De Rham-Hodge decomposition and vanishing of harmonic forms by derivation operators on the Poisson space*

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Abstract

We construct differential forms of all orders and a covariant derivative together with its adjoint on the probability space of a standard Poisson process, using derivation operators. In this framewok we derive a de Rham-Hodge-Kodaira decomposition as well as Weitzenböck and Clark-Ocone formulae for random differential forms. As in the Wiener space setting, this construction provides two distinct approaches to the vanishing of harmonic differential forms.

Keywords: de Rham-Hodge-Kodaira decomposition, Weitzenböck identity, Clark-Ocone formula, Malliavin calculus.

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1 Introduction

The Weitzenböck formula

$$\Delta_n = L + R_n$$

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relates the Hodge Laplacian $\Delta_n = \mathrm{d}^{n-1}\mathrm{d}^{(n-1)*} + \mathrm{d}^{n*}\mathrm{d}^n$ on differential n-forms to the Bochner Laplacian $L = -\nabla^*\nabla$ through a zeroth-order curvature term R_n. Here, d^n is the exterior derivative on n-forms with adjoint d^{n*}, and ∇ is the covariant derivative with adjoint ∇^*. For one-forms, the Weitzenböck curvature R_1 reduces to the usual Ricci tensor. The Weitzenböck formula has been established on the standard Wiener space in [25] using the operators of the Malliavin calculus, cf. § II-6.7 of [13], in which case the curvature tensor R_n is the identity operator on the Cameron-Martin space. Weitzenböck formulas have also been established on path and loop spaces over Lie groups, cf. [10] and [11], and also on the path space over a Riemannian manifold [7]. The Weitzenböck formula can be used to prove the vanishing of harmonic differential forms, cf. [1] in the case of loop groups.

On the other hand, the Clark-Ocone formula

$$F = \mathbb{E}[F] + \int_0^T \mathbb{E}[D_t F \mid \mathcal{F}_t] dB_t,$$

on the Wiener space, cf. [6], [15], decomposes a square-integrable function into the sum of a constant and a martingale, where D_t denotes the Malliavin gradient. The Clark-Ocone formula has been extended to the decomposition of n-forms into the sum of an exact form and a martingale, see [26] on the Wiener space, and [9] on the path space over a Riemannian manifold. It provides an alternative proof of the result of [25] on the vanishing of harmonic differential forms as well as explicit decompositions for closed differential forms, and it also admits a dual version that applies to the representation of co-closed forms.

The above framework has been extended in [24] to the more general setting of normal martingales using multiple stochastic integral chaos expansions, including compensated Poisson processes via finite difference operators, also covering the results of [26] on the Clark-Ocone formula for differential forms on the Wiener space.

In this paper we present another example of extension of this construction beyond chaos expansions, based on a natural geometry on the Poisson space over the half line

$\mathbb{R}_+ = [0, \infty)$, using a gradient operator which has the derivation property, cf. [5], [8], [16]. Our proof of the Weitzenböck formula is inspired by the arguments of [10] and [11] on path and loop groups, and it extends the results of [18], [19], [20] which are stated in the case of one-forms. In particular we construct a Hodge Laplacian

$$\Delta_n = \mathrm{d}^{n-1}\mathrm{d}^{(n-1)*} + \mathrm{d}^{n*}\mathrm{d}^n,$$

on differential n-forms, with domain $\mathrm{Dom}(\Delta_n)$, and we prove the Weitzenböck identity

$$\Delta_n = n\mathrm{Id}_{H^{\wedge n}} + \nabla^*\nabla, \qquad n \geq 1,$$

cf. Theorem 5.3, using a space of smooth random forms valued in a space H_n of $\mathcal{C}_c^1(\mathbb{R}_+^n; \mathbb{R})$ functions with compact support, that vanish in a neighborhood of the diagonals of $\mathbb{R}_+^n$. As a consequence we deduce that $\mathrm{Ker}\,\Delta_n = \{0\}$, with the de Rham-Hodge decomposition

$$L^2(\Omega; \hat{H}^{\wedge n}) = \mathrm{Im}\,\mathrm{d}^{n-1} \oplus \mathrm{Im}\,\mathrm{d}^{n*}, \qquad n \geq 1, \tag{1.1}$$

where $\hat{H}^{\wedge n}$ denotes the completed antisymmetric n-th tensor power of H, cf. Proposition 4.2 and Corollaries 5.5, 6.5. On the other hand, we recover the de Rham-Hodge decomposition (1.1) from the Clark-Ocone formula of Theorem 6.3, which also shows the exactness of the sequence

$$\mathrm{Dom}(\mathrm{d}^n) \xrightarrow{\mathrm{d}^n} \mathrm{Im}\,(\mathrm{d}^n) = \mathrm{Ker}\,(\mathrm{d}^{n+1}) \xrightarrow{\mathrm{d}^{n+1}} \mathrm{Im}\,(\mathrm{d}^{n+1}), \qquad n \in \mathbb{N},$$

and the complementarity of the Weitzenböck and Clark-Ocone approaches.

We refer the reader to [2], [3] for a different construction of differential forms on the configuration space over a Riemannian manifold with a Poisson measure, where n-forms were defined by a lifting of the underlying differential structure on the manifold to the configuration space. See also to [4] for a different approach to the construction of the Hodge decomposition on abstract metric spaces.

This paper is organized as follows. The Poisson space geometry used in this paper is presented in Section 2. In Sections 3 and 4 we construct the differential and

divergence operators on differential forms, including their duality and commutation relations. Section 5 and 6 presents the main results on the Weitzenböck identity and the generalised Clark-Ocone formulae, respectively. The appendix Section 7 contains alternative proofs for two lemmas used in the paper, using explicit calculations for the Poisson process.

