

# A RELATION BETWEEN THE GROSS LAPLACIAN AND TIME CHANGES ON BROWNIAN MOTION

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We show that the transformations of random functionals by time changes on Brownian motion can be expressed as the adjoints of generalized Fourier-Mehler transforms. The derivatives of one-parameter families of such transformations of Brownian functionals are computed using a weighted Gross Laplacian and a second quantized operator.

## 1 Introduction

The Fourier-Mehler transform<sup>6</sup> is a group  $(\mathcal{F}_\theta)_{\theta \in \mathbb{R}}$  of transformations of random variables that provides a natural analog on Gaussian space of the Fourier transform. The adjoint of the Fourier-Mehler transform also forms a group whose infinitesimal generator is the sum of the Gross Laplacian and the number operator. It has been extended as a two-parameter family of transformations<sup>2</sup> and as families of transformations<sup>1, 3</sup> indexed by continuous mappings on  $\mathcal{S}(\mathbb{R})$ . Moreover it has been noticed<sup>2</sup>, Cor. 4.4, that complex dilations of Gaussian measures can be expressed via the adjoint of the Fourier-Mehler transform.

In this paper we show that transformations of Brownian functionals by time changes on Brownian motion can be expressed as the adjoints of generalized Fourier-Mehler transforms. Although such transformations are not quasi-invariant they can be defined on a dense linear space of smooth Brownian functionals. We compute the infinitesimal generators of one-parameter families of such transformations using a generalized Gross Laplacian and second quantized operators. These relations are viewed as the infinitesimal statement of an Itô formula without adaptedness requirements.

In Sect. 2 we review the tools of white noise analysis that will be used in this paper. In Sect. 3 we define a family of transformations of random variables by time changes of Brownian motion. The generalized Gross Laplacian is introduced in Sect. 4. An expression of transformations of random functionals by time changes on Brownian motion is given in Sect. 5, using the adjoint of the generalized Fourier-Mehler transform. In Sect. 6 we determine the infinitesimal generators of families of such transformations using a weighted

Gross Laplacian and second quantized operators.

## 2 Notation and preliminaries

Let  $\mathcal{S}(\mathbb{R})$ ,  $\mathcal{S}'(\mathbb{R})$  denote respectively the Schwartz spaces of test functions and distributions with pairing  $\langle \cdot, \cdot \rangle$ , and let  $(\cdot, \cdot)$ ,  $|\cdot|$  denote the scalar product and norm in  $L^2(\mathbb{R}_+)$ . The white noise space  $(\mathcal{S}'(\mathbb{R}), \mu)$  is equipped with the standard Gaussian measure  $\mu$  on  $\mathcal{S}'(\mathbb{R})$  defined as

$$\int_{\mathcal{S}'(\mathbb{R})} \exp(i \langle x, \xi \rangle) d\mu(x) = \exp\left(-\frac{1}{2} \langle \xi, \xi \rangle\right), \quad \xi \in \mathcal{S}(\mathbb{R}).$$

Let  $\hat{L}^2(\mathbb{R}_+^n)$  be the space of symmetric square-integrable functions on  $\mathbb{R}_+^n$ . We denote by  $f_n \otimes g_m$  the tensor product of  $f_n \in L^2(\mathbb{R}_+^n)$ ,  $g_m \in L^2(\mathbb{R}_+^m)$ , and by  $u_1 \circ \cdots \circ u_n \in \hat{L}^2(\mathbb{R}_+^n)$  the symmetrization of  $u_1 \otimes \cdots \otimes u_n \in L^2(\mathbb{R}_+^n)$  in  $n$  variables. Let  $\mathcal{C}_c^\infty(\mathbb{R}_+)$  denote the space of  $\mathcal{C}^\infty$  functions on  $\mathbb{R}$  with compact support in  $\mathbb{R}_+$ , let  $\mathcal{F}$  denote the  $\sigma$ -algebra on  $\mathcal{S}'(\mathbb{R})$  generated by  $x \mapsto \langle x, \xi \rangle$ ,  $\xi \in \mathcal{C}_c^\infty(\mathbb{R}_+)$ , and let  $(L^2) = L^2(\mathcal{S}'(\mathbb{R}), \mathcal{F}, \mu)$ . Each  $F \in (L^2)$  has a decomposition

