big).
\end{aligned}$$

b) We have

$$\begin{aligned}
f(t,T) &= \lim_{S \searrow T} f(t,T,S) \\
&= -\frac{\partial}{\partial T}\log P(t,T) \\
&= -\frac{\partial}{\partial T}\left(-(T-t)r_t + \frac{\sigma^2}{6}(T-t)^3\right) \\
&= r_t - \frac{\sigma^2}{2}(T-t)^2.
\end{aligned}$$

c) We have

$$d_t f(t,T) = (T-t)\sigma^2 dt + \sigma dB_t.$$

d) The HJM condition (18.28) is satisfied since the drift of $d_t f(t,T)$ equals $\sigma \int_t^T \sigma ds$.

Exercise 18.8

a) We have

$$X_t = X_0 \mathrm{e}^{-bt} + \sigma \int_0^t \mathrm{e}^{-(t-s)b} dB_s^{(1)}$$

and

$$Y_t = Y_0 \mathrm{e}^{-bt} + \sigma \int_0^t \mathrm{e}^{-(t-s)b} dB_s^{(2)}, \qquad t \in \mathbb{R}_+,$$

see (17.2).

b) We have

$$\operatorname{Var}[X_t] = \operatorname{Var}[Y_t] = \frac{\sigma^2}{2b}(1 - \mathrm{e}^{-2bt}), \qquad t \in \mathbb{R}_+,$$

see page 478, and therefore

$$\begin{aligned}
\operatorname{Cov}(X_t, Y_t) &= \operatorname{Cov}\left(X_0\mathrm{e}^{-bt} + \sigma\int_0^t \mathrm{e}^{-(t-s)b} dB_s^{(1)}, Y_0\mathrm{e}^{-bt} + \sigma\int_0^t \mathrm{e}^{-(t-s)b} dB_s^{(2)}\right) \\
&= \sigma^2 \operatorname{Cov}\left(\int_0^t \mathrm{e}^{-(t-s)b} dB_s^{(1)}, \int_0^t \mathrm{e}^{-(t-s)b} dB_s^{(2)}\right)
\end{aligned}$$

$$
\begin{aligned}
&= \sigma^2 \, \mathbb{E}\left[\int_0^t \mathrm{e}^{-(t-s)b} dB_s^{(1)} \int_0^t \mathrm{e}^{-(t-s)b} dB_s^{(2)} \right] \\
&= \rho\sigma^2 \int_0^t \mathrm{e}^{-2(t-s)b} ds \\
&= \rho \frac{\sigma^2}{2b} \big(1 - \mathrm{e}^{-2bt}\big), \qquad t \in \mathbb{R}_+.
\end{aligned}
$$

c) We have

$$
\begin{aligned}
&\mathrm{Cov}(\log P(t,T_1), \log P(t,T_2)) \\
&= \mathrm{Cov}\left(\log\left(F_1(t,X_t,T_1)F_2(t,Y_t,T_2)\mathrm{e}^{\rho U(t,T_1)}\right), \log\left(F_1(t,X_t,T_1)F_2(t,Y_t,T_2)\mathrm{e}^{\rho U(t,T_2)}\right)\right) \\
&= \mathrm{Cov}\left(C_1^{T_1} + X_t A_1^{T_1} + C_2^{T_1} + Y_t A_2^{T_1}, C_1^{T_2} + X_t A_1^{T_2} + C_2^{T_2} + Y_t A_2^{T_2}\right) \\
&= \mathrm{Cov}\left(X_t A_1^{T_1} + Y_t A_2^{T_1}, X_t A_1^{T_2} + Y_t A_2^{T_2}\right) \\
&= A_1^{T_1} A_1^{T_2} \, \mathrm{Var}[X_t] + A_2^{T_1} A_2^{T_2} \, \mathrm{Var}[Y_t] + \left(A_1^{T_1} A_2^{T_2} + A_1^{T_2} A_2^{T_1}\right) \mathrm{Cov}(X_t, Y_t) \\
&= \left(A_1^{T_1} A_1^{T_2} + A_2^{T_1} A_2^{T_2} + \rho\left(A_1^{T_1} A_2^{T_2} + A_1^{T_2} A_2^{T_1}\right)\right) \mathrm{Var}[X_t].
\end{aligned}
$$

When $\mathrm{Cov}(X_t, Y_t) = \rho \, \mathrm{Var}[X_t] = \rho \, \mathrm{Var}[Y_t]$, we find the correlation

$$
\begin{aligned}
\mathrm{Cov}(\log P(t,T_1), \log P(t,T_2)) &= \frac{\mathrm{Cov}(\log P(t,T_1), \log P(t,T_2))}{\sqrt{\mathrm{Var}[\log P(t,T_1)] \, \mathrm{Var}[\log P(t,T_2)]}} \\
&= \frac{A_1^{T_1} A_1^{T_2} + A_2^{T_1} A_2^{T_2} + \rho\left(A_1^{T_1} A_2^{T_2} + A_1^{T_2} A_2^{T_1}\right)}{\sqrt{\left(A_1^{T_1}\right)^2 + \left(A_2^{T_1}\right)^2 + \rho\left(A_1^{T_1} A_2^{T_1} + A_1^{T_1} A_2^{T_1}\right)}} \\
&\quad \times \frac{1}{\sqrt{\left(A_1^{T_2}\right)^2 + \left(A_2^{T_2}\right)^2 + \rho\left(A_1^{T_2} A_2^{T_2} + A_1^{T_2} A_2^{T_2}\right)}}.
\end{aligned}
$$

When $\rho = 1$, we find

$$
\begin{aligned}
&\mathrm{Cov}(\log P(t,T_1), \log P(t,T_2)) \\
&= \frac{A_1^{T_1} A_1^{T_2} + A_2^{T_1} A_2^{T_2} + A_1^{T_1} A_2^{T_2} + A_1^{T_2} A_2^{T_1}}{\sqrt{\left(A_1^{T_1}\right)^2 + \left(A_2^{T_1}\right)^2 + A_1^{T_1} A_2^{T_1} + A_1^{T_1} A_2^{T_1}} \sqrt{\left(A_1^{T_2}\right)^2 + \left(A_2^{T_2}\right)^2 + A_1^{T_2} A_2^{T_2} + A_1^{T_2} A_2^{T_2}}} \\
&= \frac{\left(A_1^{T_1} + A_2^{T_1}\right)\left(A_1^{T_2} + A_2^{T_2}\right)}{\left|A_1^{T_1} + A_2^{T_1}\right| \left|A_1^{T_2} + A_2^{T_2}\right|} \\
&= \pm 1.
\end{aligned}
$$

For example, if $A_1^{T_1} = 4$, $A_1^{T_2} = 1$, $A_2^{T_1} = 1$ and $A_2^{T_2} = 4$, we find

$$
\mathrm{Cov}(\log P(t,T_1), \log P(t,T_2)) = \frac{8 + 17\rho}{17 + 8\rho}.
$$

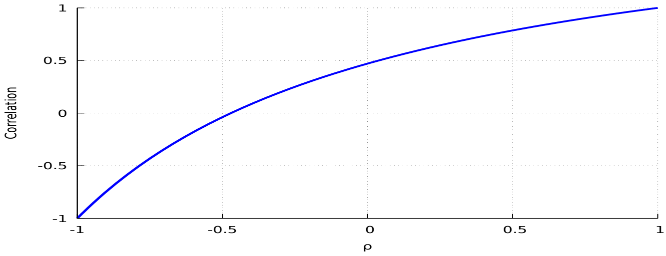


Fig. S.81: Log bond prices correlation graph in the two-factor model.

Exercise 18.9

a) We have

$$f(t,x) = f(0,x) + \alpha \int_0^t s^2 ds + \sigma \int_0^t d_s B(s,x) = r + \alpha t x^2 + \sigma B(t,x).$$

b) We have

$$r_t = f(t,0) = r + B(t,0) = r.$$

c) We have

$$\begin{aligned}
P(t,T) &= \exp\left(-\int_t^T f(t,s)ds\right) \\
&= \exp\left(-(T-t)r - \alpha t \int_0^{T-t} s^2 ds - \sigma \int_0^{T-t} B(t,x)dx\right) \\
&= \exp\left(-(T-t)r - \frac{\alpha}{3} t (T-t)^3 - \sigma \int_0^{T-t} B(t,x)dx\right), \qquad t \in [0,T].
\end{aligned}$$

d) Using (18.43), we have

$$\begin{aligned}
\mathbb{E}\left[\left(\int_0^{T-t} B(t,x)dx\right)^2\right] &= \int_0^{T-t}\int_0^{T-t} \mathbb{E}[B(t,x)B(t,y)]dxdy \\
&= t \int_0^{T-t}\int_0^{T-t} \min(x,y)dxdy \\
&= 2t \int_0^{T-t}\int_0^{y} xdxdy = \frac{1}{3} t (T-t)^3.
\end{aligned}$$

e) By Question (d) we have

$$\mathbb{E}[P(t,T)] = \mathbb{E}\left[\exp\left(-(T-t)r - \frac{\alpha}{3} t (T-t)^3 - \sigma \int_0^{T-t} B(t,x)dx\right)\right]$$

$$
\begin{aligned}
&= \exp\left(-(T-t)r - \frac{\alpha}{3}t(T-t)^3\right) \mathbb{E}\left[\exp\left(-\sigma\int_0^{T-t} B(t,x)dx\right)\right] \\
&= \exp\left(-(T-t)r - \frac{\alpha}{3}t(T-t)^3\right) \exp\left(\frac{\sigma^2}{2}\operatorname{Var}\left[\int_0^{T-t} B(t,x)dx\right]\right) \\
&= \exp\left(-(T-t)r - \frac{\alpha}{3}t(T-t)^3 + \frac{\sigma^2}{6}t(T-t)^3\right), \qquad 0 \le t \le T.
\end{aligned}
$$

f) By Question (e) we check that the required relation is satisfied if

$$
-\frac{\alpha}{3}t(T-t)^3 + \frac{\sigma^2}{6}t(T-t)^3 = 0,
$$

i.e. $\alpha = \sigma^2/2$.

Remark: In order to derive an analog of the HJM absence of arbitrage condition in this stochastic string model, one would have to check whether the discounted bond price $\mathrm{e}^{-rt}P(t,T)$ can be a martingale by doing stochastic calculus with respect to the Brownian sheet $B(t,x)$.

g) We have

$$
\begin{aligned}
&\mathbb{E}\left[\exp\left(-\int_0^T r_s ds\right)(P(T,S)-K)^+\right] \\
&= \mathrm{e}^{-rT}\,\mathbb{E}\left[\left(\exp\left(-(S-T)r - \frac{\alpha}{3}T(S-T)^3 + \sigma\int_0^{S-T} B(T,x)dx\right) - K\right)^+\right] \\
&= \mathrm{e}^{-rT}\,\mathbb{E}\left[(x\mathrm{e}^{m+X} - K)^+\right],
\end{aligned}
$$

where $x = \mathrm{e}^{-(S-T)r}$, $m = -\alpha T(S-T)^3/3$, and

$$
X = \sigma\int_0^{S-T} B(T,x)dx \simeq \mathcal{N}(0, \sigma^2 t(T-t)^3/3).
$$

Given the relation $\alpha = \sigma^2/2$, this yields

$$
\begin{aligned}
&\mathbb{E}\left[\exp\left(-\int_0^T r_s ds\right)(P(T,S)-K)^+\right] \\
&= \mathrm{e}^{-rS}\Phi\left(\sigma\sqrt{T(S-T)^3/12} + \frac{\log(\mathrm{e}^{-(S-T)r}/K)}{\sigma\sqrt{T(S-T)^3/3}}\right) \\
&\quad -K\mathrm{e}^{-rT}\Phi\left(-\sigma\sqrt{T(S-T)^3/12} + \frac{\log(\mathrm{e}^{-(S-T)r}/K)}{\sigma\sqrt{T(S-T)^3/3}}\right) \\
&= P(0,S)\Phi\left(\sigma\sqrt{T(S-T)^3/12} + \frac{\log(\mathrm{e}^{-(S-T)r}/K)}{\sigma\sqrt{T(S-T)^3/3}}\right)
\end{aligned}
$$

$$-KP(0,T)\Phi\left(-\sigma\sqrt{T(S-T)^3/12}+\frac{\log(\mathrm{e}^{-(S-T)r}/K)}{\sigma\sqrt{T(S-T)^3/3}}\right).$$

Chapter 19

Exercise 19.1

a) We price the floorlet at $t=0$, with $T_1 = 9$ months, $T_2 = 1$ year, $\kappa = 4.5\%$. The LIBOR rate $(L(t,T_1,T_2))_{t\in[0,T_1]}$ is modeled as a driftless geometric Brownian motion with volatility coefficient $\widehat{\sigma} = \sigma_{1,2}(t) = 0.1$ under the forward measure $\mathbb{P}_2$. The discount factors are given by

$$P(0,T_1) = \mathrm{e}^{-9r/12} \simeq 0.970809519$$

and

$$P(0,T_2) = \mathrm{e}^{-r} \simeq 0.961269954,$$

with $r = 3.95\%$.

b) By (19.21), the price of the floorlet is

$$\begin{aligned}&\mathbb{E}^*\left[\mathrm{e}^{-\int_0^{T_2} r_s ds}(\kappa - L(T_1,T_1,T_2))^+\right]\\ &= P(0,T_2)\left(\kappa\Phi\big(-d_-(T_1)\big) - L(0,T_1,T_2)\Phi\big(-d_+(T_1)\big)\right),\quad \text{(S.19.87)}\end{aligned}$$

where

$$d_+(T_1) = \frac{\log(L(0,T_1,T_2)/\kappa) + \sigma^2 T_1/2}{\sigma_1\sqrt{T_1}},$$

and

$$d_-(T_1) = \frac{\log(L(0,T_1,T_2)/\kappa) - \sigma^2 T_1/2}{\sigma\sqrt{T_1}},$$

are given in Proposition 19.5, and the LIBOR rate $L(0,T_1,T_2)$ is given by

$$\begin{aligned}L(0,T_1,T_2) &= \frac{P(0,T_1)-P(0,T_2)}{(T_2-T_1)P(0,T_2)}\\ &= \frac{\mathrm{e}^{-3r/4}-\mathrm{e}^{-r}}{0.25\mathrm{e}^{-r}}\\ &= 4(\mathrm{e}^{r/4}-1)\\ &\simeq 3.9695675\%.\end{aligned}$$

Hence, we have

$$d_+(T_1) = \frac{\log(0.039695675/0.045) + (0.1)^2\times 0.75/2}{0.1\times\sqrt{0.75}} \simeq -1.404927033,$$

and

$$d_-(T_1) = \frac{\log(0.039695675/0.045) - (0.1)^2 \times 0.75/2}{0.1 \times \sqrt{0.75}} \simeq -1.491529573,$$

hence

$$\begin{aligned}
&\mathbb{E}^*\left[\mathrm{e}^{-\int_0^{T_2} r_s ds}(\kappa - L(T_1, T_1, T_2))^+\right] \\
&= 0.961269954 \times (\kappa\Phi(1.491529573) - L(0, T_1, T_2) \times \Phi(1.404927033)) \\
&= 0.961269954 \times (0.045 \times 0.932089 - 0.039695675 \times 0.919979) \\
&\simeq 0.52147141\%.
\end{aligned}$$

Finally, we need to multiply (S.19.87) by the notional principal amount of \$1 million per interest rate percentage point, *i.e.* \$10,000 per percentage point or \$100 per basis point, which yields \$5214.71.

