Moments of Poisson stochastic integrals with random integrands

Nicolas Privault
Division of Mathematical Sciences
School of Physical and Mathematical Sciences
Nanyang Technological University
21 Nanyang Link
Singapore 637371

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Abstract

We show that the moment of order n of the Poisson stochastic integral of a random process $(u_x)_{x\in X}$ over a metric space X is given by the non-linear Mecke identity

$$E\left[\left(\int_X u_x(\omega)\omega(dx)\right)^n\right] = \sum_{\{P_1,\ldots,P_k\}\in\mathcal{P}_n} E\left[\int_{X^k} \varepsilon^+_{\mathfrak{s}_k}(u_{s_1}^{|P_1|}\cdots u_{s_k}^{|P_k|})\sigma(ds_1)\cdots\sigma(ds_k)\right],$$

where the sum runs over all partitions $P_1\cup\cdots\cup P_k$ of $\{1,\ldots,n\}$, $|P_i|$ denotes the cardinality of P_i, and $\varepsilon^+_{\mathfrak{s}_k}$ is the operator that acts by addition of points at $s_1,\ldots,s_k$ to Poisson configurations. This formula recovers known results in case $(u(x))_{x\in X}$ is a deterministic function on X.

Key words: Poisson stochastic integrals, moment identities, Bell polynomials, Poisson-Skorohod integral.
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1 Introduction

Let Ω^X denote the configuration space on a σ-compact metric space X with Borel σ-algebra $\mathcal{B}(X)$, i.e. Ω^X is the space of at most countable and locally finite subsets of X, defined as

$$\Omega^X = \{\omega = \{x_i\}_{i=1,\ldots,N} \subset X,\ x_i \neq x_j\ \forall i\neq j,\ N\in\mathbb{N}\cup\{\infty\}\}.$$

Each element ω of Ω^X has cardinality $\omega(X)$ and is identified with the Radon point measure

$$\omega = \sum_{i=1}^{\omega(X)} \epsilon_{x_i},$$

where ϵ_x denotes the Dirac measure at $x \in X$.

The space Ω^X is endowed with the Poisson probability measure π_σ with σ-finite diffuse intensity measure σ on X, such that for all compact disjoint subsets $A_1, \ldots, A_n$ of X, $n \geq 1$, the mapping

$$\omega \longmapsto (\omega(A_1), \ldots, \omega(A_n))$$

is a vector of independent Poisson distributed random variables on $\mathbb{N}$ with respective expectations $\sigma(A_1), \ldots, \sigma(A_n)$, cf. e.g. § 6.1 of [11].

In [2] the moment formula

$$E\left[\left(\int_X f(x)\omega(dx)\right)^n\right] = n! \sum_{\substack{r_1+2r_2+\cdots+nr_n=n \\ r_1,\ldots,r_n\geq 0}} \prod_{k=1}^n \left(\frac{1}{(k!)^{r_k} r_k!}\left(\int_X f^k(x)\sigma(dx)\right)^{r_k}\right) \tag{1.1}$$

has been proved for $f : X \longrightarrow \mathbb{R}$ a deterministic sufficiently integrable function. The proof of [2] relies on the Lévy-Khintchine representation of the Laplace transform of $\int_X u(x)\omega(dx)$, and this result can also be recovered under a different combinatorial interpretation by the Faà di Bruno formula, cf. e.g. § 2.4 and (2.4.4) page 27 of [6], from the relation

$$\begin{aligned} E\left[\left(\int_X u(x)\omega(dx)\right)^n\right] &= \sum_{\{P_1,\ldots,P_k\}\in\mathcal{P}_n} \int_X u^{|P_1|}(x)\sigma(dx)\cdots\int_X u^{|P_k|}(x)\sigma(dx) \\ &= A_n(\kappa_1, \ldots, \kappa_n), \end{aligned} \tag{1.2}$$

between the moments and the cumulants $\kappa_n = \int_X u^n(x)\sigma(dx)$, $n \geq 1$, of $\displaystyle\int_X u(x)\omega(dx)$, where the sum runs over the set $\mathcal{P}_n$ of all partitions $P_1 \cup \cdots \cup P_k$ of $\{1, \ldots, n\}$ with cardinality $|P_i|$, and

$$A_n(x_1, \ldots, x_n) = n! \sum_{\substack{r_1+2r_2+\cdots+nr_n=n \\ r_1,\ldots,r_n\geq 0}} \prod_{k=1}^n \left(\frac{1}{r_k!}\left(\frac{x_k}{k!}\right)^{r_k}\right)$$

is the Bell polynomial of degree n.

Recently, (1.1) has been applied to control the p-variation and the number of crossings of fractional Poisson and shot noise processes with deterministic kernels, cf. [4], and to insurance mathematics in [1].

