A recursive algorithm for selling at the ultimate maximum in regime-switching models

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Abstract

We propose a recursive algorithm for the numerical computation of the optimal value function $\inf_{t\le\tau\le T}\mathbb{E}\left[\sup_{0\le s\le T}Y_s/Y_\tau\big|\mathcal{F}_t\right]$ over the stopping times τ with respect to the filtration of a geometric Brownian motion Y_t with Markovian regime switching. This method allows us to determine the boundary functions of the optimal stopping set when no associated Volterra integral equation is available. It applies in particular when regime-switching drifts have mixed signs, in which case the boundary functions may not be monotone.

Key words: *Optimal stopping; Markovian regime switching; non-monotone free boundary; recursive approximation.*
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1 Introduction

The study of optimal stopping of Brownian motion as close as possible to its ultimate maximum has been initiated in Graversen, Peskir and Shiryaev [3]. For geometric Brownian motion, the optimal prediction problem

$$V_t=\inf_{t\le\tau\le T}\mathbb{E}\left[\sup_{0\le s\le T}\frac{Y_s}{Y_\tau}\Big|\mathcal{F}_t^0\right] \tag{1.1}$$

of selling at the ultimate maximum over all $(\mathcal{F}_s^t)_{s\in[t,T]}$-stopping times $\tau \in [t,T]$ has been solved in [2] by Du Toit and Peskir when the asset price $(Y_t)_{t\in\mathbb{R}_+}$ is modeled by a geometric Brownian motion and $(\mathcal{F}_s^t)_{s\in[t,T]}$ is filtration generated by $(B_s - B_t)_{s\in[t,T]}$, see [10] for background on optimal stopping and free boundary problems, and Chapter VIII therein for ultimate position and maximum problems.

This framework has recently been extended in Liu and Privault [7] to the regime-switching model

$$dY_t = \mu(\beta_t)Y_t dt + \sigma(\beta_t)Y_t dB_t, \qquad 0 \le t \le T, \tag{1.2}$$

driven by a finite-state, observable continuous-time Markov chain $(\beta_t)_{t\in\mathbb{R}_+}$ with state space $\mathcal{M} := \{1, 2, \dots, m\}$ independent of the standard Brownian motion $(B_t)_{t\in\mathbb{R}_+}$ on a filtered probability space $(\Omega, (\mathcal{F}_t)_{t\in\mathbb{R}_+}, \mathbb{P})$, where $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ is the filtration generated by $(B_t)_{t\in\mathbb{R}_+}$ and $(\beta_t)_{t\in\mathbb{R}_+}$ and $\mu : \mathcal{M} \longrightarrow \mathbb{R}$, and $\sigma : \mathcal{M} \longrightarrow (0,\infty)$ are deterministic functions.

Regime-switching models were introduced by Hamilton [5] in the framework of time series, in order to model the influence of external market factors. European options have been priced in continuous time regime-switching models by Yao, Zhang and Zhou [12] using a successive approximation algorithm. Optimal stopping for option pricing with regime switching has been dealt with in e.g. Guo [4] and Le and Wang [6].

It has been shown in particular in [2] that the boundary function $b(t)$ is nonincreasing and continuous in $t \in [0,T]$ and satisfies a Volterra integral equation of the form

$$G(t, b(t)) = J(t, b(t)) - \int_t^T K(t, r, b(t)) dr, \tag{1.3}$$

$0 \le t \le T$, with given terminal condition $b(T)$, where the functions $J(t,x)$ and $K(t,r,x)$ are specified in [2].

Under regime switching, the optimal boundary functions depend on the regime state of the system, and they may not be monotone if the drift coefficients $(\mu(i))_{i\in\mathcal{M}}$ have switching signs, cf. Figures 3 and 4 in Section 5. Essentially, a boundary function increases when there is sufficient time to switch from a state with negative drift to a state with positive drift and to remain there until maturity, and is decreasing otherwise. We refer to [8] and [9] for other optimal settings that involve non monotone boundary functions.

In the regime switching setting however, no Volterra equation such as (1.3) is available in general for boundary functions, cf. Section 5 and Remark 5.5 of [7]. In addition, the free boundary problem in the regime switching case consists in a system of interacting PDEs, making its direct solution more difficult, cf. Proposition 5.2 in [7]. In Buffington and Elliott [1] a free boundary problem has been solved under an ordering assumption on the boundary functions in the two-state case, see Assumption 3.1 therein, however this condition may not hold in general in our setting, cf. Figure 4 below, and their method is specific to American options.

In this paper we construct a recursive algorithm for the numerical solution of (1.1) in the regime-switching model (1.2), that includes the case where the drifts $(\mu(i))_{i\in\mathcal{M}}$ have nonconstant signs. Our algorithm has a linear complexity $O(n)$ in the number n of time steps, hence in the absence of regime switching it also performs faster than the resolution of the Volterra equation, which has a quadratic complexity $O(n^2)$ due to the evaluation of the integral in (1.3), cf. Section 5.

We start by recalling the main results of [7]. From Lemma 2.1 in [7], the optimal value function V_t in (1.1) can be written as

$$V_t = V(t, \hat{Y}_{0,t}/Y_t, \beta_t),$$

where the function $V : [0,T] \times [1,\infty) \times \mathcal{M} \to \mathbb{R}_+$ is given by

$$\begin{aligned} V(t,a,j) &:= \inf_{t\le\tau\le T} \mathbb{E}\left[\frac{1}{Y_\tau} \max(aY_t, \hat{Y}_{t,T}) \;\middle|\; \beta_t = j \right] && (1.4) \\ &= \inf_{t\le\tau\le T} \mathbb{E}\left[G\left(\tau, \beta_\tau, \frac{1}{Y_\tau} \max\left(aY_t, \hat{Y}_{t,\tau}\right)\right) \;\middle|\; \beta_t = j \right], && (1.5) \end{aligned}$$

for $t \in [0,T]$, $j \in \mathcal{M}$, $a \geq 1$, with $\hat{Y}_{s,t} := \max_{r\in[s,t]} Y_r$, $0 \leq s \leq t \leq T$, and

$$G(t,a,j) := \mathbb{E}\left[\max\left(a, \hat{Y}_{t,T}/Y_t\right) \;\middle|\; \beta_t = j\right], \quad t \in [0,T],\ j \in \mathcal{M}. \tag{1.6}$$

Here, the infimum is taken over all $(\mathcal{F}^t_s)_{s\in[t,T]}$-stopping times τ, where $\mathcal{F}^t_s := \sigma(B_r - B_t,\ \beta_r\ :\ t \leq r \leq s)$, $s \in [t,T]$. From Proposition 3.1 in [7], given $\beta_t = j \in \mathcal{M}$ and $\hat{Y}_{0,t}/Y_t = a \in [1,\infty)$, $t \in [0,T]$, the optimal stopping time for (1.1), or equivalently for (1.5), is the first hitting time

$$\tau_D(t,a,j) := \inf\left\{r \geq t\ :\ \left(r, \frac{\hat{Y}_{0,r}}{Y_r}, \beta_r\right) \in D\right\}$$

of the stopping set

$$D := \left\{(t,a,j) \in [0,T] \times [1,\infty) \times \mathcal{M}\ :\ V(t,a,j) = G(t,a,j)\right\} \tag{1.7}$$

by the process $(r, \hat{Y}_{0,r}/Y_r, \beta_r)_{r\in[t,T]}$.

The stopping set D defined in (1.7) is closed, and its shape can be characterized as

$$D = \{(t,y,j) \in [0,T] \times [1,\infty) \times \mathcal{M}\ :\ y \geq b_D(t,j)\}$$

in terms of the boundary functions $b_D(t,j)$ defined by

$$b_D(t,j) := \inf\{x \in [1,\infty)\ :\ (t,x,j) \in D\}, \qquad t \in [0,T], \quad j \in \mathcal{M},$$

cf. Proposition 3.2 of [7].