$$F = \sum_{n=0}^{\infty} I_n(f_n), \quad f_n \in \hat{L}^2(\mathbb{R}_+^n), \quad (2.1)$$

where  $I_n(f_n)$  is the multiple stochastic integral of the square-integrable symmetric function  $f_n \in \hat{L}^2(\mathbb{R}_+^n)$  of  $n$  variables with respect to the standard Brownian motion  $(B(t))_{t \in \mathbb{R}_+}$  defined as  $B(t) = \langle x, 1_{[0,t]} \rangle$ ,  $x \in \mathcal{S}'(\mathbb{R})$ ,  $t \in \mathbb{R}_+$ .

**Definition 2.1** Let  $\mathcal{P}$  denote the space of square-integrable random variables of the form

$$F = f(I_1(u_1), \dots, I_1(u_n)), \quad u_1, \dots, u_n \in \mathcal{C}_c^\infty(\mathbb{R}_+), \quad f \in \mathcal{C}^1(\mathbb{R}^n), \quad n \in \mathbb{N}.$$

For  $y \in \mathcal{S}'(\mathbb{R})$ , the gradient  $D_y$  is defined on  $\mathcal{P}$  as

$$D_y F(x) = \lim_{\varepsilon \rightarrow 0} \frac{F(x + \varepsilon y) - F(x)}{\varepsilon}, \quad x \in \mathcal{S}'(\mathbb{R}).$$

For  $t \in \mathbb{R}$ , the white noise gradient  $\partial_t$  is defined as  $\partial_t = D_{\delta_t}$ , where  $\delta_t$  is the Dirac distribution at  $t$ , i.e.

$$\partial_t F(x) = \sum_{n=1}^{\infty} n I_{n-1}(f_n(*, t)),$$

if  $F \in \mathcal{P}$  is written as in (2.1). Let  $(\mathcal{S})$  and  $(\mathcal{S})^*$  denote the white noise spaces of test functions and distributions. The operator  $\partial_t$  extends as a continuous

operator from  $(\mathcal{S})^*$  into  $(\mathcal{S})^*$  and its adjoint is denoted by  $\partial_t^*$ ,  $t \in \mathbb{R}_+$ . The operators  $\partial_t$  and  $\partial_t^*$  are linked by the relation

$$\dot{B}(t) = \partial_t + \partial_t^*, \quad t \in \mathbb{R}_+,$$

where  $\dot{B}(t)$  is the white noise. The exponential vector  $\phi_\xi$  is defined as

$$\phi_\xi = \sum_{n=0}^{\infty} \frac{1}{n!} I_n(\xi^{\circ n}) = \exp\left(I_1(\xi) - \frac{1}{2}|\xi|^2\right), \quad \xi \in L^2(\mathbb{R}_+),$$

and we let  $\Xi$  denote the vector space generated by  $\{\phi_\xi : \xi \in \mathcal{C}_c^\infty(\mathbb{R}_+)\}$ , which is an algebra contained in  $\mathcal{P}$  and dense in  $(L^2)$ . Finally, given a mapping  $A : L^2(\mathbb{R}_+) \rightarrow L^2(\mathbb{R}_+)$  we let  $\Gamma(A)$  denote the second quantization of  $A$ , defined on  $\Xi$  as

$$\Gamma(A)\phi_\xi = \phi_{A\xi}, \quad \xi \in L^2(\mathbb{R}_+).$$

### 3 Time changes on Brownian functionals

In this section we define a family of transformations of random functionals by time changes on Brownian motion. Let

$$\mathcal{C}_0^\infty(\mathbb{R}_+) = \{h \in \mathcal{C}^\infty(\mathbb{R}_+) : h(0) = 0 \text{ and } \lim_{t \rightarrow \infty} h(t) = +\infty\}.$$

