The Sard inequality on two non-gaussian spaces

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Abstract

We prove the Sard inequality in infinite dimensions for the exponential and uniform densities and obtain an extension of the corresponding change of variables formula.

Key words: Sard theorem, exponential density, uniform density, absolute continuity, infinite dimension.

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1 Introduction

The Sard theorem in finite dimension, cf. [12], has been extended to a (gaussian) infinite-dimensional setting via Wiener space techniques, cf. [4], [5], [15], [16]. Our aim is to show that it can be proved on other infinite-dimensional measure spaces, namely for the exponential and uniform densities, as an application of the change of variables formula proved in [10]. We obtain in turn an extension of this formula under weaker hypotheses, extending Wiener space results, cf. [5], [11], [14], [15]. The main difficulty of this extension comes from the fact that unlike the gaussian density, the exponential and uniform densities do not have full support. Consequently, the transformations that are considered here need to satisfy certain boundary conditions as supplementary hypothesis. In the exponential case, the results can be interpreted in the framework of stochastic analysis for the Poisson process as in [8]. A probabilistic interpretation of the uniform case can be found in [10], [9].

Sect. 2 contains preliminaries and definitions related to the stochastic calculus of variations and integration by parts. The main results are stated in Sect. 3 and proved in Sect. 4. Some lemmas that are usually stated for the Gaussian density can be applied here since their proofs do not use any particular property of the underlying measure.

2 Calculus of variations and integration by parts

We consider a separable Banach space $B = \mathbb{R}^\infty$ with a metric and Borel σ-algebra $\mathcal{B}$ such that a probability P is defined on $(B, \mathcal{B})$ via its expression on cylinder sets:

$$P(\{x \in B \ : \ (x_0, \ldots, x_n) \in E_n\}) = \lambda^{\otimes n+1}(E_n),$$

E_n Borel set in $\mathbb{R}^{n+1}$, $n \in \mathbb{N}$, where λ is a Gaussian, exponential or uniform probability measure on an interval $]a, b[$ respectively equal to $]-\infty, +\infty[$, $]0, \infty[$, $]-1, 1[$, i.e. $d\lambda(x) = e^{-\alpha_1 x - \frac{1}{2}\alpha_2 x^2} dx/\alpha_3$, with $(\alpha_1, \alpha_2, \alpha_3) = (0, 1, 1)$, $(\alpha_1, \alpha_2, \alpha_3) = (1, 0, 1)$, or $(\alpha_1, \alpha_2, \alpha_3) = (0, 0, 2)$. The coordinate functionals

$$\theta_k : B \longrightarrow \mathbb{R}, \quad k \in \mathbb{N},$$

are independent identically λ-distributed random variables. We denote by $B_{[a,b]}$ and $B_{]a,b[}$ the subsets of B defined as

$$B_{[a,b]} = \{\omega \in B \ : \ a \leq \omega_k \leq b, \quad k \in \mathbb{N}\},$$

$$B_{]a,b[} = \{\omega \in B \ : \ a < \omega_k < b, \quad k \in \mathbb{N}\}.$$

Let $\mathcal{S}$ be the dense set in $L^2(B)$ of functionals of the form $f(\theta_{k_1}, ..., \theta_{k_n})$ on $B_{[a,b]}$ where $n \in \mathbb{N}$, $k_1, ..., k_n \in \mathbb{N}$, and f is a polynomial or $f \in \mathcal{C}_c^\infty([a, b]^n)$. We denote by $(e_k)_{k \geq 0}$ the canonical basis of $H = l^2(\mathbb{N})$. Let X be a real separable Hilbert space with orthonormal basis $(h_i)_{i \in \mathbb{N}}$, and let $H \otimes X$ denote the completed Hilbert-Schmidt tensor product of H with X. Define two sets of smooth vector-valued functionals as

$$\mathcal{S}(X) = \left\{ \sum_{i=0}^{i=n} Q_i h_i \ : \ Q_0, \ldots, Q_n \in \mathcal{S}, \ \ n \in \mathbb{N} \right\},$$

which is dense in $L^2(B; X)$,

$$\mathcal{U}(X) = \left\{ v \in \mathcal{S}(H \otimes X) \ : \ (v, e_k)_{l^2(\mathbb{N})} = 0 \ on \ \theta_k^{-1}(\{a, b\}), \ k \in \mathbb{N} \right\},$$

which is dense in $L^2(B; H \otimes X)$, cf. [10], and let $\mathcal{U}$=$\mathcal{U}(\mathbb{R})$. In the Gaussian case, $\mathcal{U}(X) = \mathcal{S}(H \otimes X)$ since $]a, b[=]-\infty, +\infty[$. We define the gradient and divergence operators $D : \mathcal{S}(X) \to L^2(B \times \mathbb{N}; X)$ and $\delta : \mathcal{U}(X) \to L^2(B; X)$ by

$$(DF(\omega), h)_H = \lim_{\varepsilon \to 0} \frac{F(\omega + \varepsilon h) - F(\omega)}{\varepsilon}, \quad \omega \in B, \ h \in H,$$

and

$$\delta(u) = \sum_{k\in\mathbb{N}} (\alpha_1 + \alpha_2\theta_k)u_k - D_k u_k, \quad u \in \mathcal{U}.$$

