Selling at the ultimate maximum in a regime-switching model

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February 9, 2017

Abstract

This paper deals with optimal prediction in a regime-switching model driven by a continuous-time Markov chain. We extend existing results for geometric Brownian motion by deriving optimal stopping strategies that depend on the current regime state, and prove a number of continuity properties relating to optimal value and boundary functions. Our approach replaces the use of closed form expressions, which are not available in our setting, with PDE arguments that also simplify the approach of du Toit & Peskir (2009) in the classical Brownian case.

Key words: *Optimal stopping; ultimate maximum; regime-switching models; free boundary problems; diffusion processes.*
Mathematics Subject Classification (2010): 60G40; 35R35; 93E20; 60J28; 91G80.

1 Introduction

Regime-switching models have been introduced by Hamilton (1989) in discrete time and are among the most popular and effective risky asset models. The regime-switching property is reflected in the changes of states of a Markov chain β_t, which stands for the influence of external market factors.

European options have been priced in continuous-time regime-switching models by Yao, Zhang & Zhou (2006) via a recursive algorithm, and in Liu, Zhang Yin (2006) using the fast Fourier transform. Optimal stopping for option pricing in regime-switching models has been considered in Guo (2001), Guo & Zhang (2005), Le & Wang (2010), and in Ly Vath & H. Pham (2007) with optimal switching. Optimal selling under threshold rules has been dealt with in Eloe, Liu, Yatsuki, Yin & Zhang (2008) in an exponential Gaussian diffusion model with regime switching. We refer to Shiryaev (1978) and Peskir & Shiryaev (2006) for related background on the characterization of optimal stopping times and rewards.

The problem of selling a stock at the ultimate maximum has been considered by du Toit & Peskir (2009) as an extension of the results of Shiryaev, Xu & Zhou (2008). In this paper we extend the result of du Toit & Peskir (2009) to the framework of Markovian regime switching. Some of our results are natural extensions of those of du Toit & Peskir (2009) by averaging over the regime-switching component, however the regime-switching case presents notable differences and additional difficulties compared with the classical Brownian case. For example, the optimal boundary functions depend on the regime state of the system, and they may not be monotone if the drift coefficients have switching signs. In addition we can no longer rely on closed form expressions as in du Toit & Peskir (2009) and instead we use PDE arguments, cf. e.g. Lemma 4.3, that also simplify the original approach.

In Lemma 2.1 we write the optimal value of the problem as a function of time, the regime state, and the relative maximum of the underlying asset. In the general case of real-valued drifts $\mu(i) \in \mathbb{R}$, $i \in \mathcal{M}$, we identify the optimal stopping time τ_D in Proposition 3.1, and in Proposition 3.2 we determine the structure of the optimal stopping set via its boundary functions $b(t, j)$ for i in the state space $\mathcal{M}$ of the regime-switching chain.

In Proposition 5.1 we show that immediate exercise is optimal when all drift parameters $\mu(i)$ are negative, $i \in \mathcal{M}$, while exercise at maturity becomes optimal when

$\mu(i) \geq \sigma^2(i)$ for all $i \in \mathcal{M}$, where $\sigma(i)$ are the volatility parameters.

When the drift parameters $(\mu(i))_{i\in\mathcal{M}}$ of the regime-switching chain are nonnegative we prove the continuity and monotonicity of boundary functions $b(t, j)$ in Proposition 5.2, by extending arguments of du Toit & Peskir (2009) to the regime-switching setting. Those results are illustrated in Figures 1 and 2 by the plotting of value functions that yield the optimal stopping boundaries.

In Proposition 5.3 we derive a Volterra type integral equation (5.7) which is satisfied by the boundary function $b(t, j)$ of the stopping set. Such an equation is difficult to solve because, unlike in the classical setting du Toit & Peskir (2009), it also relies on the knowledge of the optimal value function, cf. Remark 5.4. In addition the associated free boundary problem (5.12a)-(5.12b) consists in a system of interacting PDEs that cannot be solved without additional assumptions, cf. e.g. Buffington & Elliott (2002) for a solution under an ordering condition on boundary functions in the case of American options.

A treatment of drifts coefficients $(\mu(i))_{i\in\mathcal{M}}$ with switching signs has been proposed in of Liu & Privault (2017) via a recursive algorithm that does not rely on a Volterra equation. In this case it turns out that the boundary functions $b(t, i)$ may not be decreasing in $t \in [0, T]$.

We proceed as follows. In Section 2 we formulate the optimal prediction problem using optimal value functions. In Section 3 we derive the optimal stopping strategies in terms of the hitting time of the boundary function of a stopping set. Section 4 is devoted to continuity lemmas, which are used to prove the continuity of boundary functions. In Section 5 we also derive the Volterra integral equation which is satisfied by the boundary functions when the drift coefficients are nonnegative. Finally we study particular exercise strategies and we present a numerical simulation of boundary functions.

2 Problem formulation

Given a standard Brownian motion $(B_t)_{t\in\mathbb{R}_+}$ independent of the Markov chain $(\beta_t)_{t\in\mathbb{R}_+}$, we consider an asset price $(Y_t)_{t\in\mathbb{R}_+}$ modeled by a geometric Brownian motion

$$dY_t = \mu(\beta_t)Y_t dt + \sigma(\beta_t)Y_t dB_t, \qquad 0 \le t \le T, \tag{2.1}$$

with regime switching driven by a time-homogeneous continuous-time Markov chain $(\beta_t)_{t\in\mathbb{R}_+}$ with state space $\mathcal{M} := \{1, 2, \dots, m\}$ and infinitesimal generator $Q = (q_{ij})_{1\le i,j\le m}$, where $\mu : \mathcal{M} \longrightarrow \mathbb{R}$, and $\sigma : \mathcal{M} \longrightarrow (0, \infty)$ are deterministic functions. In the sequel we let the filtration $(\mathcal{F}_s^t)_{s\in[t,T]}$ be defined for all $t \in [0, T]$ by

$$\mathcal{F}_s^t := \sigma(B_r - B_t,\ \beta_r\ :\ t \le r \le s), \qquad s \in [t, T]. \tag{2.2}$$

In particular, $(\mathcal{F}_t^0)_{t\in\mathbb{R}_+}$ is the filtration generated by $(B_t)_{t\in\mathbb{R}_+}$ and $(\beta_t)_{t\in\mathbb{R}_+}$.

In this paper we deal with the optimal prediction problem

$$V_t = \inf_{t\le\tau\le T} E\left[\sup_{0\le s\le T} \frac{Y_s}{Y_\tau}\middle|\mathcal{F}_t^0\right], \tag{2.3}$$

introduced in du Toit & Peskir (2009) for geometric Brownian motion, in which the infimum of expected values over all $(\mathcal{F}_s^t)_{s\in[t,T]}$-stopping times τ minimizes the "regret" of the stopping decision.

The next Lemma 2.1 shows that the optimal value function V_t in (2.3) can be written as a function of $\big(t, \beta_t, \hat{Y}_{0,t}/Y_t\big)$, where $\hat{Y}_{0,t}$ is defined by

$$\hat{Y}_{t,s} := \max_{t\le r\le s} Y_r, \qquad 0 \le t \le s \le T. \tag{2.4}$$

Lemma 2.1 *The optimal value function V_t in (2.3) takes the form*

$$V_t = V\big(t, \hat{Y}_{0,t}/Y_t, \beta_t\big), \tag{2.5}$$

where the function $V : [0, T] \times [1, +\infty) \times \mathcal{M} \to \mathbb{R}_+$ is given by

$$V(t, x, j) = \inf_{t\le\tau\le T} E\left[\frac{1}{Y_\tau} \max\Big(xY_t, \hat{Y}_{t,T}\Big)\ \Big|\ \beta_t = j\right], \tag{2.6}$$

$0 \le t \le T$, $x \ge 1$, $j \in \mathcal{M}$.

Proof. Given $t \in [0,T]$, using the drifted Brownian motion

$$\hat{B}_u^t := B_{u+t} - B_t + \int_t^{t+u} \left(\frac{\mu(\beta_s)}{\sigma(\beta_s)} - \frac{\sigma(\beta_s)}{2} \right) ds, \quad u \in [0, T-t], \tag{2.7}$$

we rewrite the solution of (2.1) as

$$Y_s = Y_t \exp\left(\int_0^{s-t} \sigma(\beta_{u+t}) d\hat{B}_u^t \right), \qquad s \in [t,T], \tag{2.8}$$

and define

$$\hat{S}_s^t := \sup_{0 \le r \le s} \int_0^r \sigma(\beta_{u+t}) d\hat{B}_u^t, \qquad s \in [0, T-t]. \tag{2.9}$$

By the definition of $\hat{Y}_{t,s}$ in (2.4) and expression (2.8), and from the conditional independence of $\left((\hat{B}_{r-t}^t)_{r\in[t,T]}, (\hat{S}_{r-t}^t)_{r\in[t,T]} \right)$ with $\mathcal{F}_t^0$ given β_t we have, for any $(\mathcal{F}_s^t)_{s\in[t,T]}$-stopping time τ with values in $[t,T]$, letting $a \vee b = \max(a,b)$,

$$\begin{aligned}
E\left[\sup_{0\le s\le T} \frac{Y_s}{Y_\tau} \Big| \mathcal{F}_t^0 \right] &= E\left[\frac{\hat{Y}_{0,t}}{Y_\tau} \vee \frac{\hat{Y}_{t,T}}{Y_\tau} \Big| \mathcal{F}_t^0 \right] \\
&= E\left[\left(\frac{\hat{Y}_{0,t}}{Y_t} \, \mathrm{e}^{-\int_0^{\tau-t} \sigma(\beta_{u+t}) d\hat{B}_u^t} \right) \vee \mathrm{e}^{\hat{S}_{T-t}^t - \int_0^{\tau-t} \sigma(\beta_{u+t}) d\hat{B}_u^t} \Big| \mathcal{F}_t^0 \right] \\
&= E\left[\left(\frac{\hat{Y}_{0,t}}{Y_t} \, \mathrm{e}^{-\int_0^{\tau-t} \sigma(\beta_{u+t}) d\hat{B}_u^t} \right) \vee \mathrm{e}^{\hat{S}_{T-t}^t - \int_0^{\tau-t} \sigma(\beta_{u+t}) d\hat{B}_u^t} \;\Big|\; \beta_t, \frac{\hat{Y}_{0,t}}{Y_t} \right] \\
&= E\left[\frac{\hat{Y}_{0,t} \vee \hat{Y}_{t,T}}{Y_\tau} \;\Big|\; \beta_t, \frac{\hat{Y}_{0,t}}{Y_t} \right] \\
&= E\left[\frac{1}{Y_\tau} \max\left(xY_t, \hat{Y}_{t,T} \right) \;\Big|\; \beta_t \right]_{x = \hat{Y}_{0,t}/Y_t}
\end{aligned}$$

where the last line follows from the conditional independence between $\hat{Y}_{0,t}/Y_t$ and

$$\left(\frac{Y_\tau}{Y_t}, \frac{\hat{Y}_{t,T}}{Y_\tau} \right) = \left(\exp\left(\int_0^{\tau-t} \sigma(\beta_{u+t}) d\hat{B}_u^t \right), \exp\left(\hat{S}_{T-t}^t - \int_0^{\tau-t} \sigma(\beta_{u+t}) d\hat{B}_u^t \right) \right) \tag{2.10}$$

given β_t. Therefore by definition (2.3) and expression (2.6), we obtain

$$\begin{aligned}
V_t &= \inf_{t\le\tau\le T} E\left[\sup_{0\le s\le T} \frac{Y_s}{Y_\tau} \Big| \mathcal{F}_t^0 \right] \\
&= \inf_{t\le\tau\le T} E\left[\frac{1}{Y_\tau} \max\left(xY_t, \hat{Y}_{t,T} \right) \;\Big|\; \beta_t \right]_{x=\hat{Y}_{0,t}/Y_t} \\
&= V\left(t \frac{\hat{Y}_{0,t}}{Y_t}, \beta_t, \right).
\end{aligned}$$

□

In the next lemma we rewrite the optimal stopping problem (2.3) in the standard form (2.12) below, using the function

$$G(t,x,i) := E\left[\max\left(x, \hat{Y}_{t,T}/Y_t\right) \ \Big| \ \beta_t = i\right], \quad t \in [0,T], \ i \in \mathcal{M}, \ x \geq 1, \tag{2.11}$$

with $G(T,x,i) = x$, $x \geq 1$.