**Definition 3.1** Given  $\nu \in \mathcal{C}_0^\infty(\mathbb{R}_+)$ , we define  $R_\nu : \mathcal{C}_c^\infty(\mathbb{R}_+) \rightarrow \mathcal{C}_c^\infty(\mathbb{R}_+)$  and  $\Lambda(R_\nu) : \mathcal{P} \rightarrow \mathcal{P}$  as

$$R_\nu f(t) = f(\nu(t)), \quad t \in \mathbb{R}_+, \quad f \in \mathcal{C}_c^\infty(\mathbb{R}_+),$$

and

$$\Lambda(R_\nu)F = f(I_1(R_\nu u_1), \dots, I_1(R_\nu u_n)), \quad F = f(I_1(u_1), \dots, I_1(u_n)) \in \mathcal{P}.$$

Since  $R_\nu$  is not continuous we need to show that this definition is independent of the particular representation  $F = f(I_1(u_1), \dots, I_1(u_n))$  chosen for  $F \in \mathcal{P}$ .

**Proposition 3.1** Let  $F, G \in \mathcal{P}$  be written as

$$F = f(I_1(u_1), \dots, I_1(u_n)), \quad u_1, \dots, u_n \in \mathcal{C}_c^\infty(\mathbb{R}_+), \quad f \in \mathcal{C}^1(\mathbb{R}^n),$$

and

$$G = g(I_1(v_1), \dots, I_1(v_m)), \quad v_1, \dots, v_m \in \mathcal{C}_c^\infty(\mathbb{R}_+), \quad g \in \mathcal{C}^1(\mathbb{R}^m).$$

If  $F = G$  a.s. then  $\Lambda(R_\nu)F = \Lambda(R_\nu)G$ , a.s.

*Proof.* Let  $e_1, \dots, e_k \in C_c^\infty(\mathbb{R}_+)$  be orthonormal vectors in  $L^2(\mathbb{R}_+)$  that generate  $u_1, \dots, u_n, v_1, \dots, v_m$ , with  $u_i = \sum_{j=1}^{j=n} \alpha_i^j e_j$  and  $v_i = \sum_{j=1}^{j=m} \beta_i^j e_j$ . Then  $F$  and  $G$  are also represented as  $F = f(I_1(e_1), \dots, I_1(e_k))$ , and  $G = \tilde{g}(I_1(e_1), \dots, I_1(e_k))$ , with

$$\tilde{f}(y_1, \dots, y_k) = f\left(\sum_{j=1}^{j=k} \alpha_1^j y_j, \dots, \sum_{j=1}^{j=k} \alpha_n^j y_j\right), \quad y_1, \dots, y_k \in \mathbb{R},$$

and

$$\tilde{g}(y_1, \dots, y_k) = g\left(\sum_{j=1}^{j=k} \beta_1^j y_j, \dots, \sum_{j=1}^{j=k} \beta_n^j y_j\right), \quad y_1, \dots, y_k \in \mathbb{R}.$$

Since  $F = G$  and  $I_1(e_1), \dots, I_1(e_k)$  are independent, we have  $\tilde{f} = \tilde{g}$ . Moreover by linearity of  $I_1$  and  $R_\nu$  we get

$$\Lambda(R_\nu)F = \tilde{f}(I_1(R_\nu e_1), \dots, I_1(R_\nu e_k)),$$

and

$$\Lambda(R_\nu)G = \tilde{g}(I_1(R_\nu e_1), \dots, I_1(R_\nu e_k)),$$

hence  $\Lambda(R_\nu)F = \Lambda(R_\nu)G$ .  $\square$

Next we show that if  $\nu \in C_0^\infty(\mathbb{R}_+)$  is strictly increasing, then the action of  $\Lambda(R_\nu)$  is to evaluate a given smooth functional on time-changed trajectories  $(B(\nu^{-1}(t)))_{t \in \mathbb{R}_+}$  of  $(B(t))_{t \in \mathbb{R}_+}$ .

