QUASI-INVARIANCE FORMULAS FOR COMPONENTS OF QUANTUM LÉVY PROCESSES[0]

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ABSTRACT. A general method for deriving Girsanov or quasi-invariance formulas for classical stochastic processes with independent increments obtained as components of Lévy processes on real Lie algebras is presented. Letting a unitary operator arising from the associated factorizable current representation act on an appropriate commutative subalgebra, a second commutative subalgebra is obtained. Under certain conditions the two commutative subalgebras lead to two classical processes such that the law of the second process is absolutely continuous w.r.t. to the first. Examples include the Girsanov formula for Brownian motion as well as quasi-invariance formulas for the Poisson process, the Gamma process [TVY00, TVY01], and the Meixner process.

1. INTRODUCTION

Lévy processes, i.e. stochastic processes with independent and stationary increments, are used as models for random fluctuations, e.g., in physics, finance, etc. In quantum physics so-called quantum noises or quantum Lévy processes occur, e.g., in the description of quantum systems coupled to a heat bath [GZ00] or in the theory of continuous measurement [Hol01]. Motivated by a model introduced for lasers [Wal84], Schürmann et al. [ASW88, Sch93] have developed the theory of Lévy processes on involutive bialgebras. This theory generalizes, in a sense, the theory of factorizable representations of current groups and current algebras as well as the theory of classical Lévy processes with values in Euclidean space or, more generally, semigroups. For a historical survey on the theory of factorizable representations and its relation to quantum stochastic calculus, see [Str00, Section 5].

Many interesting classical stochastic processes arise as components of these quantum Lévy processes, cf. [Sch91, Bia98, Fra99, AFS02]. In this Note we will demonstrate on several examples how quasi-invariance formulas can be obtained in such a situation. Our examples include the Girsanov formula for Brownian motion as well as a quasi-invariance formula for the Gamma process [TVY00, TVY01], which actually appeared first in the context of factorizable representations of current groups [VGG83]. We also present a new quasi-invariance formula for the Meixner process.

We will consider Lévy processes on real Lie algebras in this Note, but the general idea is more widely applicable. Furthermore, we restrict ourselves to commututative subalgebras of the current algebra that have dimension one at every point, see Subsection 2.3. This allows us to use techniques familiar from the representation theory of Lie algebras and groups to get explicit expressions for the two sides of our quasi-invariance formulas.

In Section 2, we present the basic facts about Lévy processes on real Lie algebras and we recall how classical increment processes can be associated to them. The general idea of our construction is outlined in Section 3. Finally, in Section 4, we present explicit calculations for several classical increment processes related to the oscillator algebra and the Lie algebra $sl(2, \mathbb{R})$ of real 2×2 matrices with trace zero. In Subsections 4.1 and 4.4, we study representations of the Lie algebras themselves and obtain quasi-invariance formulas for random variables arising from these representations. In Subsections 4.2, 4.3, 4.5, and 4.6, we turn to Lévy processes on these Lie algebras and derive quasi-invariance or Girsanov formulas for Brownian motion, the Poisson process, the Gamma process, and the Meixner process.

2. Lévy processes on Real Lie Algebras

In this section we recall the definition and the main results concerning Lévy processes on real Lie algebras, see also [AFS02]. This is a special case of the theory of Lévy processes on involutive bialgebras, cf. [Sch93],[Mey95, Chapter VII],[FS99].

2.1. Definition and construction.

Definition 2.1. Let $\mathfrak{g}$ be a real Lie algebra, H a pre-Hilbert space, and $\Omega \in H$ a unit vector. We call a family $\left(j_{st} : \mathfrak{g} \to \mathcal{L}(H)\right)_{0 \leq s \leq t}$ of representations of $\mathfrak{g}$ by anti-hermitian operators (i.e. $j_{st}(X)^* = -j_{st}(X)$ for all $0 \leq s \leq t$, $X \in \mathfrak{g}$) a *Lévy process on* $\mathfrak{g}$ over H (with respect to Ω), if the following conditions are satisfied.

(1) (Increment property) We have
$$j_{st}(X) + j_{tu}(X) = j_{su}(X)$$
for all $0 \leq s \leq t \leq u$ and all $X \in \mathfrak{g}$.

(2) (Independence) We have $[j_{st}(X), j_{s't'}(Y)] = 0$ for all $X, Y \in \mathfrak{g}$, $0 \leq s \leq t \leq s' \leq t'$ and
$$\langle \Omega, j_{s_1 t_1}(X_1)^{k_1} \cdots j_{s_n t_n}(X_n)^{k_n} \Omega \rangle = \langle \Omega, j_{s_1 t_1}(X_1)^{k_1} \Omega \rangle \cdots \langle \Omega, j_{s_n t_n}(X_n)^{k_n} \Omega \rangle$$
for all $n, k_1, \ldots, k_n \in \mathbb{N}$, $0 \leq s_1 \leq t_1 \leq s_2 \leq \cdots \leq t_n$, $X_1, \ldots, X_n \in \mathfrak{g}$.

(3) (Stationarity) For all $n \in \mathbb{N}$ and all $X \in \mathfrak{g}$, the moments
$$m_n(X; s, t) = \langle \Omega, j_{st}(X)^n \Omega \rangle$$
depend only on the difference $t - s$.

(4) (Weak continuity) We have $\lim_{t \searrow s}\langle \Omega, j_{st}(X)^n \Omega \rangle = 0$ for all $n \geq 1$ and all $X \in \mathfrak{g}$.

Two Lévy processes are called *equivalent*, if they have same moments.

For the classification and construction of these processes we introduce the notion of Schürmann triples.