The operators D and δ are closable, with

$$E\left[(DF, u)_{H\otimes X}\right] = E\left[(\delta(u), F)_X\right], \quad u \in \mathcal{U}(X), \ F \in \mathcal{S}(X), \tag{1}$$

as follows from finite-dimensional integration by parts on $]a, b[^n$ with the boundary conditions imposed on elements of $\mathcal{U}(X)$. The operators D and δ are local. Let $Dom(\delta; X)$ denote the domain of the closed extension of δ for $p = 2$. For $p \geq 1$, we call $D_{p,1}(X)$ the completion of $\mathcal{S}(X)$ with respect to the norm

$$\parallel F \parallel_{D_{p,1}(X)} = \parallel\mid F \mid_X\parallel_{L^p(B)} + \parallel\mid DF \mid_{H\otimes X}\parallel_{L^p(B)},$$

and $D^{\mathcal{U}}_{p,1}(H)$ the completion of $\mathcal{U}$ with respect to the norm $\parallel \cdot \parallel_{D_{p,1}(H)}$, which for $p = 2$ is equivalent to

$$\parallel F \parallel_{D^{\mathcal{U}}_{2,1}(H)} = \parallel\mid DF \mid_H\parallel_{L^2(B)} + \alpha_2 \parallel F \parallel_{L^2(B)},$$

on $D^{\mathcal{U}}_{2,1}(H)$, cf. [10]. For $1 \leq p \leq \infty$, we say that $F \in D^{loc}_{p,1}(X)$, resp. $D^{\mathcal{U},loc}_{p,1}(H)$ if there is a sequence $(F_n, A_n)_{n\in\mathbb{N}}$ such that $F_n \in D_{p,1}(X)$, resp. $F_n \in D^{\mathcal{U}}_{p,1}(X)$, A_n is measurable, $\bigcup_{n\in\mathbb{N}} A_n = B$ a.s., and $F_n = F$ a.s. on A_n, $n \in \mathbb{N}$.

3 The Sard theorem

If K is a Hilbert-Schmidt operator with eigenvalues $(\lambda_k)_{k\in\mathbb{N}}$, counted with their multiplicities, then the Carleman-Fredholm determinant of $I_H + K$ is defined as

$$\det\nolimits_2(I_H + K) = \prod_{i=0}^{\infty} (1 + \lambda_i)\exp(-\lambda_i),$$

cf. [2]. Since the operator δ is continuous from $D^{\mathcal{U}}_{2,1}(H)$ to $L^2(B)$, cf. [10], [8], [11], we can define

$$\Lambda_F = \det\nolimits_2(I_H + DF)\exp\left(-\delta(F) - \frac{\alpha_2}{2} \mid F \mid^2_H\right), \quad F \in D^{\mathcal{U},loc}_{2,1}(H). \tag{2}$$

Definition 1 *A random variable $F : B \to H$ is $H - C^1_{loc}$ if there is a random variable r with $r > 0$ a.s. such that $h \to F(\omega + h)$ is continuously differentiable on*

$$\left\{h \in H \ : \mid h \mid_H < r(\omega) \ \text{and} \ \omega + h \in B_{[a,b]}\right\}, \quad \omega \in B_{[a,b]}.$$

We will prove the following Sard Lemma and Theorem, which extend the result of [15] to the exponential and uniform densities.

Lemma 1 *Let* $F \in \mathcal{H} - \mathcal{C}^1_{loc}$ *with* $F(k) = 0$ *on* $\theta_k^{-1}(\{a,b\})$, $k \in \mathbb{N}$, *and* $(I_B + F)\left(B_{]a,b[}\right) \subset B_{]a,b[}$, *and let* $Q = \{r > 0\}$. *Then*

$$P((I_B + F)(A \cap Q)) \leq \int_{A \cap Q} \mid \Lambda_F \mid dP,$$

for all $A \in \mathcal{B}$.