In this paper we extend the above formula (1.1) to random integrands. Namely, we state that given $u : \Omega^X \times X \longrightarrow \mathbb{R}$ a sufficiently integrable random process we have

$$E\left[\left(\int_X u_x(\omega)\omega(dx)\right)^n\right] = \sum_{\{P_1,\ldots,P_k\}\in\mathcal{P}_n} E\left[\int_{X^k} \varepsilon^+_{\mathfrak{s}_k}\left(u_{s_1}^{|P_1|}\cdots u_{s_k}^{|P_k|}\right)\sigma(ds_1)\cdots\sigma(ds_k)\right], \tag{1.3}$$

cf. Proposition 3.1 below, where

$$\mathfrak{s}_k = (s_1,\ldots,s_k) \in X^k, \quad k \geq 1,$$

the addition operator $\varepsilon^+_{\mathfrak{s}_k}$ is defined on any random variable $F : \Omega^X \longrightarrow \mathbb{R}$ by

$$\varepsilon^+_{\mathfrak{s}_k} F(\omega) = F(\omega \cup \{s_1,\ldots,s_k\}), \qquad \omega \in \Omega^X, \quad s_1,\ldots,s_k \in X,$$

and the sum (1.3) runs over the set $\mathcal{P}_n$ of all (disjoint) partitions $P_1 \cup \cdots \cup P_k$ of $\{1,\ldots,n\}$, $k = 1,\ldots,n$. As expected, when $(u(x))_{x\in X}$ is a deterministic function we have

$$\varepsilon^+_{\mathfrak{s}_k} u(s_i) = u(s_i), \qquad 1 \leq i \leq k,$$

in which case (1.3) recovers (1.2). For $n = 1$, (1.3) is known as the Mecke identity, cf. [7].

Examples

In the case of second order moments, (1.3) yields

$$E\left[\left(\int_X u_s(\omega)\omega(ds)\right)^2\right] = E\left[\int_X \varepsilon^+_{\mathfrak{s}}|u_s|^2\sigma(ds)\right] + E\left[\int_{X^2} \varepsilon^+_{\mathfrak{s}_1}\varepsilon^+_{\mathfrak{s}_2}(u_{s_1}u_{s_2})\sigma(ds_1)\sigma(ds_2)\right]. \tag{1.4}$$

Concerning third order moments, (1.3) shows that

$$E\left[\left(\int_X u_s(\omega)\omega(ds)\right)^3\right] = E\left[\int_X \varepsilon_s^+ u_s^3 \sigma(ds)\right] + 3E\left[\int_{X^2} \varepsilon_{s_1}^+ \varepsilon_{s_2}^+ (|u_{s_1}|^2 u_{s_2}) \sigma(ds_1)\sigma(ds_2)\right]$$
$$+E\left[\int_{X^3} \varepsilon_{s_1}^+ \varepsilon_{s_2}^+ \varepsilon_{s_3}^+ (u_{s_1} u_{s_2} u_{s_3}) \sigma(ds_1)\sigma(ds_2)\sigma(ds_3)\right]. \tag{1.5}$$

We proceed as follows. In Section 2 we rewrite a result of [14] into a new formula for the moments of compensated Poisson-Skorohod integrals in the language of set partitions, without involving cancellations of terms. In Section 3, by means of the binomial inversion we deduce formulas for non-compensated integrals of random integrands In the case of deterministic integrands and indicator functions, in Section 4 we recover and extend known relations between the moments of the Poisson distribution and Stirling numbers.

The case of Wiener and Itô stochastic integrals follows another type of combinatorics based on pair partitions in the case of deterministic integrands, cf. [13] for the case of random integrands.

2 Poisson-Skorohod integrals

We start with a moment identity for compensated Poisson-Skorohod integrals, obtained by rewriting Theorem 5.1 of [14] in terms of set partitions. By saying that $u : \Omega^X \times X \longrightarrow \mathbb{R}$ has a compact support in X we mean that there exists a compact subset K of X such that $u_x(\omega) = 0$ for all $\omega \in \Omega^X$ and $x \in X \setminus K$.

Our proof of moment identities relies on the Poisson-Skorohod integral operator δ which is defined on any measurable process $u : \Omega^X \times X \longrightarrow \mathbb{R}$ by the expression

$$\delta(u)(\omega) = \int_X u_x(\omega \setminus \{x\})\omega(dx) - \int_X u_x(\omega)\sigma(dx), \qquad \omega \in \Omega^X, \tag{2.1}$$

provided $E\left[\int_X |u_x(\omega)|\sigma(dx)\right] < \infty$, cf. Corollary 1 of [9]. In (2.1), $\omega \setminus \{x\}$ denotes the configuration $\omega \in \Omega^X$ after removal of the point x in case $x \in \omega$.