Definition 2.2. Let $\mathfrak{g}$ be a real Lie algebra. A *Schürmann triple on* $\mathfrak{g}$ over some pre-Hilbert space D is a triple (ρ, η, L) consisting of

- a representation ρ of $\mathfrak{g}$ on D by anti-hermitian operators,
- a ρ-1-cocycle η, i.e. a linear map $\eta : \mathfrak{g} \to D$ such that

$$\eta\big([X, Y]\big) = \rho(X)\eta(Y) - \rho(Y)\eta(X),$$

  for all $X, Y \in \mathfrak{g}$, and
- a linear functional $L : \mathfrak{g} \to i\mathbb{R} \subseteq \mathbb{C}$ that has the map $\mathfrak{g} \wedge \mathfrak{g} \ni X \wedge Y \to \langle \eta(X), \eta(Y) \rangle - \langle \eta(Y), \eta(X) \rangle \in \mathbb{C}$ as a coboundary, i.e.

$$L\big([X, Y]\big) = \langle \eta(X), \eta(Y) \rangle - \langle \eta(Y), \eta(X) \rangle$$

  for all $X, Y \in \mathfrak{g}$.

A Schürmann triple (ρ, η, L) is called *surjective*, if $\eta(\mathfrak{g})$ is cyclic for ρ.

The following theorem can be traced back to the works of Araki, Streater, etc., in the form given here it is a special case of Schürmann's representation theorem for Lévy processes on involutive bialgebras, cf. [Sch93].

Theorem 2.3. *Let $\mathfrak{g}$ be a real Lie algebra. Then there is a one-to-one correspondence between Lévy processes on $\mathfrak{g}$ (modulo equivalence) and Schürmann triples on $\mathfrak{g}$.*

Let (ρ, η, L) be a Schürmann triple on $\mathfrak{g}$ over D, then

$$(2.1) \qquad j_{st}(X) = \Lambda_{st}\big(\rho(X)\big) + A_{st}^*\big(\eta(X)\big) - A_{st}\big(\eta(X)\big) + (t - s)L(X)\mathrm{id}$$

for $0 \leq s \leq t$ and $X \in \mathfrak{g}$ defines Lévy process on $\mathfrak{g}$ over a dense subspace $H \subseteq \Gamma\big(L^2(\mathbb{R}_+, D)\big)$ w.r.t. the vacuum vector Ω.

2.2. **Regularity assumptions.** In order to justify our calculations, we have to impose stronger conditions on Lévy processes than those stated in Definition 2.1. We will from now on assume that $(j_{st})_{0 \leq s \leq t}$ is defined as in (2.1) and that the representation ρ in the Schürmann triple can be exponentiated to a continuous unitary representation of the Lie group associated to $\mathfrak{g}$. By Nelson's theorem this implies that D contains a dense subspace whose elements are analytic vectors for all $\rho(X)$, $X \in \mathfrak{g}$. Furthermore, we will assume that $\eta(\mathfrak{g})$ consists of analytic vectors. These assumptions guarantee that j_{st} can also be exponentiated to a continuous unitary group representation and therefore, by Nelson's theorem, any finite set of operators of the form $ij_{st}(X)$, $0 \leq s \leq t$, $X \in \mathfrak{g}$, is essentially selfadjoint on some common domain. Furthermore, the vacuum vector Ω is an

analytic vector for all $j_{st}(X)$, $0 \le s \le t$, $X \in \mathfrak{g}$. These assumptions are satisfied in all the examples considered in Section 4.

Denote by $g = e^X$ an element of the simply connected Lie group G associated to $\mathfrak{g}$. Our assumptions guarantee that $\eta(g)$ and $L(g)$ can be defined for X is a sufficiently small neighborhood of 0. Then the group representation U_{st} that we get by exponentiating j_{st} can also be obtained by solving the quantum stochastic differential equations

$$U_{st}(g) = \mathrm{id} + \int_s^t U_{s\tau}(g)\mathrm{d}\Big(\Lambda_\tau\big(\rho(g)\big) + A_\tau^*\big(\eta(g)\big) - A_\tau\big(\eta(g^{-1})\big) + (t-s)L(g)\mathrm{d}\tau\Big)$$

for $0 \le s \le t$. For an explicit expression for the action of $U_{st}(g)$ on exponential vectors, see also [Sch93, Proposition 4.1.2].

2.3. **Classical processes.** Denote by $\mathfrak{g}^{\mathbb{R}_+}$ the space of simple step functions with values in $\mathfrak{g}$,

$$\mathfrak{g}^{\mathbb{R}_+} = \Big\{ X = \sum_{k=1}^n X_k \mathbf{1}_{[s_k,t_k[} \Big| 0 \le s_1 \le t_1 \le s_2 \le \cdots \le t_n < \infty, X_1, \ldots, X_n \in \mathfrak{g} \Big\}.$$

Then $\mathfrak{g}^{\mathbb{R}_+}$ is a real Lie algebra with the pointwise Lie bracket and any Lévy process on $\mathfrak{g}$ defines a representation π of $\mathfrak{g}^{\mathbb{R}_+}$ via

$$(2.2) \qquad\qquad \pi(X) = \sum_{k=1}^n j_{s_k t_k}(X_k)$$

for $X = \sum_{k=1}^n X_k \mathbf{1}_{[s_k,t_k[} \in \mathfrak{g}^{\mathbb{R}_+}$.

By choosing a commutative subalgebra of $\pi(\mathfrak{g}^{\mathbb{R}_+})$ we can get a classical process.