Lemma 2.2 *The function $V : [0,T] \times [1,+\infty) \times \mathcal{M} \to \mathbb{R}_+$ defined by (2.5) admits the expression*

$$V(t,x,j) = \inf_{t \leq \tau \leq T} E\left[G\left(\tau, X_\tau^{t,x}, \beta_\tau\right) \ \Big| \ \beta_t = j\right], \tag{2.12}$$

for $t \in [0,T], j \in \mathcal{M}, x \geq 1$, where

$$X_r^{t,x} := \frac{1}{Y_r} \max\left(xY_t, \hat{Y}_{t,r}\right), \qquad r \in [t,T], \quad x \geq 1. \tag{2.13}$$

Proof. By a conditional independence argument as in the proof of Lemma 2.1, for any $s \in [t,T]$ we have

$$\begin{aligned}
E\left[\frac{\hat{Y}_{0,T}}{Y_s} \ \Big| \ \mathcal{F}_s^0\right] &= E\left[\frac{\hat{Y}_{0,s} \vee \hat{Y}_{s,T}}{Y_s} \ \Big| \ \mathcal{F}_s^0\right] \\
&= E\left[\frac{\hat{Y}_{0,s} \vee \hat{Y}_{s,T}}{Y_s} \ \Big| \ \frac{\hat{Y}_{0,s}}{Y_s}, \beta_s\right] \\
&= E\left[y \vee \frac{\hat{Y}_{s,T}}{Y_s} \ \Big| \ \beta_s\right]_{y=\hat{Y}_{0,s}/Y_s} \\
&= G\left(s, \frac{\hat{Y}_{0,s}}{Y_s}, \beta_s\right).
\end{aligned} \tag{2.14}$$

Next, we extend the above relation (2.14) to $(\mathcal{F}_s^t)_{s\in[t,T]}$-stopping times τ written as the limit of a decreasing sequence of discrete stopping times by checking that for any discrete $(\mathcal{F}_s^t)_{s\in[t,T]}$-stopping time $\tau = \sum_{i=1}^n s_i \mathbf{1}_{\{\tau=s_i\}}$, $s_1, ..., s_n \in [t,T]$, $n \geq 1$, by (2.14) we have

$$\begin{aligned}
E\left[\frac{\hat{Y}_{0,T}}{Y_\tau} \ \Big| \ \mathcal{F}_\tau^0\right] &= \sum_{i=1}^n E\left[\frac{\hat{Y}_{0,T}}{Y_\tau}\mathbf{1}_{\{\tau=s_i\}} \ \Big| \ \mathcal{F}_\tau^0\right] \\
&= \sum_{i=1}^n E\left[\frac{\hat{Y}_{0,T}}{Y_{s_i}}\mathbf{1}_{\{\tau=s_i\}} \ \Big| \ \mathcal{F}_{s_i}^0\right]
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{n} E\left[\frac{\hat{Y}_{0,T}}{Y_{s_i}} \;\middle|\; \mathcal{F}^0_{s_i} \right] \mathbf{1}_{\{\tau = s_i\}} \\
&= \sum_{i=1}^{n} G\left(s_i, \frac{\hat{Y}_{0,s_i}}{Y_{s_i}}, \beta_{s_i} \right) \mathbf{1}_{\{\tau = s_i\}} \\
&= G\left(\tau, \frac{\hat{Y}_{0,\tau}}{Y_\tau}, \beta_\tau \right).
\end{aligned}
$$

Taking the conditional expectation $E[\,\cdot \mid \beta_t = j, \hat{Y}_{0,t}/Y_t = x]$ on both sides of the above equality, we obtain

$$
E\left[\frac{\hat{Y}_{0,T}}{Y_\tau} \;\middle|\; \beta_t = j, \frac{\hat{Y}_{0,t}}{Y_t} = x \right] = E\left[G\left(\tau, \frac{\hat{Y}_{0,\tau}}{Y_\tau}, \beta_\tau \right) \;\middle|\; \beta_t = j, \frac{\hat{Y}_{0,t}}{Y_t} = x \right]. \tag{2.15}
$$

By (2.13) and the conditional independence between $\hat{Y}_{0,t}/Y_t$ and $\left(Y_t/Y_\tau, \hat{Y}_{t,\tau}/Y_\tau \right)$ given $\beta_t = j$ we find

$$
\begin{aligned}
E\left[\frac{1}{Y_\tau} \max\left(xY_t, \hat{Y}_{t,T} \right) \;\middle|\; \beta_t = j \right] &= E\left[\frac{\hat{Y}_{0,T}}{Y_\tau} \;\middle|\; \beta_t = j, \frac{\hat{Y}_{0,t}}{Y_t} = x \right] \\
&= E\left[G\left(\tau, \frac{\hat{Y}_{0,\tau}}{Y_\tau}, \beta_\tau \right) \;\middle|\; \beta_t = j, \frac{\hat{Y}_{0,t}}{Y_t} = x \right] \\
&= E\left[G\left(\tau, \frac{(xY_t) \vee \hat{Y}_{t,\tau}}{Y_\tau}, \beta_\tau \right) \;\middle|\; \beta_t = j, \frac{\hat{Y}_{0,t}}{Y_t} = x \right] \\
&= E\left[G\left(\tau, X^{t,x}_\tau, \beta_\tau \right) \;\middle|\; \beta_t = j \right],
\end{aligned} \tag{2.16}
$$

which completes the proof by (2.6). □

3 Stopping set and boundary functions

In this section we apply Corollary 2.9 in Peskir & Shiryaev (2006) in the framework of the regime-switching model (2.1) with $\mu(i) \in \mathbb{R}$, $i \in \mathcal{M}$, in order to specify the stopping set and optimal stopping time associated to the optimal stopping problem (2.3), cf. Proposition 3.1 below. In order to deal with the existence of an optimal stopping time for (2.3) rewritten as (2.12), we define the set

$$
D := \{ (t, x, j) \in [0, T] \times [1, \infty) \times \mathcal{M} \;:\; V(t, x, j) = G(t, x, j) \}. \tag{3.1}
$$

From the relation $V(T,x,j) = G(T,x,j) = x$, $j \in \mathcal{M}$, $x \geq 1$, we check that $\{T\} \times [1,\infty) \times \mathcal{M} \subset D$, which is consistent with the fact that the infimum in (2.3) is over $(\mathcal{F}_s^t)_{s\in[t,T]}$-stopping times $\tau \in [t,T]$.

Proposition 3.1 *Let $t \in [0,T]$. Given $\beta_t = j \in \mathcal{M}$ and $\hat{Y}_{0,t}/Y_t = x \in [1,\infty)$, the $(\mathcal{F}_s^t)_{s\in[t,T]}$-stopping time*

$$\tau_D(t,x,j) := \inf\left\{ r \in [t,T] \ : \ \left(r, \frac{\hat{Y}_{0,r}}{Y_r}, \beta_r \right) \in D \right\} \tag{3.2}$$

is an optimal stopping time for (2.3), or equivalently for (2.12), provided that it is a.s. finite.

Proof. By Corollary 2.9 in Peskir & Shiryaev (2006) the optimal stopping time for problem (2.12) exists and is equal to $\tau_D(t,x,j)$ in (3.2) provided that we check that for all $t \in [0,T]$ we have:

a) $G(t,x,j)$ is lower semicontinuous with respect to x, as follows directly from the definition (2.11) of $G(t,x,j)$.

b) $V(t,x,j)$ is upper semicontinuous with respect to x, as follows from the continuity Lemma 4.5 below.

c) We have $E\left[\sup_{t\leqslant s\leqslant T} |G(s,X_s^{t,x},\beta_s)|\right] < \infty$. Indeed, from (2.13) and (2.8) we have

$$\begin{aligned} X_s^{t,x} &= \frac{1}{Y_s} \max\left(xY_t, \hat{Y}_{t,s}\right) \\ &= Y_t^{-1}\, \mathrm{e}^{-\int_0^{s-t} \sigma(\beta_{u+t}) d\hat{B}_u^t} \max\left(xY_t, Y_t\, \mathrm{e}^{\hat{S}_{s-t}^t}\right) \\ &= \mathrm{e}^{\max\left(\log x, \hat{S}_{s-t}^t\right) - \int_0^{s-t} \sigma(\beta_{u+t}) d\hat{B}_u^t}, \quad s \in [t,T], \quad x \geq 1, \end{aligned} \tag{3.3}$$

where $\hat{S}_{s-t}^t$ is defined in (2.9). Hence by (2.11) and the conditional independence between $X_s^{t,x} = \max\left(xY_t/Y_s, \hat{Y}_{t,s}/Y_s\right)$ and $\hat{Y}_{s,T}/Y_s$ given β_s, we find that

$$\begin{aligned} &G(s,X_s^{t,x},\beta_s) = E\left[y \vee \frac{\hat{Y}_{s,T}}{Y_s} \;\Big|\; \beta_s \right]_{y=X_s^{t,x}} \\ &= E\left[X_s^{t,x} \vee \frac{\hat{Y}_{s,T}}{Y_s} \;\Big|\; \beta_s, X_s^{t,x} \right] \end{aligned}$$

$$= E\left[\mathrm{e}^{-\int_0^{s-t}\sigma(\beta_{u+t})d\hat{B}_u^t}\left(\mathrm{e}^{\max\left(\log x, \sup_{t\le r\le s}\int_0^{r-t}\sigma(\beta_{u+t})d\hat{B}_u^t\right)} \vee \mathrm{e}^{\sup_{s\le r\le T}\int_0^{r-t}\sigma(\beta_{u+t})d\hat{B}_u^t}\right) \Big| \beta_s, X_s^{t,x}\right]$$

$$= E\left[\mathrm{e}^{\max\left(\log x,\, \hat{S}_{T-t}^t\right)-\int_0^{s-t}\sigma(\beta_{u+t})d\hat{B}_u^t} \mid \beta_s, X_s^{t,x}\right]. \tag{3.4}$$

Letting

$$\begin{aligned}\check{S}_{T-t}^t : &= \inf_{t\leqslant s\leqslant T}\int_0^{s-t}\sigma(\beta_{u+t})d\hat{B}_u^t \\ &= \inf_{t\leqslant s\leqslant T}\left[\int_0^{s-t}\sigma(\beta_{u+t})dB_{t+u}+\int_0^{s-t}(\mu(\beta_{u+t})-\sigma^2(\beta_{u+t})/2)du\right],\end{aligned} \tag{3.5}$$

we conclude that

$$\begin{aligned}
E\left[\sup_{t\leqslant s\leqslant T}\left|G(s, X_s^{t,x}, \beta_s)\right|\right] &= E\left[\sup_{t\leqslant s\leqslant T} E\left[\mathrm{e}^{\max\left(\log x,\, \hat{S}_{T-t}^t\right)-\int_0^{s-t}\sigma(\beta_{u+t})d\hat{B}_u^t} \Big| \beta_s, X_s^{t,x}\right]\right] \\
&\le xE\left[\sup_{t\leqslant s\leqslant T} E\left[\mathrm{e}^{\hat{S}_{T-t}^t-\int_0^{s-t}\sigma(\beta_{u+t})d\hat{B}_u^t} \Big| \beta_s, X_s^{t,x}\right]\right] \\
&\le xE\left[\sup_{t\leqslant s\leqslant T} E\left[\mathrm{e}^{\hat{S}_{T-t}^t-\inf_{t\leqslant r\leqslant T}\int_0^{r-t}\sigma(\beta_{u+t})d\hat{B}_u^t} \Big| \beta_s, X_s^{t,x}\right]\right] \\
&= xE\left[\mathrm{e}^{\hat{S}_{T-t}^t-\inf_{t\leqslant r\leqslant T}\int_0^{r-t}\sigma(\beta_{u+t})d\hat{B}_u^t}\right] \\
&= xE\left[\mathrm{e}^{\hat{S}_{T-t}^t-\check{S}_{T-t}^t}\right] \\
&\le x\sqrt{E\left[\mathrm{e}^{2\hat{S}_{T-t}^t}\right]E\left[\mathrm{e}^{-2\check{S}_{T-t}^t}\right]} \\
&\le x\,\mathrm{e}^{\max_{i\in\mathcal{M}}|\sigma^2(i)-2\mu(i)|(T-t)}\sqrt{E\left[\mathrm{e}^{2\hat{S}_{T-t}^t}\right]E\left[\mathrm{e}^{2\hat{S}_{T-t}^t}\right]} \\
&\le xE\left[\mathrm{e}^{2\hat{S}_{T-t}^t}\right]\mathrm{e}^{\max_{i\in\mathcal{M}}|\sigma^2(i)-2\mu(i)|(T-t)} \\
&< \infty.
\end{aligned} \tag{3.6}$$

