Mixing of Poisson random measures under interacting transformations

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Abstract

We derive sufficient conditions for the mixing of all orders of interacting transformations of a spatial Poisson point process, under a zero-type condition in probability and a generalized adaptedness condition. This extends a classical result in the case of deterministic transformations of Poisson measures. The approach relies on moment and covariance identities for Poisson stochastic integrals with random integrands.

Key words: Poisson random measures; interacting transformations; mixing; ergodicity; Malliavin calculus.
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1 Introduction

The ergodicity and mixing properties of Poisson random measures under deterministic transformations have been considered by several authors, cf. e.g. [7], [5], [14]. This paper investigates mixing beyond the deterministic case by considering interacting, i.e. configuration dependent, transformations of Poisson samples.

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Consider a σ-compact metric space X with Borel σ-algebra $\mathcal{B}(X)$ and let Ω denote the configuration space on $(X, \mathcal{B}(X))$, i.e.

$$\Omega = \left\{ \omega = (x_i)_{i=1}^{N} \subset X, \ x_i \neq x_j \ \forall i \neq j, \ N \in \mathbb{N} \cup \{\infty\} \right\},$$

is the space of at most countable subsets of X, whose elements $\omega \in \Omega$ are identified to the point measures

$$\omega(dy) = \sum_{x \in \omega} \delta_x(dy), \tag{1.1}$$

where δ_x denotes the Dirac measure at $x \in X$. The space Ω is endowed with the Poisson probability measure π_σ with σ-finite diffuse intensity $\sigma(dx)$ on X and its associated σ-algebra $\mathcal{F}$ generated by $\omega \mapsto \omega(A)$ for $A \in \mathcal{B}(X)$ such that $\sigma(A) < \infty$. In particular, $\pi_\sigma(d\omega)$-almost surely, $\omega \in \Omega$ is locally finite on compact sets and (1.1) is a Radon measure.

Given a measurable random transformation

$$\tau : X \times \Omega \longrightarrow X,$$

of X and an element ω of Ω of the form (1.1), let $\tau_*(\omega)$ denote the transformation of $\omega \in \Omega$ by $\tau(\cdot, \omega) : X \longrightarrow X$, i.e.

$$\tau_*(\omega) := \sum_{x \in \omega} \delta_{\tau(x,\omega)}, \qquad \omega \in \Omega, \tag{1.2}$$

is the image measure of $\omega(dy)$ by $\tau(\cdot, \omega) : X \longrightarrow X$. In other words, the transformation

$$\tau_* : \Omega \longrightarrow \Omega \tag{1.3}$$

shifts every configuration point $x \in \omega$ according to $x \longmapsto \tau(x, \omega)$, and in the deterministic case τ_* is also called the Poisson suspension over $\tau : X \longrightarrow X$, cf. § 9.1 of [2].

In Theorem 4.8 of [14] it is shown, using the moment generating function of Poisson random measures, that a conservative deterministic dynamical system $(\Omega, \pi_\sigma, \sigma, \tau)$

where $\tau : X \longrightarrow X$ leaves σ invariant is mixing of all orders if and only if $\tau : X \longrightarrow X$ is of zero type, i.e.

$$\lim_{n\to\infty} \langle h, h \circ \tau^n \rangle_{L^2_\sigma(X)} = 0,$$

for all $h \in L^2_\sigma(X)$, cf. also § 14.2 of [2] for the Gaussian case.

In Theorem 3.1 below we show that an interacting transformation $\tau(\cdot, \omega) : X \longrightarrow X$ leaving σ invariant $\pi_\sigma(d\omega)$-a.s. is mixing of all orders provided the family of transformations $\tau^{(n)} : X \times \Omega \longrightarrow \Omega$, $n \in \mathbb{N}$, inductively defined by $\tau^{(0)}(x, \omega) := x$ and

$$\tau^{(n)}(x, \omega) := \tau^{(n-1)}(\tau(x, \omega), \tau_* \omega), \qquad n \geq 1, \tag{1.4}$$

$\omega \in \Omega$, $x \in X$, satisfies the zero-type condition

$$\lim_{n\to\infty} \langle g, h \circ \tau^{(n)} \rangle_{L^2_\sigma(X)} = 0 \tag{1.5}$$

in probability for all $g, h \in \mathcal{C}_c(X)$, as well as the vanishing gradient condition (3.1) below that plays the role of an adaptedness condition in the absence of time ordering.

When $\tau : X \longrightarrow X$ is deterministic, Condition (3.1) below is always satisfied and we have

$$\tau^{(n)}(x, \omega) = \tau^n(x), \qquad \omega \in \Omega, \quad x \in X, \quad n \geq 1,$$

hence Theorem 3.1 recovers the classical mixing conditions on the Poisson space as it suffices to state Condition (1.5) for $g = h$, in which case it becomes equivalent to the deterministic zero-type condition

$$\lim_{n\to\infty} \langle h, h \circ \tau^n \rangle_{L^2_\sigma(X)} = 0, \qquad h \in \mathcal{C}_c(X).$$

Our proof uses joint moments identities for Poisson stochastic integrals with random integrands, cf. [13], and [4] for an extension to point processes.

Related arguments have been previously applied on the Wiener space using the Skorohod integral, cf. [10], [15], [16].

This paper is organized as follows. In Section 2 we state and recall some preliminary results on invariance of Poisson random measures and joint moment identities for Poisson stochastic integrals. In Section 3 we present and prove our main result on the mixing property of interacting transformations. In Section 4 we consider a family of examples based on transformations conditioned by the random boundary of a convex Poisson hull. The invariance of such transformations with respect to the Poisson measure is consistent with the intuitive fact that the distribution of the inside points remains Poisson when they are shifted within its convex hull according to the data of the vertices, cf. [3].

2 Invariance and joint moment identities

In this section we recall some preliminary results on invariance of Poisson random measures under interacting transformations, and we derive joint moment identities for the Poisson stochastic integral $\int_X u(x,\omega)\omega(dx)$ of a random integrand $u : X \times \Omega \longrightarrow \mathbb{R}$.