Denote by $\mathcal{S}(\mathbb{R}_+)$ the algebra of real-valued simple step functions on $\mathbb{R}$,

$$\mathcal{S}(\mathbb{R}_+) = \Big\{ f = \sum_{k=1}^n f_k \mathbf{1}_{[s_k,t_k[} \Big| 0 \le s_1 \le t_1 \le s_2 \le \cdots \le t_n < \infty, f_1, \ldots, f_n \in \mathbb{R} \Big\},$$

then the product fX of an element $X \in \mathfrak{g}^{\mathbb{R}_+}$ with a function $f \in \mathcal{S}(\mathbb{R}_+)$ is again in $\mathfrak{g}^{\mathbb{R}_+}$.

Theorem 2.4. *Let $(j_{st})_{0 \le s \le t}$ be a Lévy process on a real Lie algebra $\mathfrak{g}$ and let π be as in Equation (2.2). Choose $X \in \mathfrak{g}^{\mathbb{R}_+}$ and define*

$$X(f) = i\pi(fX)$$

for $f \in \mathcal{S}(\mathbb{R}_+)$.

Then there exists a classical stochastic process $(\hat{X}_t)_{t \ge 0}$ with independent increments that has the same finite distributions as X, i.e.

$$\langle \Omega, g_1\big(X(f_1)\big) \cdots g_n\big(X(f_n)\big)\Omega \rangle = \mathbb{E}\Big(g_1\big(\hat{X}(f_1)\big) \cdots g_n\big(\hat{X}(f_n)\big)\Big)$$

for all $n \in \mathbb{N}$, $f_1, \ldots, f_n \in \mathcal{S}(\mathbb{R}_+)$, $g_1, \ldots, g_n \in \mathcal{C}_0(\mathbb{R})$, where

$$\hat{X}(f) = \int_{\mathbb{R}_+} f(t)\mathrm{d}\hat{X}_t = \sum_{k=1}^{n} f_k(\hat{X}_{t_k} - \hat{X}_{s_k})$$

for $f = \sum_{k=1}^{n} f_k \mathbf{1}_{[s_k, t_k[} \in \mathcal{S}(\mathbb{R}_+)$.

The existence of $(\hat{X}_t)_{t \geq 0}$ follows as in [AFS02, Section 4]. Thanks to the regularity assumptions in 2.2, the operators $X(f_1), \ldots, X(f_n)$ are essentially self-adjoint and $g_1\big(X(f_1)\big), \ldots, g_n\big(X(f_n)\big)$ can be defined by the usual functional calculus for self-adjoint operators.

3. Quasi-invariance Formulas for Components of Quantum Lévy Processes

Let $(j_{st})_{0 \leq s \leq t}$ be a Lévy process on a real Lie algebra $\mathfrak{g}$ and and fix $X \in \mathfrak{g}^{\mathbb{R}_+}$ with classical version $(\hat{X}_t)_{t \geq 0}$. In order to get quasi-invariance formulas for $(\hat{X}_t)_{t \geq 0}$, we simply choose another element $Y \in \mathfrak{g}^{\mathbb{R}_+}$ that does not commute with X and let the unitary operator $U = e^{\pi(Y)}$ act on the algebra

$$\mathcal{A}_X = \mathrm{alg}\{X(f)|f \in \mathcal{S}(\mathbb{R}_+)\}$$

generated by X. We can describe this action in two different ways.

First, we can let U act on $\mathcal{A}_X$ directly. This gives another algebra

$$\mathcal{A}_{X'} = \mathrm{alg}\{UX(f)U^*|f \in \mathcal{S}(\mathbb{R}_+)\},$$

generated by $X'(f) = UX(f)U^*$, $f \in \mathcal{S}(X)$. Since this algebra is again commutative, there exists a classical process $(\hat{X}'_t)_{t \geq 0}$ that has the same expectation values as X' w.r.t. Ω, i.e.

$$\langle \Omega, g_1\big(X'(f_1)\big) \cdots g_n\big(X'(f_n)\big)\Omega \rangle = \mathbb{E}\Big(g_1\big(\hat{X}'(f_1)\big) \cdots g_n\big(\hat{X}'(f_n)\big)\Big)$$

for all $n \in \mathbb{N}$, $g_1, \ldots, g_n \in \mathcal{C}_0(\mathbb{R})$, $f_1, \ldots, f_n \in \mathcal{S}(\mathbb{R}_+)$, where $\hat{X}'(f) = \int_{\mathbb{R}_+} f(t)\mathrm{d}\hat{X}'_t$ for $f = \sum_{k=1}^{n} f_k \mathbf{1}_{[s_k, t_k[} \in \mathcal{S}(\mathbb{R}_+)$.

If $X'(f)$ is a function of $X(f)$, then $\mathcal{A}_X$ is invariant under the action of U. In this case the classical process $(\hat{X}'_t)_{t \geq 0}$ can be obtained from $(\hat{X}_t)_{t \geq 0}$ by a pathwise transformation, see 4.2 and 4.5. But even if this is not the case, we can still get a quasi-invariance formula that states that the law of $(\hat{X}'_t)_{t \geq 0}$ is absolutely continuous w.r.t. the law of $(\hat{X}_t)_{t \geq 0}$.

