Stochastic Finance: An Introduction with Market Examples, First Edition

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Solutions Manual

Chapter 1

Exercise 1.1

1. The possible values of R are a and b.
2. We have

$$\begin{aligned}\mathbb{E}^*[R] &= a\mathbb{P}^*(R=a)+b\mathbb{P}^*(R=b)\\ &= a\frac{b-r}{b-a}+b\frac{r-a}{b-a}\\ &= r.\end{aligned}$$

3. By Theorem 1.1, there do not exist arbitrage opportunities in this market since there exists a risk-neutral measure $\mathbb{P}^*$ from Question 2.
4. The risk-neutral measure is unique hence the market model is complete by Theorem 1.2.
5. Taking

$$\eta = \frac{\alpha(1+b)-\beta(1+a)}{\pi_1(b-a)} \quad \text{and} \quad \xi = \frac{\beta-\alpha}{S_0(b-a)},$$

we check that

$$\begin{cases}\eta\pi_1+\xi S_0(1+a)=\alpha\\ \\ \eta\pi_1+\xi S_0(1+b)=\beta,\end{cases}$$

which shows that

$$\eta\pi_1+\xi S_1 = C.$$

6. We have

$$\begin{aligned}\pi(C) &= \eta\pi_0+\xi S_0\\ &= \frac{\alpha(1+b)-\beta(1+a)}{(1+r)(b-a)}+\frac{\alpha-\beta}{a-b}\\ &= \frac{\alpha(1+b)-\beta(1+a)-(1+r)(\alpha-\beta)}{(1+r)(b-a)}\\ &= \frac{\alpha b-\beta a-r(\alpha-\beta)}{(1+r)(b-a)}. \qquad (S.1.1)\end{aligned}$$

7. We have

$$\begin{aligned}\mathbb{E}^*[C] &= \alpha\mathbb{P}^*(R=a)+\beta\mathbb{P}^*(R=b)\\ &= \alpha\frac{b-r}{b-a}+\beta\frac{r-a}{b-a}.\end{aligned} \tag{S.1.2}$$

8. Comparing (S.1.1) and (S.1.2) above we do obtain

$$\pi(C)=\frac{1}{1+r}\mathbb{E}^*[C]$$

9. The initial value $\pi(C)$ of the portfolio is interpreted as the arbitrage price of the option contract and it equals the expected value of the discounted payoff.
10. We have

$$C=(K-S_1)^+=(11-S_1)^+=\begin{cases}11-S_1 & \text{if } K>S_1,\\ \\ 0 & \text{if } K\le S_1.\end{cases}$$

11. We have

$$\xi=\frac{-(11-(1+a))}{b-a}=-\frac{2}{3},\qquad \eta=\frac{(1+b)(11-(1+a))}{(1+r)(b-a)}=\frac{8}{1.05}.$$

12. The arbitrage price $\pi(C)$ of the contingent claim C is

$$\pi(C)=\eta\pi_0+\xi S_0=6.952.$$

Chapter 2

Exercise 2.1

1. The possible values of R_t are a and b.
2. We have

$$\begin{aligned}\mathbb{E}^*[R_{t+1}\mid\mathcal{F}_t] &= a\mathbb{P}^*(R_{t+1}=a\mid\mathcal{F}_t)+b\mathbb{P}^*(R_{t+1}=b\mid\mathcal{F}_t)\\ &= a\frac{b-r}{b-a}+b\frac{r-a}{b-a}=r.\end{aligned}$$

3. We have

$$\mathbb{E}^*[S_{t+k}\mid\mathcal{F}_t]=\sum_{i=0}^{k}\left(\frac{r-a}{b-a}\right)^i\left(\frac{b-r}{b-a}\right)^{k-i}\binom{k}{i}(1+b)^i(1+a)^{k-i}S_t$$

$$
\begin{aligned}
&= S_t \sum_{i=0}^{k} \binom{k}{i} \left(\frac{r-a}{b-a}(1+b) \right)^i \left(\frac{b-r}{b-a}(1+a) \right)^{k-i} \\
&= S_t \left(\frac{r-a}{b-a}(1+b) + \frac{b-r}{b-a}(1+a) \right)^k \\
&= (1+r)^k S_t.
\end{aligned}
$$

Assuming that the formula holds for $k=1$, its extension to $k \geq 2$ can also be proved recursively from the "tower property" (16.22) of conditional expectations, as follows:

$$
\begin{aligned}
\mathbb{E}^*[S_{t+k} \mid \mathcal{F}_t] &= \mathbb{E}^*[\mathbb{E}^*[S_{t+k} \mid \mathcal{F}_{t+k-1}] \mid \mathcal{F}_t] \\
&= (1+r)\, \mathbb{E}^*[S_{t+k-1} \mid \mathcal{F}_t] \\
&= (1+r)\, \mathbb{E}^*[\mathbb{E}^*[S_{t+k-1} \mid \mathcal{F}_{t+k-2}] \mid \mathcal{F}_t] \\
&= (1+r)^2\, \mathbb{E}^*[S_{t+k-2} \mid \mathcal{F}_t] \\
&= (1+r)^2\, \mathbb{E}^*[\mathbb{E}^*[S_{t+k-2} \mid \mathcal{F}_{t+k-3}] \mid \mathcal{F}_t] \\
&= (1+r)^3\, \mathbb{E}^*[S_{t+k-3} \mid \mathcal{F}_t] \\
&= \cdots \\
&= (1+r)^{k-2}\, \mathbb{E}^*[S_{t+2} \mid \mathcal{F}_t] \\
&= (1+r)^{k-2}\, \mathbb{E}^*[\mathbb{E}^*[S_{t+2} \mid \mathcal{F}_{t+1}] \mid \mathcal{F}_t] \\
&= (1+r)^{k-1}\, \mathbb{E}^*[S_{t+1} \mid \mathcal{F}_t] \\
&= (1+r)^k S_t.
\end{aligned}
$$

Chapter 3

Exercise 3.1

1. The condition $V_N = C$ reads

$$
\begin{cases}
\eta_N \pi_N + \xi_N (1+a) S_{N-1} = (1+a) S_{N-1} - K \\
\\
\eta_N \pi_N + \xi_N (1+b) S_{N-1} = (1+b) S_{N-1} - K
\end{cases}
$$

from which we deduce $\xi_N = 1$ and $\eta_N = -K(1+r)^{-N}/\pi_0$.

2. We have

$$
\begin{cases}
\eta_{N-1} \pi_{N-1} + \xi_{N-1} (1+a) S_{N-1} = \eta_N \pi_{N-1} + \xi_N (1+a) S_{N-1} \\
\\
\eta_{N-1} \pi_{N-1} + \xi_{N-1} (1+b) S_{N-1} = \eta_N \pi_{N-1} + \xi_N (1+b) S_{N-1},
\end{cases}
$$

which yields $\xi_{N-1} = \xi_N = 1$ and $\eta_{N-1} = \eta_N = -K(1+r)^{-N}/\pi_0$. Similarly, solving the self-financing condition

$$\begin{cases} \eta_t\pi_t + \xi_t(1+a)S_t = \eta_{t+1}\pi_t + \xi_{t+1}(1+a)S_t \\ \\ \eta_t\pi_t + \xi_t(1+b)S_t = \eta_{t+1}\pi_t + \xi_{t+1}(1+b)S_t, \end{cases}$$

at time t yields $\xi_t = 1$ and $\eta_t = -K(1+r)^{-N}/\pi_0$, $t = 1, 2, \ldots, N$.

3. We have

$$\pi_t(C) = V_t = \eta_t\pi_t + \xi_t S_t = S_t - K(1+r)^{-N}\pi_t/\pi_0 = S_t - K(1+r)^{-(N-t)}.$$

4. For all $t = 0, 1, \ldots, N$ we have

$$\begin{aligned} (1+r)^{-(N-t)}\,\mathbb{E}^*[C \mid \mathcal{F}_t] &= (1+r)^{-(N-t)}\,\mathbb{E}^*[S_N - K \mid \mathcal{F}_t], \\ &= (1+r)^{-(N-t)}\,\mathbb{E}^*[S_N \mid \mathcal{F}_t] - (1+r)^{-(N-t)}\,\mathbb{E}^*[K \mid \mathcal{F}_t] \\ &= (1+r)^{-(N-t)}(1+r)^{N-t}S_t - K(1+r)^{-(N-t)} \\ &= S_t - K(1+r)^{-(N-t)} \\ &= V_t = \pi_t(C). \end{aligned}$$

Exercise 3.2

1. This model admits a unique risk-neutral measure $\mathbb{P}^*$ because we have $a < r < b$. We have

$$\mathbb{P}^*(R_t = a) = \frac{b-r}{b-a} = \frac{0.07 - 0.05}{0.07 - (-0.02)},$$

and

$$\mathbb{P}(R_t = b) = \frac{r-a}{b-a} = \frac{0.05 - (-0.02)}{0.07 - (-0.02)},$$

$t = 1, \ldots, N$.

2. There are no arbitrage opportunities in this model, due to the existence of a risk-neutral measure.
3. This market model is complete because the risk-neutral measure is unique.
4. We have

$$C = (S_N)^2,$$

hence

$$H = (S_N)^2/(1+r)^N = h(X_N),$$

with

$$h(x) = x^2(1+r)^{-N}.$$

Now we have

$$V_t = v_t(X_t),$$

where the function $v_t(x)$ is given by

$$\begin{aligned}
v_t(x) &= \sum_{k=0}^{N-t} \frac{(N-t)!}{k!(N-t-k)!} \\
&\quad \times \left(\frac{r-a}{b-a}\right)^k \left(\frac{b-r}{b-a}\right)^{N-t-k} h\left(x\left(\frac{1+b}{1+r}\right)^k \left(\frac{1+a}{1+r}\right)^{N-t-k}\right) \\
&= x^2(1+r)^{-N} \sum_{k=0}^{N-t} \frac{(N-t)!}{k!(N-t-k)!} \\
&\quad \times \left(\frac{r-a}{b-a}\right)^k \left(\frac{b-r}{b-a}\right)^{N-t-k} \left(\frac{1+b}{1+r}\right)^{2k} \left(\frac{1+a}{1+r}\right)^{2(N-t-k)} \\
&= x^2(1+r)^{-N} \sum_{k=0}^{N-t} \frac{(N-t)!}{k!(N-t-k)!} \\
&\quad \times \left(\frac{(r-a)(1+b)^2}{(b-a)(1+r)^2}\right)^k \left(\frac{(b-r)(1+a)^2}{(b-a)(1+r)^2}\right)^{N-t-k} \\
&= x^2(1+r)^{-N} \left(\frac{(r-a)(1+b)^2}{(b-a)(1+r)^2} + \frac{(b-r)(1+a)^2}{(b-a)(1+r)^2}\right)^{N-t} \\
&= \frac{x^2\left((r-a)(1+b)^2+(b-r)(1+a)^2\right)^{N-t}}{(1+r)^{N-2t}(b-a)^{N-t}} \\
&= \frac{x^2\left((r-a)(1+2b+b^2)+(b-r)(1+2a+a^2)\right)^{N-t}}{(1+r)^{N-2t}(b-a)^{N-t}} \\
&= \frac{x^2\left(r(1+2b+b^2)-a(1+2b+b^2)+b(1+2a+a^2)-r(1+2a+a^2)\right)^{N-t}}{(1+r)^{N-2t}(b-a)^{N-t}} \\
&= x^2\frac{(1+r(2+a+b)-ab)^{N-t}}{(1+r)^{N-2t}}.
\end{aligned}$$

5. We have

$$\begin{aligned}
\xi_t^1 &= \frac{v_t\left(\frac{1+b}{1+r}X_{t-1}\right) - v_t\left(\frac{1+a}{1+r}X_{t-1}\right)}{X_{t-1}(b-a)/(1+r)} \\
&= X_{t-1}\frac{\left(\frac{1+b}{1+r}\right)^2 - \left(\frac{1+a}{1+r}\right)^2}{(b-a)/(1+r)} \frac{(1+r(2+a+b)-ab)^{N-t}}{(1+r)^{N-2t}} \\
&= S_{t-1}(2+b+a)\frac{(1+r(2+a+b)-ab)^{N-t}}{(1+r)^{N-t}}, \qquad t=1,\ldots,N,
\end{aligned}$$

representing the quantity of the risky asset to be present in the portfolio at time t. On the other hand we have

$$\xi_t^0 = \frac{V_t - \xi_t^1 X_t}{X_t^0}$$

$$
\begin{aligned}
&= \frac{V_t - \xi_t^1 X_t}{\pi_0} \\
&= X_t(1 + r(2 + a + b) - ab)^{N-t} \frac{X_t - X_{t-1}(2 + b + a)/(1 + r)}{\pi_0(1 + r)^{N-2t}} \\
&= S_t(1 + r(2 + a + b) - ab)^{N-t} \frac{S_t - S_{t-1}(2 + b + a)}{\pi_0(1 + r)^N} \\
&= -(S_{t-1})^2(1 + r(2 + a + b) - ab)^{N-t} \frac{(1 + a)(1 + b)}{\pi_0(1 + r)^N},
\end{aligned}
$$

$t = 1, \ldots, N$.

6. Let us check that the portfolio is self-financing. We have

$$
\begin{aligned}
&\bar{\xi}_{t+1} \cdot \bar{S}_t = \xi_{t+1}^0 S_t^0 + \xi_{t+1}^1 S_t^1 \\
&= -(S_t)^2(1 + r(2 + a + b) - ab)^{N-t-1} \frac{(1 + a)(1 + b)}{\pi_0(1 + r)^N} S_t^0 \\
&\quad +(S_t)^2(2 + b + a) \frac{(1 + r(2 + a + b) - ab)^{N-t-1}}{(1 + r)^{N-t-1}} \\
&= (S_t)^2 \frac{(1 + r(2 + a + b) - ab)^{N-t-1}}{(1 + r)^{N-t}} \\
&\quad \times ((2 + b + a)(1 + r) - (1 + a)(1 + b)) \\
&= (X_t)^2(1 + r(2 + a + b) - ab)^{N-t} \frac{1}{(1 + r)^{N-3t}} \\
&= (1 + r)^t V_t \\
&= \bar{\xi}_t \cdot \bar{S}_t, \qquad t = 1, \ldots, N.
\end{aligned}
$$

Exercise 3.3

1. We have

$$
\begin{aligned}
V_t &= \xi_t S_t + \eta_t \pi_t \\
&= \xi_t(1 + R_t)S_{t-1} + \eta_t(1 + r)\pi_{t-1}.
\end{aligned}
$$

2. We have

$$
\begin{aligned}
\mathbb{E}^*[R_t|\mathcal{F}_{t-1}] &= a\mathbb{P}^*(R_t = a \mid \mathcal{F}_{t-1}) + b\mathbb{P}^*(R_t = b \mid \mathcal{F}_{t-1}) \\
&= a\frac{b - r}{b - a} + b\frac{r - a}{b - a} \\
&= b\frac{r}{b - a} - a\frac{r}{b - a} \\
&= r.
\end{aligned}
$$

3. By the result of Question 1 we have

$$
\mathbb{E}^*[V_t \mid \mathcal{F}_{t-1}] = \mathbb{E}^*[\xi_t(1 + R_t)S_{t-1} \mid \mathcal{F}_{t-1}] + \mathbb{E}^*[\eta_t(1 + r)\pi_{t-1} \mid \mathcal{F}_{t-1}]
$$

$$
\begin{aligned}
&= \xi_t S_{t-1} \mathbb{E}^*[1+R_t \mid \mathcal{F}_{t-1}] + (1+r)\, \mathbb{E}^*[\eta_t \pi_{t-1} \mid \mathcal{F}_{t-1}] \\
&= (1+r)\xi_t S_{t-1} + (1+r)\eta_t \pi_{t-1} \\
&= (1+r)\xi_t S_t + (1+r)\eta_t \pi_t \\
&= (1+r) V_{t-1},
\end{aligned}
$$

where we used the self-financing condition.

4. We have

$$
\begin{aligned}
V_{t-1} &= \frac{1}{1+r} \mathbb{E}^*[V_t \mid \mathcal{F}_{t-1}] \\
&= \frac{3}{1+r} \mathbb{P}^*(R_t = a \mid \mathcal{F}_{t-1}) + \frac{8}{1+r} \mathbb{P}^*(R_t = b \mid \mathcal{F}_{t-1}) \\
&= \frac{1}{1+0.15} \left(3 \frac{0.25-0.15}{0.25-0.05} + 8 \frac{0.15-0.05}{0.25-0.05} \right) \\
&= \frac{1}{1.15} \left(\frac{3}{2} + \frac{8}{2} \right) \\
&= 4.78.
\end{aligned}
$$

Chapter 4

Exercise 4.1

1. We need to check whether the four properties of the definition of Brownian motion are satisfied. Checking Conditions (i) to (iii) does not pose any particular problem since the time changes $t \mapsto c+t$, $t \mapsto t/c^2$ and $t \mapsto ct^2$ are deterministic, continuous, and increasing. As for Condition (iv), $B_{c+t} - B_{c+s}$ clearly has a centered Gaussian distribution with variance t, and the same property holds for cB_{t/c^2} since

$$
\mathrm{Var}(c(B_{t/c^2} - B_{s/c^2})) = c^2 \mathrm{Var}(B_{t/c^2} - B_{s/c^2}) = c^2(t-s)/c^2 = t-s.
$$

As a consequence, (a) and (b) are standard Brownian motions.

Concerning (c), we note that B_{ct^2} is a centered Gaussian random variable with variance ct^2 - not t, hence $(B_{ct^2})_{t \in \mathbb{R}_+}$ is not a standard Brownian motion.

2. We have $\int_0^T 2 dB_t = 2(B_T - B_0) = 2B_T$, which has a Gaussian law with mean 0 and variance $4T$. On the other hand,

$$
\int_0^T (2 \times \mathbf{1}_{[0,T/2]}(t) + \mathbf{1}_{(T/2,T]}(t)) dB_t = 2(B_{T/2} - B_0) + (B_T - B_{T/2}) = B_T + B_{T/2},
$$

which has a Gaussian law with mean 0 and variance $4(T/2) + T/2 = 5T/2$.

3. The stochastic integral $\int_0^{2\pi} \sin(t)\, dB_t$ has a Gaussian distribution with mean 0 and variance
$$\int_0^{2\pi} \sin^2(t)dt = \int_0^{2\pi} \frac{1-\cos(2t)}{2} dt = \pi.$$
4. If $0 \le s \le t$ we have
$$\mathbb{E}[B_t B_s] = \mathbb{E}[(B_t - B_s)B_s] + \mathbb{E}[B_s^2] = \mathbb{E}[(B_t - B_s)]\, \mathbb{E}[B_s] + \mathbb{E}[B_s^2] = 0 + s = s,$$
and similarly we obtain $\mathbb{E}[B_t B_s] = t$ when $0 \le t \le s$, hence in general we have $\mathbb{E}[B_t B_s] = \min(s,t)$, $s, t \ge 0$.
5. We have
$$\begin{aligned} d(f(t)B_t) &= f(t)dB_t + B_t df(t) + df(t) \cdot dB_t \\ &= f(t)dB_t + B_t f'(t)dt + f'(t)dt \cdot dB_t \\ &= f(t)dB_t + B_t f'(t)dt, \end{aligned}$$
and by integration on both sides we get
$$\begin{aligned} 0 &= f(T)B_T - f(0)B_0 \\ &= \int_0^T d(f(t)B_t) \\ &= \int_0^T f(t)dB_t + \int_0^T B_t f'(t)dt, \end{aligned}$$
hence the conclusion.

Exercise 4.2 Let $f \in L^2([0,T])$. We have
$$E\left[e^{\int_0^T f(s)dB_s} \Big| \mathcal{F}_t \right] = \exp\left(\int_0^t f(s)dB_s + \frac{1}{2}\int_t^T |f(s)|^2 ds \right), \qquad 0 \le t \le T.$$

Exercise 4.3 We have
$$\begin{aligned} E\left[\exp\left(\beta \int_0^T B_t dB_t \right) \right] &= E\left[\exp\left(\beta (B_T^2 - T)/2 \right) \right] \\ &= e^{-\beta T/2} E\left[\exp\left(e^{\beta (B_T)^2/2} \right) \right] \\ &= \frac{e^{-\beta T/2}}{\sqrt{2\pi T}} \int_{-\infty}^{\infty} e^{(\beta - \frac{1}{T})\frac{x^2}{2}} dx \\ &= \frac{e^{-\beta T/2}}{\sqrt{1 - \beta T}}. \end{aligned}$$
for all $\beta < 1/T$.

Exercise 4.4 We have $f(t) = f(0)e^{ct}$ (interest rate compounding) and $S_t = S_0 e^{\sigma B_t - \sigma^2 t/2 + rt}$, $t \in \mathbb{R}_+$, (geometric Brownian motion).

Exercise 4.5

1. By (4.24) we have

$$d(X_t^T/(T-t)) = \frac{dX_t^T}{T-t} + \frac{X_t^T}{(T-t)^2}dt = \sigma \frac{dB_t}{T-t},$$

   hence by integration using the initial condition $X_0 = 0$ we have

$$\frac{X_t^T}{T-t} = \sigma \int_0^t \frac{1}{T-s} dB_s, \qquad t \in [0,T].$$

2. We have

$$\mathbb{E}[X_t^T] = \sigma(T-t)\,\mathbb{E}\left[\int_0^t \frac{1}{T-s} dB_s\right] = 0.$$

3. Using the Itô isometry we have

$$\begin{aligned}\operatorname{Var}[X_t^T] &= \sigma^2 (T-t)^2 \operatorname{Var}\left[\int_0^t \frac{1}{T-s} dB_s\right] \\ &= \sigma^2 (T-t)^2 \int_0^t \frac{1}{(T-s)^2} ds \\ &= \sigma^2 (T-t)^2 \left(\frac{1}{T-t} - \frac{1}{T}\right) \\ &= \sigma^2 (1 - t/T).\end{aligned}$$

4. We have $\operatorname{Var}[X_T^T] = 0$ hence $X_T^T = \mathbb{E}[X_T^T] = 0$ by Question 2.

Exercise 4.6 Exponential Vasicek model.

1. We have $z_t = e^{-at} z_0 + \sigma \int_0^t e^{-a(t-s)} dB_s$.
2. We have $y_t = e^{-at} y_0 + \frac{\theta}{a}(1 - e^{-at}) + \sigma \int_0^t e^{-a(t-s)} dB_s$.
3. We have $dx_t = x_t \left(\theta + \frac{\sigma^2}{2} - a \log x_t\right) dt + \sigma x_t dB_t$.
4. We have $r_t = \exp\left(e^{-at} \log r_0 + \frac{\theta}{a}(1 - e^{-at}) + \sigma \int_0^t e^{-a(t-s)} dB_s\right)$, with $\eta = \theta + \sigma^2/2$.
5. We have

$$\mathbb{E}[r_t] = \exp\left(e^{-at} \log r_0 + \frac{\theta}{a}(1 - e^{-at}) + \frac{\sigma^2}{4a}(1 - e^{-2at})\right).$$

6. We have $\lim_{t\to\infty} \mathbb{E}[r_t] = \exp\left(\frac{\theta}{a} + \frac{\sigma^2}{4a}\right)$.

Exercise 4.7 Cox-Ingersoll-Ross (CIR) model.

1. We have $r_t = r_0 + \int_0^t (\alpha - \beta r_s) ds + \sigma \int_0^t \sqrt{r_s} dB_s$.
2. Using the fact that the expectation of the stochastic integral with respect to Brownian motion is zero, we get, taking expectations on both sides of the above integral equation: $u'(t) = \alpha - \beta u(t)$.
3. Apply Itô's formula to

$$r_t^2 = f\left(r_0 + \int_0^t (\alpha - \beta r_s) ds + \sigma \int_0^t \sqrt{r_s} dB_s\right),$$

with $f(x) = x^2$, to obtain

$$d(r_t)^2 = r_t(\sigma^2 + 2\alpha - 2\beta r_t)dt + 2r_t\sigma\sqrt{r_t}dB_t. \tag{S.4.3}$$

4. Taking again the expectation on both sides of (S.4.3) we get

$$\mathbb{E}[r_t^2] = \mathbb{E}[r_0^2] + \int_0^t (\sigma^2\, \mathbb{E}[r_t] + 2\alpha\, \mathbb{E}[r_t] - 2\beta\, \mathbb{E}[r_t^2])dt,$$

and after differentiation with respect to t this yields

$$v_t' = (\sigma^2 + 2\alpha)u(t) - 2\beta v(t).$$

Exercise 4.8

1. We have

$$\begin{aligned} S_t &= e^{X_t} \\ &= e^{X_0} + \int_0^t u_s e^{X_s} dB_s + \int_0^t v_s e^{X_s} ds + \frac{1}{2}\int_0^t u_s^2 e^{X_s} ds \\ &= e^{X_0} + \sigma \int_0^t e^{X_s} dB_s + \nu \int_0^t e^{X_s} ds + \frac{\sigma^2}{2}\int_0^t e^{X_s} ds \\ &= S_0 + \sigma \int_0^t S_s dB_s + \nu \int_0^t S_s ds + \frac{\sigma^2}{2}\int_0^t S_s ds. \end{aligned}$$

2. Let $r > 0$. The process $(S_t)_{t\in\mathbb{R}_+}$ satisfies the stochastic differential equation

$$dS_t = rS_t dt + \sigma S_t dB_t$$

when $r = \nu + \sigma^2/2$.
3. Let the process $(S_t)_{t\in\mathbb{R}_+}$ be defined by $S_t = S_0 e^{\sigma B_t + \nu t}$, $t \in \mathbb{R}_+$. Using the decomposition $S_T = S_t e^{\sigma(B_T - B_t) + \nu\tau}$, we have

$$\begin{aligned} \mathbb{P}(S_T > K \mid S_t = x) &= \mathbb{P}(S_t e^{\sigma(B_T - B_t)+\nu(T-t)} > K \mid S_t = x) \\ &= \mathbb{P}(x e^{\sigma(B_T - B_t)+\nu(T-t)} > K) \\ &= \mathbb{P}(e^{\sigma(B_T - B_t)} > K e^{-\nu(T-t)}/x) \end{aligned}$$

$$= \Phi\left(\frac{-\log(Ke^{-\nu(T-t)}/x)}{\sigma\sqrt{\tau}}\right)$$
$$= \Phi\left(\frac{\log(x/K)+\nu\tau}{\sigma\sqrt{\tau}}\right),$$

where $\tau = T - t$.