The moment identities in this paper are stated for bounded random variables F and processes u with compact support, however they can be extended by assuming suitable conditions ensuring that the right hand side of the formula is finite. In the next proposition we let

$$\mathcal{N}_{\mathfrak{L}_k} = \frac{1}{l_1(l_1+l_2)\cdots(l_1+\cdots+l_k)} \frac{n!}{(l_1-1)!\cdots(l_k-1)!}, \tag{2.2}$$

which represents the number of partitions of a set of $n = l_1 + \cdots + l_k$ elements into k subsets of sizes $l_1, \ldots, l_k \geq 1$, cf. e.g. Lemma 3.1 of [12] or Lemma 4.5 of [13].

Proposition 2.1 *Let $F : \Omega^X \longrightarrow \mathbb{R}$ be a bounded random variable and let $u : \Omega^X \times X \longrightarrow \mathbb{R}$ be a bounded process with compact support in X. For all $n \geq 0$ we have*

$$E\left[\delta(u)^n F\right] = \sum_{c=0}^{n} (-1)^c \binom{n}{c} \sum_{k=0}^{n-c} \sum_{\substack{l_1+\cdots+l_k=n-c \\ l_1,\ldots,l_k \geq 1 \\ l_{k+1}=1,\ldots,l_{k+c}=1}} \mathcal{N}_{\mathfrak{L}_k} E\left[\int_{X^{k+c}} \varepsilon^+_{\mathfrak{s}_k} F \prod_{p=1}^{k+c} \varepsilon^+_{\mathfrak{s}_k \setminus \{s_p\}} u_{s_p}^{l_p} \, d\sigma^{k+c}(\mathfrak{s}_{k+c})\right],$$

where $d\sigma^b(\mathfrak{s}_b) = \sigma(ds_1)\cdots\sigma(ds_b)$, $\mathfrak{L}_k = (l_1, \ldots, l_k)$.

Proof. The proof of this formula relies on the identity

$$E\left[\delta(u)^n F\right] = \sum_{k=0}^{n} \sum_{b=k}^{n} (-1)^{b-k} \sum_{\substack{l_1+\cdots+l_k=n-(b-k) \\ l_1,\ldots,l_k \geq 1 \\ l_{k+1}=1,\ldots,l_b=1}} C_{\mathfrak{L}_k,b} E\left[\int_{X^b} \varepsilon^+_{\mathfrak{s}_k} F \prod_{p=1}^{b} \varepsilon^+_{\mathfrak{s}_k \setminus \{s_p\}} u_{s_p}^{l_p} \, d\sigma^b(\mathfrak{s}_b)\right] \tag{2.3}$$

for the moments of the compensated Poisson-Skorohod integral $\delta(u)$, cf. Theorem 5.1 of [14] and Theorem 1 of [10], where

$$C_{\mathfrak{L}_k,k+c} = \sum_{0=r_{c+1}<\cdots<r_0=k+c+1} \prod_{q=0}^{c} \prod_{p=r_{q+1}+q-c+1}^{r_q+q-c-1} \binom{l_1+\cdots+l_p+q-1}{l_1+\cdots+l_{p-1}+q}. \tag{2.4}$$

Next we note that $C_{\mathfrak{L}_k,k+c}$ defined in (2.4) represents the number of partitions of a set of $n = l_1 + \cdots + l_k + c$ elements into k subsets of lengths $l_1, \ldots, l_k$ and c singletons, hence when $l_1 + \cdots + l_k = n - c$ we have

$$C_{\mathfrak{L}_k,k+c} = \binom{n}{c} \mathcal{N}_{\mathfrak{L}_k},$$

since $\mathcal{N}_{\mathfrak{L}_k}$ is the number of partitions of a set of $l_1 + \cdots + l_k = n - c$ elements into k subsets of lengths $l_1, \ldots, l_k$. Hence we have, by the substitution $b = c + k$ and changing the order of summation,

$$E[\delta(u)^n F] = \sum_{c=0}^{n} \sum_{k=0}^{n-c} (-1)^c \sum_{\substack{l_1+\cdots+l_k=n-c \\ l_1,\ldots,l_k \geq 1 \\ l_{k+1}=1,\ldots,l_{k+c}=1}} C_{\mathfrak{L}_k,k+c} E\left[\int_{X^{k+c}} \varepsilon^+_{\mathfrak{s}_k} F \prod_{p=1}^{k+c} \varepsilon^+_{\mathfrak{s}_k \setminus \{s_p\}} u^{l_p}_{s_p} \, d\sigma^{k+c}(\mathfrak{s}_{k+c})\right]$$

$$= \sum_{c=0}^{n} (-1)^c \binom{n}{c} \sum_{k=0}^{n-c} \sum_{\substack{l_1+\cdots+l_k=n-c \\ l_1,\ldots,l_k \geq 1 \\ l_{k+1}=1,\ldots,l_{k+c}=1}} \mathcal{N}_{\mathfrak{L}_k} E\left[\int_{X^{k+c}} \varepsilon^+_{\mathfrak{s}_k} F \prod_{p=1}^{k+c} \varepsilon^+_{\mathfrak{s}_k \setminus \{s_p\}} u^{l_p}_{s_p} \, d\sigma^{k+c}(\mathfrak{s}_{k+c})\right].$$