□

Define

$$F(t, x, j) := V(t, x, j) - G(t, x, j) \le 0, \tag{3.7}$$

which is nonpositive by (2.12), $t \in [0, T]$, $j \in \mathcal{M}$, $x \ge 1$, so that we have

$$D = \{(t, x, j) \in [0, T] \times [1, \infty) \times \mathcal{M} \ : \ F(t, x, j) = 0\}, \tag{3.8}$$

hence D is closed from the continuity of $(t, x) \longmapsto V(t, x, j)$ and $(t, x) \longmapsto G(t, x, j)$ on $[0, T] \times [1, \infty)$, cf. Lemmas 4.5 and Lemmas 4.6 below, respectively. The continuation set $C = D^c$ is an open set that can be written as

$$C = \{(t, x, j) \in [0, T] \times [1, \infty) \times \mathcal{M} \ : \ F(t, x, j) < 0\}. \tag{3.9}$$

In the next Proposition 3.2 we characterize the shape of the stopping set D defined in (3.1) in terms of the boundary function $b(t, j)$ defined by

$$b(t, j) := \inf\{x \in [1, \infty) \ : \ (t, x, j) \in D\}, \tag{3.10}$$

where we set $b(t, j) := +\infty$ if $\{x \in [1, \infty) \ : \ (t, x, j) \in D\} = \emptyset$. From the relation $\{T\} \times [1, \infty) \times \mathcal{M} \subset D$ we deduce the terminal condition $b(T, j) = 1$, $j \in \mathcal{M}$, cf. also Proposition 5.2 for sufficient conditions for the finiteness of $b(t, j)$.

Proposition 3.2 *For any $(t, x, j) \in [0, T] \times [1, \infty) \times \mathcal{M}$ such that $(t, x, j) \in D$ we have*

$$\{t\} \times [x, \infty) \times \{j\} \subset D. \tag{3.11}$$

and

$$D = \{(t, y, j) \in [0, T] \times [1, \infty) \times \mathcal{M} \ : \ y \geq b(t, j)\}. \tag{3.12}$$

Proof. Let $y := \sup\{z \in [x, \infty) \ : \ \{t\} \times [x, z] \times \{j\} \subset D\}$. If $y < \infty$ then we have $(t, y, j) \in D$ by the closedness of D, and from the monotonicity property of $F(t, x, j)$ stated in Lemma 3.3, $(t, y, j) \in D$ admits a right neighborhood of the form

$$\{t\} \times [x, x + \eta] \times \{j\} \subset D \tag{3.13}$$

for some $\eta > 0$, which leads to a contradiction. Hence $y = +\infty$ and (3.11) holds. Relation (3.12) follows from the equivalence

$$(t, x, j) \in D \iff \{t\} \times [x, \infty) \times \{j\} \subset D \iff x \geq b(t, j) \tag{3.14}$$

that follows from (3.10). □

The following lemma has been used in the proof of Proposition 3.2.

Lemma 3.3 *For any $(t, x, j) \in D$, we have*

$$\liminf_{\varepsilon \searrow 0} \frac{F(t, x + \varepsilon, j) - F(t, x, j)}{\varepsilon} \geq 0. \tag{3.15}$$

Proof. We split the proof into two parts.

(i) From (3.4) we have

$$G(s, X_s^{t,x}, \beta_s) = E\left[X_s^{t,x} \vee \frac{\hat{Y}_{s,T}}{Y_s}\bigg|\beta_s, X_s^{t,x}\right] = E\left[X_s^{t,x} \vee \frac{\hat{Y}_{s,T}}{Y_s}\bigg|\mathcal{F}_s^0\right]$$
$$= E\left[\mathrm{e}^{\max\left(\log x,\ \hat{S}_{T-t}^t\right) - \int_0^{s-t}\sigma(\beta_{u+t})d\hat{B}_u^t}\ \bigg|\ \mathcal{F}_s^0\right], \quad s \in [t, T],$$

which extends to any $(\mathcal{F}_s^t)_{s\in[t,T]}$-stopping time $\tau \in [t, T]$ as

$$G(\tau, X_\tau^{t,x}, \beta_\tau) = E\left[\mathrm{e}^{\max\left(\log x,\ \hat{S}_{T-t}^t\right) - \int_0^{\tau-t}\sigma(\beta_{u+t})d\hat{B}_u^t}\ \bigg|\ \mathcal{F}_\tau^0\right], \tag{3.16}$$

as in (2.14)-(2.15) above. For all $x \geq 1$ and $\varepsilon > 0$, consider the $(\mathcal{F}_s^t)_{s\in[t,T]}$-stopping time

$$\tau_\varepsilon^+ := \tau_D(t, x+\varepsilon, j) \in [t, T] \tag{3.17}$$

defined by (3.2), which solves the optimal stopping problem

$$V(t, x+\varepsilon, j) = \inf_{t\leq\tau\leq T} E\left[G\left(\tau, X_\tau^{t,x+\varepsilon}, \beta_\tau\right)\ \bigg|\ \beta_t = j\right] = E\left[G\left(\tau_\varepsilon^+, X_{\tau_\varepsilon^+}^{t,x+\varepsilon}, \beta_{\tau_\varepsilon^+}\right)\ \bigg|\ \beta_t = j\right], \tag{3.18}$$

cf. (2.12). The following argument relies on the fact that for any $(t, x, j) \in D$ we have

$$\lim_{\varepsilon\to 0} \tau_D(t, x+\varepsilon, j) = t, \tag{3.19}$$

as will be shown in part (ii) below. Relations (2.11), (2.12), (3.16) and (3.19) imply

$$\begin{aligned}
&\liminf_{\varepsilon\searrow 0} \frac{V(t, x+\varepsilon, j) - V(t, x, j)}{\varepsilon} \\
&\quad\geq \liminf_{\varepsilon\searrow 0} \frac{1}{\varepsilon} E\left[G(\tau_\varepsilon^+, X_{\tau_\varepsilon^+}^{t,x+\varepsilon}, \beta_{\tau_\varepsilon^+}) - G(\tau_\varepsilon^+, X_{\tau_\varepsilon^+}^{t,x}, \beta_{\tau_\varepsilon^+}) \mid \beta_t = j\right] \\
&\quad= \liminf_{\varepsilon\searrow 0} \frac{1}{\varepsilon} E\left[E\left[\mathrm{e}^{\log(x+\varepsilon)\vee\hat{S}_{T-t}^t - \int_0^{\tau_\varepsilon^+-t}\sigma(\beta_{u+t})d\hat{B}_u^t} - \mathrm{e}^{\log x\vee\hat{S}_{T-t}^t - \int_0^{\tau_\varepsilon^+-t}\sigma(\beta_{u+t})d\hat{B}_u^t}\bigg|\mathcal{F}_{\tau_\varepsilon^+}^0\right]\bigg|\beta_t = j\right] \\
&\quad= \liminf_{\varepsilon\searrow 0} \frac{1}{\varepsilon} E\left[\mathrm{e}^{\log(x+\varepsilon)\vee\hat{S}_{T-t}^t - \int_0^{\tau_\varepsilon^+-t}\sigma(\beta_{u+t})d\hat{B}_u^t} - \mathrm{e}^{\log x\vee\hat{S}_{T-t}^t - \int_0^{\tau_\varepsilon^+-t}\sigma(\beta_{u+t})d\hat{B}_u^t}\bigg|\beta_t = j\right] \\
&\quad= \liminf_{\varepsilon\searrow 0} \frac{1}{\varepsilon} E\left[\mathrm{e}^{\log(x+\varepsilon)\vee\hat{S}_{T-t}^t} - \mathrm{e}^{\log x\vee\hat{S}_{T-t}^t}\ \bigg|\ \beta_t = j\right] \\
&\quad= \frac{\partial}{\partial x} E\left[\mathrm{e}^{\max\left(\log x, \hat{S}_{T-t}^t\right)}\ \bigg|\ \beta_t = j\right] \\
&\quad= \frac{\partial G}{\partial x}(t, x, j),
\end{aligned} \tag{3.20}$$

hence we conclude to (3.15). Here we used the dominated convergence theorem with the bound

$$\frac{1}{\varepsilon}\Big| e^{\log(x+\varepsilon)\vee \hat{S}^t_{T-t}-\int_0^{\tau^+_\varepsilon-t}\sigma(\beta_{u+t})d\hat{B}^t_u} - e^{\log x\vee \hat{S}^t_{T-t}-\int_0^{\tau^+_\varepsilon-t}\sigma(\beta_{u+t})d\hat{B}^t_u}\Big|$$
$$\le \left|\frac{e^{\log(x+\varepsilon)}-e^{\log x}}{\varepsilon}\right| e^{-\inf_{0\le s\le T-t}\int_0^s \sigma(\beta_{t+u})d\hat{B}^t_u} = e^{-\check{S}^t_{T-t}},$$

where $\check{S}^t_{T-t}$ is defined in (3.5) and the righthand side is integrable as in the derivation of (3.6).

(ii) We turn to the proof of (3.19). From the expression (2.6) in Lemma 2.1, we have

$$\begin{aligned} V(t,x,j) &= \inf_{t\le\tau\le T} E\left[\frac{xY_t\vee \hat{Y}_{t,T}}{Y_\tau}\ \Big|\ \beta_t=j,\mathcal{F}^0_t\right] \\ &= \inf_{t\le\tau\le T} E\left[e^{-\int_0^{\tau-t}\sigma(\beta_{u+t})d\hat{B}^t_u}\left(x\vee e^{\hat{S}^t_{T-t}}\right)\ \Big|\ \beta_t=j,\mathcal{F}^0_t\right]. \end{aligned} \tag{3.21}$$

From (3.21) and

$$X^{t,x+\varepsilon}_r = e^{-\int_0^{r-t}\sigma(\beta_{u+t})d\hat{B}^t_u}(x+\varepsilon\vee e^{\hat{S}^t_{r-t}}) \tag{3.22}$$

cf. (3.3), we obtain

$$\begin{aligned} V(r,X^{t,x+\varepsilon}_r,\beta_r) &= \inf_{r\le\tau\le T} E\left[e^{-\int_0^{\tau-r}\sigma(\beta_{u+r})d\hat{B}^r_u}\left(y\vee e^{\hat{S}^r_{T-r}}\right)\ \Big|\ \mathcal{F}^0_r\right]_{y=X^{t,x+\varepsilon}_r} \\ &= \inf_{r\le\tau\le T} E\left[e^{-\int_0^{\tau-r}\sigma(\beta_{u+r})d\hat{B}^r_u}\left(X^{t,x+\varepsilon}_r\vee e^{\hat{S}^r_{T-r}}\right)\ \Big|\ \mathcal{F}^0_r\right]. \end{aligned} \tag{3.23}$$

Next, from the definition (3.2) of $\tau_D(t,x+\varepsilon,j)$ and (3.23) we have, on the event $\{\beta_t=j\}$,

$$\begin{aligned} &\tau_D(t,x+\varepsilon,j) = \inf\{r\in[t,T]\ :\ (r,X^{t,x+\varepsilon}_r,\beta_r)\in D\} \qquad (3.24)\\ &= \inf\left\{r\in[t,T]\ :\ \inf_{r\le\tau\le T} E\left[e^{-\int_0^{\tau-r}\sigma(\beta_{u+r})d\hat{B}^r_u}\left(X^{t,x+\varepsilon}_r\vee e^{\hat{S}^r_{T-r}}\right)\ \Big|\ \mathcal{F}^0_r\right] = E\left[X^{t,x+\varepsilon}_r\vee e^{\hat{S}^r_{T-r}}\ \Big|\ \mathcal{F}^0_r\right]\right\} \\ &\le \inf\left\{r\in[t,T]\ :\ \inf_{r\le\tau\le T} E\left[e^{-\int_0^{\tau-r}\sigma(\beta_{u+r})d\hat{B}^r_u}\left(X^{t,x}_r\vee e^{\hat{S}^r_{T-r}}\right)\ \Big|\ \mathcal{F}^0_r\right] \ge E\left[X^{t,x+\varepsilon}_r\vee e^{\hat{S}^r_{T-r}}\ \Big|\ \mathcal{F}^0_r\right]\right\} \\ &\le \inf\left\{r\in[t,T]\ :\ \inf_{r\le\tau\le T} E\left[e^{-\int_0^{\tau-r}\sigma(\beta_{u+r})d\hat{B}^r_u}\left(X^{t,x}_r\vee e^{\hat{S}^r_{T-r}}\right)\ \Big|\ \mathcal{F}^0_r\right] \ge e^{\varepsilon}\, E\left[X^{t,x}_r\vee e^{\hat{S}^r_{T-r}}\ \Big|\ \mathcal{F}^0_r\right]\right\}, \end{aligned}$$

where we applied the inequality

$$X^{t,x+\varepsilon}_r = e^{-\int_0^{r-t}\sigma(\beta_{u+t})d\hat{B}^t_u}\left(e^{\log(x+\varepsilon)}\vee e^{\hat{S}^t_{r-t}}\right) \le e^{-\int_0^{r-t}\sigma(\beta_{u+t})d\hat{B}^t_u+\varepsilon}\left(e^{\log(x)}\vee e^{\hat{S}^t_{r-t}}\right) = e^{\varepsilon}\, X^{t,x}_r, \tag{3.25}$$