2 Differential geometry of the standard Poisson space

We consider the probability space $(\Omega, \mathcal{F}, P)$ where $\Omega = \mathbb{R}^{\mathbb{N}}$ is endowed with the probability measure P and σ-algebra $\mathcal{F}$ generated by the countable sequence of independent exponentially distributed random variables $(\tau_k)_{k\geq 1}$ built as the coordinate mappings

$$\Omega \ni \omega = (\omega_n)_{n\in\mathbb{N}} \longmapsto \tau_k(\omega) := \omega_k, \qquad k \geq 1,$$

cf. [16] for details. This defines the sequence

$$T_k = \tau_1 + \cdots + \tau_k, \qquad k \geq 1, \tag{2.1}$$

of jump times of a standard Poisson process

$$N_t = \sum_{k=1}^{\infty} \mathbf{1}_{[T_k,\infty)}(t), \qquad t \in \mathbb{R}_+,$$

generating the filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ on $(\Omega, \mathcal{F}, P)$. We consider the space $\mathcal{S}$ of cylindrical functionals of the form

$$F(\omega) = f(T_1, \ldots, T_d), \qquad f \in \mathcal{C}_b^1(\mathbb{R}_+^d; \mathbb{R}), \quad d \geq 1, \tag{2.2}$$

where $\mathcal{C}_b^1(\mathbb{R}_+^d; \mathbb{R})$ denotes the set of bounded continuously differentiable real-valued functions on $\mathbb{R}_+^d$.

We will use the following gradient operator, cf. [16], which is a modification of the operator introduced in [5]. The notation ∂_k denotes the partial derivative of f with respect to its k-th variable.

Definition 2.1. *Given $F \in \mathcal{S}$ of the form $F = f(T_1, \ldots, T_d)$, let*

$$D_t F := -\sum_{k=1}^{d} \mathbf{1}_{[0,T_k]}(t) \partial_k f(T_1, \ldots, T_d), \qquad t \in \mathbb{R}_+. \tag{2.3}$$

We also let

$$D_v F = \langle v, DF \rangle_{L^2(\mathbb{R}_+)}, \quad F \in \mathcal{S}, \quad v \in \mathcal{S} \otimes L^2(\mathbb{R}_+). \tag{2.4}$$

Note that the operator D satisfies the chain rule of derivation

$$D(FG) = FDG + GDF, \qquad F, G \in \mathcal{S}. \tag{2.5}$$

The divergence operator $\delta : L^2(\Omega \times \mathbb{R}_+) \longrightarrow L^2(\Omega)$ is defined on $\mathcal{S} \otimes L^2(\mathbb{R}_+)$ by

$$\delta(hG) = G \int_0^\infty h(t)(dN_t - dt) - \langle h, DG \rangle_{L^2(\mathbb{R}_+)}, \quad G \in \mathcal{S}, \quad h \in L^2(\mathbb{R}_+), \tag{2.6}$$

and the operators D and δ satisfy the duality relation

$$\mathbb{E}[\langle u, DF \rangle_{L^2(\mathbb{R}_+)}] = \mathbb{E}[F\delta(u)], \qquad F \in \mathcal{S}, \quad \mathcal{S} \otimes L^2(\mathbb{R}_+), \tag{2.7}$$

from which D can be extended to a closed operator

$$D : \mathrm{Dom}\,(D) \longrightarrow L^2(\Omega \times \mathbb{R}_+)$$

with domain $\mathrm{Dom}\,(D) \subset L^2(\Omega)$, and δ can be extended to a closed operator

$$\delta : \mathrm{Dom}\,(\delta) \longrightarrow L^2(\Omega)$$

with domain $\mathrm{Dom}\,(\delta) \subset L^2(\Omega \times \mathbb{R}_+)$, cf. Proposition 3.1.2 and Proposition 7.2.6 of [23]. This duality condition corresponds to Assumption A2 in [24], and as a consequence these operators are extended to their respective closed domains $\mathrm{Dom}\,(D)$ and $\mathrm{Dom}\,(\delta)$.

In addition, the operator δ coincides with the compensated Poisson stochastic integral with respect to $(N_t - t)_{t \in \mathbb{R}_+}$ on the $(\mathcal{F}_t)_{t \in \mathbb{R}_+}$-adapted square-integrable processes, cf. e.g. Proposition 7.2.9 of [23].

Lie bracket

In the sequel, $L^2(\mathbb{R}_+)$ will be seen as a tangent space to Ω, in which the gradient D takes its values. On the other hand, the Lie bracket and covariant derivatives will be defined on the subspace $H = \mathcal{C}_c^\infty((0,\infty);\mathbb{R})$ of smooth vectors in $L^2(\mathbb{R}_+)$, made of continuously differentiable functions with compact support in $(0,\infty)$, and endowed with the scalar product inherited from $L^2(\mathbb{R}_+)$.

The Lie bracket $\{f,g\}$ of $f,g \in H$ is the element $\{f,g\}$ of H given by

$$\{f,g\}(t) := f'(t)\int_0^t g(s)ds - g'(t)\int_0^t f(s)ds, \qquad t \in \mathbb{R}_+,$$

whose definition is justified by the following proposition.

Proposition 2.2. *The bracket $w = \{f,g\}$ is the unique function in H satisfying the condition*

$$(D_f D_g - D_g D_f)F = D_w F, \qquad f,g \in H, \quad F \in \mathcal{S}, \tag{2.8}$$

where D_f is defined in (2.4).

Proof. First we note that by the chain rule of derivation (2.5) for D, Condition (2.8) is equivalent to

$$D_w T_k = (D_f D_g - D_g D_f)T_k, \qquad k \geq 1. \tag{2.9}$$

Next, for any $k \geq 1$ we have $D_w T_k = -\int_0^{T_k} w(s)ds$, and

$$\begin{aligned}(D_f D_g - D_g D_f)T_k &= -D_f \int_0^{T_k} g(s)ds + D_g \int_0^{T_k} f(s)ds \\ &= g(T_k)\int_0^{T_k} f(s)ds - f(T_k)\int_0^{T_k} g(s)ds,\end{aligned}$$

hence (2.9) reads

$$-\int_0^{T_k} w(s)ds = g(T_k)\int_0^{T_k} f(s)ds - f(T_k)\int_0^{T_k} g(s)ds, \qquad k \geq 1,$$

and by differentiation with respect to T_k this relation can be satisfied only by taking

$$w(t) := f'(t)\int_0^t g(s)ds - g'(t)\int_0^t f(s)ds, \qquad t \in \mathbb{R}_+,$$

hence $\{f,g\}$ is the only function $\omega = \{f,g\}$ in H for which (2.9) holds for all $k \geq 1$.

