Skorohod stochastic integration with respect to non-adapted processes on Wiener space

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Abstract

We define a Skorohod type anticipative stochastic integral that extends the Itô integral not only with respect to the Wiener process, but also with respect to a wide class of stochastic processes satisfying certain homogeneity and smoothness conditions, without requirements relative to filtrations such as adaptedness. Using this integral, a change of variable formula that extends the classical and Skorohod Itô formulas is obtained.

Key words: Itô calculus, Malliavin calculus, Skorohod integral.
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1 Introduction

The Skorohod integral, defined by creation on Fock space, is an extension of the Itô integral with respect to the Wiener or Poisson processes, depending on the probabilistic interpretation chosen for the Fock space. This means that it coincides with the Itô integral on square-integrable adapted processes. It can also extend the stochastic integral with respect to certain normal martingales, cf. [10], but it always acts with respect to the underlying process relative to a given probabilistic interpretation. The Skorohod integral is also linked to the Stratonovich, forward and backward integrals, and allows to extend the Itô formula to a class of Skorohod integral processes which contains non-adapted processes that have a certain structure.

In this paper we introduce an extended Skorohod stochastic integral on Wiener space that can act with respect to processes other than Brownian motion. It allows in particular to write a Itô formula for a class of stochastic processes that do not need to own any property with respect to filtrations. As a counterpart they are assumed to satisfy some smoothness and homogeneity conditions. The construction can be extended by means of an approximation procedure. In the particular case of integration with respect to Brownian motion our integral coincides with the classical

Skorohod integral. The definition of a new integral is justified if this integral plays a role in an analog of the "fundamental theorem of calculus", i.e. in a change of variables formula. If the underlying process has absolutely continuous trajectories, then only one choice of integral makes sense. If the trajectories of the process have less regularity, e.g. in the adapted Brownian case, then at least two notions of integral coexist (Itô and Stratonovich) and give rise to different change of variable formulas. For processes $(X_t)_{t\in[0,1]}$ that satisfy certain smoothness conditions (but no condition relative to the Brownian filtration) we obtain in Th. 1 the change of variable formula

$$f(X_t) = f(X_0) + \int_0^t f'(X_s)dX_s^u + \frac{1}{2}\int_0^t (u^s DX_s, DX_s)f''(X_s)ds, \quad t \in [0,1],$$

where D is the gradient on Wiener space, $(u_s^t)_{s,t\in[0,1]}$ is a function of two variables and dX_s^u is an anticipating stochastic differential with zero expectation, defined in Def. 8. The processes considered in Th. 1 have absolutely continuous trajectories but this is not necessarily restrictive for applications since the "non-zero quadratic variation" property can sometimes be viewed as a limiting case not attained in physical situations. The case of processes with non zero quadratic variation is considered in Corollary 1.

The main existing approaches to anticipating Itô calculus can be compared as follows to our construction.

- The Itô formula is extended to forward integral processes in [1], and to more general anticipating processes in [16], using the forward, backward and Stratonovich integrals constructed by Riemann sums. However, the stochastic integrals used in this framework do not have zero expectation in the anticipating case.

- The fact that the adapted Itô integral has zero expectation is essential in stochastic calculus and the Skorohod integral carries this property to the anticipative setting. The Itô formula for Skorohod integral processes was developed in [13], [18], cf. also [11] for a list of recent references. Skorohod integral processes represent a very particular class of stochastic integral processes, which is less natural from the point of view of applications than the processes considered in trajectorial approaches in e.g. [15].

The extension of the Skorohod integral introduced in this paper aims to combine the advantages of the trajectorial and Skorohod integrals. Namely, it has zero expectation and at the same time it yields a change of variable formula which is not restricted to Skorohod integral processes. The purpose of this construction is not to replace the Skorohod integral with another anticipating integral, since the Skorohod integral plays in fact an essential role in the definition of our extension. Rather, we suggest to modify the Skorohod integral in order to adapt it to the treatment of a larger class of processes.

This paper is organized as follows. In Sect. 2 the basic tools relative to the Fock space and its creation and annihilation operators are introduced. Sect. 3 is concerned with the definition of the extended Skorohod stochastic integral and its properties. A gradient operator (which coincides with the Malliavin calculus gradient in the case of integration with respect to Brownian motion) is also constructed as the adjoint of this Skorohod integral, and an integration by parts formula is obtained. In Sect. 4 we define the tools of our stochastic calculus, namely the analogs of the Itô and Stratonovich differentials and "quadratic covariation" without using the notion of filtration. This covariation is linked to the "carré du champ" operator associated to the Gross Laplacian on the Wiener space. Sect. 5 contains the main results of this paper. The Itô formula for our extended stochastic integral is stated in Th. 1, Th. 2, and Corollary 1. Our "quadratic covariation" bracket has several properties that make it different from its trajectorial analogues. In particular it is not symmetric and may be non zero even for processes with absolutely continuous trajectories, but in some cases (e.g. for Brownian motion) it coincides with its classical counterpart. This difference comes from the fact that even in the absolutely continuous case, our formula provides a decomposition of the process into a zero expectation "stochastic integral" part and a quadratic variation term. Since this stochastic integral term differs from the trajectorial forward integral, the quadratic variation terms also have to differ from their classical analogues. In Sect. 6 we examine the relationship between our change of variable formula and the Itô formula for the Skorohod integral. In particular we show that the Itô formula for Skorohod integral processes can be proved as a consequence of our result, which thus also extends the classical adapted Itô formula for Brownian motion. In Sect. 7 we deal with a class of non-Markovian processes that are not covered by the Skorohod change of variable formula and includes fractional Brownian motion with Hurst parameter in $]-1,1[$.

2 Notation and preliminaries

In this section we introduce the basic operators used in this paper, including the Skorohod integral. Let (W, H, μ) be the classical Wiener space with Brownian motion $(B_t)_{t\in[0,1]}$, where $H = L^2([0,1])$ has inner product $(\cdot,\cdot)$. As a convention, any function $f \in L^2([0,1])$ is extended to a function defined on $\mathbb{R}$ with $f(x) = 0$, $\forall x \notin [0,1]$. The Fock space $\Gamma(H)$ on a normed vector space H, is defined as the direct sum

$$\Gamma(H) = \bigoplus_{n\geq 0} H^{\odot n},$$

where "$\odot$" denotes the symmetrization of the completed tensor product of Hilbert spaces "$\otimes$". The symmetric tensor product $H^{\odot n}$ is endowed with the norm

$$\| \cdot \|^2_{H^{\odot n}} = n! \| \cdot \|^2_{H^{\otimes n}}, \quad n \in \mathbb{N}.$$

The annihilation and creation operators $D : \Gamma(H) \longrightarrow \Gamma(H) \otimes H$ and $\delta : \Gamma(H) \otimes H \longrightarrow \Gamma(H)$ are densely defined by linearity and polarization as

$$Dh^{\odot n} = nh^{\odot n-1} \otimes h, \quad \textit{and} \quad \delta\left(h^{\odot n} \otimes g\right) = h^{\odot n} \odot g, \quad n \in \mathbb{N}.$$

Throughout this paper, $\Gamma(H)$ is identified to $L^2(W)$ via the Itô-Wiener multiple stochastic integral isometric isomorphism. Namely, any $h_n \in H^{\odot n}$ is associated to its multiple stochastic integral $I_n(h_n)$ defined as

$$I_n(h_n) = n! \int_0^1 \int_0^{t_n} \cdots \int_0^{t_2} h_n(t_1, \ldots, t_n) dB_{t_1} \cdots dB_{t_n}.$$

Under this identification, D becomes a derivation operator whose domain is denoted by $\mathbb{D}_{2,1}$. We denote by $\mathbb{D}_{2,2}$ the set of functionals $F \in \mathbb{D}_{2,1}$ such that $D_t F \in \mathbb{D}_{2,1}$, dt-a.e., with

$$E\left[\int_0^1 \int_0^1 (D_s D_t F)^2 ds dt\right] < \infty.$$

Let $\mathcal{C}_b^2(\mathbb{R})$ denote the set of twice continuously differentiable real functions that are bounded together with their derivatives. Let $\mathcal{P}(\mathbb{R})$ denote the space of real polynomials, and let $\mathcal{C}_c^1([0,1])$ denote the set of continuously differentiable functions on $[0,1]$ that vanish on $\{0,1\}$.

Definition 1 *Let $\mathcal{S}$ denote the set of compositions of functions in $\mathcal{C}_b^2(\mathbb{R}) \cup \mathcal{P}(\mathbb{R})$ with elements of the vector space generated by*

$$\{I_n(f_1 \odot \cdots \odot f_n), \quad f_1, \ldots, f_n \in \mathcal{C}_c^1([0,1])\}.$$

We denote by $\mathcal{U}$ the set of processes $v \in L^2(W) \otimes L^2([0,1])$ such that $v_t \in \mathcal{S}$, dt-a.e. and $(v_t)_{t\in[0,1]}$ has bounded support.