Invariance of Poisson random measures

Let now D_x, $x \in X$, denote the finite difference gradient defined for all $\omega \in \Omega$ and $x \in X$ as

$$D_x F(\omega) = F(\omega \cup \{x\}) - F(\omega),$$

for any random variable $F : \Omega \longrightarrow \mathbb{R}$, cf. e.g. Theorem 6.5 page 21 of [6]. Given $\Theta = \{x_{k_1}, \ldots, x_{k_l}\} \subset \{x_1, \ldots, x_n\}$ and $u : X^n \times \Omega \longrightarrow \mathbb{R}$, we define the iterated gradient

$$D_\Theta u(x_1, \ldots, x_n, \omega) := D_{x_{k_1}} \cdots D_{x_{k_l}} u(x_1, \ldots, x_n, \omega), \qquad x_1, \ldots, x_n \in X. \tag{2.1}$$

Recall that by Theorem 3.3 of [12] or [11], or Theorem 5.2 of [1], $\tau_* : \Omega \longrightarrow \Omega$ leaves π_σ invariant, i.e. $\tau_* \pi_\sigma = \pi_\sigma$, provided that for π_σ-a.s. $\omega \in \Omega$ the random transformation $\tau(\cdot, \omega) : X \longrightarrow X$ leaves $\sigma(dx)$ invariant and satisfies the vanishing condition

$$D_{\Theta_1} \tau(x_1, \omega) \cdots D_{\Theta_m} \tau(x_m, \omega) = 0, \tag{2.2}$$

for every family $\{\Theta_1,\ldots,\Theta_m\}$ of (non empty) subsets such that $\Theta_1\cup\cdots\cup\Theta_m = \{x_1,\ldots,x_m\}$, for all $x_1,\ldots,x_m\in X$, $\pi_\sigma(d\omega)-a.s.$, $m\geq 1$.

Condition (2.2) is interpreted by saying that for $\omega\in\Omega$ and $x_1,\ldots,x_m\in X$ there exists $l\in\{1,\ldots,m\}$ such that

$$D_{x_l}\tau(x_{(l+1)\bmod m},\omega)=0,\quad i.e.\quad \tau(x_{(l+1)\bmod m},\omega\cup\{x_l\})=\tau(x_{(l+1)\bmod m},\omega),\quad (2.3)$$

where $(l \mod m)=l$, $1\leq l\leq m$, and $(m+1 \mod m)=1$, i.e. the m-tuples

$$(\tau(x_2,\omega\cup\{x_1\}),\tau(x_3,\omega\cup\{x_2\}),\ldots,\tau(x_m,\omega\cup\{x_{m-1}\}),\tau(x_1,\omega\cup\{x_m\}))$$

and $(\tau(x_2,\omega),\tau(x_3,\omega),\ldots,\tau(x_m,\omega),\tau(x_1,\omega))$ coincide on at least one component in X^m, cf. page 1074 of [9]. When $m=1$, Condition (2.2) reads $D_x\tau(x,\omega)=0$, i.e. $\tau(x,\omega\cup\{x\})=\tau(x,\omega)$, $x\in X$, $\pi_\sigma(d\omega)$-a.s.

Condition (2.2) is known to hold when $\tau: X\times\Omega\longrightarrow X$ is predictable with respect to a total binary relation $\preceq$ on X, which is the case in particular when X is of the form $X=\mathbb{R}_+\times Z$ and $\tau: X\times\Omega\longrightarrow X$ is predictable with respect to the canonical filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ generated on $X=\mathbb{R}_+\times Z$, cf. Section 4 of [12].

Joint moment identities

For any random variable $F:\Omega\longrightarrow\mathbb{R}$, we let $\varepsilon^+_{x_1,\ldots,x_k}$ denote the addition operator defined as

$$\varepsilon^+_{x_1,\ldots,x_k}F(\omega):=F(\omega\cup\{x_1,\ldots,x_k\}),\qquad \omega\in\Omega,\quad x_1,\ldots,x_k\in X.$$

Next, given $u: X\times\Omega\longrightarrow X$ a measurable process, we define the Poisson stochastic integral of u as

$$\int_X u(x,\omega)\omega(dx)=\sum_{x\in\omega}u(x,\omega),$$

provided the sum converges absolutely, $\pi_\sigma(d\omega)$-a.s. In the next proposition we extend Proposition 3.1 of [13] to a joint moment identity using an induction argument.

Proposition 2.1 *Let $u : X \times \Omega \longrightarrow X$ be a measurable process and $n = n_1 + \cdots + n_p$, $p \geq 1$. We have*

$$
\begin{aligned}
& E\left[\left(\int_X u_1(x,\omega)\omega(dx)\right)^{n_1} \cdots \left(\int_X u_p(x,\omega)\omega(dx)\right)^{n_p}\right] \qquad (2.4)\\
&= \sum_{k=1}^{n} \sum_{P_1^n,\ldots,P_k^n} E\left[\int_{X^k} \varepsilon^+_{x_1,\ldots,x_k}\left(\prod_{j=1}^{k}\prod_{i=1}^{p} u_i^{l^n_{i,j}}(x_j,\omega)\right)\sigma(dx_1)\cdots\sigma(dx_k)\right],
\end{aligned}
$$

where the sum runs over all partitions $P_1^n, \ldots, P_k^n$ of $\{1,\ldots,n\}$ and the power $l^n_{i,j}$ is the cardinal

$$
l^n_{i,j} := |P^n_j \cap (n_1 + \cdots + n_{i-1}, n_1 + \cdots + n_i]|, \qquad i = 1,\ldots,k, \quad j = 1,\ldots,p,
$$

for any $n \geq 1$ such that all terms in the right hand side of (2.4) are integrable.

Proof. We will show the modified identity

$$
\begin{aligned}
& E\left[F\left(\int_X u_1(x,\omega)\omega(dx)\right)^{n_1} \cdots \left(\int_X u_p(x,\omega)\omega(dx)\right)^{n_p}\right] \qquad (2.5)\\
&= \sum_{k=1}^{n} \sum_{P_1^n,\ldots,P_k^n} E\left[\int_{X^k} \varepsilon^+_{x_1,\ldots,x_k}\left(F\prod_{j=1}^{k}\prod_{i=1}^{p} u_i^{l^n_{i,j}}(x_j,\omega)\right)\sigma(dx_1)\cdots\sigma(dx_k)\right],
\end{aligned}
$$

for F a sufficiently integrable random variable, where $n = n_1 + \cdots + n_p$. For $p = 1$ the identity is Proposition 3.1 of [13]. Next we assume that the identity holds at the rank $p \geq 1$. Replacing F with $F\left(\int_X u_{p+1}(x,\omega)\omega(dx)\right)^{n_{p+1}}$ in (2.5) we get

