Smoothness of Wigner densities on the affine algebra

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Abstract - The non-commutative Malliavin calculus on the Heisenberg-Weyl algebra [4], [5] is extended to the affine algebra. A differential calculus is established, which generalizes the corresponding commutative integration by parts formulas. As an application we obtain sufficient conditions for the smoothness of Wigner type laws of non-commutative random variables with gamma and continuous binomial marginals.

Régularité de densités de Wigner sur l'algèbre affine

Résumé - Le calcul de Malliavin non-commutatif sur l'algèbre de Heisenberg-Weyl [4], [5] est étendu à l'algèbre affine. Un calcul différentiel non-commutatif qui généralise les formules d'intégration par parties classiques est établi. Comme application nous obtenons des conditions suffisantes pour la régularité de lois de Wigner pour des variables aléatoires non-commutatives de lois marginales gamma et binomiale continue.

Version française abrégée

Dans [5] un calcul de Malliavin non-commutatif a été introduit sur l'algèbre de Heisenberg-Weyl $\{\mathbf{p}, \mathbf{q}, I\}$, avec $[\mathbf{p}, \mathbf{q}] = 2iI$, en généralisant aux densités de Wigner le calcul de Malliavin par rapport aux variables gaussiennes. En particulier ceci permet de prouver la régularité de densités de Wigner [9] ayant des marginales gaussiennes. Dans cette Note nous traitons d'autres lois de probabilité dans un cadre plus général, voir [2] pour des références sur les applications de ces densités de Wigner généralisées. En particulier nous considérons des variables aléatoires non-commutatives ayant des marginales de loi gamma et binomiale continue. Pour cela nous utilisons la construction de telles variables aléatoires sur les algèbres de Lie, à partir de résultats généraux de [2]. En utilisant une représentation de l'algèbre affine sur un espace de Hilbert $\mathcal{H}$ donnée par $X_1 = -\frac{i}{2}P$ et $X_2 = i(Q+M)$ avec $[X_1, X_2] = X_2$, nous obtenons une expression de la densité jointe de $(P, Q+M)$ à l'aide de fonctions de Wigner, et calculons la fonction caractéristique:

$$\langle \phi, e^{iuP+iv(Q+M)}\psi\rangle_{\mathcal{H}} = \int_{\mathbb{R}} e^{iv\omega \text{sinch}\, u}\overline{\phi}(\omega e^{u})\psi(\omega e^{-u})e^{-|\omega|\cosh u}\frac{|\omega|^{\beta-1}}{\Gamma(\beta)}d\omega, \quad \phi, \psi \in \mathcal{H}.$$

Nous montrons ensuite qu'un opérateur O satisfaisant

$$O(e^{-iu\xi_1 - iv\xi_2}) = e^{-\frac{i}{2}uP+iv(Q+M)}$$

peut être étendu par continuité à $L^2_{\mathbb{C}}(\mathbb{R}^2; \frac{d\xi_1 d\xi_2}{2\pi|\xi_2|})$ avec

$$O(f) = \int_{\mathbb{R}^2} (\mathcal{F}f)(x_1, x_2)e^{-\frac{i}{2}x_1 P + ix_2(Q+M)}dx_1 dx_2,$$

où $\mathcal{F}$ représente la transformée de Fourier. A l'aide de l'opérateur gradient

$$D_x F = -\frac{i}{2} x_1 [P, F] + \frac{i}{2} x_2 [Q + M, F], \qquad x = (x_1, x_2) \in \mathbb{R}^2,$$

qui agit sur une classe d'opérateurs suffisamment réguliers de $\mathcal{H}$, nous obtenons la formule d'entrelacement

$$D_{(x_1, 2x_2)} O(f) = O(x_1 \xi_2 \partial_1 f(\xi_1, \xi_2) - x_2 \xi_2 \partial_2 f(\xi_1, \xi_2)), \quad x_1, x_2 \in \mathbb{R},$$

qui permet d'établir la régularité de la loi jointe de $(P, Q + M)$ par intégration par parties non commutative. Nous définissons aussi un opérateur

$$\delta(F \otimes x) = \frac{x_1}{2} \{Q + \alpha(M - \beta), F\} + \frac{x_2}{2} \{P, F\} - D_x F, \qquad x = (x_1, x_2) \in \mathbb{R}^2,$$

analogue de l'intégrale de Skorohod, et qui satisfait une formule d'intégration par parties.