**Proposition 3.2** *We have for  $F = f(I_1(u_1), \dots, I_1(u_n)) \in \mathcal{P}$ :*

$$\Lambda(R_\nu)F = f\left(\int_0^\infty u_1(t)dB(\nu^{-1}(t)), \dots, \int_0^\infty u_n(t)dB(\nu^{-1}(t))\right), \quad a.s.$$

*Proof.* Since  $u_i$  and  $R_\nu u_i$  are  $C_c^\infty$  functions, the stochastic integrals  $\int_0^\infty u_i(t)dB(t)$  and  $\int_0^\infty u_i(\nu(t))dB(t)$  can be defined for every path of  $(B(t))_{t \in \mathbb{R}_+}$ , with

$$\begin{aligned} \int_0^\infty u_i(\nu(t))dB(t) &= - \int_0^\infty \nu'_i(t)u'_i(\nu(t))B(t)dt = - \int_0^\infty u'_i(t)B(\nu^{-1}(t))dt \\ &= \int_0^\infty u_i(t)dB(\nu^{-1}(t)), \quad a.s., \quad i = 1, \dots, n. \end{aligned}$$

It remains to use the multiplicativity of  $\Lambda(R_\nu)$  which follows from Prop. 3.1:

$$\Lambda(R_\nu)f(F_1, \dots, F_n) = f(\Lambda(R_\nu)F_1, \dots, \Lambda(R_\nu)F_n),$$

$F_1, \dots, F_n \in \mathcal{P}$ ,  $f \in C^1(\mathbb{R}^n)$ .  $\square$

#### 4 Generalized Gross Laplacian

Given a mapping  $K : \mathcal{S}(\mathbb{R}) \rightarrow \mathcal{S}(\mathbb{R})$ , let  $\tau(K)$  denote the trace operator associated to  $K$ , and defined as

$$\langle \tau(K), \xi \otimes \eta \rangle = \langle K\xi, \eta \rangle, \quad \xi, \eta \in \mathcal{S}(\mathbb{R}).$$

Let  $\Delta_G(K)$  denote the generalized Gross Laplacian associated to  $K$ , cf. Def. 3.1. of Chung and Ji<sup>1</sup>, defined here on  $\mathcal{P}$  as

$$\Delta_G(K) = \int_{T^2} \tau(K)(s, t) \partial_s \partial_t ds dt,$$

i.e.

$$\Delta_G(K)\phi_\xi = \langle K\xi, \xi \rangle \phi_\xi, \quad \xi \in \mathcal{C}_c^\infty(\mathbb{R}_+).$$

In this section we introduce a particular generalization of the Gross Laplacian. Let  $h \in \mathcal{C}^\infty(\mathbb{R}_+)$ , and let  $K_h$  denote the operator  $K_h : \mathcal{C}_c^\infty(\mathbb{R}_+) \rightarrow \mathcal{C}_c^\infty(\mathbb{R}_+)$  defined as

$$K_h \xi(t) = h(t) \xi'(t), \quad t \in \mathbb{R}_+, \quad \xi \in \mathcal{C}_c^\infty(\mathbb{R}_+).$$

**Definition 4.1** Let  $t \in \mathbb{R}_+$ . We define  $\partial_t^1$  on  $\mathcal{P}$  as  $\partial_t^1 = -D_{\delta_t^1}$ , where  $\delta_t^1$  is the first distributional derivative of  $\delta_t$ , i.e.

$$\partial_t^1 F = \frac{d}{dt} \partial_t F, \quad t \in \mathbb{R}_+, \quad F \in \mathcal{P},$$

The generalized Gross Laplacian  $\Delta_G(K_h)$  associated to  $K_h$  can be expressed as

$$\Delta_G(K_h)F = \int_0^\infty h(s) \partial_s \partial_s^1 F ds. \quad (4.1)$$

The following proposition expresses  $\Delta_G(K_h)$  as a *weighted* Laplacian.