4. We have

$$\eta^2 = \mathrm{Var}[X] = \mathrm{Var}[\sigma(B_T - B_t)] = \sigma^2 \mathrm{Var}[B_T - B_t] = \sigma^2(T-t),$$

hence $\eta = \sigma\sqrt{T-t}$.

Chapter 5

Exercise 5.1

1. We have
$$S_t = S_0 e^{\alpha t} + \sigma \int_0^t e^{\alpha(t-s)} dB_s.$$
2. We have $\alpha_M = r$.
3. After computing the conditional expectation
$$C(t,x) = e^{-r(T-t)} \exp\left(xe^{r(T-t)} + \frac{\sigma^2}{4r}(e^{2r(T-t)}-1)\right).$$
4. Here we need to note that the usual Black-Scholes argument applies and yields $\zeta_t = \partial C(t,S_t)/\partial x$, that is
$$\zeta_t = \exp\left(S_t e^{r(T-t)} + \frac{\sigma^2}{4r}(e^{2r(T-t)}-1)\right).$$

Exercise 5.2

1. We have, counting approximately 46 days to maturity,
$$\begin{aligned} d_- &= \frac{(r-\frac{1}{2}\sigma^2)(T-t)+\log\frac{S_t}{K}}{\sigma\sqrt{T-t}} \\ &= \frac{(0.04377-\frac{1}{2}(0.9)^2)(46/365)+\log\frac{17.2}{36.08}}{0.9\sqrt{46/365}} \\ &= -2.46, \end{aligned}$$
and
$$d_+ = d_- + 0.9\sqrt{46/365} = -2.14.$$
From the attached table we get

$$\Phi(d_+) = \Phi(-2.14) = 0.0162$$

and

$$\Phi(d_-) = \Phi(-2.46) = 0.0069,$$

hence

$$\begin{aligned} f(t, S_t) &= S_t \Phi(d_+) - K e^{-r(T-t)} \Phi(d_-) \\ &= 17.2 \times 0.0162 - 36.08 * e^{-0.04377 \times 46/365} \times 0.0069 \\ &= \text{HK\$ } 0.031. \end{aligned}$$

2. We have
$$\eta_t = \frac{\partial f}{\partial x}(t, S_t) = \Phi(d_+) = \Phi(-2.14) = 0.0162,$$
hence one should only hold a fractional quantity 16.2 of the risky asset in order to hedge 1000 such call options when $\sigma = 0.90$.
3. From the curve it turns out that when $f(t, S_t) = 10 \times 0.023 = \text{HK\$ } 0.23$, the volatility σ is approximately equal to $\sigma = 122\%$.

   This approximate value of implied volatility can be found under the column "Implied Volatility (IV.)" on this set of market data from the Hong Kong Stock Exchange:

Updated: 6 November 2008

Basic Data

DW Code	Issuer	UL	Call /Put	DW Type	Listing (D-M-Y)	Maturity (D-M-Y)	Strike	Entitle-ment Ratio^
01897	FB	00066	Call	Standard	18-12-2007	23-12-2008	36.08	10

Market Data

Total Issue Size	O/S (%)	Delta (%)	IV. (%)	Day High ($)	Day Low ($)	Closing Price # ($)	T/O ('000)	UL Price ($)
138,000,000	16.43	0.780	125.375	0.000	0.000	0.023	0	17.200

Remark: a typical value for the volatility in standard market conditions would be around 20%. The observed volatility value $\sigma = 1.22$ per year is actually quite high.

Exercise 5.3

1. We find $h(x) = x - K$.
2. Letting $g(t, x)$, the PDE rewrites as
$$r(x - \alpha(t)) = -\alpha'(t) + rx,$$
hence $\alpha(t) = \alpha(0)e^{rt}$ and $g(t, x) = x - \alpha(0)e^{-rt}$. The final condition

$$g(T,x) = h(x) = x - K$$

yields $\alpha(0) = Ke^{rT}$ and $g(t,x) = x - Ke^{-r(T-t)}$.

3. We have

$$\xi_t = \frac{\partial g}{\partial x}(t, S_t) = 1.$$

Exercise 5.4

1. We have

$$\begin{aligned}
C_t &= e^{-r(T-t)} E[S_T - K \mid \mathcal{F}_t] \\
&= e^{-r(T-t)} E[S_T \mid \mathcal{F}_t] - Ke^{-r(T-t)} \\
&= e^{rt} E[e^{-rT} S_T \mid \mathcal{F}_t] - Ke^{-r(T-t)} \\
&= e^{rt} e^{-rt} S_t - Ke^{-r(T-t)} \\
&= S_t - Ke^{-r(T-t)}.
\end{aligned}$$

We can check that the function $g(x,t) = x - Ke^{-r(T-t)}$ satisfies the Black-Scholes PDE

$$rg(x,t) = \frac{\partial g}{\partial t}(x,t) + rx\frac{\partial g}{\partial x}(x,t) + \frac{\sigma^2}{2}x^2\frac{\partial g}{\partial x}(x,t)$$

with terminal condition $g(x,T) = x - K$, since $\partial g(x,t)/\partial t = -rKe^{-r(T-t)}$ and $\partial g(x,t)/\partial x = 1$.

2. We simply take $\xi_t = 1$ and $\eta_t = -Ke^{-rT}$ in order to have

$$C_t = \xi_t S_t + \eta_t e^{rt} = S_t - Ke^{-r(T-t)}, \qquad t \in [0,T].$$

Hint: Find the quantity ξ_t of the risky asset S_t and the quantity η_t of the riskless asset e^{rt} such that the equality

$$C_t = \xi_t S_t + \eta_t e^{rt}$$

holds at any time $t \in [0,T]$.

Remark: This hedging strategy is *constant* over time, and the relation $\xi_t = \partial g(S_t,t)/\partial x$ for the delta is satisfied.

Chapter 6

Exercise 6.1

1. For all $t \in [0,T]$ we have

$$\begin{aligned} C(t,S_t) &= e^{-r(T-t)} S_t^2 \, \mathbb{E}\left[\frac{S_T^2}{S_t^2}\right] \\ &= e^{-r(T-t)} S_t^2 \, \mathbb{E}\left[e^{2\sigma(B_T-B_t)-\sigma^2(T-t)+2r(T-t)}\right] \\ &= S_t^2 e^{(r+\sigma^2)(T-t)}. \end{aligned}$$

2. For all $t \in [0,T]$ we have

$$\xi_t = \frac{\partial C}{\partial x}(t,x)_{|x=S_t} = 2S_t e^{(r+\sigma^2)(T-t)},$$

and

$$\begin{aligned} \eta_t = \frac{C(t,S_t) - \xi_t S_t}{A_t} &= \frac{e^{-rt}}{A_0}(S_t^2 e^{(r+\sigma^2)(T-t)} - 2S_t^2 e^{(r+\sigma^2)(T-t)}) \\ &= -\frac{S_t^2}{A_0} e^{\sigma^2(T-t)+r(T-2t)}. \end{aligned}$$

3. We have

$$\begin{aligned} dC(t,S_t) &= d(S_t^2 e^{(r+\sigma^2)(T-t)}) \\ &= -(r+\sigma^2)e^{(r+\sigma^2)(T-t)} S_t^2 dt + e^{(r+\sigma^2)(T-t)} d(S_t^2) \\ &= -(r+\sigma^2)e^{(r+\sigma^2)(T-t)} S_t^2 dt + e^{(r+\sigma^2)(T-t)} (2S_t dS_t + \sigma^2 S_t^2 dt) \\ &= -re^{(r+\sigma^2)(T-t)} S_t^2 dt + 2S_t e^{(r+\sigma^2)(T-t)} dS_t, \end{aligned}$$

and

$$\begin{aligned} \xi_t dS_t + \eta_t dA_t &= 2S_t e^{(r+\sigma^2)(T-t)} dS_t - r\frac{S_t^2}{A_0} e^{\sigma^2(T-t)+r(T-2t)} A_t dt \\ &= 2S_t e^{(r+\sigma^2)(T-t)} dS_t - rS_t^2 e^{\sigma^2(T-t)+r(T-t)} dt, \end{aligned}$$

hence we can check that the strategy is self-financing since $dC(t,S_t) = \xi_t dS_t + \eta_t dA_t$.

Exercise 6.2

1. We have

$$S_t = S_0 e^{rt} + \sigma \int_0^t e^{r(t-s)} dB_s.$$

2. We have

$$\tilde{S}_t = S_0 + \sigma \int_0^t e^{-rs} dB_s,$$

which is a martingale, being a stochastic integral with respect to Brownian motion.

This fact can also be proved directly by computing the conditional expectation $E[\tilde{S}_t \mid \mathcal{F}_s]$ and showing it is equal to $\tilde{S}_s$:

$$\begin{aligned}
E[\tilde{S}_t \mid \mathcal{F}_s] &= E\left[S_0 + \sigma \int_0^t e^{-ru} dB_u \mid \mathcal{F}_s\right] \\
&= E[S_0] + \sigma E\left[\int_0^t e^{-ru} dB_u \mid \mathcal{F}_s\right] \\
&= S_0 + \sigma E\left[\int_0^s e^{-ru} dB_u \mid \mathcal{F}_s\right] + \sigma E\left[\int_s^t e^{-ru} dB_u \mid \mathcal{F}_s\right] \\
&= S_0 + \sigma \int_0^s e^{-ru} dB_u + \sigma E\left[\int_s^t e^{-ru} dB_u\right] \\
&= S_0 + \sigma \int_0^s e^{-ru} dB_u \\
&= \tilde{S}_s.
\end{aligned}$$

3. We have

$$\begin{aligned}
&C(t, S_t) = e^{-r(T-t)} E[\exp(S_T) | \mathcal{F}_t] \\
&= e^{-r(T-t)} E\left[\exp\left(e^{rT} S_0 + \sigma \int_0^T e^{r(T-u)} dB_u\right) \Big| \mathcal{F}_t\right] \\
&= e^{-r(T-t)} E\left[\exp\left(e^{rT} S_0 + \sigma \int_0^t e^{r(T-u)} dB_u + \sigma \int_t^T e^{r(T-u)} dB_u\right) \Big| \mathcal{F}_t\right] \\
&= \exp\left(-r(T-t) + e^{r(T-t)} S_t\right) E\left[\exp\left(\sigma \int_t^T e^{r(T-u)} dB_u\right) \Big| \mathcal{F}_t\right] \\
&= \exp\left(-r(T-t) + e^{r(T-t)} S_t\right) E\left[\exp\left(\sigma \int_t^T e^{r(T-u)} dB_u\right)\right] \\
&= \exp\left(-r(T-t) + e^{r(T-t)} S_t\right) \exp\left(\frac{\sigma^2}{2} \int_t^T (e^{r(T-u)})^2 du\right) \\
&= \exp\left(-r(T-t) + e^{r(T-t)} S_t + \frac{\sigma^2}{4r}(e^{2r(T-t)} - 1)\right).
\end{aligned}$$

4. We have

$$\xi_t = \frac{\partial C}{\partial x}(t, S_t) = \exp\left(S_t e^{r(T-t)} + \frac{\sigma^2}{4r}(e^{2r(T-t)} - 1)\right)$$

and

$$\begin{aligned}
\eta_t &= \frac{C(t, S_t) - \xi_t S_t}{A_t} \\
&= \frac{e^{-r(T-t)}}{A_t} \exp\left(S_t e^{r(T-t)} + \frac{\sigma^2}{4r}(e^{2r(T-t)} - 1)\right) \\
&\quad - \frac{S_t}{A_t} \exp\left(S_t e^{r(T-t)} + \frac{\sigma^2}{4r}(e^{2r(T-t)} - 1)\right).
\end{aligned}$$

5. We have

$$
\begin{aligned}
dC(t,S_t) &= re^{-r(T-t)}\exp\left(S_te^{r(T-t)}+\frac{\sigma^2}{4r}(e^{2r(T-t)}-1)\right)dt \\
&\quad -rS_t\exp\left(S_te^{r(T-t)}+\frac{\sigma^2}{4r}(e^{2r(T-t)}-1)\right)dt \\
&\quad -\frac{\sigma^2}{2}e^{r(T-t)}\exp\left(S_te^{r(T-t)}+\frac{\sigma^2}{4r}(e^{2r(T-t)}-1)\right)dt \\
&\quad +\exp\left(S_te^{r(T-t)}+\frac{\sigma^2}{4r}(e^{2r(T-t)}-1)\right)dS_t \\
&\quad +\frac{1}{2}e^{r(T-t)}\exp\left(S_te^{r(T-t)}+\frac{\sigma^2}{4r}(e^{2r(T-t)}-1)\right)\sigma^2dt \\
&= re^{-r(T-t)}\exp\left(S_te^{r(T-t)}+\frac{\sigma^2}{4r}(e^{2r(T-t)}-1)\right)dt \\
&\quad -rS_t\exp\left(S_te^{r(T-t)}+\frac{\sigma^2}{4r}(e^{2r(T-t)}-1)\right)dt \\
&\quad +\xi_tdS_t.
\end{aligned}
$$

On the other hand we have

$$
\begin{aligned}
\xi_tdS_t+\eta_tdA_t &= \xi_tdS_t \\
&\quad +re^{-r(T-t)}\exp\left(S_te^{r(T-t)}+\frac{\sigma^2}{4r}(e^{2r(T-t)}-1)\right)dt \\
&\quad -rS_t\exp\left(S_te^{r(T-t)}+\frac{\sigma^2}{4r}(e^{2r(T-t)}-1)\right)dt,
\end{aligned}
$$

showing that

$$
dC(t,S_t)=\xi_tdS_t+\eta_tdA_t,
$$

and confirming that the strategy $(\xi_t,\eta_t)_{t\in\mathbb{R}_+}$ is self-financing.

Exercise 6.3

1. We have

$$
\frac{\partial f}{\partial t}(t,x)=(r-\sigma^2/2)f(t,x), \qquad \frac{\partial f}{\partial x}(t,x)=\sigma f(t,x),
$$

and

$$
\frac{\partial^2 f}{\partial x^2}(t,x)=\sigma^2 f(t,x),
$$

hence

$$
\begin{aligned}
dS_t &= df(t,B_t) \\
&= \frac{\partial f}{\partial t}(t,B_t)dt+\frac{\partial f}{\partial x}(t,B_t)dB_t+\frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t,B_t)dt
\end{aligned}
$$

$$\begin{aligned}
&= \left(r - \frac{1}{2}\sigma^2\right) f(t, B_t)dt + \sigma f(t, B_t)dB_t + \frac{1}{2}\sigma^2 f(t, B_t)dt \\
&= r f(t, B_t)dt + \sigma f(t, B_t)dB_t \\
&= rS_t dt + \sigma S_t dB_t.
\end{aligned}$$

2. We have

$$\begin{aligned}
E[e^{\sigma B_T}|\mathcal{F}_t] &= E[e^{\sigma(B_T - B_t + B_t)}|\mathcal{F}_t] \\
&= e^{\sigma B_t} E[e^{\sigma(B_T - B_t)}|\mathcal{F}_t] \\
&= e^{\sigma B_t} E[e^{\sigma(B_T - B_t)}] \\
&= e^{\sigma B_t + \sigma^2(T-t)/2}.
\end{aligned}$$

3. We have

$$\begin{aligned}
E[S_T|\mathcal{F}_t] &= E[e^{\sigma B_T + rT - \sigma^2 T/2}|\mathcal{F}_t] \\
&= e^{rT - \sigma^2 T/2} E[e^{\sigma B_T}|\mathcal{F}_t] \\
&= e^{rT - \sigma^2 T/2} e^{\sigma B_t + \sigma^2(T-t)/2} \\
&= e^{rT + \sigma B_t - \sigma^2 t/2} \\
&= e^{r(T-t) + \sigma B_t + rt - \sigma^2 t/2} \\
&= e^{r(T-t)} S_t.
\end{aligned}$$

4. We have

$$\begin{aligned}
V_t &= e^{-r(T-t)} E[C|\mathcal{F}_t] \\
&= e^{-r(T-t)} E[S_T - K|\mathcal{F}_t] \\
&= e^{-r(T-t)} E[S_T|\mathcal{F}_t] - e^{-r(T-t)} E[K|\mathcal{F}_t] \\
&= S_t - e^{-r(T-t)} K.
\end{aligned}$$

5. We take $\xi_t = 1$ and $\eta_t = -Ke^{-rT}/A_0$, $t \in [0, T]$.
6. We have
$$V_T = E[C \mid \mathcal{F}_T] = C.$$

Exercise 6.4 Digital options.

1. By definition of the indicator functions $\mathbf{1}_{[K,\infty)}$ and $\mathbf{1}_{[0,K]}$ we have

$$\mathbf{1}_{[K,\infty)}(x) = \begin{cases} 1 & \text{if } x \geq K, \\ \\ 0 & \text{if } x < K, \end{cases} \qquad \text{resp.} \qquad \mathbf{1}_{[0,K]}(x) = \begin{cases} 1 & \text{if } x \leq K, \\ \\ 0 & \text{if } x > K, \end{cases}$$

which shows the claimed result by the definition of C_d and P_d.
2. We have

$$\pi_t(C_d) + \pi_t(P_d) = e^{-r(T-t)} \mathbb{E}[C_d \mid \mathcal{F}_t] + e^{-r(T-t)} \mathbb{E}[P_d \mid \mathcal{F}_t]$$

$$\begin{aligned}
&= e^{-r(T-t)}\,\mathbb{E}[C_d + P_d \mid \mathcal{F}_t] \\
&= e^{-r(T-t)}\,\mathbb{E}[\mathbf{1}_{[K,\infty)}(S_T) + \mathbf{1}_{[0,K]}(S_T) \mid \mathcal{F}_t] \\
&= e^{-r(T-t)}\,\mathbb{E}[\mathbf{1}_{[0,\infty)}(S_T) \mid \mathcal{F}_t] \\
&= e^{-r(T-t)}\,\mathbb{E}[1 \mid \mathcal{F}_t] \\
&= e^{-r(T-t)}, \qquad 0 \le t \le T,
\end{aligned}$$

since $\mathbb{P}(S_T = K) = 0$.

3. We have

$$\begin{aligned}
\pi_t(C_d) &= e^{-r(T-t)}\,\mathbb{E}[C_d \mid \mathcal{F}_t] \\
&= e^{-r(T-t)}\,\mathbb{E}[\mathbf{1}_{[K,\infty)}(S_T) \mid S_t] \\
&= e^{-r(T-t)} P(S_T \ge K \mid S_t) \\
&= C_d(t, S_t).
\end{aligned}$$

4. We have

$$\begin{aligned}
C_d(t,x) &= e^{-r(T-t)} P(S_T > K \mid S_t = x) \\
&= e^{-r(T-t)} \Phi\left(\frac{r\tau - \sigma^2\tau/2 + \log(x/K)}{\sigma\sqrt{\tau}}\right),
\end{aligned}$$

where $\tau = T - t$.

5. We have

$$\begin{aligned}
\pi_t(C_d) &= C_d(t, S_t) \\
&= e^{-r(T-t)} \Phi\left(\frac{r\tau - \sigma^2\tau/2 + \log(S_t/K)}{\sigma\sqrt{\tau}}\right) \\
&= e^{-r(T-t)} \Phi\left(d_-\right),
\end{aligned}$$

where

$$d_- = \frac{(r - \sigma^2/2)\tau + \log(S_t/K)}{\sigma\sqrt{\tau}}.$$

6. We have

$$\begin{aligned}
\pi_t(P_d) &= e^{-r(T-t)} - \pi_t(C_d) \\
&= e^{-r(T-t)} - e^{-r(T-t)} \Phi\left(\frac{r\tau - \sigma^2\tau/2 + \log(x/K)}{\sigma\sqrt{\tau}}\right) \\
&= e^{-r(T-t)}(1 - \Phi(d_-)) \\
&= e^{-r(T-t)} \Phi(-d_-).
\end{aligned}$$

7. We have

$$\xi_t = \frac{\partial C_d}{\partial x}(t, S_t)$$

$$= e^{-r(T-t)}\frac{\partial}{\partial x}\Phi\left(\frac{r\tau-\sigma^2\tau/2+\log(x/K)}{\sigma\sqrt{\tau}}\right)_{x=S_t}$$
$$= e^{-r(T-t)}\frac{1}{\sigma\sqrt{2\pi\tau}S_t}e^{-(d_-)^2/2}$$
$$> 0.$$

The Black-Scholes hedging strategy of such a call option does not involve short-selling because $\xi_t > 0$ for all t.

8. Here we have

$$\xi_t = \frac{\partial P_d}{\partial x}(t, S_t)$$
$$= e^{-r(T-t)}\frac{\partial}{\partial x}\Phi\left(-\frac{r\tau-\sigma^2\tau/2+\log(x/K)}{\sigma\sqrt{\tau}}\right)_{x=S_t}$$
$$= -e^{-r(T-t)}\frac{1}{\sigma\sqrt{2\pi\tau}S_t}e^{-(d_-)^2/2}$$
$$< 0.$$

The Black-Scholes hedging strategy of such a call option does involve short-selling because $\xi_t < 0$ for all t.

Chapter 8

Exercise 8.1

1. We have

$$P(\tau_a \geq t) = P(X_t > a) = \int_a^\infty \varphi_{X_t}(x)dx = \sqrt{\frac{2}{\pi t}}\int_y^\infty e^{-x^2/(2t)}dx, \qquad y > 0.$$

2. We have

$$\varphi_{\tau_a}(t) = \frac{d}{dt}P(\tau_a \leq t)$$
$$= \frac{d}{dt}\int_a^\infty \varphi_{X_t}(x)dx$$
$$= -\frac{1}{2}\sqrt{\frac{2}{\pi}}t^{-3/2}\int_a^\infty e^{-x^2/(2t)}dx + \frac{1}{2}\sqrt{\frac{2}{\pi}}t^{-3/2}\int_a^\infty \frac{x^2}{t}e^{-x^2/(2t)}dx$$
$$= \frac{1}{2}\sqrt{\frac{2}{\pi}}t^{-3/2}\left(-\int_a^\infty e^{-x^2/(2t)}dx + ae^{-a^2/(2t)} + \int_a^\infty e^{-x^2/(2t)}dx\right)$$
$$= \frac{a}{\sqrt{2\pi t^3}}e^{-a^2/(2t)}, \qquad t > 0.$$

3. We have

$$\begin{aligned} E[(\tau_a)^{-2}] &= \frac{a}{\sqrt{2\pi}}\int_0^\infty t^{-5/2}e^{-a^2/(2t)}dt \\ &= \frac{2a}{\sqrt{2\pi}}\int_0^\infty x^2 e^{-a^2x^2/2}dx \\ &= \frac{1}{a^2}, \end{aligned}$$

by the change of variable $x = t^{-1/2}$, $x^2 = 1/t$, $t = x^{-2}$, $dt = -2x^{-3}dx$.

Remark: We have

$$E[\tau_a] = \frac{a}{\sqrt{2\pi}}\int_0^\infty t^{-1/2}e^{-a^2/(2t)}dt = +\infty.$$

Exercise 8.2 Barrier options.

1. By (8.30) and (8.18) we find

$$\begin{aligned} \xi_t = \frac{\partial g}{\partial y}(t, S_t) &= \Phi\left(\delta_+^{T-t}\left(\frac{S_t}{K}\right)\right) - \Phi\left(\delta_+^{T-t}\left(\frac{S_t}{B}\right)\right) \\ &+\frac{K}{B}e^{-r(T-t)}\left(1-\frac{2r}{\sigma^2}\right)\left(\frac{S_t}{B}\right)^{-2r/\sigma^2}\left(\Phi\left(\delta_-^{T-t}\left(\frac{B^2}{KS_t}\right)\right) - \Phi\left(\delta_-^{T-t}\left(\frac{B}{S_t}\right)\right)\right) \\ &+\frac{2r}{\sigma^2}\left(\frac{S_t}{B}\right)^{-1-2r/\sigma^2}\left(\Phi\left(\delta_+^{T-t}\left(\frac{B^2}{KS_t}\right)\right) - \Phi\left(\delta_+^{T-t}\left(\frac{B}{S_t}\right)\right)\right) \\ &-\frac{2}{\sigma\sqrt{2\pi(T-t)}}\left(1-\frac{K}{B}\right)\exp\left(-\frac{1}{2}\left(\delta_+^{T-t}\left(\frac{S_t}{B}\right)\right)^2\right), \end{aligned}$$

$0 < S_t \leq B$, $0 \leq t \leq T$, cf. also Exercise 7.1-(ix) of **?** and Figure 8.13.