Second, we can let U act on the vacuum state Ω, this gives us a new state vector $\Omega' = U^*\Omega$. If Ω is cyclic for $\mathcal{A}_X$, then Ω' can be approximated by elements of the form $G\Omega$ with $G \in \mathcal{A}_X$. It is actually possible to find an element G which is affiliated to the von Neumann algebra generated by $\mathcal{A}_X$ such that $G\Omega = \Omega'$. This follows from the BT theorem, see [Sak71, Theorem 2.7.14]. Then the following

calculation shows that the finite marginal distributions of $(\hat{X}'_t)_{t\geq 0}$ are absolutely continuous w.r.t. those of $(\hat{X}_t)$,

$$
\begin{aligned}
\mathbb{E}\Big(g\big(\hat{X}'(f)\big)\Big) &= \langle\Omega, g\big(X'(f)\big)\Omega\rangle = \langle\Omega, g\big(UX(f)U^*\big)\Omega\rangle \\
&= \langle\Omega, Ug\big(X(f)\big)U^*\Omega\rangle = \langle U^*\Omega, g\big(X(f)\big)U^*\Omega\rangle \\
&= \langle\Omega', g\big(X(f)\big)\Omega'\rangle = \langle G\Omega, g\big(X(f)\big)G\Omega\rangle \\
&= \mathbb{E}\Big(g\big(X(f)\big)|\hat{G}|^2\Big).
\end{aligned}
$$

Here G was a "function" of X and $\hat{G}$ is obtained from G by replacing X by $\hat{X}$. This is possible, because $\mathcal{A}_X$ is commutative, and requires only standard functional calculus.

The density relating the law of $(\hat{X}'_t)_{t\geq 0}$ and that of $(\hat{X}_t)_{t\geq 0}$ is therefore given by $|\hat{G}|^2$.

The same calculation applies of course also to the finite joint distributions, i.e. we also have

$$
\mathbb{E}\Big(g_1\big(\hat{X}'(f_1)\big)\cdots g_n\big(\hat{X}'(f_n)\big)\Big) = \mathbb{E}\Big(g_1\big(\hat{X}(f_1)\big)\cdots g_n\big(\hat{X}(f_n)\big)|\hat{G}|^2\Big).
$$

for all $n\in\mathbb{N}$, $f_1,\ldots,f_n\in\mathcal{S}(\mathbb{R}_+)$, $g_1,\ldots,g_n\in\mathcal{C}_0(\mathbb{R})$.

4. Examples

In this section we give the explicit calculation of the density $|G|^2$ for several examples.

We define our real Lie algebras as complex Lie algebras with an involution, because the relations can be given in a more convenient form for the complexifications. The real Lie algebra can be recovered as the real subspace of anti-hermitian elements.

4.1. Gaussian and Poisson random variables.
The oscillator Lie algebra is the four dimensional Lie algebra *osc* with basis $\{N, A^+, A^-, E\}$ and the Lie bracket given by

$$
[N, A^\pm] = \pm A^\pm, \quad [A^-, A^+] = E, \quad [E, N] = [E, A^\pm] = 0.
$$

We equip *osc* furthermore with the involution $N^* = N$, $(A^+)^* = A^-$, and $E^* = E$.

A general hermitian element of *osc* can be written in the form $X_{\alpha,\zeta,\beta} = \alpha N + \zeta A^+ + \overline{\zeta}A^- + \beta E$ with $\alpha,\beta\in\mathbb{R}$, $\zeta\in\mathbb{C}$.

Let $Y = i(wA^+ + \overline{w}A^-)$. We can compute the adjoint action of $g_t = e^{tY}$ on $X_{\alpha,\zeta,\beta}$ by solving the differential equation

$$
\dot{X}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Ad}_{g_t}(X) = [Y, X(t)].
$$

Writing $X(t) = a(t)N + z(t)A^+ + \overline{z}(t)A^- + b(t)E$, we get the system of ode's

$$\begin{aligned}
\dot{a}(t) &= 0, \\
\dot{z}(t) &= -i\alpha w, \\
\dot{b}(t) &= i(\overline{w}z - w\overline{z}),
\end{aligned}$$

with initial conditions $a(0) = \alpha$, $z(0) = \zeta$, $b(0) = \beta$. We get

(4.1) $X(t) = \alpha N + (\zeta - i\alpha wt)A^+ + (\overline{\zeta} + i\alpha\overline{w}t)A^- + \big(\beta + 2t\Im(w\overline{\zeta}) + \alpha|w|^2t^2\big)E,$

where $\Im(z)$ denotes the imaginary part of z.

A representation ρ of osc on ℓ^2 is defined by

$$\begin{aligned}
\rho(N)|n\rangle &= n|n\rangle, \\
\rho(A^+)|n\rangle &= \sqrt{n+1}|n+1\rangle, \\
\rho(A^-)|n-1\rangle &= \sqrt{n}|n-1\rangle, \\
\rho(E)|n\rangle &= |n\rangle,
\end{aligned}$$

where $|0\rangle, |1\rangle, \ldots$ is an orthonormal basis of ℓ^2.

Proposition 4.1. *The distribution of $\rho(X_{\alpha,\zeta,\beta})$ in the vacuum vector $|0\rangle$ is given by the characteristic function*

$$\langle 0| \exp\big(i\lambda\rho(X_{\alpha,\zeta,\beta})\big)|0\rangle = \begin{cases} \exp\left(i\lambda\beta - \frac{\lambda^2}{2}|\zeta|^2\right) & \text{for} \quad \alpha = 0, \\ \exp\left(i\lambda\left(\beta - \frac{|\zeta|^2}{\alpha}\right) + \frac{|\zeta|^2}{\alpha^2}\left(e^{i\lambda\alpha} - 1\right)\right) & \text{for} \quad \alpha \neq 0, \end{cases}$$

i.e. it is either a Gaussian random variable with variance $|\zeta|^2$ and mean β or Poisson random variable with "jump size" α, intensity $\frac{|\zeta|^2}{\alpha^2}$, and drift $\beta - \frac{|\zeta|^2}{\alpha}$.