If the condition $\mu(j) \geq 0$ is not satisfied for all $j \in \mathcal{M}$, then $t \mapsto b_D(t,j)$ may not be decreasing, cf. Figure 4 below. On the other hand, $\mu(j) \leq 0$ for all $j \in \mathcal{M}$ leads to $b_D(t,j) = 1$, $t \in [0,T]$, $j \in \mathcal{M}$, which corresponds to immediate exercise, cf. Proposition 5.3 in [7].

In this paper we construct a recursive algorithm for the numerical solution of the optimal stopping problem (1.1), by determining the stopping set D from the values of $V(t,a,j)$ and $G(t,a,j)$, cf. Theorem 2.1 and Lemma 4.1 below. As this approach does not rely on the Volterra equation (1.3), it allows us in particular to determine the boundary function $b_D(t,j)$ without requiring the condition $\mu(j) \geq 0$ for all $j \in \mathcal{M}$ as in [7], cf. for example Figure 4 below. In addition we do not rely on closed form expressions as in [2] as they are no longer available in the regime-switching setting.

Our algorithm extends the method of [12] as it applies not only to the computation of expectations, but also to optimal stopping problems. However it differs from [12], even when restricted to expectations $\mathbb{E}[\phi(Y_T)]$ of payoff functions $\phi(Y_T)$, where $(Y_t)_{t\in[0,T]}$ follows (1.2). In particular, the recursion of [12] is based on the jump times of the Markov chain $(\beta_t)_{t\in\mathbb{R}_+}$ whereas we apply a discretization of the time interval $[0,T]$, and our algorithm requires the Monte Carlo method only for the estimation of (2.3) below.

2 Main results

In the sequel we let $\delta_n := T/n$, $t_k^n := k\delta_n$, $k = 0,1,\ldots,n$, $\mathcal{T}_n := (t_0^n, t_1^n, \ldots, t_n^n)$, and

$$\lceil s \rceil_n := \min \big\{ t \in \mathcal{T}_n \ : \ t \geq s \big\}, \quad s \in [0,T], \quad n \geq 1.$$

In the following Theorem 2.1, which is proved in Section 3, the function $V_n(t,a,j)$ is computed by the backward induction (2.3) starting from the terminal time T.

Theorem 2.1 *(i) For all $t \in [0,T]$, $j = 1,2,\ldots,m$ and $a \geq 1$, the solution $V(t,a,j)$ of (1.4) satisfies*

$$V(t,a,j) = \lim_{n\to\infty} V_n(\lceil t \rceil_n, a, j), \tag{2.1}$$

where $V_n(t_k^n,a,j)$ is the discrete infimum

$$V_n(t_k^n,a,j) := \inf_{t_{k+1}^n \leq \tau_n \leq T} \mathbb{E}\left[\frac{\hat{Y}_{0,T}}{Y_{\tau_n}} \ \Big| \ \frac{\hat{Y}_{0,t_k^n}}{Y_{t_k^n}} = a, \ \beta_{t_k^n} = j \right], \quad k = 0,1,\ldots,n-1, \tag{2.2}$$

taken over all $\mathcal{T}_n$-valued stopping times τ_n, and $V_n(T,a,j) := V(T,a,j) = a$.

(ii) The value of $V_n(t_k^n,a,j)$ in (2.2) can be computed by the backward induction

$$V_n\left(t_{k-1}^n,a,j\right) = \mathbb{E}\left[G\left(t_k^n, \frac{\hat{Y}_{0,t_k^n}}{Y_{t_k^n}}, \beta_{t_k^n}\right) \wedge V_n\left(t_k^n, \frac{\hat{Y}_{0,t_k^n}}{Y_{t_k^n}}, \beta_{t_k^n}\right) \ \Big| \ \frac{\hat{Y}_{0,t_{k-1}^n}}{Y_{t_{k-1}^n}} = a, \ \beta_{t_{k-1}^n} = j \right], \tag{2.3}$$

for $k = 1, 2, \ldots, n$, under the terminal condition $V_n(T, a, j) = G(T, a, j) = a$, $a \geq 1$, where $G(t, a, j)$ in defined in (1.6).

In addition, by the following Theorem 2.2 proved in Section 4, we provide a way to approximate the function $G(t, a, j)$ used in (2.3). In the sequel we denote by

$$\varphi_r(x, y) := \sqrt{\frac{2}{\pi}} \frac{(2y - x)}{r^{3/2}} e^{-(2y-x)^2/2r}, \qquad 0 \leq x \leq y, \; r \in (0, T], \tag{2.4}$$

the joint probability density function of $\left(B_r, \sup_{0 \leq s \leq r} B_s\right)$, and we let $Q := [q_{i,j}]_{1 \leq i,j \leq m}$ denote the infinitesimal generator of $(\beta_t)_{t \in [0,T]}$, and define

$$u(j) := \mu(j)/\sigma(j) - \sigma(j)/2, \qquad j \in \mathcal{M}. \tag{2.5}$$

Next, we show in Theorem 2.2 that G is approximated by a limiting sequence $(G_n)_{n \in \mathbb{N}}$ given by the backward induction (2.6) below.

Theorem 2.2 *For any $t \in [0, T]$ and $j \in \mathcal{M}$ we have*

$$G(t, a, j) = \lim_{n \to \infty} G_n(\lceil t \rceil_n, a, j),$$

where the limit is uniform in $a \geq 1$ and $G_n(t_k^n, a, j)$ is defined by the backward induction

$$\begin{aligned} &G_n(t_{k-1}^n, a, j) = \\ &e^{q_{j,j}\delta_n} \int_0^\infty \int_{-\infty}^y e^{(u(j)+\sigma(j))x - u^2(j)T/(2n)} G_n\left(t_k^n, a \vee (\sigma(j)y) - \sigma(j)x, j\right) \varphi_{\delta_n}(x, y) dx dy \\ &+ \sum_{\substack{i=1 \\ i \neq j}}^m q_{j,i} \int_0^{\delta_n} e^{q_{j,j}r} \int_0^\infty \int_{-\infty}^y e^{(u(j)+\sigma(j))x - u^2(j)r/2} G_n\left(t_k^n, a \vee (\sigma(j)y) - \sigma(j)x, i\right) \varphi_r(x, y) dx dy dr, \end{aligned} \tag{2.6}$$

$k = 1, 2, \ldots, n$, with the terminal condition $G_n(T, a, j) = a$, $j \in \mathcal{M}$, $a \geq 1$.

In the particular case of constant drift μ and volatility σ cf, Theorems 2.1 and 2.2 also provide an alternative numerical solution in the geometric Brownian motion model of [2]. In this case, $(V_n(t_{k-1}^n, a))_{k=1,2,\ldots,n}$ is computed from (2.3) by the backward induction

$$V_n\left(t_{k-1}^n, a\right) = \int_0^\infty \int_{-\infty}^y G\left(t_k^n, e^{\sigma\left(\frac{\log a}{\sigma} \vee y - x\right)}\right) \wedge V_n\left(t_k^n, e^{\sigma\left(\frac{\log a}{\sigma} \vee y - x\right)}\right) e^{\lambda x - \lambda^2 \delta_n/2} \varphi_{\delta_n}(x, y) dx dy,$$

with

$$G(t,a) = \mathbb{E}\left[a \vee e^{\sigma S^\lambda_{T-t}}\right] = \int_0^\infty \int_{-\infty}^y e^{(\log a)\vee(\sigma y)+\lambda x-\lambda^2(T-t)/2}\varphi_{T-t}(x,y)dxdy, \quad (2.7)$$

for all $t \in [0,T]$ and $a \geq 1$, where $S^\lambda_t := \max_{0\leq s\leq t}(B_s + \lambda s)$, $\lambda := \mu/\sigma - \sigma/2$, and $\varphi_r(x,y)$ is given by (2.4). In the general regime switching setting, the function $G(t,a)$ in (2.7) can be estimated by Monte Carlo, while in the absence of regime switching it can be computed in closed form, cf. (2.7) in [2].