$$
\begin{aligned}
& E\left[F\left(\int_X u_1(x,\omega)\omega(dx)\right)^{n_1} \cdots \left(\int_X u_{p+1}(x,\omega)\omega(dx)\right)^{n_{p+1}}\right]\\
&= \sum_{k=1}^{n} \sum_{P_1^n,\ldots,P_k^n} \int_{X^k} \sigma(dx_1)\cdots\sigma(dx_k)\\
&\quad E\left[\varepsilon^+_{x_1,\ldots,x_k}\left(F\left(\int_X u_{p+1}(x,\omega)\omega(dx)\right)^{n_{p+1}}\prod_{j=1}^{k}\prod_{i=1}^{p} u_i^{l^n_{i,j}}(x_j,\omega)\right)\right]\\
&= \sum_{k=1}^{n} \sum_{P_1^n,\ldots,P_k^n} \int_{X^k} E\Bigg[\left(\int_X \varepsilon^+_{x_1,\ldots,x_k} u_{p+1}(x,\omega)\omega(dx) + \sum_{i=1}^{k} \varepsilon^+_{x_1,\ldots,x_k} u_{p+1}(x_i,\omega)\right)^{n_{p+1}}\\
&\quad \varepsilon^+_{x_1,\ldots,x_k}\left(F\prod_{j=1}^{k}\prod_{i=1}^{p} u_i^{l^n_{i,j}}(x_j,\omega)\right)\Bigg]\sigma(dx_1)\cdots\sigma(dx_k)
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{k=1}^{n} \sum_{P_1^n,\dots,P_k^n} \sum_{a_0+\cdots+a_k=n_{p+1}} \frac{n_{p+1}!}{a_0!\cdots a_k!} \int_{X^k} E\Bigg[\left(\int_X \varepsilon^+_{x_1,\dots,x_k} u_{p+1}(x,\omega)\omega(dx) \right)^{a_0} \\
&\quad \varepsilon^+_{x_1,\dots,x_k} \left(F \prod_{j=1}^{k} \left(u_{p+1}^{a_j}(x_j,\omega) \prod_{i=1}^{p} u_i^{l_{i,j}^n}(x_j,\omega) \right) \right) \Bigg] \sigma(dx_1)\cdots\sigma(dx_k) \\
&= \sum_{k=1}^{n} \sum_{P_1^n,\dots,P_k^n} \sum_{a_0+\cdots+a_k=n_{p+1}} \frac{n_{p+1}!}{a_0!\cdots a_k!} \sum_{j=1}^{a_0} \int_{X^{k+a_0}} E\Bigg[\sum_{Q_j^{a_0},\dots,Q_j^{a_0}} \\
&\quad \varepsilon^+_{x_1,\dots,x_{k+a_0}} \left(F \prod_{q=k+1}^{k+a_0} u_{p+1}^{|Q_q^{a_0}|}(x_q,\omega) \prod_{j=1}^{k} \left(u_{p+1}^{a_j}(x_j,\omega) \prod_{i=1}^{p} u_i^{l_{i,j}^n}(x_j,\omega) \right) \right) \Bigg] \sigma(dx_1)\cdots\sigma(dx_{k+a_0}) \\
&= \sum_{k=1}^{n+n_{p+1}} \sum_{P_1^{n+n_{p+1}},\dots,P_k^{n+n_{p+1}}} E\left[\int_{X^k} \varepsilon^+_{x_1,\dots,x_k} \left(F \prod_{l=1}^{k} \prod_{i=1}^{p+1} u_i^{l_{i,j}^{n+n_{p+1}}}(x_l,\omega) \right) \sigma(dx_1)\cdots\sigma(dx_k) \right],
\end{aligned}
$$

where the summation over the partitions $P_1^{n+n_{p+1}},\dots,P_k^{n+n_{p+1}}$ of $\{1,\dots,n+n_{p+1}\}$, is obtained by combining the partitions of $\{1,\dots,n\}$ with the partitions $Q_j^{a_0},\dots,Q_j^{a_0}$ of $\{1,\dots,a_0\}$ and $a_1,\dots,a_k$ elements of $\{1,\dots,n_{p+1}\}$ which are counted according to $n_{p+1}!/(a_0!\cdots a_k!)$, with

$$
l_{p+1,j}^{n+n_{p+1}} = l_{i,j}^n + a_j, \quad 1 \le j \le k, \qquad l_{p+1,j}^{n+n_{p+1}} = l_{i,j}^n + |Q_q^{a_0}|, \quad k+1 \le j \le k+a_0,.
$$

□

Note that when $n=1$, (2.4) coincides with the classical Mecke [8] identity

$$
E\left[\int_X u(x,\omega)\omega(dx) \right] = E\left[\int_X \varepsilon_x^+ u(x,\omega)\sigma(dx) \right]. \tag{2.6}
$$

3 Mixing of interacting transformations

Theorem 3.1 is the main result of this paper. The vanishing condition (3.1) below is stated in the sense of (2.3) above.

Theorem 3.1 *Assume that $\tau(\cdot,\omega) : X \longrightarrow X$ leaves $\sigma(dx)$ invariant for π_σ-a.s. $\omega \in \Omega$, and*

$$
D_{\Theta_1}\tau^{(k_1)}(x_1,\omega)\cdots D_{\Theta_m}\tau^{(k_m)}(x_m,\omega) = 0, \tag{3.1}
$$

for every family $\{\Theta_1,\dots,\Theta_m\}$ of (non empty) subsets such that $\Theta_1\cup\cdots\cup\Theta_m = \{x_1,\dots,x_m\}$, $x_1,\dots,x_m \in X$ and all $\pi_\sigma(d\omega)$-a.s., $k_1,\dots,k_m \ge 1$, $m \ge 1$. Then *the measure-preserving transformation $\tau_* : \Omega \longrightarrow \Omega$ is mixing of all orders $m \geq 1$ provided the zero-type condition*

$$\lim_{n\to\infty} \langle g, h \circ \tau^{(n)} \rangle = 0 \tag{3.2}$$

is satisfied in probability for all $g, h \in \mathcal{C}_c(X)$.

Proof. Let $k_{i,n} := p_{1,n} + \cdots + p_{i,n}$, $i = 1, \ldots, m$, where $(p_{1,n})_{n\geq 1}, \ldots, (p_{m,n})_{n\geq 1}$ is a family of m strictly increasing sequences of integers. In order to prove mixing of order m we need to show that for all $h_1, \ldots, h_m \in \mathcal{C}_c^+(X)$ nonnegative continuous functions bounded by 1 with compact support and all $l_1, \ldots, l_m \geq 1$, the joint moments

$$\begin{aligned} & E\left[\left(\int_X h_1(x)\omega(dx) \right)^{l_1} \circ \tau_*^{k_{1,n}} \cdots \left(\int_X h_m(x)\omega(dx) \right)^{l_m} \circ \tau_*^{k_{m,n}} \right] \\ = \ & E\left[\left(\int_X h_1(\tau^{(k_{1,n})}(x,\omega))\omega(dx) \right)^{l_1} \cdots \left(\int_X h_m(\tau^{(k_{m,n})}(x,\omega))\omega(dx) \right)^{l_m} \right], \end{aligned} \tag{3.3}$$

converge to

$$E\left[\left(\int_X h_1(x)\omega(dx) \right)^{l_1} \right] \cdots E\left[\left(\int_X h_m(x)\omega(dx) \right)^{l_m} \right]$$

as n goes to infinity.