□

The Lie bracket $\{\cdot,\cdot\}$ is extended to $u = F \otimes f, v = G \otimes g \in \mathcal{S} \otimes H$ by

$$\{F \otimes f, G \otimes g\}(t) = FG \otimes \{f,g\}(t) + FD_f G \otimes g(t) - GD_g F \otimes f(t), \quad t \in \mathbb{R}_+, \tag{2.10}$$

and (2.8) extends similarly as

$$(D_u D_v - D_v D_u)F = D_{\{u,v\}}F$$

to all $u, v \in \mathcal{S} \otimes H$, $F \in \mathcal{S}$.

Covariant derivative

In the sequel we will refer to stochastic processes of the form

$$u(t,\omega) = \sum_{i=1}^{n} F_i(\omega) \otimes h_i(t), \qquad t \in \mathbb{R}_+, \quad \omega \in \Omega, \tag{2.11}$$

with $F_1, \ldots, F_n \in \mathcal{S}$, $h_1, \ldots, h_n \in L^2(\mathbb{R}_+)$, as a simple (vector-valued) vector fields. We extend D to a closable operator on the domain $\mathbb{D}_{2,1}(L^2(\mathbb{R}_+)) \subset L^2(\Omega; L^2(\mathbb{R}_+))$ by defining

$$D_s u(t,\omega) := \sum_{i=1}^{n} D_s F_i(\omega) \otimes h_i(t), \qquad s,t \in \mathbb{R}_+, \quad \omega \in \Omega,$$

on simple vector fields, as in e.g. Remark 2 page 31 of [14] on the Wiener space.

The covariant derivative operator ∇ will be defined on the space

$$\mathrm{Dom}(\nabla) := \left\{ u \in \mathbb{D}_{2,1}(L^2(\mathbb{R}_+)) \ : \ E\left[\int_0^\infty |u'(t,\omega)|^2 \, t \, dt \right] < \infty \right\}$$

where $u'(t,\omega)$ denotes the partial time derivative of $t \mapsto u(t,\omega)$ in the Sobolev sense, $P(d\omega)$-a.e. Given $u \in \mathrm{Dom}(\nabla)$ the covariant derivative ∇u of u is the process defined as

$$\nabla_s u(t) := D_s u(t) - \mathbf{1}_{[0,t]}(s) u'(t), \qquad s,t \in \mathbb{R}_+, \tag{2.12}$$

cf. Section 3 of [19].

The use of the time derivative in (2.12) is a major difference from chaos-based settings which involve a vanishing covariant derivative operator ∇ on H and finite

differences operators which do not have the derivation property in the Poisson case, cf. [24].

With respect to the construction of [2], [3], the operator ∇ defined in (2.12) has the advantage to satisfy the simple commutation relation (2.16) below between the gradient D and the divergence δ, which takes the same form as on the Wiener and Lie-Wiener path spaces.

The operator ∇ is closed with domain $\mathrm{Dom}(\nabla)$, and continuous with respect to the norm

$$\|u\|^2_{\mathrm{Dom}(\nabla)} := E\left[\int_0^\infty \int_0^\infty |D_s u(t)|^2 ds dt\right] + E\left[\int_0^\infty t|u'(t)|^2 dt\right], \quad u \in \mathrm{Dom}(\nabla).$$

The covariant derivative $\nabla_v u$ of u of the form (2.11) in the direction of $v \in L^2(\mathbb{R}_+)$ is defined by

$$\begin{aligned} \nabla_v u(t) &:= \langle v, \nabla u(t)\rangle \\ &= \int_0^\infty v(s)\nabla_s u(t) ds \\ &= \sum_{i=1}^n h_i(t) D_v F_i - F_i h_i'(t) \int_0^t v(s) ds, \qquad t \in \mathbb{R}_+, \end{aligned}$$

$u, v \in \mathcal{S}\otimes H$, where $D_v F$ is defined in (2.4). We also note that the operator ∇ satisfies

$$\nabla_t f(s) = 0, \qquad 0 < s < t, \quad f \in H, \tag{2.13}$$

by the definition (2.12), as in Assumption A4 of [24].

Vanishing of torsion

The next proposition, cf. Proposition 3.1 of [19], Proposition 3.2 of [20], or Proposition 7.6.3 of [23], corresponds to Assumption A1 in [24].

Proposition 2.3. *The connection defined by ∇ has a vanishing torsion, i.e. the Lie bracket $\{\cdot,\cdot\}$ satisfies*

$$\{u, v\} = \nabla_u v - \nabla_v u, \qquad u, v \in \mathcal{S} \otimes H. \tag{2.14}$$

Proof. By (2.10) it suffices to check that

$$\begin{aligned}\{f,g\}(t) &= -g'(t)\int_0^t f(s)ds + f'(t)\int_0^t g(s)ds\\ &= -\int_0^\infty f(s)g'(t)\mathbf{1}_{[0,t]}(s)ds + \int_0^\infty g(s)f'(t)\mathbf{1}_{[0,t]}(s)ds\\ &= \int_0^\infty f(s)\nabla_s g(t)ds - \int_0^\infty g(s)\nabla_s f(t)ds\\ &= \nabla_f g(t) - \nabla_g f(t), \qquad t\in\mathbb{R}_+,\end{aligned}$$

for all $f,g\in H$. □

From (2.8) the vanishing of torsion (2.14) can be written as

$$\begin{aligned}&\int_0^\infty \langle h, DD_tF - D_tDF\rangle_{L^2(\mathbb{R}_+)} g(t)dt \qquad (2.15)\\ &\qquad = \int_0^\infty h(t)\langle \nabla_t g, DF\rangle_{L^2(\mathbb{R}_+)}dt - \int_0^\infty g(t)\langle \nabla_t h, DF\rangle_{L^2(\mathbb{R}_+)}dt,\end{aligned}$$

for $F\in\mathcal{S}$, $f,g\in H$.