In Sections 3 and 4 we prove Theorems 2.1 and 2.2. Numerical illustrations are presented in Section 5 with and without regime switching. We observe in particular that boundary functions may not be monotone when the drift coefficients $\mu(j)$, $j \in \mathcal{M}$, have different signs.

3 Proof of Theorem 2.1

(i) First, we note that for any $(\mathcal{F}^t_s)_{s\in[t,T]}$-stopping time $\tau \in [t,T]$ we have

$$\begin{aligned}\mathbb{E}\left[\frac{(aY_t)\vee \hat{Y}_{t,T}}{Y_\tau} \;\Big|\; \beta_t = j\right] &= \mathbb{E}\left[\frac{a \vee \left(\hat{Y}_{t,T}/Y_t\right)}{Y_\tau/Y_t} \;\Big|\; \frac{\hat{Y}_{0,t}}{Y_t} = a,\ \beta_t = j\right] \\ &= \mathbb{E}\left[\frac{\hat{Y}_{0,T}}{Y_\tau} \;\Big|\; \frac{\hat{Y}_{0,t}}{Y_t} = a,\ \beta_t = j\right], \qquad (3.1)\end{aligned}$$

$t \in [0,T]$, $j \in \mathcal{M}$, $a \in [1,\infty)$, since $\hat{Y}_{0,t}/Y_t$ is conditionally independent of

$$\left(\frac{Y_t}{Y_\tau}, \frac{\hat{Y}_{t,T}}{Y_\tau}\right) = \left(\exp\left(-\int_t^\tau \sigma(\beta_r)d\tilde{B}_r\right), \exp\left(\sup_{t\leq v\leq T}\int_t^v \sigma(\beta_r)d\tilde{B}_r - \int_t^\tau \sigma(\beta_t)d\tilde{B}_r\right)\right)$$

given β_t, where $(\tilde{B}_v)_{v\in[0,T]}$ is the drifted Brownian motion

$$\tilde{B}_v := B_v + \int_0^v u(\beta_r)dr, \qquad v \in [0,T], \qquad (3.2)$$

and $u(j) := \mu(j)/\sigma(j) - \sigma(j)/2$, $j \in \mathcal{M}$, is defined in (2.5). Hence by (1.4) we have

$$V(t^n_k, a, j) = \inf_{t^n_k\leq\tau\leq T} \mathbb{E}\left[\frac{\hat{Y}_{0,T}}{Y_\tau} \;\Big|\; \frac{\hat{Y}_{0,t^n_k}}{Y_{t^n_k}} = a,\ \beta_{t^n_k} = j\right]$$

$$
\begin{aligned}
&\leq \inf_{t_k^n \leq \tau_n \leq T} \mathbb{E}\left[\frac{\hat{Y}_{0,T}}{Y_{\tau_n}} \;\Big|\; \frac{\hat{Y}_{0,t_k^n}}{Y_{t_k^n}} = a,\ \beta_{t_k^n} = j \right] \\
&\leq \inf_{t_{k+1}^n \leq \tau_n \leq T} \mathbb{E}\left[\frac{\hat{Y}_{0,T}}{Y_{\tau_n}} \;\Big|\; \frac{\hat{Y}_{0,t_k^n}}{Y_{t_k^n}} = a,\ \beta_{t_k^n} = j \right] \\
&= V_n(t_k^n, a, j),
\end{aligned}
$$

$k = 0, 1, \ldots, n-1$, $j \in \mathcal{M}$, $a \geq 1$, where we used (2.2) and the infimum is taken over all $\mathcal{T}_n$-valued discrete stopping times τ_n. Therefore by the continuity of $V(t, a, j)$ with respect to t, cf. e.g. [10], Chap III, §7.1.1 page 130 and §7.4.1 pages 135-136, we obtain

$$
V(t, a, j) = \lim_{n \to \infty} V(\lceil t \rceil_n, a, j) \leq \liminf_{n \to \infty} V_n(\lceil t \rceil_n, a, j). \tag{3.3}
$$

(ii) On the other hand, by (3.1) we have

$$
\begin{aligned}
\limsup_{n \to \infty} V_n(\lceil t \rceil_n, a, j) &= \limsup_{n \to \infty} \inf_{\lceil t \rceil_n + \delta_n \leq \tau_n \leq T} \mathbb{E}\left[\frac{\hat{Y}_{0,T}}{Y_{\tau_n}} \;\Big|\; \frac{\hat{Y}_{0,\lceil t \rceil_n}}{Y_{\lceil t \rceil_n}} = a,\ \beta_{\lceil t \rceil_n} = j \right] \\
&= \limsup_{n \to \infty} \inf_{\lceil t \rceil_n + \delta_n \leq \tau_n \leq T} \mathbb{E}\left[\frac{(aY_{\lceil t \rceil_n}) \vee \hat{Y}_{\lceil t \rceil_n, T}}{Y_{\tau_n}} \;\Big|\; \beta_{\lceil t \rceil_n} = j \right] \\
&= \limsup_{n \to \infty} \sum_{l=1}^m \left[e^{(\lceil t \rceil_n - t)Q} \right]_{j,l} \inf_{\lceil t \rceil_n + \delta_n \leq \tau_n \leq T} \mathbb{E}\left[\frac{(aY_{\lceil t \rceil_n}) \vee \hat{Y}_{\lceil t \rceil_n, T}}{Y_{\tau_n}} \;\Big|\; \beta_{\lceil t \rceil_n} = l \right] \\
&\leq \limsup_{n \to \infty} \inf_{\lceil t \rceil_n + \delta_n \leq \tau_n \leq T} \sum_{l=1}^m \left[e^{(\lceil t \rceil_n - t)Q} \right]_{j,l} \mathbb{E}\left[\frac{(aY_{\lceil t \rceil_n}) \vee \hat{Y}_{\lceil t \rceil_n, T}}{Y_{\tau_n}} \;\Big|\; \beta_{\lceil t \rceil_n} = l \right] \\
&= \limsup_{n \to \infty} \inf_{\lceil t \rceil_n + \delta_n \leq \tau_n \leq T} \mathbb{E}\left[\frac{(aY_{\lceil t \rceil_n}) \vee \hat{Y}_{\lceil t \rceil_n, T}}{Y_{\tau_n}} \;\Big|\; \beta_t = j \right],
\end{aligned} \tag{3.4}
$$

$t \in [0, T - \delta_n]$, $j \in \mathcal{M}$, $a \in [1, \infty)$, where $Q = [q_{i,j}]_{1 \leq i,j \leq m}$ is the infinitesimal generator of $(\beta_t)_{t \in [0,T]}$. Next, we note that for every stopping time $\tau \in [t, T]$ we have $|\lceil \tau \rceil_n - \tau| < 1/n$, hence $(\lceil \tau \rceil_n)_{n \geq 1}$ converges to τ uniformly in $L^\infty(\Omega)$ and pointwise. Hence we have

$$
\lim_{n \to \infty} \mathbb{E}\left[\frac{(aY_{\lceil t \rceil_n}) \vee \hat{Y}_{\lceil t \rceil_n, T}}{Y_{\lceil \tau \vee (t + \delta_n) \rceil_n}} \;\Big|\; \beta_t = j \right] = \mathbb{E}\left[\lim_{n \to \infty} \frac{(aY_{\lceil t \rceil_n}) \vee \hat{Y}_{\lceil t \rceil_n, T}}{Y_{\lceil \tau \vee (t + \delta_n) \rceil_n}} \;\Big|\; \beta_t = j \right], \tag{3.5}
$$