We have that $\mathcal{S}$ is dense in $L^2(W)$ and $\mathcal{U}$ is dense in $L^2(W) \otimes L^2([0,1])$. From the multiplication formula between n-th and first order Wiener integrals

$$I_n(f^{\odot n})I_1(g) = I_{n+1}(g \odot f^{\odot n}) + n(f,g)I_{n-1}(f^{\odot(n-1)}), \tag{1}$$

$f, g \in L^2([0,1])$, we check that $\mathcal{S}$ is an algebra, with $\mathcal{S} \subset \cap_{p\geq 2} L^p(W)$. The operator δ is identified to the Skorohod integral, cf. [17]. It satisfies

$$\| \delta(u) \|^2_{L^2(W)} = \| u \|^2_{L^2(W)\otimes L^2([0,1])} + E\left[\int_0^1 \int_0^1 D_s u_t D_t u_s ds dt\right], \quad v \in \mathcal{U},$$

hence in particular $\mathcal{U} \subset \mathrm{Dom}(\delta)$. Let ∂ denote the operator of differentiation of functions of real variable. We now introduce a weighted Gross Laplacian.

Definition 2 *Let $\mathcal{D}$ denote the set of functionals $F \in \mathbb{D}_{2,2}$ such that $\lim_{s\uparrow t} D_s D_t F$ exists $dP \otimes dt$ a.e. and belongs to $L^2(W) \otimes L^2([0,1])$. For $u \in \mathcal{C}^1_c([0,1])$ we define on $\mathcal{D}$ the operator $\mathcal{G}_u : L^2(W) \to L^2(W)$ as*

$$\mathcal{G}_u F = \frac{1}{2} \int_0^1 \lim_{s\uparrow t} D_s D_t F \partial_t u_t dt.$$

If $u_t = t$, $t \in [0,1]$, then $\mathcal{G}_u$ is identical to the classical Gross Laplacian, cf. [8], and it acts on compositions of smooth functions with elements of $\mathcal{S} \subset \mathcal{D}$ in the following way.

Proposition 1 *We have for $f \in \mathcal{C}^2_b(\mathbb{R}) \cup \mathcal{P}(\mathbb{R})$ and $u \in \mathcal{C}^1_c([0,1])$:*

$$\mathcal{G}_u f(F) = f'(F)\mathcal{G}_u F + \frac{1}{2} f''(F)(DF, DF\partial u), \quad F \in \mathcal{S}. \tag{2}$$

Proof. We have

$$\frac{1}{2} D_s D_t f(F) = \frac{1}{2} f'(F) D_s D_t F + \frac{1}{2} f''(F) D_s F D_t F, \quad s,t \in [0,1].$$

Moreover, $(D_s F)_{s\in[0,1]}$ has continuous trajectories since $F \in \mathcal{S}$.

□

This weighted Gross Laplacian can be viewed as an infinite dimensional realization of the generator of Brownian motion, from the relation

$$\mathcal{G}_u [f(B_t)] = \left[\frac{1}{2}\partial^2\right] f(B_t),$$

if $\int_0^t u_s ds = 1$. We also define the "carré du champ" operator $\Gamma^u : \mathcal{S} \times \mathcal{S} \to \mathcal{S}$ associated to $\mathcal{G}_u$ as

$$\Gamma^u(F,G) = \mathcal{G}_u(FG) - F\mathcal{G}_u - G\mathcal{G}_u F, \quad F,G \in \mathcal{S}.$$

This operator bilinear and symmetric but not necessarily positive. We have from Prop. 1:

$$\Gamma^u(F,G) = (\partial u DF, DG), \quad F,G \in \mathcal{S}.$$

As a consequence of the identity

$$\delta(uF) = F\delta(u) - (u, DF), \quad F \in \mathcal{S},\ u \in \mathcal{U}, \tag{3}$$

cf. [12], we have the following result which will be essential in the proof of our extended Itô formula. Let $\mathcal{C}_0^1([0,1]) = \{u \in \mathcal{C}_c^1([0,1]) \ : \ u_0 = 0\}$.

Proposition 2 *Let $u \in \mathcal{C}_0^1([0,1])$ and $f \in \mathcal{C}_b^2(\mathbb{R}) \cup \mathcal{P}(\mathbb{R})$. We have*

$$\delta(u\partial Df(F)) = f'(F)\delta(u\partial DF) + \frac{1}{2}f''(F)(DF, DF\partial u), \quad F \in \mathcal{S}.$$

Proof. We have

$$\begin{aligned} \delta(u\partial Df(F)) &= \delta(f'(F)u\partial DF) = f'(F)\delta(u\partial DF) - (Df'(F), \partial DF\partial u) \\ &= f'(F)\delta(u\partial DF) + \frac{1}{2}f''(F)(\partial u DF, DF). \quad \square \end{aligned}$$

We note that as a consequence of Prop. 1 and Prop. 2, $F \mapsto \delta(u\partial DF) - \mathcal{G}_u F$ is a derivation operator and for $f \in \mathcal{C}_b^2(\mathbb{R}) \cup \mathcal{P}(\mathbb{R})$,

$$\delta(u\partial Df(F)) - \mathcal{G}_u f(F) = f'(F)(\delta(u\partial DF) - \mathcal{G}_u F), \quad F \in \mathcal{S}. \tag{4}$$

Moreover, from (3), the "carré du champ" Γ^u also satisfies

$$\begin{aligned} \Gamma^u(F,G) &= (\partial u DF, DG) = -(uDF, \partial DG) - (uDG, \partial DF) \\ &= \delta(u\partial D(FG)) - F\delta(u\partial DG) - G\delta(u\partial DF), \quad F,G \in \mathcal{S}. \end{aligned} \tag{5}$$

Hence $\Gamma^u(F,F)$ and $-\Gamma^u(F,F)$ are exactly the terms that compensate each-other so that $F \mapsto \delta(u\partial DF) - \mathcal{G}_u F$ becomes a derivation. Let $h \in \mathcal{C}_0^1([0,1])$ with $\| h \|_\infty < 1$, and $\nu_h(t) = t + h_t$, $t \in [0,1]$. For F in the vector space $\mathcal{A}$ generated by

$$\{I_n(h_1 \odot \cdots \odot h_n) \ : \ h_1, \ldots, h_n \in L^2([0,1]),\ n \in \mathbb{N}\},$$

let

$$F = f(I_1(g_1), \ldots, I_1(g_m))$$

be a polynomial in single stochastic integrals. We define

$$\mathcal{U}_h F = f(I_1(g_1 \circ \nu_h), \ldots, I_1(g_m \circ \nu_h)).$$

The definition of the operator $\mathcal{U}_h$ extends to $\mathcal{D}$ by linearity since polynomial in single stochastic integrals form a basis of $\mathcal{D}$. It also extends to functionals of the form $f(F)$, $f \in \mathcal{C}(\mathbb{R})$, $F \in \mathcal{A}$, and to $\mathcal{S}$. If F is a random variable defined for every trajectory of $(B_t)_{t\in[0,1]}$, let $\mathcal{T}_h F$ denote the functional $F \in \mathcal{S}$ evaluated at time-changed trajectories are given by the time-changed Brownian motion $B^h_{\nu_h(t)} = B_t$, $t \in [0,1]$. Single Wiener stochastic integrals can be defined everywhere provided their integrand belongs to $\mathcal{C}^1_c([0,1])$ and multiple stochastic integrals in $\mathcal{S}$ can be expressed as polynomials in single stochastic integrals, hence for any $F \in \mathcal{S}$ there is a version $\hat{F}$ of F such that $\mathcal{T}_{\varepsilon h}\hat{F} = \mathcal{U}_{\varepsilon h} F$, $\varepsilon \in [-1,1]$, a.s. We are using $\mathcal{U}_h$ instead of $\mathcal{T}_h$ because the former is defined on a set of L^2 functionals, whereas $\mathcal{T}_h$ is not.

Lemma 1 *Let $u \in \mathcal{C}^1_0([0,1])$. We have*

$$\frac{d}{d\varepsilon}\mathcal{U}_{\varepsilon u} F_{|\varepsilon=0} = \delta(u\partial DF) - \mathcal{G}_u F, \; a.s., \; F \in \mathcal{S}. \tag{6}$$

Proof. Given (4) it suffices to notice that (6) holds for a single stochastic integral $F = I_1(h) \in \mathcal{S}$:

$$\frac{d}{d\varepsilon}\mathcal{U}_{\varepsilon u} I_1(h)_{|\varepsilon=0} = \frac{d}{d\varepsilon} I_1(h \circ \nu_{\varepsilon u})_{|\varepsilon=0} = I_1(u\partial h), \; a.s. \quad \square$$

The Wick product of $F, G \in \mathcal{S}$ is defined by linearity from

$$I_n(f_n) \diamond I_m(g_m) = I_{n+m}(f_n \odot g_m), \qquad n, m \in \mathbb{N}.$$

Using the smoothed Brownian motion and white noise respectively written as

$$B^\phi_t = \int_0^1 \phi^t_s dB_s \;\; and \;\; W^\phi_t = -\int_0^1 \partial_s \phi^t_s dB_s, \quad \phi \in \mathcal{C}^1([0,1]), \tag{7}$$

where ϕ has support in $]-\infty, 1]$ and $\phi^t_s = \phi(s-t)$, $s, t \in [0,1]$, we have

$$\delta(\phi * v) = \int_0^1 v_s \diamond W^\phi_s ds,$$

cf. [7], and if $W_t = I_1(\delta_t)$, $t \in [0,1]$, denotes white noise in the sense of Hida distributions, then

$$\delta(v) = \int_0^1 v_s \diamond W_s ds.$$

3 Skorohod integration with respect to non-adapted processes

In this section we construct a Skorohod type integral that acts with respect to a class of not necessarily Markovian or adapted processes. If $u = (u_s^t)_{s,t\in[0,1]}$ is a family of functions we adopt the conventions $u^t = (u_s^t)_{s\in[0,1]}$, $t \in [0,1]$, and $u_s = (u_s^t)_{t\in[0,1]}$, $s \in [0,1]$.