2. We find

$$\mathbb{P}(Y_T \leq a \ \&\ B_T \geq b) = \mathbb{P}(B_T \leq 2a - b), \qquad a < b < 0,$$

hence

$$f_{Y_T,B_T}(a,b) = \frac{d\mathbb{P}(Y_T \leq a \ \&\ B_T \leq b)}{dadb} = -\frac{d\mathbb{P}(Y_T \leq a \ \&\ B_T \geq b)}{dadb}, \qquad a, b \in \mathbb{R}.,$$

satisfies

$$f_{Y_T,B_T}(a,b) = \sqrt{\frac{2}{\pi T}}\mathbf{1}_{(-\infty, b\wedge 0]}(a)\frac{(b-2a)}{T}e^{-(2a-b)^2/(2T)}$$

$$= \begin{cases} \sqrt{\frac{2}{\pi T}} \frac{(b-2a)}{T} e^{-(2a-b)^2/(2T)}, & a < b \wedge 0, \\ \\ 0, & a > b \wedge 0. \end{cases}$$

3. We find

$$f_{\tilde{Y}_T, \tilde{B}_T}(a,b) = \mathbf{1}_{(-\infty, b\wedge 0]}(a) \frac{1}{T} \sqrt{\frac{2}{\pi T}} (b-2a) e^{-\mu^2 T/2 + \mu b - (2a-b)^2/(2T)}$$

$$= \begin{cases} \frac{1}{T} \sqrt{\frac{2}{\pi T}} (2a-b) e^{-\mu^2 T/2 + \mu b - (2a-b)^2/(2T)}, & a < b \wedge 0, \\ \\ 0, & a > b \wedge 0. \end{cases}$$

4. The function $g(t,x)$ is given in Relations (8.12) and (8.13).

Exercise 8.3 Lookback options. By (8.21) and (8.22) we find

$$\begin{aligned} \xi_t &= \frac{\partial f}{\partial x}(t, S_t, M_0^t) \\ &= -1 + \left(1 + \frac{2r}{\sigma^2}\right) \Phi\left(\delta_+^{T-t}\left(\frac{S_t}{M_0^t}\right)\right) \\ &\quad + e^{-r(T-t)} \left(\frac{M_0^t}{S_t}\right)^{2r/\sigma^2} \left(1 - \frac{\sigma^2}{2r}\right) \Phi\left(-\delta_-^{T-t}\left(\frac{M_0^t}{S_t}\right)\right), \end{aligned}$$

$t \in [0,T]$, and

$$\begin{aligned} &\eta_t A_t = f(t, S_t, M_0^t) - \xi_t S_t \\ &= M_0^t e^{-r(T-t)} \Phi\left(-\delta_-^{T-t}\left(\frac{S_t}{M_0^t}\right)\right) - e^{-r(T-t)} \left(\frac{M_0^t}{S_t}\right)^{-1+2r/\sigma^2} \Phi\left(-\delta_-^{T-t}\left(\frac{M_0^t}{S_t}\right)\right). \end{aligned}$$

Exercise 8.4 We have

$$\begin{aligned} &e^{-r(T-t)} \mathbb{E}\left[\left(\frac{1}{T}\int_0^T S_u du - \kappa\right)^+ \Big| \mathcal{F}_t\right] = e^{-r(T-t)} \mathbb{E}\left[\frac{1}{T}\int_0^T S_u du - \kappa \Big| \mathcal{F}_t\right] \\ &= e^{-r(T-t)} \mathbb{E}\left[\frac{1}{T}\int_0^T S_u du \Big| \mathcal{F}_t\right] - \kappa e^{-r(T-t)} \\ &= e^{-r(T-t)} \frac{1}{T} \mathbb{E}\left[\int_0^t S_u du \Big| \mathcal{F}_t\right] + e^{-r(T-t)} \frac{1}{T} \mathbb{E}\left[\int_t^T S_u du \Big| \mathcal{F}_t\right] - \kappa e^{-r(T-t)} \\ &= e^{-r(T-t)} \frac{1}{T} \int_0^t S_u du + e^{-r(T-t)} \frac{1}{T} \mathbb{E}\left[\int_t^T S_u du \Big| \mathcal{F}_t\right] - \kappa e^{-r(T-t)} \end{aligned}$$

$$
\begin{aligned}
&= e^{-r(T-t)}\frac{1}{T}\int_0^t S_u du + e^{-r(T-t)}\frac{1}{T}\int_t^T \mathbb{E}[S_u \mid \mathcal{F}_t]du - \kappa e^{-r(T-t)} \\
&= e^{-r(T-t)}\frac{1}{T}\int_0^t S_u du + e^{-r(T-t)}\frac{1}{T}\int_t^T S_t e^{r(u-t)}du - \kappa e^{-r(T-t)} \\
&= e^{-r(T-t)}\frac{1}{T}\int_0^t S_u du + e^{-r(T-t)}\frac{S_t}{T}\int_0^{T-t} e^{ru}du - \kappa e^{-r(T-t)} \\
&= e^{-r(T-t)}\frac{1}{T}\int_0^t S_u du + e^{-r(T-t)}\frac{S_t}{rT}(e^{r(T-t)}-1) - \kappa e^{-r(T-t)} \\
&= e^{-r(T-t)}\frac{1}{T}\int_0^t S_u du + S_t\frac{1-e^{-r(T-t)}}{rT} - \kappa e^{-r(T-t)},
\end{aligned}
$$

$t \in [0,T]$, cf. **?** page 361.

We check that the function $f(t,x,y) = e^{-r(T-t)}(y/T - \kappa) + x(1 - e^{-r(T-t)})/(rT)$ satisfies the PDE

$$
rf(t,x,y) = \frac{\partial f}{\partial t}(t,x,y) + x\frac{\partial f}{\partial y}(t,x,y) + rx\frac{\partial f}{\partial x}(t,x,y) + \frac{1}{2}x^2\sigma^2\frac{\partial^2 f}{\partial x^2}(t,x,y),
$$

$t,x > 0$, and the boundary conditions $f(t,0,y) = e^{-r(T-t)}(y/T - \kappa)$, $0 \le t \le T$, $y \in \mathbb{R}_+$, and $f(T,x,y) = y/T - \kappa$, $x,y \in \mathbb{R}_+$. However, the condition $\lim_{y\to-\infty} f(t,x,y) = 0$ is not satisfied because we need to take $y > 0$ in the above calculation.

Exercise 8.5 The Asian option price can be written as

$$
\begin{aligned}
e^{-r(T-t)}\,\mathbb{E}^*\left[\left(\frac{1}{T}\int_0^T S_u du - K\right)^+ \middle| \mathcal{F}_t\right] &= S_t\hat{\mathbb{E}}\left[(U_T)^+ \mid U_t\right] \\
&= S_t h(t,U_t) = S_t g(t,Z_t),
\end{aligned}
$$

which shows that

$$
g(t,Z_t) = h(t,U_t),
$$

and it remains to use the relation

$$
U_t = \frac{1-e^{-r(T-t)}}{rT} + e^{-r(T-t)}Z_t, \qquad t \in [0,T].
$$

Chapter 9

Exercise 9.1 Stopping times.

1. For any $t \in \mathbb{R}_+$, the question "is $\tau > t$?" can be answered based on the observation of the paths of $(B_s)_{0\le s\le t}$ and of the (deterministic) curve $(\alpha e^{-s/2})_{0\le s\le t}$ up to the time t. Therefore τ is a stopping time.

2. Since τ is a stopping time and $(B_t)_{t\in\mathbb{R}_+}$ is a martingale, the *stopping time theorem* shows that $(e^{B_{t\wedge\tau}-(t\wedge\tau)/2})_{t\in\mathbb{R}_+}$ is also a martingale and in particular its expectation $E[e^{B_{t\wedge\tau}-(t\wedge\tau)/2}] = E[e^{B_{0\wedge\tau}-(0\wedge\tau)/2}] = E[e^{B_0-0/2}] = 1$ is constantly equal to 1 for all t. This shows that

$$E[e^{B_\tau-\tau/2}] = E\left[\lim_{t\to\infty} e^{B_{t\wedge\tau}-(t\wedge\tau)/2}\right] = \lim_{t\to\infty} E[e^{B_{t\wedge\tau}-(t\wedge\tau)/2}] = 1.$$

Next, we note that we have $e^{B_\tau} = \alpha e^{-\tau/2}$, hence

$$\alpha E[e^{-\tau}] = E[e^{B_\tau-\tau/2}] = 1,$$

i.e.

$$\alpha E[e^{-\tau}] = 1/\alpha \leq 1.$$

Remark: note that this argument fails when $\alpha < 1$ because in that case τ is not *a.s.* finite.

3. When $0 \leq t < 1$ the question "is $\nu > t$?" cannot be answered at time t without waiting to know the value of B_1 at time 1. Therefore ν is *not* a stopping time.

Exercise 9.2

1. Letting $A_0 = 0$,

$$A_{n+1} = A_n + \mathbb{E}[M_{n+1} - M_n \mid \mathcal{F}_n], \quad n \geq 0,$$

and

$$N_n = M_n - A_n, \qquad n \in \mathbb{N}, \tag{S.9.4}$$

we have,

(i) for all $n \in \mathbb{N}$,

$$\begin{aligned}
\mathbb{E}[N_{n+1} \mid \mathcal{F}_n] &= \mathbb{E}[M_{n+1} - A_{n+1} \mid \mathcal{F}_n] \\
&= \mathbb{E}[M_{n+1} - A_n - \mathbb{E}[M_{n+1} - M_n \mid \mathcal{F}_n] \mid \mathcal{F}_n] \\
&= \mathbb{E}[M_{n+1} - A_n \mid \mathcal{F}_n] - \mathbb{E}[\mathbb{E}[M_{n+1} - M_n \mid \mathcal{F}_n] \mid \mathcal{F}_n] \\
&= \mathbb{E}[M_{n+1} - A_n \mid \mathcal{F}_n] - \mathbb{E}[M_{n+1} - M_n \mid \mathcal{F}_n] \\
&= -\mathbb{E}[A_n \mid \mathcal{F}_n] + \mathbb{E}[M_n \mid \mathcal{F}_n] \\
&= M_n - A_n \\
&= N_n,
\end{aligned}$$

hence $(N_n)_{n\in\mathbb{N}}$ is a martingale with respect to $(\mathcal{F}_n)_{n\in\mathbb{N}}$.

(ii) We have

$$\begin{aligned}
A_{n+1} - A_n &= \mathbb{E}[M_{n+1} - M_n \mid \mathcal{F}_n] \\
&= \mathbb{E}[M_{n+1} \mid \mathcal{F}_n] - \mathbb{E}[M_n \mid \mathcal{F}_n]
\end{aligned}$$

$$= \mathbb{E}[M_{n+1} \mid \mathcal{F}_n] - M_n \geq 0, \qquad n \in \mathbb{N},$$

since $(M_n)_{n\in\mathbb{N}}$ is a submartingale.

(iii) By induction we have

$$A_n = A_{n-1} + \mathbb{E}[M_n - M_{n-1} \mid \mathcal{F}_{n-1}], \quad n \geq 1,$$

which is $\mathcal{F}_{n-1}$-measurable provided A_n is $\mathcal{F}_{n-1}$-measurable, $n \geq 1$.

(iv) This property is obtained by construction in (S.9.4).

2. For all bounded stopping times σ and τ such that $\sigma \leq \tau$ a.s., we have

$$\begin{aligned}\mathbb{E}[M_\sigma] &= \mathbb{E}[N_\sigma] + \mathbb{E}[A_\sigma] \\ &\leq \mathbb{E}[N_\sigma] + \mathbb{E}[A_\tau] \\ &= \mathbb{E}[N_\tau] + \mathbb{E}[A_\tau] \\ &= \mathbb{E}[M_\tau],\end{aligned}$$

by (9.11), since $(M_n)_{n\in\mathbb{N}}$ is a martingale and $(A_n)_{n\in\mathbb{N}}$ is non-decreasing.

Exercise 9.3 American digital options.

1. The optimal strategy is as follows:

   (i) if $S_t \geq K$, then exercise immediately.
   (ii) if $S_t < K$, then wait.

2. The optimal strategy is as follows:

   (i) if $S_t > K$, then wait.
   (ii) if $S_t \leq K$, exercise immediately.

3. Based on the answers to Question 1 we set

$$C_d^{\mathrm{Am}}(t, K) = 1, \qquad 0 \leq t < T,$$

and

$$C_d^{\mathrm{Am}}(T, x) = 0, \qquad 0 \leq x < K.$$

4. Based on the answers to Question 2, we set

$$P_d^{\mathrm{Am}}(t, K) = 1, \qquad 0 \leq t < T,$$

and

$$P_d^{\mathrm{Am}}(T, x) = 0, \qquad x > K.$$

5. Starting from $S_t \leq K$, the maximum possible payoff is clearly reached as soon as S_t hits the level K before the expiration date T, hence the discounted optimal payoff of the option is $e^{-r(\tau_K - t)}\mathbf{1}_{\{\tau_K < T\}}$.
6. From Relation (8.7) we find

$$\mathbb{P}(\tau_a \leq u) = \Phi\left(\frac{a-\mu u}{\sqrt{u}}\right) - e^{2\mu a}\Phi\left(\frac{-a-\mu u}{\sqrt{u}}\right),$$

and by differentiation with respect to u this yields the probability density function

$$f_{\tau_a}(u) = \frac{\partial}{\partial u}\mathbb{P}(\tau_a \leq u) = \frac{a}{\sqrt{2\pi u^3}}e^{-\frac{(a-\mu u)^2}{2u}}$$

of the first hitting time of level a by Brownian motion with drift μ. Given the relation

$$S_u = S_t e^{\sigma(B_u - B_t) - \sigma^2(u-t)/2 + \mu(u-t)}, \qquad u \geq t,$$

we find that the probability density function of the first hitting time of level K after time t by $(S_u)_{u\in[t,\infty)}$ is given by

$$u \longmapsto \frac{a}{\sqrt{2\pi(u-t)^3}}e^{-\frac{(a-\mu(u-t))^2}{2(u-t)}}, \qquad u \geq t,$$

with $\mu = \sigma^{-1}(r - \sigma^2/2)$ and

$$a = \frac{1}{\sigma}\log\frac{K}{x},$$

given that $S_t = x$. Hence for $x \in (0, K)$ we have, letting $\tau = T - t$,

$$\begin{aligned}
C_d^{\mathrm{Am}}(t,x) &= \mathbb{E}[e^{-r(\tau_K - t)}\mathbf{1}_{\{\tau_K < T\}} \mid S_t = x] \\
&= \int_t^T e^{-r(s-t)}\frac{a}{\sqrt{2\pi(s-t)^3}}e^{-\frac{(a-\mu(s-t))^2}{2(s-t)}}ds \\
&= \int_0^\tau e^{-rs}\frac{a}{\sqrt{2\pi s^3}}e^{-\frac{(a-\mu s)^2}{2s}}ds \\
&= \int_0^\tau \frac{\log(K/x)}{\sigma\sqrt{2\pi s^3}}\exp\left(-rs - \frac{1}{2\sigma^2 s}\left(-\left(r - \frac{\sigma^2}{2}\right)s + \log\frac{K}{x}\right)^2\right)ds \\
&= \left(\frac{K}{x}\right)^{\left(\frac{r}{\sigma^2}-\frac{1}{2}\right)\pm\left(\frac{r}{\sigma^2}+\frac{1}{2}\right)} \\
&\quad \times \int_0^\tau \frac{\log(K/x)}{\sigma\sqrt{2\pi s^3}}\exp\left(-\frac{1}{2\sigma^2 s}\left(\pm\left(r + \frac{\sigma^2}{2}\right)s + \log\frac{K}{x}\right)^2\right)ds \\
&= \frac{1}{\sqrt{2\pi}}\frac{x}{K}\int_{y_-}^\infty e^{-y^2/2}dy + \frac{1}{\sqrt{2\pi}}\left(\frac{K}{x}\right)^{2r/\sigma^2}\int_{y_+}^\infty e^{-y^2/2}dy \\
&= \frac{x}{K}\Phi\left(\frac{(r+\sigma^2/2)\tau + \log(x/K)}{\sigma\sqrt{\tau}}\right) \\
&\quad + \left(\frac{x}{K}\right)^{-2r/\sigma^2}\Phi\left(\frac{-(r+\sigma^2/2)\tau + \log(x/K)}{\sigma\sqrt{\tau}}\right), \qquad 0 < x < K,
\end{aligned}$$

where

$$y_\pm = \frac{1}{\sigma\sqrt{\tau}}\left(\pm\left(r+\frac{\sigma^2}{2}\right)\tau + \log\frac{K}{x}\right),$$

and used the decomposition

$$\log\frac{K}{x} = \frac{1}{2}\left(\left(r+\frac{\sigma^2}{2}\right)s + \log\frac{K}{x}\right) + \frac{1}{2}\left(-\left(r+\frac{\sigma^2}{2}\right)s + \log\frac{K}{x}\right).$$

We check that

$$C_d^{\mathrm{Am}}(t,K) = \Phi(\infty) + \Phi(-\infty) = 1,$$

and

$$C_d^{\mathrm{Am}}(T,x) = \frac{x}{K}\Phi(-\infty) + \left(\frac{x}{K}\right)^{-2r/\sigma^2}\Phi(-\infty) = 0, \qquad x < K,$$

since $\tau = 0$, which is consistent with the answers to Question 3.

7. Starting from $S_t \geq K$, the maximum possible payoff is clearly reached as soon as S_t hits the level K before the expiration date T, hence the discounted optimal payoff of the option is $e^{-r(\tau_K - t)}\mathbf{1}_{\{\tau_K < T\}}$.
8. Using the notation and answer to Question 6, for $x > K$ we find, letting $\tau = T - t$,

$$\begin{aligned}
P_d^{\mathrm{Am}}(t,x) &= \mathbb{E}[e^{-r(\tau_K - t)}\mathbf{1}_{\{\tau_K<T\}} \mid S_t = x] \\
&= \int_0^\tau e^{-rs}\frac{a}{\sqrt{2\pi s^3}}e^{-\frac{(a-\mu s)^2}{2s}}ds \\
&= \int_0^\tau \frac{\log(x/K)}{\sigma\sqrt{2\pi s^3}}\exp\left(-rs - \frac{1}{2\sigma^2 s}\left(\left(r-\frac{\sigma^2}{2}\right)s + \log\frac{x}{K}\right)^2\right)ds \\
&= \left(\frac{K}{x}\right)^{\left(\frac{r}{\sigma^2}-\frac{1}{2}\right)\pm\left(\frac{r}{\sigma^2}+\frac{1}{2}\right)} \\
&\quad \times \int_0^\tau \frac{\log(x/K)}{\sigma\sqrt{2\pi s^3}}\exp\left(-\frac{1}{2\sigma^2 s}\left(\mp\left(r+\frac{\sigma^2}{2}\right)s + \log\frac{x}{K}\right)^2\right)ds \\
&= \frac{1}{\sqrt{2\pi}}\frac{x}{K}\int_{y_-}^\infty e^{-y^2/2}dy + \frac{1}{\sqrt{2\pi}}\left(\frac{x}{K}\right)^{2r/\sigma^2}\int_{y_+}^\infty e^{-y^2/2}dy \\
&= \frac{x}{K}\Phi\left(\frac{-(r+\sigma^2/2)\tau - \log(x/K)}{\sigma\sqrt{\tau}}\right) \\
&\quad + \left(\frac{x}{K}\right)^{-2r/\sigma^2}\Phi\left(\frac{(r+\sigma^2/2)\tau - \log(x/K)}{\sigma\sqrt{\tau}}\right), \qquad x > K,
\end{aligned}$$

with

$$y_\pm = \frac{1}{\sigma\sqrt{\tau}}\left(\mp\left(r+\frac{\sigma^2}{2}\right)\tau + \log\frac{x}{K}\right),$$

We check that

$$P_d^{\mathrm{Am}}(t,K) = \Phi(-\infty) + \Phi(\infty) = 1,$$

and

$$P_d^{\mathrm{Am}}(T,x) = \frac{x}{K}\Phi(-\infty) + \left(\frac{x}{K}\right)^{-2r/\sigma^2} \Phi(-\infty) = 0, \qquad 0 < x < K,$$

since $\tau = 0$, which is consistent with the answers to Question 3.

9. The call-put parity does not hold for American digital options since for $x \in (0,K)$ we have

$$\begin{aligned} C_d^{\mathrm{Am}}(t,x) + P_d^{\mathrm{Am}}(t,x) &= 1 + \frac{x}{K}\Phi\left(\frac{(r+\sigma^2/2)\tau + \log(x/K)}{\sigma\sqrt{\tau}}\right) \\ &\quad + \left(\frac{x}{K}\right)^{-2r/\sigma^2} \Phi\left(\frac{-(r+\sigma^2/2)\tau + \log(x/K)}{\sigma\sqrt{\tau}}\right), \end{aligned}$$

while for $x > K$ we find

$$\begin{aligned} C_d^{\mathrm{Am}}(t,x) + P_d^{\mathrm{Am}}(t,x) &= 1 + \frac{x}{K}\Phi\left(\frac{-(r+\sigma^2/2)\tau - \log(x/K)}{\sigma\sqrt{\tau}}\right) \\ &\quad + \left(\frac{x}{K}\right)^{-2r/\sigma^2} \Phi\left(\frac{(r+\sigma^2/2)\tau - \log(x/K)}{\sigma\sqrt{\tau}}\right). \end{aligned}$$

Exercise 9.4 American forward contracts. Consider $(S_t)_{t\in\mathbb{R}_+}$ an asset price process given by

$$\frac{dS_t}{S_t} = rdt + \sigma dB_t,$$

where $(B_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion.

1. For all stopping times τ such that $t \le \tau \le T$ we have

$$\begin{aligned} \mathbb{E}^*\left[e^{-r(\tau-t)}(K - S_\tau)\Big|S_t\right] &= K\,\mathbb{E}^*\left[e^{-r(\tau-t)}\Big|S_t\right] - \mathbb{E}^*\left[e^{-r(\tau-t)}S_\tau\Big|S_t\right] \\ &= e^{-r(\tau-t)}K - S_t, \end{aligned}$$

since $\tau \in [t,T]$ is bounded and $(e^{-rt}S_t)_{t\in\mathbb{R}_+}$ is a martingale, and the above quantity is clearly maximized by taking $\tau = t$. Hence we have

$$f(t,S_t) = \sup_{\substack{t\le\tau\le T \\ \tau \text{ stopping time}}} \mathbb{E}^*\left[e^{-r(\tau-t)}(K - S_\tau)\Big|S_t\right] = K - S_t,$$

and the optimal strategy is to exercise immediately at time t.

2. Similarly we have

$$\begin{aligned} \mathbb{E}^*\left[e^{-r(\tau-t)}(S_\tau - K)\Big|S_t\right] &= \mathbb{E}^*\left[e^{-r(\tau-t)}S_\tau\Big|S_t\right] - K\,\mathbb{E}^*\left[e^{-r(\tau-t)}\Big|S_t\right] \\ &= S_t - K\,\mathbb{E}^*\left[e^{-r(\tau-t)}\Big|S_t\right], \end{aligned}$$

since $\tau \in [t, T]$ is bounded and $(e^{-rt}S_t)_{t\in\mathbb{R}_+}$ is a martingale, and the above quantity is clearly maximized by taking $\tau = T$. Hence we have

$$f(t, S_t) = \sup_{\substack{t\leq\tau\leq T \\ \tau \text{ stopping time}}} \mathbb{E}^*\left[e^{-r(\tau-t)}(S_\tau - K)\Big|S_t\right] = S_t - e^{-r(T-t)}K,$$

and the optimal strategy is to exercise at time T.

3. Concerning the perpetual American call forward contract, since $u \mapsto e^{-r(u-t)}S_u$ is a martingale, for all stopping times τ we have*

$$\begin{aligned}\mathbb{E}^*\left[e^{-r(\tau-t)}(S_\tau - K)\Big|S_t\right] &= \mathbb{E}^*\left[e^{-r(\tau-t)}S_\tau\Big|S_t\right] - K\,\mathbb{E}^*\left[e^{-r(\tau-t)}\Big|S_t\right] \\ &\leq S_t - K\,\mathbb{E}^*\left[e^{-r(\tau-t)}\Big|S_t\right] \\ &\leq S_t, \qquad t \geq 0.\end{aligned}$$

On the other hand, for all fixed $T > 0$ we have

$$\begin{aligned}\mathbb{E}^*\left[e^{-r(T-t)}(S_T - K)\Big|S_t\right] &= \mathbb{E}^*\left[e^{-r(T-t)}S_T\Big|S_t\right] - K\,\mathbb{E}^*\left[e^{-r(T-t)}\Big|S_t\right] \\ &= S_t - e^{-r(T-t)}K, \qquad t \geq 0,\end{aligned}$$

hence

$$\sup_{\substack{\tau\geq t \\ \tau \text{ stopping time}}} \mathbb{E}^*\left[e^{-r(\tau-t)}(S_\tau - K)\Big|S_t\right] \geq \left(S_t - e^{-r(T-t)}K\right), \qquad T \geq t,$$

and letting $T \to \infty$ we get

$$\begin{aligned}\sup_{\substack{\tau\geq t \\ \tau \text{ stopping time}}} \mathbb{E}^*\left[e^{-r(\tau-t)}(S_\tau - K)\Big|S_t\right] &\geq \lim_{T\to\infty}\left(S_t - e^{-r(T-t)}K\right) \\ &= S_t,\end{aligned}$$

hence we have

$$f(t, S_t) = \sup_{\substack{t\leq\tau\leq T \\ \tau \text{ stopping time}}} \mathbb{E}^*\left[e^{-r(\tau-t)}(S_\tau - K)\Big|S_t\right] = S_t,$$

and the optimal strategy $\tau^* = +\infty$ is to wait indefinitely.