$t \in [0, T - \delta_n]$, for any stopping time $\tau \in [t, T]$, where we applied Lebesgue's dominated convergence theorem based on the bounds (3.8) and (3.9) stated at the end of this section. Hence from (3.4) and (3.5) we find, for any stopping time $\tau \in [t, T]$,

$$\begin{aligned}
\limsup_{n\to\infty} V_n(\lceil t\rceil_n, a, j) &\leq \limsup_{n\to\infty} \inf_{\lceil t\rceil_n+\delta_n \leq \tau_n \leq T} \mathbb{E}\left[\frac{(aY_{\lceil t\rceil_n}) \vee \hat{Y}_{\lceil t\rceil_n,T}}{Y_{\tau_n}} \;\Big|\; \beta_t = j\right] \\
&\leq \lim_{n\to\infty} \mathbb{E}\left[\frac{(aY_{\lceil t\rceil_n}) \vee \hat{Y}_{\lceil t\rceil_n,T}}{Y_{\lceil \tau \vee (t+\delta_n)\rceil_n}} \;\Big|\; \beta_t = j\right] \\
&= \mathbb{E}\left[\lim_{n\to\infty} \frac{(aY_{\lceil t\rceil_n}) \vee \hat{Y}_{\lceil t\rceil_n,T}}{Y_{\lceil \tau \vee (t+\delta_n)\rceil_n}} \;\Big|\; \beta_t = j\right] \\
&= \mathbb{E}\left[\frac{(aY_t) \vee \hat{Y}_{t,T}}{Y_\tau} \;\Big|\; \beta_t = j\right] \\
&= \mathbb{E}\left[\frac{\hat{Y}_{0,T}}{Y_\tau} \;\Big|\; \frac{\hat{Y}_{0,t}}{Y_t} = a,\ \beta_t = j\right],
\end{aligned}$$

where we applied (3.1) and the pathwise continuity of $(Y_t)_{t\in[0,T]}$. Hence by (2.2), we obtain

$$\limsup_{n\to\infty} V_n(\lceil t\rceil_n, a, j) \leq \inf_{t\leq\tau\leq T} \mathbb{E}\left[\frac{\hat{Y}_{0,T}}{Y_\tau} \;\Big|\; \frac{\hat{Y}_{0,t}}{Y_t} = a,\ \beta_t = j\right] = V(t, a, j),$$

$t \in [0, T - \delta_n]$, which completes the proof of (2.1) by (3.3).

(iii) In order to prove (2.3) for $0 \leq k \leq l \leq n$, we consider an optimal stopping time $\tau_n^{(t_l^n)}$ such that

$$\mathbb{E}\left[\frac{\hat{Y}_{0,T}}{Y_{\tau_n^{(t_l^n)}}} \;\Big|\; \frac{\hat{Y}_{0,t_k^n}}{Y_{t_k^n}} = a,\ \beta_{t_k^n} = j\right] = \inf_{t_l^n \leq \tau_n \leq T} \mathbb{E}\left[\frac{\hat{Y}_{0,T}}{Y_{\tau_n}} \;\Big|\; \frac{\hat{Y}_{0,t_k^n}}{Y_{t_k^n}} = a,\ \beta_{t_k^n} = j\right], \tag{3.6}$$

where the infimum is taken over the discrete $\mathcal{T}_n$-valued stopping times τ_n, and the existence of $\tau_n^{(h)}$ is guaranteed by Corollary 2.9 of [10] as in Proposition 3.1 of [7]. We note the induction

$$\begin{aligned}
\mathbb{E}\left[\frac{\hat{Y}_{0,T}}{Y_{\tau_n^{(t_k^n)}}} \;\Big|\; \frac{\hat{Y}_{0,t_k^n}}{Y_{t_k^n}},\ \beta_{t_k^n}\right] &= \mathbb{E}\left[\frac{\hat{Y}_{0,T}}{Y_{t_k^n}} \;\Big|\; \frac{\hat{Y}_{0,t_k^n}}{Y_{t_k^n}},\ \beta_{t_k^n}\right] \wedge \mathbb{E}\left[\frac{\hat{Y}_{0,T}}{Y_{\tau_n^{(t_{k+1}^n)}}} \;\Big|\; \frac{\hat{Y}_{0,t_k^n}}{Y_{t_k^n}},\ \beta_{t_k^n}\right] \\
&= G\left(t_k^n, \frac{\hat{Y}_{0,t_k^n}}{Y_{t_k^n}}, \beta_{t_k^n}\right) \wedge V_n\left(t_k^n, \frac{\hat{Y}_{0,t_k^n}}{Y_{t_k^n}}, \beta_{t_k^n}\right),
\end{aligned} \tag{3.7}$$

$k = 0, 1, \ldots, n-1$, $a \geq 1$, where V_n and G are defined in (2.2) and (1.6) respectively. By (3.6), this yields

$$
\begin{aligned}
&V_n\left(t_{k-1}^n, a, j\right) = \mathbb{E}\left[\frac{\hat{Y}_{0,T}}{Y_{\tau_n^{(t_k^n)}}} \;\middle|\; \frac{\hat{Y}_{0,t_{k-1}^n}}{Y_{t_{k-1}^n}} = a,\ \beta_{t_{k-1}^n} = j\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[\frac{\hat{Y}_{0,T}}{Y_{\tau_n^{(t_k^n)}}} \;\middle|\; \frac{\hat{Y}_{0,t_k^n}}{Y_{t_k^n}}, \frac{\hat{Y}_{0,t_{k-1}^n}}{Y_{t_{k-1}^n}} = a,\ \beta_{t_k^n},\ \beta_{t_{k-1}^n} = j\right] \;\middle|\; \frac{\hat{Y}_{0,t_{k-1}^n}}{Y_{t_{k-1}^n}} = a,\ \beta_{t_{k-1}^n} = j\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[\frac{\hat{Y}_{0,T}}{Y_{\tau_n^{(t_k^n)}}} \;\middle|\; \frac{\hat{Y}_{0,t_k^n}}{Y_{t_k^n}},\ \beta_{t_k^n}\right] \;\middle|\; \frac{\hat{Y}_{0,t_{k-1}^n}}{Y_{t_{k-1}^n}} = a,\ \beta_{t_{k-1}^n} = j\right] \\
&= \mathbb{E}\left[G\left(t_k^n, \frac{\hat{Y}_{0,t_k^n}}{Y_{t_k^n}},\ \beta_{t_k^n}\right) \wedge V_n\left(t_k^n, \frac{\hat{Y}_{0,t_k^n}}{Y_{t_k^n}},\ \beta_{t_k^n}\right) \;\middle|\; \frac{\hat{Y}_{0,t_{k-1}^n}}{Y_{t_{k-1}^n}} = a,\ \beta_{t_{k-1}^n} = j\right],
\end{aligned}
$$

$k = 1, 2, \ldots, n$, where we applied (3.7), the Markov property of $(\hat{Y}_{0,t}/Y_t, \beta_t)_{t\in[0,T]}$ and the relation $V_n(T, a, j) = G(T, a, j)$. $\square$

We close this section with the proof of the two bounds used for (3.5) above.

(a) Letting $\check{Y}_{t,T} := \min_{t\leq v\leq T} Y_v$, we check that, for any stopping time τ and $a \geq 1$, we have the bound

$$
\max\left(\frac{aY_{\lceil t\rceil_n}}{Y_{\lceil \tau\vee(t+\delta_n)\rceil_n}}, \frac{\hat{Y}_{\lceil t\rceil_n,T}}{Y_{\lceil \tau\vee(t+\delta_n)\rceil_n}}\right) \leq \frac{a\hat{Y}_{t,T}}{Y_{\lceil \tau\vee(t+\delta_n)\rceil_n}} \leq \frac{a\hat{Y}_{t,T}}{\check{Y}_{t,T}}, \tag{3.8}
$$

in which the right hand side is integrable for all $t \in [0, T-\delta_n]$.