$x \geq 1$, $\varepsilon \geq 0$, $r \in [t, T]$. This implies

$$\begin{aligned}
&\lim_{\varepsilon \to 0} \tau_D(t, x+\varepsilon, j) && (3.26)\\
&\leq \lim_{\varepsilon \to 0} \inf \left\{ r \in [t,T] \ : \ \inf_{r \leq \tau \leq T} E\left[\mathrm{e}^{-\int_0^{\tau - r} \sigma(\beta_{u+r}) d\hat{B}^r_u} \left(X^{t,x}_r \vee \mathrm{e}^{\hat{S}^r_{T-r}} \right) \Bigm| \mathcal{F}^0_r \right] \geq \mathrm{e}^{\varepsilon} E\left[X^{t,x}_r \vee \mathrm{e}^{\hat{S}^r_{T-r}} \Bigm| \mathcal{F}^0_r \right] \right\} \\
&= \inf \left\{ r \in [t,T] \ : \ \inf_{r \leq \tau \leq T} E\left[\mathrm{e}^{-\int_0^{\tau - r} \sigma(\beta_{u+r}) d\hat{B}^r_u} \left(X^{t,x}_r \vee \mathrm{e}^{\hat{S}^r_{T-r}} \right) \Bigm| \mathcal{F}^0_r \right] \geq E\left[X^{t,x}_r \vee \mathrm{e}^{\hat{S}^r_{T-r}} \Bigm| \mathcal{F}^0_r \right] \right\} \\
&= \inf \left\{ r \in [t,T] \ : \ \inf_{r \leq \tau \leq T} E\left[\mathrm{e}^{-\int_0^{\tau - r} \sigma(\beta_{u+r}) d\hat{B}^r_u} \left(X^{t,x}_r \vee \mathrm{e}^{\hat{S}^r_{T-r}} \right) \Bigm| \mathcal{F}^0_r \right] = E\left[X^{t,x}_r \vee \mathrm{e}^{\hat{S}^r_{T-r}} \Bigm| \mathcal{F}^0_r \right] \right\} \\
&= \inf \left\{ r \in [t,T] \ : \ (r, X^{t,x}_r, \beta_r) \in D \right\} \\
&= t, && (3.27)
\end{aligned}$$

since $(t, x, j) \in D$, $\beta_t = j$ and $X^{t,x}_t = x$. Since $\tau_D(t, x+\varepsilon, j) \geq t$ we conclude to (3.19).

□

4 Continuity lemmas

The following property of smooth fit, namely the continuity of the function $y \longmapsto \frac{\partial V}{\partial y}(t, y, j)$ over the optimal stopping boundary ∂C, will be needed in the proof of Proposition 5.3 below.

Lemma 4.1 *For any $(t, y, j) \in \partial C$, $y > 1$, we have*

$$\frac{\partial V}{\partial y}(t, y+, j) = \frac{\partial V}{\partial y}(t, y-, j). \tag{4.1}$$

Proof. For any $\varepsilon \in (0, y-1)$, let $\tau^-_\varepsilon = \tau_D(t, y-\varepsilon, j) \in [t, T]$, cf. (3.2). Since $(t, y, j) \in \partial C$ and D is closed we have $(t, y, j) \in D$. Similarly to (3.24) to (3.26), τ^-_ε converges to t a.s. when ε tends to 0. By the same approach as in (3.20), replacing $y+\varepsilon$ with $y-\varepsilon$ shows that

$$\frac{\partial G}{\partial y}(t, y, j) \leq \liminf_{\varepsilon \searrow 0} \frac{V(t, y-\varepsilon, j) - V(t, y, j)}{\varepsilon}. \tag{4.2}$$

On the other hand, since $(t, y, j) \in \partial C \subset D$, we have

$$\limsup_{\varepsilon \searrow 0} \frac{V(t, y-\varepsilon, j) - V(t, y, j)}{\varepsilon} \leq \lim_{\varepsilon \searrow 0} \frac{G(t, y-\varepsilon, j) - G(t, y, j)}{\varepsilon} = \frac{\partial G}{\partial y}(t, y, j), \tag{4.3}$$

hence

$$\frac{\partial V}{\partial y}(t, y-, j) = \frac{\partial G}{\partial y}(t, y, j). \tag{4.4}$$

Finally the fact that $V = G$ on the closed set D implies

$$\frac{\partial V}{\partial y}(t, y-, j) = \frac{\partial V}{\partial y}(t, y+, j) = \frac{\partial G}{\partial y}(t, y, j). \tag{4.5}$$

□

In the next proposition, which will be used in the proof of Proposition 5.3, we show the normal reflection of the free boundary problem by proving that the right derivative of the value function $V(t, y, j)$ vanishes at $y = 1$, cf. also page 264 of Peskir & Shiryaev (2006) without regime switching.

Lemma 4.2 *For any $t \in [0, T]$ and $j \in \mathcal{M}$ we have*

$$\frac{\partial V}{\partial y}(t, 1+, j) = 0. \tag{4.6}$$

Proof. For convenience of notation we set $\tau_0 = \tau_D(t, 1, j)$, and note that

$$\begin{aligned}
&\limsup_{\varepsilon \searrow 0} \frac{V(t, 1+\varepsilon, j) - V(t, 1, j)}{\varepsilon} \\
&\leq \limsup_{\varepsilon \searrow 0} \frac{1}{\varepsilon} E[G(\tau_0, X_{\tau_0}^{t,1+\varepsilon}, \beta_{\tau_0}) - G(\tau_0, X_{\tau_0}^{t,1}, \beta_{\tau_0}) \mid \beta_t = j] \\
&= \limsup_{\varepsilon \searrow 0} \frac{1}{\varepsilon} E\left[\mathrm{e}^{\log(1+\varepsilon) \vee \hat{S}_{T-t}^t - \int_0^{\tau_0 - t} \sigma(\beta_{t+r}) d\hat{B}_r^t} - \mathrm{e}^{\hat{S}_{T-t}^t - \int_0^{\tau_0 - t} \sigma(\beta_{t+r}) d\hat{B}_r^t} \;\middle|\; \beta_t = j \right] \\
&= \limsup_{\varepsilon \searrow 0} E\left[\frac{1}{\varepsilon} \left(\mathrm{e}^{\log(1+\varepsilon) \vee \hat{S}_{T-t}^t - \int_0^{\tau_0 - t} \sigma(\beta_{t+r}) d\hat{B}_r^t} - \mathrm{e}^{\hat{S}_{T-t}^t - \int_0^{\tau_0 - t} \sigma(\beta_{t+r}) d\hat{B}_r^t} \right) \mathbf{1}_{\{\hat{S}_{T-t}^t < \log(1+\varepsilon)\}} \;\middle|\; \beta_t = j \right] \\
&= E\left[\limsup_{\varepsilon \searrow 0} \frac{\mathrm{e}^{\log(1+\varepsilon) \vee \hat{S}_{T-t}^t - \int_0^{\tau_0 - t} \sigma(\beta_{t+r}) d\hat{B}_r^t} - \mathrm{e}^{\hat{S}_{T-t}^t - \int_0^{\tau_0 - t} \sigma(\beta_{t+r}) d\hat{B}_r^t}}{\varepsilon} \mathbf{1}_{\{\hat{S}_{T-t}^t < \log(1+\varepsilon)\}} \;\middle|\; \beta_t = j \right] \\
&= 0,
\end{aligned}$$

since $\lim_{\varepsilon \searrow 0} \mathbf{1}_{\{\hat{S}_{T-t}^t < \log(1+\varepsilon)\}} = 0$, where we applied the dominated convergence theorem as in the proof of Lemma 3.3 with the same dominating function as in (3.21). Since $V(t, y, j)$ is nondecreasing in $y \in [1, \infty)$, we have

$$\liminf_{\varepsilon \searrow 0} \frac{V(t, 1+\varepsilon, j) - V(t, 1, j)}{\varepsilon} \geq 0, \tag{4.7}$$

which shows that

$$\frac{\partial V}{\partial y}(t,1+,j)=\lim_{\varepsilon\searrow 0}\frac{V(t,1+\varepsilon,j)-V(t,1,j)}{\varepsilon}=0. \tag{4.8}$$

□

Next, we note that $(s,X_s^{t,x},\beta_s)_{s\in[t,T]}$ is a Markov process, cf. Lemma 1 of Yao, Zhang & Zhou (2006), and we consider its infinitesimal generator

$$\mathbb{L}f(s,x,j)=\left(\frac{\partial}{\partial s}+x(\sigma^2(j)-\mu(j))\frac{\partial}{\partial x}+\frac{1}{2}\sigma^2(j)x^2\frac{\partial^2}{\partial x^2}\right)f(s,x,j)+\sum_{i=1}^{m}q_{j,i}f(s,x,i), \tag{4.9}$$

where $Q=(q_{ij})_{1\le i,j\le m}$ is the infinitesimal matrix generator of the Markov chain $(\beta_t)_{t\in[0,T]}$, for f a sufficiently differentiable function of $(s,y,j)\in[0,T]\times[1,\infty)\times\mathcal{M}$, cf. Lemma 4.7 below.

The following lemmas will be used in the proof of Proposition 5.1 below. In Lemma 4.3 we replace the use of closed form expressions for $\mathbb{L}G(t,x,j)$, which are no longer available in our setting, with the differential expression (4.11).

Lemma 4.3 *We have*

$$\frac{\partial G}{\partial x}(t,1+,j)=0, \qquad t\in[0,T], \tag{4.10}$$

and

$$\mathbb{L}G(t,x,j)=x\sigma^2(j)\frac{\partial G}{\partial x}(t,x,j)-\mu(j)G(t,x,j), \qquad t\in[0,T], \tag{4.11}$$

with $\mathbb{L}G(T,x,j)=-\mu(j)x$, $j\in\mathcal{M}$, $x\in[1,\infty)$. *In particular, for any* $(t,x,j)\in[0,T)\times[1,\infty)\times\mathcal{M}$ *we have*

$$\begin{cases}\mathbb{L}G(t,x,j)>0, & \text{when } \mu(j)\le 0,\\[2ex] \mathbb{L}G(t,x,j)<0, & \text{when } \mu(j)\ge\sigma^2(j).\end{cases} \tag{4.12}$$

In addition, $\mathbb{L}G(t,x,j)$ *is nondecreasing and continuous in* t *for all* $x\ge 1$ *when* $\mu(j)\ge 0$.