Exercise 19.2

a) We price the swaption at $t = 0$, with $T_1 = 4$ years, $T_2 = 5$ years, $T_3 = 6$ years, $T_4 = 7$ years, $\kappa = 5\%$, and the swap rate $(S(t, T_1, T_4))_{t\in[0,T_1]}$ is modeled as a driftless geometric Brownian motion with volatility coefficient $\widehat{\sigma} = \sigma_{1,4}(t) = 0.2$ under the forward swap measure $\mathbb{P}_{1,4}$. The discount factors are given by $P(0, T_1) = \mathrm{e}^{-4r}$, $P(0, T_2) = \mathrm{e}^{-5r}$, $P(0, T_3) = \mathrm{e}^{-6r}$, $P(0, T_4) = \mathrm{e}^{-7r}$, where $r = 5\%$.

b) By Proposition 19.17 the price of the swaption is

$$\begin{aligned}
&(P(0, T_1) - P(0, T_4))\Phi(d_+(T_1 - t)) \\
&\qquad -\kappa\Phi(d_-(T_1))(P(0, T_2) + P(0, T_3) + P(0, T_4)),
\end{aligned}$$

where $d_+(T_1)$ and $d_-(T_1)$ are given in Proposition 19.17, and the LIBOR swap rate $S(0, T_1, T_4)$ is given by

$$\begin{aligned}
S(0, T_1, T_4) &= \frac{P(0, T_1) - P(0, T_4)}{P(0, T_1, T_4)} \\
&= \frac{P(0, T_1) - P(0, T_4)}{P(0, T_2) + P(0, T_3) + P(0, T_4)} \\
&= \frac{\mathrm{e}^{-4r} - \mathrm{e}^{-7r}}{\mathrm{e}^{-5r} + \mathrm{e}^{-6r} + \mathrm{e}^{-7r}} \\
&= \frac{\mathrm{e}^{3r} - 1}{\mathrm{e}^{2r} + \mathrm{e}^{r} + 1} \\
&= \frac{\mathrm{e}^{0.15} - 1}{\mathrm{e}^{0.1} + \mathrm{e}^{0.05} + 1} \\
&= 0.051271096.
\end{aligned}$$

By Proposition 19.17 we also have

$$d_+(T_1) = \frac{\log(0.051271096/0.05) + (0.2)^2 \times 4/2}{0.2\sqrt{4}} = 0.526161682,$$

and

$$d_-(T_1) = \frac{\log(0.051271096/0.05) - (0.2)^2 2 \times 4/2}{0.2\sqrt{4}} = 0.005214714,$$

Hence, the price of the swaption is given by

$$\begin{aligned}
&(\mathrm{e}^{-4r} - \mathrm{e}^{-7r})\Phi(0.526161682) \\
&\quad -\kappa\Phi(0.005214714)(\mathrm{e}^{-5r} + \mathrm{e}^{-6r} + \mathrm{e}^{-7r}) \\
&= (0.818730753 - 0.70468809) \times 0.700612 \\
&\quad -0.05 \times 0.50208 \times (0.818730753 + 0.740818221 + 0.70468809) \\
&= 2.3058251\%.
\end{aligned} \tag{S.19.88}$$

Finally, we need to multiply (S.19.88) by the notional principal amount of \$10 million, *i.e.* \$100,000 by interest percentage point, or \$1,000 by basis point, which yields \$230,582.51.

Exercise 19.3 Taking $t = 0$, we have $T_1 = 3$, $T_2 = 4$ and $T_3 = 5$. The LIBOR swap rate $S(t, T_1, T_3)$ is modeled as a driftless geometric Brownian motion with volatility $\sigma = 0.1$ under the forward swap measure $\widehat{\mathbb{P}}_{i,j}$. The receiver swaption is priced using the Black-Scholes formula as

$$\begin{aligned}
&\mathbb{E}^*\left[\mathrm{e}^{-\int_t^{T_1} r_s ds} P(T_1, T_1, T_3) \left(\kappa - S(T_1, T_1, T_3)\right)^+ \Big| \mathcal{F}_t\right] \\
&= \kappa\Phi(-d_-(T_1 - t)) \sum_{k=1}^{2} (T_{k+1} - T_k) P(t, T_{k+1}) \\
&\quad -(P(t, T_1) - P(t, T_3))\Phi(-d_+(T_1 - t)),
\end{aligned}$$

where $\kappa = 5\%$, $r = 2\%$ and $P(t, T_1) = \mathrm{e}^{-3r} = 0.9417$, $P(t, T_2) = \mathrm{e}^{-4r} = 0.9231$, $P(t, T_3) = \mathrm{e}^{-5r} = 0.9048$. Hence,

$$P(t, T_1, T_3) = P(t, T_2) + P(t, T_3) = 0.92311 + 0.90483 = 1.82794$$

and

$$S(t, T_1, T_3) = \frac{P(t, T_1) - P(t, T_3)}{P(t, T_1, T_3)} = \frac{0.9417 - 0.9048}{1.82794} = 0.02018.$$

We also have

$$d_+(T_1 - t) = \frac{\log(S(t, T_1, T_3)/\kappa) + \sigma^2(T_1 - t)/2}{\sigma\sqrt{T_1 - t}}$$

$$= \frac{\log(2.018/5) + 0.1^2 \times 3/2}{0.1\sqrt{3}} = -5.1518$$

and

$$d_-(T_1 - t) = d_+(T_1 - t) - \sigma\sqrt{T_1 - t} = -5.3250,$$

hence

$$\begin{aligned}
&\mathbb{E}^*\left[\mathrm{e}^{-\int_t^{T_1} r_s ds} P(T_1, T_1, T_3)\left(\kappa - S(T_1, T_1, T_3)\right)^+ \Big| \mathcal{F}_t\right] \\
&= 0.05 \times 1.82794 \times \Phi(5.3250) - (0.9417 - 0.9048) \times \Phi(5.1518) \\
&= 0.05 \times 1.82794 \times 0.9999999 - (0.9417 - 0.9048) \times 0.9999999 \\
&= 0.054496 \\
&= 5.4496\% \\
&= 544.96 \text{ bp},
\end{aligned}$$

which yields \$54, 496 after multiplication by the \$10, 000 notional principal.

Exercise 19.4

a) We have

$$\begin{aligned}
d\left(\frac{P(t,T_2)}{P(t,T_1)}\right) &= \frac{dP(t,T_2)}{P(t,T_1)} + P(t,T_2)d\left(\frac{1}{P(t,T_1)}\right) + dP(t,T_2) \bullet d\left(\frac{1}{P(t,T_1)}\right) \\
&= \frac{dP(t,T_2)}{P(t,T_1)} + P(t,T_2)\left(-\frac{dP(t,T_1)}{(P(t,T_1))^2} + \frac{dP(t,T_1) \bullet dP(t,T_1)}{(P(t,T_1))^3}\right) \\
&\quad - \frac{dP(t,T_1) \bullet dP(t,T_2)}{(P(t,T_1))^2} \\
&= \frac{1}{P(t,T_1)}\big(r_t P(t,T_2)dt + \zeta_2(t)P(t,T_2)dW_t\big) \\
&\quad - \frac{P(t,T_2)}{(P(t,T_1))^2}\big(r_t P(t,T_1)dt + \zeta_1(t)P(t,T_1)dW_t\big) \\
&\quad + \frac{P(t,T_2)}{(P(t,T_1))^3}\big((r_t P(t,T_1)dt + \zeta_1(t)P(t,T_1)dW_t)^2\big) \\
&\quad - \frac{1}{(P(t,T_1))^2}\big((r_t P(t,T_1)dt + \zeta_1(t)P(t,T_1)dW_t) \bullet (r_t P(t,T_2)dt + \zeta_2(t)P(t,T_2)dW_t)\big) \\
&= \zeta_2(t)\frac{P(t,T_2)}{P(t,T_1)}dW_t - \zeta_1(t)\frac{P(t,T_2)}{P(t,T_1)}dW_t \\
&\quad + (\zeta_1(t))^2\frac{P(t,T_2)}{P(t,T_1)}dt - \zeta_1(t)\zeta_2(t)\frac{P(t,T_2)}{P(t,T_1)}dt \\
&= -\frac{P(t,T_2)}{P(t,T_1)}\zeta_1(t)(\zeta_2(t) - \zeta_1(t))dt + \frac{P(t,T_2)}{P(t,T_1)}(\zeta_2(t) - \zeta_1(t))dW_t
\end{aligned}$$

$$= \frac{P(t,T_2)}{P(t,T_1)}(\zeta_2(t)-\zeta_1(t))(dW_t-\zeta_1(t)dt)$$
$$= (\zeta_2(t)-\zeta_1(t))\frac{P(t,T_2)}{P(t,T_1)}d\widehat{W}_t = (\zeta_2(t)-\zeta_1(t))\frac{P(t,T_2)}{P(t,T_1)}d\widehat{W}_t,$$

where $d\widehat{W}_t = dW_t - \zeta_1(t)dt$ is a *standard Brownian motion* under the T_1-forward measure $\widehat{\mathbb{P}}$.

b) From Question (a) or (19.7) we have

$$\begin{aligned} P(T_1,T_2) &= \frac{P(T_1,T_2)}{P(T_1,T_1)} \\ &= \frac{P(t,T_2)}{P(t,T_1)}\exp\left(\int_t^{T_1}(\zeta_2(s)-\zeta_1(s))d\widehat{W}_s - \frac{1}{2}\int_t^{T_1}(\zeta_2(s)-\zeta_1(s))^2ds\right) \\ &= \frac{P(t,T_2)}{P(t,T_1)}\mathrm{e}^{X-v^2/2}, \end{aligned}$$

where X is a centered Gaussian random variable with variance $v^2 = \int_t^{T_1}(\zeta_2(s)-\zeta_1(s))^2ds$, independent of $\mathcal{F}_t$ under $\widehat{\mathbb{P}}$. Hence by the hint (19.43) with $x := P(t,T_2)/P(t,T_1)$ and $\kappa := K/x$, we find

$$\begin{aligned} &\mathbb{E}^*\left[\mathrm{e}^{-\int_0^{T_1} r_s ds}(K-P(T_1,T_2))^+ \,\middle|\, \mathcal{F}_t\right] = P(t,T_1)\widehat{\mathbb{E}}\left[(K-P(T_1,T_2))^+ \mid \mathcal{F}_t\right] \\ &= P(t,T_1)\left(K\Phi\left(\frac{v}{2}+\frac{1}{v}\log\frac{K}{x}\right) - \frac{P(t,T_2)}{P(t,T_1)}\Phi\left(-\frac{v}{2}+\frac{1}{v}\log\frac{K}{x}\right)\right) \\ &= KP(t,T_1)\Phi\left(\frac{v}{2}+\frac{1}{v}\log\frac{K}{x}\right) - P(t,T_2)\Phi\left(-\frac{v}{2}+\frac{1}{v}\log\frac{K}{x}\right). \end{aligned}$$

Exercise 19.5

a) The forward measure $\widehat{\mathbb{P}}_S$ is defined from the numéraire $N_t := P(t,S)$ and this gives
$$F_t = P(t,S)\widehat{\mathbb{E}}[(\kappa - L(T,T,S))^+ \mid \mathcal{F}_t].$$

b) The LIBOR rate $L(t,T,S)$ is a driftless geometric Brownian motion with volatility σ under the forward measure $\widehat{\mathbb{P}}_S$. Indeed, the LIBOR rate $L(t,T,S)$ can be written as the forward price $L(t,T,S) = \widehat{X}_t = X_t/N_t$ where $X_t = (P(t,T)-P(t,S))/(S-T)$ and $N_t = P(t,S)$. Since both discounted bond prices $\mathrm{e}^{-\int_0^t r_s ds}P(t,T)$ and $\mathrm{e}^{-\int_0^t r_s ds}P(t,S)$ are martingales under $\mathbb{P}^*$, the same is true of X_t. Hence $L(t,T,S) = X_t/N_t$ becomes a martingale under the forward measure $\widehat{\mathbb{P}}_S$ by Proposition 16.4, and computing its dynamics under $\widehat{\mathbb{P}}_S$ amounts to removing any "dt" term in (19.44) after rewriting the equation in terms of the standard Brownian motion $(\widehat{W}_t)_{t\in\mathbb{R}_+}$ under $\widehat{\mathbb{P}}_S$, *i.e.* we have

$$dL(t,T,S) = \sigma L(t,T,S) d\widehat{W}_t,$$

which is solved as $L(t,T,S) = L(0,T,S)\mathrm{e}^{\sigma\widehat{W}_t - \sigma^2 t/2}$, $0 \le t \le T$.