(b) On the other hand we have $\mathbb{E}\left[\hat{Y}_{t,T}/\check{Y}_{t,T} \;\middle|\; \beta_t = j\right] < \infty$ since, using the drifted Brownian motion $(\tilde{B}_v)_{v\in[0,T]}$ defined in (3.2) we have, using the Cauchy-Schwarz inequality,

$$
\begin{aligned}
\mathbb{E}\left[\frac{\hat{Y}_{t,T}}{\check{Y}_{t,T}} \;\middle|\; \beta_t = j\right] &= \mathbb{E}\left[e^{\sup_{t\leq r\leq T}\int_t^r \sigma(\beta_v)d\tilde{B}_v - \inf_{t\leq r\leq T}\int_t^r \sigma(\beta_v)d\tilde{B}_v} \;\middle|\; \beta_t = j\right] \\
&\leq \sqrt{\mathbb{E}\left[e^{2\sup_{t\leq r\leq T}\int_t^r \sigma(\beta_v)d\tilde{B}_v} \;\middle|\; \beta_t = j\right]\mathbb{E}\left[e^{-2\inf_{t\leq r\leq T}\int_t^r \sigma(\beta_v)d\tilde{B}_v} \;\middle|\; \beta_t = j\right]} \\
&\leq \sqrt{\mathbb{E}\left[e^{2\sup_{t\leq r\leq T}\int_t^r \sigma(\beta_v)d\tilde{B}_v} \;\middle|\; \beta_t = j\right]} \\
&< \infty,
\end{aligned} \tag{3.9}
$$

where we conclude to finiteness by conditioning and use of the density (2.4).

4 Proof of Theorem 2.2

We start with two lemmas.

Lemma 4.1 *For all $k = 1, 2, \ldots, n$, $j \in \mathcal{M}$ and $a \geq 1$, we have*

$$
\begin{aligned}
&G(t_{k-1}^n, a, j) = \qquad (4.1)\\
&e^{q_{j,j}\delta_n} \int_0^\infty \int_{-\infty}^y e^{(u(j)+\sigma(j))x - u^2(j)T/(2n)} G\left(t_k^n, a \vee (\sigma(j)y) - \sigma(j)x, j\right) \varphi_{\delta_n}(x, y) dx dy \\
&+ \sum_{\substack{i=1 \\ i \neq j}}^m q_{j,i} \int_0^{\delta_n} e^{q_{j,j}r} \int_0^\infty \int_{-\infty}^y e^{(u(j)+\sigma(j))x - u^2(j)r/2} G\left(t_{k-1}^n + r, a \vee (\sigma(j)y) - \sigma(j)x, i\right) \varphi_r(x, y) dx dy dr.
\end{aligned}
$$

Proof. Let $\tilde{\mathbb{P}}$ denote the probability measure defined by

$$
\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} := \exp\left(- \int_0^T u(\beta_r) dB_r - \frac{1}{2} \int_0^T u^2(\beta_r) dr \right),
$$

where $u(j)$, $j \in \mathcal{M}$, is defined in (2.5), and $(\tilde{B}_r)_{r \in [0,T]}$ is the standard Brownian motion under $\tilde{\mathbb{P}}$ defined in (3.2). From the definition (1.6) of $G(t, a, j)$ we have

$$
\begin{aligned}
G(t, a, j) &= \mathbb{E}\left[a \vee \exp\left(\sup_{t \leq s \leq T} \int_t^s \sigma(\beta_r) d\tilde{B}_r \right) \,\middle|\, \beta_t = j \right] \qquad (4.2)\\
&= \tilde{\mathbb{E}}\left[e^{(\log a) \vee \sup_{t \leq s \leq T} \int_t^s \sigma(\beta_r) d\tilde{B}_r + \int_t^T u(\beta_r) dB_r + \frac{1}{2} \int_t^T u^2(\beta_r) dr} \,\middle|\, \beta_t = j \right] \\
&= \tilde{\mathbb{E}}\left[e^{(\log a) \vee \sup_{t \leq s \leq T} \int_t^s \sigma(\beta_r) d\tilde{B}_r + \int_t^T u(\beta_r) d\tilde{B}_r - \frac{1}{2} \int_t^T u^2(\beta_r) dr} \,\middle|\, \beta_t = j \right] \\
&= \mathbb{E}\left[e^{(\log a) \vee \sup_{t \leq s \leq T} \int_t^s \sigma(\beta_r) dB_r + \int_t^T u(\beta_r) dB_r - \frac{1}{2} \int_t^T u^2(\beta_r) dr} \,\middle|\, \beta_t = j \right], \qquad (4.3)
\end{aligned}
$$

which allows us to remove the drift component in the supremum $\sup_{t \leq s \leq T} \int_t^s \sigma(\beta_r) dB_r$. Next, using (4.3) we write

$$
G(t_k^n, a, j) = \Phi_n(t_k^n, a, j) + \Upsilon_n(t_k^n, a, j), \qquad j \in \mathcal{M}, \quad a \geq 1, \qquad (4.4)
$$

where

$$
\Phi_n(t_k^n, a, j) := \mathbb{E}\left[e^{(\log a) \vee \sup_{t_k^n \leq s \leq T} \int_{t_k^n}^s \sigma(\beta_r) dB_r + \int_{t_k^n}^T u(\beta_r) dB_r - \frac{1}{2} \int_{t_k^n}^T u^2(\beta_r) dr} \mathbf{1}_{\{T_1(t_k^n) > t_{k+1}^n\}} \,\middle|\, \beta_{t_k^n} = j \right], \qquad (4.5)
$$

with $T_1(t) := \inf\{s \geq t : \beta_s \neq \beta_t\}$ for any $t \in \mathbb{R}_+$, and

$$\Upsilon_n(t_k^n, a, j) := \mathbb{E}\left[e^{(\log a) \vee \sup_{t_k^n \leq s \leq T} \int_{t_k^n}^s \sigma(\beta_r) dB_r + \int_{t_k^n}^T u(\beta_r) dB_r - \frac{1}{2} \int_{t_k^n}^T u^2(\beta_r) dr} \mathbf{1}_{\{T_1(t_k^n) \leq t_{k+1}^n\}} \Big| \beta_{t_k^n} = j \right]. \tag{4.6}$$

By (4.5) we have, for any $k = 0, 1, \ldots, n-1$, $j \in \mathcal{M}$, and $a \geq 1$,

$$\Phi_n(t_k^n, a, j) = e^{q_{j,j}\delta_n} \int_0^\infty \int_{-\infty}^y$$

$$\mathbb{E}\left[e^{(\log a) \vee \left(\sigma(j) y \vee \left(\sigma(j) x + \sup_{t_{k+1}^n \leq s \leq T} \int_{t_{k+1}^n}^s \sigma(\beta_r) dB_r \right) \right) + \int_{t_{k+1}^n}^T u(\beta_r) dB_r + u(j) x - \frac{1}{2} \int_{t_{k+1}^n}^T u^2(\beta_r) dr - u^2(j)\delta_n/2} \right]$$

$$\times \varphi_{\delta_n}(x, y) dx dy$$

$$= e^{q_{j,j}\delta_n} \int_0^\infty \int_{-\infty}^y$$

$$\mathbb{E}\left[e^{(((\log a) \vee (\sigma(j) y) - \sigma(j) x) \vee \sup_{t_{k+1}^n \leq s \leq T} \int_{t_{k+1}^n}^s \sigma(\beta_r) dB_r + \sigma(j) x + \int_{t_{k+1}^n}^T u(\beta_r) dB_r + u(j) x - \frac{1}{2} \int_{t_{k+1}^n}^T u^2(\beta_r) dr - u^2(j)\delta_n/2)} \right]$$