Proof. For all $j \in \mathcal{M}$ we let

$$f(t,y,z,j) := yG\left(t,\frac{z}{y},j\right) = E\left[\max\left(z, y\frac{\hat{Y}_{t,T}}{Y_t}\right) \;\Big|\; \beta_t = j\right], \quad t \in [0,T], \quad y,z > 0. \tag{4.13}$$

By (2.1) and the Itô formula we have

$$\begin{aligned} df\big(t,Y_t,\hat{Y}_{0,t},\beta_t\big) =& \frac{\partial f}{\partial t}\big(t,Y_t,\hat{Y}_{0,t},\beta_t\big)dt + \mu(\beta_t)Y_t\frac{\partial f}{\partial x}\big(t,Y_t,\hat{Y}_{0,t},\beta_t\big)dt \\ &+ \sigma(\beta_t)Y_t\frac{\partial f}{\partial x}\big(t,Y_t,\hat{Y}_{0,t},\beta_t\big)dB_t + \frac{1}{2}\sigma^2(\beta_t)Y_t^2\frac{\partial^2 f}{\partial x^2}\big(t,Y_t,\hat{Y}_{0,t},\beta_t\big)dt \\ &+ \frac{\partial f}{\partial y}\big(t,Y_t,\hat{Y}_{0,t},\beta_t\big)d\hat{Y}_{0,t} + f\big(t,Y_t,\hat{Y}_{0,t},\beta_t\big) - f\big(t,Y_t,\hat{Y}_{0,t},\beta_{t-}\big), \end{aligned}$$

and given that

$$f\big(t,Y_t,\hat{Y}_{0,t},\beta_t\big) = E\big[\,\hat{Y}_{0,T} \mid \beta_t,\ Y_t,\ \hat{Y}_{0,t}\,\big] = E\big[\,\hat{Y}_{0,T} \mid \mathcal{F}_t^0\,\big], \qquad t \in [0,T], \tag{4.14}$$

is a martingale and $\big(\hat{Y}_{0,t}\big)_{t\in[0,T]}$ has finite variation, we find

$$\frac{\partial f}{\partial t}(t,y,z,j) + \mu(j)y\frac{\partial f}{\partial x}(t,y,z,j) + \frac{1}{2}\sigma^2(j)y^2\frac{\partial^2 f}{\partial x^2}(t,y,z,j) + \sum_{i=1}^m q_{j,i}f(t,y,z,i) = 0, \tag{4.15}$$

and $\dfrac{\partial f}{\partial y}(t,x,y,j)_{x=y} = 0$. Substituting (4.13) into (4.15) shows that

$$\begin{aligned} &y\frac{\partial G}{\partial t}\left(t,\frac{z}{y},j\right) + \mu(j)y\left(G\left(t,\frac{z}{y},j\right) + y\frac{\partial G}{\partial x}\left(t,\frac{z}{y},j\right)\left(-\frac{z}{y^2}\right)\right) \\ &\quad + \frac{1}{2}\sigma^2(j)y^2\left(\frac{\partial G}{\partial x}\left(t,\frac{z}{y},j\right)\left(-\frac{z}{y^2}\right) + \frac{z}{y^2}\frac{\partial G}{\partial x}\left(t,\frac{z}{y},j\right) + \frac{z^2}{y^3}\frac{\partial^2 G}{\partial x^2}\left(t,\frac{z}{y},j\right)\right) \\ &\quad + \sum_{i=1}^m q_{j,i}yG\left(t,\frac{z}{y},i\right) = 0, \end{aligned}$$

which shows that the function $G(t,x,j)$ satisfies the PDE

$$\mu(j)G(t,x,j) + \frac{\partial G}{\partial t}(t,x,j) - \mu(j)x\frac{\partial G}{\partial x}(t,x,j) + \frac{1}{2}\sigma^2(j)x^2\frac{\partial^2 G}{\partial x^2}(t,x,j) + \sum_{i=1}^m q_{j,i}G(t,x,i) = 0, \tag{4.16}$$

and we conclude to (4.11) by (4.9). Next we note that (4.10) follows from

$$\frac{\partial G}{\partial x}(t,x,j) = P\left(\frac{\hat{Y}_{t,T}}{Y_t} < x \;\Big|\; \beta_t = j\right) \le 1, \quad (t,x,j) \in [0,T]\times[1,\infty)\times\mathcal{M}, \tag{4.17}$$

cf. the definition (2.11) of G. Next, by (2.11) and Lemma 4.3, for any $(t,x,j) \in [0,T] \times [1,\infty) \times \mathcal{M}$, we find

$$\begin{aligned}
\mathbb{L}G(t,x,j) &= x\sigma^2(j)P\left(\frac{\hat{Y}_{t,T}}{Y_t} < x \;\middle|\; \beta_t = j\right) - \mu(j)E\left[\max\left(x, \hat{Y}_{t,T}/Y_t\right) \;\middle|\; \beta_t = j\right] \\
&= E\left[x\sigma^2(j)\mathbf{1}_{\{\hat{Y}_{t,T}/Y_t < x\}} - \mu(j)\left(x \vee \frac{\hat{Y}_{t,T}}{Y_t}\right) \;\middle|\; \beta_t = j\right] \qquad (4.18) \\
&= E\left[x(\sigma^2(j) - \mu(j))\mathbf{1}_{\{\hat{Y}_{t,T}/Y_t < x\}} \;\middle|\; \beta_t = j\right] - E\left[\mu(j)\left(x \vee \frac{\hat{Y}_{t,T}}{Y_t}\right)\mathbf{1}_{\{\hat{Y}_{t,T}/Y_t \geq x\}} \;\middle|\; \beta_t = j\right],
\end{aligned}$$

which shows (4.12), and implies by (4.18) that $\mathbb{L}G(t,x,j)$ is nondecreasing and continuous in $t \in [0,T]$ when $\mu(j) \geq 0$. □

The proof of the next lemma, which will be used in Proposition 5.1 below, extends the argument of du Toit & Peskir (2009) page 993 to the regime-switching setting.

Lemma 4.4 *We have*

$$\{(t,x,j) \in [0,T) \times [1,\infty) \times \mathcal{M} \;:\; \mathbb{L}G(t,x,j) < 0\} \subset C, \qquad (4.19)$$

where $C = D^c$ is the continuation set.

Proof. By Lemma 4.7 below and Lemma 1 in Yao, Zhang & Zhou (2006) we have

$$E[G(s, X_s^{t,x}, \beta_s) \mid \beta_t = j] = G(t,x,j) + E\left[\int_t^s \mathbb{L}G(r, X_r^{t,x}, \beta_r)dr \;\middle|\; \beta_t = j\right], \qquad (4.20)$$

$s \in [t,T]$. Assume now that $(t,x,j) \in [0,T) \times [1,\infty) \times \mathcal{M}$ is such that $\mathbb{L}G(t,x,j) < 0$. By the continuity of $\mathbb{L}G(t,x,j)$ with respect to t, which follows from (3.4), the time homogeneity of $(\beta_t)_{t\in\mathbb{R}_+}$ and the path continuity of $(Y_t)_{t\in[0,T]}$, there exists an open neighbourhood $U \subset [0,T) \times [1,\infty)$ of (t,x), depending on U and such that $\mathbb{L}G(s,y,j) < 0$ for all $(s,y) \in U$. Substituting the variable s in (4.20) with the first exist time τ_U of U when $(X_s^{t,x}, \beta_s)_{s\in[t,T]}$ is started at (x,j) at time t, Relation (4.20) above shows by optional sampling that

$$E[G(\tau_U, X_{\tau_U}^{t,x}, \beta_{\tau_U}) \mid \beta_t = j] = G(t,x,j) + E\left[\int_t^{\tau_U} \mathbb{L}G(r, X_r^{t,x}, \beta_r)dr \;\middle|\; \beta_t = j\right].$$

Since $\tau_U > t$ a.s. and $\mathbb{L}G(r, X_r^{t,x}, \beta_r) < 0$ when $r \in (t, \tau_U)$, the right hand side is strictly smaller than $G(t, x, j)$, while we have

$$E[G(\tau_U, X_{\tau_U}^{t,x}, \beta_{\tau_U}) \mid \beta_t = j] \geq V(t, x, j), \tag{4.21}$$

showing that $V(t, x, j) < G(t, x, j)$, which implies that $(t, x, j) \in C$. □

Next we derive the following continuity result wich has been used in the proof of Proposition 3.1.

Lemma 4.5 *For any $j \in \mathcal{M}$, the mapping $(t, x) \longmapsto V(t, x, j)$ is jointly continuous on $[0, T] \times [1, \infty)$.*

Proof. We proceed in two steps. (i) We show that the mapping $t \longmapsto V(t, x, j)$ is continuous on $[0, T]$ for every fixed $x \geq 1$ and any $j \in \mathcal{M}$. By (2.6) we have

$$\begin{aligned}
V(t, x, j) &= \inf_{t \leq \tau \leq T} E\left[\frac{(xY_t) \vee \hat{Y}_{t,T}}{Y_\tau} \,\middle|\, \beta_t = j \right] \\
&= \inf_{0 \leq \tau \leq T-t} E\left[\frac{x \vee \mathrm{e}^{\max\limits_{0 \leq r \leq T-t}((\mu(\beta_r) - \sigma^2(\beta_r)/2)r + \sigma(\beta_r)B_r)}}{\mathrm{e}^{(\mu(\beta_\tau) - \sigma^2(\beta_\tau)/2)\tau + \sigma(\beta_\tau)B_\tau}} \,\middle|\, \beta_0 = j \right] \\
&= \inf_{0 \leq \tau \leq T-t} E\left[U(t, \tau) \,\middle|\, \beta_0 = j \right], \quad t \in [0, T], \; j \in \mathcal{M}, \; x \in [1, \infty),
\end{aligned}$$

where

$$U(t, s) := \frac{x \vee \mathrm{e}^{\max\limits_{0 \leq r \leq T-t}((\mu(\beta_r) - \sigma^2(\beta_r)/2)r + \sigma(\beta_r)B_r)}}{\mathrm{e}^{(\mu(\beta_s) - \sigma^2(\beta_s)/2)\tau + \sigma(\beta_s)B_s}}, \qquad s, t \in [0, T]. \tag{4.22}$$

For any $(\mathcal{F}_s^t)_{s \in [t,T]}$-stopping time $\tau \in [0, T - t]$ we have

$$\begin{aligned}
0 &\leq E\left[U(t, \tau) - U(t + s, \tau) \,\middle|\, \beta_0 = j \right] \\
&\leq \sqrt{E\left[\left(\mathrm{e}^{\max\limits_{0 \leq r \leq T-t}((\mu(\beta_r) - \sigma^2(\beta_r)/2)r + \sigma(\beta_r)B_r)} - \mathrm{e}^{\max\limits_{0 \leq r \leq T-t-s}((\mu(\beta_r) - \sigma^2(\beta_r)/2)r + \sigma(\beta_r)B_r)} \right)^2 \,\middle|\, \beta_0 = j \right]} \\
&\quad \times \sqrt{E\left[\mathrm{e}^{-2(\mu(\beta_\tau) - \sigma^2(\beta_\tau)/2)\tau - 2\sigma(\beta_\tau)B_\tau} \,\middle|\, \beta_0 = j \right]} \\
&\leq \sqrt{E\left[\left(\mathrm{e}^{\max\limits_{0 \leq r \leq T-t}((\mu(\beta_r) - \sigma^2(\beta_r)/2)r + \sigma(\beta_r)B_r)} - \mathrm{e}^{\max\limits_{0 \leq r \leq T-t-s}((\mu(\beta_r) - \sigma^2(\beta_r)/2)r + \sigma(\beta_r)B_r)} \right)^2 \,\middle|\, \beta_0 = j \right]}
\end{aligned}$$

$$\times \sqrt{\mathrm{e}^{(T-t)\max_{i\in\mathcal{M}}|3\sigma(i)-2\mu(i)|} E\left[\mathrm{e}^{-2\sigma^2(\beta_\tau)\tau-2\sigma(\beta_\tau)B_\tau} \;\middle|\; \beta_0=j\right]}$$

$$\leq \sqrt{E\left[\left(\mathrm{e}^{\max_{0\leq r\leq T-t}((\mu(\beta_r)-\sigma^2(\beta_r)/2)r+\sigma(\beta_r)B_r)} - \mathrm{e}^{\max_{0\leq r\leq T-t-s}((\mu(\beta_r)-\sigma^2(\beta_r)/2)r+\sigma(\beta_r)B_r)}\right)^2 \;\middle|\; \beta_0=j\right]}$$

$$\times\, \mathrm{e}^{(T-t)\max_{i\in\mathcal{M}}|3\sigma(i)-2\mu(i)|/2}, \tag{4.23}$$

where we applied the optional sampling theorem. Letting s tend to 0 on both sides of (4.23), we get

$$\lim_{s\searrow 0} E\big[U(t+s,\tau) \mid \beta_0=j\big] = E\big[U(t,\tau) \mid \beta_0=j\big], \tag{4.24}$$

and since the convergence is uniform on all $(\mathcal{F}_s^0)_{s\in[0,T]}$-stopping times $\tau\in[0,T]$, we obtain

$$\begin{aligned}\liminf_{s\searrow 0} \inf_{0\leq\tau\leq T-t-s} E\big[U(t+s,\tau) \mid \beta_0=j\big] &\geq \liminf_{s\searrow 0} \inf_{0\leq\tau\leq T-t} E\big[U(t+s,\tau) \mid \beta_0=j\big] \\ = \inf_{0\leq\tau\leq T-t} \lim_{s\searrow 0} E\big[U(t+s,\tau) \mid \beta_0=j\big] &= \inf_{0\leq\tau\leq T-t} E\big[U(t,\tau) \mid \beta_0=j\big]. \end{aligned} \tag{4.25}$$