Concerning the perpetual American put forward contract we have

$$\begin{aligned}f(t, S_t) &= \sup_{\substack{t\leq\tau\leq T \\ \tau \text{ stopping time}}} \mathbb{E}^*\left[e^{-r(\tau-t)}(K - S_\tau)\Big|S_t\right] \\ &\leq \sup_{\substack{t\leq\tau\leq T \\ \tau \text{ stopping time}}} \mathbb{E}^*\left[e^{-r(\tau-t)}(K - S_\tau)^+\Big|S_t\right]\end{aligned}$$

* by Fatou's Lemma.

$$= f_{L^*}(S_t).$$

On the other hand, for $\tau = \tau_{L^*}$ we have

$$(K - S_{\tau_{L^*}}) = (K - L^*) = (K - L^*)^*$$

since $0 < L^* = 2Kr/(2r + \sigma^2) < K$, hence

$$\begin{aligned}
f_{L^*}(S_t) &= \mathbb{E}^*\left[e^{-r(\tau-t)}(K - S_{\tau_{L^*}})^+ \Big| S_t\right] \\
&= \mathbb{E}^*\left[e^{-r(\tau-t)}(K - S_{\tau_{L^*}}) \Big| S_t\right] \\
&\le \sup_{\substack{t\le\tau\le T \\ \tau \text{ stopping time}}} \mathbb{E}^*\left[e^{-r(\tau-t)}(K - S_\tau) \Big| S_t\right] \\
&= f(t, S_t),
\end{aligned}$$

which shows that

$$f(t, S_t) = f_{L^*}(S_t),$$

i.e. the perpetual American put forward contract has same price and exercise strategy as the perpetual American put option.

Exercise 9.5

1. The option payoff equals $(\kappa - S_t)^p$ if $S_t \le L$.
2. We have

$$\begin{aligned}
f_L(S_t) &= \mathbb{E}^*\left[e^{-r(\tau_L-t)}((\kappa - S_{\tau_L})^+)^p \Big| S_t\right] \\
&= \mathbb{E}^*\left[e^{-r(\tau_L-t)}((\kappa - L)^+)^p \Big| S_t\right] \\
&= (\kappa - L)^p\, \mathbb{E}^*\left[e^{-r(\tau_L-t)} \Big| S_t\right].
\end{aligned}$$

3. We have

$$\begin{aligned}
f_L(x) &= \mathbb{E}^*\left[e^{-r(\tau_L-t)}(\kappa - S_{\tau_L})^+ \Big| S_t = x\right] \\
&= \begin{cases} (\kappa - x)^p, & 0 < x \le L, \\ \\ (\kappa - L)^p \left(\dfrac{x}{L}\right)^{-2r/\sigma^2}, & x \ge L. \end{cases}
\end{aligned} \tag{S.9.5}$$

4. By differentiating $\frac{d}{dx}(\kappa - x)^p = -p(\kappa - x)^{p-1}$ we find

$$f'_{L^*}(L^*) = -\frac{2r}{\sigma^2}(\kappa - L^*)^p \frac{(L^*)^{-2r/\sigma^2-1}}{(L^*)^{-2r/\sigma^2}} = -p(\kappa - L^*)^{p-1},$$

i.e.

$$\frac{2r}{\sigma^2}(\kappa - L^*) = pL^*,$$

or

$$L^* = \frac{2r}{2r + p\sigma^2}\kappa < \kappa.$$

5. By (S.9.5) the price can be computed as

$$f(t, S_t) = f_{L^*}(S_t) = \begin{cases} (\kappa - S_t)^p, & 0 < S_t \le L^*, \\ \\ \left(\frac{p\sigma^2\kappa}{2r + p\sigma^2}\right)^p \left(\frac{2r + p\sigma^2}{2r}\frac{S_t}{\kappa}\right)^{-2r/\sigma^2}, & S_t \ge L^*. \end{cases}$$

Exercise 9.6

1. The payoff will be $\kappa - (S_t)^p$.
2. We have

$$\begin{aligned} f_L(S_t) &= \mathbb{E}^*\left[e^{-r(\tau_L - t)}(\kappa - (S_{\tau_L})^p)\Big|S_t\right] \\ &= \mathbb{E}^*\left[e^{-r(\tau_L - t)}(\kappa - L^p)\Big|S_t\right] \\ &= (\kappa - L^p)\mathbb{E}^*\left[e^{-r(\tau_L - t)}\Big|S_t\right]. \end{aligned}$$

3. We have

$$f_L(x) = \mathbb{E}^*\left[e^{-r(\tau_L - t)}(\kappa - (S_{\tau_L})^p)\Big|S_t = x\right]$$

$$= \begin{cases} \kappa - x^p, & 0 < x \le L, \\ \\ (\kappa - L^p)\left(\frac{x}{L}\right)^{-2r/\sigma^2}, & x \ge L. \end{cases}$$

4. We have

$$f'_{L^*}(L^*) = -\frac{2r}{\sigma^2}(\kappa - (L^*)^p)\frac{(L^*)^{-2r/\sigma^2 - 1}}{(L^*)^{-2r/\sigma^2}} = -p(L^*)^{p-1},$$

i.e.

$$\frac{2r}{\sigma^2}(\kappa - (L^*)^p) = p(L^*)^p,$$

or

$$L^* = \left(\frac{2r\kappa}{2r + p\sigma^2}\right)^{1/p} < (\kappa)^{1/p}. \tag{S.9.6}$$

<u>Remark</u>: We may also compute L^* by maximizing $L \mapsto f_L(x)$ for all fixed x. The derivative $\partial f_L(x)/\partial L$ can be computed as

$$\frac{\partial f_L(x)}{\partial L} = \frac{\partial}{\partial L}\left((\kappa - L^p)\left(\frac{L}{x}\right)^{2r/\sigma^2}\right)$$

$$= -pL^{p-1}\left(\frac{L}{x}\right)^{2r/\sigma^2} + \frac{2r}{\sigma^2}L^{-1}(\kappa - L^p)\left(\frac{L}{x}\right)^{2r/\sigma^2},$$

and equating $\partial f_L(x)/\partial L$ to 0 at $L = L^*$ yields

$$-p(L^*)^{p-1} + \frac{2r}{\sigma^2}(L^*)^{-1}(\kappa - (L^*)^p) = 0,$$

which recovers (S.9.6).

5. The price can be computed as

$$f(t, S_t) = f_{L^*}(S_t) = \begin{cases} \kappa - (S_t)^p, & 0 < S_t \leq L^*, \\ \\ (\kappa - (L^*)^p)\dfrac{(S_t)^{-2r/\sigma^2}}{(L^*)^{-2r/\sigma^2}}, & S_t \geq L^* \end{cases}$$

$$= \begin{cases} \kappa - (S_t)^p, & 0 < S_t \leq L^*, \\ \\ \dfrac{\sigma^2}{2r}p(S_t)^{-2r/\sigma^2}(L^*)^{p+2r/\sigma^2}, & S_t \geq L^*, \end{cases}$$

$$= \begin{cases} \kappa - (S_t)^p, & 0 < S_t \leq L^*, \\ \\ \dfrac{p\sigma^2\kappa}{2r + p\sigma^2}\left(\dfrac{2r + p\sigma^2}{2r}\dfrac{S_t^p}{\kappa}\right)^{-2r/(p\sigma^2)} < \kappa, & S_t \geq L^*. \end{cases}$$

Exercise 9.7

1. We have that

$$Z_t := \left(\frac{S_t}{S_0}\right)^{\lambda} e^{-(r-a)\lambda t + \lambda\sigma^2 t/2 - \lambda^2\sigma^2 t/2} = e^{\lambda\sigma\tilde{B}_t - \lambda^2\sigma^2 t/2}, \qquad t \in \mathbb{R}_+,$$

is a geometric Brownian motion without drift under the risk-neutral probability measure $\mathbb{P}^*$, hence it is a martingale.

2. By the stopping time theorem we have

$$\mathbb{E}^*[Z_{\tau_L}] = \mathbb{E}^*[Z_0] = 1,$$

which rewrites as

$$\mathbb{E}^*\left[\left(\frac{S_{\tau_L}}{S_0}\right)^{\lambda} e^{-((r-a)\lambda - \lambda\sigma^2/2 + \lambda^2\sigma^2/2)\tau_L}\right] = 1,$$

or, given the relation $S_{\tau_L} = L$,

$$\left(\frac{L}{S_0}\right)^{\lambda}\mathbb{E}^*\left[e^{-((r-a)\lambda - \lambda\sigma^2/2 + \lambda^2\sigma^2/2)\tau_L}\right] = 1,$$

i.e.

$$\mathbb{E}^*\left[e^{-r\tau_L}\right] = \left(\frac{S_0}{L}\right)^\lambda,$$

provided we choose λ such that

$$-((r-a)\lambda - \lambda\sigma^2/2 + \lambda^2\sigma^2/2) = -r, \tag{S.9.7}$$

i.e.

$$0 = \lambda^2\sigma^2/2 + \lambda(r-a-\sigma^2/2) - r.$$

This equation admits two solutions

$$\lambda = \frac{-(r-a-\sigma^2/2) \pm \sqrt{(r-a-\sigma^2/2)^2 + 4r\sigma^2/2}}{\sigma^2},$$

and we choose the negative solution

$$\lambda = \frac{-(r-a-\sigma^2/2) - \sqrt{(r-a-\sigma^2/2)^2 + 4r\sigma^2/2}}{\sigma^2}$$

since $S_0/L = x/L > 1$ and the expectation $\mathbb{E}^*[e^{-r\tau_L}] < 1$ is lower than 1 as $r \geq 0$.

3. Noting that $\tau_L = 0$ if $S_0 \leq L$, for all $L \in (0, K)$ we have

$$\begin{aligned}
&\mathbb{E}^*\left[e^{-r\tau_L}(K - S_{\tau_L})^+ \Big| S_0 = x\right] \\
&= \begin{cases} K - x, & 0 < x \leq L, \\ E\left[e^{-r\tau_L}(K-L)^+ \Big| S_0 = x\right], & x \geq L. \end{cases} \\
&= \begin{cases} K - x, & 0 < x \leq L, \\ (K-L)E\left[e^{-r\tau_L} \Big| S_0 = x\right], & x \geq L. \end{cases} \\
&= \begin{cases} K - x, & 0 < x \leq L, \\ (K-L)\left(\frac{x}{L}\right)^{\frac{-(r-a-\sigma^2/2)-\sqrt{(r-a-\sigma^2/2)^2+4r\sigma^2/2}}{\sigma^2}}, & x \geq L. \end{cases}
\end{aligned}$$

4. In order to compute L^* we observe that, geometrically, the slope of $f_L(x)$ at $x = L^*$ is equal to -1, *i.e.*

$$f'_{L^*}(L^*) = \lambda(K - L^*)\frac{(L^*)^{\lambda-1}}{(L^*)^\lambda} = -1,$$

or

$$\lambda(K - L^*) = L^*,$$

or

$$L^* = \frac{\lambda}{\lambda - 1} K < K.$$

5. For $x \geq L$ we have

$$\begin{aligned} f_{L^*}(x) &= (K - L^*) \left(\frac{x}{L^*} \right)^\lambda \\ &= \left(K - \frac{\lambda}{\lambda - 1} K \right) \left(\frac{x}{\frac{\lambda}{\lambda-1} K} \right)^\lambda \\ &= \left(- \frac{K}{\lambda - 1} \right) \left(\frac{x(\lambda - 1)}{\lambda K} \right)^\lambda \\ &= \left(- \frac{K}{\lambda - 1} \right) \left(\frac{x}{-\lambda} \right)^\lambda \left(\frac{\lambda - 1}{-K} \right)^\lambda \\ &= \left(\frac{x}{-\lambda} \right)^\lambda \left(\frac{\lambda - 1}{-K} \right)^{\lambda - 1} \\ &= \left(\frac{x}{K} \right)^\lambda \left(\frac{\lambda - 1}{\lambda} \right)^\lambda \frac{K}{1 - \lambda}. \end{aligned} \tag{S.9.8}$$

6. Let us check that the relation

$$f_{L^*}(x) \geq (K - x)^+ \tag{S.9.9}$$

holds. For all $x \leq K$ we have

$$\begin{aligned} f_{L^*}(x) - (K - x) &= \left(\frac{x}{K} \right)^\lambda \left(\frac{\lambda - 1}{\lambda} \right)^\lambda \frac{K}{1 - \lambda} + x - K \\ &= K \left(\left(\frac{x}{K} \right)^\lambda \left(\frac{\lambda - 1}{\lambda} \right)^\lambda \frac{1}{1 - \lambda} + \frac{x}{K} - 1 \right). \end{aligned}$$

Hence it suffices to take $K = 1$ and to show that for all

$$L^* = \frac{\lambda}{\lambda - 1} \leq x \leq 1$$

we have

$$\begin{aligned} f_{L^*}(x) - (1 - x) &= \frac{x^\lambda}{1 - \lambda} \left(\frac{\lambda - 1}{\lambda} \right)^\lambda + x - 1 \\ &\geq 0. \end{aligned}$$

Equality to 0 holds for $x = \lambda/(\lambda - 1)$. By differentiation of this relation we get

$$\begin{aligned} f'_{L^*}(x) - (1-x)' &= \lambda x^{\lambda-1} \left(\frac{\lambda-1}{\lambda}\right)^{\lambda} \frac{1}{1-\lambda} + 1 \\ &= x^{\lambda-1} \left(\frac{\lambda-1}{\lambda}\right)^{\lambda-1} + 1 \\ &\geq 0, \end{aligned}$$

hence the function $f_{L^*}(x) - (1-x)$ is non-decreasing and the inequality holds throughout the interval $[\lambda/(\lambda-1), K]$.

On the other hand, using (S.9.7) it can be checked by hand that f_{L^*} given by (S.9.8) satisfies the equality

$$(r-a)xf'_{L^*}(x) + \frac{1}{2}\sigma^2 x^2 f''_{L^*}(x) = rf_{L^*}(x) \tag{S.9.10}$$

for $x \geq L^* = \dfrac{\lambda}{\lambda-1}K$. In case

$$0 \leq x \leq L^* = \frac{\lambda}{\lambda-1}K < K,$$

we have

$$f_{L^*}(x) = K - x = (K-x)^+,$$

hence the relation

$$\left(rf_{L^*}(x) - (r-a)xf'_{L^*}(x) - \frac{1}{2}\sigma^2 x^2 f''_{L^*}(x)\right)(f_{L^*}(x) - (K-x)^+) = 0$$

always holds. On the other hand, in that case we also have

$$(r-a)xf'_{L^*}(x) + \frac{1}{2}\sigma^2 x^2 f''_{L^*}(x) = -(r-a)x,$$

and to conclude we need to show that

$$(r-a)xf'_{L^*}(x) + \frac{1}{2}\sigma^2 x^2 f''_{L^*}(x) \leq rf_{L^*}(x) = r(K-x), \tag{S.9.11}$$

which is true if

$$ax \leq rK.$$

Indeed by (S.9.7) we have

$$\begin{aligned} (r-a)\lambda &= r + \lambda(\lambda-1)\sigma^2/2 \\ &\geq r, \end{aligned}$$

hence

$$a\frac{\lambda}{\lambda-1} \leq r,$$

since $\lambda < 0$, which yields

$$ax \leq aL^* \leq a\frac{\lambda}{\lambda - 1}K \leq rK.$$

7. By Itô's formula and the relation

$$dS_t = (r-a)S_t dt + \sigma S_t d\tilde{B}_t$$

we have

$$\begin{aligned} d(\tilde{f}_{L^*}(S_t)) &= -re^{-rt} f_{L^*}(S_t)dt + e^{-rt} df_{L^*}(S_t) \\ &= -re^{-rt} f_{L^*}(S_t)dt + e^{-rt} f'_{L^*}(S_t)dS_t + \frac{1}{2}e^{-rt}\sigma^2 S_t^2 f''_{L^*}(S_t) \\ &= e^{-rt}\left(-rf_{L^*}(S_t) + (r-a)S_t f'_{L^*}(S_t) + \frac{1}{2}\sigma^2 S_t^2 f''_{L^*}(S_t)\right) dt \\ &\quad + e^{-rt}\sigma S_t f'_{L^*}(S_t) d\tilde{B}_t, \end{aligned}$$

and from Equations (S.9.10) and (S.9.11) we have

$$(r-a)xf'_{L^*}(x) + \frac{1}{2}\sigma^2 x^2 f''_{L^*}(x) \leq rf_{L^*}(x),$$

hence

$$t \mapsto e^{-rt} f_{L^*}(S_t)$$

is a supermartingale.

8. By the supermartingale property of

$$t \mapsto e^{-rt} f_{L^*}(S_t),$$

for all stopping times τ we have

$$f_{L^*}(S_0) \geq \mathbb{E}^*\left[e^{-r\tau} f_{L^*}(S_\tau)\Big| S_0\right] \geq \mathbb{E}^*\left[e^{-r\tau}(K - S_\tau)^+ \Big| S_0\right],$$

by (S.9.9), hence

$$f_{L^*}(S_0) \geq \sup_{\tau \text{ stopping time}} \mathbb{E}^*\left[e^{-r\tau}(K - S_\tau)^+ \Big| S_0\right]. \qquad \text{(S.9.12)}$$

9. The stopped process

$$t \mapsto e^{-rt\wedge\tau_{L^*}} f_{L^*}(S_{t\wedge\tau_{L^*}})$$

is a martingale since it has vanishing drift up to time τ_{L^*} by (S.9.10), and it is constant after time τ_{L^*}, hence by the martingale stopping time Theorem (9.1) we find

$$
\begin{aligned}
f_{L^*}(S_0) &= \mathbb{E}^*\left[e^{-r\tau} f_{L^*}(S_{\tau_{L^*}})\Big| S_0\right] \\
&= \mathbb{E}^*\left[e^{-r\tau} f_{L^*}(L^*)\Big| S_0\right] \\
&= \mathbb{E}^*\left[e^{-r\tau} (K - S_{\tau_{L^*}})^+\Big| S_0\right] \\
&\leq \sup_{\tau \text{ stopping time}} \mathbb{E}^*\left[e^{-r\tau} (K - S_\tau)^+\Big| S_0\right].
\end{aligned}
$$

10. By combining the above results and conditioning at time t instead of time 0 we deduce that

$$
\begin{aligned}
f_{L^*}(S_t) &= \mathbb{E}^*\left[e^{-r(\tau_{L^*}-t)}(K - S_{\tau_{L^*}})^+\Big| S_t\right] \\
&= \begin{cases} K - S_t, & 0 < S_t \leq \dfrac{\lambda}{\lambda-1}K, \\[2ex] \left(\dfrac{\lambda-1}{-K}\right)^{\lambda-1}\left(\dfrac{-S_t}{\lambda}\right)^{\lambda}, & S_t \geq \dfrac{\lambda}{\lambda-1}K, \end{cases}
\end{aligned}
$$

for all $t \in \mathbb{R}_+$, where

$$
\tau_{L^*} = \inf\{u \geq t \ : \ S_u \leq L\}.
$$

We note that the perpetual put option price does not depend on the value of $t \geq 0$.

Exercise 9.8

1. By the definition (9.36) of $S_1(t)$ and $S_2(t)$ we have

$$
\begin{aligned}
Z_t &= e^{-rt} S_2(t) \left(\frac{S_1(t)}{S_2(t)}\right)^{\alpha} \\
&= e^{-rt} S_1(t)^{\alpha} S_2(t)^{1-\alpha} \\
&= S_1(0)^{\alpha} S_2(0)^{1-\alpha} e^{(\alpha\sigma_1 + (1-\alpha)\sigma_2)W_t - \sigma_2^2 t/2},
\end{aligned}
$$

which is a martingale when

$$
\sigma_2^2 = (\alpha\sigma_1 + (1-\alpha)\sigma_2)^2,
$$

i.e.

$$
\alpha\sigma_1 + (1-\alpha)\sigma_2 = \pm\sigma_2,
$$

which yields either $\alpha = 0$ or

$$
\alpha = \frac{2\sigma_2}{\sigma_2 - \sigma_1} > 1,
$$

since $0 \leq \sigma_1 < \sigma_2$.

2. We have

$$\begin{aligned}\mathbb{E}[e^{-r\tau_L}(S_1(\tau_L)-S_2(\tau_L))^+] &= \mathbb{E}[e^{-r\tau_L}(LS_2(\tau_L)-S_2(\tau_L))^+]\\ &= (L-1)^+\,\mathbb{E}[e^{-r\tau_L}S_2(\tau_L)]. \quad \text{(S.9.13)}\end{aligned}$$

3. Since $\tau_L \wedge t$ is a bounded stopping time we can write

$$S_2(0)\left(\frac{S_1(0)}{S_2(0)}\right)^\alpha = \mathbb{E}\left[e^{-r(\tau_L\wedge t)}S_2(\tau_L\wedge t)\left(\frac{S_1(\tau_L\wedge t)}{S_2(\tau_L\wedge t)}\right)^\alpha\right] \quad \text{(S.9.14)}$$

$$= \mathbb{E}\left[e^{-r\tau_L}S_2(\tau_L)\left(\frac{S_1(\tau_L)}{S_2(\tau_L)}\right)^\alpha \mathbf{1}_{\{\tau_L<t\}}\right] + \mathbb{E}\left[e^{-rt}S_2(t)\left(\frac{S_1(t)}{S_2(t)}\right)^\alpha \mathbf{1}_{\{\tau_L>t\}}\right]$$

We have

$$e^{-rt}S_2(t)\left(\frac{S_1(t)}{S_2(t)}\right)^\alpha \mathbf{1}_{\{\tau_L>t\}} \leq e^{-rt}S_2(t)L^\alpha\mathbf{1}_{\{\tau_L>t\}} \leq e^{-rt}S_2(t)L^\alpha,$$

hence by a uniform integrability argument,

$$\lim_{t\to\infty}\mathbb{E}\left[e^{-rt}S_2(t)\left(\frac{S_1(t)}{S_2(t)}\right)^\alpha \mathbf{1}_{\{\tau_L>t\}}\right] = 0,$$

and letting t go to infinity in (S.9.14) shows that

$$S_2(0)\left(\frac{S_1(0)}{S_2(0)}\right)^\alpha = \mathbb{E}\left[e^{-r\tau_L}S_2(\tau_L)\left(\frac{S_1(\tau_L)}{S_2(\tau_L)}\right)^\alpha\right] = L^\alpha\,\mathbb{E}\left[e^{-r\tau_L}S_2(\tau_L)\right],$$

since $S_1(\tau_L)/S_2(\tau_L) = L/L = 1$. The conclusion

$$\mathbb{E}[e^{-r\tau_L}(S_1(\tau_L)-S_2(\tau_L))^+] = (L-1)^+L^{-\alpha}S_2(0)\left(\frac{S_1(0)}{S_2(0)}\right)^\alpha \quad \text{(S.9.15)}$$

then follows by an application of (S.9.13).

4. In order to maximize (S.9.15) as a function of L we consider the derivative

$$\frac{\partial}{\partial L}\frac{L-1}{L^\alpha} = \frac{1}{L^\alpha} - \alpha(L-1)L^{-\alpha-1} = 0,$$

which vanishes for

$$L^* = \frac{\alpha}{\alpha-1},$$

and we substitute L in (S.9.15) with the value of L^*.

5. In addition to $r = \sigma_2^2/2$ it is sufficient to let $S_1(0) = \kappa$ and $\sigma_1 = 0$ which yields $\alpha = 2$, $L^* = 2$, and we find

$$\sup_{\tau \text{ stopping time}} \mathbb{E}[e^{-r\tau}(\kappa - S_2(\tau))^+] = \frac{1}{S_2(0)}\left(\frac{\kappa}{2}\right)^2,$$

which coincides with the result of Proposition 9.4.