1 Random variables on the affine algebra

Let a^-, a^+ denote the boson annihilation and creation operators and let $\mathbf{q} = a^- + a^+$, $\mathbf{p} = i(a^- - a^+)$, with $[\mathbf{p}, \mathbf{q}] = 2iI$. The joint law of $(\mathbf{p}, \mathbf{q})$ is called a Wigner law [9], and has Gaussian marginals in the vacuum state. Moreover, $\{\mathbf{p}, \mathbf{q}, I\}$, with $[\mathbf{p}, \mathbf{q}] = 2iI$, yield a representation of the Heisenberg-Weyl algebra.

Let now $\tilde{a}_\tau^- = \tau \partial_\tau$, i.e. $\tilde{a}_\tau^- f(\tau) = \tau f'(\tau)$, $f \in \mathcal{C}_b^\infty(\mathbb{R})$. The adjoint $\tilde{a}_\tau^+$ of $\tilde{a}_\tau^-$ with respect to the gamma density $\gamma_\beta(\tau) = 1_{\{\tau \geq 0\}} \frac{\tau^{\beta-1}}{\Gamma(\beta)} e^{-\tau}$ on $\mathbb{R}_+$ satisfies

$$\int_0^\infty g(\tau) \tilde{a}_\tau^- f(\tau) \gamma_\beta(\tau) d\tau = \int_0^\infty f(\tau) \tilde{a}_\tau^+ g(\tau) \gamma_\beta(\tau) d\tau, \quad f, g \in \mathcal{C}_b^\infty(\mathbb{R}), \tag{1.1}$$

and is given by $\tilde{a}_\tau^+ = (\tau - \beta) - \tilde{a}_\tau^-$. The multiplication operator $\tilde{a}^- + \tilde{a}^+ = \tau - \beta$ has a compensated gamma law (or spectral measure) in the vacuum state in $L^2_{\mathbb{C}}(\mathbb{R}_+, \gamma_\beta(\tau) d\tau)$. In [8] it has been noticed that when $\beta = 1$, $i(\tilde{a}^- - \tilde{a}^+)$ has the continuous binomial density $(2 \cosh \pi \xi_1 / 2)^{-1}$, in relation to a representation of the subgroup of upper-triangular matrices of sl_2. This type of law can be studied for all $\beta > 0$ in the general framework of [1], starting from a representation (M, B^-, B^+) of sl_2:

$$[B^-, B^+] = M, \quad [M, B^-] = -2B^-, \quad [M, B^+] = 2B^+,$$

which can be constructed as

$$M = \beta + 2\tilde{a}_\tau^\circ, \quad B^- = \tilde{a}_\tau^- - \tilde{a}_\tau^\circ, \quad B^+ = \tilde{a}_\tau^+ - \tilde{a}_\tau^\circ,$$

with $\tilde{a}_\tau^\circ = \tilde{a}_\tau^+ \partial_\tau = -(\beta - \tau)\partial - \tau \partial^2$. Letting $Q = B^- + B^+ = \tilde{a}^- + \tilde{a}^+ - 2\tilde{a}^\circ$ and $P = i(B^- - B^+) = i(\tilde{a}^- - \tilde{a}^+)$, we have $Q + M = \tau$ and more generally $Q + \alpha M$

has a gamma law when $\alpha = \pm 1$, whereas $P = i(\tilde{a}^- - \tilde{a}^+)$ has a continuous binomial distribution with parameter β. The Heisenberg-Weyl Malliavin calculus of [4], [5] relies on a functional calculus which allows to define the composition of a function on $\mathbb{R}^2$ with a couple of non-commutative random variables, and on a covariance identity which plays the role of integration by parts formula. Here, $\{-\frac{i}{2}P, i(Q+M)\}$ form a representation of the affine algebra: $[-\frac{i}{2}P, i(Q+M)] = i(Q+M)$. In order to extend the construction of [4], [5] to the gamma and continuous binomial laws we will use the formalism of [2] which provides in particular a functional calculus on the affine algebra.