Gradient-divergence intertwinning relation

Noting that

$$\delta(h) = \sum_{k=1}^\infty h(T_k) - \int_0^\infty h(s)ds, \qquad h\in H,$$

the operators ∇, δ and D can be shown to satisfy the commutation relation

$$D_t\delta(h) = h(t) + \delta(\nabla_t h), \qquad t\in\mathbb{R}_+, \quad h\in H, \qquad (2.16)$$

cf. Relation (3.6) and Proposition 3.3 in [19], or Lemma 7.6.6 page 276 of [23]. Next we extend the commutation relation (2.16) to cylindrical random processes in the next proposition, cf. Assumption A5 in [24], which will be needed in Section 5 on the Weitzenböck identity.

Lemma 2.4. *(Intertwining relation). For all $u\in\mathcal{S}\otimes H$ of the form $u = F\otimes h$ we have*

$$\langle g, D\delta(u)\rangle_{L^2(\mathbb{R}_+)} = \langle g,u\rangle_{L^2(\mathbb{R}_+)} + \delta(\nabla_g u) + \langle \nabla_h g, DF\rangle_{L^2(\mathbb{R}_+)}, \qquad g\in H. \qquad (2.17)$$

Proof. By Relation (2.6) together with (2.16) and the derivation rule (2.5) we get

$$\begin{aligned} D_t\delta(F\otimes h) &= D_t(F\delta(h)-\langle h, DF\rangle_{L^2(\mathrm{R}_+)}) \\ &= \delta(h)D_tF + FD_t\delta(h) - D_t\langle h, DF\rangle_{L^2(\mathrm{R}_+)} \\ &= \delta(h)D_tF + Fh(t) + F\delta(\nabla_t h) - D_t\langle h, DF\rangle_{L^2(\mathrm{R}_+)} \\ &= Fh(t) + \delta(hD_tF) + \langle h, DD_tF\rangle_{L^2(\mathrm{R}_+)} + \delta(F\nabla_t h) \\ &\quad +\langle \nabla_t h(\cdot), D_\cdot F\rangle_{L^2(\mathrm{R}_+)} - \langle h, D_tDF\rangle_{L^2(\mathrm{R}_+)} \\ &= u(t) + \delta(\nabla_t u) + \langle h, DD_tF\rangle_{L^2(\mathrm{R}_+)} \\ &\quad -\langle h, D_tDF\rangle_{L^2(\mathrm{R}_+)} + \langle \nabla_t h(\cdot), D_\cdot F\rangle_{L^2(\mathrm{R}_+)}, \end{aligned} \tag{2.18}$$

which yields (2.17) by the vanishing torsion identity (2.15) and Relation (2.12) written as

$$\nabla_t u = D_tF\otimes h + F\otimes\nabla_t h, \qquad t\in\mathbb{R}_+.$$

□

We note in particular that Lemma 2.4 can be used to extend (2.16) to simple adapted vector field $u = F\otimes h\in\mathcal{S}\otimes H$ as

$$D_t\delta(u) = u(t) + \delta(\nabla_t u), \qquad t\in\mathbb{R}_+, \tag{2.19}$$

provided $h(s) = 0$ for $s\leq t$, since we have $h(r)D_rF = 0$, $r\in\mathbb{R}_+$, when F is $\mathcal{F}_t$-measurable, see e.g. Lemma 7.2.3 of [23].

3 Differential forms and exterior derivative

Exterior product

The exterior product $\wedge$ is defined as

$$h_1\wedge\cdots\wedge h_n := \mathcal{A}_n(h_1\otimes\cdots\otimes h_n), \quad h_1,\ldots,h_n\in H,$$

where $\mathcal{A}_n$ denotes the antisymmetrization map on n-tensors given by

$$\mathcal{A}_n(h_1\otimes\cdots\otimes h_n) = \sum_{\sigma\in\Sigma_n}\mathrm{sign}(\sigma)(h_{\sigma(1)}\otimes\cdots\otimes h_{\sigma(n)}),$$

and the summation is over $n!$ elements of the symmetric group Σ_n, consisting of all permutations of $\{1,\ldots,n\}$.

We denote by $\hat{H}^{\otimes n}$ the n-th tensor power of H, and by $\hat{H}^{\circ n}$, resp. $\hat{H}^{\wedge n}$, its subspaces of symmetric, resp. and skew-symmetric tensors, completed using the Hilbert space cross norm inherited from $L^2(\mathbb{R}_+)$. We also equip $\hat{H}^{\wedge n}$ with the inner product

$$\langle f_n, g_n\rangle_{\hat{H}^{\wedge n}} := \frac{1}{n!}\langle f_n, g_n\rangle_{\hat{H}^{\otimes n}}, \qquad f_n, g_n \in \hat{H}^{\wedge n}.$$

In the sequel we will work on the space $\mathcal{S}\otimes\hat{H}^{\wedge n}$ of elementary (random) n-forms that can be written as a linear combination of terms of the form

$$u_n = F\otimes h_1\wedge\cdots\wedge h_n \in \mathcal{S}\otimes\hat{H}^{\wedge n}, \qquad F\in\mathcal{S}, \quad h_1,\ldots,h_n\in H. \tag{3.1}$$

The operator D is extended to $u_n\in\mathcal{S}\otimes\hat{H}^{\wedge n}$ as in (3.1) by the pointwise equality

$$D_t u_n = (D_t F)\otimes(h_1\wedge\cdots\wedge h_n), \qquad t\in\mathbb{R}_+,$$

i.e. $Du_n\in H\otimes\mathcal{S}\otimes\hat{H}^{\wedge n}$.