Chapter 10

Exercise 10.1

1. We have

$$\begin{aligned}
d\hat{X}_t &= d\left(\frac{X_t}{N_t}\right) = \frac{X_0}{N_0} d\left(e^{(\sigma-\eta)B_t-(\sigma^2-\eta^2)t/2}\right) \\
&= \frac{X_0}{N_0}(\sigma-\eta)e^{(\sigma-\eta)B_t-(\sigma^2-\eta^2)t/2}dB_t + \frac{X_0}{2N_0}(\sigma-\eta)^2 e^{(\sigma-\eta)B_t-(\sigma^2-\eta^2)t/2}dt \\
&\quad -\frac{X_0}{2N_0}(\sigma^2-\eta^2)e^{(\sigma-\eta)B_t-(\sigma^2-\eta^2)t/2}dt \\
&= -\frac{X_t}{2N_t}(\sigma^2-\eta^2)dt + \frac{X_t}{N_t}(\sigma-\eta)dB_t + \frac{X_t}{2N_t}(\sigma-\eta)^2 dt \\
&= -\frac{X_t}{N_t}\eta(\sigma-\eta)dt + \frac{X_t}{N_t}(\sigma-\eta)dB_t \\
&= \frac{X_t}{N_t}(\sigma-\eta)(dB_t - \eta dt) \\
&= (\sigma-\eta)\frac{X_t}{N_t}d\hat{B}_t = (\sigma-\eta)\hat{X}_t d\hat{B}_t,
\end{aligned}$$

   where $d\hat{B}_t = dB_t - \eta dt$ is a *standard Brownian motion* under $\hat{\mathbb{P}}$.

2. By the result of Question 1, $\hat{X}_t$ is a driftless geometric Brownian motion with volatility $\sigma - \eta$ under $\hat{\mathbb{P}}$, hence

$$\hat{\mathbb{E}}[(\hat{X}_T - \lambda)^+] = \hat{X}_0 \Phi\left(\frac{\log(\hat{X}_0/\lambda)}{\hat{\sigma}\sqrt{T}} + \frac{\hat{\sigma}\sqrt{T}}{2}\right) - \lambda\Phi\left(\frac{\log(\hat{X}_0/\lambda)}{\hat{\sigma}\sqrt{T}} - \frac{\hat{\sigma}\sqrt{T}}{2}\right)$$

   is given by the Black-Scholes formula with zero interest rate and volatility parameter $\hat{\sigma} = \sigma - \eta$, which shows (10.30) by multiplication by N_0 and the relation $X_0 = N_0\hat{X}_0$.

   Hint 1: We have the change of numéraire identity $\mathbb{E}[(X_T - \lambda N_T)^+] = N_0\hat{\mathbb{E}}[(\hat{X}_T - \lambda)^+]$.

3. We have $\hat{\sigma} = \sigma - \eta$.

Exercise 10.2 Bond options.

1. Itô's formula yields

$$\begin{aligned}
d\left(\frac{P(t,S)}{P(t,T)}\right) &= \frac{P(t,S)}{P(t,T)}(\zeta^S(t) - \zeta^T(t))(dW_t - \zeta^T(t)dt) \\
&= \frac{P(t,S)}{P(t,T)}(\zeta^S(t) - \zeta^T(t))d\hat{W}_t, \qquad \text{(S.10.16)}
\end{aligned}$$

where $(\hat{W}_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion under $\hat{\mathbb{P}}$ by the Girsanov theorem.

2. From (S.10.16) we have

$$\frac{P(t,S)}{P(t,T)} = \frac{P(0,S)}{P(0,T)} \exp\left(\int_0^t (\zeta^S(s) - \zeta^T(s))d\hat{W}_s - \frac{1}{2}\int_0^t |\zeta^S(s) - \zeta^T(s)|^2 ds\right),$$

hence

$$\frac{P(u,S)}{P(u,T)} = \frac{P(t,S)}{P(t,T)} \exp\left(\int_t^u (\zeta^S(s) - \zeta^T(s))d\hat{W}_s - \frac{1}{2}\int_t^u |\zeta^S(s) - \zeta^T(s)|^2 ds\right),$$

$u \geq t$, and for $u = T$ this yields

$$P(T,S) = \frac{P(t,S)}{P(t,T)} \exp\left(\int_t^T (\zeta^S(s) - \zeta^T(s))d\hat{W}_s - \frac{1}{2}\int_t^T |\zeta^S(s) - \zeta^T(s)|^2 ds\right),$$

since $P(T,T) = 1$. Let $\hat{\mathbb{P}}$ denote the forward measure associated to the numéraire

$$N_t := P(t,T), \qquad 0 \leq t \leq T.$$

3. For all $S \geq T > 0$ we have

$$\begin{aligned}
&\mathbb{E}\left[e^{-\int_t^T r_s ds}(P(T,S) - K)^+ \Big| \mathcal{F}_t\right] \\
&= P(t,T)\hat{\mathbb{E}}\left[\left(\frac{P(t,S)}{P(t,T)} \exp\left(X - \frac{1}{2}\int_t^T |\zeta^S(s) - \zeta^T(s)|^2 ds\right) - K\right)^+ \Big| \mathcal{F}_t\right] \\
&= P(t,T)\hat{\mathbb{E}}\left[\left(e^{X+m(t,T,S)} - K\right)^+ \Big| \mathcal{F}_t\right],
\end{aligned}$$

where X is a centered Gaussian random variable with variance

$$v^2(t,T,S) = \int_t^T |\zeta^S(s) - \zeta^T(s)|^2 ds$$

given $\mathcal{F}_t$, and

$$m(t,T,S) = -\frac{1}{2}v^2(t,T,S) + \log\frac{P(t,S)}{P(t,T)}.$$

Recall that when X is a centered Gaussian random variable with variance v^2, the expectation of $(e^{m+X} - K)^+$ is given, as in the standard Black-Scholes formula, by

$$\mathbb{E}[(e^{m+X} - K)^+] = e^{m+\frac{v^2}{2}}\Phi(v + (m - \log K)/v) - K\Phi((m - \log K)/v),$$

where

$$\Phi(z) = \int_{-\infty}^{z} e^{-y^2/2} \frac{dy}{\sqrt{2\pi}}, \qquad z \in \mathbb{R},$$

denotes the Gaussian cumulative distribution function and for simplicity of notation we dropped the indices t, T, S in $m(t,T,S)$ and $v^2(t,T,S)$.

Consequently we have

$$\mathbb{E}\left[e^{-\int_t^T r_s ds}(P(T,S)-K)^+ \Big| \mathcal{F}_t\right]$$
$$= P(t,S)\Phi\left(\frac{v}{2} + \frac{1}{v}\log\frac{P(t,S)}{KP(t,T)}\right) - KP(t,T)\Phi\left(-\frac{v}{2} + \frac{1}{v}\log\frac{P(t,S)}{KP(t,T)}\right).$$

4. The self-financing hedging strategy that hedges the bond option is obtained by holding a (possibly fractional) quantity

$$\Phi\left(\frac{v}{2} + \frac{1}{v}\log\frac{P(t,S)}{KP(t,T)}\right)$$

of the bond with maturity S, and by shorting a quantity

$$K\Phi\left(-\frac{v}{2} + \frac{1}{v}\log\frac{P(t,S)}{KP(t,T)}\right)$$

of the bond with maturity T.

Exercise 10.3

1. The process

$$e^{-rt}S_2(t) = S_2(0)e^{\sigma_2 W_t + (\mu - r)t}$$

is a martingale if

$$r - \mu = \frac{1}{2}\sigma_2^2.$$

2. We note that

$$\begin{aligned} e^{-rt}X_t &= e^{-rt}e^{(r-\mu)t-\sigma_1^2 t/2}S_1(t) \\ &= e^{-rt}e^{(\sigma_2^2-\sigma_1^2)t/2}S_1(t) \\ &= e^{-\mu t-\sigma_1^2 t/2}S_1(t) \\ &= S_1(0)e^{\mu t-\sigma_1^2 t/2}e^{\sigma_1 W_t+\mu t} \\ &= S_1(0)e^{\sigma_1 W_t-\sigma_1^2 t/2} \end{aligned}$$

is a martingale, where

$$X_t = e^{(r-\mu)t-\sigma_1^2 t/2}S_1(t) = e^{(\sigma_2^2-\sigma_1^2)t/2}S_1(t).$$

3. By (10.32) we have

$$\begin{aligned}
\hat{X}(t) &= \frac{X_t}{N_t} \\
&= e^{(\sigma_2^2-\sigma_1^2)t/2}\frac{S_1(t)}{S_2(t)} \\
&= \frac{S_1(0)}{S_2(0)}e^{(\sigma_2^2-\sigma_1^2)t/2+(\sigma_1-\sigma_2)W_t} \\
&= \frac{S_1(0)}{S_2(0)}e^{(\sigma_2^2-\sigma_1^2)t/2+(\sigma_1-\sigma_2)\hat{W}_t+\sigma_2(\sigma_1-\sigma_2)t} \\
&= \frac{S_1(0)}{S_2(0)}e^{(\sigma_1-\sigma_2)\hat{W}_t+\sigma_2\sigma_1 t-(\sigma_2^2+\sigma_1^2)t/2} \\
&= \frac{S_1(0)}{S_2(0)}e^{(\sigma_1-\sigma_2)\hat{W}_t-(\sigma_1-\sigma_2)^2 t/2},
\end{aligned}$$

where

$$\hat{W}_t := W_t - \sigma_2 t$$

is a standard Brownian motion under the forward measure $\hat{\mathbb{P}}$ defined by

$$\begin{aligned}
\frac{d\hat{\mathbb{P}}}{d\mathbb{P}} &= e^{-\int_0^T r_s ds}\frac{N_T}{N_0} \\
&= e^{-rT}\frac{S_2(T)}{S_2(0)} \\
&= e^{-rT}e^{\sigma_2 W_T+\mu T} \\
&= e^{\sigma_2 W_T+(\mu-r)T} \\
&= e^{\sigma_2 W_T-\sigma_2^2 t/2}.
\end{aligned}$$

4. Given that $X_t = e^{(\sigma_2^2-\sigma_1^2)t/2}S_1(t)$ and $\hat{X}(t) = X_t/N_t = X_t/S_2(t)$, we have

$$\begin{aligned}
&e^{-rT}\,\mathbb{E}[(S_1(T)-\kappa S_2(T))^+] = e^{-rT}\,\mathbb{E}[(e^{-(\sigma_2^2-\sigma_1^2)T/2}X_T-\kappa S_2(T))^+] \\
&= e^{-rT}e^{-(\sigma_2^2-\sigma_1^2)T/2}\,\mathbb{E}[(X_T-\kappa e^{(\sigma_2^2-\sigma_1^2)T/2}S_2(T))^+] \\
&= S_2(0)e^{-(\sigma_2^2-\sigma_1^2)T/2}\hat{\mathbb{E}}[(\hat{X}_T-\kappa e^{(\sigma_2^2-\sigma_1^2)T/2})^+] \\
&= S_2(0)e^{-(\sigma_2^2-\sigma_1^2)T/2}\hat{\mathbb{E}}[(\hat{X}_0 e^{(\sigma_1-\sigma_2)\hat{W}_T-(\sigma_1-\sigma_2)^2T/2}-\kappa e^{(\sigma_2^2-\sigma_1^2)T/2})^+] \\
&= S_2(0)e^{-(\sigma_2^2-\sigma_1^2)T/2}\left(\hat{X}_0\Phi_+^0(T,\hat{X}_0)-\kappa e^{(\sigma_2^2-\sigma_1^2)T/2}\Phi_-^0(T,\hat{X}_0)\right) \\
&= S_2(0)e^{-(\sigma_2^2-\sigma_1^2)T/2}\hat{X}_0\Phi_+^0(T,\hat{X}_0) \\
&\quad -\kappa S_2(0)e^{-(\sigma_2^2-\sigma_1^2)T/2}e^{(\sigma_2^2-\sigma_1^2)T/2}\Phi_-^0(T,\hat{X}_0) \\
&= e^{-(\sigma_2^2-\sigma_1^2)T/2}X_0\Phi_+^0(T,\hat{X}_0)-\kappa S_2(0)\Phi_-^0(T,\hat{X}_0) \\
&= S_1(0)\Phi_+^0(T,\hat{X}_0)-\kappa S_2(0)\Phi_-^0(T,\hat{X}_0),
\end{aligned}$$

where

$$\Phi_+^0(T,x) = \Phi\left(\frac{\log(x/\kappa)}{|\sigma_1-\sigma_2|\sqrt{T}} + \frac{(\sigma_1-\sigma_2)^2-(\sigma_2^2-\sigma_1^2)}{2|\sigma_1-\sigma_2|}\sqrt{T}\right)$$
$$= \begin{cases} \Phi\left(\dfrac{\log(x/\kappa)}{|\sigma_1-\sigma_2|\sqrt{T}} + \sigma_1\sqrt{T}\right), & \sigma_1 > \sigma_2, \\ \\ \Phi\left(\dfrac{\log(x/\kappa)}{|\sigma_1-\sigma_2|\sqrt{T}} - \sigma_1\sqrt{T}\right), & \sigma_1 < \sigma_2, \end{cases}$$

and

$$\Phi_-^0(T,x) = \Phi\left(\frac{\log(x/\kappa)}{|\sigma_1-\sigma_2|\sqrt{T}} - \frac{(\sigma_1-\sigma_2)^2+(\sigma_2^2-\sigma_1^2)}{2|\sigma_1-\sigma_2|}\sqrt{T}\right)$$
$$= \begin{cases} \Phi\left(\dfrac{\log(x/\kappa)}{|\sigma_1-\sigma_2|\sqrt{T}} + \sigma_2\sqrt{T}\right), & \sigma_1 > \sigma_2, \\ \\ \Phi\left(\dfrac{\log(x/\kappa)}{|\sigma_1-\sigma_2|\sqrt{T}} - \sigma_2\sqrt{T}\right), & \sigma_1 < \sigma_2, \end{cases}$$

if $\sigma_1 \neq \sigma_2$. In case $\sigma_1 = \sigma_2$ we find

$$\begin{aligned} e^{-rT}\,\mathbb{E}[(S_1(T)-\kappa S_2(T))^+] &= e^{-rT}\,\mathbb{E}[S_1(T)(1-\kappa S_2(0)/S_1(0))^+] \\ &= (1-\kappa S_2(0)/S_1(0))^+ e^{-rT}\,\mathbb{E}[S_1(T)] \\ &= (S_1(0)-\kappa S_2(0))\mathbf{1}_{\{S_1(0)>\kappa S_2(0)\}}. \end{aligned}$$

Exercise 10.4

1. It suffices to check that the definition of $(W_t^N)_{t\in\mathbb{R}_+}$ implies the correlation identity $dW_t^S \cdot dW_t^N = \rho dt$ by Itô's calculus.
2. We let
$$\hat{\sigma}_t = \sqrt{(\sigma_t^S)^2 - 2\rho\sigma_t^R\sigma_t^S + (\sigma_t^R)^2}$$
and
$$dW_t^X = \frac{\sigma_t^S - \rho\sigma_t^N}{\hat{\sigma}_t}dW_t^S - \sqrt{1-\rho^2}\frac{\sigma_t^N}{\hat{\sigma}_t}dW_t, \qquad t\in\mathbb{R}_+,$$
which defines a standard Brownian motion under $\mathbb{P}^*$ due to the definition of $\hat{\sigma}_t$.

Exercise 10.5

1. We have $\hat{\sigma} = \sqrt{(\sigma^S)^2 - 2\rho\sigma^R\sigma^S + (\sigma^R)^2}$.
2. Letting $\tilde{X}_t = e^{-rt}X_t = e^{(a-r)t}S_t/R_t$, $t\in\mathbb{R}_+$, we have

$$\mathbb{E}^*\left[\left(\frac{S_T}{R_T}-\kappa\right)^+ \Big| \mathcal{F}_t\right] = e^{-aT}\,\mathbb{E}^*\left[\left(X_T - e^{aT}\kappa\right)^+ \Big| \mathcal{F}_t\right]$$

$$= e^{-(a-r)T}\,\mathbb{E}^*\left[\left(\tilde{X}_T - e^{(a-r)T}\kappa\right)^+ \Big| \mathcal{F}_t\right]$$
$$= e^{-(a-r)T}\left(\tilde{X}_t\Phi\left(\frac{(r-a+\hat{\sigma}^2/2)(T-t)}{\hat{\sigma}\sqrt{T-t}} + \frac{1}{\hat{\sigma}\sqrt{T-t}}\log\frac{S_t}{\kappa R_t}\right)\right.$$
$$\left. -\kappa e^{(a-r)T}\Phi\left(\frac{(r-a-\hat{\sigma}^2/2)(T-t)}{\hat{\sigma}\sqrt{T-t}} + \frac{1}{\hat{\sigma}\sqrt{T-t}}\log\frac{S_t}{\kappa R_t}\right)\right)$$
$$= \frac{S_t}{R_t}e^{(r-a)(T-t)}\Phi\left(\frac{(r-a+\hat{\sigma}^2/2)(T-t)}{\hat{\sigma}\sqrt{T-t}} + \frac{1}{\hat{\sigma}\sqrt{T-t}}\log\frac{S_t}{\kappa R_t}\right)$$
$$-\kappa\Phi\left(\frac{(r-a-\hat{\sigma}^2/2)(T-t)}{\hat{\sigma}\sqrt{T-t}} + \frac{1}{\hat{\sigma}\sqrt{T-t}}\log\frac{S_t}{\kappa R_t}\right),$$

hence the price of the quanto option is

$$e^{-r(T-t)}\,\mathbb{E}^*\left[\left(\frac{S_T}{R_T} - \kappa\right)^+ \Big| \mathcal{F}_t\right]$$
$$= \frac{S_t}{R_t}e^{-a(T-t)}\Phi\left(\frac{(r-a+\hat{\sigma}^2/2)(T-t)}{\hat{\sigma}\sqrt{T-t}} + \frac{1}{\hat{\sigma}\sqrt{T-t}}\log\frac{S_t}{\kappa R_t}\right)$$
$$-\kappa e^{-r(T-t)}\Phi\left(\frac{(r-a-\hat{\sigma}^2/2)(T-t)}{\hat{\sigma}\sqrt{T-t}} + \frac{1}{\hat{\sigma}\sqrt{T-t}}\log\frac{S_t}{\kappa R_t}\right).$$

Chapter 11

Exercise 11.1

1. We have $r_t = r_0 + at + B_t$, and

$$F(t, r_t) = F(t, r_0 + at + \sigma B_t),$$

hence by Proposition 11.2 the PDE satisfied by $F(t,x)$ is

$$-xF(t,x) + \frac{\partial F}{\partial t}(t,x) + a\frac{\partial F}{\partial x}(t,x) + \frac{1}{2}\sigma^2\frac{\partial^2 F}{\partial x^2}(t,x) = 0, \qquad \text{(S.11.17)}$$

with terminal condition $F(T,x) = 1$.

2. We have $r_t = r_0 + at + B_t$ and

$$F(t, r_t) = \mathbb{E}^*\left[\exp\left(-\int_t^T r_s ds\right)\Big|\mathcal{F}_t\right]$$
$$= \mathbb{E}^*\left[\exp\left(-r_0(T-t) - a\int_t^T s ds - \int_t^T B_s ds\right)\Big|\mathcal{F}_t\right]$$
$$= \mathbb{E}^*\left[e^{-r_0(T-t)-a(T^2-t^2)/2}\exp\left(-(T-t)B_t - \int_t^T (T-s)dB_s\right)\Big|\mathcal{F}_t\right]$$

$$= e^{-r_0(T-t)-a(T^2-t^2)/2-(T-t)B_t}\, \mathbb{E}^*\left[\exp\left(-\int_t^T (T-s)dB_s\right)\Big|\mathcal{F}_t\right]$$
$$= e^{-r_0(T-t)-a(T-t)(T+t)/2-(T-t)B_t}\, \mathbb{E}^*\left[\exp\left(-\int_t^T (T-s)dB_s\right)\right]$$
$$= \exp\left(-(T-t)r_t - a(T-t)^2/2 + \frac{1}{2}\int_t^T (T-s)^2 ds\right)$$
$$= \exp\left(-(T-t)r_t - a(T-t)^2/2 + (T-t)^3/6\right),$$

hence $F(t,x) = \exp\left(-(T-t)x - a(T-t)^2/2 + (T-t)^3/6\right)$.

Note that the PDE (S.11.17) can also be solved by looking for a solution of the form $F(t,x) = e^{A(T-t)+xC(T-t)}$, in which case one would find $A(s) = -as^2/2 + s^3/6$ and $C(s) = -s$.

3. We check that the function $F(t,x)$ of Question 2 satisfies the PDE (S.11.17) of Question 1, since $F(T,x) = 1$ and

$$-xF(t,x) + \left(x + a(T-t) - \frac{(T-t)^2}{2}\right)F(t,x) - a(T-t)F(t,x)$$
$$+\frac{1}{2}\sigma^2(T-t)^2F(t,x) = 0.$$

4. We have

$$f(t,T,S) = \frac{1}{S-T}(\log P(t,T) - \log P(t,S))$$
$$= \frac{1}{S-T}\left(\left(-(T-t)r_t + \frac{\sigma^2}{6}(T-t)^3\right) - \left(-(S-t)r_t + \frac{\sigma^2}{6}(S-t)^3\right)\right)$$
$$= r_t + \frac{1}{S-T}\frac{\sigma^2}{6}((T-t)^3 - (S-t)^3).$$

5. We have
$$f(t,T) = -\frac{\partial}{\partial T}\log P(t,T) = r_t - \frac{\sigma^2}{2}(T-t)^2.$$

6. We have
$$d_t f(t,T) = \sigma^2(T-t)dt + \sigma dW_t.$$

7. The HJM condition (11.33) is satisfied since the drift of $d_t f(t,T)$ equals $\sigma\int_t^T \sigma ds$.

Exercise 11.2

1. We have

$$P(t,T) = P(s,T)\exp\left(\int_s^t r_u du + \int_s^t \sigma_u^T dB_u - \frac{1}{2}\int_s^t |\sigma_u^T|^2 du\right),$$

$0 \le s \le t \le T$.

2. We have
$$d\left(e^{-\int_0^t r_s ds} P(t,T)\right) = e^{-\int_0^t r_s ds} \sigma_t^T P(t,T) dB_t,$$
which gives a martingale after integration, from the properties of the Itô integral.
3. By the martingale property of the previous question we have
$$\begin{aligned}\mathbb{E}\left[e^{-\int_0^T r_s ds}\Big|\mathcal{F}_t\right] &= \mathbb{E}\left[P(T,T)e^{-\int_0^T r_s ds}\Big|\mathcal{F}_t\right] \\ &= P(t,T)e^{-\int_0^t r_s ds}, \qquad 0 \le t \le T.\end{aligned}$$
4. By the previous question we have
$$\begin{aligned}P(t,T) &= e^{\int_0^t r_s ds}\,\mathbb{E}\left[e^{-\int_0^T r_s ds}\Big|\mathcal{F}_t\right] \\ &= \mathbb{E}\left[e^{\int_0^t r_s ds}e^{-\int_0^T r_s ds}\Big|\mathcal{F}_t\right] \\ &= \mathbb{E}\left[e^{-\int_t^T r_s ds}\Big|\mathcal{F}_t\right], \qquad 0 \le t \le T,\end{aligned}$$
since $e^{-\int_0^t r_s ds}$ is an $\mathcal{F}_t$-measurable random variable.
5. We have
$$\begin{aligned}\frac{P(t,S)}{P(t,T)} &= \frac{P(s,S)}{P(s,T)} \exp\left(\int_s^t (\sigma_u^S - \sigma_u^T) dB_u - \frac{1}{2}\int_s^t (|\sigma_u^S|^2 - |\sigma_u^T|^2) du\right) \\ &= \frac{P(s,S)}{P(s,T)} \exp\left(\int_s^t (\sigma_u^S - \sigma_u^T) dB_u^T - \frac{1}{2}\int_s^t (\sigma_u^S - \sigma_u^T)^2 du\right),\end{aligned}$$
$0 \le t \le T$, hence letting $s = t$ and $t = T$ in the above expression we have
$$P(T,S) = \frac{P(t,S)}{P(t,T)} \exp\left(\int_t^T (\sigma_s^S - \sigma_s^T) dB_s^T - \frac{1}{2}\int_t^T (\sigma_s^S - \sigma_s^T)^2 ds\right).$$
6. We have
$$\begin{aligned}&P(t,T)\,\mathbb{E}_T\left[(P(T,S)-\kappa)^+\right] \\ &= P(t,T)\,\mathbb{E}_T\left[\left(\frac{P(t,S)}{P(t,T)} e^{\int_t^T (\sigma_s^S - \sigma_s^T) dB_s^T - \frac{1}{2}\int_t^T (\sigma_s^S - \sigma_s^T)^2 ds} - \kappa\right)^+\right] \\ &= P(t,T)\,\mathbb{E}[(e^X - \kappa)^+ \mid \mathcal{F}_t] \\ &= P(t,T) e^{m_t + v_t^2/2} \Phi\left(\frac{v_t}{2} + \frac{1}{v_t}(m_t + v_t^2/2 - \log\kappa)\right) \\ &\quad -\kappa P(t,T) \Phi\left(-\frac{v_t}{2} + \frac{1}{v_t}(m_t + v_t^2/2 - \log\kappa)\right),\end{aligned}$$
with

$$m_t = \log(P(t,S)/P(t,T)) - \frac{1}{2}\int_t^T (\sigma_s^S - \sigma_s^T)^2 ds$$

and

$$v_t^2 = \int_t^T (\sigma_s^S - \sigma_s^T)^2 ds,$$

i.e.

$$P(t,T)\,\mathbb{E}_T\left[(P(T,S)-\kappa)^+\right]$$
$$= P(t,S)\Phi\left(\frac{v_t}{2} + \frac{1}{v_t}\log\frac{P(t,S)}{\kappa P(t,T)}\right) - \kappa P(t,T)\Phi\left(-\frac{v_t}{2} + \frac{1}{v_t}\log\frac{P(t,S)}{\kappa P(t,T)}\right).$$

Exercise 11.3

1. We check that $P(T,T) = e^{X_T^T} = 1$.
2. We have

$$\begin{aligned}
f(t,T,S) &= -\frac{1}{S-T}\left(X_t^S - X_t^T - \mu(S-T)\right)\\
&= \mu - \sigma\frac{1}{S-T}\left((S-t)\int_0^t \frac{1}{S-s}dB_s - (T-t)\int_0^t \frac{1}{T-s}dB_s\right)\\
&= \mu - \sigma\frac{1}{S-T}\int_0^t \left(\frac{S-t}{S-s} - \frac{T-t}{T-s}\right)dB_s\\
&= \mu - \sigma\frac{1}{S-T}\int_0^t \frac{(T-s)(S-t)-(T-t)(S-s)}{(S-s)(T-s)}dB_s\\
&= \mu + \frac{\sigma}{S-T}\int_0^t \frac{(s-t)(S-T)}{(S-s)(T-s)}dB_s.
\end{aligned}$$

3. We have

$$f(t,T) = \mu - \sigma\int_0^t \frac{t-s}{(T-s)^2}dB_s.$$

4. We note that

$$\lim_{T\searrow t} f(t,T) = \mu - \sigma\int_0^t \frac{1}{t-s}dB_s$$

does not exist in $L^2(\Omega)$.