c) We find

$$\begin{aligned}
F_t &= P(t,S)\widehat{\mathbb{E}}[(\kappa - L(T,T,S))^+ \mid \mathcal{F}_t] \\
&= P(t,S)\widehat{\mathbb{E}}[(\kappa - L(t,T,S)\mathrm{e}^{-(T-t)\sigma^2/2 + (\widehat{W}_T - \widehat{W}_t)\sigma})^+ \mid \mathcal{F}_t] \\
&= P(t,S)(\kappa \Phi(-d_-(T-t)) - \widehat{X}_t \Phi(-d_+(T-t))) \\
&= \kappa P(t,S)\Phi(-d_-(T-t)) - P(t,S)L(t,T,S)\Phi(-d_+(T-t)) \\
&= \kappa P(t,S)\Phi(-d_-(T-t)) - (P(t,T) - P(t,S))\Phi(-d_+(T-t))/(S-T),
\end{aligned}$$

where $\mathrm{e}^m = L(t,T,S)\mathrm{e}^{-(T-t)\sigma^2/2}$, $v^2 = (T-t)\sigma^2$, and

$$d_+(T-t) = \frac{\log(L(t,T,S)/\kappa)}{\sigma\sqrt{T-t}} + \frac{\sigma\sqrt{T-t}}{2},$$

and

$$d_-(T-t) = \frac{\log(L(t,T,S)/\kappa)}{\sigma\sqrt{T-t}} - \frac{\sigma\sqrt{T-t}}{2},$$

because $L(t,T,S)$ is a driftless geometric Brownian motion with volatility σ under the forward measure $\widehat{\mathbb{P}}_S$.

Exercise 19.6

a) We have

$$P(T_i, T_i, T_j) = \sum_{l=i}^{j-1} c_{l+1} P(T_i, T_{l+1}).$$

b) It suffices to let $\tilde{c}_l = 1$, $l = i+1, \ldots, j-1$. and $\tilde{c}_j = c_j + 1/\kappa$.

c) The swaption can be priced as

$$\begin{aligned}
&\mathbb{E}^*\left[\mathrm{e}^{-\int_t^{T_i} r_s ds}\big(P(T_i,T_i) - P(T_i,T_j) - \kappa P(T_i,T_i,T_j)\big)^+ \;\Big|\; \mathcal{F}_t\right] \\
&= \mathbb{E}^*\left[\mathrm{e}^{-\int_t^{T_i} r_s ds}\left(1 - \kappa \sum_{l=i}^{j-1} \tilde{c}_{l+1} P(T_i, T_{l+1})\right)^+ \;\Big|\; \mathcal{F}_t\right] \\
&= \kappa\, \mathbb{E}^*\left[\mathrm{e}^{-\int_t^{T_i} r_s ds}\left(\sum_{l=i}^{j-1} \tilde{c}_{l+1} F_{l+1}(T_i, \gamma_\kappa) - \sum_{l=i}^{j-1} \tilde{c}_{l+1} F_{l+1}(T_i, r_{T_i})\right)^+ \;\Big|\; \mathcal{F}_t\right] \\
&= \kappa\, \mathbb{E}^*\left[\mathrm{e}^{-\int_t^{T_i} r_s ds}\left(\sum_{l=i}^{j-1} \tilde{c}_{l+1} \left(F_{l+1}(T_i, \gamma_\kappa) - F_{l+1}(T_i, r_{T_i})\right)\right)^+ \;\Big|\; \mathcal{F}_t\right]
\end{aligned}$$

$$
\begin{aligned}
&= \kappa\, \mathbb{E}^*\left[\mathrm{e}^{-\int_t^{T_i} r_s ds} \mathbb{1}_{\{r_{T_i} \le \gamma_\kappa\}} \sum_{l=i}^{j-1} \tilde{c}_{l+1}(F_{l+1}(T_i, \gamma_\kappa) - F_{l+1}(T_i, r_{T_i})) \,\Big|\, \mathcal{F}_t \right] \\
&= \kappa\, \mathbb{E}^*\left[\mathrm{e}^{-\int_t^{T_i} r_s ds} \sum_{l=i}^{j-1} \tilde{c}_{l+1} \left(F_{l+1}(T_i, \gamma_\kappa) - F_{l+1}(T_i, r_{T_i})\right)^+ \,\Big|\, \mathcal{F}_t \right] \\
&= \kappa \sum_{l=i}^{j-1} \tilde{c}_{l+1}\, \mathbb{E}^*\left[\mathrm{e}^{-\int_t^{T_i} r_s ds} \left(F_{l+1}(T_i, \gamma_\kappa) - P(T_i, T_{l+1})\right)^+ \,\Big|\, \mathcal{F}_t \right] \\
&= \kappa \sum_{l=i}^{j-1} \tilde{c}_{l+1} P(t, T_i) \widehat{\mathbb{E}}_i\left[(F_{l+1}(T_i, \gamma_\kappa) - P(T_i, T_{l+1}))^+ \,\big|\, \mathcal{F}_t \right],
\end{aligned}
$$

which is a weighted sum of bond put option prices with strike prices $F_{l+1}(T_i, \gamma_\kappa)$, $l = i, i+1, \ldots, j-1$.

Exercise 19.7

a) We have

$$
\frac{dP(t, T_i)}{P(t, T_i)} = r_t dt + \zeta^{(i)}(t) dB_t, \qquad i = 1, 2,
$$

and

$$
P(T, T_i) = P(t, T_i) \exp\left(\int_t^T r_s ds + \int_t^T \zeta^{(i)}(s) dB_s - \frac{1}{2} \int_t^T |\zeta^{(i)}(s)|^2 ds \right),
$$

$0 \le t \le T \le T_i$, $i = 1, 2$, hence

$$
\log P(T, T_i) = \log P(t, T_i) + \int_t^T r_s ds + \int_t^T \zeta^{(i)}(s) dB_s - \frac{1}{2} \int_t^T |\zeta^{(i)}(s)|^2 ds,
$$

$0 \le t \le T \le T_i$, $i = 1, 2$, and

$$
d \log P(t, T_i) = r_t dt + \zeta^{(i)}(t) dB_t - \frac{1}{2} |\zeta^{(i)}(t)|^2 dt, \qquad i = 1, 2.
$$

In the present model, we have

$$
dr_t = \sigma dB_t,
$$

where $(B_t)_{t \in \mathbb{R}_+}$ is a standard Brownian motion under $\mathbb{P}$, by the solution of Exercise 17.4 and (17.25) we have

$$
\zeta^{(i)}(t) = -(T_i - t)\sigma, \qquad 0 \le t \le T_i, \quad i = 1, 2.
$$

Letting

$$
dB_t^{(i)} = dB_t - \zeta^{(i)}(t) dt,
$$

defines a standard Brownian motion under $\mathbb{P}_i$, $i = 1, 2$, and we have

$$\begin{aligned}\frac{P(T,T_1)}{P(T,T_2)} &= \frac{P(t,T_1)}{P(t,T_2)}\exp\left(\int_t^T(\zeta^{(1)}(s)-\zeta^{(2)}(s))dB_s - \frac{1}{2}\int_t^T(|\zeta^{(1)}(s)|^2-|\zeta^{(2)}(s)|^2)ds\right)\\ &= \frac{P(t,T_1)}{P(t,T_2)}\exp\left(\int_t^T(\zeta^{(1)}(s)-\zeta^{(2)}(s))dB_s^{(2)} - \frac{1}{2}\int_t^T(\zeta^{(1)}(s)-\zeta^{(2)}(s))^2ds\right),\end{aligned}$$

which is an $\mathcal{F}_t$-martingale under $\mathbb{P}_2$ and under $\mathbb{P}_{1,2}$, and

$$\frac{P(T,T_2)}{P(T,T_1)} = \frac{P(t,T_2)}{P(t,T_1)}\exp\left(-\int_t^T(\zeta^{(1)}(s)-\zeta^{(2)}(s))dB_s^{(1)} - \frac{1}{2}\int_t^T(\zeta^{(1)}(s)-\zeta^{(2)}(s))^2ds\right),$$

which is an $\mathcal{F}_t$-martingale under $\mathbb{P}_1$.

b) We have

$$\begin{aligned}f(t,T_1,T_2) &= -\frac{1}{T_2-T_1}(\log P(t,T_2)-\log P(t,T_1))\\ &= r_t + \frac{1}{T_2-T_1}\frac{\sigma^2}{6}((T_1-t)^3-(T_2-t)^3).\end{aligned}$$

c) We have

$$\begin{aligned}&df(t,T_1,T_2) = -\frac{1}{T_2-T_1}d\log\frac{P(t,T_2)}{P(t,T_1)}\\ &= -\frac{1}{T_2-T_1}\left((\zeta^{(2)}(t)-\zeta^{(1)}(t))dB_t - \frac{1}{2}(|\zeta^{(2)}(t)|^2-|\zeta^{(1)}(t)|^2)dt\right)\\ &= -\frac{1}{T_2-T_1}\left((\zeta^{(2)}(t)-\zeta^{(1)}(t))(dB_t^{(2)}+\zeta^{(2)}(t)dt) - \frac{1}{2}(|\zeta^{(2)}(t)|^2-|\zeta^{(1)}(t)|^2)dt\right)\\ &= -\frac{1}{T_2-T_1}\left((\zeta^{(2)}(t)-\zeta^{(1)}(t))dB_t^{(2)} - \frac{1}{2}(\zeta^{(2)}(t)-\zeta^{(1)}(t))^2dt\right).\end{aligned}$$

d) We have

$$\begin{aligned}&f(T,T_1,T_2) = -\frac{1}{T_2-T_1}\log\frac{P(T,T_2)}{P(T,T_1)}\\ &= f(t,T_1,T_2) - \frac{1}{T_2-T_1}\left(\int_t^T(\zeta^{(2)}(s)-\zeta^{(1)}(s))dB_s - \frac{1}{2}(|\zeta^{(2)}(s)|^2-|\zeta^{(1)}(s)|^2)ds\right)\\ &= f(t,T_1,T_2) - \frac{1}{T_2-T_1}\left(\int_t^T(\zeta^{(2)}(s)-\zeta^{(1)}(s))dB_s^{(2)} - \frac{1}{2}\int_t^T(\zeta^{(2)}(s)-\zeta^{(1)}(s))^2ds\right)\\ &= f(t,T_1,T_2) - \frac{1}{T_2-T_1}\left(\int_t^T(\zeta^{(2)}(s)-\zeta^{(1)}(s))dB_s^{(1)} + \frac{1}{2}\int_t^T(\zeta^{(2)}(s)-\zeta^{(1)}(s))^2ds\right).\end{aligned}$$

Hence $f(T,T_1,T_2)$ has a Gaussian distribution given $\mathcal{F}_t$ with conditional mean

$$m_1 := f(t,T_1,T_2) - \frac{1}{2}\int_t^T(\zeta^{(2)}(s)-\zeta^{(1)}(s))^2ds$$

under $\mathbb{P}_1$, resp.

$$m_2 := f(t,T_1,T_2) + \frac{1}{2}\int_t^T (\zeta^{(2)}(s)-\zeta^{(1)}(s))^2 ds$$

under $\mathbb{P}_2$, and variance

$$v^2 = \frac{1}{(T_2-T_1)^2}\int_t^T (\zeta^{(2)}(s)-\zeta^{(1)}(s))^2 ds.$$

Hence, we have

$$\begin{aligned}
&(T_2-T_1)\,\mathbb{E}^*\left[\mathrm{e}^{-\int_t^{T_2} r_s ds}(f(T_1,T_1,T_2)-\kappa)^+\Big|\mathcal{F}_t\right] \\
&= (T_2-T_1)P(t,T_2)\,\mathbb{E}_2\left[(f(T_1,T_1,T_2)-\kappa)^+ \mid \mathcal{F}_t\right] \\
&= (T_2-T_1)P(t,T_2)\,\mathbb{E}_2\left[(m_2+X-\kappa)^+\big|\mathcal{F}_t\right] \\
&= (T_2-T_1)P(t,T_2)\left(\frac{v}{\sqrt{2\pi}}\mathrm{e}^{-(\kappa-m_2)^2/(2v^2)} + (m_2-\kappa)\Phi((m_2-\kappa)/v)\right).
\end{aligned}$$

e) We have

$$\begin{aligned}
&L(T,T_1,T_2) = S(T,T_1,T_2) \\
&= \frac{1}{T_2-T_1}\left(\frac{P(T,T_1)}{P(T,T_2)}-1\right) \\
&= \frac{1}{T_2-T_1} \\
&\quad\times\left(\frac{P(t,T_1)}{P(t,T_2)}\exp\left(\int_t^T(\zeta^{(1)}(s)-\zeta^{(2)}(s))dB_s - \frac{1}{2}\int_t^T(|\zeta^{(1)}(s)|^2-|\zeta^{(2)}(s)|^2)ds\right)-1\right) \\
&= \frac{1}{T_2-T_1} \\
&\quad\times\left(\frac{P(t,T_1)}{P(t,T_2)}\exp\left(\int_t^T(\zeta^{(1)}(s)-\zeta^{(2)}(s))dB_s^{(2)} - \frac{1}{2}\int_t^T(\zeta^{(1)}(s)-\zeta^{(2)}(s))^2ds\right)-1\right) \\
&= \frac{1}{T_2-T_1} \\
&\quad\times\left(\frac{P(t,T_1)}{P(t,T_2)}\exp\left(\int_t^T(\zeta^{(1)}(s)-\zeta^{(2)}(s))dB_s^{(1)} + \frac{1}{2}\int_t^T(\zeta^{(1)}(s)-\zeta^{(2)}(s))^2ds\right)-1\right),
\end{aligned}$$