Covariant derivative of differential forms

Using the partial covariant derivative $\nabla_s^{(j)}$ defined on $h_n\in\hat{H}^{\wedge n}$ as

$$\nabla_s^{(j)} h_n(t_1,\ldots,t_n) := -\mathbf{1}_{[0,t_j]}(s)\frac{\partial h_n}{\partial t_j}(t_1,\ldots,t_n), \qquad j=1,\ldots,n,$$

$t_1,\ldots,t_n\in\mathbb{R}_+$, we extend the definition of ∇ to deterministic tensors $h_n(t_1,\ldots,t_n)$ in $\hat{H}^{\wedge n}$ by letting

$$\nabla_s h_n(t_1,\ldots,t_n) := \sum_{j=1}^n \nabla_s^{(j)} h_n(t_1,\ldots,t_n) = -\sum_{j=1}^n \mathbf{1}_{[0,t_j]}(s)\frac{\partial h_n}{\partial t_j}(t_1,\ldots,t_n),$$

$t_1,\ldots,t_n\in\mathbb{R}_+$. Given $g\in H$ we also define $\nabla_g(h_1\wedge\cdots\wedge h_n)\in\hat{H}^{\wedge n}$ by

$$\begin{aligned}\nabla_g(h_1\wedge\cdots\wedge h_n) &= \int_0^\infty g(t)\nabla_t(h_1\wedge\cdots\wedge h_n)dt\\ &= \sum_{j=1}^n\int_0^\infty g(t)(h_1\wedge\cdots\wedge h_{j-1}\wedge\nabla_t h_j\wedge h_{j+1}\wedge\cdots\wedge h_n)dt\end{aligned}$$

$$= \sum_{j=1}^{n}(h_1 \wedge \cdots \wedge h_{j-1} \wedge \nabla_g h_j \wedge h_{j+1} \wedge \cdots \wedge h_n).$$

The definition of ∇_s also extends to random forms $u_n = F \otimes f_n \in \mathcal{S} \otimes \hat{H}^{\wedge n}$ by

$$\nabla_s(u_n(t_1, \ldots, t_n)) = (D_s F) \otimes f_n(t_1, \ldots, t_n) + F \otimes \nabla_s f_n(t_1, \ldots, t_n),$$

$s, t_1, \ldots, t_n \in \mathbb{R}_+$, i.e.

$$\nabla_s u_n(t_1, \ldots, t_n) = D_s u_n(t_1, \ldots, t_n) - \sum_{j=1}^{n} \mathbf{1}_{[0,t_j]}(s) \frac{\partial u_n}{\partial t_j}(t_1, \ldots, t_n).$$

Exterior derivative

The exterior derivative $\mathrm{d}^n(h_1 \wedge \cdots \wedge h_n)$ of the n-form $h_1 \wedge \cdots \wedge h_n \in \hat{H}^{\wedge n}$ is the $(n+1)$-form

$$\begin{aligned}
&\mathrm{d}^n_{t_{n+1}}(h_1 \wedge \cdots \wedge h_n(t_1, \ldots, t_n)) = \mathrm{d}^n(h_1 \wedge \cdots \wedge h_n)(t_1, \ldots, t_{n+1}) \\
&= \sum_{j=1}^{n+1}(-1)^{j-1} \nabla_{t_j}(h_1 \wedge \cdots \wedge h_n)(t_1, \ldots, t_{j-1}, t_{j+1}, \ldots, t_{n+1}) \\
&= \sum_{j=1}^{n+1}(-1)^{j-1} \sum_{i=1}^{n} \nabla^{(i)}_{t_j}(h_1 \wedge \cdots \wedge h_n)(t_1, \ldots, t_{j-1}, t_{j+1}, \ldots, t_{n+1}) \\
&= \sum_{j=1}^{n+1}(-1)^{j-1} \sum_{i=1}^{n}(h_1 \wedge \cdots \wedge \nabla_{t_j} h_i \wedge \cdots \wedge h_n)(t_1, \ldots, t_{j-1}, t_{j+1}, \ldots, t_{n+1})
\end{aligned}$$

$t_1, \ldots, t_{n+1} \in \mathbb{R}_+$, in $\hat{H}^{\wedge(n+1)}$. We now can define the exterior derivative of elementary forms of the form (3.1) by

$$\begin{aligned}
&\mathrm{d}^n_{t_{n+1}}(F \otimes h_1 \wedge \cdots \wedge h_n(t_1, \ldots, t_n)) := \mathrm{d}^n(F \otimes h_1 \wedge \cdots \wedge h_n)(t_1, \ldots, t_{n+1}) \\
&= ((D_\cdot F) \wedge h_1 \wedge \cdots \wedge h_n)(t_1, \ldots, t_{n+1}) + F \otimes \mathrm{d}^n(h_1 \wedge \cdots \wedge h_n)(t_1, \ldots, t_{n+1}).
\end{aligned}$$

In other words, for a random n-form $u_n \in \mathcal{S} \otimes \hat{H}^{\wedge n}$ we have

$$\begin{aligned}
\mathrm{d}^n u_n(t_1, \ldots, t_{n+1}) &= \mathrm{d}^n_{t_{n+1}}(u_n(t_1, \ldots, t_n)) \\
&= \frac{1}{n!} \mathcal{A}_{n+1}(\nabla_\cdot u_n)(t_1, \ldots, t_{n+1})
\end{aligned}$$

$$= \quad (D.F \wedge f_n)(t_1, \ldots, t_{n+1}) + \frac{1}{n!} F \otimes \mathcal{A}_{n+1}(\nabla . f_n)(t_1, \ldots, t_{n+1}),$$

which is also equal to

$$\sum_{j=1}^{n+1} (-1)^{j-1} \nabla_{t_j} u_n(t_1, \ldots, t_{j-1}, t_{j+1}, \ldots, t_{n+1})$$

in $\hat{H}^{\wedge(n+1)}$, for $u_n \in \mathcal{S} \otimes \hat{H}^{\wedge n}$. In this torsion free setting we also have

$$\langle h_1, \mathrm{d}^n u_n \wedge \cdots \wedge h_{n+1} \rangle_{\hat{H}^{\wedge(n+1)}} = \sum_{k=1}^{n+1} (-1)^{k-1} \langle h_1, \nabla_{h_k} u_n \wedge \cdots \wedge h_{k-1} \wedge h_{k+1} \wedge \cdots \wedge h_{n+1} \rangle_{\hat{H}^{\wedge n}}, \tag{3.2}$$

where $h_1, \ldots, h_{n+1} \in H$, i.e. $\mathrm{d}^n = \dfrac{1}{n!} \mathcal{A}_{n+1} \nabla$.