Definition 3 *Let $\mathcal{V}$ denote the set of couples (X,u) where $X = (X_t)_{t\in[0,1]}$ is a family of random variables contained in $\mathcal{D}$ and $u = (u_s^t)_{s,t\in[0,1]}$ is a family of functions such that $u^t \in \mathcal{C}_0^1([0,1])$, $0 \le t \le 1$.*

We now define the extended Skorohod integral with respect to a given process $(X,u) \in \mathcal{V}$. The interpretation of this operator as a stochastic integral will result from the change of variable formula of Th. 1.

Definition 4 *For $(X,u) \in \mathcal{V}$ such that $(X_t)_{t\in[0,1]} \subset \mathcal{S}$ we define the unbounded operator $\delta^{X,u} : L^2(W) \otimes L^2([0,1]) \to L^2(W)$ on $\mathcal{U}$ as*

$$\delta^{X,u}(v) = -\int_0^1 \delta(v_s u^s \partial D X_s) ds, \quad v \in \mathcal{U}. \tag{8}$$

From (3) we have

$$\delta^{X,u}(v) = -\int_0^1 v_s \delta(u^s \partial D X_s) ds + \int_0^1 (D v_s, u^s \partial D X_s) ds, \quad v \in \mathcal{U}. \tag{9}$$

We note that if v is of the form $v_s = f'(X_s)$, $f \in \mathcal{C}_b^1(\mathbb{R}) \cup \mathcal{P}(\mathbb{R})$, then $t \mapsto \delta^{X,u}(f'(X.)1_{[0,t]}(\cdot))$ is absolutely continuous with

$$d\delta^{X,u}(f'(X.)1_{[0,t]}) = -\delta(u^s \partial D f(X_t)) dt. \tag{10}$$

Using the notation $\delta(v) = \int_0^1 v_t \delta B_t$ we may also write

$$\delta^{X.u}(v) = -\int_0^1 (v u_t, \partial_t D_t X) \delta B_t = -\int_0^1 \int_0^1 v_s u_s^t \partial_t D_t X_s ds \delta B_t.$$

In the particular case where the process X is written as $X_t = \int_0^1 h_s^t dB_s$, i.e. X_t belongs to the first Wiener chaos, $t \in [0,1]$, then

$$\delta^{X,u}(v) \;=\; -\int_0^1 \delta(v_s u^s \partial D X_s) ds = -\int_0^1 \delta(v_s u^s \partial h^s) ds = -\int_0^1 v_s \diamond I_1(u^s \partial h^s) ds,$$

and if $X = B^\phi$ is the approximation (7) of Brownian motion obtained by convolution, then

$$\delta^{B^\phi,u}(v) \quad = \quad - \int_0^1 v_s \diamond I_1(u^s \partial \phi^s) ds.$$

If further $\phi_s = 1$, $s \leq 0$, and $u^s_s = 1$, $s \in [0,1]$, then $u^s \partial \phi^s = \partial \phi^s$ and $-I_1(u^s \partial \phi^s) = W_{\phi^s}$, hence

$$\delta^{B^\phi,u}(v) = - \int_0^1 v_s \diamond W_{\phi^s} ds = \delta(\phi * v),$$

and as ϕ approaches $1_{]-\infty,0]}$ in distribution, $\delta^{B^\phi,u}(v)$ converges in the sense of Hida distributions to the Skorohod integral $\delta(v)$ of v, cf. [7] and the references therein.

We now turn to the definition of the gradient operator $D^{X,u}$ adjoint of $\delta^{X,u}$.

Definition 5 *Let $(X,u) \in \mathcal{V}$. We define the operator $D^{X,u} : L^2(W) \to L^2(W) \otimes L^2([0,1])$ on $\mathcal{S}$ as*

$$D^{X,u}_s F = (\partial (u^s DF), DX_s), \quad s \in [0,1], \ F \in \mathcal{S}.$$

We remark that if $(X_t)_{t\in[0,1]} \subset \mathcal{S}$, then by integration by parts,

$$D^{X,u}_s F = -(u^s DF, \partial DX_s), \quad s \in [0,1], \ F \in \mathcal{S}. \tag{11}$$

If $X = B$ and $u^t_t = 1$, $t0 \leq t \leq 1$, then $D^{B,u}$ is the gradient D of the Malliavin calculus since

$$D^{B,u}_t F = \int_0^t \partial_s (u^t_s D_s F) ds = u^t_t D_t F - u^t_0 D_0 F = D_t F, \quad F \in \mathcal{S}, \ 0 \leq t \leq 1. \tag{12}$$

Proposition 3 *Let $(X,u) \in \mathcal{V}$ such that $(X_t)_{t\in[0,1]} \subset \mathcal{S}$. The operators $D^{X,u}$ and $\delta^{X,u}$ are closable and the following duality relation holds:*

$$E[F\delta^{X,u}(v)] = E[(D^{X,u}F, v)], \quad F \in \mathcal{S}, \ v \in \mathcal{U}. \tag{13}$$

Proof. If $(X_t)_{t\in[0,1]} \subset \mathcal{S}$ we may integrate by parts and obtain, since $u^s_0 = 0$, $s \in [0,1]$:

$$\begin{aligned}
E[(D^{X,u}F, v)] &= E\left[\int_0^1 v_s \int_0^1 \partial_t (u^s_t D_t F) D_t X_s dt ds\right] \\
&= -E\left[\int_0^1 v_s \int_0^1 u^s_t D_t F \partial_t D_t X_s dt ds\right] \\
&= -E\left[\int_0^1 D_t F \int_0^1 v_s u^s_t \partial_t D_t X_s ds dt\right] \\
&= -E\left[F\delta\left(\int_0^1 v_s u^s \partial DX_s ds\right)\right] = E[F\delta^{X,u}(v)],
\end{aligned}$$

hence (13). The closability of $\delta^{X,u}$ and $D^{X,u}$ follows from this duality relation and the fact that $\delta^{X,u}$ and $D^{X,u}$ have dense domains.

□

We now extend the definition of $\delta^{X,u}$ as a closable operator to $(X,u) \in \mathcal{V}$.

Definition 6 *For $(X,u) \in \mathcal{V}$, let $\mathrm{Dom}(\delta^{X,u})$ be the set of $v \in L^2(W) \otimes L^2([0,1])$ such that there is a constant $C > 0$ such that*

$$| E[(D^{X,u}F, v)] | \leq C \parallel F \parallel^2_{L^2(W)}, \quad F \in \mathcal{S}.$$

For $v \in \mathrm{Dom}(\delta^{X,u})$ we denote by $\delta^{X,u}(v)$ the random variable that satisfies

$$E[(D^{X,u}F, v)] = E[F\delta^{X,u}(v)], \quad F \in \mathcal{S}.$$

From Prop. 3, if $X \in \mathcal{V}$ satisfies $(X_t)_{t\in[0,1]} \subset \mathcal{S}$, then $\mathcal{U} \subset \mathrm{Dom}(\delta^{X,u})$. Moreover if $X = B$ and $u^t_t = 1$, $0 \leq t \leq 1$, $\delta^{B,u}$ is the Skorohod integral since $D^{B,u} = D$ from (12). The following result follows from the fact that $D^{X,u}$ is a derivation operator adjoint of $\delta^{X,u}$.

Proposition 4 *For $(X,u) \in \mathcal{V}$ such that $(X_t)_{t\in[0,1]} \subset \mathcal{S}$, we have for any $v \in \mathrm{Dom}(\delta^{X,u})$ and $F \in \mathcal{S}$ such that $F\delta^{X,u}(v) - (v, D^{X,u}F) \in L^2(W)$:*

$$\delta^{X,u}(vF) = F\delta^{X,u}(v) - (v, D^{X,u}F), \quad F \in \mathcal{S}.$$

Proof. It suffices to prove that for $F, G \in \mathcal{S}$ and $v \in \mathcal{U}$,

$$\begin{aligned} E[G\delta^{X,u}(vF)] &= E[(v, FD^{X,u}G)] = E[(v, D^{X,u}(FG) - GD^{X,u}F)] \\ &= E[G(F\delta^{X,u}(v) - (v, D^{X,u}F))]. \quad \square \end{aligned}$$

4 Stochastic differentials, quadratic covariation and the "carré du champ"

We define a class of families of random variables which will play the role of Itô processes in our construction and introduce the notions of Itô differential, Stratonovich differential, and "quadratic covariation" of such processes, in connection to the "carré du champ" Γ^u. These operators have been introduced in [14] in the case of Lévy processes.