and, by Itô calculus,

$$\begin{aligned}
&dS(t,T_1,T_2) = \frac{1}{T_2-T_1}d\left(\frac{P(t,T_1)}{P(t,T_2)}\right) \\
&= \frac{1}{T_2-T_1}\frac{P(t,T_1)}{P(t,T_2)}\left((\zeta^{(1)}(t)-\zeta^{(2)}(t))dB_t + \frac{1}{2}(\zeta^{(1)}(t)-\zeta^{(2)}(t))^2dt\right. \\
&\quad\left.-\frac{1}{2}(|\zeta^{(1)}(t)|^2-|\zeta^{(2)}(t)|^2)dt\right)
\end{aligned}$$

$$= \left(\frac{1}{T_2 - T_1} + S(t, T_1, T_2)\right)\left((\zeta^{(1)}(t) - \zeta^{(2)}(t))dB_t + \zeta^{(2)}(t)(\zeta^{(2)}(t) - \zeta^{(1)}(t))dt)dt\right)$$
$$= \left(\frac{1}{T_2 - T_1} + S(t, T_1, T_2)\right)\left((\zeta^{(1)}(t) - \zeta^{(2)}(t))dB_t^{(1)} + (|\zeta^{(2)}(t)|^2 - |\zeta^{(1)}(t)|^2)dt\right)$$
$$= \left(\frac{1}{T_2 - T_1} + S(t, T_1, T_2)\right)(\zeta^{(1)}(t) - \zeta^{(2)}(t))dB_t^{(2)}, \qquad t \in [0, T_1],$$

hence $t \mapsto \frac{1}{T_2 - T_1} + S(t, T_1, T_2)$ is a geometric Brownian motion, with

$$\begin{aligned}
&\frac{1}{T_2 - T_1} + S(T, T_1, T_2) \\
&= \left(\frac{1}{T_2 - T_1} + S(t, T_1, T_2)\right) \\
&\quad \times \exp\left(\int_t^T (\zeta^{(1)}(s) - \zeta^{(2)}(s))dB_s^{(2)} - \frac{1}{2}\int_t^T (\zeta^{(1)}(s) - \zeta^{(2)}(s))^2 ds\right),
\end{aligned}$$

$0 \leq t \leq T \leq T_1$.

f) We have

$$\begin{aligned}
&(T_2 - T_1)\, \mathbb{E}^*\left[\mathrm{e}^{-\int_t^{T_2} r_s ds}(L(T_1, T_1, T_2) - \kappa)^+ \Big| \mathcal{F}_t\right] \\
&= (T_2 - T_1)\, \mathbb{E}^*\left[\mathrm{e}^{-\int_t^{T_1} r_s ds} P(T_1, T_2)(L(T_1, T_1, T_2) - \kappa)^+ \Big| \mathcal{F}_t\right] \\
&= P(t, T_1, T_2)\, \mathbb{E}_{1,2}\left[(S(T_1, T_1, T_2) - \kappa)^+ \big| \mathcal{F}_t\right].
\end{aligned}$$

The forward measure $\mathbb{P}_2$ is defined by

$$\mathbb{E}^*\left[\frac{d\mathbb{P}_2}{d\mathbb{P}} \Big| \mathcal{F}_t\right] = \frac{P(t, T_2)}{P(0, T_2)}\mathrm{e}^{-\int_0^t r_s ds}, \qquad 0 \leq t \leq T_2,$$

and the forward swap measure is defined by

$$\mathbb{E}^*\left[\frac{d\mathbb{P}_{1,2}}{d\mathbb{P}} \Big| \mathcal{F}_t\right] = \frac{P(t, T_2)}{P(0, T_2)}\mathrm{e}^{-\int_0^t r_s ds}, \qquad 0 \leq t \leq T_1,$$

hence $\mathbb{P}_2$ and $\mathbb{P}_{1,2}$ coincide up to time T_1 and $\left(B_t^{(2)}\right)_{t \in [0, T_1]}$ is a standard Brownian motion until time T_1 under $\mathbb{P}_2$ and under $\mathbb{P}_{1,2}$, consequently under $\mathbb{P}_{1,2}$ we have

$$\begin{aligned}
&L(T, T_1, T_2) = S(T, T_1, T_2) \\
&= -\frac{1}{T_2 - T_1} + \left(\frac{1}{T_2 - T_1} + S(t, T_1, T_2)\right)\mathrm{e}^{\int_t^T (\zeta^{(1)}(s) - \zeta^{(2)}(s))dB_s^{(2)} - \frac{1}{2}\int_t^T (\zeta^{(1)}(s) - \zeta^{(2)}(s))^2 ds},
\end{aligned}$$

has same distribution as

$$\frac{1}{T_2 - T_1}\left(\frac{P(t,T_1)}{P(t,T_2)}\mathrm{e}^{X - \mathrm{Var}[X]/2} - 1\right),$$

where X is a centered Gaussian random variable with variance

$$\int_t^{T_1} (\zeta^{(1)}(s) - \zeta^{(2)}(s))^2 ds$$

given $\mathcal{F}_t$. Hence, we have

$$\begin{aligned}
&(T_2 - T_1)\,\mathbb{E}^*\left[\mathrm{e}^{-\int_t^{T_2} r_s ds}(L(T_1,T_1,T_2) - \kappa)^+ \,\Big|\, \mathcal{F}_t\right] \\
&= P(t,T_1,T_2) \\
&\quad \times \mathrm{Bl}\left(\frac{1}{T_2 - T_1} + S(t,T_1,T_2), \frac{\int_t^{T_1}(\zeta^{(1)}(s) - \zeta^{(2)}(s))^2 ds}{T_1 - t}, \kappa + \frac{1}{T_2 - T_1}, T_1 - t\right).
\end{aligned}$$

Exercise 19.8

a) The LIBOR rate $L(t,T,S)$ is a driftless geometric Brownian motion with deterministic volatility function $\sigma(t)$ under the forward measure $\widehat{\mathbb{P}}_S$.

   Explanation: The LIBOR rate $L(t,T,S)$ can be written as the forward price $L(t,T,S) = \widehat{X}_t = X_t/N_t$ where $X_t = (P(t,T) - P(t,S))/(S-T)$ and $N_t = P(t,S)$. Since both discounted bond prices $\mathrm{e}^{-\int_0^t r_s ds}P(t,T)$ and $\mathrm{e}^{-\int_0^t r_s ds}P(t,S)$ are martingales under $\mathbb{P}^*$, the same is true of X_t. Hence $L(t,T,S) = X_t/N_t$ becomes a martingale under the forward measure $\widehat{\mathbb{P}}_S$ by Proposition 16.4, and computing its dynamics under $\widehat{\mathbb{P}}_S$ amounts to removing any "dt" term in the original SDE defining $L(t,T,S)$, *i.e.* we find

$$dL(t,T,S) = \sigma(t)L(t,T,S)d\widehat{W}_t, \qquad 0 \le t \le T,$$

   hence

$$L(t,T,S) = L(0,T,S)\exp\left(\int_0^t \sigma(s)d\widehat{W}_s - \int_0^t \sigma^2(s)ds/2\right),$$

   where $(\widehat{W}_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion under $\widehat{\mathbb{P}}_S$.

b) Choosing the annuity numéraire $N_t = P(t,S)$, we have

$$\begin{aligned}
\mathbb{E}^*\left[\mathrm{e}^{-\int_t^S r_s ds}\phi(L(T,T,S))\big|\mathcal{F}_t\right] &= \mathbb{E}^*\left[\mathrm{e}^{-\int_t^S r_s ds}N_S\phi(L(T,T,S))\big|\mathcal{F}_t\right] \\
&= N_t\widehat{\mathbb{E}}\left[\phi(L(T,T,S))\big|\mathcal{F}_t\right] \\
&= P(t,S)\widehat{\mathbb{E}}[\phi(L(T,T,S)) \mid \mathcal{F}_t].
\end{aligned}$$

c) Given the solution

$$\begin{aligned}L(T,T,S) &= L(0,T,S)\exp\left(\int_0^T \sigma(s)d\widehat{W}_s - \int_0^T \sigma^2(s)ds/2\right)\\ &= L(t,T,S)\exp\left(\int_t^T \sigma(s)d\widehat{W}_s - \int_t^T \sigma^2(s)ds/2\right),\end{aligned}$$

we find

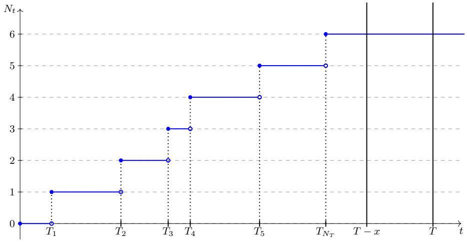
$$\begin{aligned}&P(t,S)\widehat{\mathbb{E}}\left[\phi(L(T,T,S)) \mid \mathcal{F}_t\right]\\ &= P(t,S)\widehat{\mathbb{E}}\left[\phi\left(L(t,T,S)\mathrm{e}^{\int_t^T \sigma(s)d\widehat{W}_s - \int_t^T \sigma^2(s)ds/2}\right) \Big| \mathcal{F}_t\right]\\ &= P(t,S)\int_{-\infty}^{\infty} \phi\big(L(t,T,S)\mathrm{e}^{x-\eta^2/2}\big)\mathrm{e}^{-x^2/(2\eta^2)}\frac{dx}{\sqrt{2\pi\eta^2}},\end{aligned}$$

because $\displaystyle\int_t^T \sigma(s)d\widehat{W}_s$ is a centered Gaussian variable with variance $\eta^2 := \displaystyle\int_t^T \sigma^2(s)ds$, independent of $\mathcal{F}_t$ under the forward measure $\widehat{\mathbb{P}}$.

Chapter 20

Exercise 20.1 For any $x \in [0,T]$, we have

$$\begin{aligned}\mathbb{P}(T - T_{N_T} > x \mid N_T \geq 1) &= \frac{\mathbb{P}(T - T_{N_T} > x \text{ and } N_T \geq 1)}{\mathbb{P}(N_T \geq 1)}\\ &= \frac{\mathbb{P}(N_T - N_{T-x} = 0 \text{ and } N_T \geq 1)}{\mathbb{P}(N_T \geq 1)}\\ &= \frac{\mathbb{P}(N_T - N_{T-x} = 0 \text{ and } N_{T-x} \geq 1)}{\mathbb{P}(N_T \geq 1)}\\ &= \frac{\mathbb{P}(N_T - N_{T-x} = 0)\mathbb{P}(N_{T-x} \geq 1)}{\mathbb{P}(N_T \geq 1)}\\ &= \frac{\mathrm{e}^{-(T-(T-x))\lambda}(1-\mathrm{e}^{-(T-x)\lambda})}{1-\mathrm{e}^{-\lambda T}}\\ &= \frac{\mathrm{e}^{-(T-(T-x))\lambda} - \mathrm{e}^{-\lambda T}}{1-\mathrm{e}^{-\lambda T}}\\ &= \frac{\mathrm{e}^{-\lambda x} - \mathrm{e}^{-\lambda T}}{1-\mathrm{e}^{-\lambda T}}, \qquad 0 \leq t \leq T.\end{aligned}$$



We note that

$$\mathbb{P}(T - T_{N_T} > 0 \mid N_T \geq 1) = 1 \quad \text{and} \quad \mathbb{P}(T - T_{N_T} > T \mid N_T \geq 1) = 0.$$

Exercise 20.2

a) When $t \in [0, T_1)$, the equation reads

$$dS_t = -\eta\lambda S_{t^-} dt = -\eta\lambda S_t dt,$$

which is solved as $S_t = S_0 \mathrm{e}^{-\eta\lambda t}$, $0 \leq t < T_1$. Next, at the first jump time $\underline{t = T_1}$ we have

$$\varDelta S_t := S_t - S_{t^-} = \eta S_{t^-} dN_t = \eta S_{t^-},$$

which yields $S_t = (1+\eta) S_{t^-}$, hence $S_{T_1} = (1+\eta) S_{T_1^-} = S_0 (1+\eta) \mathrm{e}^{-\eta\lambda T_1}$. Repeating this procedure over the N_t jump times contained in the interval $[0, t]$ we get

$$S_t = S_0 (1+\eta)^{N_t} \mathrm{e}^{-\lambda\eta t}, \qquad t \geq 0.$$

b) When $t \in [0, T_1)$ the equation reads

$$dS_t = -\eta\lambda S_{t^-} dt = -\eta\lambda S_t dt,$$

which is solved as $S_t = S_0 \mathrm{e}^{-\eta\lambda t}$, $0 \leq t < T_1$. Next, at the first jump time $\underline{t = T_1}$ we have

$$dS_t = S_t - S_{t^-} = dN_t = 1,$$

which yields $S_t = 1 + S_{t^-}$, hence $S_{T_1} = 1 + S_{T_1^-} = 1 + S_0 \mathrm{e}^{-\eta\lambda T_1}$, and for $t \in [T_1, T_2)$ we will find

$$S_t = (1 + S_0 \mathrm{e}^{-\eta\lambda T_1}) \mathrm{e}^{-(t-T_1)\eta\lambda}, \qquad T_1 \leq t < T_2.$$