**Proposition 4.1** Let  $h \in \mathcal{C}_c^\infty(\mathbb{R}_+)$ . We have

$$\Delta_G(K_h)F = -\frac{1}{2} \int_0^\infty h'(s) \partial_s \partial_s F ds, \quad F \in \mathcal{P}. \quad (4.2)$$

*Proof.* For  $F = f(I_1(u_1), \dots, I_1(u_n)) \in \mathcal{P}$  we have

$$\partial_s \partial_t f(I_1(u_1), \dots, I_1(u_n)) = \sum_{i,j=1}^n u_i(s) u_j(t) \frac{\partial^2 f}{\partial y_i \partial y_j}(I_1(u_1), \dots, I_1(u_n)),$$

$s, t \geq 0$ . Hence by integration by parts on  $\mathbb{R}_+$ , using the condition  $h(0) = 0$ ,

$$\begin{aligned}\Delta_G(K_h)F &= \sum_{i,j=1}^n (h, u_i u_j') \frac{\partial^2 f}{\partial y_i \partial y_j}(I_1(u_1), \dots, I_1(u_n)) \\ &= -\frac{1}{2} \sum_{i,j=1}^n (h' u_i, u_j) \frac{\partial^2 f}{\partial y_i \partial y_j}(I_1(u_1), \dots, I_1(u_n)).\end{aligned}$$

□

## 5 Time changes and the Fourier-Mehler transform

We now present a relation between the operator  $\Lambda(R_\nu)$  and the adjoint of a generalized Fourier-Mehler transform. This relation can be viewed as the integrated form of the relation proved in the next section. In the case of complex dilations of Gaussian measures this type of result has been obtained in Cor 4.4-(v) of Chung and Ji<sup>2</sup>. For  $A$  and  $B$  two continuous linear mappings on  $\mathcal{S}(\mathbb{R})$ , the transform  $\mathcal{G}(A, B)$  has been defined in Lemma 4.1 of Chung and Ji<sup>1</sup> as

$$\mathcal{G}(A, B) = \Gamma(B) \exp(\Delta_G(A)),$$

and shown to be equal to the adjoint of a generalized Fourier-Mehler transform. We note that this definition is still possible on the space  $\Xi$  without continuity assumptions on  $A, B : \mathcal{C}_c^\infty(\mathbb{R}_+) \rightarrow \mathcal{C}_c^\infty(\mathbb{R}_+)$ , since the expression of  $F \in \Xi$  as

$$F = \sum_{i=1}^{i=n} \alpha_i \phi_{\xi_i}, \quad (5.1)$$

is unique whenever  $\xi_i \neq \xi_j, i \neq j$ .

**Proposition 5.1** *Assume that  $\nu \in \mathcal{C}_0^\infty(\mathbb{R}_+)$  is bijective on  $\mathbb{R}_+$  and written as*

$$\nu^{-1}(t) = t + h(t), \quad t \in \mathbb{R}_+,$$

*for some  $h \in \mathcal{C}^\infty(\mathbb{R}_+)$ . Then  $\Lambda(R_\nu) = \mathcal{G}(-K_h, R_\nu)$ .*