Next, according to Proposition 3.1 there exists an optimal $(\mathcal{F}_s^0)_{s\in[0,T]}$-stopping time $\tau_t^*\in[0,T-t]$ such that

$$\inf_{0\leq\tau\leq T-t} E\big[U(t,\tau) \mid \beta_0=j\big] = E\big[U(t,\tau_t^*) \mid \beta_0=j\big], \tag{4.26}$$

hence we have

$$\begin{aligned}\inf_{0\leq\tau\leq T-t-s} E\big[U(t+s,\tau) \mid \beta_0=j\big] &\leq \inf_{0\leq\tau\leq T-t-s} E\big[U(t,\tau) \mid \beta_0=j\big] \\ &\leq E\big[U(t,\tau_t^*\wedge(T-t-s)) \mid \beta_0=j\big]. \end{aligned} \tag{4.27}$$

Since $U(t,s)$ is nonnegative for any $s,t\in[0,T]$, we have

$$\begin{aligned} U(t,\tau_t^*\wedge(T-t-s)) &\leq U(t,\tau_t^*)+U(t,T-t-s) \\ &= U(t,\tau_t^*) + \frac{x\vee \mathrm{e}^{\max_{0\leq r\leq T-t}((\mu(\beta_r)-\sigma^2(\beta_r)/2)r+\sigma(\beta_r)B_r)}}{\mathrm{e}^{(\mu(\beta_{T-t-s})-\sigma^2(\beta_{T-t-s})/2)(T-t-s)+\sigma(\beta_{T-t-s})B_{T-t-s}}} \\ &\leq U(t,\tau_t^*) + \frac{x\vee \mathrm{e}^{\max_{0\leq r\leq T-t}((\mu(\beta_r)-\sigma^2(\beta_r)/2)r+\sigma(\beta_r)B_r)}}{\mathrm{e}^{\inf_{i\in\mathcal{M},r\in[0,T-t]}(\mu(i)-\sigma^2(i)/2)r+\inf_{i\in\mathcal{M},r\in[0,T-t]}(\sigma(i)B_r)}}, \end{aligned}$$

which is integrable by (4.26). By the reverse Fatou Lemma we have

$$\limsup_{s\searrow 0} E\big[U(t,\tau_t^*\wedge(T-t-s))\mid\beta_0=j\big]\leq E\big[\limsup_{s\searrow 0} U(t,\tau_t^*\wedge(T-t-s))\mid\beta_0=j\big]$$
$$= E\big[U(t,\tau_t^*)\mid\beta_0=j\big]. \tag{4.28}$$

Combining (4.26), (4.27), (4.28) and (4.25) we find

$$\lim_{s\searrow 0}\inf_{0\leq\tau\leq T-t-s} E\left[U(t+s,\tau)\,\middle|\,\beta_0=j\right]=\inf_{0\leq\tau\leq T-t} E\left[U(t,\tau)\,\middle|\,\beta_0=j\right]. \tag{4.29}$$

Similarly we have

$$\lim_{s\searrow 0}\inf_{0\leq\tau\leq T-t+s} E\left[U(t-s,\tau)\,\middle|\,\beta_0=j\right]=\inf_{0\leq\tau\leq T-t} E\left[U(t,\tau)\,\middle|\,\beta_0=j\right], \tag{4.30}$$

hence $t\longmapsto V(t,x,j)$ is continuous on $[0,T]$.

(ii) We show that $x\longmapsto V(t,x,j)$ is continuous on $[1,\infty)$, uniformly in $t\in[0,T]$, extending the argument of du Toit & Peskir (2009) page 995 to the regime-switching setting. By Relation (4.17) and the mean value theorem, for all $y\in[x,\infty)$ there exists a (random) $\eta\in[X_{t+\tau}^{t,x},X_{t+\tau}^{t,y}]$ such that for any $(\mathcal{F}_s^0)_{s\in[0,T]}$-stopping time $\tau\in[0,T-t]$ we have

$$G(t+\tau,X_{t+\tau}^{t,y},\beta_{t+\tau})-G(t+\tau,X_{t+\tau}^{t,x},\beta_{t+\tau})=\frac{\partial G}{\partial x}(t+\tau,\eta)(X_{t+\tau}^{t,y}-X_{t+\tau}^{t,x},\beta_{t+\tau})$$
$$\leq (y-x)\frac{Y_t}{Y_{t+\tau}}, \tag{4.31}$$

since $X_{t+\tau}^{t,y}-X_{t+\tau}^{t,x}\leq(y-x)Y_t/Y_{t+\tau}$ by (2.13). Let now $(t,x,j)\in[0,T]\times[1,\infty)\times\mathcal{M}$ and consider $\tau_x:=\tau(t,x,j)$ given by (3.2). By Lemma 2.2 we have

$$V(t,y,j)-V(t,x,j)\leq E\left[G(t+\tau_x,X_{t+\tau_x}^{t,y},\beta_{t+\tau_x})-G(t+\tau_x,X_{t+\tau_x}^{t,x},\beta_{t+\tau_x})\mid\beta_t=j\right]. \tag{4.32}$$

Since $E\big[Y_t/Y_{t+\tau}\mid\beta_t=j\big]$ is uniformly bounded as in (3.6), taking expectation on both sides of (4.31) yields

$$\lim_{y\to x} E\left[G(t+\tau,X_{t+\tau}^{t,y},\beta_{t+\tau})-G(t+\tau,X_{t+\tau}^{t,x},\beta_{t+\tau})\mid\beta_t=j\right]=0, \tag{4.33}$$

uniformly in $t\in[0,T]$ and in the $(\mathcal{F}_s^0)_{s\in[0,T]}$-stopping times $\tau\in[0,T-t]$. Since $V(t,x,j)$ is increasing in $x\in[1,\infty)$, (4.32) and (4.33) yield

$$0\leq\lim_{y\to x}(V(t,y,j)-V(t,x,j))\leq 0, \tag{4.34}$$

which shows the continuity of $x \longmapsto V(t,x,j)$, uniformly in $t \in [0,T]$, for all $j \in \mathcal{M}$.

From (i) and (ii) we conclude to the joint continuity of $(t,x) \longmapsto V(t,x,j)$ on $[0,T] \times [1,\infty)$ by classical arguments. $\square$

Lemma 4.6 *The mapping $(t,x) \longmapsto G(t,x,j)$ is jointly continuous on $[0,T]\times[1,\infty)$.*

Proof. By Relation (4.17) and the mean value theorem, for all $y \in [x,\infty)$ there exists an $\eta \in [x,y]$ such that for any $t \in [0,T]$ we have

$$0 \le G(t,y,j) - G(t,x,j) = (y-x)\frac{\partial G}{\partial x}(t,\eta,j) \le y-x, \tag{4.35}$$

which shows the continuity of $x \longmapsto G(t,x,j)$, uniformly in $t \in [0,T]$. On the other hand, we have by (2.11) that $t \longmapsto G(t,x,j)$ is continuous on $[0,T]$ for every $x \ge 1$. We conclude to the joint continuity of $(t,x) \longmapsto G(t,x,j)$ on $[0,T] \times [1,\infty)$ by a classical argument. $\square$

We close this section with the following three lemmas.

Lemma 4.7 *The Markov process $(s, X_s^{t,x}, \beta_s)_{s\in[t,T]}$ has the infinitesimal generator*

$$\mathbb{L}f(s,y,j) = \left(\frac{\partial}{\partial s} + y(\sigma^2(j)-\mu(j))\frac{\partial}{\partial y} + \frac{1}{2}\sigma^2(j)y^2\frac{\partial^2}{\partial y^2}\right)f(s,y,j) + \sum_{i=1}^m q_{j,i}f(s,y,i), \tag{4.36}$$

$s \in [0,T]$, $j \in \mathcal{M}$, $y \in [1,\infty)$, for $f \in \mathrm{Dom}\,(\mathbb{L})$ satisfying $\frac{\partial f}{\partial y}(s,1+,j) = 0$.

Proof. Letting

$$Z_s^{t,x} := \log x \vee \hat{S}_{s-t}^t - \int_0^{s-t} \sigma(\beta_{u+t})d\hat{B}_u^t, \tag{4.37}$$

$s \in [t,T]$, $x \ge 1$, from (3.3) we have $X_s^{t,x} = \exp(Z_s^{t,x})$, $s \in [t,T]$, $x \ge 1$. Since $(\hat{S}_r^t)_{r\in[0,T-t]}$ is nondecreasing it has finite variation, hence

$$d\langle Z_r^{t,x}, Z_r^{t,x}\rangle = \sigma^2(\beta_r)\langle d\hat{B}_{r-t}^t, d\hat{B}_{r-t}^t\rangle = \sigma^2(\beta_r)d\langle B_r, B_r\rangle = \sigma^2(\beta_r)dr, \tag{4.38}$$

which shows that

$$dX_s^{t,x} = X_s^{t,x}dZ_s^{t,x} + \frac{1}{2}X_s^{t,x}d\langle Z_s^{t,x}, Z_s^{t,x}\rangle$$

$$= X_s^{t,x} dZ_s^{t,x} + \frac{1}{2}\sigma^2(\beta_s) X_s^{t,x} ds$$
$$= X_s^{t,x} d(\log x \vee \hat{S}_{s-t}^t) - \sigma(\beta_s) X_s^{t,x} d\hat{B}_{s-t}^t + \frac{1}{2}\sigma^2(\beta_s) X_s^{t,x} ds. \tag{4.39}$$

Given that $\frac{\partial f}{\partial y}(s, 1+, j) = 0$ for $(s, y, j) \in [0, T] \times [1, \infty) \times \mathcal{M}$, we have

$$\begin{aligned}
\frac{\partial f}{\partial y}(s, X_s^{t,x}, \beta_s)\, d(\log x \vee \hat{S}_{s-t}^t) &= \frac{\partial f}{\partial y}(s, X_s^{t,x}, \beta_s) \mathbf{1}_{\{X_s^{t,x} > 1\}} d(\log x \vee \hat{S}_{s-t}^t) \\
&= \frac{\partial f}{\partial y}(s, X_s^{t,x}, \beta_s) \mathbf{1}_{\{Z_s^{t,x} > 0\}} d(\log x \vee \hat{S}_{s-t}^t) \\
&= 0,
\end{aligned}$$

since $d(\log x \vee \hat{S}_{s-t}^t) = 0$ when $Z_s^{t,x} > 0$, $s \in [t, T]$. From (4.39) this shows that

$$\frac{\partial f}{\partial y}(s, X_s^{t,x}, \beta_s)\, dX_s^{t,x} = \frac{\partial f}{\partial y}(s, X_s^{t,x}, \beta_s) \left(-\sigma(\beta_s) X_s^{t,x} d\hat{B}_{s-t}^t + \frac{1}{2}\sigma^2(\beta_s) X_s^{t,x} dr \right), \tag{4.40}$$

and we conclude the proof by Itô's calculus. □

The next two lemmas will be used in the proof of Proposition 5.2 below.

Lemma 4.8 *Let $j \in \mathcal{M}$ such that $\mu(j) \geq 0$. The function $h(t, j)$ defined by*

$$h(t, j) := \inf\{x \in [1, \infty) \ : \ \mathbb{L}G(t, y, j) \geq 0, \quad \forall y \in [x, \infty)\}, \tag{4.41}$$

is nonincreasing and continuous in $t \in [0, T]$ and satisfies $h(T, j) = 1$, for all $j \in \mathcal{M}$.