More generally, the equation can be solved by letting $Y_t := \mathrm{e}^{\eta\lambda t} S_t$ and noting that $(Y_t)_{t\in\mathbb{R}_+}$ satisfies $dY_t = \mathrm{e}^{\lambda\eta t} dN_t$, which has the solution

$$Y_t = Y_0 + \int_0^t \mathrm{e}^{\eta\lambda s} dN_s, \qquad t \geq 0,$$

hence in general we have

$$S_t = \mathrm{e}^{-\eta\lambda t} S_0 + \int_0^t \mathrm{e}^{-(t-s)\eta\lambda} dN_s, \qquad t \geq 0,$$

Exercise 20.3

a) Taking expectations on both sides of (20.40), we have

$$\begin{aligned} u(t) &= \mathbb{E}[S_t] \\ &= \mathbb{E}\left[S_0 + \mu\int_0^t S_s ds + \sigma\int_0^t S_s dB_s + \eta\int_0^t S_{s^-} dY_s\right] \\ &= \mathbb{E}[S_0] + \mathbb{E}\left[\mu\int_0^t S_s ds\right] + \mathbb{E}\left[\sigma\int_0^t S_s dB_s\right] + \mathbb{E}\left[\eta\int_0^t S_{s^-} dY_s\right] \\ &= S_0 + \mu\int_0^t \mathbb{E}[S_s]ds + 0 + \eta\lambda\,\mathbb{E}[Z]\int_0^t \mathbb{E}[S_{s^-}]ds \\ &= S_0 + \mu\int_0^t u(s)ds + \eta\lambda\,\mathbb{E}[Z]\int_0^t u(s)ds, , \qquad t \geq 0. \end{aligned}$$

b) The above equation can be rewritten in differential form as

$$u'(t) = \mu u(t) + \eta\lambda\,\mathbb{E}[Z]u(t)$$

with $u(0) = S_0$, which admits the solution

$$\mathbb{E}[S_t] = u(t) = u(0)\mathrm{e}^{(\mu+\eta\lambda\,\mathbb{E}[Z])t} = S_0\mathrm{e}^{(\mu+\eta\lambda\,\mathbb{E}[Z])t}, \qquad t \geq 0.$$

Exercise 20.4

a) We have

$$X_t = \begin{cases} X_0\mathrm{e}^{\alpha t}, & 0 \leq t < T_1, \\ \left(X_0\mathrm{e}^{\alpha T_1} + \sigma\right)\mathrm{e}^{(t-T_1)\alpha} = X_0\mathrm{e}^{\alpha t} + \sigma\mathrm{e}^{(t-T_1)\alpha}, & T_1 \leq t < T_2, \\ \left(\left(X_0\mathrm{e}^{\alpha T_1} + \sigma\right)\mathrm{e}^{(T_2-T_1)\alpha} + \sigma\right)\mathrm{e}^{(t-T_2)\alpha} & \\ \qquad = X_0\mathrm{e}^{\alpha t} + \sigma\mathrm{e}^{(t-T_1)\alpha} + \sigma\mathrm{e}^{(t-T_2)\alpha}, & T_2 \leq t < T_3, \end{cases}$$

and more generally the solution $(X_t)_{t\in\mathbb{R}_+}$ can be written as

$$X_t = X_0\mathrm{e}^{\alpha t} + \sigma\sum_{k=1}^{N_t}\mathrm{e}^{(t-T_k)\alpha} = X_0\mathrm{e}^{\alpha t} + \sigma\int_0^t \mathrm{e}^{(t-s)\alpha}dN_s, \qquad t \geq 0. \tag{S.20.89}$$

b) Letting $f(t) := \mathbb{E}[X_t]$ and taking expectation on both sides of the stochastic differential equation $dX_t = \alpha X_t dt + \sigma dN_t$ we find

$$df(t) = \alpha f(t)dt + \sigma\lambda dt,$$

or

$$f'(t) = \alpha f(t) + \sigma\lambda.$$

Letting $g(t) = f(t)\mathrm{e}^{-\alpha t}$, we check that

$$g'(t) = \sigma\lambda\mathrm{e}^{-\alpha t},$$

hence

$$g(t) = g(0) + \int_0^t g'(s)ds = g(0) + \sigma\lambda \int_0^t \mathrm{e}^{-\alpha s}ds = f(0) + \sigma\frac{\lambda}{\alpha}(1 - \mathrm{e}^{-\alpha t}),$$

and

$$\begin{aligned} f(t) &= \mathbb{E}[X_t] \\ &= g(t)\mathrm{e}^{\alpha t} \\ &= f(0)\mathrm{e}^{\alpha t} + \sigma\frac{\lambda}{\alpha}(\mathrm{e}^{\alpha t} - 1) \\ &= X_0\mathrm{e}^{\alpha t} + \sigma\frac{\lambda}{\alpha}(\mathrm{e}^{\alpha t} - 1), \qquad t \geq 0. \end{aligned}$$

We could also take the expectation on both sides of (S.20.89) and directly find

$$f(t) = \mathbb{E}[X_t] = X_0\mathrm{e}^{\alpha t} + \sigma\lambda \int_0^t \mathrm{e}^{(t-s)\alpha}ds = X_0\mathrm{e}^{\alpha t} + \sigma\frac{\lambda}{\alpha}(\mathrm{e}^{\alpha t} - 1), \qquad t \geq 0.$$

Exercise 20.5

a) We have $X_t = X_0 \prod_{k=1}^{N_t}(1+\sigma) = X_0(1+\sigma)^{N_t} = (1+\sigma)^{N_t}$, $t \in \mathbb{R}_+$.

b) By stochastic calculus and using the relation $dX_t = \sigma X_{t^-} dN_t$, we have

$$\begin{aligned} dS_t &= d\left(S_0 X_t + rX_t \int_0^t X_s^{-1} ds \right) = S_0 dX_t + rd\left(X_t \int_0^t X_s^{-1} ds \right) \\ &= S_0 dX_t + rX_t d\left(\int_0^t X_s^{-1} ds \right) + r\left(\int_0^t X_s^{-1} ds \right) dX_t + rdX_t \bullet d\left(\int_0^t X_s^{-1} ds \right) \\ &= S_0 dX_t + rX_t X_t^{-1} dt + r\left(\int_0^t X_s^{-1} ds \right) dX_t + rdX_t \bullet (X_t^{-1} dt) \\ &= S_0 dX_t + rdt + r\left(\int_0^t X_s^{-1} ds \right) dX_t = rdt + \left(S_0 + r\int_0^t X_s^{-1} ds \right) dX_t \\ &= rdt + \sigma\left(S_0 X_{t^-} + rX_{t^-} \int_0^t X_s^{-1} ds \right) dN_t = rdt + \sigma S_{t^-} dN_t. \end{aligned}$$

c) We have

$$
\begin{aligned}
\mathbb{E}[X_t/X_s] &= \mathbb{E}[(1+\sigma)^{N_t-N_s}] \\
&= \sum_{k\geq 0}(1+\sigma)^k \mathbb{P}(N_t - N_s = k) \\
&= \mathrm{e}^{-(t-s)\lambda}\sum_{k\geq 0}(1+\sigma)^k \frac{((t-s)\lambda)^k}{k!} = \mathrm{e}^{-(t-s)\lambda}\sum_{k\geq 0}\frac{((t-s)(1+\sigma)\lambda)^k}{k!} \\
&= \mathrm{e}^{-(t-s)\lambda}\mathrm{e}^{(t-s)(1+\sigma)\lambda} = \mathrm{e}^{(t-s)\lambda\sigma}, \qquad 0 \leq s \leq t.
\end{aligned}
$$

Remarks: We could also let $f(t) = \mathbb{E}[X_t]$ and take expectation in the equation $dX_t = \sigma X_{t^-} dN_t$ to get $f'(t) = \sigma\lambda f(t)dt$ and $f(t) = \mathbb{E}[X_t] = f(0)\mathrm{e}^{\lambda\sigma t} = \mathrm{e}^{\lambda\sigma t}$. Note that the relation $\mathbb{E}[X_t/X_s] = \mathbb{E}[X_t]/\mathbb{E}[X_s]$, which happens to be true here, is *wrong* in general.

d) We have

$$
\begin{aligned}
\mathbb{E}[S_t] &= \mathbb{E}\left[S_0 X_t + rX_t\int_0^t X_s^{-1}ds\right] = S_0\,\mathbb{E}[X_t] + r\int_0^t \mathbb{E}[X_t/X_s]ds \\
&= S_0\mathrm{e}^{\lambda\sigma t} + r\int_0^t \mathrm{e}^{(t-s)\lambda\sigma}ds = S_0\mathrm{e}^{\lambda\sigma t} + r\int_0^t \mathrm{e}^{\lambda\sigma s}ds \\
&= S_0\mathrm{e}^{\lambda\sigma t} + \frac{(\mathrm{e}^{\lambda\sigma t}-1)r}{\lambda\sigma}, \qquad t \geq 0.
\end{aligned}
$$

Exercise 20.6

a) Since $\mathbb{E}[N_t] = \lambda t$, the expectation $\mathbb{E}[N_t - 2\lambda t] = -\lambda t$ is a decreasing function of $t \in \mathbb{R}_+$, and $(N_t - 2\lambda t)_{t\in\mathbb{R}_+}$ is a *super*martingale.

b) We have

$$
S_t = S_0\mathrm{e}^{rt-\lambda\sigma t}(1+\sigma)^{N_t}, \qquad t \geq 0.
$$

c) The stochastic differential equation

$$
dS_t = rS_t dt + \sigma S_{t^-}(dN_t - \lambda dt)
$$

contains a martingale component $(dN_t - \lambda dt)$ and a positive drift $rS_t dt$, therefore $(S_t)_{t\in\mathbb{R}_+}$ is a *sub*martingale.

d) Given that $\sigma > 0$ we have $((1+\sigma)^k - 1)^+ = (1+\sigma)^k - 1$, hence

$$
\begin{aligned}
\mathrm{e}^{-rT}\,\mathbb{E}^*[(S_T-K)^+] &= \mathrm{e}^{-rT}\,\mathbb{E}^*[(S_0\mathrm{e}^{(r-\sigma\lambda)T}(1+\sigma)^{N_T} - K)^+] \\
&= \mathrm{e}^{-rT}\,\mathbb{E}^*[(S_0\mathrm{e}^{(r-\sigma\lambda)T}(1+\sigma)^{N_T} - S_0\mathrm{e}^{(r-\lambda\sigma)T})^+] \\
&= S_0\mathrm{e}^{-\sigma\lambda T}\,\mathbb{E}^*[((1+\sigma)^{N_T}-1)^+] \\
&= S_0\mathrm{e}^{-\sigma\lambda T}\sum_{k\geq 0}((1+\sigma)^k - 1)^+\mathbb{P}(N_T = k) \\
&= S_0\mathrm{e}^{-\sigma\lambda T}\sum_{k\geq 0}((1+\sigma)^k - 1)\mathbb{P}(N_T = k)
\end{aligned}
$$

$$\begin{aligned}
&= S_0 \mathrm{e}^{-\sigma\lambda T} \sum_{k\geq 0} (1+\sigma)^k \mathbb{P}(N_T = k) - S_0 \mathrm{e}^{-\sigma\lambda T} \sum_{k\geq 0} \mathbb{P}(N_T = k) \\
&= S_0 \mathrm{e}^{-\sigma\lambda T - \lambda T} \sum_{k\geq 0} \frac{(T(1+\sigma)\lambda)^k}{k!} - S_0 \mathrm{e}^{-\sigma\lambda T} \\
&= S_0 (1 - \mathrm{e}^{-\sigma\lambda T}),
\end{aligned}$$

where we applied the exponential identity

$$\mathrm{e}^x = \sum_{k\geq 0} \frac{x^k}{k!}$$

to $x := T(1+\sigma)\lambda$.

Exercise 20.7

a) For all $k = 1, 2, \ldots, N_t$ we have

$$X_{T_k} - X_{T_k^-} = a + \sigma X_{T_k^-},$$

hence

$$X_{T_k} = a + (1+\sigma) X_{T_k^-},$$

and continuing by induction, we obtain

$$\begin{aligned}
X_{T_k} &= a + (1+\sigma)a + \cdots + (1+\sigma)^{k-1}a + X_0(1+\sigma)^k \\
&= a \frac{(1+\sigma)^k - 1}{\sigma} + X_0(1+\sigma)^k,
\end{aligned}$$

which shows that

$$\begin{aligned}
X_t &= X_{T_{N_t}} \\
&= X_0(1+\sigma)^{N_t} + a \frac{(1+\sigma)^{N_t} - 1}{\sigma} \\
&= (1+\sigma)^{N_t} \left(X_0 + \frac{a}{\sigma} \right) - \frac{a}{\sigma}, \quad t \geq 0.
\end{aligned}$$

This result can also be obtained by noting that

$$X_{T_k} + \frac{a}{\sigma} = (1+\sigma) \left(X_{T_k^-} + \frac{a}{\sigma} \right), \qquad k = 1, 2, \ldots, N_t.$$

b) We have

$$\mathbb{E}[(1+\sigma)^{N_t}] = \mathrm{e}^{-\lambda t} \sum_{n\geq 0} (1+\sigma)^k \frac{(\lambda t)^k}{k!} = \mathrm{e}^{\sigma\lambda t}, \qquad t \geq 0,$$

hence

$$\mathbb{E}[X_t] = X_0 e^{\lambda\sigma t} + a\frac{e^{\lambda\sigma t}-1}{\sigma} = e^{\lambda\sigma t}\left(X_0 + \frac{a}{\sigma}\right) - \frac{a}{\sigma}, \qquad t \geq 0.$$

Exercise 20.8 We have $S_t = S_0 e^{rt}\prod_{k=1}^{N_t}(1+\eta Z_k)$, $t \in \mathbb{R}_+$.