Definition 7 *We denote by $\mathcal{HV}$ the class of processes $(X,u) \in \mathcal{V}$ such that*

- $(X_t)_{t\in[0,1]} \subset \mathcal{S}$,

- *$t \mapsto X_t$ is differentiable in $L^2(W)$ with respect to t and satisfies the homogeneity condition*
$$\frac{d}{dt}X_t = -\frac{d}{d\varepsilon}\mathcal{U}_{\varepsilon u^t}X_t \mid_{\varepsilon=0}, \quad a.s., \ 0 \le t \le 1. \tag{14}$$

We note that $\mathcal{HV}$ is stable under the composition by functions in $\mathcal{C}_b^1(\mathbb{R}) \cup \mathcal{P}(\mathbb{R})$, i.e. if $(X, u) \in \mathcal{HV}$, then $(f(X), u) \in \mathcal{HV}$, $f \in \mathcal{C}_b^1(\mathbb{R}) \cup \mathcal{P}(\mathbb{R})$. Processes in $\mathcal{HV}$ have absolutely continuous trajectories, hence the process (B, u) does not belong to $\mathcal{HV}$.

Remark 1 *The approximation (B^ϕ, u) of (B, u) defined at the end of Sect. 2 does belong to $\mathcal{HV}$, provided $\phi_s = 1$, $s \le 0$, and $u^t \in \mathcal{C}_0^1([0,1])$ satisfies $u_s^t = 1$, $1 \ge s \ge t/2$, since in this case*
$$\mathcal{U}_{\varepsilon u^t} f(B_t^\phi) = f(B_{t-\varepsilon}^\phi), \ a.s., \ 0 \le \varepsilon \le t/2, \ f \in \mathcal{C}(\mathbb{R}).$$

The following is an analytic definition of the tools of stochastic calculus. Terms that do not necessarily coincide with their classical definitions are quoted.

Definition 8 *Let $(X, u) \in \mathcal{V}$.*

- *We define the "Itô differential" dX_t^u and its associated stochastic integral as*
$$\int_0^1 v_s dX_s^u = \delta^{X,u}(v) + \int_0^1 v_s \mathcal{G}_{u^s} X_s ds, \quad v \in \mathrm{Dom}(\delta^{X,u}).$$

- *We define the "quadratic covariation" of (X, u) and $(Y, v) \in \mathcal{V}$ with $(X_t)_{t\in[0,1]} \subset \mathcal{S}$ to be the process $([X, Y]_t^u)_{t\in[0,1]}$ satisfying $[X, Y]_0^u = 0$ and*
$$d[X, Y]_t^u = -2(u^t \partial DX_t, DY_t)dt = 2D_t^{X,u} Y_t dt, \quad t \in [0, 1].$$

- *If $(X, u) \in \mathcal{V}$ with $(X_t)_{t\in[0,1]} \subset \mathcal{S}$ then we define the Stratonovich differential $\circ dX_t^u$ and its associated stochastic integral as*
$$\int_0^1 v_s \circ dX_s^u = \int_0^1 v_s(\mathcal{G}_{u^s} X_s - \delta(u^s \partial DX_s))ds, \quad v \in \mathcal{U}.$$

We make the following remarks.

- The "quadratic covariation" $[X, Y]_t^u$ depends on X, Y and u, but not on v. The application $((X, u), (Y, v)) \mapsto [X, Y]^u$ is bilinear but not symmetric. If $u = v$, then $[X, Y]_t^u + [Y, X]_t^u$ is defined for $(X, u), (Y, v) \in \mathcal{V}$ (without requiring X or Y to take values in $\mathcal{S}$), with
$$d[X, Y]_t^u + d[Y, X]_t^u = 2(DX_t, DY_t \partial u^t)dt = 2D_t^{X,u} Y_t dt + 2D_t^{Y,u} X_t dt, \quad t \in [0, 1],$$

and in particular,

$$d[X,X]_t^u = (DX_t, DX_t \partial u^t)dt = 2D_t^{X,u} X_t dt, \quad t \in [0,1].$$

The relationship between the covariation bracket $[X,Y]_t^u$ and the "carré du champ" Γ^u is

$$\Gamma^{u^s}(X_s, Y_s)ds = \frac{1}{2}(d[X,Y]_s^u + d[Y,X]_s^u),$$

i.e. $\Gamma^{u^s}(X_s, Y_s)$ is the density of the symmetrization of the covariation bracket of X and Y, and in particular,

$$\Gamma^{u^s}(X_s, X_s)ds = d[X,X]_s^u.$$

- We have for $(X,u), (Y,v) \in \mathcal{V}$ with $(X_t)_{t\in[0,1]} \subset \mathcal{S}$:

$$d[f(X), g(Y)]_t^u = f'(X_t)g'(Y_t)d[X,Y]_t^u, \quad t \in [0,1], \quad f,g \in \mathcal{C}_b^1(\mathbb{R}),$$

since D is a derivation operator.

- We already noticed that Γ^u can be defined equivalently from $\mathcal{G}_u$ or from $F \mapsto \delta(u\partial DF)$, with the same formula. In some sense, Γ^u measures the "difference" between such operators and derivation operators. In particular, (5) can be rewritten as

$$\Gamma^{u^s}(f(X_s), g(Y_s))ds = d(f(X_s)g(Y_s))^u - f(X_s)dg(Y_s)^u - g(Y_s)df(X_s)^u, \quad (15)$$

$f,g \in \mathcal{C}_b^1(\mathbb{R}) \cup \mathcal{P}(\mathbb{R})$, $(X,u) \in \mathcal{HV}$, $(Y,u) \in \mathcal{HV}$, where

$$f(X_s)dg(Y_s)^u = -\delta(u^s f(X_s)\partial Dg(Y_s))ds + f(X_s)\mathcal{G}_{u^s} g(Y_s), \quad (16)$$

$$g(Y_s)df(X_s)^u = -\delta(u^s g(Y_s)\partial Df(X_s))ds + g(Y_s)\mathcal{G}_{u^s} f(X_s), \quad (17)$$

and

$$d(f(X_s)g(Y_s))^u = -\delta(u^s \partial D(f(X_s)g(Y_s)))ds + \mathcal{G}_{u^s}(f(X_s)g(Y_s))ds.$$

Hence $\Gamma^{u^s}(f(X_s), g(Y_s))$ is the density of the correction term in the product of "Itô differential", and this property is directly linked to the analytic definition of Γ^u from the weighted Gross Laplacian $\mathcal{G}_u$. This fact can be viewed as an infinite-dimensional realization, for a *finite dimensional* process, of a well-known situation in stochastic calculus and potential theory, cf. e.g. Ch. XV of [4] and the references therein.

- One has to be careful here that the differential dX^u behaves differently in general from classical stochastic differentials. In particular, in (15), $f(X_s)dg(Y_s)^u$ and $g(Y_s)df(X_s)^u$, unlike their Stratonovich counterparts, have no interpretation as a pointwise product of a process and a differential, but hold in the sense of (16) and (17). In short, the differential dX^u acts on the integrand as an operator, and not necessarily as a multiplication operator.

- For processes $(X,u)\in\mathcal{V}$ and $(Y,u)\in\mathcal{V}$ such that $(X_t)_{t\in[0,1]}\subset\mathcal{S}$ and $(Y_t)_{t\in[0,1]}\subset\mathcal{S}$, the operator Γ^u can be expressed with ordinary differentials of absolutely continuous processes from (5) and (10):

$$\Gamma^{u^s}(X_s,Y_s)ds = d\delta^{XY,u}(1_{[0,s]}) - X_s d\delta^{Y,u}(1_{[0,s]}) - Y_s d\delta^{X,u}(1_{[0,s]}).$$

If $(X,u),(Y,v)\in\mathcal{HV}$, then $(X_t)_{t\in[0,1]}$ and $(Y_t)_{t\in[0,1]}$ have absolutely continuous trajectories, hence zero quadratic covariation. Therefore $[X,Y]^u_t$ is an object which is in general different from the classical quadratic covariation $[X,Y]_t$. Moreover $[X,X]^u_t$ may be negative, depending on u. Both notions may coincide, e.g. if $X=B$ is the Brownian motion and $u^t_t=1$, $0\le t\le 1$. The reason why $[X,X]^u$ may differ in general from the trajectorial quadratic variation is that the stochastic term of our Itô formula is constrained to have zero expectation, unlike e.g. the forward integral used in [1] and [16]. For $(X,u)\in\mathcal{HV}$, i.e. in the absolutely continuous case, there is only one natural notion of differential which is the Stratonovich differential $\circ dX^u_t$, and dX^u_t differs from it. The relation between the differentials dX^u_t and $\circ dX^u_t$ is given by the following proposition.