Proof. By Lemma 4.3 the function $\mathbb{L}G(t, x, j)$ is nondecreasing in t for all $x \geq 1$ since $\mu(j) \geq 0$ and it follows from the definition (4.41) of $h(t, j)$ that $t \longmapsto h(t, j)$ is nonincreasing in $t \in [0, T]$. For any $t_0 \in [0, T)$ and decreasing sequence $(t_n)_{n \geq 1} \subset (t_0, T]$ converging to t_0 from the right hand side we have $\lim_{n\to\infty} h(t_n, j) \leq h(t_0, j)$ and $\lim_{n\to\infty} h(t_n, j) \geq h(t_k, j)$ for any $k \geq 1$, hence $\lim_{n\to\infty} h(t_n, j) \geq h(t_0, j)$ as by the continuity of $t \to \mathbb{L}G(t, x, j)$ we have

$$\mathbb{L}G(t_0, \lim_{n\to\infty} h(t_n, j), j) = \lim_{k\to\infty} \mathbb{L}G(t_k, \lim_{n\to\infty} h(t_n, j), j) \geq 0, \tag{4.42}$$

and this proves that $\lim_{t \searrow t_0} h(t, j) = h(t_0, j)$. On the other hand we have $h(t_0-, j) := \lim_{t \uparrow t_0} h(t, j) \geq h(t_0, j)$ for any $t_0 \in [0, T]$, $j \in \mathcal{M}$. In case $h(t_0-, j) > h(t_0, j)$ we have $\mathbb{L}G(t_0, x, j) \geq 0$ for all $x \in [h(t_0, j), \infty]$. In addition, for any $t \in [0, t_0)$ and

$x \in [h(t_0, j), h(t_0-, j))$ we have $\mathbb{L}G(t, x, j) < 0$ since $h(t, j) \geq h(t_0-, j)$, hence $\mathbb{L}G(t_0, x, j) = 0$ for all $x \in [h(t_0, j), h(t_0-, j))$ by the continuity of $t \longmapsto \mathbb{L}G(t, x, j)$. By Lemma 4.3 we would have

$$x\sigma^2(j)\frac{\partial G}{\partial x}(t_0, x, j) = \mu(j)G(t_0, x, j), \qquad x \in [h(t_0, j), h(t_0-, j)), \tag{4.43}$$

which shows that $G(t_0, x, j) = C(t_0, j)x^{\mu(j)/\sigma^2(j)}$, where $C(t_0, j)$ depends only on t_0 and $j \in \mathcal{M}$. This is a contradiction since $\frac{\partial}{\partial x}G(t_0, x, j) = P\left(\hat{Y}_{t_0,T}/Y_{t_0} < x \mid \beta_{t_0} = j\right) = C(t_0, j)\mu(j)x^{-1+\mu(j)/\sigma^2(j)}/\sigma^2(j)$ for $x \in [h(t_0, j), h(t_0-, j))$ cannot hold when $\mu(j) < \sigma^2(j)$, and more generally $\hat{Y}_{t_0,T}/Y_{t_0}$ cannot have a power law, even locally. □

Similarly to (3.22)-(3.23) in du Toit & Peskir (2009), we now show that $F(t, x, j)$ defined by (3.7) is nondecreasing in $t \in [0, T]$ for all $j \in \mathcal{M}$ and $x \in [1, \infty)$, as in the following Lemma 4.9 which will be used for Proposition 5.2, and whose proof follows the lines of du Toit & Peskir (2009) page 994.

Lemma 4.9 *Under the condition $\mu(j) \geq 0$ for all $j \in \mathcal{M}$, the function*

$$t \longmapsto F(t, x, j) = V(t, x, j) - G(t, x, j) \tag{4.44}$$

is nondecreasing in $t \in [0, T]$, for any $(j, x) \in \mathcal{M} \times [1, \infty)$.

Proof. For any $r, s \in [0, T - t]$ with $r < s$, let $\tau_s := \tau_D(s, x, j) - s \in [0, T - s]$ by the definition (3.2) of τ_D. Replacing s with τ_s and t with r in the formula (4.20) and using optional sampling, we have

$$\begin{aligned} F(r, x, j) &= V(r, x, j) - G(r, x, j) \\ &\leq E[G(r + \tau_s, X^{r,x}_{r+\tau_s}, \beta_{r+\tau_s}) \mid \beta_r = j] - G(r, x, j) \\ &= E\left[\int_r^{r+\tau_s} \mathbb{L}G(v, X^{r,x}_v, \beta_v)dv \;\middle|\; \beta_r = j\right] \\ &= E\left[\int_0^{\tau_s} \mathbb{L}G(v + r, X^{r,x}_{v+r}, \beta_{v+r})dv \;\middle|\; \beta_r = j\right] \\ &= E\left[\int_0^{\tau_s} \mathbb{L}G(v + r, X^{0,x}_v, \beta_v)dv \;\middle|\; \beta_0 = j\right], \qquad 0 \leq r < s \leq T - t. \end{aligned} \tag{4.45}$$

Combining (4.45) with

$$F(s, x, j) = V(s, x, j) - G(s, x, j) = E[G(s + \tau_s, X^{r,x}_{s+\tau_s}, \beta_{s+\tau_s}) \mid \beta_r = j] - G(s, x, j)$$

$$= \quad E\left[\int_0^{\tau_s} \mathbb{L}G(v+s, X_v^{0,x}, \beta_v)dv \;\Big|\; \beta_0 = j\right],$$

we have

$$\begin{aligned}
&F(s,x,j) - F(r,x,j) \qquad (4.46)\\
&\geq \quad E\left[\int_0^{\tau_s} \mathbb{L}G(v+s, X_v^{0,x}, \beta_v)dv \;\Big|\; \beta_0 = j\right] - E\left[\int_0^{\tau_s} \mathbb{L}G(v+r, X_v^{0,x}, \beta_v)dv \;\Big|\; \beta_0 = j\right]\\
&= \quad E\left[\int_0^{\tau_s} \mathbb{L}G(v+s, X_v^{0,x}, \beta_v) - \mathbb{L}G(v+r, X_v^{0,x}, \beta_v)dv \;\Big|\; \beta_0 = j\right].
\end{aligned}$$

Since by (4.18) the function $t \mapsto \mathbb{L}G(t,x,i)$ is nondecreasing in t when $\mu(i) \geq 0$, we find that the right hand side of (4.46) is nonnegative, thereby $F(t,x,j)$ is nondecreasing in $t \in [0,T]$. □

5 Solution of the free boundary problem

In this section we turn to the solution of the free boundary problem (2.6). First, we note that the stopping set D has a simple form in two special situations.

Proposition 5.1 *We have the following special cases of optimal stopping sets D.*

i) Immediate exercise. Under the condition $\mu(j) \leq 0$ for all $j \in \mathcal{M}$, we have $D = [0,T] \times [1,\infty) \times \mathcal{M}$.

ii) Exercise at maturity. Under the condition $\mu(j) \geq \sigma^2(j)$ for all $j \in \mathcal{M}$, we have $D = \{T\} \times [1,\infty) \times \mathcal{M}$.

Proof. Replacing s in (4.20) with τ_D defined in (3.2) and using optional sampling, we find

$$V(t,x,j) = G(t,x,j) + E\left[\int_t^{\tau_{D(t,x,j)}} \mathbb{L}G(r, X_r^{t,x}, \beta_r)dr \;\Big|\; \beta_t = j\right], \quad t \in [0,T], \quad j \in \mathcal{M}, \tag{5.1}$$

where $(X_r^{t,x})_{r\in[t,T]}$ is defined in (2.13).

i) In case $\mu(j) \leq 0$ for all $j \in \mathcal{M}$, by Lemma 4.3, we have $\mathbb{L}G(t,x,i) > 0$ for all $(t,x,i) \in [0,T) \times [1,\infty) \times \mathcal{M}$, hence (5.1) implies $\tau_D(t,x,i) = 0$ a.s., otherwise it contradicts the fact that $V(t,x,i) \leq G(t,x,i)$ because of (5.1). This implies $[0,T] \times [1,\infty) \times \mathcal{M} \subset D$.

ii) In case $\mu(j) \geq \sigma^2(j)$ for all $j \in \mathcal{M}$, by Lemma 4.3 we have $\mathbb{L}G(t,x,i) < 0$ for all $(t,x,i) \in [0,T) \times [1,\infty) \times \mathcal{M}$, and applying Lemma 4.4, we see that $[0,T) \times [1,\infty) \times \mathcal{M} \subset C$, which means $D = \{T\} \times [1,\infty) \times \mathcal{M}$.

□

Next, we provide sufficient conditions on the drift coefficients $(\mu(j))_{j \in \mathcal{M}}$ for the boundary function $b(t,j)$ defined by (3.10) to be nonincreasing and continuous in $t \in [0,T]$.

Proposition 5.2 *Assume that $\mu(j) \geq 0$ for all $j \in \mathcal{M}$. Then the boundary function $b(t,j)$ defined by (3.10) is nonincreasing in $t \in [0,T]$. If in addition $\mu(j) \in (0, \sigma^2(j))$ for all $j \in \mathcal{M}$, then $b(t,j)$ is finite and continuous in $t \in [0,T]$.*

Proof. (i) Monotonicity. Let $(t,x,j) \in D$ and $s \in [t,T]$. We have $F(t,x,j) = 0$ and $F(s,x,j) = 0$ since $F(t,x,j)$ is nondecreasing in t by Lemma 4.9, hence

$$[t,T] \times \{x\} \times \{j\} \subset D, \tag{5.2}$$

showing that $(t,x,j) \in D \iff [t,T] \times \{x\} \times \{j\} \subset D$. Then for any $s \in (t,T]$, we have $(s, b(t,j), j) \in D$ since $(t, b(t,j), j) \in D$. By Proposition 3.2 and noting that $(s, b(s,j), j) \in D$, we conclude that $b(s,j) \leq b(t,j)$.

(ii) Finiteness. Since the function $h(t,j)$ defined in Lemma 4.8 is nonincreasing and continuous in $t \in [0,T]$ with $h(T,j) = 1$, we can repeat the argument on pages 994-995 of du Toit & Peskir (2009) as by (4.18) and the condition $\mu(j) \in (0, \sigma^2(j))$ for all $j \in \mathcal{M}$, the function $\mathbb{L}G(t,x,j)$ satisfies $\lim_{x\to\infty} \mathbb{L}G(t,x,j) = \infty$, $t \in [0,T]$, $j \in \mathcal{M}$, and

$$\inf \left\{ \mathbb{L}G(t,x,j) \ : \ (t,x,j) \in [0,T] \times [1,\infty) \times \mathcal{M} \right\} > -\infty. \tag{5.3}$$

(iii) Right continuity. Given $(t, b(t,j), j) \in D$, consider a strictly decreasing sequence $(t_n)_{n\geq 1}$ such that $\lim_{n\to\infty} t_n = t$. By part (i) above we know that $b(t_n, j) \leq b(t,j)$, $n \geq 1$, and $\lim_{n\to\infty} b(t_n, j) \leq b(t,j)$. Next, by Proposition 3.2 we have

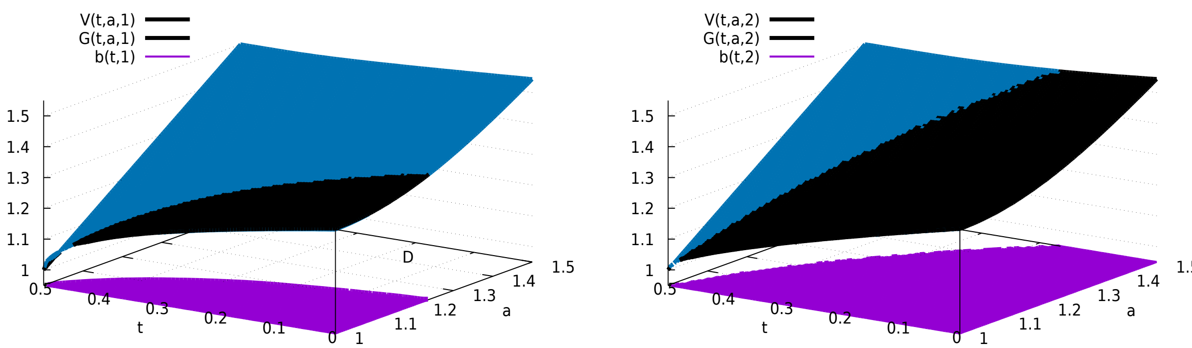
$$[t,T] \times [b(t,j), \infty) \times \{j\} \subset D, \tag{5.4}$$

and since $(t_n, j, b(t_n, j)) \in D$, $n \geq 1$, and D is closed, we have $\left(t, \lim_{n\to\infty} b(t_n,j), j\right) \in D$, hence $\lim_{n\to\infty} b(t_n, j) \geq b(t,j)$.