□

The proof of (2.3), given in [14], relies on the duality relation

$$E[\langle DF, u \rangle_{L^2(X)}] = E[F\delta(u)], \tag{2.5}$$

cf. [9] and Proposition 2.3 of [14], between δ and the finite difference gradient

$$D_x F(\omega) = \varepsilon^+_x F(\omega) - F(\omega), \qquad \omega \in \Omega^X, \quad x \in X, \tag{2.6}$$

for all F and u in the respective closed L^2 domains $\mathrm{Dom}\,(\delta) \subset L^2(\Omega^X \times X, \pi_\sigma \otimes \sigma)$ and $\mathrm{Dom}\,(D) \subset L^2(\Omega^X, \pi_\sigma)$ of D and δ, cf. also Proposition 6.4.3 of [11] and references therein.

3 Pathwise integrals

The next Proposition 3.1 is the main result of this paper and, unlike (2.1), does not involve cancellations of terms. The proof of this result, which yields (1.3) and follows directly by binomial inversion of Proposition 2.1, is stated due to the additional presence of expectations.

Proposition 3.1 *Let $F : \Omega^X \longrightarrow \mathbb{R}$ be a bounded random variable, and let $u : \Omega^X \times X \longrightarrow \mathbb{R}$ be a bounded random process with compact support in X. For all $n \geq 0$ we have*

$$E\left[F\left(\int_X u_x(\omega)\omega(dx)\right)^n\right] = \sum_{\{P_1,\ldots,P_k\} \in \mathcal{P}_n} E\left[\int_{X^k} \varepsilon^+_{\mathfrak{s}_k}(F u^{|P_1|}_{s_1} \cdots u^{|P_k|}_{s_k}) \sigma(ds_1) \cdots \sigma(ds_k)\right]. \tag{3.1}$$

Proof. Applying Proposition 2.1 at the rank $n-i$ to the process $\varepsilon^+ u = (\varepsilon_x^+ u_x)_{x\in X}$ and the random variable $F\left(\int_X \varepsilon_x^+ u_x \sigma(dx)\right)^i$ we have, using the substitution $a = i + c$,

$$
\begin{aligned}
& E\left[F\left(\int_X u_x(\omega)\omega(dx)\right)^n\right] = E\left[F\left(\delta(\varepsilon^+ u) + \int_X \varepsilon_x^+ u_x \sigma(dx)\right)^n\right] \\
&= \sum_{i=0}^n \binom{n}{i} E\left[F\left(\int_X \varepsilon_x^+ u_x \sigma(dx)\right)^i (\delta(\varepsilon^+ u))^{n-i}\right] \\
&= \sum_{i=0}^n \binom{n}{i} \sum_{c=0}^{n-i} (-1)^c \binom{n-i}{c} \sum_{k=0}^{n-i-c} \\
&\quad \sum_{\substack{l_1+\cdots+l_k=n-i-c \\ l_1,\ldots,l_k\geq 1}} \mathcal{N}_{\mathfrak{L}_k} E\left[\int_{X^k} \varepsilon_{\mathfrak{s}_k}^+ \left(F\left(\int_X \varepsilon_x^+ u_x \sigma(dx)\right)^i \left(\int_X \varepsilon_x^+ u_x \sigma(dx)\right)^c\right) \prod_{p=1}^k \varepsilon_{\mathfrak{s}_k \setminus \{s_p\}}^+ \varepsilon_{s_p}^+ u_{s_p}^{l_p} \, d\sigma^k(\mathfrak{s}_k)\right] \\
&= \sum_{a=0}^n \sum_{i=0}^a \binom{n}{i} (-1)^{a-i} \binom{n-i}{a-i} \sum_{k=0}^{n-a} \\
&\quad \sum_{\substack{l_1+\cdots+l_k=n-a \\ l_1,\ldots,l_k\geq 1}} \mathcal{N}_{\mathfrak{L}_k} E\left[\int_{X^k} \varepsilon_{\mathfrak{s}_k}^+ F\left(\int_X \varepsilon_x^+ u_x \sigma(dx)\right)^a \prod_{p=1}^k \varepsilon_{\mathfrak{s}_k}^+ u_{s_p}^{l_p} \, d\sigma^k(\mathfrak{s}_k)\right] \\
&= \sum_{a=0}^n \binom{n}{a} \sum_{k=0}^{n-a} \sum_{i=0}^a (-1)^{a-i} \binom{a}{i} \\
&\quad \sum_{\substack{l_1+\cdots+l_k=n-a \\ l_1,\ldots,l_k\geq 1}} \mathcal{N}_{\mathfrak{L}_k} E\left[\int_{X^k} \varepsilon_{\mathfrak{s}_k}^+ \left(F\left(\int_X \varepsilon_x^+ u_x \sigma(dx)\right)^a\right) \varepsilon_{\mathfrak{s}_k}^+ \prod_{p=1}^k u_{s_p}^{l_p} \, d\sigma^k(\mathfrak{s}_k)\right] \\
&= \sum_{k=0}^n \sum_{\substack{l_1+\cdots+l_k=n \\ l_1,\ldots,l_k\geq 1}} \mathcal{N}_{\mathfrak{L}_k} E\left[\int_{X^k} \varepsilon_{\mathfrak{s}_k}^+ \left(F \prod_{p=1}^k u_{s_p}^{l_p}\right) d\sigma^k(\mathfrak{s}_k)\right] \qquad (3.2) \\
&= \sum_{\{P_1,\ldots,P_k\}\in\mathcal{P}_n} E\left[\int_{X^k} \varepsilon_{\mathfrak{s}_k}^+ (F u_{s_1}^{|P_1|} \cdots u_{s_k}^{|P_k|}) \, \sigma(ds_1)\cdots\sigma(ds_k)\right],
\end{aligned}
$$