As a consequence, Th. 3 below can be extended as follows:

Theorem 1 *Let* $F \in H - C^1_{loc}$ *with* $F(k) = 0$ *on* $\theta_k^{-1}(\{a,b\})$, $k \in \mathbb{N}$, $(I_B + F)\left(B_{]a,b[}\right) \subset B_{]a,b[}$, *and let* $N = card((I_B + F)^{-1}(\omega))$, $\omega \in B$. *Then*

$$E\left[fN\right] = E\left[f \circ (I_B + F) \mid \Lambda_F \mid\right],$$

for f *bounded and measurable.*

We then obtain the extension of the Sard theorem:

Theorem 2 *Let* $F \in \mathcal{H} - \mathcal{C}^1_{loc}$ *with*
(1) $F(k) = 0$ *on* $\theta_k^{-1}(\{a,b\})$, $k \in \mathbb{N}$,
(2) $(I_B + F)\left(B_{]a,b[}\right) \subset B_{]a,b[}$.
Then

$$P((I_B + F)(A)) \leq \int_A \mid \Lambda_F \mid dP, \quad A \in \mathcal{B}.$$

4 Proofs

The following result, proved in [10], [8], extends the anticipating Girsanov theorem of [5], [6], [11], [14], to the exponential and uniform densities.

Theorem 3 *Let* $F \in H - C^1_{loc}$ *with*
(1) $F(k) = 0$ *on* $\theta_k^{-1}(\{a,b\})$, $k \in \mathbb{N}$,
(2) $(I_B + F)\left(B_{]a,b[}\right) \subset B_{]a,b[}$,
and let $M = \left\{\omega \in B_{[a,b]} \ : \det_2(I_H + DF) \neq 0\right\}$. *Then*

$$E\left[f(\omega) \sum_{\theta \in (I_B+F)^{-1}(\omega) \cap M} g(\theta)\right] = E\left[f \circ (I_B + F) \mid \Lambda_F \mid g\right]$$

for f, g *bounded and measurable.*

The results of [10], [8] were in fact proven in the case where $g = 1$. The above extension can easily be proven as in [14] from the decomposition of B into a partition of sets where $I_B + F$ is injective in the proof of [10], [8].

Proposition 1 *([15]) Let $F : B \to H$ be a measurable mapping and let $A \in \mathcal{B}$. Then $(I_B + F)(A)$ is universally measurable, i.e. it belongs to the intersection for all probabilities on $(B, \mathcal{B})$ of the completions of $\mathcal{B}$ with null sets.*

Proof. The proof of this proposition is not dependent on the nature of the measure chosen on $(B, \mathcal{B})$, hence the proof of [15], cf. also [1], applies here. □

The following lemma is taken from [10], [8].

Lemma 2 *Let $G : B \to H$ measurable with*

$$G(k) = 0 \text{ on } \theta_k^{-1}(\{a, b\}),\ k \in \mathbb{N}, \quad \text{and} \quad (I_B + G)(B) \subset B_{[a,b]},$$

and for some $c > 0$

$$| G(\omega + h) - G(\omega) |_H < c \mid h |_H,$$

$h \in H$, and $\omega, \omega + h \in B_{[a,b]}$. Then $G \in D^{\mathcal{U}}_{\infty,1}$, and there is a sequence $(G_n)_{n \in \mathbb{N}} \subset \mathcal{U}$ that converges to G in $D^{\mathcal{U}}_{2,1}(H)$ with
(i) $\| | G_n |_H \|_\infty \leq \| | G |_H \|_\infty$,
(ii) $\| | DG_n |_{H \otimes H} \|_\infty \leq c$,
(iii) $(I_B + G_n)(B) \subset B_{[a,b]}$, $n \in \mathbb{N}$.

Let $\pi_n : B \to H$ be defined as $\pi_n(\omega) = (1_{\{k \leq n\}} \omega_k)_{k \in \mathbb{N}}$, and let $\pi_n^\perp = I_B - \pi_n$.