Lemma 4.2. *The vacuum vector $|0\rangle$ is cyclic for $\rho(X_{\alpha,\zeta,\beta})$, as long as $\zeta \neq 0$, i.e.*

$$\overline{\text{span}\left\{\rho(X_{\alpha,\zeta,\beta})^k|0\rangle : k = 0, 1, \ldots\right\}} = \ell^2.$$

Proof. Due to the creation operator A^+ in the definition of $\rho(X_{\alpha,\zeta,\beta})$, we have

$$\rho(X_{\alpha,\zeta,\beta})^k|0\rangle = \zeta^k\sqrt{k!}|k\rangle + \sum_{\ell=0}^{k-1} c_\ell|\ell\rangle$$

with some coefficients $c_\ell \in \mathbb{C}$. Therefore

$$\text{span}\left\{|0\rangle, \rho(X_{\alpha,\zeta,\beta})|0\rangle, \ldots, \rho(X_{\alpha,\zeta,\beta})^k|0\rangle\right\} = \text{span}\left\{|0\rangle, \ldots, |k\rangle\right\}.$$

for all $k \in \mathbb{N}$, if $\zeta \neq 0$. $\square$

Therefore $v(t) = \exp\big(-t\rho(Y)\big)|0\rangle$ can be written in the form

$$v(t) = \sum_{k=0}^{\infty} c_k(t)\rho(X_{\alpha,\zeta,\beta})^k|0\rangle = G(X_{\alpha,\zeta,\beta}, t)|0\rangle.$$

In order to compute the function G, we consider

$$\frac{\mathrm{d}}{\mathrm{d}t}v(t) = -\exp\big(-t\rho(Y)\big)\rho(Y)|0\rangle = -i\exp\big(-t\rho(Y)\big)w|1\rangle.$$

We can rewrite this as

$$\frac{\mathrm{d}}{\mathrm{d}t}v(t) = -\frac{iw}{\tilde{\zeta}(t)}\exp\big(-t\rho(Y)\big)\left(\rho(X_{\alpha,\tilde{\zeta}(t),\tilde{\beta}(t)})|0\rangle - \tilde{\beta}(t)|0\rangle\right)$$

$$= -\frac{iw}{\tilde{\zeta}(t)}\left(\rho(X_{\alpha,\zeta,\beta}) - \tilde{\beta}(t)\right)\exp\big(-t\rho(Y)\big)|0\rangle = -\frac{iw}{\tilde{\zeta}(t)}\left(\rho(X_{\alpha,\zeta,\beta}) - \tilde{\beta}(t)\right)v(t),$$

where

$$\begin{aligned}\tilde{\zeta}(t) &= \zeta - i\alpha wt, \\ \tilde{\beta}(t) &= \beta + 2t\Im(w\overline{\zeta}) + \alpha|w|^2t^2.\end{aligned}$$

This is satisfied provided $G(x,t)$ satisfies the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}G(x,t) = -\frac{iw}{\tilde{\zeta}(t)}\left(x - \tilde{\beta}(t)\right)G(x,t)$$

with initial condition $G(x,0) = 1$. We get

$$G(x,t) = \exp\left(-iw\int_0^t\left(\frac{x - \tilde{\beta}(s)}{\tilde{\zeta}(s)}\right)\mathrm{d}s\right).$$

Evaluating the integral, this can be written as

$$G(x,t) = \left(1 - i\frac{\alpha wt}{\zeta}\right)^{\frac{x-\beta}{\alpha} + \frac{|\zeta|^2}{\alpha^2}}\exp\left(i\frac{w\overline{\zeta}}{\alpha} - \frac{t^2}{2}|w|^2\right),$$

and therefore

$$|G(x,t)|^2 = \left(1 + 2t\alpha\Im\left(\frac{w}{\zeta}\right) + t^2\alpha^2\frac{|w|^2}{|\zeta|^2}\right)^{\frac{x-\beta}{\alpha} + \frac{|\zeta|^2}{\alpha^2}}e^{-\frac{|\zeta|^2}{\alpha^2}\left(2t\alpha\Im\frac{w}{\zeta} + t^2\alpha^2\frac{|w|^2}{|\zeta|^2}\right)}.$$

After letting α go to 0 we get

$$|G(x,t)|^2 = e^{2t(x-\beta)\Im\frac{w}{\zeta} - \frac{t^2}{2}|\zeta|^2\left(2\Im\frac{w}{\zeta}\right)^2}.$$

Note that the classical analog of this limiting procedure is

$$(1+\alpha)^{\alpha\lambda(N_\alpha - \lambda/\alpha^2) + \frac{\lambda^2}{\alpha}} \xrightarrow{\alpha \to 0} e^{\lambda X - \frac{1}{2}\lambda^2},$$

where N_α is a Poisson random variable with intensity $\lambda > 0$ and $\lambda(N_\alpha - \lambda/\alpha^2)$ converges in distribution to a standard Gaussian variable X. No such normalization is needed in the quantum case.

Proposition 4.3. *We have*

$$\mathbb{E}\Big[g\big(X(t)\big)\Big] = \mathbb{E}\left[g(X_{\alpha,\zeta,\beta})\big|G(X_{\alpha,\zeta,\beta},t)\big|^2\right]$$

for all $g \in \mathcal{C}_0(\mathbb{R})$.

For $\alpha = 0$, this identity gives the relative density of two Gaussian random variables with the same variance, but different means. For $\alpha \neq 0$, it gives the relative density of two Poisson random variables with different intensities.