*Proof.* We will prove the following relation on  $\Xi$ :

$$\Lambda(R_\nu) = \Gamma(R_\nu) \exp(-\Delta_G(K_h)). \quad (5.2)$$

We have

$$\begin{aligned}
\Lambda(R_\nu)\phi_\xi &= \Lambda(R_\nu) \exp\left(I_1(\xi) - \frac{1}{2}|\xi|^2\right) \\
&= \exp\left(I_1(R_\nu\xi) - \frac{1}{2}|\xi|^2\right) \\
&= \exp\left(\frac{1}{2}(|R_\nu\xi|^2 - |\xi|^2)\right) \exp\left(I_1(R_\nu\xi) - \frac{1}{2}|R_\nu\xi|^2\right) \\
&= \exp\left(\frac{1}{2}(|R_\nu\xi|^2 - |\xi|^2)\right) \phi_{R_\nu\xi} \\
&= \exp\left(\frac{1}{2}(|R_\nu\xi|^2 - |\xi|^2)\right) \Gamma(R_\nu)\phi_\xi \\
&= \exp\left(-\int_0^\infty \xi(t)\xi'(t)(\nu^{-1}(t) - t)dt\right) \Gamma(R_\nu)\phi_\xi \\
&= \exp\left(-\int_0^\infty \xi(t)\xi'(t)h(t)dt\right) \Gamma(R_\nu)\phi_\xi \\
&= \Gamma(R_\nu) \exp(-\Delta_G(K_h)) \phi_\xi, \quad \xi \in L^2(\mathbb{R}_+).
\end{aligned}$$

□

Similarly, if  $\tilde{h} \in \mathcal{C}^\infty(\mathbb{R}_+)$  satisfies  $\nu(t) = t + \tilde{h}(t)$ ,  $t \in \mathbb{R}_+$ , we can prove that

$$\Lambda(R_\nu) = \exp(\Delta_G(K_{\tilde{h}})) \Gamma(R_\nu),$$

as follows:

$$\begin{aligned}
\Lambda(R_\nu)\phi_\xi &= \Lambda(R_\nu) \exp\left(I_1(\xi) - \frac{1}{2}|\xi|^2\right) \\
&= \exp\left(\frac{1}{2}(|R_\nu\xi|^2 - |\xi|^2)\right) \Gamma(R_\nu)\phi_\xi \\
&= \exp\left(\int_0^\infty \nu'(t)\xi(\nu(t))\xi'(\nu(t))(\nu(t) - t)dt\right) \Gamma(R_\nu)\phi_\xi \\
&= \exp\left(\int_0^\infty \nu'(t)\xi(\nu(t))\xi'(\nu(t))\tilde{h}(t)dt\right) \Gamma(R_\nu)\phi_\xi \\
&= \exp(\Delta_G(K_{\tilde{h}})) \Gamma(R_\nu)\phi_\xi, \quad \xi \in L^2(\mathbb{R}_+).
\end{aligned}$$

## 6 Derivatives of transformations induced by time changes

It has been shown<sup>4</sup> that the generator of the adjoint of the Fourier-Mehler transform is given by the sum

$$i \int_0^\infty ds \partial_s^* \partial_s + \frac{i}{2} \int_0^\infty ds \partial_s \partial_s \quad (6.1)$$

of a second quantized operator (the number operator) and the Gross Laplacian. The formulas 6.2-6.5 obtained below are an extension of these results one-parameter families  $(\Lambda(R_{\varepsilon h}))_{\varepsilon \in \mathbb{R}}$  of transformations of Brownian functionals induced by time changes on  $(B(t))_{t \in \mathbb{R}_+}$ . The operator  $\int_0^\infty ds \partial_s^* \partial_s^1$  in the next proposition is in fact the differential second quantization of the operator of differentiation of Fock kernels, which differs from the number operator  $\int_0^\infty ds \partial_s^* \partial_s$ .