$$\times \varphi_{\delta_n}(x, y) dx dy$$

$$= e^{q_{j,j}\delta_n} \int_0^\infty \int_{-\infty}^y e^{(u(j)+\sigma(j))x - u^2(j)T/(2n)} G(t_{k+1}^n, e^{(\log a) \vee (\sigma(j) y) - \sigma(j) x}, j) \varphi_{\delta_n}(x, y) dx dy, \tag{4.7}$$

where in the first equality we used the fact that the time to the first jump of $(\beta_s)_{s \in [t, \infty)}$ after t is exponentially distributed with parameter $-q_{j,j} > 0$ given $\beta_t = j$, cf. e.g. § 10.4 in [11]. Next, for $\Upsilon_n(t_k^n, a, j)$, $k = 0, 1, \ldots, n-1$, $j \in \mathcal{M}$, and $a \geq 1$, by (4.6) we see that

$$\Upsilon_n(t_k^n, a, j) = \sum_{\substack{i=1 \\ i \neq j}}^m q_{j,i} \int_0^{\delta_n} e^{q_{j,j} r} \int_0^\infty \int_{-\infty}^y \tag{4.8}$$

$$\mathbb{E}\left[e^{(\log a) \vee (\sigma(j) y) \vee (\sigma(j) x + \sup_{t_k^n + r \leq s \leq T} \int_{t_k^n + r}^s \sigma(\beta_z) dB_z + \int_{t_k^n + r}^T u(\beta_z) dB_z + u(j) x - \frac{1}{2} \int_{t_k^n + r}^T u^2(\beta_z) dz - u^2(j) r/2)} \Big| \beta_{t_k^n + r} = i \right]$$

$$\varphi_r(x, y) dx dy dr$$

$$= \sum_{\substack{i=1 \\ i \neq j}}^m q_{j,i} \int_0^{\delta_n} e^{q_{j,j} r} \int_0^\infty \int_{-\infty}^y e^{(u(j)+\sigma(j))x - u^2(j) r/2} G(t_k^n + r, e^{(\log a) \vee (\sigma(j) y) - \sigma(j) x}, i) \varphi_r(x, y) dx dy dr,$$

where we used the conditional probability distribution

$$\mathbb{P}(T_1 \in dt,\ \beta_{T_1} = i \mid \beta_0 = j) = \mathbf{1}_{[0,\infty)}(t) q_{j,i} e^{q_{j,j} t} dt, \qquad i \neq j \in \mathcal{M},$$

computed from the exponential distribution with parameter $-q_{i,i}$ of the first jump time T_1 of the Markov chain $(\beta_t)_{t\in\mathbb{R}_+}$ started at $i \in \mathcal{M}$ and the transition matrix $(-q_{i,j}\mathbf{1}_{\{i\neq j\}}/q_{i,i})_{i,j\in\mathcal{M}}$ of the embedded Markov chain, cf. e.g. § 10.7 of [11]. Hence we conclude to (4.1) by (4.4), (4.7) and (4.8). $\square$

Lemma 4.2 *For any $j \in \mathcal{M}$, the function $t \mapsto G(t, a, j)$ is uniformly continuous in $t \in [0, T]$, uniformly in $a \geq 1$, i.e.*

$$\lim_{\varepsilon\to 0} \sup_{|t-s|\leq\varepsilon} \sup_{a\geq 1} |G(t, a, j) - G(s, a, j)| = 0, \qquad j \in \mathcal{M}. \tag{4.9}$$

Proof. By (4.2), for all $a \geq 1$ we have

$$\begin{aligned}
&|G(t, a, j) - G(s, a, j)| \\
&= \left| \mathbb{E}\left[a \vee \exp\left(\sup_{t\leq v\leq T} \int_t^v \sigma(\beta_r)d\tilde{B}_r \right) \;\middle|\; \beta_t = j \right] - \mathbb{E}\left[a \vee \exp\left(\sup_{s\leq v\leq T} \int_s^v \sigma(\beta_r)d\tilde{B}_r \right) \;\middle|\; \beta_s = j \right] \right| \\
&\leq \left| \mathbb{E}\left[\exp\left(\sup_{t\leq v\leq T} \int_t^v \sigma(\beta_r)d\tilde{B}_r \right) \;\middle|\; \beta_t = j \right] - \mathbb{E}\left[\exp\left(\sup_{s\leq v\leq T} \int_s^v \sigma(\beta_r)d\tilde{B}_r \right) \;\middle|\; \beta_s = j \right] \right|,
\end{aligned} \tag{4.10}$$

hence it suffices to show the continuity in $t \in [0, T]$ of the above bound. Similarly to (4.3), we have

$$\mathbb{E}\left[\exp\left(\sup_{t\leq v\leq T} \int_t^v \sigma(\beta_r)d\tilde{B}_r \right) \;\middle|\; \beta_t = j \right] = \mathbb{E}\left[e^{\sup_{t\leq v\leq T} \int_t^v \sigma(\beta_r)dB_r + \int_t^T u(\beta_r)dB_r - \frac{1}{2}\int_t^T u^2(\beta_r)dr} \;\middle|\; \beta_t = j \right],$$

$t \in [0, T]$, $j \in \mathcal{M}$. Next, for any $n \geq 1$ we have

$$\begin{aligned}
\sup_{\lceil t\rceil_n\leq s\leq T} \int_{\lceil t\rceil_n}^s \sigma(\beta_r)dB_r &= \sup_{\lceil t\rceil_n\leq s\leq T} \int_t^s \sigma(\beta_r)dB_r - \int_t^{\lceil t\rceil_n} \sigma(\beta_r)dB_r \\
&\leq \sup_{t\leq s\leq T} \int_t^s \sigma(\beta_r)dB_r - \inf_{t\leq s\leq T} \int_t^s \sigma(\beta_r)dB_r,
\end{aligned}$$

and similarly by replacing $\sigma(\beta_r)$ with $u(\beta_r)$, thus

$$\exp\left(\sup_{\lceil t\rceil_n\leq s\leq T} \int_{\lceil t\rceil_n}^s \sigma(\beta_r)dB_r + \int_{\lceil t\rceil_n}^T u(\beta_r)dB_r - \frac{1}{2}\int_{\lceil t\rceil_n}^T u^2(\beta_r)dr \right)$$

is upper bounded by

$$\exp\left(\sup_{t\leq s\leq T} \int_t^s (2\sigma(\beta_r) + u(\beta_r))dB_r - \inf_{t\leq s\leq T} \int_t^s (2\sigma(\beta_r) + u(\beta_r))dB_r \right),$$

which is $\mathbb{P}$-integrable as in (3.9). Therefore, by dominated convergence we find

$$\begin{aligned}
&\lim_{s\searrow t}\mathbb{E}\left[e^{\sup\limits_{s\le v\le T}\int_s^v\sigma(\beta_r)dB_r+\int_s^T u(\beta_r)dB_r-\frac{1}{2}\int_s^T u^2(\beta_r)dr}\;\Big|\;\beta_s=j\right]\\
&= \lim_{s\searrow t}\sum_{l=1}^m\left[e^{(s-t)Q}\right]_{j,l}\mathbb{E}\left[e^{\sup\limits_{s\le v\le T}\int_s^v\sigma(\beta_r)dB_r+\int_s^T u(\beta_r)dB_r-\frac{1}{2}\int_s^T u^2(\beta_r)dr}\;\Big|\;\beta_s=l\right]\\
&= \lim_{s\searrow t}\mathbb{E}\left[e^{\sup\limits_{s\le v\le T}\int_s^v\sigma(\beta_r)dB_r+\int_s^T u(\beta_r)dB_r-\frac{1}{2}\int_s^T u^2(\beta_r)dr}\;\Big|\;\beta_t=j\right]\\
&= \mathbb{E}\left[e^{\sup\limits_{t\le v\le T}\int_t^v\sigma(\beta_r)dB_r+\int_t^T u(\beta_r)dB_r-\frac{1}{2}\int_t^T u^2(\beta_r)dr}\;\Big|\;\beta_t=j\right]
\end{aligned}$$

and similarly,

$$\begin{aligned}
&\lim_{s\nearrow t}\mathbb{E}\left[e^{\sup\limits_{s\le v\le T}\int_s^v\sigma(\beta_r)dB_r+\int_s^T u(\beta_r)dB_r-\frac{1}{2}\int_s^T u^2(\beta_r)dr}\;\Big|\;\beta_s=j\right]\\
&= \mathbb{E}\left[e^{\sup\limits_{t\le v\le T}\int_t^v\sigma(\beta_r)dB_r+\int_t^T u(\beta_r)dB_r-\frac{1}{2}\int_t^T u^2(\beta_r)dr}\;\Big|\;\beta_t=j\right].
\end{aligned}\tag{4.11}$$