2 Functional calculus on the affine algebra

The affine group can be constructed as the group of 2×2 matrices of the form

$$g = \begin{pmatrix} e^{x_1} & x_2 e^{\frac{x_1}{2}} \operatorname{sinch} \frac{x_1}{2} \\ 0 & 1 \end{pmatrix} = e^{x_1 X_1 + x_2 X_2}, \quad x_1, x_2 \in \mathbb{R},$$

where $\operatorname{sinch} x = (\sinh x)/x$, $x \in \mathbb{R}$, and $X_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $X_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, generate the affine algebra, with $[X_1, X_2] = X_2$. Consider the representation of the affine group on $\mathcal{H} = L^2_{\mathbb{C}}(\mathbb{R}, \gamma_\beta(|\tau|)d\tau)$ defined by

$$(\hat{U}(g)\phi)(\tau) = \phi(a\tau)e^{ib\tau}e^{-(a-1)|\tau|/2}a^{\beta/2}, \quad \phi \in L^2_{\mathbb{C}}(\mathbb{R}, \gamma_\beta(|\tau|)d\tau), \quad g = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}.$$

We have

$$\hat{U}(X_1) = -\frac{i}{2}P \quad \text{and} \quad \hat{U}(X_2) = i(Q+M).$$

Given $\phi, \psi \in \mathcal{H}$, let

$$W_{|\phi\rangle\langle\psi|}(\xi_1, \xi_2) = \int_{\mathbb{R}} \phi\left(\frac{\xi_2 e^{-\frac{x}{2}}}{\operatorname{sinch}\frac{x}{2}}\right) \frac{|\xi_2| e^{-ix\xi_1}}{\operatorname{sinch}\frac{x}{2}} \overline{\psi}\left(\frac{\xi_2 e^{\frac{x}{2}}}{\operatorname{sinch}\frac{x}{2}}\right) e^{-|\xi_2|\frac{\cosh\frac{x}{2}}{\operatorname{sinch}\frac{x}{2}}} \left(\frac{|\xi_2|}{\operatorname{sinch}\frac{x}{2}}\right)^{\beta-1} \frac{dx}{\Gamma(\beta)}, \tag{2.1}$$

$\xi_1, \xi_2 \in \mathbb{R}$, denote the Wigner function on the affine algebra, cf. (102) of [2]. The next two propositions are obtained by computing the action of $e^{-\frac{i}{2}uP + iv(Q+M)} = \hat{U}(e^{uX_1 + vX_2})$ in two different ways, using results of [2], see [6].

Proposition 1 *Let $\phi, \psi \in \mathcal{H}$. We have*

$$\langle\psi|e^{\frac{i}{2}uP - iv(Q+M)}\phi\rangle_{\mathcal{H}} = \int_{\mathbb{R}^2} e^{iu\xi_1 + iv\xi_2} W_{|\phi\rangle\langle\psi|}(\xi_1, \xi_2) \frac{d\xi_1 d\xi_2}{2\pi|\xi_2|}, \qquad u, v \in \mathbb{R}.$$

As a consequence, the joint density of $(\frac{1}{2}P, -(Q+M))$ in the state $|\phi\rangle\langle\psi|$ is given as

$$\tilde{W}_{|\phi\rangle\langle\psi|}(\xi_1, \xi_2) = \frac{1}{2\pi|\xi_2|} W_{|\phi\rangle\langle\psi|}(\xi_1, \xi_2), \quad \xi_1, \xi_2 \in \mathbb{R}.$$

Proposition 2 *The characteristic function of $(P, Q+M)$ in the state $|\phi\rangle\langle\psi|$ is given by*

$$\langle\psi, e^{iuP+iv(Q+M)}\phi\rangle_{\mathcal{H}} = \int_{\mathbb{R}} e^{iv\omega\mathrm{sinch}\, u}\overline{\psi}(\omega e^{u})\phi(\omega e^{-u})e^{-|\omega|\cosh u}\frac{|\omega|^{\beta-1}}{\Gamma(\beta)}d\omega, \quad u,v\in\mathbb{R}.$$

In the vacuum state $\Omega = 1_{\mathbb{R}_+}$ we have

$$\langle\Omega, e^{iuP+iv(Q+M)}\Omega\rangle_{\mathcal{H}} = \frac{1}{(\cosh u - iv\mathrm{sinch}\, u)^{\beta}}.$$