Lemma 3 *Let $G : B \to H$ be measurable such that*

- *there is $c \in]0, 1[$ such that*

$$| G(\omega + h) - G(\omega) |_H \leq c \mid h |_H, \tag{3}$$

  for $h \in H$, $\omega, \omega + h \in B_{[a,b]}$,

- *$G(k) = 0$ on $\theta_k^{-1}(\{a, b\})$, $k \in \mathbb{N}$,*

- *$(I_B + G)(B) \in B_{[a,b]}$, a.s.*

Then

(i) $I_B + G$ *is almost surely bijective,*

(ii) *its inverse can be written as* $I_B + U$, *where* $U \in D^{\mathcal{U}}_{2,1}$,

(iii) *and we have the absolute continuity relations*

$$E[f \circ (I_B + G) \mid \Lambda_G \mid] = E[f] \quad \textit{and} \quad E[f \circ (I_B + U) \mid \Lambda_U \mid] = E[f], \tag{4}$$

for f *bounded and measurable.*

Proof. We refer to [13] for the gaussian case. The injectivity of $I_B + G$ follows from the contractivity hypothesis (3). After modification of G with $G = 0$ on $B \setminus B_{[a,b]}$, consider a sequence $(G_n)_{n\in\mathbb{N}} \subset \mathcal{U}$ given by Lemma 2, converging to G in $D^{\mathcal{U}}_{2,1}(H)$ with $G_n = 0$ on $B \setminus B_{[a,b]}$, such that $\pi_n^{\perp} G_n(k) = 0$, G_n depending only on $\theta_0, \ldots, \theta_n$, $n \in \mathbb{N}$. The mapping $I_B + G_n$ is bijective on B and its inverse $I_B + U_n$ satisfies

$$U_n = -G_n \circ (I_B + U_n),$$

and from Lemma 2-(iii), $(I_B + U_n)(B_{[a,b]}) = B_{[a,b]}$. Moreover,

$$\mid DU_n \mid_{H\otimes H} \leq c/(1-c), \tag{5}$$

and it is shown in e.g. the proof of Prop. 11 in [10] that the sequence $(\Lambda_{G_n})_{n\in\mathbb{N}}$ is uniformly integrable, and that U_n, G_n, G, $n \in \mathbb{N}$, satisfy the hypothesis of Th. 3. Hence from Th. 3,

$$E[f \circ (I_B + U_n) \mid \Lambda_{U_n} \mid] = E[f], \quad E[f \circ (I_B + G_n) \mid \Lambda_{G_n} \mid] = E[f], \quad n \in \mathbb{N},$$

and

$$E[f \circ (I_B + G) \mid \Lambda_G \mid] = E[f],$$

for f bounded and measurable. We may now proceed exactly as in [13], p. 89, to show that $(U_n)_{n\in\mathbb{N}}$ converges in probability to an element U of $\mathbb{D}^{\mathcal{U}}_{2,1}(H)$ that satisfies $(i), (ii), (iii)$. $\square$

We recall that a sufficient condition for F to be in $D^{\mathcal{U},loc}_{\infty,1}$ is that $F \in \mathcal{H} - \mathcal{C}^1_{loc}$ with $F_k = 0$ on $\theta_k^{-1}(\{a, b\})$, $k \in \mathbb{N}$, cf. Prop. 5 in [8].

Lemma 4 *Let* $V, F, U \in D^{\mathcal{U},loc}_{2,1}(H)$ *such that* $(I_B + U)^* P$ *is absolutely continuous with respect to* P, *with* $I_B + V = (I_B + F) \circ (I_B + U)$. *Then* $\Lambda_V = \Lambda_F \circ (I_B + U)\Lambda_U$.

Proof. The proof of this result, (cf. [5], [7] on the Wiener space) relies here on the identity

$$\delta(\pi_n F) \circ (I_B + U) = \delta(\pi_n V) + \delta(U) + trace(DU \cdot (D\pi_n F) \circ (I_B + U)),$$

cf. Prop. 4 of [8], and on the fact that $(\delta(\pi_n F))_{n\in\mathbb{N}}$, $(\delta(\pi_n F) \circ (I_B + U))_{n\in\mathbb{N}}$, $(\delta(\pi_n V))_{n\in\mathbb{N}}$ converge in probability as n goes to infinity respectively to $\delta(F)$, $\delta(F) \circ (I_B + U)$ and $\delta(V)$. $\square$

Lemma 5 *Let $F \in \mathcal{H} - \mathcal{C}^1_{loc}$ with $(I_B+F)(B_{[a,b]}) \subset B_{[a,b]}$, and $F_k = 0$ on $\theta_k^{-1}(\{a,b\})$, $k \in \mathbb{N}$. Let $Q = \{r > 0\}$. There exists a partition $(B_{n,m})_{n,m}$ of $B_{[a,b]} \cap Q$ and two families $(G_{n,m})_{n,m}$ and $(K_{n,m})_{n,m}$ in $D^{\mathcal{U}}_{2,1}$ with $G_{n,m} = \pi_n^{\perp} F$ on $B_{n,m}$, and such that*
(i) *the mapping $S_{n,m}$ defined as*

$$I_B + S_{n,m} = (I_B + F) \circ (I_B + K_{n,m})$$

has range in $\pi_n B$ and is Lipschitz on $(I_B + G_{n,m})(B_{n,m})$,
(ii) *$G_{n,m}(k) = 0$ on $\theta_k^{-1}(\{a,b\})$, $k \in \mathbb{N}$,*
(iii) *$I_B + K_{n,m} = (I_B + G_{n,m})^{-1}$ on $E = (I_B + G_{n,m})(B_{n,m})$,*
(iv) *$\mid DG_{n,m} \mid_2$, $\mid DK_{n,m} \mid_2 < 1/2$, a.s., $n, m \in \mathbb{N}$.*