Combining (4.10) and (4.11) we conclude to Lemma 4.2 by a classical uniform continuity argument. □

Finally, we proceed to the proof of Theorem 2.2. Let

$$\Delta_k^n := \max_{j\in\mathcal{M}}\sup_{a\ge 1}|G_n(t_k^n,a,j)-G(t_k^n,a,j)|,\qquad k=0,1,\ldots,n,\tag{4.12}$$

with $\Delta_n^n=0$. By (2.6), (4.1) and (4.12) we have

$$\begin{aligned}
\Delta_{k-1}^n &\le e^{q_{j,j}\delta_n}\Delta_k^n\max_{j\in\mathcal{M}}\int_0^\infty\int_{-\infty}^y e^{(u(j)+\sigma(j))x-u^2(j)\delta_n/2}\varphi_{\delta_n}(x,y)dxdy \qquad (4.13)\\
&+\max_{j\in\mathcal{M}}\sup_{a\ge 1}\sum_{\substack{i=1\\ i\ne j}}^m q_{j,i}\int_0^{\delta_n}e^{q_{jj}r}\int_0^\infty\int_{-\infty}^y e^{(u(j)+\sigma(j))x-u^2(j)r/2}\\
&\times|G_n(t_k^n,a\vee(\sigma(j)y)-\sigma(j)x,i)-G(t_{k-1}^n+r,a\vee(\sigma(j)y)-\sigma(j)x,i)|\varphi_{\delta_n}(x,y)dxdydr,
\end{aligned}$$

$k=1,2,\ldots,n$, where

$$\begin{aligned}
&|G_n(t_k^n,a\vee(\sigma(j)y)-\sigma(j)x,i)-G(t_{k-1}^n+r,a\vee(\sigma(j)y)-\sigma(j)x,i)|\\
&\le |G_n(t_k^n,a\vee(\sigma(j)y)-\sigma(j)x,i)-G(t_k^n,a\vee(\sigma(j)y)-\sigma(j)x,i)|\\
&\quad+|G(t_k^n,a\vee(\sigma(j)y)-\sigma(j)x,i)-G(t_{k-1}^n+r,a\vee(\sigma(j)y)-\sigma(j)x,i)|
\end{aligned}$$

$$\leq \quad \Delta_k^n + \varepsilon_{k-1}^n, \qquad k = 1, 2, \ldots, n, \quad a \geq 1,$$

with

$$\varepsilon_k^n := \max_{i \in \mathcal{M}} \sup_{\substack{a \geq 1 \\ t_k^n \leq s < t \leq t_{k+1}^n}} |G(t, a, i) - G(s, a, i)|, \qquad k = 0, 1, \ldots, n-1.$$

Combining (4.13) and (4.14) yields

$$\begin{aligned}
\Delta_{k-1}^n \quad &\leq \quad \max_{j \in \mathcal{M}} e^{(q_{j,j}+\mu(j))\delta_n} \Delta_k^n \\
&\quad + (\Delta_k^n + \varepsilon_{k-1}^n) \max_{j \in \mathcal{M}} \sum_{\substack{i=1 \\ i \neq j}}^m q_{j,i} \int_0^{\delta_n} e^{q_{jj} r} \int_0^\infty \int_{-\infty}^y e^{(u(j)+\sigma(j))x - u^2(j)r/2} \varphi_{\delta_n}(x, y) dx dy dr \\
&= \quad e^{(q_{j,j}+\mu(j))\delta_n} \Delta_k^n + (\Delta_k^n + \varepsilon_{k-1}^n) \max_{j \in \mathcal{M}} \int_0^{\delta_n} \mathbb{E}\left[e^{(u(j)+\sigma(j))B_{\delta_n} - u^2(j)r/2 + q_{jj} r} \right] dr \sum_{\substack{i=1 \\ i \neq j}}^m q_{j,i} \\
&= \quad e^{(q_{j,j}+\mu(j))\delta_n} \Delta_k^n + (\Delta_k^n + \varepsilon_{k-1}^n) \max_{j \in \mathcal{M}} \int_0^{\delta_n} e^{(u(j)+\sigma(j))^2 \delta_n/2 - u^2(j)r/2 + q_{jj} r} dr \sum_{\substack{i=1 \\ i \neq j}}^m q_{j,i} \\
&\leq \quad c \left(\Delta_k^n + \varepsilon_{k-1}^n \delta_n \right), \qquad k = 1, 2, \ldots, n-1,
\end{aligned}$$

for some constant $c > 0$ independent of $n \geq 1$, hence

$$\Delta_k^n \leq c \left(\delta_n \sum_{i=k}^{n-1} \varepsilon_i^n \right), \qquad k = 0, 1, \ldots, n,$$

and

$$\max_{k=0,1,\ldots,n} \Delta_k^n = \max_{\substack{k=0,1,\ldots,n \\ j \in \mathcal{M}}} \sup_{a \geq 1} |G_n(t_k^n, a, j) - G(t_k^n, a, j)| \leq c \left(\max_{k=0,\ldots,n-1} \varepsilon_k^n \right)$$

which tends to 0 as n tends to infinity by (4.9) in Lemma 4.2. Consequently we have

$$\lim_{n \to \infty} \sup_{a \geq 1} |G(\lceil t \rceil_n, a, j) - G_n(\lceil t \rceil_n, a, j)| = 0$$

for any $0 \leq t \leq T$, $j \in \mathcal{M}$, and by Lemma 4.2 it follows that

$$G(t, a, j) = \lim_{n \to \infty} G(\lceil t \rceil_n, a, j) = \lim_{n \to \infty} G_n(\lceil t \rceil_n, a, j),$$

uniformly in $a \geq 1$, for all $j \in \mathcal{M}$ and $t \in [0, T]$. $\square$

5 Numerical results

In this section we present numerical estimates obtained from Theorems 2.1 and 2.2 for the boundary functions

$$b_D(t,j) := \inf\{x \in [1,\infty) \ : \ (t,x,j) \in D\}, \quad t \in [0,T], \quad j \in \mathcal{M},$$

of the stopping set D defined in (1.7), in the case of two-state Markov chains with $\mathcal{M} = \{1,2\}$.

(i) Constant drift.

In the absence of regime switching, the recursive algorithm of Theorems 2.1 and 2.2 is applied in Figure 1 to the computation of the value functions $V(t,a,j)$ and $G(t,a,j)$ with $T=1$, $\sigma=0.5$, $\mu=0.2$, $n=50$, and $\delta_n = T/n = 0.01$.