By Proposition 2.1 and the relation

$$\begin{aligned} \varepsilon^+_{x_1,\ldots,x_k}(u_1(x_1,\omega)\cdots u_k(x_k,\omega)) &= (I + D_{x_1})\cdots(I + D_{x_k})(u_1(x_1,\omega)\cdots u_k(x_k,\omega)) \\ &= \sum_{\Theta\subset\{1,\ldots,k\}} D_\Theta(u_1(x_1,\omega)\cdots u_k(x_k,\omega)), \end{aligned} \tag{3.4}$$

where $D_\Theta = D_{x_1}\cdots D_{x_l}$ when $\Theta = \{x_1, \ldots, x_l\}$, we can express the joint moment (3.3) as a finite sum of terms of the form

$$E\left[\int_{X^k} D_\Theta \left(\prod_{i_1\in Q_1} h_{i_1}^{l^N_{1,i_1}}(\tau^{(k_{i_1,n})}(x_1,\omega)) \cdots \prod_{i_k\in Q_k} h_{i_k}^{l^N_{k,i_k}}(\tau^{(k_{i_k,n})}(x_k,\omega)) \right) \sigma(dx_1)\cdots\sigma(dx_k) \right], \tag{3.5}$$

where, with $N = l_1 + \cdots + l_m$, $l^N_{j,i} := |P^N_j \cap (l_1 + \cdots + l_{i-1}, l_1 + \cdots + l_i]|$ and

$$Q_j = \left\{ i \in \{1, \ldots, m\} \ : \ l^N_{j,i} \geq 1 \right\}, \qquad j = 1, \ldots, k, \tag{3.6}$$

and $\Theta \subset \{x_1, \ldots, x_k\}$. Note that when $\Theta = \{x_1, \ldots, x_k\}$, Condition (3.1) shows the vanishing of (3.5) due to the relation

$$D_{x_1} \cdots D_{x_k} \big(u_1(x_1, \omega) \cdots u_k(x_k, \omega) \big) = \sum_{\Theta_1 \cup \cdots \cup \Theta_k = \{1, \ldots, k\}} D_{\Theta_1} u_1(x_1, \omega) \cdots D_{\Theta_k} u_k(x_k, \omega), \tag{3.7}$$

where the above sum includes all (possibly empty) sets $\Theta_1, \ldots, \Theta_k$ whose union is $\{x_1, \ldots, x_k\}$. Hence in the sequel we can assume that $\Theta = \{x_1, \ldots, x_l\} \subset \{x_1, \ldots, x_{k-1}\}$,

The proof is split in four steps that are based on the evaluation of (3.5).

Step 1. The term (3.5) vanishes as n tends to infinity if $|Q_k| \geq 2$.

If Q_k contains at least two distinct indexes a, b with $1 \leq a < b \leq m$ and $l < k$, we have

$$\begin{aligned}
\int_X \prod_{j \in Q_k} h_j(\tau^{(k_{j,n})}(x_k, \omega)) \sigma(dx_k) &\leq \int_X h_a(\tau^{(k_{a,n})}(x_k, \omega)) h_b(\tau^{(k_{b,n})}(x_k, \omega)) \sigma(dx_k) \\
&= \int_X h_a(\tau^{(k_{a,n}-1)}(\tau(x_k, \omega), \tau_* \omega)) h_b(\tau^{(k_{b,n}-1)}(\tau(x_k, \omega), \tau_* \omega)) \sigma(dx_k) \\
&= \int_X h_a(\tau^{(k_{a,n}-1)}(x_k, \tau_* \omega)) h_b(\tau^{(k_{b,n}-1)}(x_k, \tau_* \omega)) \sigma(dx_k) \\
&= \int_X h_a(x_k) h_b(\tau^{(k_{b,n}-k_{a,n})}(x_k, \tau_*^{k_{a,n}} \omega)) \sigma(dx_k),
\end{aligned} \tag{3.8}$$

$\omega \in \Omega$, where used the invariance of σ under $\tau(\cdot, \omega) : X \longrightarrow X$. By (3.8), this shows that for all $p \geq 1$ we have

$$\begin{aligned}
& E\left[\left(\int_X \prod_{j \in Q_k} h_j(\tau^{(k_{j,n})}(x_k, \omega)) \sigma(dx_k) \right)^p \right] \\
&\leq E\left[\left(\int_X h_a(x_k) h_b(\tau^{(k_{b,n}-k_{a,n})}(x_k, \tau_*^{k_{a,n}} \omega)) \sigma(dx_k) \right)^p \right] \\
&= E\left[\left(\int_X h_a(x_k) h_b(\tau^{(k_{b,n}-k_{a,n})}(x_k, \omega)) \sigma(dx_k) \right)^p \right],
\end{aligned}$$

while

$$\int_X h_a(x_k) h_b(\tau^{(k_{b,n}-k_{a,n})}(x_k, \omega)) \sigma(dx_k)$$

is a.s. bounded by $\int_X h_a(x_k)\sigma(dx_k)$ and tends to zero in probability by (3.2) as n goes to infinity since h_a has compact support and $\lim_{n\to\infty} k_{b,n} - k_{a,n} = +\infty$. Hence

$$\lim_{n\to\infty} \int_X \prod_{j\in Q_r} h_j(\tau^{(k_{j,n})}(x_r,\omega))\sigma(dx_r) = 0 \tag{3.9}$$

in $L^p(\Omega)$, $p \geq 1$.

From (3.9) and the fact that $\Theta = \{x_1,\ldots,x_l\} \subset \{x_1,\ldots,x_{k-1}\}$ it is apparent that (3.5) will tend to zero as n tends to infinity, however to conclude Step 1 we need to an integrability argument.