**Proposition 6.1** *Let  $h \in \mathcal{C}_c^\infty(\mathbb{R}_+)$ . For all  $\varepsilon > 0$ , define  $\nu_\varepsilon \in \mathcal{C}_0^\infty(\mathbb{R}_+)$  as*

$$\nu_\varepsilon(t) = t + \varepsilon h(t), \quad t \in \mathbb{R}_+.$$

*Then  $\varepsilon \mapsto \Lambda(R_{\nu_\varepsilon})F$  is differentiable in  $(L^2)$  for all  $F \in \mathcal{P}$  and we have the equalities:*

$$\frac{d}{d\varepsilon} \Lambda(R_{\nu_\varepsilon})|_{\varepsilon=0} = d\Gamma(K_h) + \Delta_G(K_h) \quad (6.2)$$

$$= \int_0^\infty ds h(s) \partial_s^* \partial_s^1 + \int_0^\infty ds h(s) \partial_s \partial_s^1 \quad (6.3)$$

$$= \int_0^\infty ds h(s) \partial_s^* \partial_s^1 - \frac{1}{2} \int_0^\infty ds h'(s) \partial_s \partial_s, \quad (6.4)$$

$$= \int_0^\infty ds h(s) \dot{B}(s) \partial_s^1. \quad (6.5)$$

*Proof.* On the space  $\Xi$  of exponential vectors these formulas follow directly by differentiation of (5.2) under the hypothesis of Prop. 5.1. We need to proceed differently in order to do the proof on  $\mathcal{P}$ . We will show that

$$\frac{d}{d\varepsilon} \Lambda(R_{\nu_\varepsilon})F|_{\varepsilon=0} = \int_0^\infty h(s) \partial_s^* \partial_s^1 F ds - \frac{1}{2} \int_0^\infty h'(s) \partial_s \partial_s F ds. \quad (6.6)$$

The expressions (6.2) and (6.4) follow then from Prop. 4.1, and (6.5) follows from the relation  $\dot{B}(t) = \partial_t^* + \partial_t$ .

a) We start by proving (6.6) for  $F = I_1(u)$ ,  $u \in \mathcal{C}_c^\infty(\mathbb{R}_+)$ . Given that

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} (R_{\nu_\varepsilon} u - u) \stackrel{L^2(\mathbb{R}_+)}{=} hu',$$



we have

$$\begin{aligned} \frac{d}{d\varepsilon} \Lambda(R_{\nu_\varepsilon}) F|_{\varepsilon=0} &\stackrel{(L^2)}{=} \frac{d}{d\varepsilon} I_1(R_{\nu_\varepsilon} u)|_{\varepsilon=0} = I_1(hu') = \int_0^\infty h(s) \partial_s^* u'(s) ds \\ &= \int_0^\infty h(s) \partial_s^* \partial_s^1 I_1(u) ds = \left( \int_0^\infty h(s) \partial_s^* \partial_s^1 ds + \Delta_G(K_h) \right) I_1(u), \end{aligned}$$

since  $\Delta_G(K_h) I_1(u) = 0$ .

b) Next we show the chain rule

$$\begin{aligned} &\left( \int_0^\infty h(s) \partial_s^* \partial_s^1 ds + \Delta_G(K_h) \right) f(F_1, \dots, F_n) \\ &= \sum_{i=1}^{i=n} \frac{\partial f}{\partial y_i}(F_1, \dots, F_n) \left( \int_0^\infty h(s) \partial_s^* \partial_s^1 ds + \Delta_G(K_h) \right) F_i, \end{aligned} \quad (6.7)$$

$F_1, \dots, F_n \in \mathcal{P}$ ,  $f \in C_b^\infty(\mathbb{R}^n)$ , using the relation

$$\partial_t^*(FG) = F \partial_t^* G - G \partial_t F = G \partial_t^* F - F \partial_t G, \quad t \in \mathbb{R}_+, \quad F, G \in \mathcal{P}.$$