Note that $\tilde{W}_{|\psi\rangle\langle\phi|}$ has the correct marginals, since:

$$\int_{\mathbb{R}} \tilde{W}_{|\phi\rangle\langle\psi|}(\xi_1,\xi_2)d\xi_1 = \gamma_\beta(|\xi_2|)\overline{\phi}(\xi_2)\psi(\xi_2), \qquad \xi_2\in\mathbb{R},$$

and

$$\int_{\mathbb{R}} \tilde{W}_{|\psi\rangle\langle\phi|}(\xi_1,\xi_2)d\xi_2 \;=\; \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{-i\xi_1 x}\overline{\phi}(\omega e^{x/2})\psi(\omega e^{-x/2})e^{-|\omega|\cosh\frac{x}{2}}\frac{|\omega|^{\beta-1}}{\Gamma(\beta)}dxd\omega,$$

$\xi_1\in\mathbb{R}$. In the vacuum state $\Omega = 1_{\mathbb{R}_+}$, this yields respectively a Gamma law and the density

$$\int_{\mathbb{R}} W_{|\Omega\rangle\langle\Omega|}(\xi_1,\xi_2)\frac{d\xi_2}{2\pi\xi_2} \;=\; c\left|\Gamma\left(\frac{\beta}{2}+\frac{i}{2}\xi_1\right)\right|^2,$$

where c is a normalization constant and Γ is the Gamma function, which gives the expected hyperbolic cosine density when $\beta = 1$.

Definition 1 *For f in the Schwartz space $\mathcal{S}(\mathbb{R}^2)$, let the operator $O(f)$ be defined as*

$$O(f) = \int_{\mathbb{R}^2}(\mathcal{F}f)(x_1,x_2)e^{-\frac{i}{2}x_1P+ix_2(Q+M)}dx_1dx_2.$$

The following proposition extends the definition of $O(f)$ by continuity to a map from $L^2_{\mathbb{C}}(\mathbb{R}^2;\frac{d\xi_1 d\xi_2}{2\pi|\xi_2|})$ into the space $\mathcal{B}_2(\mathcal{H})$ of Hilbert-Schmidt operators on $\mathcal{H}$. It is obtained from the isometry given by the representation theorem of square-integrable representations of [3].

Proposition 3 *We have the bound*

$$\|O(f)\|_{\mathcal{B}_2(\mathcal{H})} \le \|f\|_{L^2_{\mathbb{C}}(\mathbb{R}^2;\frac{d\xi_1 d\xi_2}{2\pi|\xi_2|})}.$$

Note that we have

$$\langle\psi, O(f)\phi\rangle_{\mathcal{H}} = \int_{\mathbb{R}^2}\tilde{W}_{|\phi\rangle\langle\psi|}(\xi_1,\xi_2)f(\xi_1,\xi_2)d\xi_1 d\xi_2, \quad f\in L^2_{\mathbb{C}}\left(\mathbb{R}^2;\frac{d\xi_1 d\xi_2}{2\pi|\xi_2|}\right),$$

and $O(e^{-iu\xi_1-iv\xi_2}) = e^{-\frac{i}{2}uP+iv(Q+M)} = \hat{U}(e^{uX_1+vX_2})$, $u,v\in\mathbb{R}$.

3 Malliavin calculus on the affine algebra

We define a gradient operator which will be useful in showing the smoothness of Wigner densities. Let $\mathcal{S}_{\mathcal{H}}$ denote the algebra of operators on $\mathcal{H}$ that leave $\mathcal{S}(\mathbb{R})$ invariant.

Definition 2 *Fix $\kappa \in \mathbb{R}$ and let $x = (x_1, x_2) \in \mathbb{R}^2$. The gradient operator D_x is defined as*

$$D_x F = -\frac{i}{2} x_1 [P, F] + \frac{i}{2} x_2 [Q + \kappa M, F], \quad F \in \mathcal{S}_{\mathcal{H}}.$$

The following intertwining relation is the non-commutative analog of the integration by parts (1.1), and is proved using the covariance identity of [2].