4.2. Brownian motion and the Girsanov formula.
Let now $(j_{st})_{0 \leq s \leq t}$ be the Lévy process on *osc* with the Schürmann triple defined by $D = \mathbb{C}$,

$$\rho(N) = 1, \quad \rho(A^{\pm}) = \rho(E) = 0,$$
$$\eta(A^+) = 1, \quad \eta(N) = \eta(A^-) = \eta(E) = 0,$$
$$L(N) = L(A^{\pm}) = 0, \quad L(E) = 1.$$

Taking for X the constant function with value $-i(A^+ + A^-)$, we get

$$X(f) = A^*(f) + A(f)$$

and the associated classical process $(\hat{X}_t)_{t \geq 0}$ is Brownian motion.

We choose for $Y = h(A^+ - A^-)$, with $h \in \mathcal{S}(\mathbb{R}_+)$. A similar calculation as in the previous subsection yields

$$X'(\mathbf{1}_{[0,t]}) = e^Y X(\mathbf{1}_{[0,t]})e^{-Y} = X(\mathbf{1}_{[0,t]}) - 2\int_0^t h(s)\mathrm{d}s$$

i.e. $\mathcal{A}_X$ is invariant under e^Y and $(\hat{X}'_t)_{t \geq 0}$ is obtained from $(\hat{X}_t)_{t \geq 0}$ by adding a drift.

$e^{\pi(Y)}$ is a Weyl operator and gives an exponential vector, if it acts on the vacuum, i.e.

$$e^{\pi(Y)}\Omega = e^{-||h||^2/2}\mathcal{E}(h)$$

see, e.g., [Par92, Mey95]. But - up to the normalisation - we can create the same exponential vector also by acting on Ω with $e^{X(h)}$,

$$e^{X(h)}\Omega = e^{||h||^2/2}\mathcal{E}(h).$$

Therefore we get $G = \exp\big(X(h) - ||h||^2\big)$ and the well-known Girsanov formula

$$\mathbb{E}\Big(g\big(\hat{X}'(f)\big)\Big) = \mathbb{E}\left(g\big(\hat{X}(f)\big)\exp\left(2X(h) - 2\int_0^\infty h^2(s)\mathrm{d}s\right)\right).$$

4.3. **Poisson process and the Girsanov formula.** Taking for X the constant function with value $-i(N + \nu A^+ + \nu A^- + \nu^2 E)$, we get

$$X(f) = N(f) + \nu A^*(f) + \nu A(f) + \nu^2 \int_0^\infty f(s)ds$$

and the associated classical process $(\hat{X}_t)_{t\geq 0}$ is a non-compensated Poisson process with intensity ν^2 and jump size 1. Given $h \in \mathcal{S}(\mathbb{R}_+)$ of the form

$$h(t) = \sum_{k=1}^n h_k 1_{[s_k, t_k[}(t),$$

with $h_k > -\nu^2$, let

$$w(t) = i(\sqrt{\nu^2 + h(t)} - \nu),$$

and $Y = w(A^+ - A^-)$. The calculations of Subsection 4.1 show that

$$X'(1_{[0,t]}) = e^Y X(1_{[0,t]})e^{-Y}$$

is a non-compensated Poisson process with intensity $\nu^2 + h(t)$. We have the Girsanov formula

$$\begin{aligned}
\mathbb{E}\Big(g\big(\hat{X}'(f)\big)\Big) &= \mathbb{E}\left(g\big(\hat{X}(f)\big)\prod_{k=1}^n \left(1 + \frac{h_k}{\nu^2}\right)^{\hat{X}(1_{[s_k,t_k[})} e^{-\nu^2(t_k - s_k)h_k}\right) \\
&= \mathbb{E}\left(g\big(\hat{X}(f)\big)e^{-\nu^2 \int_0^\infty h(s)ds}\prod_{k=1}^n \left(1 + \frac{h_k}{\nu^2}\right)^{\hat{X}(1_{[s_k,t_k[})}\right) \\
&= \mathbb{E}\left(g\big(\hat{X}(f)\big)\exp\left(\hat{X}\left(\log\left(1 + \frac{h}{\nu^2}\right)\right) - \nu^2 \int_0^\infty h(s)\mathrm{d}s\right)\right).
\end{aligned}$$

4.4. $sl(2,\mathbb{R})$ **and the Meixner, Gamma, and Pascal random variables.** Let us now consider the three-dimensional Lie algebra $sl(2;\mathbb{R})$, with basis B^+, B^-, M, Lie bracket

$$[M, B^\pm] = \pm 2B^\pm, \qquad [B^-, B^+] = M,$$

and the involution $(B^+)^* = B^-$, $M^* = M$.

For $\beta \in \mathbb{R}$, we set $X_\beta = B^+ + B^- + \beta M$. Furthermore, we choose $Y = B^- - B^+$. We compute $[Y, X_\beta] = 2\beta B^+ + 2\beta B^- + 2M = 2\beta X_{1/\beta}$ and

$$e^{tY/2} X_\beta e^{-tY/2} = e^{\frac{t}{2}\mathrm{ad}Y} X_\beta = \big(\cosh(t) + \beta \sinh(t)\big)X_{\gamma(\beta,t)},$$

where

$$\gamma(\beta, t) = \frac{\beta \cosh(t) + \sinh(t)}{\cosh(t) + \beta \sinh(t)}.$$

For $c > 0$ we can define representations of $sl(2;\mathbb{R})$ on ℓ^2 by

$$\begin{aligned}
\rho_c(B^+)|k\rangle &= \sqrt{(k+c)(k+1)}|k+1\rangle, \\
\rho_c(M)|k\rangle &= (2k+c)|k\rangle, \\
\rho_c(B^-)|k\rangle &= \sqrt{k(k+c-1)}|k\rangle,
\end{aligned}$$

where $|0\rangle, |1\rangle, \ldots$ is an orthonormal basis of ℓ^2.