5. By Itô's calculus we have

$$\begin{aligned}
\frac{dP(t,T)}{P(t,T)} &= \sigma dB_t + \frac{1}{2}\sigma^2 dt + \mu dt - \frac{X_t^T}{T-t}dt\\
&= \sigma dB_t + \frac{1}{2}\sigma^2 dt - \frac{\log P(t,T)}{T-t}dt, \qquad t\in[0,T].
\end{aligned}$$

6. Let

$$r_t^S = \mu + \frac{1}{2}\sigma^2 - \frac{X_t^S}{S-t}$$

$$= \mu + \frac{1}{2}\sigma^2 - \sigma \int_0^t \frac{1}{S-s} dB_s,$$

and apply the result of Exercise 11.11.5-(4).

7. We have

$$\mathbb{E}\left[\frac{d\mathbb{P}_T}{d\mathbb{P}}\Big|\mathcal{F}_t\right] = e^{\sigma B_t - \sigma^2 t/2}.$$

8. By the Girsanov theorem, the process $\tilde{B}_t := B_t - \sigma t$ is a standard Brownian motion under $\mathbb{P}_T$.

9. We have

$$\begin{aligned}
\log P(T,S) &= -\mu(S-T) + \sigma(S-T)\int_0^T \frac{1}{S-s} dB_s \\
&= -\mu(S-T) + \sigma(S-T)\int_0^t \frac{1}{S-s} dB_s + \sigma(S-T)\int_t^T \frac{1}{S-s} dB_s \\
&= \frac{S-T}{S-t}\log P(t,S) + \sigma(S-T)\int_t^T \frac{1}{S-s} dB_s \\
&= \frac{S-T}{S-t}\log P(t,S) + \sigma(S-T)\int_t^T \frac{1}{S-s} d\tilde{B}_s + \sigma^2(S-T)\int_t^T \frac{1}{S-s} ds \\
&= \frac{S-T}{S-t}\log P(t,S) + \sigma(S-T)\int_t^T \frac{1}{S-s} d\tilde{B}_s + \sigma^2(S-T)\log\frac{S-t}{S-T},
\end{aligned}$$

$0 < T < S$.

10. We have

$$\begin{aligned}
&P(t,T)\,\mathbb{E}_T\left[(P(T,S)-K)^+\Big|\mathcal{F}_t\right] \\
&= P(t,T)\,\mathbb{E}[(e^X-\kappa)^+ \mid \mathcal{F}_t] \\
&= P(t,T)e^{m_t+v_t^2/2}\Phi\left(\frac{v_t}{2} + \frac{1}{v_t}(m_t + v_t^2/2 - \log\kappa)\right) \\
&\quad -\kappa P(t,T)\Phi\left(-\frac{v_t}{2} + \frac{1}{v_t}(m_t + v_t^2/2 - \log\kappa)\right) \\
&= P(t,T)e^{m_t+v_t^2/2}\Phi\left(v_t + \frac{1}{v_t}(m_t - \log\kappa)\right) - \kappa P(t,T)\Phi\left(\frac{1}{v_t}(m_t - \log\kappa)\right),
\end{aligned}$$

with

$$m_t = \frac{S-T}{S-t}\log P(t,S) + \sigma^2(S-T)\log\frac{S-t}{S-T}$$

and

$$\begin{aligned}
v_t^2 &= \sigma^2(S-T)^2\int_t^T \frac{1}{(S-s)^2} ds \\
&= \sigma^2(S-T)^2\left(\frac{1}{S-T} - \frac{1}{S-t}\right)
\end{aligned}$$

$$= \sigma^2(S-T)\frac{(T-t)}{(S-t)},$$

hence

$$\begin{aligned}
&P(t,T)\,\mathbb{E}_T\left[(P(T,S)-K)^+\Big|\mathcal{F}_t\right]\\
&= P(t,T)\,(P(t,S))^{(S-T)(S-t)}\left(\frac{S-t}{S-T}\right)^{\sigma^2(S-T)} e^{v_t^2/2}\\
&\times\Phi\left(v_t+\frac{1}{v_t}\log\left(\frac{(P(t,S))^{(S-T)(S-t)}}{\kappa}\left(\frac{S-t}{S-T}\right)^{\sigma^2(S-T)}\right)\right)\\
&-\kappa P(t,T)\Phi\left(\frac{1}{v_t}\log\left(\frac{(P(t,S))^{(S-T)(S-t)}}{\kappa}\left(\frac{S-t}{S-T}\right)^{\sigma^2(S-T)}\right)\right).
\end{aligned}$$

Exercise 11.4 From Proposition 11.2 the bond pricing PDE is

$$\begin{cases}\dfrac{\partial F}{\partial t}(t,x) = xF(t,x)-(\alpha-\beta x)\dfrac{\partial F}{\partial x}(t,x)-\dfrac{1}{2}\sigma^2x^2\dfrac{\partial^2 F}{\partial x^2}(t,x)\\[2ex] F(T,x)=1.\end{cases}$$

Let us search for a solution of the form

$$F(t,x) = e^{A(T-t)-xB(T-t)},$$

with $A(0)=B(0)=0$, which implies

$$\begin{cases}A'(s)=0\\[2ex] B'(s)+\beta B(s)+\frac{1}{2}\sigma^2B^2(s)=1.\end{cases}$$

hence in particular $A(s)=0$, $s\in\mathbb{R}$, and $B(s)$ solves a Riccatti equation, whose solution is easily checked to be

$$B(s) = \frac{2(e^{\gamma s}-1)}{2\gamma+(\beta+\gamma)(e^{\gamma s}-1)},$$

with $\gamma=\sqrt{\beta^2+2\sigma^2}$.

Chapter 12

Exercise 12.1

1. The forward measure $\hat{\mathbb{P}}_S$ is defined from the numéraire $N_t := P(t,S)$ and this gives
$$F_t = P(t,S)\hat{\mathbb{E}}[(\kappa - L(T,T,S))^+ \mid \mathcal{F}_t].$$
2. The LIBOR rate $L(t,T,S)$ is a driftless geometric Brownian motion with volatility σ under the forward measure $\hat{\mathbb{P}}_S$. Indeed, the LIBOR rate $L(t,T,S)$ can be written as the forward price $L(t,T,S) = \hat{X}_t = X_t/N_t$ where $X_t = (P(t,T) - P(t,S))/(S-T)$ and $N_t = P(t,S)$. Since both discounted bond prices $e^{-\int_0^t r_s ds}P(t,T)$ and $e^{-\int_0^t r_s ds}P(t,S)$ are martingales under $\mathbb{P}^*$, the same is true of X_t. Hence $L(t,T,S) = X_t/N_t$ becomes a martingale under the forward measure $\hat{\mathbb{P}}_S$ by Proposition 2.1, and computing its dynamics under $\hat{\mathbb{P}}_S$ amounts to removing any "dt" term in (12.19), i.e.
$$dL(t,T,S) = \sigma L(t,T,S)d\hat{W}_t, \qquad 0 \le t \le T,$$
hence $L(t,T,S) = L(0,T,S)e^{\sigma \hat{W}_t - \sigma^2 t/2}$, where $(\hat{W}_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion under $\hat{\mathbb{P}}_S$.
3. We find
$$\begin{aligned}
F_t &= P(t,S)\hat{\mathbb{E}}[(\kappa - L(T,T,S))^+ \mid \mathcal{F}_t] \\
&= P(t,S)\hat{\mathbb{E}}[(\kappa - L(t,T,S)e^{-\sigma^2(T-t)/2+\sigma(\hat{W}_T-\hat{W}_t)})^+ \mid \mathcal{F}_t] \\
&= P(t,S)(\kappa\Phi(-d_-) - \hat{X}_t\Phi(-d_+)) \\
&= \kappa P(t,S)\Phi(-d_-) - P(t,S)L(t,T,S)\Phi(-d_+) \\
&= \kappa P(t,S)\Phi(-d_-) - (P(t,T) - P(t,S))\Phi(-d_+)/(S-T),
\end{aligned}$$
where $e^m = L(t,T,S)e^{-\sigma^2(T-t)/2}$, $v^2 = (T-t)\sigma^2$, and
$$d_+ = \frac{\log(L(t,T,S)/\kappa)}{\sigma\sqrt{T-t}} + \frac{\sigma\sqrt{T-t}}{2},$$
and
$$d_- = \frac{\log(L(t,T,S)/\kappa)}{\sigma\sqrt{T-t}} - \frac{\sigma\sqrt{T-t}}{2},$$
because $L(t,T,S)$ is a driftless geometric Brownian motion with volatility σ under the forward measure $\hat{\mathbb{P}}_S$.

Exercise 12.2

1. We have
$$\frac{dP(t,T_i)}{P(t,T_i)} = r_t dt + \zeta_t^i dB_t, \qquad i = 1,2,$$
and
$$P(T,T_i) = P(t,T_i)\exp\left(\int_t^T r_s ds + \int_t^T \zeta_s^i dB_s - \frac{1}{2}\int_t^T (\zeta_s^i)^2 ds\right),$$

$0 \leq t \leq T \leq T_i$, $i = 1, 2$, hence

$$\log P(T, T_i) = \log P(t, T_i) + \int_t^T r_s ds + \int_t^T \zeta_s^i dB_s - \frac{1}{2} \int_t^T (\zeta_s^i)^2 ds,$$

$0 \leq t \leq T \leq T_i$, $i = 1, 2$, and

$$d \log P(t, T_i) = r_t dt + \zeta_t^i dB_t - \frac{1}{2} (\zeta_t^i)^2 dt, \qquad i = 1, 2.$$

In the present model

$$dr_t = \sigma dB_t,$$

where $(B_t)_{t \in \mathbb{R}_+}$ is a standard Brownian motion under $\mathbb{P}$, we have

$$\zeta_t^i = -\sigma(T_i - t), \qquad 0 \leq t \leq T_i, \quad i = 1, 2.$$

Letting

$$dB_t^i = dB_t - \zeta_t^i dt,$$

defines a standard Brownian motion under $\mathbb{P}_i$, $i = 1, 2$, and we have

$$\begin{aligned}
\frac{P(T, T_1)}{P(T, T_2)} &= \frac{P(t, T_1)}{P(t, T_2)} \exp\left(\int_t^T (\zeta_s^1 - \zeta_s^2) dB_s - \frac{1}{2} \int_t^T ((\zeta_s^1)^2 - (\zeta_s^2)^2) ds \right) \\
&= \frac{P(t, T_1)}{P(t, T_2)} \exp\left(\int_t^T (\zeta_s^1 - \zeta_s^2) dB_s^2 - \frac{1}{2} \int_t^T (\zeta_s^1 - \zeta_s^2)^2 ds \right),
\end{aligned}$$

which is an $\mathcal{F}_t$-martingale under $\mathbb{P}_2$ and under $\mathbb{P}_{1,2}$, and

$$\frac{P(T, T_2)}{P(T, T_1)} = \frac{P(t, T_2)}{P(t, T_1)} \exp\left(-\int_t^T (\zeta_s^1 - \zeta_s^2) dB_s^1 - \frac{1}{2} \int_t^T (\zeta_s^1 - \zeta_s^2)^2 ds \right),$$

which is an $\mathcal{F}_t$-martingale under $\mathbb{P}_1$.

2. We have

$$\begin{aligned}
f(t, T_1, T_2) &= -\frac{1}{T_2 - T_1} (\log P(t, T_2) - \log P(t, T_1)) \\
&= r_t + \frac{1}{T_2 - T_1} \frac{\sigma^2}{6} ((T_1 - t)^3 - (T_2 - t)^3).
\end{aligned}$$

3. We have

$$\begin{aligned}
df(t, T_1, T_2) &= -\frac{1}{T_2 - T_1} d \log \left(P(t, T_2) / P(t, T_1) \right) \\
&= -\frac{1}{T_2 - T_1} \left((\zeta_t^2 - \zeta_t^1) dB_t - \frac{1}{2} ((\zeta_t^2)^2 - (\zeta_t^1)^2) dt \right) \\
&= -\frac{1}{T_2 - T_1} \left((\zeta_t^2 - \zeta_t^1)(dB_t^2 + \zeta_t^2 dt) - \frac{1}{2} ((\zeta_t^2)^2 - (\zeta_t^1)^2) dt \right)
\end{aligned}$$

$$= -\frac{1}{T_2 - T_1}\left((\zeta_t^2 - \zeta_t^1)dB_t^2 - \frac{1}{2}(\zeta_t^2 - \zeta_t^1)^2 dt\right).$$

4. We have

$$\begin{aligned}
&f(T,T_1,T_2) = -\frac{1}{T_2 - T_1}\log\left(P(T,T_2)/P(T,T_1)\right) \\
&= f(t,T_1,T_2) - \frac{1}{T_2 - T_1}\left(\int_t^T (\zeta_s^2 - \zeta_s^1)dB_s - \frac{1}{2}((\zeta_s^2)^2 - (\zeta_s^1)^2)ds\right) \\
&= f(t,T_1,T_2) - \frac{1}{T_2 - T_1}\left(\int_t^T (\zeta_s^2 - \zeta_s^1)dB_s^2 - \frac{1}{2}\int_t^T (\zeta_s^2 - \zeta_s^1)^2 ds\right) \\
&= f(t,T_1,T_2) - \frac{1}{T_2 - T_1}\left(\int_t^T (\zeta_s^2 - \zeta_s^1)dB_s^1 + \frac{1}{2}\int_t^T (\zeta_s^2 - \zeta_s^1)^2 ds\right).
\end{aligned}$$

Hence $f(T,T_1,T_2)$ has a Gaussian distribution given $\mathcal{F}_t$ with conditional mean

$$m = f(t,T_1,T_2) + \frac{1}{2}\int_t^T (\zeta_s^2 - \zeta_s^1)^2 ds$$

under $\mathbb{P}_2$, resp.

$$m = f(t,T_1,T_2) - \frac{1}{2}\int_t^T (\zeta_s^2 - \zeta_s^1)^2 ds$$

under $\mathbb{P}_1$, and variance

$$v^2 = \frac{1}{(T_2 - T_1)^2}\int_t^T (\zeta_s^2 - \zeta_s^1)^2 ds.$$

Hence

$$\begin{aligned}
&(T_2 - T_1)\,\mathbb{E}\left[e^{-\int_t^{T_2} r_s ds}(f(T_1,T_1,T_2) - \kappa)^+ \Big| \mathcal{F}_t\right] \\
&= (T_2 - T_1)P(t,T_2)\,\mathbb{E}_2\left[(f(T_1,T_1,T_2) - \kappa)^+ \Big| \mathcal{F}_t\right] \\
&= (T_2 - T_1)P(t,T_2)\,\mathbb{E}_2\left[(m + X - \kappa)^+ \Big| \mathcal{F}_t\right] \\
&= (T_2 - T_1)P(t,T_2)\left(\frac{v}{\sqrt{2\pi}}e^{-\frac{(\kappa - m)^2}{2v^2}} + (m - \kappa)\Phi((m - \kappa)/v)\right).
\end{aligned}$$

5. We have

$$\begin{aligned}
&L(T,T_1,T_2) = S(T,T_1,T_2) \\
&= \frac{1}{T_2 - T_1}\left(\frac{P(T,T_1)}{P(T,T_2)} - 1\right) \\
&= \frac{1}{T_2 - T_1}\left(\frac{P(t,T_1)}{P(t,T_2)}\exp\left(\int_t^T (\zeta_s^1 - \zeta_s^2)dB_s - \frac{1}{2}\int_t^T ((\zeta_s^1)^2 - (\zeta_s^2)^2)ds\right) - 1\right)
\end{aligned}$$

$$= \frac{1}{T_2 - T_1}\left(\frac{P(t,T_1)}{P(t,T_2)}\exp\left(\int_t^T(\zeta_s^1-\zeta_s^2)dB_s^2 - \frac{1}{2}\int_t^T(\zeta_s^1-\zeta_s^2)^2ds\right)-1\right)$$
$$= \frac{1}{T_2 - T_1}\left(\frac{P(t,T_1)}{P(t,T_2)}\exp\left(\int_t^T(\zeta_s^1-\zeta_s^2)dB_s^1 + \frac{1}{2}\int_t^T(\zeta_s^1-\zeta_s^2)^2ds\right)-1\right),$$

and by Itô calculus,

$$dS(t,T_1,T_2) = \frac{1}{T_2-T_1}d\left(\frac{P(t,T_1)}{P(t,T_2)}\right)$$
$$= \frac{1}{T_2-T_1}\frac{P(t,T_1)}{P(t,T_2)}\left((\zeta_t^1-\zeta_t^2)dB_t + \frac{1}{2}(\zeta_t^1-\zeta_t^2)^2dt - \frac{1}{2}((\zeta_t^1)^2-(\zeta_t^2)^2)dt\right)$$
$$= \left(\frac{1}{T_2-T_1}+S(t,T_1,T_2)\right)((\zeta_t^1-\zeta_t^2)dB_t + \zeta_t^2(\zeta_t^2-\zeta_t^1)dt)dt)$$
$$= \left(\frac{1}{T_2-T_1}+S(t,T_1,T_2)\right)((\zeta_t^1-\zeta_t^2)dB_t^1 + ((\zeta_t^2)^2-(\zeta_t^1)^2)dt)$$
$$= \left(\frac{1}{T_2-T_1}+S(t,T_1,T_2)\right)(\zeta_t^1-\zeta_t^2)dB_t^2, \qquad t\in[0,T_1],$$

hence $\frac{1}{T_2-T_1}+S(t,T_1,T_2)$ is a geometric Brownian motion, with

$$\frac{1}{T_2-T_1}+S(T,T_1,T_2)$$
$$= \left(\frac{1}{T_2-T_1}+S(t,T_1,T_2)\right)\exp\left(\int_t^T(\zeta_s^1-\zeta_s^2)dB_s^2 - \frac{1}{2}\int_t^T(\zeta_s^1-\zeta_s^2)^2ds\right),$$

$0\le t\le T\le T_1$.

6. We have

$$(T_2-T_1)\,\mathbb{E}\left[e^{-\int_t^{T_2}r_sds}(L(T_1,T_1,T_2)-\kappa)^+\Big|\mathcal{F}_t\right]$$
$$= (T_2-T_1)\,\mathbb{E}\left[e^{-\int_t^{T_1}r_sds}P(T_1,T_2)(L(T_1,T_1,T_2)-\kappa)^+\Big|\mathcal{F}_t\right]$$
$$= P(t,T_1,T_2)\,\mathbb{E}_{1,2}\left[(S(T_1,T_1,T_2)-\kappa)^+\Big|\mathcal{F}_t\right].$$

The forward measure $\mathbb{P}_2$ is defined by

$$\mathbb{E}\left[\frac{d\mathbb{P}_2}{d\mathbb{P}}\Big|\mathcal{F}_t\right] = \frac{P(t,T_2)}{P(0,T_2)}e^{-\int_0^t r_sds}, \qquad 0\le t\le T_2,$$

and the forward swap measure is defined by

$$\mathbb{E}\left[\frac{d\mathbb{P}_{1,2}}{d\mathbb{P}}\Big|\mathcal{F}_t\right] = \frac{P(t,T_2)}{P(0,T_2)}e^{-\int_0^t r_sds}, \qquad 0\le t\le T_1,$$

hence $\mathbb{P}_2$ and $\mathbb{P}_{1,2}$ coincide up to time T_1 and $(B_t^2)_{t\in[0,T_1]}$ is a standard Brownian motion until time T_1 under $\mathbb{P}_2$ and under $\mathbb{P}_{1,2}$, consequently under $\mathbb{P}_{1,2}$ we have

$$\begin{aligned} L(T,T_1,T_2) &= S(T,T_1,T_2) \\ &= -\frac{1}{T_2-T_1} + \left(\frac{1}{T_2-T_1} + S(t,T_1,T_2)\right) e^{\int_t^T (\zeta_s^1-\zeta_s^2)dB_s^2 - \frac{1}{2}\int_t^T (\zeta_s^1-\zeta_s^2)^2 ds}, \end{aligned}$$

has same law as

$$\frac{1}{T_2-T_1}\left(\frac{P(t,T_1)}{P(t,T_2)} e^{X-\frac{1}{2}\mathrm{Var}[X]} - 1\right),$$

where X is a centered Gaussian random variable with variance

$$\int_t^{T_1} (\zeta_s^1-\zeta_s^2)^2 ds$$

given $\mathcal{F}_t$. Hence

$$\begin{aligned} &(T_2-T_1)\,\mathbb{E}\left[e^{-\int_t^{T_2} r_s ds}(L(T_1,T_1,T_2)-\kappa)^+ \Big| \mathcal{F}_t\right] \\ &= P(t,T_1,T_2) \\ &\quad \times \mathrm{Bl}\left(\frac{1}{T_2-T_1} + S(t,T_1,T_2), \frac{\int_t^{T_1}(\zeta_s^1-\zeta_s^2)^2 ds}{T_1-t}, \kappa + \frac{1}{T_2-T_1}, T_1-t\right). \end{aligned}$$

Exercise 12.3

(i) We have

$$L(t,T_1,T_2) = L(0,T_1,T_2) e^{\int_0^t \gamma_1(s)dW_s^2 - \frac{1}{2}\int_0^t |\gamma_1(s)|^2 ds}, \qquad 0 \le t \le T_1,$$

and $L(t,T_2,T_3) = b$. Note that we have $P(t,T_2)/P(t,T_3) = 1+\delta b$ hence $\mathbb{P}_2 = \mathbb{P}_3 = \mathbb{P}_{1,2}$ up to time T_1.