Proposition 4 *Let $x_1, x_2 \in \mathbb{R}$. We have for $\kappa = 1$:*

$$D_{(x_1, 2x_2)} O(f) = [x_1 X_1 + x_2 X_2, O(f)] = O(x_1 \xi_2 \partial_1 f(\xi_1, \xi_2) - x_2 \xi_2 \partial_2 f(\xi_1, \xi_2)).$$

We turn to showing the smoothness of the Wigner density $\tilde{W}_{|\psi\rangle\langle\phi|}(\xi_1, \xi_2)$. Let $H^\sigma_{1,2}(\mathbb{R} \times (0, \infty))$ denote the Sobolev space with respect to the norm

$$\begin{aligned} &\|f\|^2_{H^\sigma_{1,2}(\mathbb{R}\times(0,\infty))} \\ &= \int_0^\infty \frac{1}{\xi_2} \int_{\mathbb{R}} |f(\xi_1, \xi_2)|^2 d\xi_1 d\xi_2 + \int_0^\infty \xi_2 \int_{\mathbb{R}} (|\partial_1 f(\xi_1, \xi_2)|^2 + |\partial_2 f(\xi_1, \xi_2)|^2) d\xi_1 d\xi_2. \end{aligned}$$

Theorem 1 *Let $\phi, \psi \in \mathrm{Dom}\, X_1 \cap \mathrm{Dom}\, X_2$. Then*

$$1_{\mathbb{R}\times(0,\infty)} W_{|\phi\rangle\langle\psi|} \in H^\sigma_{1,2}(\mathbb{R} \times (0, \infty)).$$

Proof. We have, for $f \in \mathcal{C}^\infty_c(\mathbb{R} \times (0, \infty))$ and $x_1, x_2 \in \mathbb{R}$:

$$\begin{aligned} &\left| \int_{\mathbb{R}^2} (x_1 \partial_1 f(\xi_1, \xi_2) + x_2 \partial_2 f(\xi_1, \xi_2)) \overline{W}_{|\phi\rangle\langle\psi|}(\xi_1, \xi_2) d\xi_1 d\xi_2 \right| && (3.1) \\ &= 2\pi \, |\langle \phi | O(x_1 \xi_2 \partial_1 f(\xi_1, \xi_2) - x_2 \xi_2 \partial_2 f(\xi_1, \xi_2)) \psi \rangle_{\mathcal{H}}| \\ &= 2\pi \, |\langle \phi | [x_1 X_1 + x_2 X_2, O(f)] \psi \rangle_{\mathcal{H}}| && (3.2) \\ &\leq \sqrt{2\pi} \|\phi\|_{\mathcal{H}} \|(x_1 X_1 + x_2 X_2) \psi\| \|f\|_{L^2_{\mathbb{C}}(\mathbb{R}^2; \frac{d\xi_1 d\xi_2}{|\xi_2|})}. && (3.3) \end{aligned}$$

□

Under the same hypothesis we can show that $1_{\mathbb{R}\times(-\infty,0)} W_{|\phi\rangle\langle\psi|}$ belongs to the Sobolev space $H^\sigma_{1,2}(\mathbb{R} \times (-\infty, 0))$ which is defined in a way similar to (3.1). We now define the analog of a Skorohod integral operator.

Definition 3 *Fix $\alpha \in \mathbb{R}$ and let for $F \in \mathcal{S}_{\mathcal{H}}$:*

$$\delta(F \otimes x) = \frac{x_1}{2} \{Q + \alpha(M - \beta), F\} + \frac{x_2}{2} \{P, F\} - D_x F,$$

with $x = (x_1, x_2) \in \mathbb{R}^2$.

Given $F, U, V \in \mathcal{S}_{\mathcal{H}}$, let

$$U \overleftarrow{D}_x^F = -\frac{i}{2} x_1 [P, U] F + \frac{i}{2} x_2 [Q, U] F, \quad \overrightarrow{D}_x^F V = -\frac{i}{2} x_1 F [P, V] + \frac{i}{2} x_2 F [Q, V],$$

and define a two-sided gradient as $U \overleftrightarrow{D}_x^F V = U \overleftarrow{D}_x^F V + U \overrightarrow{D}_x^F V$. Let $E[X] = \langle \Omega, X\Omega \rangle_{\mathcal{H}}$ denote the expectation of X when $\Omega = 1_{\mathbb{R}_+}$ is the vacuum state in $\mathcal{H}$. The integration by parts formulas given below generalizes the classical integration by parts formula (1.1) on $\mathbb{R}$. It follows from the relations

$$E[D_x F] = \frac{1}{2} E\left[x_1 \{Q, F\} + x_2 \{P, F\}\right], \quad \text{and} \quad E\left[\delta(F \otimes x)\right] = 0, \quad x = (x_1, x_2) \in \mathbb{R}^2.$$

Although δ is not the adjoint of D, we have the following analog of the commutative integration by parts formula.