Proof. The proof of this lemma consists in the part of lemma 3.2 of [15] that does not depend on the nature of the underlying measure, but only on the normed vector space structures of B and H. The fact that $G_{n,m}$, $K_{n,m}$ belongs to $D^{\mathcal{U}}_{2,1}$ instead of $D_{2,1}$ can be easily verified using Lemma 2. $\square$

Proof of Lemma 1. The proof is done here in the exponential and uniform cases. Let P_n and $P_n^{\perp}$ denote respectively the image measures of P by π_n and $\pi_n^{\perp}$, and let $E_{n,m} = (I_B + G_{n,m})(B_{n,m})$, $n, m \in \mathbb{N}$. We have $E_{nm}, B_{n,m} \subset B_{[a,b]}$, $n, m \in \mathbb{N}$. Now from Lemma 3, $(I_B + G_{n,m})^{-1}$ can be written as $I_B + U_{n,m}$, and

$$E[f \circ (I_B + G_{n,m})^{-1} \mid \Lambda_{U_{n,m}} \mid] = E[f],$$

for f bounded and measurable. Using Lemma 5 and Th. 3.2.3 of [3] and omitting the indices n,m, we obtain

$$\begin{aligned} &P((I_B + S)(I_B + G)(B)) \\ &\quad = \int_{B_n^{\perp}} P_n((I_{\mathbb{R}^{n+1}} + S(\omega + \cdot))\,((E - \omega) \cap \pi_n H))\, P_n^{\perp}(d\omega) \end{aligned}$$

$$\begin{aligned}
&\leq \int_{B_n^\perp} P_n^\perp(d\omega) \int_{(E-\omega)\cap\pi_n H} \mid \det(I_{\mathrm{R}^{n+1}} + DS(\omega+\tilde{\omega})) \mid e^{-(S(\tilde{\omega}+\omega),\tilde{\omega})} P_n(d\tilde{\omega}) \\
&= \int_{B_n^\perp} P_n(d\omega) \int_{(E-\tilde{\omega})\cap\pi_n H} \mid \Lambda_S(\omega+\tilde{\omega}) \mid P_n(d\tilde{\omega}) \\
&= \int_E \mid \Lambda_S \mid dP(\omega) \\
&= \int_E \mid \Lambda_F \circ (I_B + U) \mid\mid \Lambda_U \mid dP.
\end{aligned}$$

For the last equality we used Lemma 4 and the locality property of D and δ. We can now end the proof as in [15]:

$$\begin{aligned}
P((I_B+S)(I_B+G)(B_{n,m})) &\leq \int_{(I_B+G)(B_{n,m})} \mid \Lambda_F \circ (I_B+U) \mid\mid \Lambda_U \mid dP \\
&= \int_B (1_{B_{n,m}} \Lambda_F) \circ (I_B+G)^{-1} \mid \Lambda_K \mid dP \\
&= \int_B 1_{B_{n,m}} \mid \Lambda_F \mid dP.
\end{aligned}$$

□

Proof of Th. 1: Let $Q = \{r > 0\}$ and $N(A) = card((I_B+F)^{-1}(\omega) \cap A)$, $\omega \in B$, $A \subset B$. We have $1_{T(M^c\cap Q)^c}(\omega) = 1_{\{(I_B+F)^{-1}(\omega)\cap M^c\cap Q=\emptyset\}}$, $\omega \in B$, and

$$P(T(M^c \cap Q)) \leq \int_{M^c\cap Q} \mid \Lambda_F \mid dP = 0,$$

from Lemma 1, hence from Th. 3,

$$\begin{aligned}
E[f \circ (I_B+F) \mid \Lambda_F \mid] &= E[1_Q f \circ (I_B+F) \mid \Lambda_F \mid] \\
&= E[fN(M \cap Q)] \\
&= E[f 1_{T(M^c\cap Q)^c} N(Q)] = E[fN].
\end{aligned}$$

□

Th. 2 now follows from Th. 1 as in [15].

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