(ii) We have

$$\begin{aligned} &\mathbb{E}\left[e^{-\int_t^{T_2} r_s ds}(L(T_1,T_1,T_2)-\kappa)^+ \Big| \mathcal{F}_t\right] \\ &= P(t,T_2)\hat{\mathbb{E}}_2\left[(L(T_1,T_1,T_2)-\kappa)^+ \mid \mathcal{F}_t\right] \\ &= P(t,T_2)\hat{\mathbb{E}}_2\left[(L(t,T_1,T_2) e^{\int_t^{T_1} \gamma_1(s)dW_s^2 - \frac{1}{2}\int_t^{T_1}|\gamma_1(s)|^2 ds} - \kappa)^+ \mid \mathcal{F}_t\right] \\ &= P(t,T_2)\mathrm{Bl}(\kappa, L(t,T_1,T_2), \sigma_1(t), 0, T_1-t), \end{aligned}$$

where

$$\sigma_1^2(t) = \frac{1}{T_1-t}\int_t^{T_1} |\gamma_1(s)|^2 ds.$$

(iii) We have

$$\begin{aligned}\frac{P(t,T_1)}{P(t,T_1,T_3)} &= \frac{P(t,T_1)}{\delta P(t,T_2)+\delta P(t,T_3)}\\ &= \frac{P(t,T_1)}{\delta P(t,T_2)}\frac{1}{1+P(t,T_3)/P(t,T_2)}\\ &= \frac{1+\delta b}{\delta(\delta b+2)}(1+\delta L(t,T_1,T_2)), \qquad 0\le t\le T_1,\end{aligned}$$

and

$$\begin{aligned}\frac{P(t,T_3)}{P(t,T_1,T_3)} &= \frac{P(t,T_3)}{P(t,T_2)+P(t,T_3)}\\ &= \frac{1}{1+P(t,T_2)/P(t,T_3)}\\ &= \frac{1}{\delta}\frac{1}{2+\delta b}, \qquad 0\le t\le T_2. \end{aligned} \tag{S.12.18}$$

(iv) We have

$$\begin{aligned}S(t,T_1,T_3) &= \frac{P(t,T_1)}{P(t,T_1,T_3)} - \frac{P(t,T_3)}{P(t,T_1,T_3)}\\ &= \frac{1+\delta b}{\delta(2+\delta b)}(1+\delta L(t,T_1,T_2)) - \frac{1}{\delta(2+\delta b)}\\ &= \frac{1}{2+\delta b}(b+(1+\delta b)L(t,T_1,T_2)), \qquad 0\le t\le T_2.\end{aligned}$$

We have

$$\begin{aligned}dS(t,T_1,T_3) &= \frac{1+\delta b}{2+\delta b}L(t,T_1,T_2)\gamma_1(t)dW_t^2\\ &= \left(S(t,T_1,T_3)-\frac{b}{2+\delta b}\right)\gamma_1(t)dW_t^2\\ &= S(t,T_1,T_3)\sigma_{1,3}(t)dW_t^2, \qquad 0\le t\le T_2,\end{aligned}$$

with

$$\begin{aligned}\sigma_{1,3}(t) &= \left(1-\frac{b}{S(t,T_1,T_3)(2+\delta b)}\right)\gamma_1(t)\\ &= \left(1-\frac{b}{b+(1+\delta b)L(t,T_1,T_2)}\right)\gamma_1(t)\\ &= \frac{(1+\delta b)L(t,T_1,T_2)}{b+(1+\delta b)L(t,T_1,T_2)}\gamma_1(t)\\ &= \frac{(1+\delta b)L(t,T_1,T_2)}{(2+\delta b)S(t,T_1,T_3)}\gamma_1(t).\end{aligned}$$

(v) The process $(W^2)_{t\in\mathbb{R}_+}$ is a standard Brownian motion under $\mathbb{P}_2$ and

$$P(t,T_1,T_3)\hat{\mathbb{E}}_{1,3}\left[(S(T_1,T_1,T_3)-\kappa)^+ \mid \mathcal{F}_t\right]$$
$$= P(t,T_2)\mathrm{Bl}(\kappa,S(t,T_1,T_2),\tilde{\sigma}_{1,3}(t),0,T_1-t),$$

where $|\tilde{\sigma}_{1,3}(t)|^2$ is the approximation of the volatility

$$\frac{1}{T_1-t}\int_t^{T_1}|\sigma_{1,3}(s)|^2ds = \frac{1}{T_1-t}\int_t^{T_1}\left(\frac{(1+\delta b)L(s,T_1,T_2)}{(2+\delta b)S(s,T_1,T_3)}\right)^2\gamma_1(s)ds$$

obtained by freezing the random component of $\sigma_{1,3}(s)$ at time t, i.e.

$$\tilde{\sigma}_{1,3}^2(t) = \frac{1}{T_1-t}\left(\frac{(1+\delta b)L(t,T_1,T_2)}{(2+\delta b)S(t,T_1,T_3)}\right)^2\int_t^{T_1}|\gamma_1(s)|^2ds.$$

Exercise 12.4

1. We have

$$\mathbb{E}\left[e^{-\int_t^T r_s ds}(P(T,S)-\kappa)^+\,\Big|\mathcal{F}_t\right] = V_T = V_0 + \int_0^T dV_t$$
$$= P(0,T)\,\mathbb{E}_T\left[(P(T,S)-\kappa)^+\right] + \int_0^t \xi_s^T dP(s,T) + \int_0^t \xi_s^S dP(s,S).$$

2. We have

$$\begin{aligned}d\tilde{V}_t &= d\left(e^{-\int_0^t r_s ds}V_t\right)\\ &= -r_t e^{-\int_0^t r_s ds}V_t dt + e^{-\int_0^t r_s ds}dV_t\\ &= -r_t e^{-\int_0^t r_s ds}(\xi_t^T P(t,T)\\ &\quad +\xi_t^S P(t,S))dt + e^{-\int_0^t r_s ds}\xi_t^T dP(t,T) + e^{-\int_0^t r_s ds}\xi_t^S dP(t,S)\\ &= \xi_t^T d\tilde{P}(t,T) + \xi_t^S d\tilde{P}(t,S).\end{aligned}$$

3. By Itô's formula we have

$$\begin{aligned}\mathbb{E}_T&\left[(P(T,S)-\kappa)^+\,|\mathcal{F}_t\right] = C(X_T,0,0)\\ &= C(X_0,T,v(0,T)) + \int_0^t \frac{\partial C}{\partial x}(X_s,T-s,v(s,T))dX_s\\ &= \mathbb{E}_T\left[(P(T,S)-\kappa)^+\right] + \int_0^t \frac{\partial C}{\partial x}(X_s,T-s,v(s,T))dX_s,\end{aligned}$$

since the process

$$t\mapsto \mathbb{E}_T\left[(P(T,S)-\kappa)^+\,|\mathcal{F}_t\right]$$

is a martingale under $\tilde{\mathbb{P}}$.

4. We have

$$d\hat{V}_t = d(V_t/P(t,T))$$

$$
\begin{aligned}
&= d\,\mathbb{E}_T\left[(P(T,S)-\kappa)^+\,|\mathcal{F}_t\right]\\
&= \frac{\partial C}{\partial x}(X_t, T-t, v(t,T))dX_t\\
&= \frac{P(t,S)}{P(t,T)}\frac{\partial C}{\partial x}(X_t, T-t, v(t,T))(\sigma_t^S-\sigma_t^T)dB_t^T.
\end{aligned}
$$

5. We have

$$
\begin{aligned}
dV_t &= d(P(t,T)\hat{V}_t)\\
&= P(t,T)d\hat{V}_t + \hat{V}_t dP(t,T) + d\hat{V}_t\cdot dP(t,T)\\
&= P(t,S)\frac{\partial C}{\partial x}(X_t, T-t, v(t,T))(\sigma_t^S-\sigma_t^T)dB_t^T + \hat{V}_t dP(t,T)\\
&\quad + P(t,S)\frac{\partial C}{\partial x}(X_t, T-t, v(t,T))(\sigma_t^S-\sigma_t^T)\sigma_t^T dt\\
&= P(t,S)\frac{\partial C}{\partial x}(X_t, T-t, v(t,T))(\sigma_t^S-\sigma_t^T)dB_t + \hat{V}_t dP(t,T).
\end{aligned}
$$

6. We have

$$
\begin{aligned}
d\tilde{V}_t &= d(e^{-\int_0^t r_s ds}V_t)\\
&= -r_t e^{-\int_0^t r_s ds}V_t dt + e^{-\int_0^t r_s ds}dV_t\\
&= \tilde{P}(t,S)\frac{\partial C}{\partial x}(X_t, T-t, v(t,T))(\sigma_t^S-\sigma_t^T)dB_t + \hat{V}_t d\tilde{P}(t,T).
\end{aligned}
$$

7. We have

$$
\begin{aligned}
d\tilde{V}_t &= \tilde{P}(t,S)\frac{\partial C}{\partial x}(X_t, T-t, v(t,T))(\sigma_t^S-\sigma_t^T)dB_t + \hat{V}_t d\tilde{P}(t,T)\\
&= \frac{\partial C}{\partial x}(X_t, T-t, v(t,T))d\tilde{P}(t,S)\\
&\quad - \frac{P(t,S)}{P(t,T)}\frac{\partial C}{\partial x}(X_t, T-t, v(t,T))d\tilde{P}(t,T) + \hat{V}_t d\tilde{P}(t,T)\\
&= \left(\hat{V}_t - \frac{P(t,S)}{P(t,T)}\frac{\partial C}{\partial x}(X_t, T-t, v(t,T))\right)d\tilde{P}(t,T)\\
&\quad + \frac{\partial C}{\partial x}(X_t, T-t, v(t,T))d\tilde{P}(t,S),
\end{aligned}
$$

hence the hedging strategy $(\xi_t^T, \xi_t^S)_{t\in[0,T]}$ of the bond option is given by

$$
\begin{aligned}
\xi_t^T &= \hat{V}_t - \frac{P(t,S)}{P(t,T)}\frac{\partial C}{\partial x}(X_t, T-t, v(t,T))\\
&= C(X_t, T-t, v(t,T)) - \frac{P(t,S)}{P(t,T)}\frac{\partial C}{\partial x}(X_t, T-t, v(t,T)),
\end{aligned}
$$

and

$$\xi_t^S = \frac{\partial C}{\partial x}(X_t, T-t, v(t,T)),$$

$t \in [0,T]$.

8. We have

$$\begin{aligned}
&\frac{\partial C}{\partial x}(x,\tau,v) \\
&= \frac{\partial}{\partial x}\left[x\Phi\left(\frac{v\sqrt{\tau}}{2} + \frac{1}{v\sqrt{\tau}}\log\frac{x}{\kappa}\right) - \kappa\Phi\left(-\frac{v\sqrt{\tau}}{2} + \frac{1}{v\sqrt{\tau}}\log\frac{x}{\kappa}\right)\right] \\
&= x\frac{\partial}{\partial x}\Phi\left(\frac{v\sqrt{\tau}}{2} + \frac{1}{v\sqrt{\tau}}\log\frac{x}{\kappa}\right) - \kappa\frac{\partial}{\partial x}\Phi\left(-\frac{v\sqrt{\tau}}{2} + \frac{1}{v\sqrt{\tau}}\log\frac{x}{\kappa}\right) \\
&\quad +\Phi\left(\frac{v\sqrt{\tau}}{2} + \frac{1}{v\sqrt{\tau}}\log\frac{x}{\kappa}\right) \\
&= x\frac{e^{-\frac{1}{2}\left(\frac{v\sqrt{\tau}}{2} + \frac{1}{v\sqrt{\tau}}\log\frac{x}{\kappa}\right)^2}}{\sqrt{2\pi}}\left(\frac{1}{v\sqrt{\tau}x}\right) - \kappa\frac{e^{-\frac{1}{2}\left(-\frac{v\sqrt{\tau}}{2} + \frac{1}{v\sqrt{\tau}}\log\frac{x}{\kappa}\right)^2}}{\sqrt{2\pi}}\left(\frac{1}{v\sqrt{\tau}x}\right) \\
&\quad +\Phi\left(\frac{v\sqrt{\tau}}{2} + \frac{1}{v\sqrt{\tau}}\log\frac{x}{\kappa}\right) \\
&= \Phi\left(\frac{\log(x/\kappa) + \tau v^2/2}{\sqrt{\tau}v}\right).
\end{aligned}$$

As a consequence we get

$$\begin{aligned}
\xi_t^T &= C(X_t, T-t, v(t,T)) - \frac{P(t,S)}{P(t,T)}\frac{\partial C}{\partial x}(X_t, T-t, v(t,T)) \\
&= \frac{P(t,S)}{P(t,T)}\Phi\left(\frac{(T-t)v^2(t,T)/2 + \log X_t}{\sqrt{T-t}v(t,T)}\right) \\
&\quad -\kappa\Phi\left(-\frac{v(t,T)}{2} + \frac{1}{v(t,T)}\log\frac{P(t,S)}{\kappa P(t,T)}\right) \\
&\quad -\frac{P(t,S)}{P(t,T)}\Phi\left(\frac{\log(X_t/\kappa) + (T-t)v^2(t,T)/2}{\sqrt{T-t}v(t,T)}\right) \\
&= -\kappa\Phi\left(\frac{\log(X_t/\kappa) - (T-t)v^2(t,T)/2}{v(t,T)\sqrt{T-t}}\right),
\end{aligned}$$

and

$$\xi_t^S = \frac{\partial C}{\partial x}(X_t, T-t, v(t,T)) = \Phi\left(\frac{\log(X_t/\kappa) + (T-t)v^2(t,T)/2}{v(t,T)\sqrt{T-t}}\right),$$

$t \in [0,T]$, and the hedging strategy is given by

$$V_T = \mathbb{E}\left[e^{-\int_t^T r_s ds}(P(T,S)-\kappa)^+ \,\Big|\, \mathcal{F}_t\right]$$

$$
\begin{aligned}
&= V_0 + \int_0^t \xi_s^T dP(s,T) + \int_0^t \xi_s^S dP(s,S) \\
&= V_0 - \kappa \int_0^t \Phi\left(\frac{\log(X_t/\kappa) - (T-t)v^2(t,T)/2}{v(t,T)\sqrt{T-t}}\right) dP(s,T) \\
&\quad + \int_0^t \Phi\left(\frac{\log(X_t/\kappa) + (T-t)v^2(t,T)/2}{v(t,T)\sqrt{T-t}}\right) dP(s,S).
\end{aligned}
$$

Consequently the bond option can be hedged by shortselling a bond with maturity T for the amount

$$
\kappa\Phi\left(\frac{\log(X_t/\kappa) - (T-t)v^2(t,T)/2}{v(t,T)\sqrt{T-t}}\right),
$$

and by buying a bond with maturity S for the amount

$$
\Phi\left(\frac{\log(X_t/\kappa) + (T-t)v^2(t,T)/2}{v(t,T)\sqrt{T-t}}\right).
$$

Exercise 12.5

1. We have

$$
S(T_i,T_i,T_j) = S(t,T_i,T_j)\exp\left(\int_t^{T_i} \sigma_{i,j}(s)dB_s^{i,j} - \frac{1}{2}\int_t^{T_i} |\sigma_{i,j}|^2(s)ds\right).
$$

2. We have

$$
\begin{aligned}
&P(t,T_i,T_j)\,\mathbb{E}_{i,j}\left[(S(T_i,T_i,T_j)-\kappa)^+ \Big| \mathcal{F}_t\right] \\
&= P(t,T_i,T_j)\,\mathbb{E}_{i,j}\left[\left(S(t,T_i,T_j)e^{\int_t^{T_i}\sigma_{i,j}(s)dB_s^{i,j} - \frac{1}{2}\int_t^{T_i}|\sigma_{i,j}|^2(s)d_s} - \kappa\right)^+ \Big| \mathcal{F}_t\right] \\
&= P(t,T_i,T_j)\mathrm{Bl}(\kappa, v(t,T_i)/\sqrt{T_i-t}, 0, T_i-t) \\
&= P(t,T_i,T_j) \\
&\quad \times \left(S(t,T_i,T_j)\Phi\left(\frac{\log(x/K)}{v(t,T_i)} + \frac{v(t,T_i)}{2}\right) - \kappa\Phi\left(\frac{\log(x/K)}{v(t,T_i)} - \frac{v(t,T_i)}{2}\right)\right),
\end{aligned}
$$

where

$$
v^2(t,T_i) = \int_t^{T_i} |\sigma_{i,j}|^2(s)ds.
$$

3. Integrate the self-financing condition (12.25) between 0 and t.
4. We have

$$
\begin{aligned}
d\tilde{V}_t &= d\left(e^{-\int_0^t r_s ds}V_t\right) \\
&= -r_t e^{-\int_0^t r_s ds V_t} dt + e^{-\int_0^t r_s ds} dV_t
\end{aligned}
$$

$$= -r_t e^{-\int_0^t r_s ds} \sum_{k=i}^{j} \xi_t^k P(t, T_k), dt + e^{-\int_0^t r_s ds} \sum_{k=i}^{j} \xi_t^k dP(t, T_k)$$

$$= \sum_{k=i}^{j} \xi_t^k d\tilde{P}(t, T_k), \qquad 0 \leq t \leq T_i.$$

since

$$\frac{d\tilde{P}(t, T_k)}{\tilde{P}(t, T_k)} = \zeta_k(t) dt, \qquad k = i, \ldots, j.$$

5. We apply the Itô formula and the fact that

$$t \mapsto \mathbb{E}_{i,j}\left[(S(T_i, T_i, T_j) - \kappa)^+ \,\middle|\, \mathcal{F}_t \right]$$

and $(S_t)_{t \in \mathbb{R}_+}$ are both martingales under $\mathbb{P}_{i,j}$.

6. Use the fact that

$$\hat{V}_t = \mathbb{E}_{i,j}\left[(S(T_i, T_i, T_j) - \kappa)^+ \,\middle|\, \mathcal{F}_t \right],$$

and apply the result of Question 5.

7. Apply the Itô rule to $V_t = P(t, T_i, T_j)\hat{V}_t$ using Relation (12.23) and the result of Question 6.

8. We have

$$\begin{aligned}
dV_t &= S_t \frac{\partial C}{\partial x}(S_t, v(t, T_i)) \\
&\quad \times \left(\sum_{k=i}^{j-1} (T_{k+1} - T_k) P(t, T_{k+1})(\zeta_i(t) - \zeta_{k+1}(t)) + P(t, T_j)(\zeta_i(t) - \zeta_j(t)) \right) dB_t \\
&\quad + \hat{V}_t dP(t, T_i, T_j) \\
&= S_t \frac{\partial C}{\partial x}(S_t, v(t, T_i)) \\
&\quad \times \left(\sum_{k=i}^{j-1} (T_{k+1} - T_k) P(t, T_{k+1})(\zeta_i(t) - \zeta_{k+1}(t)) + P(t, T_j)(\zeta_i(t) - \zeta_j(t)) \right) dB_t \\
&\quad + \hat{V}_t \sum_{k=i}^{j-1} (T_{k+1} - T_k) \zeta_{k+1}(t) P(t, T_{k+1}) dB_t \\
&= S_t \frac{\partial C}{\partial x}(S_t, v(t, T_i)) \sum_{k=i}^{j-1} (T_{k+1} - T_k) P(t, T_{k+1})(\zeta_i(t) - \zeta_{k+1}(t)) dB_t \\
&\quad + \frac{\partial C}{\partial x}(S_t, v(t, T_i)) P(t, T_j)(\zeta_i(t) - \zeta_j(t)) dB_t \\
&\quad + \hat{V}_t \sum_{k=i}^{j-1} (T_{k+1} - T_k) \zeta_{k+1}(t) P(t, T_{k+1}) dB_t
\end{aligned}$$

$$= S_t\zeta_i(t)\frac{\partial C}{\partial x}(S_t, v(t,T_i))\sum_{k=i}^{j-1}(T_{k+1}-T_k)P(t,T_{k+1})dB_t$$
$$-S_t\frac{\partial C}{\partial x}(S_t, v(t,T_i))\sum_{k=i}^{j-1}(T_{k+1}-T_k)P(t,T_{k+1})\zeta_{k+1}(t)dB_t$$
$$+\frac{\partial C}{\partial x}(S_t, v(t,T_i))P(t,T_j)(\zeta_i(t)-\zeta_j(t))dB_t$$
$$+\hat{V}_t\sum_{k=i}^{j-1}(T_{k+1}-T_k)\zeta_{k+1}(t)P(t,T_{k+1})dB_t$$
$$= S_t\zeta_i(t)\frac{\partial C}{\partial x}(S_t, v(t,T_i))\sum_{k=i}^{j-1}(T_{k+1}-T_k)P(t,T_{k+1})dB_t$$
$$+\left(\hat{V}_t - S_t\frac{\partial C}{\partial x}(S_t, v(t,T_i))\right)\sum_{k=i}^{j-1}(T_{k+1}-T_k)P(t,T_{k+1})\zeta_{k+1}(t)dB_t$$
$$+\frac{\partial C}{\partial x}(S_t, v(t,T_i))P(t,T_j)(\zeta_i(t)-\zeta_j(t))dB_t.$$

9. We have

$$d\tilde{V}_t = S_t\zeta_i(t)\frac{\partial C}{\partial x}(S_t, v(t,T_i))\sum_{k=i}^{j-1}(T_{k+1}-T_k)\tilde{P}(t,T_{k+1})dB_t$$
$$+\left(\hat{V}_t - S_t\frac{\partial C}{\partial x}(S_t, v(t,T_i))\right)\sum_{k=i}^{j-1}(T_{k+1}-T_k)\tilde{P}(t,T_{k+1})\zeta_{k+1}(t)dB_t$$
$$+\frac{\partial C}{\partial x}(S_t, v(t,T_i))\tilde{P}(t,T_j)(\zeta_i(t)-\zeta_j(t))dB_t$$
$$= (\tilde{P}(t,T_i)-\tilde{P}(t,T_j))\zeta_i(t)\frac{\partial C}{\partial x}(S_t, v(t,T_i))dB_t$$
$$+\left(\hat{V}_t - S_t\frac{\partial C}{\partial x}(S_t, v(t,T_i))\right)d\tilde{P}(t,T_i,T_j)$$
$$+\frac{\partial C}{\partial x}(S_t, v(t,T_i))\tilde{P}(t,T_j)(\zeta_i(t)-\zeta_j(t))dB_t$$
$$= (\zeta_i(t)\tilde{P}(t,T_i)-\zeta_j(t)\tilde{P}(t,T_j))\frac{\partial C}{\partial x}(S_t, v(t,T_i))dB_t$$
$$+\left(\hat{V}_t - S_t\frac{\partial C}{\partial x}(S_t, v(t,T_i))\right)d\tilde{P}(t,T_i,T_j)$$
$$= \frac{\partial C}{\partial x}(S_t, v(t,T_i))d(\tilde{P}(t,T_i)-\tilde{P}(t,T_j))$$
$$+\left(\hat{V}_t - S_t\frac{\partial C}{\partial x}(S_t, v(t,T_i))\right)d\tilde{P}(t,T_i,T_j).$$

10. We have

$$\begin{aligned}
\frac{\partial C}{\partial x}(x,\tau,v) &= \frac{\partial}{\partial x}\left[x\Phi\left(\frac{v}{2}+\frac{1}{v}\log\frac{x}{\kappa}\right)-\kappa\Phi\left(-\frac{v}{2}+\frac{1}{v}\log\frac{x}{\kappa}\right)\right] \\
&= x\frac{\partial}{\partial x}\Phi\left(\frac{v}{2}+\frac{1}{v}\log\frac{x}{\kappa}\right)-\kappa\frac{\partial}{\partial x}\Phi\left(-\frac{v}{2}+\frac{1}{v}\log\frac{x}{\kappa}\right)+\Phi\left(\frac{v}{2}+\frac{1}{v}\log\frac{x}{\kappa}\right) \\
&= x\frac{e^{-\frac{1}{2}\left(\frac{v}{2}+\frac{1}{v}\log\frac{x}{\kappa}\right)^2}}{\sqrt{2\pi}}\left(\frac{1}{vx}\right)-\kappa\frac{e^{-\frac{1}{2}\left(-\frac{v}{2}+\frac{1}{v}\log\frac{x}{\kappa}\right)^2}}{\sqrt{2\pi}}\left(\frac{1}{vx}\right) \\
&\quad +\Phi\left(\frac{v}{2}+\frac{1}{v}\log\frac{x}{\kappa}\right) \\
&= \Phi\left(\frac{\log(x/\kappa)}{v}+\frac{v}{2}\right).
\end{aligned}$$

11. We have

$$\begin{aligned}
d\tilde{V}_t &= \frac{\partial C}{\partial x}(S_t,v(t,T_i))d(\tilde{P}(t,T_i)-\tilde{P}(t,T_j)) \\
&\quad +\left(\hat{V}_t-S_t\frac{\partial C}{\partial x}(S_t,v(t,T_i))\right)d\tilde{P}(t,T_i,T_j) \\
&= \Phi\left(\frac{\log(S_t/K)}{v(t,T_i)}+\frac{v(t,T_i)}{2}\right)d(\tilde{P}(t,T_i)-\tilde{P}(t,T_j)) \\
&\quad -\kappa\Phi\left(\frac{\log(S_t/K)}{v(t,T_i)}-\frac{v(t,T_i)}{2}\right)d\tilde{P}(t,T_i,T_j).
\end{aligned}$$

12. We compare the results of Questions 4 and 11.

Chapter 13

Exercise 13.1 Defaultable bonds.