For this, using the relation

$$D_\Theta = \sum_{\eta\subset\Theta} (-1)^{|\eta|+l} \epsilon_\eta^+,$$

where ϵ_η^+ is defined as in (2.1), we rewrite (3.5) as a linear combination of terms of the form

$$E\left[\int_{X^k} \varepsilon_\eta^+ \left(\prod_{i_1\in Q_1} h_{i_1}^{l_{1,i_1}^N}(\tau^{(k_{i_1,n})}(x_1,\omega))\cdots \prod_{i_k\in Q_k} h_{i_k}^{l_{k,i_k}^N}(\tau^{(k_{i_k,n})}(x_k,\omega))\right)\sigma(dx_1)\cdots\sigma(dx_k)\right],$$

with $\eta = \{x_1,\ldots,x_l\} \subset \{x_1,\ldots,x_{k-1}\}$. Applying the first moment Mecke identity (2.6) to the variable x_1, we get

$$\begin{aligned}
&E\left[\int_{X^k} \varepsilon_\eta^+ \prod_{j=1}^k \left(\prod_{i_j\in Q_j} h_{i_j}^{l_{j,i_j}^N}(\tau^{(k_{i_j,n})}(x_j,\omega))\right)\sigma(dx_1)\cdots\sigma(dx_k)\right]\\
&\quad = E\left[\int_{X^{k-1}}\int_X \varepsilon_{\eta\setminus\{x_1\}}^+ \left(\prod_{j=1}^k \left(\prod_{i_j\in Q_j} h_{i_j}^{l_{j,i_j}^N}(\tau^{(k_{i_j,n})}(x_j,\omega))\right)\right)\omega(dx_1)\sigma(dx_2)\cdots\sigma(dx_k)\right]\\
&\quad = E\left[\int_{X^{k-1}} \varepsilon_{\eta\setminus\{x_1\}}^+ \left(\int_X \prod_{j=1}^k \left(\prod_{i_j\in Q_j} h_{i_j}^{l_{j,i_j}^N}(\tau^{(k_{i_j,n})}(x_j,\omega))\right)\omega(dx_1)\right)\sigma(dx_2)\cdots\sigma(dx_k)\right]\\
&-\sum_{r=2}^l E\left[\int_{X^{k-1}} \varepsilon_{\eta\setminus\{x_1\}}^+ \left(\prod_{i_1\in Q_1} h_{i_1}^{l_{1,i_1}^N}(\tau^{(k_{i_1,n})}(x_l,\omega))\prod_{j=2}^k\left(\prod_{i_j\in Q_j} h_{i_j}^{l_{j,i_j}^N}(\tau^{(k_{i_j,n})}(x_j,\omega))\right)\right)\sigma(dx_2)\cdots\sigma(dx_k)\right],
\end{aligned}$$

where we used the relation

$$\varepsilon_{x_2}^+\cdots\varepsilon_{x_l}^+ \int_X v(x_1,\omega)\omega(dx_1) = \int_X \varepsilon_{x_2}^+\cdots\varepsilon_{x_l}^+ v(x_1,\omega)\omega(dx_1) + \sum_{r\in\eta\setminus\{x_1\}} \varepsilon_{x_2}^+\cdots\varepsilon_{x_l}^+ v(x_r,\omega).$$

After inductively exhausting all elements of η by repeating the above argument we find that (3.5) rewrites as a linear combination of terms of the form

$$E\left[\left(\prod_{j=1}^{l'}\int_X \prod_{i_j\in R_j} h_{i_j}^{l^N_{j,i_j}}(\tau^{(k_{i_j,n})}(x_j,\omega))\omega(dx_j)\right)\left(\prod_{j=l'+1}^{k'}\int_X \prod_{i_j\in R_j} h_{i_j}^{l^N_{j,i_j}}(\tau^{(k_{i_j,n})}(x_j,\omega))\sigma(dx_j)\right)\right], \tag{3.10}$$

$1 \le l' < k'$, where $\{R_1,\dots,R_{k'}\}$ is another family of subsets of $\{1,\dots,m\}$ with $R_{k'} = Q_k$.

Denoting by $K \subset X$ a compact set containing the supports of $h_1,\dots,h_m$, all l terms in the left product in (3.10) are a.s. bounded by the random variable

$$\begin{aligned}\int_X \mathbf{1}_K(\tau^{(k_{i_j,n})}(x_j,\omega))\omega(dx_j) &= \int_X \mathbf{1}_K(\tau^{(k_{i_j,n}-1)}(\tau(x_j,\omega),\tau_*\omega))\omega(dx) \\ &= \int_X \mathbf{1}_K(\tau^{(k_{i_j,n}-1)}(x_j,\tau_*\omega))\tau_*\omega(dx),\end{aligned} \tag{3.11}$$

which has the same distribution as $\int_X \mathbf{1}_K(\tau^{(k_{i_j,n}-1)}(x,\omega))\omega(dx)$ since $\tau_* : \Omega \longrightarrow \Omega$ leaves the Poisson measures π_σ invariant by Theorem 3.3 of [12] or [11]. By decreasing induction on $k_{i_j,n}, k_{i_j,n}-1,\dots,1$, this shows that (3.11) has the Poisson distribution of $\int_X \mathbf{1}_K(x)\omega(dx) = \omega(K)$ with parameter $\sigma(K) < \infty$, in particular it has finite moments of all orders.

On the other hand, the terms of index $j = l+1,\dots,k'-1$ in the right product (3.10) are uniformly bounded in n by $\sigma(K)$ as in (3.8), and the last term of index k' converges to 0 in $L^p(\Omega)$ for all $p \ge 1$ by (3.9) since Q_k is not a singleton. Hence by Hölder's inequality, (3.5) tends to 0 as n goes to infinity.

Step 2. As a consequence of Step 1 we only need to consider terms (3.5) of the form

$$\begin{aligned}&E\left[\int_{X^k} D_\Theta\left(h_{i_1}^{l^N_{1,i_1}}(\tau^{(k_{i_1,n})}(x_1,\omega))\cdots h_{i_k}^{l^N_{k,i_k}}(\tau^{(k_{i_k,n})}(x_k,\omega))\right)\sigma(dx_1)\cdots\sigma(dx_k)\right] \\ &= E\left[\int_{X^k} D_\Theta\left(h_{i_1}^{l^N_{1,i_1}}(\tau^{(k_{i_1,n})}(x_1,\omega))\cdots h_{i_{k-1}}^{l^N_{k-1,i_{k-1}}}(\tau^{(k_{i_{k-1},n})}(x_{k-1},\omega))\right)\right. \\ &\qquad\qquad \left.\times h_{i_k}^{l^N_{k,i_k}}(\tau^{(k_{i_k,n})}(x_k,\omega))\sigma(dx_1)\cdots\sigma(dx_k)\right]\end{aligned}$$