Exercise 20.9 We have

$$\begin{aligned}
\mathrm{Var}[Y_T] &= \mathbb{E}\left[\left(\sum_{k=1}^{N_T} Z_k - \mathbb{E}[Y_T]\right)^2\right] \\
&= \sum_{n\geq 0}\mathbb{E}\left[\left(\sum_{k=1}^{N_T} Z_k - \lambda t\mathbb{E}[Z]\right)^2 \,\middle|\, N_T = k\right]\mathbb{P}(N_T = k) \\
&= e^{-\lambda t}\sum_{n\geq 0}\frac{\lambda^n t^n}{n!}\mathbb{E}\left[\left(\sum_{k=1}^{n} Z_k - \lambda t\mathbb{E}[Z]\right)^2\right] \\
&= e^{-\lambda t}\sum_{n\geq 0}\frac{\lambda^n t^n}{n!}\mathbb{E}\left[\left(\sum_{k=1}^{n} Z_k\right)^2 - 2\lambda t\mathbb{E}[Z]\sum_{k=1}^{n} Z_k + \lambda^2 t^2(\mathbb{E}[Z])^2\right] \\
&= e^{-\lambda t}\sum_{n\geq 0}\frac{\lambda^n t^n}{n!} \\
&\quad \times\mathbb{E}\left[2\sum_{1\leq k<l\leq n} Z_k Z_l + \sum_{k=1}^{n}|Z_k|^2 - 2\lambda t\mathbb{E}[Z]\sum_{k=1}^{n} Z_k + \lambda^2 t^2(\mathbb{E}[Z])^2\right] \\
&= e^{-\lambda t}\sum_{n\geq 0}\frac{\lambda^n t^n}{n!} \\
&\quad \times\left(n(n-1)(\mathbb{E}[Z])^2 + n\mathbb{E}\big[|Z|^2\big] - 2n\lambda t(\mathbb{E}[Z])^2 + \lambda^2 t^2(\mathbb{E}[Z])^2\right) \\
&= e^{-\lambda t}(\mathbb{E}[Z])^2\sum_{n\geq 2}\frac{\lambda^n t^n}{(n-2)!} + e^{-\lambda t}\mathbb{E}\big[|Z|^2\big]\sum_{n\geq 1}\frac{\lambda^n t^n}{(n-1)!} \\
&\quad -2e^{-\lambda t}\lambda t(\mathbb{E}[Z])^2\sum_{n\geq 1}\frac{\lambda^n t^n}{(n-1)!} + \lambda^2 t^2(\mathbb{E}[Z])^2) \\
&= \lambda t\mathbb{E}\big[|Z|^2\big],
\end{aligned}$$

or, using the *moment generating function* of Proposition 20.6,

$$\begin{aligned}
\mathrm{Var}[Y_T] &= \mathbb{E}\big[|Y_T|^2\big] - (\mathbb{E}[Y_T])^2 \\
&= \frac{\partial^2}{\partial\alpha^2}\mathbb{E}[e^{\alpha Y_T}]_{|\alpha=0} - \lambda^2 t^2(\mathbb{E}[Z])^2 \\
&= \lambda t\int_{-\infty}^{\infty}|y|^2\mu(dy) = \lambda t\mathbb{E}\big[|Z|^2\big].
\end{aligned}$$

Exercise 20.10

a) Applying the Itô formula (20.24) to the function $f(x) = \mathrm{e}^x$ and to the process $X_t = \mu t + \sigma W_t + Y_t$, we find

$$\begin{aligned} dS_t &= \left(\mu + \frac{1}{2}\sigma^2\right) S_t dt + \sigma S_t dW_t + (S_t - S_{t^-}) dN_t \\ &= \left(\mu + \frac{1}{2}\sigma^2\right) S_t dt + \sigma S_t dW_t + (S_0 \mathrm{e}^{\mu t + \sigma W_t + Y_t} - S_0 \mathrm{e}^{\mu t + \sigma W_t + Y_{t-}}) dN_t \\ &= \left(\mu + \frac{1}{2}\sigma^2\right) S_t dt + \sigma S_t dW_t + (S_0 \mathrm{e}^{\mu t + \sigma W_t + Y_{t^-} + Z_{N_t}} - \mathrm{e}^{\mu t + \sigma W_t + Y_{t-}}) dN_t \\ &= \left(\mu + \frac{1}{2}\sigma^2\right) S_t dt + \sigma S_t dW_t + S_{t^-} (\mathrm{e}^{Z_{N_t}} - 1) dN_t, \end{aligned}$$

hence the jumps of S_t are given by the sequence $(\mathrm{e}^{Z_k} - 1)_{k \geq 1}$.

b) The discounted process $\mathrm{e}^{-rt} S_t$ satisfies

$$d(\mathrm{e}^{-rt} S_t) = \mathrm{e}^{-rt} \left(\mu - r + \frac{1}{2}\sigma^2\right) S_t dt + \sigma \mathrm{e}^{-rt} S_t dW_t + \mathrm{e}^{-rt} S_{t^-} (\mathrm{e}^{Z_{N_t}} - 1) dN_t.$$

Hence by the Girsanov Theorem 20.20, choosing u, $\tilde{\lambda}$, $\tilde{\nu}$ such that

$$\mu - r + \frac{1}{2}\sigma^2 = \sigma u - \tilde{\lambda}\, \mathbb{E}_{\tilde{\nu}}[\mathrm{e}^Z - 1],$$

shows that

$$d(\mathrm{e}^{-rt} S_t) = \sigma \mathrm{e}^{-rt} S_t (dW_t + u dt) + \mathrm{e}^{-rt} S_{t^-} \left((\mathrm{e}^{Z_{N_t}} - 1) dN_t - \tilde{\lambda}\, \mathbb{E}_{\tilde{\nu}} \left[\mathrm{e}^Z - 1\right] dt \right)$$

is a martingale under $(\mathbb{P}_{u,\tilde{\lambda},\tilde{\nu}})$.

Exercise 20.11

a) We have

$$S_t = S_0 \mathrm{e}^{\mu t} \prod_{k=1}^{N_t} (1 + Z_k) = S_0 \exp\left(\mu t + \sum_{k=1}^{N_t} X_k\right), \qquad t \geq 0.$$

b) We have the discounted asset price process

$$\widetilde{S}_t := \mathrm{e}^{-rt} S_t = S_0 \exp\left((\mu - r)t + \sum_{k=1}^{N_t} X_k\right), \qquad t \geq 0,$$

satisfies the stochastic differential equation

$$d\widetilde{S}_t = (\mu - r)\widetilde{S}_t dt + X_{N_t} \widetilde{S}_{t^-} dN_t$$

$$= (\mu - r + \lambda\,\mathbb{E}[Z])\widetilde{S}_t dt + \big(X_{N_t} - \lambda\,\mathbb{E}[Z]\big)\widetilde{S}_{t^-} dN_t \qquad t \geq 0,$$

hence it is a martingale if

$$0 = \mu - r + \lambda\,\mathbb{E}[Z] = \mu - r + \lambda\,\mathbb{E}[\mathrm{e}^{X_k} - 1] = \mu - r + (\mathrm{e}^{\sigma^2/2} - 1)\lambda.$$

c) We have

$$\begin{aligned}
&\mathrm{e}^{-(T-t)r}\,\mathbb{E}[(S_T-\kappa)^+ \mid S_t] \\
&= \mathrm{e}^{-(T-t)r}\,\mathbb{E}\left[\left(S_0 \exp\left(\mu T + \sum_{k=1}^{N_T} X_k\right) - \kappa\right)^+ \Bigg| S_t\right] \\
&= \mathrm{e}^{-(T-t)r}\,\mathbb{E}\left[\left(S_t \exp\left((T-t)\mu + \sum_{k=N_t+1}^{N_T} X_k\right) - \kappa\right)^+ \Bigg| S_t\right] \\
&= \mathrm{e}^{-(T-t)r}\,\mathbb{E}\left[\left(x \exp\left((T-t)\mu + \sum_{k=N_t+1}^{N_T} X_k\right) - \kappa\right)^+\right]_{x=S_t} \\
&= \mathrm{e}^{-(T-t)r}\,\mathbb{E}\left[\left(x \exp\left((T-t)\mu + \sum_{k=1}^{N_T-N_t} X_k\right) - \kappa\right)^+\right]_{x=S_t} \\
&= \mathrm{e}^{-(T-t)r}\sum_{n\geq 0}\mathbb{E}\left[\left(x\mathrm{e}^{(T-t)\mu+\sum_{k=1}^n X_k} - \kappa\right)^+\right]_{x=S_t}\mathbb{P}(N_T - N_t = n) \\
&= \mathrm{e}^{-(r+\lambda)(T-t)}\sum_{n\geq 0}\mathbb{E}\left[\left(x\mathrm{e}^{(T-t)\mu+\sum_{k=1}^n X_k} - \kappa\right)^+\right]_{x=S_t}\frac{((T-t)\lambda)^n}{n!} \\
&= \mathrm{e}^{-(T-t)\lambda}\sum_{n\geq 0}\frac{((T-t)\lambda)^n}{n!}\mathrm{Bl}(S_t\mathrm{e}^{(\mu-r)(T-t)+n\sigma^2/2}, r, n\sigma^2/(T-t), \kappa, T-t) \\
&= \mathrm{e}^{-(T-t)\lambda}\sum_{n\geq 0}\left(S_t\mathrm{e}^{(\mu-r)(T-t)+n\sigma^2/2}\Phi(d_+) - \kappa\mathrm{e}^{-(T-t)r}\Phi(d_-)\right)\frac{((T-t)\lambda)^n}{n!},
\end{aligned}$$

with

$$\begin{aligned}
d_+ &= \frac{\log(S_t\mathrm{e}^{(\mu-r)(T-t)+n\sigma^2/2}/\kappa) + (T-t)r + n\sigma^2/2}{\sigma\sqrt{n}} \\
&= \frac{\log(S_t/\kappa) + (T-t)\mu + n\sigma^2}{\sigma\sqrt{n}},
\end{aligned}$$

$$d_- = \frac{\log(S_t\mathrm{e}^{(\mu-r)(T-t)+n\sigma^2/2}/\kappa) + (T-t)r - n\sigma^2/2}{\sigma\sqrt{n}}$$

$$= \frac{\log(S_t/\kappa) + (T-t)\mu}{\sigma\sqrt{n}},$$

and $\mu = r + (1 - e^{\sigma^2/2})\lambda$.

Exercise 20.12

a) We have

$$d(e^{\alpha t} S_t) = \sigma e^{\alpha t}(dN_t - \beta dt),$$

hence

$$e^{\alpha t} S_t = S_0 + \sigma \int_0^t e^{\alpha s}(dN_s - \beta d),$$

and

$$S_t = S_0 e^{-\alpha t} + \sigma \int_0^t e^{-(t-s)\alpha}(dN_s - \beta ds), \quad t \geq 0. \tag{S.20.90}$$

b) We have

$$\begin{aligned}
f(t) &= \mathbb{E}[S_t] \\
&= S_0 e^{-\alpha t} + \sigma\, \mathbb{E}\left[\int_0^t e^{-(t-s)\alpha} dN_s\right] - \beta\sigma \int_0^t e^{-(t-s)\alpha} ds \\
&= S_0 e^{-\alpha t} + \lambda\sigma \int_0^t e^{-(t-s)\alpha} ds - \beta\sigma \int_0^t e^{-(t-s)\alpha} ds \\
&= S_0 e^{-\alpha t} + (\lambda - \beta)\sigma \frac{1 - e^{-\alpha t}}{\alpha} \\
&= \sigma \frac{\lambda - \beta}{\alpha} + \left(S_0 + \sigma \frac{\beta - \lambda}{\alpha}\right) e^{-\alpha t}, \qquad t \geq 0.
\end{aligned}$$

c) By rewriting (S.20.90) as

$$\begin{aligned}
S_t &= S_0 - \alpha S_0 \int_0^t e^{-(t-s)\alpha} ds + \sigma \int_0^t e^{-(t-s)\alpha}(dN_s - \beta ds) \\
&= S_0 + \sigma \int_0^t e^{-(t-s)\alpha}(dN_s - (\beta + \alpha S_0/\sigma)ds) \\
&= S_0 + \sigma \int_0^t e^{-(t-s)\alpha}(\lambda - \beta - \alpha S_0/\sigma)ds) + \sigma \int_0^t e^{-(t-s)\alpha}(dN_s - \lambda ds),
\end{aligned}$$

$t \in \mathbb{R}_+$, we check that the process $(S_t)_{t\in\mathbb{R}_+}$ is a *sub*martingale, provided that $\lambda - \beta - \alpha S_0/\sigma \geq 0$, *i.e.* $S_0 + (\beta - \lambda)\sigma/\alpha \leq 0$. We also check that this condition makes the expectation $f(t) = \mathbb{E}[S_t]$ decreasing in Question (b).

d) Since, given that $N_T = n$ the jump times $(T_1, T_2, \ldots, T_n)$ of the Poisson process $(N_t)_{t\in\mathbb{R}_+}$ are independent uniformly distributed random variables over $[0, T]$, hence we can write

$$\mathbb{E}[\phi(S_T)] = \mathbb{E}\left[\phi\left(S_0 e^{-\alpha T} + \sigma \int_0^T e^{-(T-s)\alpha}(dN_s - \beta ds),\right)\right]$$

$$= \sum_{n\geq 0} \mathbb{P}(N_T = n)$$
$$\times \mathbb{E}\left[\phi\left(S_0 \mathrm{e}^{-\alpha T} + \sigma \int_0^T \mathrm{e}^{-(T-s)\alpha}(dN_s - \beta ds)\right) \,\Big|\, N_T = n\right]$$
$$= \mathrm{e}^{-\lambda T} \sum_{n\geq 0} \frac{(\lambda T)^n}{n!}$$
$$\times \mathbb{E}\left[\phi\left(S_0 \mathrm{e}^{-\alpha T} + \sigma \sum_{k=1}^n \mathrm{e}^{-(T-T_k)\alpha} - \sigma\beta \int_0^T \mathrm{e}^{-(T-s)\alpha} ds\right) \,\Big|\, N_T = n\right]$$
$$= \mathrm{e}^{-\lambda T} \sum_{n\geq 0} \frac{\lambda^n}{n!}$$
$$\times \int_0^T \cdots \int_0^T \phi\left(S_0 \mathrm{e}^{-\alpha T} + \sigma \sum_{k=1}^n \mathrm{e}^{-(T-s_k)\alpha} - \sigma\beta \frac{1-\mathrm{e}^{-\alpha T}}{\alpha}\right) ds_1 \cdots ds_n,$$

$T \geq 0$.