Proposition 5 *Let $x = (x_1, x_2) \in \mathbb{R}$. Assume that $x_1(Q + \alpha M) + x_2 P$ commutes with $U, V \in \mathcal{S}_{\mathcal{H}}$. We have*

$$E[U \overleftrightarrow{D}_x^F V] = E[U \delta(F \otimes x) V], \qquad F \in \mathcal{S}_{\mathcal{H}}.$$

We also have the commutation formula, for $\kappa = 0$:

$$\begin{aligned} & D_x \delta(F \otimes y) - \delta(D_x F \otimes y) \\ &= \frac{1}{2} x_1 y_1 \{M + \alpha Q, F\} + i \frac{y_1 - i y_2}{2} x_2 [M, F] - \frac{i}{2} x_1 y_2 \{M, F\} + \frac{\alpha}{2} x_2 y_1 \{P, F\}, \end{aligned}$$

and

$$\delta(GF \otimes x) = G\delta(F) - G \overleftarrow{D}_F - \frac{x_1}{2} [Q + \alpha M, G] F - \frac{x_2}{2} [P, G] F,$$

$$\delta(FG \otimes x) = \delta(F) G - \overrightarrow{D}_F G - \frac{x_1}{2} F [Q + \alpha M, G] - \frac{x_2}{2} F [P, G].$$

By standard arguments, the operators D and δ can be shown to be closable for the topology of weak convergence in the space of bounded operators on $\mathcal{H}$.

4 Relation to the commutative case

In the Gaussian interpretation of Fock space, $\mathbf{q} = a_x^- + a_x^+ = x$ is multiplication by $x \in \mathbb{R}$. Taking $\beta = 1/2$ and writing $\tau = \frac{1}{2} x^2$, we have the relations

$$\tilde{a}_\tau^- = \frac{1}{2} \mathbf{q} a_x^-, \quad \tilde{a}_\tau^+ = \frac{1}{2} a_x^+ \mathbf{q}, \quad \tilde{a}_\tau^\circ = \frac{1}{2} a_x^+ a_x^-,$$

i.e.

$$\tilde{a}_\tau^- f(\tau) = \frac{1}{2} \mathbf{q} a_x^- f\left(\frac{x^2}{2}\right), \quad \tilde{a}_\tau^+ f(\tau) = \frac{1}{2} a_x^+ \mathbf{q} f\left(\frac{x^2}{2}\right), \quad \tilde{a}_\tau^\circ f(\tau) = \frac{1}{2} a_x^+ a_x^- f\left(\frac{x^2}{2}\right),$$

see e.g. [7]. The representation (M, B^-, B^+) of sl_2 can be constructed as

$$M = \frac{1}{2}(a_x^- a_x^+ + a_x^+ a_x^-), \quad B^- = \frac{1}{2}(a_x^-)^2, \quad B^+ = \frac{1}{2}(a_x^+)^2.$$

We have

$$Q + \alpha M = \frac{\alpha+1}{2}\frac{\mathbf{p}^2}{2} + \frac{\alpha-1}{2}\frac{\mathbf{q}^2}{2}, \quad \text{and} \quad M + \alpha Q = \frac{\alpha+1}{2}\frac{\mathbf{p}^2}{2} + \frac{1-\alpha}{2}\frac{\mathbf{q}^2}{2}.$$

The commutative case is obtained with $\alpha = 1$ when considering functionals of $\frac{\mathbf{q}^2}{2}$ only, or with $\alpha = -1$ when considering functionals of $\frac{\mathbf{p}^2}{2}$ only. For example the analogs of the classical integration by parts formula (1.1) are written as

$$E[D_{(1,0)}F] = \frac{1}{2}E\left[\left\{\frac{\mathbf{p}^2}{2}, F\right\} - F\right], \qquad E[D_{(1,0)}F] = \frac{1}{2}E\left[F - \left\{\frac{\mathbf{q}^2}{2}, F\right\}\right],$$

when $\alpha = 1$ and $\alpha = -1$ respectively.

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