Proposition 5 *Let $(X,u)\in\mathcal{V}$ such that $(X_t)_{t\in[0,1]}\subset\mathcal{S}$. We have*

$$\begin{aligned}\int_0^1 v_s dX^u_s &= \int_0^1 v_s\circ dX^u_s - \int_0^1 D^{X,u}_s v_s ds\\ &= \int_0^1 v_s\circ dX^u_s - \frac{1}{2}\int_0^1 [X_s,v_s]^u_s ds, \quad v\in\mathcal{U}.\end{aligned}$$

Proof. We have from (9) and (11):

$$\begin{aligned}\int_0^1 v_s dX^u_s &= -\int_0^1 \delta(v_s u^s\partial DX_s)ds + \int_0^1 v_s\mathcal{G}_{u^s}X_s ds\\ &= -\int_0^1 v_s\delta(u^s\partial DX_s)ds + \int_0^1 (Dv_s,u^s\partial DX_s)ds + \int_0^1 v_s\mathcal{G}_{u^s}X_s ds\\ &= \int_0^1 v_s\circ dX^u_s - \int_0^1 D^{X,u}_s v_s ds, \quad v\in\mathcal{U}. \quad \square\end{aligned}$$

In particular, $\int_0^1 v_s dX^u_s$ and $\int_0^1 v_s\circ dX^u_s$ coincide if v is deterministic.

5 The Itô formula

This section contains the main results of this paper. We start by writing a Stratonovich type change of variable formula which uses the natural differential $\circ dX_t^u$ of processes $(X,u) \in \mathcal{HV}$.

Proposition 6 *(Stratonovich formula). Let $(X,u) \in \mathcal{HV}$. We have the change of variable formula*

$$f(X_t) = f(X_0) + \int_0^t f'(X_s) \circ dX_s^u, \quad f \in \mathcal{C}_b^2(\mathbb{R}) \cup \mathcal{P}(\mathbb{R}). \tag{18}$$

Proof. We apply Lemma 1 to $F = f(X_s)$ to obtain

$$\begin{aligned} f(X_t) - f(X_0) &= -\int_0^t \frac{d}{d\varepsilon} \mathcal{U}_{\varepsilon u^s} f(X_s)_{|\varepsilon=0} ds \\ &= -\int_0^t (\delta(u^s \partial D f(X_s)) - \mathcal{G}_{u^s} f(X_s)) ds \\ &= -\int_0^t f'(X_s)(\delta(u^s \partial D X_s) - \mathcal{G}_{u^s} X_s) ds = \int_0^t f'(X_s) \circ dX_s^u, \ a.s. \quad \square \end{aligned}$$

We can now write the Itô change of variable formula.

Theorem 1 *The Itô formula for $(X,u) \in \mathcal{HV}$ is written as*

$$f(X_t) = f(X_0) + \int_0^t f'(X_s) dX_s^u + \frac{1}{2} \int_0^t f''(X_s) d[X,X]_s^u, \quad t \in [0,1], \quad f \in \mathcal{C}_b^2(\mathbb{R}) \cup \mathcal{P}(\mathbb{R}).$$

Proof. We use Prop. 6 and Relation (18) (or Prop. 5 applied to $v_s = f'(X_s)$) to obtain

$$\begin{aligned} f(X_t) &= f(X_0) + \int_0^t f'(X_s) \left(\mathcal{G}_{u^s} X_s - \delta(u^s \partial D X_s) \right) ds \\ &= f(X_0) - \int_0^t \delta(f'(X_s) u^s \partial D X_s) ds + \int_0^t f'(X_s) \mathcal{G}_{u^s} X_s ds \\ &\quad + \frac{1}{2} \int_0^t (DX_s, DX_s \partial u^s) f''(X_s) ds \\ &= f(X_0) + \delta^{X,u}(1_{[0,t]}(\cdot) f'(X_\cdot)) + \int_0^t f'(X_s) \mathcal{G}_{u^s} X_s ds \\ &\quad + \frac{1}{2} \int_0^t (DX_s, DX_s \partial u^s) f''(X_s) ds. \quad \square \end{aligned}$$

The Itô correction term reads

$$\frac{1}{2} \int_0^t (DX_s, DX_s \partial u^s) f''(X_s) ds = \int_0^t D_s^{X,u} f'(X_s) ds = \int_0^t f''(X_s) D_s^{X,u} X_s ds.$$

The proof of Th. 1 generalizes easily to vector-valued processes including an absolutely continuous drift.

Theorem 2 *Let* $(X_1, u^1), \ldots, (X^n, u^n) \in \mathcal{HV}$ *and let* $Y_t = \int_0^t V_s ds$, $V \in \mathcal{U}$. *The Itô formula for* $f(X_t^1, \ldots, X_t^n, Y_t)$ *is written for* $f \in \mathcal{C}_b^2(\mathbb{R}^{n+1}, \mathbb{R})$ *as*

$$\begin{aligned} f(X_t^1, \ldots, X_t^n, Y_t) &= f(X_0^1, \ldots, X_0^n, Y_0) + \sum_{i=1}^{i=n} \int_0^t \partial_i f(X_s^1, \ldots, X_s^n, Y_s) dX_s^{i,u^i} \\ &+ \int_0^t V_s \partial_{n+1} f(X_s^1, \ldots, X_s^n, Y_s) ds \\ &+ \frac{1}{2} \sum_{i,j=1}^{n} \int_0^t \partial_i \partial_j f(X_s^1, \ldots, X_s^n, Y_s) d[X^i, X^j]_s^{u^i} \\ &+ \frac{1}{2} \sum_{i=1}^{i=n} \int_0^t \partial_i \partial_{n+1} f(X_s^1, \ldots, X_s^n, Y_s) d[X^i, Y]_s^{u^i}. \end{aligned}$$

Proof. Although it is similar to that of Th. 1, the proof of this extension is stated because it shows the important role played here by the quadratic covariation $[X^i, Y]^{u^i}$ (which always vanishes in the classical case). We have

$$\begin{aligned} & f(X_t^1, \ldots, X_t^n, Y_t) = f(X_0^1, \ldots, X_0^n, Y_0) \\ & \quad + \sum_{i=1}^{i=n} \int_0^t \partial_i f(X_s^1, \ldots, X_s^n, Y_s) \left(\mathcal{G}_{u^{i,s}} X_s^i - \delta(u^{i,s} \partial D X_s^i) \right) ds \\ & \quad + \int_0^t V_s \partial_{n+1} f(X_s^1, \ldots, X_s^n, Y_s) ds \\ = \ & f(X_0^1, \ldots, X_0^n, Y_0) - \sum_{i=1}^{i=n} \int_0^t \delta(\partial_i f(X_s^1, \ldots, X_s^n, Y_s) u^{i,s} \partial D X_s^i) ds \\ & \quad + \int_0^t V_s \partial_{n+1} f(X_s^1, \ldots, X_s^n, Y_s) ds + \sum_{i=1}^{i=n} \int_0^t \partial_i f(X_s^1, \ldots, X_s^n, Y_s) \mathcal{G}_{u^{i,s}} X_s^i ds \\ & \quad - \sum_{i,j=1}^{n} \int_0^t \partial_i \partial_j f(X_s^1, \ldots, X_s^n, Y_s) (u^{i,s} \partial D X_s^i, D X_s^j) ds \\ & \quad - \sum_{i=1}^{i=n} \int_0^t \partial_i \partial_{n+1} f(X_s^1, \ldots, X_s^n, Y_s) (u^{i,s} \partial D X_s^i, D Y_s) ds \\ = \ & f(X_0^1, \ldots, X_0^n, Y_0) + \sum_{i=1}^{i=n} \int_0^t \partial_i f(X_s^1, \ldots, X_s^n, Y_s) dX_s^{i,u^i} \\ & \quad + \int_0^t V_s \partial_{n+1} f(X_s^1, \ldots, X_s^n, Y_s) ds + \frac{1}{2} \sum_{i,j=1}^{n} \int_0^t \partial_i \partial_j f(X_s^1, \ldots, X_s^n, Y_s) d[X^i, X^j]_s^{u^i} \\ & \quad + \frac{1}{2} \sum_{i=1}^{i=n} \int_0^t \partial_i \partial_{n+1} f(X_s^1, \ldots, X_s^n, Y_s) d[X^i, Y]_s^{u^i}. \end{aligned}$$

We made use of the fact that $(f(X^1, \ldots, X^n, Y), u^i) \in \mathcal{V}$ and $f(X_s^1, \ldots, X_s^n, Y_s) \in \mathcal{S}$, $s \in \mathbb{R}_+$.