where $Q_k = \{i_k\}$ is a singleton. By invariance of $\tau(\cdot, \tilde{\omega}) : X \longrightarrow X$ for any $\tilde{\omega} \subset \omega \cup \{x_1, \ldots, x_l\} \in \Omega$, we have

$$\begin{aligned}
\int_X h_{i_k}^{l_{k,i_k}^N}(\tau^{(k_{i_k,n})}(x_k, \tilde{\omega}))\sigma(dx_k) &= \int_X h_{i_k}^{l_{k,i_k}^N}(\tau^{(k_{i_k,n}-1)}(\tau(x_k, \tilde{\omega})\tau_*\tilde{\omega}))\sigma(dx_k) \\
= & \int_X h_{i_k}^{l_{k,i_k}^N}(\tau^{(k_{i_k,n}-1)}(x_k, \tau_*\tilde{\omega}))\sigma(dx_k) = \int_X h_{i_k}^{l_{k,i_k}^N}(\tau(x_k, \tau_*^{k_{i_k,n}-1}\tilde{\omega}))\sigma(dx_k) \\
= & \int_X h_{i_k}^{l_{k,i_k}^N}(x_k)\sigma(dx_k), \qquad (3.12)
\end{aligned}$$

where the step before last is reached by induction on $1, \ldots, k_{i_k,n} - 1$. Since (3.12) is deterministic, the integral in $\sigma(dx_k)$ can then be factored out of D_Θ in (3.5) and we can reconsider (3.5) at the order $k-1$ instead of k.

Step 3. Decreasing induction on k.

After implementing Step 2, from (3.7) and Condition (3.1) we can again assume that $\Theta = \{x_1, \ldots, x_l\} \subset \{x_1, \ldots, x_{k-2}\}$, and repeating Step 2 above by further decrementing k we find that (3.5) vanishes as n tends to infinity unless Q_j is a singleton for all $j = 1, \ldots, k = m$ and Θ is empty, in which case we have

$$\begin{aligned}
\lim_{n\to\infty} E\left[\int_{X^k} \left(h_{i_1}^{l_{1,i_1}^N}(\tau^{(k_{i_1,n})}(x_1, \omega)) \cdots h_{i_k}^{l_{k,i_k}^N}(\tau^{(k_{i_k,n})}(x_k, \omega))\right) \sigma(dx_1)\cdots\sigma(dx_k)\right] \\
= \int_X h_1^{l_{1,i_1}^N}(x)\sigma(dx) \cdots \int_X h_{k_m}^{l_{m,i_m}^N}(x)\sigma(dx).
\end{aligned}$$

Step 4. To conclude, taking again $N = l_1 + \cdots + l_m$ we let

$$U_j^i := P_j^N \cap (l_1 + \cdots + l_{i-1}, l_1 + \cdots + l_i], \qquad i = 1, \ldots, m, \quad j = 1, \ldots, k,$$

and note that from (3.6) and Step 3, (3.5) vanishes as n tends to infinity, unless $\Theta = \emptyset$ and the cardinal

$$l_{j,i}^N = |U_j^i| = |P_j^N \cap (l_1 + \cdots + l_{i-1}, l_1 + \cdots + l_i]|$$

is either 0 or 1 for all $j = 1, \ldots, k$ and $i \in \{1, \ldots, m\}$.

Hence we only need to consider partitions of $\{1, \dots, k\}$ of the form

$$\{U_1^1, \dots, U_{k_1}^1, \dots, U_1^m, \dots, U_{k_m}^m\}$$

such that for all $i = 1, \dots, m$,

$$\{U_1^i, \dots, U_{k_i}^i\}$$

is a partition of $\{l_1 + \cdots + l_{i-1} + 1, \dots, l_1 + \cdots l_i\}$ made of (say) k_i non empty sets, $k_i \in \{1, \dots, l_i\}$, after a suitable re-indexing of the lower index j in U_j^i.

Then by (3.4) we have

$$\begin{aligned}
&\lim_{n\to\infty} E\left[\left(\int_X h_1(\tau^{(k_{1,n})}(x,\omega))\omega(dx)\right)^{l_1} \cdots \left(\int_X h_m(\tau^{(k_{m,n})}(x,\omega))\omega(dx)\right)^{l_m}\right] \\
&= \lim_{n\to\infty} \sum_{k=1}^N \sum_{P_1^N,\dots,P_k^N} E\left[\int_{X^k} \varepsilon^+_{x_1,\dots,x_k}\left(\prod_{j=1}^k \prod_{i=1}^m h_i^{l_{i,j}^N}(\tau^{(k_{i,n})}(x_j,\omega))\right)\sigma(dx_1)\cdots\sigma(dx_k)\right] \\
&= \lim_{n\to\infty} \sum_{k_1=1}^{l_1} \cdots \sum_{k_m=1}^{l_m} \sum_{U_1^1\cup\ldots\cup U_{k_1}^1=\{1,\dots,l_1\}} \cdots \sum_{U_1^m\cup\ldots\cup U_{k_m}^m=\{l_1+\cdots+l_{m-1}+1,\dots,l_1+\cdots+l_m\}} \\
&E\left[\int_{X^{k_1+\cdots+k_m}} \varepsilon^+_{x_1,\dots,x_{k_1+\cdots+k_m}}\left(\prod_{i=1}^m \prod_{q_i=1}^{k_i} h_i^{|U_{q_i}^i|}(\tau^{(k_{i,n})}(x_{k_1+\cdots+k_{i-1}+q_i},\omega))\right)\sigma(dx_1)\cdots\sigma(dx_{k_1+\cdots+k_m})\right] \\
&= \lim_{n\to\infty} \sum_{k_1=1}^{l_1} \cdots \sum_{k_m=1}^{l_m} \sum_{U_1^1\cup\ldots\cup U_{k_1}^1=\{1,\dots,l_1\}} \cdots \sum_{U_1^m\cup\ldots\cup U_{k_m}^m=\{l_1+\cdots+l_{m-1}+1,\dots,l_1+\cdots+l_m\}} \sum_{\Theta\subset\{1,\dots,k_1+\cdots+k_m\}} \\
&\quad E\left[\int_{X^{k_1+\cdots+k_m}} D_\Theta\left(\prod_{i=1}^m \prod_{q_i=1}^{k_i} h_i^{|U_{q_i}^i|}(\tau^{(k_{i,n})}(x_{k_1+\cdots+k_{i-1}+q_i},\omega))\right)\sigma(dx_1)\cdots\sigma(dx_{k_1+\cdots+k_m})\right] \\
&= \sum_{k_1=1}^{l_1} \cdots \sum_{k_m=1}^{l_m} \sum_{U_1^1\cup\ldots\cup U_{k_1}^1=\{1,\dots,l_1\}} \cdots \sum_{U_1^m\cup\ldots\cup U_{k_m}^m=\{1,\dots,l_m\}} \\
&\quad \int_X h_1^{|U_1^1|}(x)\sigma(dx)\cdots\int_X h_{k_1}^{|U_{k_1}^1|}(x)\sigma(dx)\cdots\int_X h_m^{|U_1^m|}(x)\sigma(dx)\cdots\int_X h_{k_m}^{|U_{k_m}^m|}(x)\sigma(dx) \\
&= E\left[\left(\int_X h_1(x)\omega(dx)\right)^{l_1}\right]\cdots E\left[\left(\int_X h_m(x)\omega(dx)\right)^{l_m}\right],
\end{aligned}$$