since $\displaystyle\sum_{i=0}^a (-1)^{a-i} \binom{a}{i} = \mathbf{1}_{\{a=0\}}$ under the convention $0^0 = 1$. □

Note that the structure of (3.1) does not apply to extend the cumulant formula $\kappa_n = \int_X u^n(x)\sigma(dx)$ to the case where u becomes a random process.

When $f : X \longrightarrow \mathbb{R}$ is a deterministic function, Proposition 3.1 yields

$$E\left[F\left(\int_X f(x)\omega(dx)\right)^n\right] = \sum_{\{P_1,\ldots,P_k\}\in\mathcal{P}_n} \int_{X^k} f^{|P_1|}(s_1)\cdots f^{|P_k|}(s_k) E\left[\varepsilon^+_{\mathfrak{s}_k} F\right] \sigma(ds_1)\cdots\sigma(ds_k), \tag{3.3}$$

which recovers (1.1) by taking $F = 1$. Furthermore,

$$\begin{aligned} &\mathrm{Cov}\left(F, \left(\int_X f(x)\omega(dx)\right)^n\right) \\ &= \sum_{\{P_1,\ldots,P_k\}\in\mathcal{P}_n} \int_{X^k} f^{|P_1|}(s_1)\cdots f^{|P_k|}(s_k) E\left[\varepsilon^+_{\mathfrak{s}_k} F - F\right] \sigma(ds_1)\cdots\sigma(ds_k). \end{aligned}$$

By (3.2), Proposition 3.1 also rewrites for compensated integrals as follows.

Proposition 3.2 *Let $F : \Omega^X \longrightarrow \mathbb{R}$ be a bounded random variable, and let $u : \Omega^X \times X \longrightarrow \mathbb{R}$ be a bounded random process with compact support in X. For all $n \geq 0$ we have*

$$\begin{aligned} E\left[F\left(\int_X u_x(\omega)(\omega(dx)-\sigma(dx))\right)^n\right] &= \sum_{c=0}^n (-1)^c \binom{n}{c} \\ \sum_{\{P_1,\ldots,P_a\}\in\mathcal{P}_{n-c}} E&\left[\int_{X^a} \varepsilon^+_{\mathfrak{s}_a}\left(F\left(\int_X u_x(\omega)\sigma(dx)\right)^c u^{|P_1|}_{s_1}\cdots u^{|P_a|}_{s_a}\right) d\sigma^a(\mathfrak{s}_a)\right]. \end{aligned} \tag{3.4}$$

Proof. By (3.2) we have

$$\begin{aligned} &E\left[F\left(\int_X u_x(\omega)(\omega(dx)-\sigma(dx))\right)^n\right] \\ &= \sum_{c=0}^n (-1)^c \binom{n}{c} E\left[F\left(\int_X u_x(\omega)\omega(dx)\right)^{n-c}\left(\int_X u_x(\omega)\sigma(dx)\right)^c\right] \\ &= \sum_{c=0}^n (-1)^c \binom{n}{c} \sum_{a=0}^{n-c} \sum_{\substack{l_1+\cdots+l_a=n-c \\ l_1,\ldots,l_a\geq 1 \\ l_{a+1}=1,\ldots,l_{a+c}=1}} \mathcal{N}_{\mathfrak{L}_a} E\left[\int_{X^{a+c}} \varepsilon^+_{\mathfrak{s}_a}\left(F\prod_{p=1}^{a+c} u^{l_p}_{s_p}\right) d\sigma^{a+c}(\mathfrak{s}_{a+c})\right] \\ &= \sum_{c=0}^n (-1)^c \binom{n}{c} \\ &\sum_{\{P_1,\ldots,P_a\}\in\mathcal{P}_{n-c}} E\left[\int_{X^a} \varepsilon^+_{\mathfrak{s}_a}\left(F\left(\int_X u_x(\omega)\sigma(dx)\right)^c u^{|P_1|}_{s_1}\cdots u^{|P_a|}_{s_a}\right) d\sigma^a(\mathfrak{s}_a)\right]. \end{aligned}$$

□

The next proposition specializes the above result to the case of deterministic integrands. We note that it can also be obtained independently as in (1.2) from the relation between the moments and the cumulants $\kappa_1 = 0$, $\kappa_n = \int_X u^n(x)\sigma(dx)$, $n \geq 2$, of $\displaystyle\int_X u(x)(\omega(dx) - \sigma(dx))$.