(iv) Left continuity. Similarly to point (ii) above we can apply the argument of du Toit & Peskir (2009) page 998 based on the fact that from Lemma 4.8 the function $h(t,j)$ is nonincreasing and continuous in $t \in [0,T]$ for all $j \in \mathcal{M}$. $\square$

Figure 1 illustrates the result of Proposition 5.2 by applying the recursive algorithm of Liu & Privault (2017) in order to plot the value functions $V(t,a,j)$ and $G(t,a,j)$ in the case $\mu(j) \in (0,\sigma^2(j))$, $j \in \mathcal{M}$. In Figure 1 we take the positive drifts $\mu(1) = 0.15$, $\mu(2) = 0.05$, with $\sigma(1) = 0.5$, $\sigma(2) = 0.3$, $T = 0.5$, $n = 100$, $\delta_n = T/n = 0.05$, and

$$\mathbf{Q} = \begin{bmatrix} -2.5 & 2.5 \\ 2 & -2 \end{bmatrix}.$$



(a) Value functions from state 1. (b) Value functions from state 2.

Figure 1: Value functions in the two-state case as functions of time and the underlying.

Figure 1 also allows us to visualize the stopping set D and the continuation set

$$C = \big\{(t,a,j) \in [0,T] \times [1,\infty) \times \mathcal{M} \ : \ V(t,a,j) < G(t,a,j)\big\}. \tag{5.5}$$

The numerical instabilities observed are due to the necessity to check the equality $V(t,a,j) = G(t,a,j)$ when $V(t,a,j)$ and $G(t,a,j)$ are very close to each other.

The boundary functions are plotted in Figure 2 based on Figure 1, with spline smoothing. We observe that starting from state 1 it is better to exercise earlier than if we start from state 2 which has a lower drift. This is due to the possibility to switch from state 1 to state 2 after the average time $1/q_{1,1} = 0.4$ and to stay at state 2 for the remaining time $T - t \leq 1/q_{2,2} = 0.5$, in which case the drift takes the lower value $\mu(2) = 0.05$. The opposite occurs if we start from state 2, for which the boundary graph is higher than if we start from state 1.

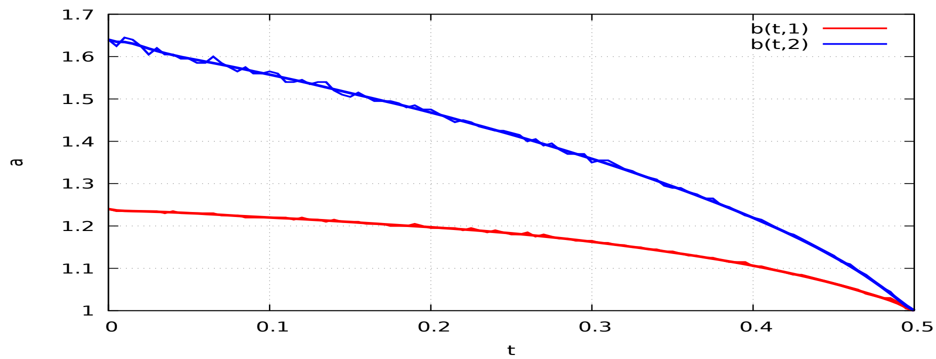


Figure 2: Boundary curves as functions of time in the two-state case.

We note that without the condition $\mu(i) \geq 0$ for all $i \in \mathcal{M}$, the function $F(t,x,j)$ in Lemma 4.9 may not be nondecreasing in $t \in [0,T]$, in which case the equivalence $(t,x,j) \in D \Longleftrightarrow [t,T] \times \{x\} \times \{j\} \subset D$ in the proof of Proposition 5.2 does not hold, and in this situation the boundary function $t \longmapsto b(t,j)$ may not be decreasing in $t \in [0,T]$, cf. e.g. Figure 4 in Liu & Privault (2017).

Next, we derive a Volterra type equation (5.7) below satisfied by the function $b(t,j)$ defined in (3.10), for the boundary curves

$$\{(t,x) \in [0,T] \times [1,\infty) \ : \ x = b(t,j)\} \tag{5.6}$$

of the optimal stopping set D in (3.1), for any $j \in \mathcal{M}$.

Proposition 5.3 *Assume that $\mu(j) \in (0, \sigma^2(j))$ for all $j \in \mathcal{M}$. Then the boundary function $b(t,j)$ satisfies the Volterra type equation*

$$G(t, b(t,j), j) = J(t, b(t,j), j) - \int_t^T K(t, r, b(t,j), j) dr, \tag{5.7}$$

$0 \leq t \leq T$, with terminal condition $b(T,j) = 1$, $j \in \mathcal{M}$, where

$$J(t,x,j) := E[X_T^{t,x} \mid \beta_t = j], \tag{5.8}$$

and

$$K(t,r,x,j) := E\left[\mathbb{L}V(r, X_r^{t,x}, \beta_r) \mathbf{1}_{\{X_r^{t,x} > b(r,\beta_r)\}} \;\middle|\; \beta_t = j \right], \tag{5.9}$$

for $0 \leq t \leq r \leq T$ and $x \geq 1$.

Proof. Noting that $V(t,x,j) \le G(t,x,j)$ for all $(t,x,j) \in [0,T] \times [1,\infty) \times \mathcal{M}$ by (2.12), the continuation set $C := D^c$ is given by

$$C = D^c = \{(t,x,j) \in [0,T] \times [1,\infty) \times \mathcal{M} \; : \; V(t,x,j) < G(t,x,j)\} \,. \tag{5.10}$$

According to Proposition 3.1, for any $(t,x,j) \in C$, we have

$$V(t,x,j) = E\left[G\left(\tau_D, X^{t,x}_{\tau_D}, \beta_{\tau_D}\right) \;\Big|\; \beta_t = j \right], \tag{5.11}$$

where $\tau_D = \tau_D(t,x,j)$ is defined by (3.2). Given that $\frac{\partial V}{\partial y}(t,1+,j) = 0$ by Lemma 4.2, by the application of Peskir & Shiryaev (2006), Chapter III, § 7.1.1, § 7.4.1 as in du Toit & Peskir (2009) § 3.5, page 996, the function V in (5.11) is $\mathcal{C}^{1,2}$ in the continuation set C in (5.10) and it solves the Cauchy-Dirichlet free boundary problem

$$\begin{cases} \mathbb{L}V(t,y,j) = 0, & (t,y,j) \in C, \qquad (5.12a) \\ V(t,y,j) = G(t,y,j), & (t,y,j) \in \partial C, \qquad (5.12b) \end{cases}$$

hence $\partial C \subset D$, where ∂C denotes the boundary of the open set C. By the local time change of variable formula of Peskir (2005), and by Lemma 4.7 below with the property $\frac{\partial V}{\partial y}(t,1+,j) = 0$ shown in Lemma 4.2 above, we have

$$\begin{aligned} & E[X^{t,x}_T \mid \beta_t = j] = E[V(T, X^{t,x}_T, \beta_T) \mid \beta_t = j] \\ & = \; V(t,x,j) + E\left[\int_t^T \mathbb{L}V(r, X^{t,x}_r, \beta_r) \mathbf{1}_{\{X^{t,x}_r \neq b(r,\beta_r)\}} dr \;\Big|\; \beta_t = j \right] \qquad (5.13) \\ & + \frac{1}{2} E\left[\int_t^T \left(\frac{\partial V}{\partial y}(r, X^{t,x}_r+, \beta_r) - \frac{\partial V}{\partial y}(r, X^{t,x}_r-, \beta_r) \right) \mathbf{1}_{\{X^{t,x}_r = b(r,\beta_r)\}} d\ell^b_r(X^{t,x}) \;\Big|\; \beta_t = j \right], \end{aligned}$$

where we applied the equality $V(T, X^{t,x}_T, \beta_T) = X^{t,x}_T$, and $(\ell^b_r(X^{t,x}))_{r\in[t,T]}$ denotes the local time of $X^{t,x}$ on the (piecewise continuous and nonincreasing by Proposition 5.2) curve $r \longmapsto b(r,\beta_r)$. By the smooth fit property shown in Lemma 4.1 above, the last term in (5.13) vanishes. By Proposition 3.2 above and the definition (3.10) of $b(t,j)$, Relation (5.12a) can be rewritten as

$$\mathbb{L}V(r,y,j)\mathbf{1}_{\{y<b(r,j)\}} = 0, \qquad r \in [0,T], \; j \in \mathcal{M}, \; y \ge 1, \tag{5.14}$$

which implies

$$E\left[\int_t^T \mathbb{L}V(r,X_r^{t,x},\beta_r)dr \mid \beta_t=j\right] = E\left[\int_t^T \mathbb{L}V(r,X_r^{t,x},\beta_r)\mathbf{1}_{\{X_r^{t,x}>b(r,\beta_r)\}}dr \,\Big|\, \beta_t=j\right]. \tag{5.15}$$

Hence, combining (5.13) and (5.15), we obtain

$$E[X_T^{t,x} \mid \beta_t=j] = V(t,x,j)+\int_t^T E\left[\mathbb{L}V(r,X_r^{t,x},\beta_r)\mathbf{1}_{\{X_r^{t,x}>b(r,\beta_r)\}} \,\Big|\, \beta_t=j\right]dr, \tag{5.16}$$

and substituting x with $b(t,j)$ in (5.16) above we find that

$$\begin{aligned}
G(t,b(t,j),j) &= V(t,b(t,j),j)\\
&= E[X_T^{t,b(t,j)} \mid \beta_t=j] - E\left[\int_t^T \mathbb{L}V(r,X_r^{t,b(t,j)},\beta_r)\mathbf{1}_{\{X_r^{t,x}\geq b(r,\beta_r)\}}dr \,\Big|\, \beta_t=j\right]\\
&= J(t,b(t,j),j) - \int_t^T K(t,r,b(t,j),j)dr,
\end{aligned}$$

where the functions J, K are defined by (5.8)-(5.9). □

Remark 5.4 *Note that the equation (5.7) also involves the optimal value function $V(r,y,j)$ and not only the function $G(r,y,j)$. Indeed, when $m\geq 2$ the equality $V(r,y,j)=G(r,y,j)$ in (5.15) for a given $(r,y,j)=(r,X_r^{t,x},\beta_r)\in D$ does not imply*

$$\mathbb{L}V(r,y,j)=\mathbb{L}G(r,y,j) \tag{5.17}$$

as in du Toit & Peskir (2009) because we may not have $V(r,y,i)=G(r,y,i)$ for all $i=1,\ldots,m$ in the summation over the states of $(\beta_t)_{t\in[0,T]}$ in the definition (4.9) of $\mathbb{L}$. In Buffington & Elliott (2002), this issue is dealt with via an ordering assumption on the boundary functions $(b(t,j))_{t\in[0,T]}$ in the two-state case $j=1,2$, see Assumption 3.1 therein, however this method applies specifically to American options and not to ultimate maximum problems, which have a more complex payoff structure. Moreover, such an ordering condition may not be satisfied in our current setting, cf. Figure 4 of Liu & Privault (2017).

In the absence of regime switching with $Y_t = Y_0\,\mathrm{e}^{(\mu-\sigma^2/2)t+\sigma B_t}$, Relation (3.10) is replaced by

$$b(t)=\inf\{x\in\mathbb{R}_+ : (t,x)\in D\}, \qquad t\in[0,T], \tag{5.18}$$

and the boundary equation (5.7) becomes

$$G(t,b(t)) = E[X_T^{t,b(t)}] - E\left[\int_t^T \mathbb{L}G(r,X_r^{b(t)})\mathbf{1}_{\left\{X_r^{t,b(t)}>b(r)\right\}}dr\right], \tag{5.19}$$

which recovers (3.50) in du Toit & Peskir (2009), with

$$\mathbb{L}G(r,x) = \left(\frac{\partial}{\partial r} + x(\sigma^2-\mu)\frac{\partial}{\partial x} + \frac{1}{2}\sigma^2 x^2\frac{\partial^2}{\partial x^2}\right)G(r,x), \qquad r\in[0,T],\ x\in\mathbb{R}_+. \tag{5.20}$$

Since the Volterra type equation (5.7) cannot be solved by standard methods under regime switching, we have applied the recursive algorithm of Liu & Privault (2017) in order to plot Figures 1 and 2.

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