showing that τ_* is mixing of all orders $n \geq 1$, by density in $L^2(\Omega, \pi_\sigma)$ of the polynomials in $\int_X h(x)\omega(dx)$, $h \in \mathcal{C}_c(X)$. □

4 Examples

We consider a family of examples satisfying the hypotheses of Theorem 3.1, based on transformations conditioned by a random boundary. We let $X = \mathbb{R}^d$ with norm $\| \cdot \|$ and for all $\omega \in \Omega$ we denote by $\omega_e \subset \omega$ denote the extremal vertices of the convex hull of $\omega \cap B(0,1)$. We also denote by $\mathcal{C}(\omega)$ the convex hull of ω, with interior $\dot{\mathcal{C}}(\omega)$.

Consider a mapping $\widehat{\tau} : X \times \Omega \longrightarrow X$ such that for all $\omega \in \Omega$,

$$\widehat{\tau}(\cdot, \omega) : X \longrightarrow X$$

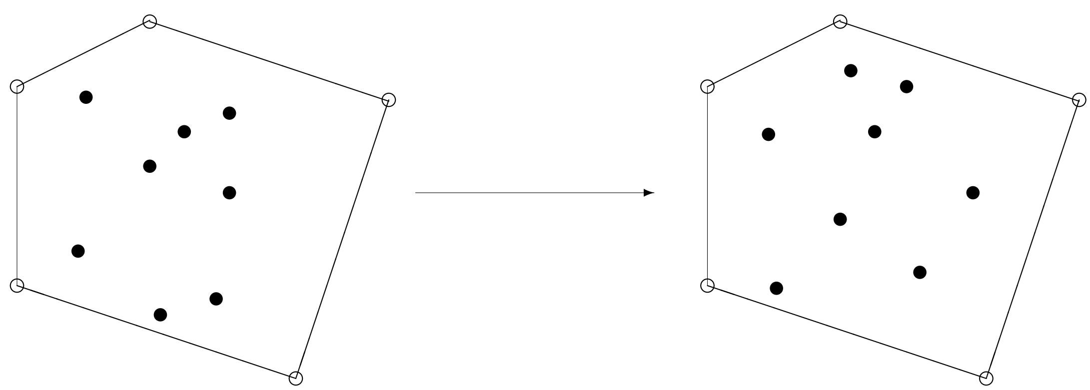
leaves $X \setminus \dot{\mathcal{C}}(\omega_e)$ invariant (including the extremal vertices ω_e of $\mathcal{C}(\omega_e)$) while

$$\widehat{\tau} : \dot{\mathcal{C}}(\omega_e) \times \Omega \longrightarrow \dot{\mathcal{C}}(\omega_e)$$

shifts the points inside $\dot{\mathcal{C}}(\omega_e)$ depending on the data of ω_e, i.e. we have

$$\widehat{\tau}(x, \omega) = \begin{cases} \widehat{\tau}(x, \omega_e), & x \in \dot{\mathcal{C}}(\omega_e), \\ \\ x, & x \in X \setminus \dot{\mathcal{C}}(\omega_e). \end{cases} \tag{4.1}$$

As shown in Proposition 4.1 below, such a transformation $\widehat{\tau}$ satisfies the vanishing condition (2.2) hence by Theorem 3.3 of [12] or [11] the mapping $\widehat{\tau}_* : \Omega \longrightarrow \Omega$ leaves π_σ invariant. The next figure shows an example of behaviour such a transformation, with a finite set of points for simplicity of illustration.



Using the mapping $\widehat{\tau} : X \times \Omega \longrightarrow X$, we will build examples of interacting transformations $\tau : X \times \Omega \longrightarrow X$ that satisfy Conditions (3.1) and (3.2).

Vanishing condition (3.1)

Proposition 4.1 *Let* $\widehat{\tau} : X \times \Omega \longrightarrow X$ *satisfy* (4.1) *and let* $f : X \longrightarrow X$ *be a bijective deterministic mapping that preserves set convexity. Then the transformation*

$$\begin{aligned} \tau \quad : X \times \Omega \quad &\longrightarrow \quad \Omega \\ (\omega, x) \quad &\longmapsto \quad \tau(x,\omega) := f(\widehat{\tau}(x,\omega)) \end{aligned} \tag{4.2}$$

satisfies the vanishing condition (3.1)*.*

Proof. In order to check that (3.1) holds for all $m \geq 1$, we note that by induction on $k \geq 1$ we have

$$\tau^{(k)}(x,\omega) = \tau^{(k)}(x,\omega_e), \qquad x \in X, \tag{4.3}$$

i.e. $\tau^{(k)}(x,\omega)$ depends only on x and on the points in ω_e. Indeed, Relation (4.3) is satisfied for $k = 1$ by (4.1) and we have

$$\tau^{(k+1)}(x,\omega) = \tau^{(k)}(\tau(x,\omega), \tau_*\omega) = \tau^{(k)}(\tau(x,\omega_e), (\tau_*\omega)_e),$$

while the positions of the points in $(\tau_*\omega)_e$ themselves depend only on ω_e through the function f, showing that $\tau^{(k+1)}(x,\omega)$ depends only on ω_e and x.