1. Use the fact that $(r_t,\lambda_t)_{t\in[0,T]}$ is a Markov process.
2. Use the "tower property" (16.22) for the conditional expectation given $\mathcal{F}_t$.
3. We have

$$\begin{aligned}
&d\left(e^{-\int_0^t(r_s+\lambda_s)ds}P(t,T)\right) \\
&= -(r_t+\lambda_t)e^{-\int_0^t(r_s+\lambda_s)ds}P(t,T)dt+e^{-\int_0^t(r_s+\lambda_s)ds}dP(t,T) \\
&= -(r_t+\lambda_t)e^{-\int_0^t(r_s+\lambda_s)ds}P(t,T)dt+e^{-\int_0^t(r_s+\lambda_s)ds}dF(t,r_t,\lambda_t) \\
&= -(r_t+\lambda_t)e^{-\int_0^t(r_s+\lambda_s)ds}P(t,T)dt+e^{-\int_0^t(r_s+\lambda_s)ds}\frac{\partial F}{\partial x}(t,r_t,\lambda_t)dr_t
\end{aligned}$$

$$
\begin{aligned}
&+e^{-\int_0^t(r_s+\lambda_s)ds}\frac{\partial F}{\partial y}(t,r_t,\lambda_t)d\lambda_t+\frac{1}{2}e^{-\int_0^t(r_s+\lambda_s)ds}\frac{\partial^2 F}{\partial x^2}(t,r_t,\lambda_t)\sigma_1^2(t,r_t)dt\\
&+\frac{1}{2}e^{-\int_0^t(r_s+\lambda_s)ds}\frac{\partial^2 F}{\partial y^2}(t,r_t,\lambda_t)\sigma_2^2(t,\lambda_t)dt\\
&+e^{-\int_0^t(r_s+\lambda_s)ds}\rho\frac{\partial^2 F}{\partial x\partial y}(t,r_t,\lambda_t)\sigma_1(t,r_t)\sigma_2(t,\lambda_t)dt+e^{-\int_0^t(r_s+\lambda_s)ds}\frac{\partial F}{\partial t}(t,r_t,\lambda_t)dt\\
=\ &e^{-\int_0^t(r_s+\lambda_s)ds}\frac{\partial F}{\partial x}(t,r_t,\lambda_t)\sigma_1(t,r_t)dB_t^{(1)}+e^{-\int_0^t(r_s+\lambda_s)ds}\frac{\partial F}{\partial y}(t,r_t,\lambda_t)\sigma_2(t,\lambda_t)dB_t^{(2)}\\
&+e^{-\int_0^t(r_s+\lambda_s)ds}\bigg(-(r_t+\lambda_t)P(t,T)+\frac{\partial F}{\partial x}(t,r_t,\lambda_t)\mu_1(t,r_t)\\
&+\frac{\partial F}{\partial y}(t,r_t,\lambda_t)\mu_2(t,\lambda_t)+\frac{1}{2}\frac{\partial^2 F}{\partial x^2}(t,r_t,\lambda_t)\sigma_1^2(t,r_t)+\frac{1}{2}\frac{\partial^2 F}{\partial y^2}(t,r_t,\lambda_t)\sigma_2^2(t,\lambda_t)\\
&+\rho\frac{\partial^2 F}{\partial x\partial y}(t,r_t,\lambda_t)\sigma_1(t,r_t)\sigma_2(t,\lambda_t)+\frac{\partial F}{\partial t}(t,r_t,\lambda_t)\bigg)\,dt,
\end{aligned}
$$

hence the bond pricing PDE is

$$
\begin{aligned}
&-(x+y)F(t,x,y)+\mu_1(t,x)\frac{\partial F}{\partial x}(t,x,y)\\
&+\mu_2(t,y)\frac{\partial F}{\partial y}(t,x,y)+\frac{1}{2}\sigma_1^2(t,x)\frac{\partial^2 F}{\partial x^2}(t,x,y)\\
&+\frac{1}{2}\sigma_2^2(t,y)\frac{\partial^2 F}{\partial y^2}(t,x,y)+\rho\sigma_1(t,x)\sigma_2(t,y)\frac{\partial^2 F}{\partial x\partial y}(t,x,y)+\frac{\partial F}{\partial t}(t,r_t,\lambda_t)=0.
\end{aligned}
$$

4. We have

$$
\begin{aligned}
\int_0^t r_s ds &= \frac{1}{a}\left(\sigma B_t^{(1)}-r_t\right)\\
&=\frac{\sigma}{a}\left(B_t^{(1)}-\int_0^t e^{-a(t-s)}dB_s^{(1)}\right)\\
&=\frac{\sigma}{a}\int_0^t(1-e^{-a(t-s)})dB_s^{(1)},
\end{aligned}
$$

hence

$$
\begin{aligned}
&\int_t^T r_s ds=\int_0^T r_s ds-\int_0^t r_s ds\\
&=\frac{\sigma}{a}\int_0^T(1-e^{-a(T-s)})dB_s^{(1)}-\frac{\sigma}{a}\int_0^t(1-e^{-a(t-s)})dB_s^{(1)}\\
&=-\frac{\sigma}{a}\left(\int_0^t(e^{-a(T-s)}-e^{-a(t-s)})dB_s^{(1)}+\int_t^T(e^{-a(T-s)}-1)dB_s^{(1)}\right)\\
&=-\frac{\sigma}{a}(e^{-a(T-t)}-1)\int_0^t e^{-a(t-s)}dB_s^{(1)}-\frac{\sigma}{a}\int_t^T(e^{-a(T-s)}-1)dB_s^{(1)}\\
&=-\frac{1}{a}(e^{-a(T-t)}-1)r_t-\frac{\sigma}{a}\int_t^T(e^{-a(T-s)}-1)dB_s^{(1)}.
\end{aligned}
$$

The answer for λ_t is similar.

5. As a consequence of the previous question we have

$$\mathbb{E}\left[\int_t^T r_s ds + \int_t^T \lambda_s ds \Big| \mathcal{F}_t\right] = C(a,t,T)r_t + C(b,t,T)\lambda_t,$$

and

$$\begin{aligned}
&\mathrm{Var}\left[\int_t^T r_s ds + \int_t^T \lambda_s ds \Big| \mathcal{F}_t\right] = \\
&= \mathrm{Var}\left[\int_t^T r_s ds \Big| \mathcal{F}_t\right] + \mathrm{Var}\left[\int_t^T \lambda_s ds \Big| \mathcal{F}_t\right] \\
&\quad +2\,\mathrm{Cov}\left(\int_t^T X_s ds, \int_t^T Y_s ds \Big| \mathcal{F}_t\right) \\
&= \frac{\sigma^2}{a^2}\int_t^T (e^{-a(T-s)}-1)^2 ds \\
&\quad +2\rho\frac{\sigma\eta}{ab}\int_t^T (e^{-a(T-s)}-1)(e^{-b(T-s)}-1) ds \\
&\quad +\frac{\eta^2}{b^2}\int_t^T (e^{-b(T-s)}-1)^2 ds \\
&= \sigma^2\int_t^T C^2(a,s,T)ds + 2\rho\sigma\eta\int_t^T C(a,s,T)C(b,s,T)ds \\
&\quad +\eta^2\int_t^T C^2(b,sT)ds,
\end{aligned}$$

from the Itô isometry.

6. We have

$$\begin{aligned}
P(t,T) &= \mathbf{1}_{\{\tau>t\}}\,\mathbb{E}\left[\exp\left(-\int_t^T r_s ds - \int_t^T \lambda_s ds\right)\Big|\mathcal{F}_t\right] \\
&= \mathbf{1}_{\{\tau>t\}}\exp\left(-\mathbb{E}\left[\int_t^T r_s ds\Big|\mathcal{F}_t\right] - \mathbb{E}\left[\int_t^T \lambda_s ds\Big|\mathcal{F}_t\right]\right) \\
&\quad \times \exp\left(\frac{1}{2}\mathrm{Var}\left[\int_t^T r_s ds + \int_t^T \lambda_s ds\Big|\mathcal{F}_t\right]\right) \\
&= \mathbf{1}_{\{\tau>t\}}\exp\left(-C(a,t,T)r_t - C(b,t,T)\lambda_t\right) \\
&\quad \times \exp\left(\frac{\sigma^2}{2}\int_t^T C^2(a,s,T)ds + \frac{\eta^2}{2}\int_t^T C^2(b,s,T)e^{-b(T-s)}ds\right) \\
&\quad \times \exp\left(\rho\sigma\eta\int_t^T C(a,s,T)C(b,s,T)ds\right).
\end{aligned}$$

7. This is a direct consequence of the answers to Questions 3 and 6.
8. The above analysis shows that

$$\mathbb{P}(\tau > T \mid \mathcal{G}_t) = \mathbf{1}_{\{\tau>t\}} \mathbb{E}\left[\exp\left(-\int_t^T \lambda_s ds\right) \Big| \mathcal{F}_t\right]$$
$$= \mathbf{1}_{\{\tau>t\}} \exp\left(-C(b,t,T)\lambda_t + \frac{\eta^2}{2}\int_t^T C^2(b,s,T)ds\right),$$

for $a = 0$ and

$$\mathbb{E}\left[\exp\left(-\int_t^T r_s ds\right) \Big| \mathcal{F}_t\right] = \exp\left(-C(a,t,T)r_t + \frac{\sigma^2}{2}\int_t^T C^2(a,s,T)ds\right),$$

for $b = 0$, and this implies

$$U(t,T) = \exp\left(\rho\sigma\eta\int_t^T C(a,s,T)C(b,s,T)ds\right)$$
$$= \exp\left(\rho\frac{\sigma\eta}{ab}\left(T - t - C(a,t,T) - C(b,t,T) + C(a+b,t,T)\right)\right).$$

9. We have

$$f(t,T) = -\mathbf{1}_{\{\tau>t\}}\frac{\partial \log P(t,T)}{\partial T}$$
$$= \mathbf{1}_{\{\tau>t\}}\left(r_t e^{-a(T-t)} - \frac{\sigma^2}{2}C^2(a,t,T) + \lambda_t e^{-b(T-t)} - \frac{\eta^2}{2}C^2(b,t,T)\right)$$
$$-\mathbf{1}_{\{\tau>t\}}\rho\sigma\eta C(a,t,T)C(b,t,T).$$

10. We use the relation

$$\mathbb{P}(\tau > T \mid \mathcal{G}_t) = \mathbf{1}_{\{\tau>t\}} \mathbb{E}\left[\exp\left(-\int_t^T \lambda_s ds\right) \Big| \mathcal{F}_t\right]$$
$$= \mathbf{1}_{\{\tau>t\}} \exp\left(-C(b,t,T)\lambda_t + \frac{\eta^2}{2}\int_t^T C^2(b,s,T)ds\right)$$
$$= \mathbf{1}_{\{\tau>t\}} e^{-\int_t^T f_2(t,u)du},$$

where $f_2(t,T)$ is the Vasiçek forward rate corresponding to λ_t, i.e.

$$f_2(t,u) = \lambda_t e^{-b(u-t)} - \frac{\eta^2}{2}C^2(b,t,u).$$

11. In this case we have $\rho = 0$ and

$$P(t,T) = \mathbb{P}(\tau > T \mid \mathcal{G}_t)\, \mathbb{E}\left[\exp\left(-\int_t^T r_s ds\right) \Big| \mathcal{F}_t\right],$$

since $U(t,T) = 0$.

Chapter 14

Exercise 14.1

1. We have $S_t = S_0 e^{-\lambda\eta t}(1+\eta)^{N_t}$, $t \in \mathbb{R}_+$.
2. We have $S_t = S_0 e^{rt} \prod_{k=1}^{N_t} (1+\eta Z_k)$, $t \in \mathbb{R}_+$.

Exercise 14.2 We have

$$
\begin{aligned}
\mathrm{Var}[Y_T] &= \mathbb{E}\left[\left(\sum_{k=1}^{N_T} Z_k - \mathbb{E}[Y_T]\right)^2\right] \\
&= \sum_{n=0}^{\infty} \mathbb{E}\left[\left(\sum_{k=1}^{N_T} Z_k - \lambda t \mathbb{E}[Z_1]\right)^2 \Big| N_T = k\right] \mathbb{P}(N_T = k) \\
&= \mathrm{e}^{-\lambda t} \sum_{n=0}^{\infty} \frac{\lambda^n t^n}{n!} \mathbb{E}\left[\left(\sum_{k=1}^{n} Z_k - \lambda t \mathbb{E}[Z_1]\right)^2\right] \\
&= \mathrm{e}^{-\lambda t} \sum_{n=0}^{\infty} \frac{\lambda^n t^n}{n!} \mathbb{E}\left[\left(\sum_{k=1}^{n} Z_k\right)^2 - 2\lambda t \mathbb{E}[Z_1] \sum_{k=1}^{n} Z_k + \lambda^2 t^2 (\mathbb{E}[Z_1])^2\right] \\
&= \mathrm{e}^{-\lambda t} \sum_{n=0}^{\infty} \frac{\lambda^n t^n}{n!} \\
&\quad \times \mathbb{E}\left[2 \sum_{1 \leq k < l \leq n} Z_k Z_l + \sum_{k=1}^{n} |Z_k|^2 - 2\lambda t \mathbb{E}[Z_1] \sum_{k=1}^{n} Z_k + \lambda^2 t^2 (\mathbb{E}[Z_1])^2\right] \\
&= \mathrm{e}^{-\lambda t} \sum_{n=0}^{\infty} \frac{\lambda^n t^n}{n!} \\
&\quad \times (n(n-1)(\mathbb{E}[Z_1])^2 + n\mathbb{E}[|Z_1|^2] - 2n\lambda t (\mathbb{E}[Z_1])^2 + \lambda^2 t^2 (\mathbb{E}[Z_1])^2) \\
&= \mathrm{e}^{-\lambda t} (\mathbb{E}[Z_1])^2 \sum_{n=2}^{\infty} \frac{\lambda^n t^n}{(n-2)!} + \mathrm{e}^{-\lambda t} \mathbb{E}[|Z_1|^2] \sum_{n=1}^{\infty} \frac{\lambda^n t^n}{(n-1)!} \\
&\quad -2\mathrm{e}^{-\lambda t} \lambda t (\mathbb{E}[Z_1])^2 \sum_{n=1}^{\infty} \frac{\lambda^n t^n}{(n-1)!} + \lambda^2 t^2 (\mathbb{E}[Z_1])^2) \\
&= \lambda t \mathbb{E}[|Z_1|^2],
\end{aligned}
$$

or, using the characteristic function of Proposition 14.3,

$$
\begin{aligned}
\mathrm{Var}[Y_T] &= \mathbb{E}[|Y_T|^2] - (\mathbb{E}[Y_T])^2 \\
&= -\frac{d^2}{d\alpha^2} \mathbb{E}[\mathrm{e}^{i\alpha Y_T}]_{|\alpha=0} - \lambda^2 t^2 (\mathbb{E}[Z_1])^2
\end{aligned}
$$

$$= \lambda t \int_{-\infty}^{\infty} |y|^2 \mu(dy) = \lambda t \mathbb{E}[|Z_1|^2].$$

Exercise 14.3

1. We have

$$\begin{aligned} dS_t &= \left(\mu + \frac{1}{2}\sigma^2\right) S_t dt + \sigma S_t dW_t + (S_t - S_{t^-}) dN_t \\ &= \left(\mu + \frac{1}{2}\sigma^2\right) S_t dt + \sigma S_t dW_t + (S_0 e^{\mu t + \sigma W_t + Y_t} - S_0 e^{\mu t + \sigma W_t + Y_{t^-}}) dN_t \\ &= \left(\mu + \frac{1}{2}\sigma^2\right) S_t dt + \sigma S_t dW_t + (S_0 e^{\mu t + \sigma W_t + Y_{t^-} + Z_{N_t}} - e^{\mu t + \sigma W_t + Y_{t^-}}) dN_t \\ &= \left(\mu + \frac{1}{2}\sigma^2\right) S_t dt + \sigma S_t dW_t + S_{t^-} (e^{Z_{N_t}} - 1) dN_t, \end{aligned}$$

   hence the jumps of S_t are given by the sequence $(e^{Z_k} - 1)_{k \geq 1}$.
2. The discounted process $e^{-rt} S_t$ satisfies

$$d(e^{-rt} S_t) = e^{-rt} \left(\mu - r + \frac{1}{2}\sigma^2\right) S_t dt + \sigma e^{-rt} S_t dW_t + e^{-rt} S_{t^-} (e^{Z_{N_t}} - 1) dN_t.$$

   Hence by the Girsanov theorem, choosing u, $\tilde{\lambda}$, $\tilde{\nu}$ such that

$$\mu - r + \frac{1}{2}\sigma^2 = u - \tilde{\lambda}\, \mathbb{E}_{\tilde{\nu}}[e^{Z_1} - 1],$$

   shows that

$$d(e^{-rt} S_t) = \sigma e^{-rt} S_t (dW_t + u dt) + e^{-rt} S_{t^-} ((e^{Z_{N_t}} - 1) dN_t - \tilde{\lambda}\, \mathbb{E}_{\tilde{\nu}}[e^{Z_1} - 1] dt)$$

   is a martingale under $(\mathbb{P}_{u, \tilde{\lambda}, \tilde{\nu}})$.

Exercise 14.4

1. We have

$$S_t = S_0 e^{\mu t} \prod_{k=1}^{N_t} (1 + Y_k) = S_0 \exp\left(\mu t + \sum_{k=1}^{N_t} X_k\right), \qquad t \in \mathbb{R}_+.$$

2. We have

$$e^{-rt} S_t = S_0 \exp\left((\mu - r) t + \sum_{k=1}^{N_t} X_k\right), \qquad t \in \mathbb{R}_+,$$

   which is a martingale if

$$0 = \mu - r + \lambda\, \mathbb{E}[Y_k] = \mu - r + \lambda\, \mathbb{E}[e^{X_k} - 1] = \mu - r + \lambda (e^{\sigma^2/2} - 1).$$

3. We have

$$
\begin{aligned}
& e^{-r(T-t)} \mathbb{E}[(S_T - \kappa)^+ \mid S_t] \\
&= e^{-r(T-t)} \mathbb{E}\left[\left(S_0 \exp\left(\mu T + \sum_{k=1}^{N_t} X_k \right) - \kappa \right)^+ \middle| S_t \right] \\
&= e^{-r(T-t)} \sum_{n=0}^{\infty} \mathbb{E}\left[\left(S_t e^{\mu(T-t)+\sum_{k=1}^{n} X_k} - \kappa \right)^+ \middle| S_t \right] \mathbb{P}(N_T - N_t = n) \\
&= e^{-(r+\lambda)(T-t)} \sum_{n=0}^{\infty} \mathbb{E}\left[\left(S_t e^{\mu(T-t)+\sum_{k=1}^{n} X_k} - \kappa \right)^+ \middle| S_t \right] \frac{(\lambda(T-t))^n}{n!} \\
&= e^{-\lambda(T-t)} \sum_{n=0}^{\infty} \mathrm{Bl}(S_t e^{(\mu-r)(T-t)+n\sigma^2/2}, r, n\sigma^2/(T-t), \kappa, T-t) \frac{(\lambda(T-t))^n}{n!} \\
&= e^{-\lambda(T-t)} \sum_{n=0}^{\infty} \left(S_t e^{(\mu-r)(T-t)+n\sigma^2/2} \Phi(d_+) - \kappa e^{-r(T-t)} \Phi(d_-) \right) \frac{(\lambda(T-t))^n}{n!},
\end{aligned}
$$

with

$$
\begin{aligned}
d_+ &= \frac{\log(S_t e^{(\mu-r)(T-t)+n\sigma^2/2}/\kappa) + r(T-t) + n\sigma^2/2}{\sqrt{n}\sigma} \\
&= \frac{\log(S_t/\kappa) + \mu(T-t) + n\sigma^2}{\sqrt{n}\sigma},
\end{aligned}
$$

$$
\begin{aligned}
d_- &= \frac{\log(S_t e^{(\mu-r)(T-t)+n\sigma^2/2}/\kappa) + r(T-t) - n\sigma^2/2}{\sqrt{n}\sigma} \\
&= \frac{\log(S_t/\kappa) + \mu(T-t)}{\sqrt{n}\sigma},
\end{aligned}
$$

and $\mu = r + \lambda(1 - e^{\sigma^2/2})$.

Chapter 15

Exercise 15.1

1. Independently of the choice of a risk-neutral measure $\mathbb{P}_{u,\tilde{\lambda},\tilde{\nu}}$ we have

$$
\begin{aligned}
e^{-r(T-t)} \mathbb{E}_{u,\tilde{\lambda},\tilde{\nu}}[S_T - K \mid \mathcal{F}_t] &= e^{rt} \mathbb{E}_{u,\tilde{\lambda},\tilde{\nu}}[e^{-rT} S_T \mid \mathcal{F}_t] - K e^{-r(T-t)} \\
&= e^{rt} e^{-rt} S_t - K e^{-r(T-t)} \\
&= S_t - K e^{-r(T-t)}
\end{aligned}
$$

$$= f(t, S_t),$$

for

$$f(t, x) = x - Ke^{-r(T-t)}, \qquad t, x > 0.$$

2. Clearly, holding one unit of the risky asset and shorting a (possibly fractional) quantity Ke^{-rT} of the riskless asset will hedge the payoff $S_T - K$, and this hedging strategy is self-financing because it is constant in time.
3. Since $\frac{\partial f}{\partial x}(t, x) = 1$ we have

$$\begin{aligned}
\xi_t &= \frac{\sigma^2 \frac{\partial f}{\partial x}(t, S_{t^-}) + \frac{a\tilde{\lambda}}{S_{t^-}}(f(t, S_{t^-}(1+a)) - f(t, S_{t^-}))}{\sigma^2 + a^2\tilde{\lambda}} \\
&= \frac{\sigma^2 + \frac{a\tilde{\lambda}}{S_{t^-}}(S_{t^-}(1+a) - S_{t^-})}{\sigma^2 + a^2\tilde{\lambda}} \\
&= 1, \qquad t \in [0, T],
\end{aligned}$$

which coincides with the result of Question 2.

Exercise 15.2

(i) We have

$$S_t = S_0 \exp\left(\mu t + \sigma B_t - \frac{1}{2}\sigma^2 t\right)(1+\eta)^{N_t}.$$

(ii) We have

$$\tilde{S}_t = S_0 \exp\left((\mu - r)t + \sigma B_t - \frac{1}{2}\sigma^2 t\right)(1+\eta)^{N_t},$$

and

$$d\tilde{S}_t = (\mu - r + \lambda\eta)\tilde{S}_t dt + \eta\tilde{S}_{t^-}(dN_t - \lambda dt) + \sigma\tilde{S}_t dW_t,$$

hence we need to take

$$\mu - r + \lambda\eta = 0,$$

since the compensated Poisson process $(N_t - \lambda t)_{t\in\mathbb{R}_+}$ is a martingale.

(iii) We have

$$\begin{aligned}
&e^{-r(T-t)}\mathbb{E}^*[(S_T - \kappa)^+ \mid S_t] \\
&= e^{-r(T-t)}\mathbb{E}^*\left[\left(S_0 \exp\left(\mu T + \sigma B_T - \frac{1}{2}\sigma^2 T\right)(1+\eta)^{N_T} - \kappa\right)^+ \middle| S_t\right] \\
&= e^{-r(T-t)}\mathbb{E}^*\left[\left(S_t e^{\mu(T-t)+\sigma(B_T-B_t)-\frac{1}{2}\sigma^2(T-t)}(1+\eta)^{N_T-N_t} - \kappa\right)^+ \middle| S_t\right]
\end{aligned}$$

$$
\begin{aligned}
&= e^{-r(T-t)} \sum_{n=0}^{\infty} \mathbb{P}(N_T - N_t = n) \\
&\quad \times \mathbb{E}^* \left[\left(S_t e^{\mu(T-t)+\sigma(B_T-B_t)-\frac{1}{2}\sigma^2(T-t)}(1+\eta)^n - \kappa \right)^+ \Big| S_t \right] \\
&= e^{-(r+\lambda)(T-t)} \sum_{n=0}^{\infty} \frac{(\lambda(T-t))^n}{n!} \\
&\quad \times \mathbb{E}^* \left[\left(S_t e^{(r-\lambda\eta)(T-t)+\sigma(B_T-B_t)-\frac{1}{2}\sigma^2(T-t)}(1+\eta)^n - \kappa \right)^+ \Big| S_t \right] \\
&= e^{-\lambda(T-t)} \sum_{n=0}^{\infty} \mathrm{Bl}(S_t e^{-\lambda\eta(T-t)}(1+\eta)^n, r, \sigma^2, T-t, \kappa) \frac{(\lambda(T-t))^n}{n!} \\
&= e^{-\lambda(T-t)} \sum_{n=0}^{\infty} \left(S_t e^{-\lambda\eta(T-t)}(1+\eta)^n \Phi(d_+) - \kappa e^{-r(T-t)} \Phi(d_-) \right) \frac{(\lambda(T-t))^n}{n!},
\end{aligned}
$$

with

$$
\begin{aligned}
d_+ &= \frac{\log(S_t e^{-\lambda\eta(T-t)}(1+\eta)^n/\kappa) + (r+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} \\
&= \frac{\log(S_t(1+\eta)^n/\kappa) + (r-\lambda\eta+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}},
\end{aligned}
$$

and

$$
\begin{aligned}
d_- &= \frac{\log(S_t e^{-\lambda\eta(T-t)}(1+\eta)^n/\kappa) + (r-\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} \\
&= \frac{\log(S_t(1+\eta)^n/\kappa) + (r-\lambda\eta-\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}.
\end{aligned}
$$

Exercise 15.3

1. The discounted process $\tilde{S}_t = e^{-rt} S_t$ satisfies the equation

$$
d\tilde{S}_t = Y_{N_t} \tilde{S}_{t^-} dN_t, ,
$$

and it is a martingale since the compound Poisson process $Y_{N_t} dN_t$ is centered with independent increments as $\mathbb{E}[Y_1] = 0$.
2. We have

$$
S_T = S_0 e^{rT} \prod_{k=1}^{N_T} (1+Y_k),
$$

hence

$$
e^{-rT}\, \mathbb{E}[(S_T - \kappa)] = e^{-rT}\, \mathbb{E}\left[\left(S_0 e^{rT} \prod_{k=1}^{N_T} (1+Y_k) - \kappa \right)^+ \right]
$$

$$
\begin{aligned}
&= e^{-rT} \sum_{n=0}^{\infty} \mathbb{E}\left[\left(S_0 e^{rT} \prod_{k=1}^{N_T} (1+Y_k) - \kappa \right)^+ \middle| N_T = n \right] \mathbb{P}(N_T = n) \\
&= e^{-rT-\lambda T} \sum_{k=0}^{\infty} \mathbb{E}\left[\left(S_0 e^{rT} \prod_{k=1}^{n} (1+Y_k) - \kappa \right)^+ \right] \frac{(\lambda T)^n}{n!} \\
&= e^{-rT-\lambda T} \sum_{k=0}^{\infty} \frac{(\lambda T)^n}{2^n n!} \int_{-1}^{1} \cdots \int_{-1}^{1} \left(S_0 e^{rT} \prod_{k=1}^{n} (1+y_k) - \kappa \right)^+ dy_1 \cdots dy_n.
\end{aligned}
$$