$\square$

At the present stage, Th. 1 may seem artificial since it addresses only processes with absolutely continuous trajectories. Its goal is in fact to provide a decomposition of $f(X_t) - f(X_0)$ that includes a zero expectation (or "martingale") term written with help of the Skorohod integral. We show in the next corollary that Th. 1 extends to processes in $\mathcal{V}$ under certain conditions. Let $I\!D_{4,1}([0,1])$ denote the Hilbert subspace of $L^4(W \times [0,1])$ which is the completion of $\mathcal{U}$ under the norm

$$\| u \|_{I\!D_{4,1}([0,1])}^4 = \| u \|_{L^4(W\times[0,1])}^4 + E\left[\int_0^1 \int_0^1 \mid D_s u_v \mid^4 dvds\right].$$

In the same way, we define $H^{4,1}([0,1])$ to be the completion of $\mathcal{C}_c^1([0,1])$ under the norm

$$\| v \|_{H^{4,1}}^4 = \| v \|_{L^4([0,1])}^4 + \int_0^1 \mid \partial_v u_v \mid^4 dv.$$

Corollary 1 *Let $(X,u) \in \mathcal{V}$ and assume that there is a sequence $(X^n, u^n)_{n\in\mathrm{N}} \subset \mathcal{HV}$ such that $(X^n)_{n\in\mathrm{N}}$ converges to X in $I\!D_{4,1}([0,1])$, and $(u^n)_{n\in\mathrm{N}}$ converges to u in $H^{4,1}$, $t \in [0,1]$. Then for $f \in \mathcal{C}_b^2(\mathbb{R})$, $1_{[0,t]}(\cdot)f'(X_\cdot) \in \mathrm{Dom}(\delta^{X,u})$ and*

$$f(X_t) = f(X_0) + \int_0^t f'(X_s)dX_s^u + \frac{1}{2}\int_0^t f''(X_s)d[X,X]_s^u, \quad t\in[0,1]. \tag{19}$$

Proof. We apply Th. 1 to (X^n, u^n):

$$\begin{aligned}\delta^{X^n,u^n}(1_{[0,t]}(\cdot)f'(X_\cdot^n)) &= f(X_t^n) - f(X_0^n) - \int_0^t f'(X_s^n)\mathcal{G}_{u^s}X_s^n ds \\ &\quad -\frac{1}{2}\int_0^t (DX_s^n, DX_s^n \partial u^s) f''(X_s^n)ds,\end{aligned}$$

$n \in \mathbb{N}$. By duality we have for $G \in \mathcal{S}$:

$$\begin{aligned}E\left[\int_0^t f'(X_s^n)D_s^{X^n,u^n}Gds\right] &= E\left[G\left(f(X_t^n) - f(X_0^n) - \int_0^t f'(X_s^n)\mathcal{G}_{u^{n,s}}X_s^n ds\right)\right. \\ &\quad \left. -\frac{1}{2}G\int_0^t (DX_s^n, DX_s^n \partial u^{n,s})f''(X_s^n)ds\right],\end{aligned}$$

As n goes to infinity $(D^{X^n,u^n}G)_{n\in\mathrm{N}}$ converges to $D^{X,u}G$ in $L^2(W\times[0,1])$, $(\mathcal{G}_{u^{n,\cdot}}X_\cdot^n)_{n\in\mathrm{N}}$ converges to $\mathcal{G}_{u^\cdot}X_\cdot$ in $L^1(W\times[0,1])$ and $((DX_\cdot^n, DX_\cdot^n\partial u^{n,\cdot}))_{n\in\mathrm{N}}$ converges to $(DX_\cdot, DX_\cdot\partial u^\cdot)$ in $L^1(W\times[0,1])$. We obtain in the limit

$$\begin{aligned}&E\left[\int_0^t f'(X_s)D_s^{X,u}Gds\right] \\ &= E\left[G\left(f(X_t) - f(X_0) - \int_0^t f'(X_s)\mathcal{G}_{u^s}X_s ds - \frac{1}{2}\int_0^t (DX_s, DX_s\partial u^s)f''(X_s)ds\right)\right],\end{aligned}$$

hence $1_{[0,t]}(\cdot)f'(X_\cdot) \in \mathrm{Dom}(\delta^{X,u})$ and the Itô formula (19) holds for (X,u).

$\square$

As Th. 2, this corollary can be extended to the vector-valued case.

Remark 2 *Corollary 1 allows to retrieve the classical Itô formula in the Brownian case. The approximation B^ϕ of B by convolution given by Remark 1 satisfies the hypothesis of Corollary 1, because*

$$d[B^\phi, B^\phi]^u_t = (\phi^t, \phi^t \partial u^t)dt = \int_0^{t/2} \partial u^t_s \phi^2(s-t)dsdt = \int_0^{t/2} \partial u^t_s dsdt = u^t_{t/2} - u^t_0 = dt,$$

and $\mathcal{G}_{u^s} B^\phi_s = 0$.

Naturally, $\delta^{B,u}(1_{[0,t]} f'(B.))$ coincides with the Skorohod integral $\delta(1_{[0,t]}(\cdot)f'(B.))$ and with the adapted Itô integral $\int_0^t f'(B_s)dB_s$ of $1_{[0,t]}(\cdot)f'(B.)$. In addition to Brownian motion, Corollary 1 can also be easily applied to a large class of non-Markovian processes, containing polynomials in Brownian motion evaluated at different functions of time. A specific analysis is required in most examples, and the case of Skorohod integral processes and fractional Brownian motion is studied in details in the next sections. In such cases, the classical Skorohod integral δ appears explicitly in final formulas.

6 The Itô-Skorohod change of variable formula

In this section we show how the Skorohod change of variable of formula, cf. Nualart-Pardoux [13], can be linked to Th. 2.

Definition 9 *Let $\mathcal{W}$ denote the class of processes $Y \in L^2(W) \otimes L^2([0,1])$ of the form*

$$Y(t,\omega) = \sum_{i=0}^{i=n-1} 1_{[a_i,a_{i+1}[}(t) F_i(\omega),$$

$0 \le a_0 < \cdots < a_n$, $F_0, \ldots, F_n \in \mathcal{S}$, $\partial D F_j = 0$ a.e. on $[a_i, a_{i+1}[\times W$, $i,j = 0, \ldots n$, $n \in \mathbb{N}$.

We note that $\mathcal{W}$ is dense in $L^2(W) \otimes L^2([0,1])$. We proceed by proving a change of variable formula for Stratonovich integrals of processes in $\mathcal{W}$ added to an absolutely continuous drift. The Skorohod formula will follow as a consequence, and both results can be extended by standard procedures because of the density of $\mathcal{W}$.

Proposition 7 *Let $U, V \in \mathcal{W}$, and*

$$X_t = \int_0^t U_s dB_s + \int_0^t V_s ds, \quad t \in [0,1].$$

Then for $f \in \mathcal{C}_b^2(\mathbb{R})$,

$$\begin{aligned} f(X_t) &= f(0) + \delta(U.f'(X.)1_{[0,t]}(\cdot)) + \int_0^t f'(X_s) D_s U_s ds + \int_0^t f'(X_s) V_s ds \qquad (20) \\ &+ \frac{1}{2}\int_0^t f''(X_s) U_s^2 ds + \int_0^t f''(X_s) U_s \left[\int_0^s D_s U_\alpha dB_\alpha + \int_0^s D_s V_u du\right] ds, \quad t \in [0,1]. \end{aligned}$$

Proof. First we will prove

$$\begin{aligned} f(X_t) &= f(X_0) + \delta(U.f'(X.)1_{[0,t]}(\cdot)) + \int_0^t f'(X_s) D_s U_s ds + \int_0^t f'(X_s) V_s ds \qquad (21) \\ &+ \frac{1}{2}\int_0^t f''(X_s) U_s^2 ds + \int_0^t f''(X_s) U_s \left[D_s X_0 + \int_0^s D_s U_\alpha dB_\alpha + \int_0^s D_s V_u du\right] ds, \end{aligned}$$

for

$$X_t = X_0 + \int_0^t U_s dB_s + \int_0^t V_s ds, \quad 0 \leq t \leq a < 1,$$

the processes U, V being of the form $U = 1_{[0,a]} F$ and $V = 1_{[0,a]} G$, where $F, G \in \mathcal{S}$ satisfy $\partial D F = \partial D G = 0$ on $[0,a] \times W$, and $X_0 \in \mathcal{S}$ with $\partial D X_0 = 0$ a.e. on $[0,a] \times W$. Let $e \in \mathcal{C}^\infty(\mathbb{R})$ such that $0 \leq e(s) \leq 1$, $s \in \mathbb{R}_+$, and $e(s) = 1$, $s \in]-\infty, 0] \cup [1, \infty[$. For any $\eta > 0$, let $e_\eta(s) = \frac{1}{\eta} e(s/\eta)$, $s \in \mathbb{R}$, and

$$X_t^\eta = Y_t^\eta + Z_t, \quad 2\eta < t < a - 2\eta < 1,$$

where

$$Y_t^\eta = X_0 + \int_{2\eta}^1 e_\eta(s-t) U_s dB_s, \qquad Z_t = \int_{2\eta}^t V_s ds, \quad t \in [0,1].$$