On the other hand we can also show by induction that

$$\tau^{(k)}(x,\omega) = f^k(x), \qquad x \in X \setminus \mathcal{C}(\omega_e), \tag{4.4}$$

Indeed this condition is satisfied for $k = 1$ by (4.1) and (4.2). Now since $f : X \longrightarrow X$ preserves set convexity we have

$$\mathcal{C}((\tau_*\omega)_e) = \mathcal{C}(f(\omega_e)) \subset f(\mathcal{C}(\omega_e)),$$

because $f(\mathcal{C}(\omega_e))$ is convex and contains $f(\omega_e)$, hence since f is bijective we get

$$\tau(x,\omega) \in \mathcal{C}((\tau_*\omega)_e) \Longrightarrow \tau(x,\omega) \in f(\mathcal{C}(\omega_e)) \Longrightarrow \widehat{\tau}(x,\omega) \in \mathcal{C}(\omega_e) \Longrightarrow x \in \mathcal{C}(\omega_e),$$

i.e.

$$x \in X \setminus \mathcal{C}(\omega_e) \Longrightarrow \tau(x,\omega) = f(x) \in X \setminus \mathcal{C}((\tau_*\omega)_e), \qquad x \in X. \tag{4.5}$$

Therefore, assuming that (4.4) holds at the rank $n \geq 1$, for every $x \in X \setminus \mathcal{C}(\omega_e)$ we get, by (4.5),

$$\tau^{(k+1)}(x,\omega) = \tau^{(k)}(\tau(x,\omega),\tau_*\omega) = f^k(\tau(x,\omega)) = f^{k+1}(x),$$

which is (4.4) at the rank $k+1$. In the remainder of this proof we will conclude from (4.3) and (4.4) as in Proposition 3.3 of [1] and [12] that the vanishing Condition (3.1) is satisfied, i.e. we show that

$$D_{\Theta_1}\tau^{(k_1)}(x_1,\omega)\cdots D_{\Theta_m}\tau^{(k_m)}(x_m,\omega) = 0, \tag{4.6}$$

for every family $\{\Theta_1,\ldots,\Theta_m\}$ of (non empty) subsets such that $\Theta_1 \cup \cdots \cup \Theta_m = \{x_1,\ldots,x_m\}$, $x_1,\ldots,x_m \in X$ and all $\pi_\sigma(d\omega)$-a.s., $k_1,\ldots,k_m \geq 1$, $m \geq 1$.

Note that whenever x_i lies inside of $\mathcal{C}(\omega) = \mathcal{C}(\omega_e)$ then by (4.3) we have

$$\begin{aligned} D_{x_i}\tau^{(k)}(x_j,\omega) &= \tau^{(k)}(x_j,\omega\cup\{x_i\}) - \tau^{(k)}(x_j,\omega) = \tau^{(k)}(x_j,(\omega\cup\{x_i\})_e) - \tau^{(k)}(x_j,\omega_e) \\ &= \tau^{(k)}(x_j,\omega_e) - \tau^{(k)}(x_j,\omega_e) = 0 \end{aligned}$$

for all $i,j = 1,\ldots,m$ and $k \geq 1$, hence $D_\eta\tau^{(k)}(x_j,\omega) = 0$ provided $\{x_i\} \subset \eta \subset \{x_1,\ldots,x_m\}$.

Consequently it suffices to consider the case where $\mathcal{C}(\omega\cup\{x_1,\ldots,x_m\})$ has (at least) one extremal point denoted x_e within $\{x_1,\ldots,x_m\}$.

Now, for all $\eta \subset \{x_1,\ldots,x_m\}$ we have

$$\tau^{(k)}(x_e,\omega\cup\eta) = \tau^{(k)}(x_e,\omega) = f^k(x_e)$$

by (4.4), hence

$$D_\Theta\tau^{(k)}(x_e,\omega) = 0,$$

for all $\Theta \subset \{x_1,\ldots,x_m\}$, due to the relation

$$D_\Theta\tau^{(k)}(x_e,\omega) = \sum_{\eta\subset\Theta}(-1)^{|\Theta|+1-|\eta|}\tau^{(k)}(x_e,\omega\cup\eta)$$

$$\begin{aligned} &= f^k(x_e)\sum_{\eta\subset\Theta}(-1)^{|\Theta|+1-|\eta|} \\ &= f^k(x_e)(1-1)^{|\Theta|+1} \\ &= 0, \end{aligned}$$

where the summation above holds over all (possibly empty) subset η of Θ. As a consequence, a factor in (4.6) has to vanish. □

Zero-type condition (3.2)

In order for the zero-type condition (3.2) to hold it suffices that

$$\lim_{n\to\infty}\|\tau^{(n)}(x,\omega)\| = \infty, \qquad \omega\in\Omega, \quad x\in\mathbb{R}^d.$$

For this we can assume for example that $\tau : X\times\Omega \longrightarrow X$ satisfies a random dilation property

$$\|\tau(x,\omega)\| \geq C(\omega)\|x\|_s, \qquad \omega\in\Omega, \quad x\in\mathbb{R}^d, \tag{4.7}$$

for a random variable

$$C : \Omega \longrightarrow (1,\infty).$$

In this case, for any $g, h \in \mathcal{C}_c(X)$ with support in $B(0,r)$ for some $r>0$, we have

$$\lim_{n\to\infty}\langle g, h\circ\tau^{(n)}\rangle_{L^2_\sigma(X)} = 0, \qquad \omega\in\Omega,$$

because the support of $x \longmapsto h(\tau^{(n)}(x,\omega))$ is in $B(0, rC^{-n}(\omega))$ by construction, for all $\omega\in\Omega$.

Condition (4.7) holds in particular when $f:\mathbb{R}^d\longrightarrow\mathbb{R}^d$ in (4.2) satisfies the dilation property

$$\|f(x)\| \geq r\|x\|, \qquad x\in\mathbb{R}^d,$$

for some $r>1$, and $\widehat{\tau} : X\times\Omega\longrightarrow X$ satisfies

$$\|\widehat{\tau}(x,\omega)\| \geq c(\omega)\|x\|, \qquad \omega\in\Omega, \quad x\in\mathbb{R}^d,$$

for some $r>1$ and $c:\Omega\longrightarrow(0,1]$ such that $\inf_{\omega\in\Omega} c(\omega) > 1/r$.

For example in case $f(x) = rUx$, $x \in \mathbb{R}^d$, where $r > 1$ and $U : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ is a linear isometry of $\mathbb{R}^d$, the intensity measure $\sigma(dx) := \|x\|^{-d}dx$ is invariant by $f : \mathbb{R}^d \longrightarrow \mathbb{R}^d$, and if $c(\omega) = 1$, the measure-preserving mapping $\widehat{\tau}(\cdot, \omega_e) : X \longrightarrow X$ can be built from any isometric transformations of $\dot{\mathcal{C}}(\omega_e)$. This includes for example any random rotation within a (random) disk contained in $\dot{\mathcal{C}}(\omega_e)$.

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