Proposition 3.3 *Let $f : X \longrightarrow \mathbb{R}$ be a bounded deterministic function with compact support on X. For all $n \geq 1$ we have*

$$E\left[\left(\int_X f(x)(\omega(dx) - \sigma(dx))\right)^n\right] = \sum_{\substack{\{P_1,\ldots,P_k\}\in\mathcal{P}_n \\ |P_1|\geq 2,\ldots,|P_k|\geq 2}} \int_X f^{|P_1|}(x)\sigma(dx)\cdots\int_X f^{|P_k|}(x)\sigma(dx),$$

where the sum runs over all partitions $P_1 \cup \cdots \cup P_k$ of $\{1,\ldots,n\}$ of size at least 2.

Proof. By (3.3) with $F = 1$ and using binomial inversion [3] and the definition (2.2) of $\mathcal{N}_{\mathfrak{L}_k}$, we have

$$\begin{aligned}
&E\left[\left(\int_X f(x)\omega(dx) - \int_X f(x)\sigma(dx)\right)^n\right] \\
&= \sum_{c=0}^{n}(-1)^c\binom{n}{c}\left(\int_X f(x)\sigma(dx)\right)^c E\left[\left(\int_X f(x)\omega(dx)\right)^{n-c}\right] \\
&= \sum_{c=0}^{n}(-1)^c\binom{n}{c}\left(\int_X f(x)\sigma(dx)\right)^c \sum_{\substack{l_1+\cdots+l_a=n-c \\ l_1,\ldots,l_a\geq 1}} \mathcal{N}_{\mathfrak{L}_a}\int_{X^a}\prod_{p=1}^{a} f^{l_p}(s_p)\,d\sigma^a(\mathfrak{s}_a) \\
&= \sum_{c=0}^{n}(-1)^c\binom{n}{c}\sum_{k=0}^{n-c}\binom{n-c}{k}\left(\int_X f(x)\sigma(dx)\right)^{k+c} \sum_{\substack{l_1+\cdots+l_a=n-c-k \\ l_1,\ldots,l_a\geq 2}} \mathcal{N}_{\mathfrak{L}_a}\int_{X^a}\prod_{p=1}^{a} f^{l_p}(s_p)\,d\sigma^a(\mathfrak{s}_a) \\
&= \sum_{b=0}^{n}\sum_{c=0}^{b}(-1)^c\binom{n}{c}\binom{n-c}{b-c}\left(\int_X f(x)\sigma(dx)\right)^b \sum_{\substack{l_1+\cdots+l_a=n-b \\ l_1,\ldots,l_a\geq 2}} \mathcal{N}_{\mathfrak{L}_a}\int_{X^a}\prod_{p=1}^{a} f^{l_p}(s_p)\,d\sigma^a(\mathfrak{s}_a) \\
&= \sum_{b=0}^{n}\binom{n}{b}\sum_{c=0}^{b}(-1)^c\binom{b}{c}\left(\int_X f(x)\sigma(dx)\right)^b \sum_{\substack{l_1+\cdots+l_a=n-b \\ l_1,\ldots,l_a\geq 2}} \mathcal{N}_{\mathfrak{L}_a}\int_{X^a}\prod_{p=1}^{a} f^{l_p}(s_p)\,d\sigma^a(\mathfrak{s}_a) \\
&= \sum_{\substack{l_1+\cdots+l_a=n \\ l_1,\ldots,l_a\geq 2}} \mathcal{N}_{\mathfrak{L}_a}\int_{X^a}\prod_{p=1}^{a} f^{l_p}(s_p)\,d\sigma^a(\mathfrak{s}_a) \\
&= \sum_{\substack{\{P_1,\ldots,P_k\}\in\mathcal{P}_n \\ |P_1|\geq 2,\ldots,|P_k|\geq 2}} \int_X f^{|P_1|}(x)\sigma(dx)\cdots\int_X f^{|P_k|}(x)\sigma(dx).
\end{aligned}$$

□

4 Indicator functions and polynomials

When $u(x) = \mathbf{1}_A(x)$ is a deterministic indicator function of $A \in \mathcal{B}(X)$ and $\sigma(A) < \infty$, then

$$Z := \int_X u(x)\omega(dx) = \int_X \mathbf{1}_A(x)\omega(dx) = \omega(A)$$

is a Poisson random variable with intensity $\lambda = \sigma(A)$, and Proposition 3.1 yields the following corollary.