We construct a family $(u^{\eta,t})_{t\in[0,1]}$ of $\mathcal{C}_c^\infty(\mathbb{R})$ functions such that $(Y_\cdot^\eta, u^\eta) \in \mathcal{HV}$. For each $t \in [0,1]$ let $u^{\eta,t} \in \mathcal{C}^\infty(\mathbb{R})$ with $\int_0^\eta \partial_s u_s^{\eta,t} ds = 1$, $\int_{t+\eta}^{t+2\eta} \partial_s u_s^{\eta,t} ds = -1$, and $u_s^{\eta,t} = 0$ for $s \in [\eta, t+\eta]$ and $1 > s \geq t + 2\eta$. We note that $\partial D F = 0$ a.e. on $[0,a] \times W$ implies $\mathcal{U}_{\varepsilon u^{\eta,t}} F = F$, $2\eta \leq t \leq a - 2\eta$. Then, for $\varepsilon < \eta$:

$$\begin{aligned} \mathcal{U}_{\varepsilon u^{\eta,t}} Y_t^\eta &= \mathcal{U}_{\varepsilon u^{\eta,t}} X_0 + \mathcal{U}_{\varepsilon u^{\eta,t}} F \mathcal{U}_{\varepsilon u^{\eta,t}} \int_{2\eta}^1 e_\eta(s-t) dB_s \\ &= X_0 + F \int_{2\eta}^1 e_\eta(s + \varepsilon u_s^{\eta,t} - t) dB_s \\ &= X_0 + F \int_{2\eta}^1 e_\eta(s - (t-\varepsilon)) dB_s = Y_{t-\varepsilon}^\eta, \quad 2\eta \leq t \leq a - 2\eta < 1, \end{aligned}$$

$0 < \varepsilon < \eta$. This implies that $(Y^\eta_\cdot, u^\eta) \in \mathcal{HV}$, hence from Th. 2, the Itô formula can be written as

$$\begin{aligned} f(X^\eta_t) &= f(X^\eta_{2\eta}) + \delta^{X^\eta, u^{\eta,s}}(U.f'(X^\eta_\cdot)1_{[2\eta,t]}) + \int_{2\eta}^t f'(X^\eta_s)\mathcal{G}_{u^{\eta,s}}X^\eta_s ds \\ &\quad - \int_{2\eta}^t f''(X^\eta_s)(\partial u^{\eta,s}DY^\eta_s, D(Y^\eta_s + Z_s))ds. \end{aligned}$$

We have

$$\begin{aligned} D_vY^\eta_s &= D_vX_0 + e_\eta(v-s)F + D_vF\int_{2\eta}^1 e_\eta(\alpha - s)dB_\alpha \\ &= D_vX_0 + e_\eta(v-s)F + \int_{2\eta}^1 D_vU_\alpha e_\eta(\alpha-s)dB_\alpha, \quad 2\eta \le s \le a - 2\eta. \end{aligned}$$

We now compute successively the terms of the Itô formula.

$$\begin{aligned} &\delta^{X^\eta,u^\eta}(f'(X^\eta_\cdot)1_{[2\eta,t]}(\cdot)) = -\int_{2\eta}^t \delta(f'(X^\eta_s)u^{\eta,s}_\cdot \partial.D.X^\eta_s)ds \\ &= -\delta\left(\int_{2\eta}^t f'(X^\eta_s)u^{\eta,s}_\cdot \partial D.X_0 ds\right) - \delta\left(\int_{2\eta}^t f'(X^\eta_s)u^{\eta,s}_\cdot e'_\eta(\cdot - s)Fds\right) \\ &\quad -\delta\left(\int_{2\eta}^t f'(X^\eta_s)u^{\eta,s}_\cdot \partial.D.F\int_{2\eta}^1 e_\eta(v-s)dB_v ds\right) \\ &\quad -\delta\left(\int_{2\eta}^t f'(X^\eta_s)u^{\eta,s}_\cdot \int_{2\eta}^s \partial.D.V_\alpha d\alpha ds\right) \\ &= -\delta\left(\int_{2\eta}^t f'(X^\eta_s)e'_\eta(\cdot - s)Fds\right) = -\delta\left(\int_{2\eta}^t f'(X^\eta_s)e'_\eta(\cdot - s)U_s ds\right), \end{aligned}$$

$2\eta \le t \le a - 2\eta$. We compute the "quadratic variation" of Y^η as

$$\begin{aligned} (\partial_v u^{\eta,s}_v DY^\eta_s, DY^\eta_s) &= \int_0^1 \partial_v u^{\eta,s}_v U^2_v e^2_\eta(v-s)dv + 2\int_0^1 \partial_v u^{\eta,s}_v FD_vX_0 e_\eta(v-s)dv \\ &\quad +2\int_0^1 e_\eta(\alpha - s)dB_\alpha \int_0^1 F\partial_v u^{\eta,s}_v e_\eta(v-s)D_vFdv \\ &\quad +\int_0^1 \partial_v u^{\eta,s}_v (D_vF)^2\left(\int_{2\eta}^1 e_\eta(\alpha-s)dB_\alpha\right)^2 dv \\ &= U^2_s + 2U_s\left(D_sX_0 + \int_0^1 D_sFe_\eta(\alpha-s)dB_\alpha\right), \end{aligned}$$

$2\eta \le s \le a - 2\eta$. The "quadratic covariation" of Y^η and Z is

$$\begin{aligned} &-2(u^{\eta,s}\partial DY^\eta, DZ) = -2D_sG\int_0^1 u^{\eta,s}_v \partial DY^\eta_v dv \\ &= 2D_sG\int_0^1 \partial_v u^{\eta,s}_v\left(D_vX_0 + e_\eta(v-s)F + D_vF\int_{2\eta}^1 e_\eta(\alpha-s)dB_\alpha\right)dv \\ &= 2FD_sG = 2U_sD_sZ_s. \end{aligned}$$

The absolutely continuous drift $\mathcal{G}_{u^{\eta,s}}Y_s^\eta ds$ is computed from

$$D_v D_v Y_s^\eta \;=\; e_\eta(v-s)D_vF + \int_{2\eta}^1 e_\eta(\alpha-s)dB_\alpha D_vD_vF + D_vFe_\eta(v-s).$$

Hence

$$\mathcal{G}_{u^{\eta,s}}Y_s^\eta \;=\; \frac{1}{2}\int_0^1 \partial_v u_v^{\eta,s} D_vD_vY^\eta s dv = D_sU_s, \quad 2\eta \le s \le a-2\eta.$$

Consequently, we have for $2\eta \le t \le a-2\eta$:

$$\begin{aligned} f(X_t^\eta) \;=\;& f(X_{2\eta}^\eta) + \delta\left(\int_{2\eta}^t e'_\eta(\cdot - s)f'(X_s^\eta)ds\right) + \int_{2\eta}^t f'(X_s^\eta)D_sU_sds \\ &+ \int_{2\eta}^t f'(X_s)V_sds + \frac{1}{2}\int_{2\eta}^t f''(X_s)U_s^2ds \\ &+ \int_{2\eta}^t f''(X_s^\eta)U_s\left(D_sX_0 + \int_0^1 e_\eta(u-s)D_sU_\alpha dB_\alpha\right)ds \\ &+ \int_{2\eta}^t f''(X_s^\eta)U_s\int_0^s D_sV_u du ds. \end{aligned}$$

As η goes to zero we obtain (21) in the limit. Relation (21) extends by density to $X_0 \in \mathcal{D}$ such that $\partial DX_0 = 0$ a.e. on $[0,a]\times W$, and then implies (20) by induction.

□

We note that this result is in fact a particular decomposition of the Stratonovich formula

$$df(X_t) = f'(X_s)\left(\mathcal{G}_{u^s}X_s - \delta(u^s\partial DX_s)\right), \quad f \in \mathcal{C}_b^2(\mathbb{R}) \cup \mathcal{P}(\mathbb{R}),$$

with a zero expectation "martingale" term which uses the Skorohod integral. We now show that it allows to retrieve the Skorohod anticipating change of variable formula.

Proposition 8 *Let $U, Z \in \mathcal{W}$ and $X_t = \delta(U1_{[0,t]}) + \int_0^t Z_s ds$. Then for $f \in \mathcal{C}_b^2(\mathbb{R})$,*

$$\begin{aligned} f(X_t) \;=\;& f(0) + \delta(U.f'(X.)1_{[0,t]}(\cdot)) + \int_0^t f'(X_s)Z_sds + \frac{1}{2}\int_0^t f''(X_s)U_s^2ds \\ &+ \int_0^t f''(X_s)U_s\left[\delta(D_sU.1_{[0,s]}(\cdot)) + \int_0^s D_sZ_vdv\right]ds. \end{aligned} \tag{22}$$

Proof. The proof of (22) from (20) is classical, cf. e.g. [12]. We apply Prop. 7 with $V_s = D_sU_s + Z_s$, $s \in [0,1]$, after checking that $V \in \mathcal{W}$, and make use of the relation

$$\delta(D_sU.1_{[0.t]}(\cdot)) = \int_0^s D_sU_\alpha dB_\alpha - \int_0^s D_\alpha D_sU_\alpha d\alpha. \quad \square$$

By density of $\mathcal{W}$, this result can be extended to processes in spaces $L^p([0,1], \mathbb{D}_{p,k})$ using the arguments of [12].