Using the Splitting Lemma for $sl(2)$, cf. [FS93, Chapter 1, 4.3.10] to write $e^{\lambda X_\beta}$ as a product $e^{\nu_+ B^+} e^{\nu_0 M} e^{\nu_- B^-}$, it is straight-forward to compute the distribution of $\rho_c(X_\beta)$ in the state vector $|0\rangle$.

Proposition 4.4. *The Fourier-Laplace transform of the distribution of $\rho_c(X_\beta)$ w.r.t. $|0\rangle$ is given by*

$$\langle 0| \exp\left(\lambda \rho_c(X_\beta)\right)|0\rangle = \left(\frac{\sqrt{\beta^2 - 1}}{\sqrt{\beta^2 - 1}\cosh\left(\lambda\sqrt{\beta^2 - 1}\right) - \beta \sinh\left(\lambda\sqrt{\beta^2 - 1}\right)} \right)^c .$$

Remark 4.5. For $|\beta| < 1$, this distribution is called the Meixner distribution. It is absolutely continuous w.r.t. to the Lebesgue measure and the density is given by

$$C \exp\left(\frac{(\pi - 2\arccos\beta)x}{2\sqrt{1 - \beta^2}} \right) \left| \Gamma\left(\frac{c}{2} + \frac{ix}{2\sqrt{1 - \beta^2}} \right) \right|^2 ,$$

see also [AFS02]. C is a normalization constant.

For $\beta = \pm 1$, we get the Gamma distribution, which has the density

$$\frac{|x|^{c-1}}{\Gamma(c)} e^{-\beta x} \mathbf{1}_{\beta \mathbb{R}_+}.$$

Finally, for $|\beta| > 1$, we get a discrete measure, the so-called Pascal distribution.

Lemma 4.6. *The lowest weight vector $|0\rangle$ is cyclic for $\rho_c(X_\beta)$ for all $\beta \in \mathbb{R}$, $c > 0$.*

Proof. On $|0\rangle$, we get

$$\rho_c(X_\beta)^k |0\rangle = \sqrt{k! c(c+1)\cdots(c+k-1)}|k\rangle + \sum_{\ell=0}^{k-1} c_\ell |\ell\rangle$$

with some coefficients $c_\ell \in \mathbb{C}$. Therefore

$$\operatorname{span}\left\{ |0\rangle, \rho_c(X_\beta)|0\rangle, \ldots, \rho_c(X_\beta)^k|0\rangle \right\} = \operatorname{span}\left\{ |0\rangle, \ldots, |k\rangle \right\}.$$

for all $k \in \mathbb{N}$, if $c > 0$. $\qquad\square$

Therefore we can write $v_t = e^{-t\rho_c(Y)/2}|0\rangle$ in the form

$$v(t) = \sum_{k=0}^{\infty} c_k(t)\rho_c(X_\beta)^k|0\rangle = G(X_\beta, t)|0\rangle.$$

In order to compute the function G, we consider

$$\frac{\mathrm{d}}{\mathrm{d}t} v(t) = -\frac{1}{2}\exp\left(-t\rho_c(Y)/2 \right)\rho_c(Y)|0\rangle = \frac{1}{2}\exp\left(-t\rho_c(Y) \right)\sqrt{c}|1\rangle.$$

As in Subsection 4.1, we introduce X_β into this equation to get an ordinary differential equation for G,

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}v(t) &= \frac{1}{2}\exp\big(-t\rho_c(Y)/2\big)\big(\rho_c(X_{\gamma(\beta,t)}) - c\gamma(\beta,t)\big)|0\rangle \\
&= \frac{1}{2}\frac{\rho_c(X_\beta) - c\big(\beta\cosh(t) + \sinh(t)\big)}{\cosh(t) + \beta\sinh(t)}\exp\big(-t\rho_c(Y)/2\big)|0\rangle
\end{aligned}
$$

This is satisfied, if

$$
\frac{\mathrm{d}}{\mathrm{d}t}G(x,t) = \frac{1}{2}\frac{x - c\big(\beta\cosh(t) + \sinh(t)\big)}{\cosh(t) + \beta\sinh(t)}G(x,t)
$$

with initial condition $G(x,0) = 1$. The solution of this ode is given by

$$
G(x,t) = \exp\frac{1}{2}\int_0^t \frac{x - c\big(\beta\cosh(s) + \sinh(s)\big)}{\cosh(s) + \beta\sinh(s)}\mathrm{d}s.
$$

For $\beta = 1$, we get $G(x,t) = \exp\frac{1}{2}\big(x(1 - e^{-t}) - ct\big)$ and we recover the following identity for a Gamma distributed random variable Z with parameter c,

$$
\mathbb{E}\big(g(e^t Z)\big) = \mathbb{E}\Big(g(Z)\exp\big(Z(1 - e^{-t}) - ct\big)\Big).
$$

If $|\beta| < 1$, then we can write G in the form

$$
G(x,t) = \exp\big(\Phi(\beta,t)x - c\Psi(\beta,t)\big),
$$

where

$$
(4.2)\quad \Phi(\beta,t) = \frac{1}{\sqrt{1-\beta^2}}\left(\arctan\left(e^t\sqrt{\frac{1+\beta}{1-\beta}}\right) - \arctan\left(\sqrt{\frac{1+\beta}{1-\beta}}\right)\right),
$$

$$
(4.3)\quad \Psi(\beta,t) = \frac{1}{2}\Big(t + \log\big(1 + \beta + e^{-2t}(1 - \beta)\big) - \log 2\Big).
$$

4.5. A quasi-invariance formula for the Gamma process.

Let now $(j_{st})_{0 \le s \le t}$ be the Lévy process on $sl(2;\mathbb{R})$ with Schürmann triple $D = \ell^2$, $\rho = \rho_2$, and

$$
\eta(B^+) = |0\rangle, \quad \eta(B^-) = \eta(M) = 0,
$$
$$
L(M) = 1, \quad L(B^\pm) = 0,
$$

cf. [AFS02, Example 3.1]. We take for X the constant function with value $-i(B^+ + B^+ + M)$, then the random variables

$$
X(\mathbf{1}_{[s,t]}) = \Lambda_{st}\big(\rho(X)\big) + A_{st}^*\big(|0\rangle\big) + A_{st}(|0\rangle) + (t - s)\mathrm{id}
$$

are Gamma distributed in the vacuum vector Ω.