Corollary 4.1 *Let $F : \Omega^X \longrightarrow \mathbb{R}$ be a bounded random variable. We have*

$$E[FZ^n] = \sum_{k=0}^{n} S(n,k) \int_{A^k} E\left[\varepsilon^+_{\mathfrak{s}_k} F\right] \sigma(ds_1)\cdots\sigma(ds_k), \qquad n \in \mathbb{N}. \tag{4.1}$$

where $S(n,k)$ denotes the Stirling number of the second kind, i.e. the number of ways to partition a set of n objects into k non-empty subsets.

Proof. By (3.3) and the definition (2.2) of $\mathcal{N}_{\mathfrak{L}_k}$, we have

$$\begin{aligned} E[FZ^n] &= \sum_{\{P_1,\ldots,P_k\}\in\mathcal{P}_n} E\left[\int_{A^k} \varepsilon^+_{\mathfrak{s}_k} F\, \sigma(ds_1)\cdots\sigma(ds_k)\right] \\ &= \sum_{k=0}^{n} \sum_{\substack{l_1+\cdots+l_k=n \\ l_1,\ldots,l_k\geq 1}} \mathcal{N}_{\mathfrak{L}_k} \int_{A^k} E\left[\varepsilon^+_{\mathfrak{s}_k} F\right] \sigma(ds_1)\cdots\sigma(ds_k), \end{aligned}$$

and it remains to note that

$$S(n,k) = \sum_{\substack{l_1+\cdots+l_k=n \\ l_1,\ldots,l_k\geq 1}} \mathcal{N}_{l_1,\ldots,l_k}, \qquad 0 \leq k \leq n.$$

$\square$

As a consequence of (4.1) we find

$$\mathrm{Cov}(F, Z^n) = \sum_{k=0}^{n} S(n,k) \int_{A^k} E\left[\varepsilon^+_{\mathfrak{s}_k} F - F\right] \sigma(ds_1)\cdots\sigma(ds_k), \qquad n \in \mathbb{N},$$

and

$$E[Fe^{tZ}] = \sum_{k=0}^{\infty} \frac{1}{k!}(e^t-1)^k \int_{A^k} E\left[\varepsilon^+_{\mathfrak{s}_k} F\right] \sigma(ds_1)\cdots\sigma(ds_k), \tag{4.2}$$

using, e.g., (3) on page 2 of [3]. Relation (4.2) also recovers the decomposition of the Fourier transform (also called $\mathcal{U}$-transform) on the Poisson space, cf. e.g. Proposition 3.2 of [5].

When F has the form $F = f(Z)$ with $f : \mathbb{N} \longrightarrow \mathbb{R}$, (4.1) also yields the following extended Chen-Stein identity

$$E\left[Z^n f(Z)\right] = \sum_{k=0}^{n} \lambda^k S(n,k) E[f(Z+k)],$$

(see e.g. Lemma 3.3.3 of [8]), and (4.2) yields

$$E[f(Z)e^{tZ}] = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!}(e^t-1)^k E\left[f(Z+k)\right].$$

In particular, in case $F = 1$, (4.1) corresponds to the classical relation

$$E[Z^n] = B_n(\lambda), \qquad n \in \mathbb{N}, \tag{4.3}$$

between the moments of a Poisson random variable Z with intensity $\lambda > 0$ and the Bell polynomials

$$B_n(\lambda) = A_n(\lambda,\ldots,\lambda) = \sum_{\{P_1,\ldots,P_k\}\in\mathcal{P}_n} \lambda^k = \sum_{k=0}^{n} \lambda^k S(n,k), \qquad n \in \mathbb{N}, \tag{4.4}$$

of order n, cf. e.g. Proposition 3.3.2 of [8] and references therein.

The comparison of (1.1) and (4.4) yields the relation

$$S(n,k) = n! \sum_{\substack{r_1+2r_2+\cdots+nr_n=n\\ r_1+r_2+\cdots+r_n=k\\ r_1,\ldots,r_n\geq 0}} \prod_{k=1}^{n} \frac{1}{(k!)^{r_k} r_k!} = n! \sum_{\substack{r_1+2r_2+\cdots+(n-k+1)r_{n-k+1}=n\\ r_1+r_2+\cdots+r_{n-k+1}=k\\ r_1,\ldots,r_n\geq 0}} \prod_{k=1}^{n} \frac{1}{(k!)^{r_k} r_k!},$$

cf. e.g. Proposition 2.3.4 of [8].