The invariant formula for differential forms (see e.g. Prop. 3.11 page 36 of [12]) shows that we have

$$\mathrm{d}^n(\mathcal{S} \otimes \hat{H}^{\wedge n}) \subset \mathrm{Dom}(\mathrm{d}^{n+1}), \qquad n \in \mathbb{N},$$

and

$$\mathrm{d}^{n+1} \mathrm{d}^n = 0, \qquad n \in \mathbb{N}. \tag{3.3}$$

In the appendix Section 7 we present an alternative derivation of (3.3) by explicit computation.

We close this section with the next lemma which, in the present framework, corresponds to Assumption A3 in [24]. Here the definition of ∇ is further extended to the higher deterministic tensor $h_n(t, t_1, \ldots, t_n)$ in $H \otimes \hat{H}^{\wedge n}$ by

$$\nabla_s h_n(t, t_1, \ldots, t_n) = -\sum_{j=1}^{n} \mathbf{1}_{[0,t_j]}(s) \frac{\partial h_n}{\partial t_j}(t, t_1, \ldots, t_n), \qquad s, t, t_1, \ldots, t_n \in \mathbb{R}_+.$$

Lemma 3.1. *The operator ∇ satisfies the condition*

$$\begin{aligned} &\int_0^\infty \cdots \int_0^\infty \langle \nabla_t f_{n+1}(t, t_1, \ldots, t_n), g_n(t_1, \ldots, t_n) \rangle_{\mathrm{R}^{\otimes n}} dt_1 \cdots dt_n dt \\ &\qquad = -\int_0^\infty \cdots \int_0^\infty \langle f_{n+1}(t, t_1, \ldots, t_n), \nabla_t g_n(t_1, \ldots, t_n) \rangle_{\mathrm{R}^{\otimes(n+1)}} dt_1 \cdots dt_n dt, \end{aligned} \tag{3.4}$$

$g_n \in \hat{H}^{\wedge n}$, $f_{n+1} \in \hat{H}^{\wedge(n+1)}$, $n \geq 1$.

Proof. By an elementary integration by parts for $\mathcal{C}^1_c(\mathbb{R}_+;\mathbb{R})$ functions we check that

$$\begin{aligned}
&\int_0^\infty \cdots \int_0^\infty f_{n+1}(t,t_1,\ldots,t_n)\nabla_t g_n(t_1,\ldots,t_n)dt_1\cdots dt_n\\
&= -\sum_{j=1}^n \int_0^\infty \cdots \int_0^\infty \mathbf{1}_{[0,t_j]}(t) f_{n+1}(t,t_1,\ldots,t_n)\frac{\partial g_n}{\partial t_j}(t_1,\ldots,t_n)dt_1\cdots dt_n\\
&= \sum_{j=1}^n \int_0^\infty \cdots \int_0^\infty f_{n+1}(t,t_1,\ldots,t_{j-1},t,t_{j+1},\ldots,t_n)\\
&\quad \times g_n(t_1,\ldots,t_{j-1},t,t_{j+1},\ldots,t_n)dt_1\cdots dt_{j-1}dt_{j+1}\cdots t_n\\
&\quad + \sum_{j=1}^n \int_0^\infty \cdots \int_0^\infty \mathbf{1}_{[0,t_j]}(t)\frac{\partial f_{n+1}}{\partial t_j}(t,t_1,\ldots,t_n) g_n(t_1,\ldots,t_n)dt_1\cdots dt_n\\
&= \sum_{j=1}^n \int_0^\infty \cdots \int_0^\infty \mathbf{1}_{[0,t_j]}(t)\frac{\partial f_{n+1}}{\partial t_j}(t,t_1,\ldots,t_n) g_n(t_1,\ldots,t_n)dt_1\cdots dt_n\\
&= -\int_0^\infty \cdots \int_0^\infty \nabla_t f_{n+1}(t,t_1,\ldots,t_n) g_n(t_1,\ldots,t_n)dt_1\cdots dt_n, \qquad t\in\mathbb{R}_+,
\end{aligned}$$

$f_{n+1}\in \hat{H}^{\wedge(n+1)}$, $g_n\in \hat{H}^{\otimes n}$, $n\geq 1$, since $f_{n+1}(t,t_1,\ldots,t_{j-1},t,t_{j+1},\ldots,t_n)=0$ by antisymmetry of $f_{n+1}\in \hat{H}^{\wedge(n+1)}$. □

4 Divergence of n-forms and duality

The divergence operator

$$\begin{aligned}
\delta : \mathcal{S}\otimes H &\longrightarrow L^2(\Omega),\\
v=(v_t)_{t\in\mathbb{R}_+} &\longmapsto \delta(v)
\end{aligned}$$

defined in (2.6) acts on stochastic processes $v\in\mathcal{S}\otimes H$ and it can be extended to elementary n-forms by letting

$$\delta(h_1\wedge\cdots\wedge h_n) := \frac{1}{n}\sum_{j=1}^n (-1)^{j-1}\delta(h_j)\otimes(h_1\wedge\cdots\wedge h_{j-1}\wedge h_{j+1}\wedge\cdots\wedge h_n)\in\mathcal{S}\otimes\hat{H}^{\wedge(n-1)},$$

and to random forms $u_n = F\otimes h_1\wedge\cdots\wedge h_n\in\mathcal{S}\otimes\hat{H}^{\wedge n}$ of the form (3.1) by

$$\begin{aligned}
\delta(u_n)(t_1,\ldots,t_{n-1}) &:= \delta(u_n(\cdot,t_1,\ldots,t_{n-1}))\\
&= \delta(F\otimes(h_1\wedge\cdots\wedge h_n))(t_1,\ldots,t_{n-1})
\end{aligned} \tag{4.1}$$

$$= \frac{1}{n}\sum_{j=1}^{n}(-1)^{j-1}\delta(F\otimes h_j)\otimes(h_1\wedge\cdots\wedge h_{j-1}\wedge h_{j+1}\wedge\cdots\wedge h_n)(t_1,\ldots,t_{n-1}).$$