Exercise 20.13

a) From the decomposition $Y_t - \lambda t(t + \mathbb{E}[Z]) = Y_t - \lambda\,\mathbb{E}[Z]t - \lambda t^2$ as the sum of a martingale and a decreasing function, we conclude that $t \mapsto Y_t - \lambda t(t + \mathbb{E}[Z])$ is a *super*martingale.

b) Writing

$$\begin{aligned} dS_t &= \mu S_t dt + \sigma S_{t^-} dY_t \\ &= r S_t dt + \sigma S_{t^-}\left(dY_t - \frac{r-\mu}{\sigma} dt\right) \\ &= r S_t dt + \sigma S_{t^-}\left(dY_t - \tilde{\lambda}\,\mathbb{E}[Z]dt\right), \qquad 0 \leq t \leq T, \end{aligned}$$

we conclude that $(S_t)_{t\in[0,T]}$ is a martingale under $\mathbb{P}_{\tilde{\lambda}}$ provided that

$$\frac{\mu - r}{\sigma} = -\tilde{\lambda}\,\mathbb{E}[Z]dt,$$

i.e.

$$\tilde{\lambda} = \frac{r-\mu}{\sigma\,\mathbb{E}[Z]}.$$

We note that $\tilde{\lambda} < 0$ if $\mu < r$, hence in this case there is no risk-neutral probability measure and the market admits arbitrage opportunities as the risky asset always overperforms the risk-free interest rate r.

c) We have

$$\begin{aligned} \mathrm{e}^{-(T-t)r}\,\mathbb{E}_{\tilde{\lambda}}[S_T - \kappa \mid \mathcal{F}_t] &= \mathrm{e}^{rt}\,\mathbb{E}_{\tilde{\lambda}}[\mathrm{e}^{-rT} S_T \mid \mathcal{F}_t] - K\mathrm{e}^{-(T-t)r} \\ &= S_t - K\mathrm{e}^{-(T-t)r}, \end{aligned}$$

since $(S_t)_{t\in[0,T]}$ is a martingale under $\mathbb{P}_{\tilde{\lambda}}$.

Exercise 20.14

a) We have

$$S_t = S_0 \mathrm{e}^{\mu t} \prod_{k=1}^{N_t} (1+Z_k), \qquad t \geq 0.$$

b) Letting $X_k = \log(1+Z_k)$, $k \geq 1$, we find that

$$\mathrm{e}^{-rt} S_t = S_0 \exp\left((\mu - r)t + \sum_{k=1}^{N_t} X_k \right), \qquad t \geq 0,$$

and (20.43) can be rewritten for the discounted price process

$$\widetilde{S}_t := \mathrm{e}^{-rt} S_t, \qquad t \geq 0,$$

as

$$d\widetilde{S}_t = (\mu - r + \lambda\,\mathbb{E}[Z])\widetilde{S}_t dt + \widetilde{S}_{t^-}(dY_t + \lambda\,\mathbb{E}[Z]),$$

which becomes a martingale if

$$0 = \mu - r + \lambda\,\mathbb{E}[Z] = \mu - r + \lambda \int_{-\infty}^{\infty} z\nu(dz).$$

c) We have

$$\begin{aligned}
&\mathrm{e}^{-(T-t)r}\,\mathbb{E}[(S_T-\kappa)^+ \mid S_t] \\
&= \mathrm{e}^{-(T-t)r}\,\mathbb{E}\left[\left(S_0 \mathrm{e}^{\mu T} \prod_{k=1}^{N_T} Z_k - \kappa \right)^+ \,\middle|\, S_t \right] \\
&= \mathrm{e}^{-(T-t)r} \sum_{n\geq 0} \mathbb{E}\left[\left(S_t \mathrm{e}^{(T-t)\mu} \prod_{k=N_t+1}^{N_T} Z_k - \kappa \right)^+ \,\middle|\, S_t \right] \mathbb{P}(N_T - N_t = n) \\
&= \mathrm{e}^{-(r+\lambda)(T-t)} \sum_{n\geq 0} \mathbb{E}\left[\left(S_t \mathrm{e}^{(T-t)\mu} \prod_{k=N_t+1}^{N_T} Z_k - \kappa \right)^+ \,\middle|\, S_t \right] \frac{((T-t)\lambda)^n}{n!} \\
&= \mathrm{e}^{-(r+\lambda)(T-t)} \sum_{n\geq 0} \frac{((T-t)\lambda)^n}{n!} \\
&\quad \times \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left(S_t \mathrm{e}^{(T-t)\mu} \prod_{k=1}^{n} z_k - \kappa \right)^+ \nu(dz_1)\cdots\nu(dz_n).
\end{aligned}$$

Exercise 20.15

a) The discounted price process $(e^{-rt}S_t)_{t\in[0,T]}$ is a martingale, hence it is both a *sub*martingale and a *super*martingale.
b) The discounted price process $(e^{-rt}S_t)_{t\in[0,T]}$ is a *super*martingale.
c) The discounted price process $(e^{-rt}S_t)_{t\in[0,T]}$ is a *sub*martingale.
d) Under the probability measure $\widetilde{\mathbb{P}}_{\tilde{\lambda}}$, the discounted price process $(e^{-rt}S_t)_{t\in[0,T]}$ is a martingale, hence it is both a *sub*martingale and a *super*martingale.

Chapter 21

Exercise 21.1

a) We have $\mathbb{E}[N_t - \alpha t] = \mathbb{E}[N_t] - \alpha t = \lambda t - \alpha t$, hence $N_t - \alpha t$ is a martingale if and only if $\alpha = \lambda$. Given that

$$d(e^{-rt}S_t) = \eta e^{-rt}S_{t^-}(dN_t - \alpha dt),$$

we conclude that the discounted price process $e^{-rt}S_t$ is a martingale if and only if $\underline{\alpha = \lambda}$.
b) Since we are pricing under the risk-neutral probability measure we take $\alpha = \lambda$. Next, we note that

$$S_T = S_0 e^{(r-\eta\lambda)T}(1+\eta)^{N_T} = S_t e^{(r-\eta\lambda)(T-t)}(1+\eta)^{N_T-N_t}, \quad 0 \le t \le T,$$

hence the price at time t of the option is

$$\begin{aligned}
& e^{-(T-t)r}\,\mathbb{E}[|S_T|^2 \mid \mathcal{F}_t] \\
&= e^{-(T-t)r}\,\mathbb{E}[|S_t|^2 e^{2(r-\eta\lambda)(T-t)}(1+\eta)^{2(N_T-N_t)} \mid \mathcal{F}_t] \\
&= |S_t|^2 e^{(r-2\eta\lambda)(T-t)}\,\mathbb{E}[(1+\eta)^{2(N_T-N_t)} \mid \mathcal{F}_t] \\
&= |S_t|^2 e^{(r-2\eta\lambda)(T-t)}\,\mathbb{E}[(1+\eta)^{2(N_T-N_t)}] \\
&= |S_t|^2 e^{(r-2\eta\lambda)(T-t)} \sum_{n\ge 0}(1+\eta)^{2n}\mathbb{P}(N_T - N_t = n) \\
&= |S_t|^2 e^{(r-2\eta\lambda-\lambda)(T-t)} \sum_{n\ge 0}(1+\eta)^{2n}\frac{(\lambda(T-t))^n}{n!} \\
&= |S_t|^2 e^{(r-2\eta\lambda-\lambda)(T-t)+(1+\eta)^2\lambda(T-t)} \\
&= |S_t|^2 e^{(r+\eta^2\lambda)(T-t)}, \qquad 0 \le t \le T.
\end{aligned}$$

Exercise 21.2

a) Regardless of the choice of a particular risk-neutral probability measure $\mathbb{P}_{u,\tilde{\lambda},\tilde{\nu}}$, we have

$$e^{-(T-t)r}\,\mathbb{E}_{u,\tilde{\lambda},\tilde{\nu}}[S_T - K \mid \mathcal{F}_t] = e^{rt}\,\mathbb{E}_{u,\tilde{\lambda},\tilde{\nu}}[e^{-rT}S_T \mid \mathcal{F}_t] - Ke^{-(T-t)r}$$

$$\begin{aligned}
&= \mathrm{e}^{rt}\mathrm{e}^{-rt}S_t - K\mathrm{e}^{-(T-t)r} \\
&= S_t - K\mathrm{e}^{-(T-t)r} \\
&= f(t, S_t),
\end{aligned}$$

for

$$f(t, x) = x - K\mathrm{e}^{-(T-t)r}, \qquad t, x > 0.$$

b) Clearly, holding one unit of the risky asset and shorting a (possibly fractional) quantity $K\mathrm{e}^{-rT}$ of the riskless asset will hedge the payoff $S_T - K$, and this (static) hedging strategy is self-financing because it is constant in time.

c) Since $\dfrac{\partial f}{\partial x}(t, x) = 1$ we have

$$\begin{aligned}
\xi_t &= \frac{\sigma^2 \dfrac{\partial f}{\partial x}(t, S_{t^-}) + \dfrac{a\widetilde{\lambda}}{S_{t^-}}(f(t, S_{t^-}(1+a)) - f(t, S_{t^-}))}{\sigma^2 + a^2\widetilde{\lambda}} \\
&= \frac{\sigma^2 + \dfrac{a\widetilde{\lambda}}{S_{t^-}}(S_{t^-}(1+a) - S_{t^-})}{\sigma^2 + a^2\widetilde{\lambda}} \\
&= 1, \qquad 0 \leq t \leq T,
\end{aligned}$$

which coincides with the result of Question (b).

Exercise 21.3

a) We have

$$S_t = S_0 \exp\left(\mu t + \sigma B_t - \frac{1}{2}\sigma^2 t\right)(1+\eta)^{N_t}.$$

b) We have

$$\widetilde{S}_t = S_0 \exp\left((\mu - r)t + \sigma B_t - \frac{1}{2}\sigma^2 t\right)(1+\eta)^{N_t},$$

and

$$d\widetilde{S}_t = (\mu - r + \lambda\eta)\widetilde{S}_t dt + \eta\widetilde{S}_{t^-}(dN_t - \lambda dt) + \sigma\widetilde{S}_t dW_t,$$

hence we need to take

$$\mu - r + \lambda\eta = 0,$$

since the compensated Poisson process $(N_t - \lambda t)_{t\in\mathbb{R}_+}$ is a martingale.

c) We have

$$\begin{aligned}
&\mathrm{e}^{-r(T-t)}\,\mathbb{E}^*[(S_T - \kappa)^+ \mid S_t] \\
&= \mathrm{e}^{-r(T-t)}\,\mathbb{E}^*\left[\left(S_0 \exp\left(\mu T + \sigma B_T - \frac{1}{2}\sigma^2 T\right)(1+\eta)^{N_T} - \kappa\right)^+ \,\middle|\, S_t\right]
\end{aligned}$$

$$
\begin{aligned}
&= \mathrm{e}^{-r(T-t)}\,\mathbb{E}^*\left[\left(S_t\mathrm{e}^{\mu(T-t)+(B_T-B_t)\sigma-(T-t)\sigma^2/2}(1+\eta)^{N_T-N_t}-\kappa\right)^+\,\Big|\,S_t\right]\\
&= \mathrm{e}^{-r(T-t)}\sum_{n\geq 0}\mathbb{P}(N_T-N_t=n)\\
&\quad\times\mathbb{E}^*\left[\left(S_t\mathrm{e}^{\mu(T-t)+(B_T-B_t)\sigma-(T-t)\sigma^2/2}(1+\eta)^{n}-\kappa\right)^+\,\Big|\,S_t\right]\\
&= \mathrm{e}^{-(r+\lambda)(T-t)}\sum_{n\geq 0}\frac{(\lambda(T-t))^n}{n!}\\
&\quad\times\mathbb{E}^*\left[\left(S_t\mathrm{e}^{(r-\lambda\eta)(T-t)+(B_T-B_t)\sigma-(T-t)\sigma^2/2}(1+\eta)^{n}-\kappa\right)^+\,\Big|\,S_t\right]\\
&= \mathrm{e}^{-\lambda(T-t)}\sum_{n\geq 0}\mathrm{Bl}(S_t\mathrm{e}^{-\lambda\eta(T-t)}(1+\eta)^n,r,\sigma^2,T-t,\kappa)\frac{(\lambda(T-t))^n}{n!}\\
&= \mathrm{e}^{-\lambda(T-t)}\sum_{n\geq 0}\left(S_t\mathrm{e}^{-\lambda\eta(T-t)}(1+\eta)^n\Phi(d_+)-\kappa\mathrm{e}^{-r(T-t)}\Phi(d_-)\right)\frac{(\lambda(T-t))^n}{n!},
\end{aligned}
$$

with

$$
\begin{aligned}
d_+ &= \frac{\log(S_t\mathrm{e}^{-\lambda\eta(T-t)}(1+\eta)^n/\kappa)+(r+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}\\
&= \frac{\log(S_t(1+\eta)^n/\kappa)+(r-\lambda\eta+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}},
\end{aligned}
$$

and

$$
\begin{aligned}
d_- &= \frac{\log(S_t\mathrm{e}^{-\lambda\eta(T-t)}(1+\eta)^n/\kappa)+(r-\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}\\
&= \frac{\log(S_t(1+\eta)^n/\kappa)+(r-\lambda\eta-\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}.
\end{aligned}
$$