For Y we choose $h(B^- - B^+)$ with $h = \sum_{k=1}^r h_k \mathbf{1}_{[s_k,t_k[} \in \mathcal{S}(\mathbb{R}_+)$, $0 \le s_1 \le t_1 \le s_2 \le \cdots \le t_n$. As in the previous subsection, we get

$$
X'(\mathbf{1}_{[s,t]}) = e^{\pi(Y)}X(\mathbf{1}_{[s,t]})e^{-\pi(Y)} = X(e^{2h}\mathbf{1}_{[s,t]}).
$$

On the other hand, using the tensor product structure of the Fock space, we can calculate

$$
\begin{aligned}
e^{-\pi(Y)}\Omega &= e^{-\sum_{k=1}^{n} h_k j_{s_k t_k}(Y)}\Omega \\
&= e^{-h_1 j_{s_1 t_1}(Y))}\Omega \otimes \cdots \otimes e^{-h_n j_{s_n t_n}(Y))}\Omega \\
&= \exp \frac{1}{2}\Big(X\big((1-e^{-2h_1})\mathbf{1}_{[s_1,t_1[}\big) - (t_1-s_1)2h\mathrm{d}s \Big)\Omega \otimes \cdots \\
&\qquad \cdots \otimes \exp \frac{1}{2}\Big(X\big((1-e^{-2h_n})\mathbf{1}_{[s_n,t_n[}\big) - (t_n-s_n)2h\mathrm{d}s \Big)\Omega \\
&= \exp \frac{1}{2}\left(X(1-e^{-2h}) - \int_{\mathbb{R}_+} 2h\mathrm{d}s \right)\Omega,
\end{aligned}
$$

since j_{st} is equivalent to ρ_{t-s}.

Proposition 4.7. *Let $n \in \mathbb{N}$, $f_1,\ldots,f_n \in \mathcal{S}(\mathbb{R}_+)$, $g_1,\ldots,g_n \in \mathcal{C}_0(\mathbb{R})$, then we have*

$$
\mathbb{E}\Big(g_1\big(X'(f_1)\big)\cdots g_n\big(X'(f_n)\big)\Big)
$$

$$
= \mathbb{E}\left(g_1\big(X(f_1)\big)\cdots g_n\big(X(f_n)\big) \exp\left(X(1-e^{-2h}) - \int_{\mathbb{R}_+} 2h\mathrm{d}s \right)\right).
$$

4.6. **A quasi-invariance formula for the Meixner process.** We consider again the same Lévy process on $sl(2;\mathbb{R})$ as in the previous subsection. Let $\varphi,\beta \in \mathcal{S}(\mathbb{R}_+)$ with $|\beta(t)| < 1$ for all $t \in \mathbb{R}_+$, and set

$$
X_{\varphi,\beta} = \varphi(B^+ + B^- + \beta M) \in sl(2;\mathbb{R})^{\mathbb{R}_+}.
$$

Let Y again be given by $Y = h(B^- - B^+)$, $h \in \mathcal{S}(\mathbb{R}_+)$. Then we get

$$
X'(t) = e^{Y(t)}X(t)e^{-Y(t)} = \varphi(t)\big(\cosh(2h) + \beta(t)\sinh(2h)\big)\Big(B^+ + B^- + \gamma\big(\beta(t),2h\big)M\Big)
$$

i.e. $X' = X_{\varphi',\beta'}$ with

$$
\begin{aligned}
\varphi'(t) &= \varphi(t)\Big(\cosh\big(2h(t)\big) + \beta(t)\sinh\big(2h(t)\big)\Big), \\
\beta'(t) &= \gamma\big(\beta(t), 2h(t)\big).
\end{aligned}
$$

As in the previous subsection, we can also calculate the function G,

$$
e^{\pi(Y)}\Omega = \exp\frac{1}{2}\left(X_{\Phi(\beta,2h),\beta} - \int_{\mathbb{R}_+} \Psi\big(\beta(t),2h(t)\big)\right)\Omega,
$$

where Φ, Ψ are defined as in Equations (4.2) and (4.3).

Proposition 4.8. *The finite joint distributions of $X_{\varphi',\beta'}$ are absolutely continuous w.r.t. to those of $X_{\varphi,\beta}$, and the mutual density is given by*

$$
\exp\left(X_{\Phi(\beta,2h),\beta} - \int_{\mathbb{R}_+} \Psi\big(\beta(t),2h(t)\big)\right).
$$

5. Conclusion

The results stated in Subsections 4.2, 4.3, 4.5, and 4.6 have been proved with our methods for the finite joint distributions. They can be extended to the distribution of the processes either using continuity arguments for the states and endomorphisms on our operator algebras or by the use of standard tightness arguments of classical probability.

Of course the general idea also applies to classical processes obtained by a different choice of the commutative subalgebra, e.g. as in [Bia98], and to more general classes of quantum stochastic processes. The restrictions in this Note were motivated by the fact that they simplify the explicit calculations.

Acknowledgements

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