Similarly, Proposition 3.3 applied to $u(x) = \mathbf{1}_A(x)$ recovers the fact that the centered moments of a Poisson random variable can be written as

$$E[(Z-\lambda)^n] = \sum_{k=0}^{n} \lambda^k S_2(n,k), \qquad n \in \mathbb{N},$$

using the 2-associated Stirling numbers $S_2(n,k)$ of the second kind, which count the partitions of a set of size n into k non-singleton subsets, cf. [12] and Proposition 3.3.6 of [8].

In addition we can check that if $\lambda := \sigma(X) < \infty$, and taking $u_s(\omega) = \omega(X)$, $s \in X$, (1.4) recovers (4.3), i.e.

$$\begin{aligned}
E\left[(\omega(X))^6\right] &= E\left[\left(\int_X u_s(\omega)\omega(ds)\right)^3\right] \\
&= E\left[\int_X \varepsilon_s^+ (\omega(X))^2 \sigma(ds)\right] + E\left[\int_{X^2} \varepsilon_{s_1}^+ \varepsilon_{s_2}^+ (\omega(X))^2 \sigma(ds_1)\sigma(ds_2)\right] \\
&= \lambda E\left[(\omega(X)+1)^2\right] + \lambda^2 E\left[(\omega(X)+2)^2\right] \\
&= \lambda + 7\lambda^2 + 6\lambda^3 + \lambda^4 \\
&= B_4(\lambda).
\end{aligned}$$

Similarly, (1.5) yields

$$\begin{aligned}
& E\left[(\omega(X))^6\right] = E\left[\left(\int_X u_s(\omega)\omega(ds)\right)^3\right] \\
&= E\left[\int_X \varepsilon_s^+ (\omega(X))^3 \sigma(ds)\right] + 3E\left[\int_{X^2} \varepsilon_{s_1}^+ \varepsilon_{s_2}^+ ((\omega(X))^2 \omega(X)) \sigma(ds_1)\sigma(ds_2)\right] \\
&\quad + E\left[\int_{X^3} \varepsilon_{s_1}^+ \varepsilon_{s_2}^+ \varepsilon_{s_3}^+ (\omega(X))^3 \sigma(ds_1)\sigma(ds_2)\sigma(ds_3)\right] \\
&= \lambda E\left[(\omega(X)+1)^3\right] + 3\lambda^2 E\left[(\omega(X)+2)^2(\omega(X)+2)\right] \\
&\quad + \lambda^3 E\left[(\omega(X)+3)^3\right] \\
&= \lambda + 31\lambda^2 + 90\lambda^3 + 65\lambda^4 + 15\lambda^5 + \lambda^6 \\
&= B_6(\lambda),
\end{aligned}$$

where B_6 is the Bell polynomial of order 6.

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References

[1] Ph. Barbe and W.P. McCormick. Ruin probabilities in tough times - Part 1 - Heavy-traffic approximation for fractionally integrated random walks in the domain of attraction of a non-Gaussian stable distribution. Preprint arXiv:1101.4437, 2011.

[2] B. Bassan and E. Bona. Moments of stochastic processes governed by Poisson random measures. *Comment. Math. Univ. Carolin.*, 31(2):337–343, 1990.

[3] M. Bernstein and N. J. A. Sloane. Some canonical sequences of integers. *Linear Algebra Appl.*, 226/228:57–72, 1995.

[4] H. Biermé, Y. Demichel, and A. Estrade. Fractional Poisson field on a finite set. Preprint hal-00597722, 2011.

[5] Y. Ito. Generalized Poisson functionals. *Probab. Theory Related Fields*, 77:1–28, 1988.

[6] E. Lukacs. *Characteristic functions*. Hafner Publishing Co., New York, 1970. Second edition, revised and enlarged.

[7] J. Mecke. Stationäre zufällige Masse auf lokalkompakten Abelschen Gruppen. *Z. Wahrscheinlichkeitstheorie Verw. Geb.*, 9:36–58, 1967.

[8] G. Peccati and M. Taqqu. *Wiener Chaos: Moments, Cumulants and Diagrams: A survey with Computer Implementation*. Bocconi & Springer Series. Springer, 2011.

[9] J. Picard. Formules de dualité sur l'espace de Poisson. *Ann. Inst. H. Poincaré Probab. Statist.*, 32(4):509–548, 1996.

[10] N. Privault. Moment identities for Poisson-Skorohod integrals and application to measure invariance. *C. R. Math. Acad. Sci. Paris*, 347:1071–1074, 2009.

[11] N. Privault. *Stochastic Analysis in Discrete and Continuous Settings*, volume 1982 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 309 pp., 2009.

[12] N. Privault. Generalized Bell polynomials and the combinatorics of Poisson central moments. *Electron. J. Combin.*, 18(1):Research Paper 54, 10, 2011.

[13] N. Privault. Cumulant operators for Lie-Wiener-Itô-Poisson stochastic integrals. Preprint, 2012.

[14] N. Privault. Invariance of Poisson measures under random transformations. Preprint arXiv:1004.2588v3, to appear in Ann. Inst. H. Poincaré Probab. Statist., 2012.