We have

$$\begin{aligned} \int_0^\infty h(s) \partial_s^* \partial_s^1 f(F_1, \dots, F_n) ds &= \sum_{i=1}^{i=n} \int_0^\infty h(s) \partial_s^* \left( \frac{\partial f}{\partial y_i}(F_1, \dots, F_n) \partial_s^1 F_i \right) ds \\ &= \sum_{i=1}^{i=n} \frac{\partial f}{\partial y_i}(F_1, \dots, F_n) \int_0^\infty h(s) \partial_s^* \partial_s^1 F_i ds - \int_0^\infty h(s) (\partial_s^1 F_i) \partial_s f(F_1, \dots, F_n) ds \\ &= \sum_{i=1}^{i=n} \frac{\partial f}{\partial y_i}(F_1, \dots, F_n) \int_0^\infty h(s) \partial_s^* \partial_s^1 F_i ds \\ &\quad - \sum_{i,j=1}^{i=n} \frac{\partial^2 f}{\partial y_i \partial y_j}(F_1, \dots, F_n) \int_0^\infty h(s) (\partial_s^1 F_i) (\partial_s F_j) ds \\ &= \sum_{i=1}^{i=n} \frac{\partial f}{\partial y_i}(F_1, \dots, F_n) \int_0^\infty h(s) \partial_s^* \partial_s^1 F_i ds \\ &\quad + \frac{1}{2} \sum_{i,j=1}^{i=n} \frac{\partial^2 f}{\partial y_i \partial y_j}(F_1, \dots, F_n) \int_0^\infty (\partial_s F_i) (\partial_s F_j) h'(s) ds. \end{aligned} \quad (6.8)$$

On the other hand, using the fact that  $\partial_s$  is a derivation operator on  $\mathcal{P}$  it can be shown exactly as for the classical Gross Laplacian<sup>5</sup>, Th. 6.18, that

$$\Delta_G(K_h)f(F_1, \dots, F_n) \quad (6.9)$$

$$\begin{aligned} &= \sum_{i=1}^{i=n} \frac{\partial f}{\partial y_i}(F_1, \dots, F_n) \Delta_G(K_h)F_i \\ &\quad - \frac{1}{2} \sum_{i,j=1}^{i=n} \frac{\partial^2 f}{\partial y_i \partial y_j}(F_1, \dots, F_n) \int_0^\infty (\partial_s F_i)(\partial_s F_j) h'(s) ds. \end{aligned} \quad (6.10)$$

Combining (6.8) and (6.9) we obtain (6.7) on  $\mathcal{P}$ . Since  $\int_0^\infty h(s) \partial_s^* \partial_s^1 ds + \Delta_G(K_h)$  and  $d\Lambda(R_{\nu_\varepsilon})/d\varepsilon|_{\varepsilon=0}$  satisfy the same chain rule of derivation and coincide on first chaos random variables, they coincide on  $\mathcal{P}$ .  $\square$

If  $h$  is defined by  $h(t) = -t$ ,  $t \in \mathbb{R}_+$ , then  $\Delta_G(K_h) = \Delta_G$  is the classical Gross Laplacian. On the other hand, the operator  $\Delta_G(K_h)$  can be interpreted as an infinite-dimensional realization of the generator of Brownian motion: indeed for all fixed  $T > 0$  and  $h \in C_0^\infty(\mathbb{R}_+)$  such that  $h(T) = -1$  we have the relation

$$\Delta_G(K_h)(f(B(T))) = \frac{1}{2} f''(B(T)), \quad f \in C_b^2(\mathbb{R}).$$

The computation of the derivative of one-parameter families of transformations associated to time changes:

$$\frac{d}{d\varepsilon} \Lambda(R_{\nu_\varepsilon})|_{\varepsilon=0} = d\Gamma(K_h) + \Delta_G(K_h) = \int_0^\infty ds h(s) \partial_s^* \partial_s^1 - \frac{1}{2} \int_0^\infty ds h'(s) \partial_s \partial_s,$$

can be viewed as an elementary non-adapted Itô formula in which the finite variation term and the stochastic integral term correspond respectively to the Gross Laplacian and to the second quantization of the derivation of Fock kernels.

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