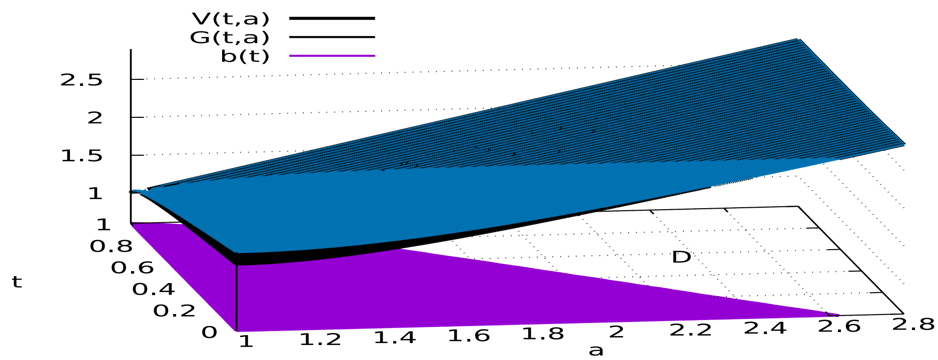

Figure 1: Value functions computed from Theorems 2.1 and 2.2.

Figure 1 allows us in particular to visualize the stopping set D defined in (1.7) and the continuation set $C = \big\{(t,a) \in [0,T] \times [1,\infty) \ : \ V(t,a) < G(t,a)\big\}$.

In Figure 2 the recursive method is compared to the solution of the Volterra integral equation (1.3) by dichotomy for the computation of the boundary function $b(t)$.

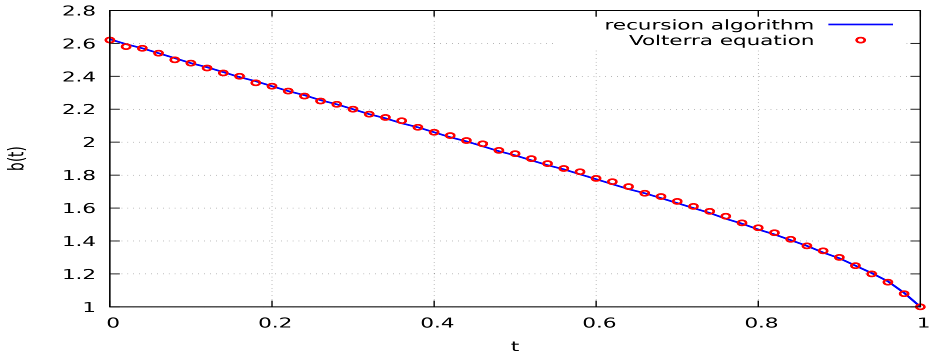

Figure 2: Boundary function computed from Theorems 2.1 and 2.2 *vs* (1.3).

As shown in Figure 2, the recursive and Volterra equation methods yield similar levels of precision. However, increasing the number n of time steps will make the Volterra equation method perform slower relative to the recursion method, due to the quadratic complexity of the former and to the linear complexity of the latter.

(ii) Drifts with switching signs.

Figure 3 presents the graphs of the value functions obtained from the recursive algorithm of Theorems 2.1 and 2.2 with $\mu(1) = 0.2$, $\mu(2) = -0.2$, $\sigma(1) = 0.5$, $\sigma(2) = 0.3$, $T = 0.5$, $n = 100$, $\delta_n = T/n = 0.05$, and

$$\mathbf{Q} = \begin{bmatrix} q_{1,1} & q_{1,2} \\ q_{2,1} & q_{2,2} \end{bmatrix} = \begin{bmatrix} -2.5 & 2.5 \\ 2 & -2 \end{bmatrix}.$$

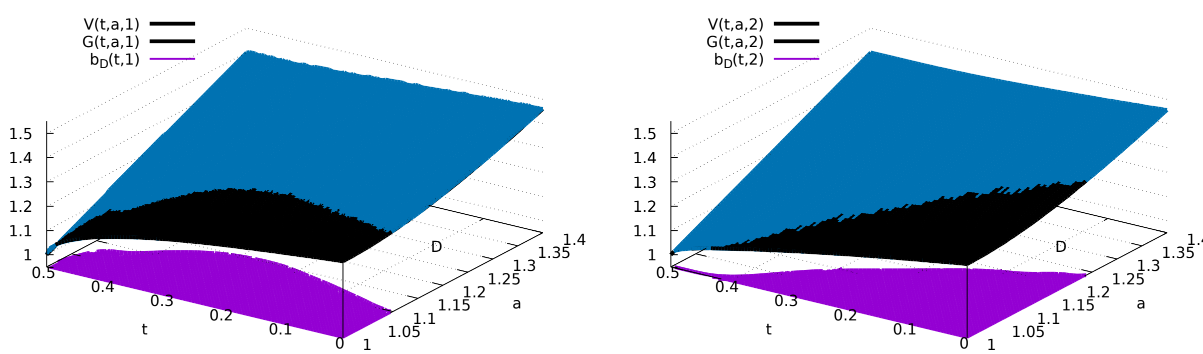

Figure 3: Value functions under drifts of mixed signs.

Figure 3 also allows us to visualize the stopping set D and the continuation set

$$C = \big\{(t, a, j) \in [0, T] \times [1, \infty) \times \mathcal{M} \ : \ V(t, a, j) < G(t, a, j)\big\}.$$

The numerical instabilities observed are due to the necessity to check the equality $V(t,a,j) = G(t,a,j)$ when $V(t,a,j)$ and $G(t,a,j)$ are very close to each other.

The boundary functions are plotted in Figure 4 with spline smoothing. Starting from state 2 with $\mu(2) = -0.2$ we observe the usual decreasing boundary function $t \mapsto b_D(t,2)$, which here becomes close to 0 before time T, since in this case we should exercise immediately as the average time $1/q_{2,1} = 0.5$ to switch to state 1 with $\mu(1) = 0.2$ exceeds the remaining time $T - t$ until maturity.

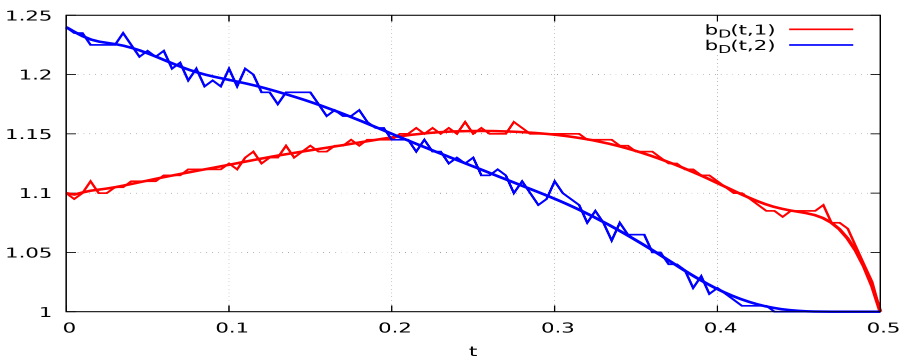

Figure 4: Boundary functions under drifts of mixed signs.

We observe that the boundary function $t \mapsto b_D(t,1)$ starting from state 1 is not monotone. Precisely, when time t is close to 0 it is better to exercise early because one may switch to state 2 after the average time $1/q_{1,2} = 0.4$, in which case the drift will take the negative value $\mu(2) = -0.2$. On the other hand, when t increases up to 0.3 the function $t \mapsto b_D(t,1)$ tends to increase as it makes more sense to wait since we may remain at state 1 with $\mu(1) = 0.2$ for the average time $1/q_{1,2} = 0.4$, which is now higher than the remaining time $T - t$.

Until time 0.2 we should exercise immediately when switching from state 2 to state 1 at a time t such that $b(t,1) < \hat{Y}_{0,t}/Y_t = a < b_D(t,2)$, while after time 0.2 the strategy is the opposite if $b(t,2) < \hat{Y}_{0,t}/Y_t = a < b_D(t,1)$.

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