The divergence operator d^{n*} on $(n+1)$-forms $u_{n+1}\in\mathcal{S}\otimes\hat{H}^{\wedge(n+1)}$ of the form (3.1) is defined by

$$\mathrm{d}^{n*}u_{n+1}(t_1,\ldots,t_n):=\delta(F\otimes f_{n+1}(\cdot,t_1,\ldots,t_n))-F\,\mathrm{trace}\,(\nabla_{\cdot}f_{n+1}(\cdot,t_1,\ldots,t_n))$$

where

$$\mathrm{trace}\,(\nabla_{\cdot}f_{n+1}(\cdot,t_1,\ldots,t_n)):=\int_0^\infty\nabla_t f_{n+1}(t,t_1,\ldots,t_n)dt.$$

In other words we have

$$\mathrm{d}^{n*}u_{n+1}(t_1,\ldots,t_n) \;=\; \delta(F\otimes f_{n+1}(\cdot,t_1,\ldots,t_n))-F\int_0^\infty\nabla_t f_{n+1}(t,t_1,\ldots,t_n)dt, \tag{4.2}$$

which belongs to $\mathcal{S}\otimes\hat{H}^{\wedge n}$ from (4.1), with

$$\mathrm{d}^{0*}u_1=\delta(u_1),\qquad u_1\in\mathcal{S}\otimes H,$$

since $\nabla_t f_1(t)=0$ as $f_1(t)$ is regarded here as a 0-form. Relation (2.6) for δ extends to d^{n*} as the divergence formula

$$\mathrm{d}^{n*}u_{n+1}(t_1,\ldots,t_n)=F\delta(f_{n+1}(\cdot,t_1,\ldots,t_n))-\int_0^\infty\nabla_t(Ff_{n+1}(t,t_1,\ldots,t_n))dt.$$

The next result is a consequence of Proposition 2.3 and Lemmas 2.4 and 3.1.

Proposition 4.1. (Duality). *For any $u_n\in\mathcal{S}\otimes\hat{H}^{\wedge n}$ and $v_{n+1}\in\mathcal{S}\otimes\hat{H}^{\wedge(n+1)}$ we have*

$$\langle\mathrm{d}^n u_n,v_{n+1}\rangle_{L^2(\Omega,\hat{H}^{\wedge(n+1)})}=\langle u_n,\mathrm{d}^{n*}v_{n+1}\rangle_{L^2(\Omega,\hat{H}^{\wedge n})}. \tag{4.3}$$

Proof. Assuming that $u_n=F\otimes f_n\in\mathcal{S}\otimes\hat{H}^{\wedge n}$ and $v_{n+1}=G\otimes g_{n+1}\in\mathcal{S}\otimes\hat{H}^{\wedge(n+1)}$ have the form (3.1) we have, using successively the definition (3.2) of d^n, the duality relation (2.7), the antisymmetry of g_{n+1}, the antisymmetry condition (3.4), and the definition (4.2) of d^{n*}, we have

$$\langle\mathrm{d}^n_{t_{n+1}}(Ff_n(t_1,\ldots,t_n)),Gg_{n+1}(t_1,\ldots,t_{n+1})\rangle_{L^2(\Omega,\hat{H}^{\wedge(n+1)})}$$

$$
\begin{aligned}
&= \sum_{j=1}^{n+1} \frac{(-1)^{j-1}}{(n+1)!} \langle \nabla_{t_j}(F f_n(t_1,\ldots,t_{j-1},t_{j+1},\ldots,t_{n+1})), G g_{n+1}(t_1,\ldots,t_{n+1}) \rangle_{L^2(\Omega,\hat{H}^{\otimes(n+1)})} \\
&= \sum_{j=1}^{n+1} \frac{(-1)^{j-1}}{(n+1)!} \langle D_{t_j} F f_n(t_1,\ldots,t_{j-1},t_{j+1},\ldots,t_{n+1}), G g_{n+1}(t_1,\ldots,t_{n+1}) \rangle_{L^2(\Omega,\hat{H}^{\otimes(n+1)})} \\
&+ \sum_{j=1}^{n+1} \frac{(-1)^{j-1}}{(n+1)!} \langle F \nabla_{t_j} f_n(t_1,\ldots,t_{j-1},t_{j+1},\ldots,t_{n+1}), G g_{n+1}(t_1,\ldots,t_{n+1}) \rangle_{L^2(\Omega,\hat{H}^{\otimes(n+1)})} \\
&= \frac{1}{(n+1)!} \sum_{j=1}^{n+1} (-1)^{j-1} \langle F f_n(t_1,\ldots,t_{j-1},t_{j+1},\ldots,t_{n+1}), \delta(G g_{n+1}(t_1,\ldots,t_{j-1},\cdot,t_{j+1},\ldots,t_{n+1})) \rangle_{L^2(\Omega,\hat{H}^{\otimes n})} \\
&- \frac{1}{(n+1)!} \sum_{j=1}^{n+1} (-1)^{j-1} \langle F f_n(t_1,\ldots,t_{j-1},t_{j+1},\ldots,t_{n+1}), G \int_0^\infty \nabla_t g_{n+1}(t_1,\ldots,t_{j-1},t,t_{j+1},\ldots,t_{n+1}) dt \rangle_{L^2(\Omega,\hat{H}^{\otimes n})} \\
&= \frac{1}{(n+1)!} \sum_{j=1}^{n+1} \langle F f_n(t_1,\ldots,t_{j-1},t_{j+1},\ldots,t_{n+1}), \delta(G g_{n+1}(\cdot,t_1,\ldots,t_{j-1},t_{j+1},\ldots,t_{n+1})) \rangle_{L^2(\Omega,\hat{H}^{\otimes n})} \\
&- \frac{1}{(n+1)!} \sum_{j=1}^{n+1} \langle F f_n(t_1,\ldots,t_{j-1},t_{j+1},\ldots,t_{n+1}), G \int_0^\infty \nabla_t g_{n+1}(t,t_1,\ldots,t_{j-1},t_{j+1},\ldots,t_{n+1}) dt \rangle_{L^2(\Omega,\hat{H}^{\otimes n})} \\
&= \langle F f_n(t_1,\ldots,t_n), \mathrm{d}^{n*}(G g_{n+1})(t_1,\ldots,t_n) \rangle_{L^2(\Omega,\hat{H}^{\wedge n})}.
\end{aligned}
$$