Exercise 21.4

a) The discounted process $\widetilde{S}_t=\mathrm{e}^{-rt}S_t$ satisfies the equation

$$
d\widetilde{S}_t = Y_{N_t}\widetilde{S}_{t^-}dN_t,,
$$

and it is a martingale since the compound Poisson process $Y_{N_t}dN_t$ is centered with independent increments as $\mathbb{E}[Y_1]=0$.

b) We have

$$
S_T = S_0\mathrm{e}^{rT}\prod_{k=1}^{N_T}(1+Y_k),
$$

hence

$$
\begin{aligned}
\mathrm{e}^{-rT}\,\mathbb{E}\left[(S_T-\kappa)^+\right] &= \mathrm{e}^{-rT}\,\mathbb{E}\left[\left(S_0\mathrm{e}^{rT}\prod_{k=1}^{N_T}(1+Y_k)-\kappa\right)^+\right]\\
&= \mathrm{e}^{-rT}\sum_{n\geq 0}\mathbb{E}\left[\left(S_0\mathrm{e}^{rT}\prod_{k=1}^{N_T}(1+Y_k)-\kappa\right)^+ \;\middle|\; N_T=n\right]\mathbb{P}(N_T=n)\\
&= \mathrm{e}^{-rT-\lambda T}\sum_{k\geq 0}\mathbb{E}\left[\left(S_0\mathrm{e}^{rT}\prod_{k=1}^{n}(1+Y_k)-\kappa\right)^+\right]\frac{(\lambda T)^n}{n!}\\
&= \mathrm{e}^{-rT-\lambda T}\sum_{k\geq 0}\frac{(\lambda T)^n}{2^n n!}\int_{-1}^{1}\cdots\int_{-1}^{1}\left(S_0\mathrm{e}^{rT}\prod_{k=1}^{n}(1+y_k)-\kappa\right)^+ dy_1\cdots dy_n.
\end{aligned}
$$

Exercise 21.5

a) We find $\alpha=\lambda$ where λ is the intensity of the Poisson process $(N_t)_{t\in\mathbb{R}_+}$.
b) We have

$$
\begin{aligned}
\mathrm{e}^{-(T-t)r}\,\mathbb{E}[S_T-\kappa\mid\mathcal{F}_t] &= \mathrm{e}^{rt}\,\mathbb{E}[\mathrm{e}^{-rT}S_T\mid\mathcal{F}_t]-\mathrm{e}^{-(T-t)r}\,\mathbb{E}[\kappa\mid\mathcal{F}_t]\\
&= \mathrm{e}^{rt}\,\mathbb{E}[\mathrm{e}^{-rt}S_t\mid\mathcal{F}_t]-\mathrm{e}^{-(T-t)r}\kappa\\
&= S_t-\mathrm{e}^{-(T-t)r}\kappa,
\end{aligned}
$$

since the process $(\mathrm{e}^{-rt}S_t)_{t\in\mathbb{R}_+}$ is a martingale.

Exercise 21.6

a) We have

$$
S_t=S_0\mathrm{e}^{(r-\lambda\alpha)t}(1+\alpha)^{N_t},\qquad t\geq 0.
$$

b) We have

$$
\begin{aligned}
&\mathrm{e}^{-(T-t)r}\,\mathbb{E}^*[\phi(S_T)\mid\mathcal{F}_t] = \mathrm{e}^{-(T-t)r}\,\mathbb{E}^*\left[\phi\left(x\frac{S_T}{S_t}\right)\right]_{|x=S_t}\\
&= \mathrm{e}^{-(T-t)r}\,\mathbb{E}^*\left[\phi(x\mathrm{e}^{(r-\lambda\alpha)(T-t)}(1+\alpha)^{N_T-N_t})\right]_{|x=S_t}\\
&= \mathrm{e}^{-(r+\lambda)(T-t)}\sum_{k=0}^{\infty}\frac{((T-t)\lambda)^k}{k!}\phi\big(S_t\mathrm{e}^{(r-\lambda\alpha)(T-t)}(1+\alpha)^k\big),\quad 0\leq t\leq T.
\end{aligned}
$$

c) We have

$$
\begin{aligned}
dV_t &= r\eta_t\mathrm{e}^{rt}dt+\xi_t dS_t\\
&= r\eta_t\mathrm{e}^{rt}dt+\xi_t(rS_t dt+\alpha S_{t^-}(dN_t-\lambda dt))\\
&= rV_t dt+\alpha\xi_t S_{t^-}(dN_t-\lambda dt)\\
&= rf(t,S_t)dt+\alpha\xi_t S_{t^-}(dN_t-\lambda dt). \qquad \text{(S.21.91)}
\end{aligned}
$$

d) By the Itô formula with jumps we have

$$\begin{aligned}
d(\mathrm{e}^{-rt}f(t,S_t)) &= -r\mathrm{e}^{-rt}f(t,S_t)dt + \mathrm{e}^{-rt}df(t,S_t)\\
&= -r\mathrm{e}^{-rt}f(t,S_t)dt\\
&\quad +\mathrm{e}^{-rt}\left(\frac{\partial f}{\partial t}(t,S_t)dt + rS_t\frac{\partial f}{\partial x}(t,S_t)dt - \lambda S_t\frac{\partial f}{\partial x}(t,S_t)dt\right.\\
&\qquad \left. +\big(f(t,(1+\alpha)S_{t^-}) - f(t,S_{t^-})\big)dN_t\right)\\
&= -r\mathrm{e}^{-rt}f(t,S_t)dt\\
&\quad +\mathrm{e}^{-rt}\left(\frac{\partial f}{\partial t}(t,S_t) + rS_t\frac{\partial f}{\partial x}(t,S_t) - \lambda S_t\frac{\partial f}{\partial x}(t,S_t)\right.\\
&\qquad \left. +\lambda\mathrm{e}^{-rt}(f(t,(1+\alpha)S_t) - f(t,S_t))\right)dt\\
&\quad +\mathrm{e}^{-rt}\big(f(t,(1+\alpha)S_{t^-}) - f(t,S_{t^-})\big)dN_t\\
&\quad -\lambda\mathrm{e}^{-rt}\,\mathbb{E}[f(t,(1+\alpha)x) - f(t,x)]_{|x=S_{t^-}}dt.
\end{aligned}$$

Since the discounted price process $(\mathrm{e}^{-rt}f(t,S_t))_{t\in\mathbb{R}_+}$ is a martingale under the risk-neutral measure, $d(\mathrm{e}^{-rt}f(t,S_t))$ must reduce to its martingale component, *i.e.* the sum of "dt" terms vanishes, and we get

$$\begin{aligned}
&d(\mathrm{e}^{-rt}f(t,S_t))\\
&= \mathrm{e}^{-rt}\big(f(t,(1+\alpha)S_{t^-}) - f(t,S_{t^-})\big)dN_t - \lambda\mathrm{e}^{-rt}\,\mathbb{E}[f(t,(1+\alpha)x) - f(t,x)]_{|x=S_{t^-}}dt\\
&= \mathrm{e}^{-rt}\big(f(t,(1+\alpha)S_{t^-}) - f(t,S_{t^-})\big)dN_t - \lambda\mathrm{e}^{-rt}\big(f(t,(1+\alpha)S_{t^-}) - f(t,S_{t^-})\big)dt,
\end{aligned}$$

or equivalently

$$df(t,S_t) = rf(t,S_t)dt + \big(f(t,(1+\alpha)S_{t^-}) - f(t,S_{t^-})\big)(dN_t - \lambda)dt. \quad \text{(S.21.92)}$$

Finally, by identification of the terms in the above formula (S.21.92) with those appearing in (S.21.91), we obtain

$$\alpha\xi_t S_{t^-}(dN_t - \lambda dt) = \big(f(t,(1+\alpha)S_{t^-}) - f(t,S_{t^-})\big)(dN_t - \lambda dt),$$

which yields the hedging strategy

$$\xi_t = \frac{1}{\alpha S_{t^-}}\big(f(t,(1+\alpha)S_{t^-}) - f(t,S_{t^-})\big), \quad 0 < t \leq T.$$

Exercise 21.7

a) We have

$$\begin{aligned}
\mathbb{E}[N_t \mid \mathcal{F}_s] &= \mathrm{e}^{\theta Y_s - sm(\theta)}\,\mathbb{E}\left[\mathrm{e}^{(Y_t-Y_s)\theta-(t-s)m(\theta)} \,\middle|\, \mathcal{F}_s\right]\\
&= \mathrm{e}^{\theta Y_s - sm(\theta)}\,\mathbb{E}\left[\mathrm{e}^{(Y_t-Y_s)\theta-(t-s)m(\theta)}\right] = N_s, \quad 0 \leq s \leq t.
\end{aligned}$$

b) We have

$$\begin{aligned}\mathbb{E}^\theta\left[\mathrm{e}^{-rt}S_t \mid \mathcal{F}_s\right] &= \mathbb{E}\left[\mathrm{e}^{Y_t}\frac{N_T}{N_s}\,\Big|\,\mathcal{F}_s\right]\\ &= \mathbb{E}\left[\mathrm{e}^{Y_t}\frac{N_t}{N_s}\,\Big|\,\mathcal{F}_s\right]\\ &= \mathrm{e}^{Y_s}\,\mathbb{E}\left[\mathrm{e}^{Y_t-Y_s}\mathrm{e}^{(Y_t-Y_s)\theta-(t-s)m(\theta)} \mid \mathcal{F}_s\right]\\ &= \mathrm{e}^{Y_s}\mathrm{e}^{-(t-s)m(\theta)}\,\mathbb{E}\left[\mathrm{e}^{(1+\theta)(Y_t-Y_s)}\right]\\ &= \mathrm{e}^{Y_s}\mathrm{e}^{-(t-s)m(\theta)}\mathrm{e}^{(t-s)m(\theta+1)},\end{aligned}$$

hence we should have $m(\theta) = m(\theta+1)$. For example, when $(Y_t)_{t\in\mathbb{R}_+} = (N_t - t)_{t\in\mathbb{R}_+}$ is a compensated Poisson process we have $m(\theta) = \mathrm{e}^\theta - \theta - 1$ and the condition reads $\mathrm{e}^\theta + 1 = \mathrm{e}^{\theta+1}$, *i.e.* $\theta = -\log(\mathrm{e}-1)$.

c) We have

$$\begin{aligned}&\mathrm{e}^{-(T-t)r}\,\mathbb{E}^\theta[(S_T-K)^+ \mid \mathcal{F}_t] = \mathrm{e}^{-(T-t)r}\,\mathbb{E}\left[(S_T-K)^+\frac{N_T}{N_t}\,\Big|\,\mathcal{F}_t\right]\\ &= \mathrm{e}^{-(T-t)((1+r)\theta+m(\theta))}\,\mathbb{E}\left[(S_T-K)^+\left(\frac{S_T}{S_t}\right)^\theta\,\Bigg|\,\mathcal{F}_t\right].\end{aligned}$$

Chapter 22

Exercise 22.1 For all $j = 1, 2, \ldots, M-1$ we have

$$B_{j,j-1} + B_{j,j} + B_{j,j+1} = 1 + r\Delta t,$$

hence when the terminal condition is a constant $\phi(T,x) = c > 0$ we get

$$\phi(t_i,x) = c(1+r\Delta t)^{-(N-i)} = c\left(1+r\frac{T}{N}\right)^{-(N-i)}, \qquad i = 0,\ldots,N.$$

In particular, when the number N of discretization steps tends to infinity, denoting by $[x]$ the integer part of $x \in \mathbb{R}$ we find

$$\begin{aligned}\phi(s,x) &= \lim_{N\to\infty}\phi\big(t_{[Ns/T]},x\big)\\ &= c\lim_{N\to\infty}\left(1+r\frac{T}{N}\right)^{-(N-[Ns/T])}\\ &= c\lim_{N\to\infty}\left(1+r\frac{T}{N}\right)^{-[N(T-s)/T]}\end{aligned}$$

$$
\begin{aligned}
&= c \lim_{N\to\infty} \left(1 + r\frac{T}{N}\right)^{-(T-s)/T} \\
&= c\mathrm{e}^{-r(T-s)},
\end{aligned}
$$

for all $s \in [0, T]$, as expected.

Exercise 22.2

a) We have

$$
\widehat{X}^N_{t_{k+1}} = \widehat{X}^N_{t_k} + r\widehat{X}^N_{t_k}(t_{k+1} - t_k) + \sigma\widehat{X}^N_{t_k}(W_{t_{k+1}} - W_{t_k}),
$$

which yields

$$
\widehat{X}^N_{t_k} = \widehat{X}^N_{t_0} \prod_{i=1}^{k} \big(1 + r(t_i - t_{i-1}) + (W_{t_i} - W_{t_{i-1}})\sigma\big), \quad k = 0, 1, \ldots, N.
$$

b) We have

$$
\begin{aligned}
\widehat{X}^N_{t_{k+1}} &= \widehat{X}^N_{t_k} + (r - \sigma^2/2)\widehat{X}^N_{t_k}(t_{k+1} - t_k) + \sigma\widehat{X}^N_{t_k}(W_{t_{k+1}} - W_{t_k}) \\
&\quad + \frac{1}{2}\sigma^2\widehat{X}^N_{t_k}(W_{t_{k+1}} - W_{t_k})^2,
\end{aligned}
$$

which yields

$$
\widehat{X}^N_{t_k} = \widehat{X}^N_{t_0} \prod_{i=1}^{k} \left(1 + (r - \sigma^2/2)(t_i - t_{i-1}) + (W_{t_i} - W_{t_{i-1}})\sigma + \frac{1}{2}(W_{t_i} - W_{t_{i-1}})^2\sigma^2\right).
$$

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