7 An application to fractional Brownian motion

We show that non-Markovian processes of the family of fractional Brownian motion can be treated via our approach. We refer to [3] for a study of fractional Brownian motion in a different framework. For $h \in L^2([0,1] \times [0,1])$ we define the process $(X_t^h)_{t\in[0,1]}$ as $X_t^h = \int_0^1 h(t,s)dB_s$, $t \in [0,1]$.

Lemma 2 *Let $h \in L^2([0,1] \times [0,1])$, such that $t \mapsto h(t,s)$ is absolutely continuous, ds-a.e., with $\partial_1 h, h\partial_1 h \in L^1([0,1] \times [0,1])$. Let $X_t = \int_0^1 h(t,s)dB_s$, $0 \le t \le 1$. We have*

$$f(X_t) = f(X_0) + \delta\left(\int_0^t \partial_s h(s,\cdot) f'(X_s)ds\right) + \int_0^t f''(X_s)\int_0^1 h(s,v)\partial_s h(s,v)dvds,$$

$0 \le t \le 1$, $f \in \mathcal{C}_b^2(\mathbb{R})$.

Proof. We start by assuming that $h \in \mathcal{C}^\infty([0,1] \times [0,1])$ and $\partial_t h(t,s) = 0$ whenever $\partial_s h(t,s) = 0$, $s,t \in [0,1]$. Let u_s^t be defined as

$$u_s^t = -\frac{\partial_t h(t,s)}{\partial_s h(t,s)}$$

if $\partial_s h(t,s) \neq 0$, and $u_s^t = 0$ if $\partial_s h(t,s) = 0$, with $0 \le s,t \le 1$. Then

$$\frac{d}{dt}h(t,\cdot) = -\frac{d}{d\varepsilon}h\left(t,\cdot + \varepsilon u_\cdot^t\right)_{|\varepsilon=0}, \quad a.s.,\ 0 \le t \le 1,$$

hence

$$\frac{d}{dt}X_t = -\frac{d}{d\varepsilon}\mathcal{T}_{\varepsilon u^t}X_{t|\varepsilon=0}, \quad a.s.,\ 0 \le t \le 1,$$

i.e. condition (14) of Def. 7 is fulfilled, and $(X,u) \in \mathcal{HV}$. Hence from Th. 1,

$$\begin{aligned} f(X_t) &= f(X_0) + \delta^{X,u}(f'(X_\cdot)1_{[0,t]}(\cdot)) + \frac{1}{2}\int_0^t (\partial u^s DX_s, DX_s)f''(X_s)ds \\ &= f(X_0) - \delta\left(\int_0^t f'(X_s)u_\cdot^s \partial_\cdot D_\cdot X_s ds\right) - \int_0^t f''(X_s)\int_0^1 u_v^s h(s,v)\partial_v h(s,v)dvds \\ &= f(X_0) + \delta\left(\int_0^t \partial_s h(s,\cdot)f'(X_s)ds\right) + \int_0^t f''(X_s)\int_0^1 h(s,v)\partial_s h(s,v)dvds. \end{aligned}$$

A density argument concludes the proof and shows that $v \mapsto \int_0^t \partial_s h(s,v)f'(X_s)ds \in \mathrm{Dom}(\delta)$.

□

Remark. Using Th. 2, the Itô formula can be written is this way for a large class of processes whose value at time t is a polynomial in multiple stochastic integrals.

The fractional Brownian motion with Hurst parameter $H \in]0,1[$ can be constructed as

$$X_t = \int_0^t h(t,s)dB_s, \quad 0 \le t \le 1,$$

with

$$h(t,s) = 1_{[0,t]}(s)\frac{s^{1/2-H}}{\gamma(H-1/2)}\int_s^t \alpha^{H-1/2}(\alpha-s)^{H-3/2}d\alpha, \quad 0 \le s,t \le 1,$$

if $H > 1/2$, and

$$h(t,s) = \frac{(t-s)^{H-1/2}}{\gamma(H+1/2)}F(1/2-H, H-1/2, H+1/2, 1-t/s)1_{[0,t]}(s), \tag{23}$$

if $H < 1/2$, γ being the gamma function and F the hypergeometric function (we use the notation of [3]). In the case $H > 1/2$, $t \mapsto h(t,s)$ is absolutely continuous, in fact

$$h(t,s) = \frac{s^{1/2-H}}{\gamma(H-1/2)}\int_0^t 1_{[s,\infty[}(\alpha)\alpha^{H-1/2}(\alpha-s)^{H-3/2}d\alpha, \quad 0 \le s,t \le 1.$$

and the functions $\partial_1 h$ and $h\partial_1 h$ are integrable on $[0,1]\times[0,1]$, hence Lemma 2 can be directly applied to fractional Brownian motion with $H > 1/2$. (In the case $H > 1/2$, the function $h(t,s) = 1_{[0,t]}(s)(t-s)^{H-3/2}$ defines another type of fractional Brownian motion and also satisfies the hypothesis of Lemma 2). If $H < 1/2$, then h is no longer absolutely continuous and we need to proceed differently. Let

$$\mathcal{C}_0^\infty([0,1]\times[0,1]) = \{h \in \mathcal{C}^\infty([0,1]\times[0,1]) \; : \; h(1,r) = h(0,r) = 0, \; r \in [0,1]\},$$

and denote by $H^1([0,1], L^2([0,1]))$ the completion of $\mathcal{C}_0^\infty([0,1]\times[0,1])$ under the norm

$$\| u \|_{H^1}^2 = \| u \|_{L^2([0,1]\times[0,1])}^2 + \int_0^1 \partial_s \int_0^1 u^2(s,r)drds.$$

Let $h \in H^1([0,1], L^2([0,1]))$.

Definition 10 *We say that $u \in L^2(W)\otimes L^2([0,1])$ is X^h-integrable if for any sequence $(h_n)_{n\in\mathbb{N}} \subset \mathcal{C}_0^\infty([0,1]\times[0,1])$ converging to h in $H^1([0,1], L^2([0,1]))$, we have $\int_0^1 \partial_s h_n(s,\cdot)u_s ds \in \mathrm{Dom}(\delta)$, $n \in \mathbb{N}$, and the limit*

$$\int_0^1 u_s dX^h := \lim_{n\to\infty} \delta\left(\int_0^1 \partial_s h(s,\cdot)u_s ds\right)$$

exists in $L^2(W)$ and is independent of the choice of the sequence $(h_n)_{n\in\mathbb{N}}$.

With this definition we can state the following corollary of Lemma 2 and apply it to fractional Brownian motion with parameter $H < 1/2$.

Corollary 2 *Let $h \in H^1([0,1], L^2([0,1]))$. For $f \in \mathcal{C}_b^2(\mathbb{R})$, $f'(X_\cdot^h)1_{[0,t]}(\cdot)$ is X^h-integrable and*

$$f(X_t^h) = f(0) + \int_0^t f'(X_s^h)dX_s^h + \frac{1}{2}\int_0^t f''(X_s^h)\partial_s \int_0^1 h^2(s,r)drds, \quad 0 \leq t \leq 1.$$

Proof. Let $(h_n)_{n\in\mathbb{N}} \subset \mathcal{C}_0^\infty([0,1]\times[0,1])$ be a sequence converging in $H^1([0,1], L^2([0,1]))$ to h. From Lemma 2 we have

$$\int_0^t f'(X_s^{h_n})dX_s^{h_n} = f(X_t^{h_n}) - f(0) - \frac{1}{2}\int_0^t f''(X_s^{h_n})\partial_s \int_0^1 h_n^2(s,r)drds.$$

The conclusion follows then from Def. 10.

□

If h is given by (23), i.e. X^h is the fractional Brownian motion with parameter $H < 1/2$, then $h \in H^1([0,1], L^2([0,1]))$ and we obtain

$$f(X_t^h) = f(0) + \int_0^t f'(X_s^h)dX_s^h + \frac{\gamma(2-2H)\cos(\pi H)}{\pi(1-2H)}\int_0^t f''(X_s^h)s^{2H-1}ds,$$

since the variance of X_t^h is

$$E[(X_t^h)^2] = \int_0^t \int_0^1 h^2(s,r)dsdr = \frac{\gamma(2-2H)\cos(\pi H)}{\pi H(1-2H)}t^{2H}, \quad t \in [0,1].$$

Note that for $H \neq 1/2$, $\int_0^t f'(X_s^h)dX_s^h$ differs from the forward integral with respect to X^h defined by Riemann sums, cf. [2], [5], [6], [9], because the latter does not have zero expectation, due to the dependence property between the increments of X^h. If $H = 1/2$, then $\int_0^t f'(X_s^h)dX_s^h$ coincides with the Itô integral of $f'(B_\cdot)1_{[0,t]}(\cdot)$ with respect to $(B_t)_{t\in[0,1]}$.

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