□

As in (2.7) above we note that the duality (4.3) implies that d^n can be extended to a closed operator

$$
\mathrm{d}^n : \mathrm{Dom}(\mathrm{d}^n) \longrightarrow L^2(\Omega; \hat{H}^{\wedge(n+1)})
$$

with domain $\mathrm{Dom}(\mathrm{d}^n) \subset L^2(\Omega; \hat{H}^{\wedge n})$, and d^{n*}, $n \in \mathbb{N}$, can be extended to a closed operator

$$
\mathrm{d}^{n*} : \mathrm{Dom}(\mathrm{d}^{n*}) \longrightarrow L^2(\Omega; \hat{H}^{\wedge n})
$$

with domain $\mathrm{Dom}(\mathrm{d}^{n*}) \subset L^2(\Omega; \hat{H}^{\wedge(n+1)})$, by the same argument as in Proposition 3.1.2 of [23]. When $n = 0$, the statement of Proposition 4.1 reduces to (2.7).

Relation (3.3) shows that

$$
\mathrm{Im}\, \mathrm{d}^n \subset \mathrm{Ker}\, \mathrm{d}^{n+1}, \qquad n \in \mathbb{N}. \tag{4.4}
$$

In addition, the duality and the coboundary condition (3.3) imply

$$
\mathrm{d}^{n*} \mathrm{d}^{(n+1)*} = 0, \qquad n \in \mathbb{N}, \tag{4.5}
$$

and

$$\mathrm{Im}\,\mathrm{d}^{(n+1)*} \subset \mathrm{Ker}\,\mathrm{d}^{n*}, \qquad n \in \mathbb{N}. \tag{4.6}$$

De Rham-Hodge decomposition

Based on (4.4) and (4.6) we define the usual Hodge Laplacian Δ_n on differential n-forms as

$$\Delta_n = \mathrm{d}^{n-1}\mathrm{d}^{(n-1)*} + \mathrm{d}^{n*}\mathrm{d}^n,$$

and call harmonic n-forms the elements of the kernel $\mathrm{Ker}\,\Delta_n$ of Δ_n. Note that from the closedness of d^n and d^{n*}, the Hodge Laplacian Δ_n extends to a closed operator

$$\Delta_n : \mathrm{Dom}\,(\Delta_n) \longrightarrow L^2(\Omega; \hat{H}^{\wedge n})$$

with domain $\mathrm{Dom}\,(\Delta_n) \subset L^2(\Omega; \hat{H}^{\wedge n})$.

Proposition 4.2. *We have the de Rham-Hodge decomposition*

$$L^2(\Omega; \hat{H}^{\wedge n}) = \mathrm{Im}\,\mathrm{d}^{n-1} \oplus \mathrm{Im}\,\mathrm{d}^{n*} \oplus \mathrm{Ker}\,\Delta_n, \qquad n \geq 1. \tag{4.7}$$

Proof. The spaces of exact and co-exact forms $\mathrm{Im}\,\mathrm{d}^{n-1}$ and $\mathrm{Im}\,\mathrm{d}^{n*}$ are mutually orthogonal in $L^2(\Omega; \hat{H}^{\wedge n})$ by (3.3) or (4.5) and the duality of Proposition 4.1. In addition, the orthogonal complement $(\mathrm{Ker}\,\mathrm{d}^{(n-1)*}) \cap (\mathrm{Ker}\,\mathrm{d}^n)$ of $\mathrm{Im}\,\mathrm{d}^{n-1} \oplus \mathrm{Im}\,\mathrm{d}^{n*}$ in $L^2(\Omega; \hat{H}^{\wedge n})$ is made of n-forms u_n that are both closed ($d^n u_n = 0$) and co-closed ($d^{(n-1)*} u_n = 0$), hence it is contained in (and equal to) $\mathrm{Ker}\,\Delta_n$. □

The next lemma extends the intertwining relation of Lemma 2.4 to differential forms, and will be used in the proof of the Weitzenböck identity in Theorem 5.3.

Lemma 4.3. *For any $u_n \in \mathcal{S} \otimes \hat{H}^{\wedge n}$ of the form $u_n = F \otimes f_n$, with $F \in \mathcal{S}$ and $f_n, g_n \in \hat{H}^{\wedge n}$, we have*

$$\begin{aligned}
\langle \mathrm{d}^{n-1}\mathrm{d}^{(n-1)*} u_n(t_1, \ldots, t_n), g_n(t_1, \ldots, t_n)\rangle_{\hat{H}^{\wedge n}} &= nF\langle f_n(t_1, \ldots, t_n), g_n(t_1, \ldots, t_n)\rangle_{\hat{H}^{\wedge n}} \\
&+ \sum_{j=1}^n \langle \delta(\nabla_{t_j}(F \otimes f_n(t_1, \ldots, t_{j-1}, \cdot, t_{j+1}, \ldots, t_n))), g_n(t_1, \ldots, t_n)\rangle_{\hat{H}^{\wedge n}} \\
&- \Big\langle \mathrm{d}_{t_n}^{n-1}(F \otimes \mathrm{trace}\,\nabla . f_n(\cdot, t_1, \ldots, t_{n-1})), g_n(t_1, \ldots, t_n)\Big\rangle_{\hat{H}^{\wedge n}}
\end{aligned}$$

$$+\sum_{j=1}^{n}\int_0^\infty \langle D_t F\otimes f_n(t_1,\ldots,t_n),\nabla_{t_j}^{(j)} g_n(t_1,\ldots,t_{j-1},t,t_{j+1},\ldots,t_n)\rangle_{\hat{H}^{\wedge n}}dt. \tag{4.8}$$

Proof. By (2.17) we have

$$\begin{aligned}&\mathrm{d}_{t_1}^0\mathrm{d}^{0*}u_1 = D_{t_1}\delta(u_1)\\ &= u_1(t_1)+\delta(\nabla_{t_1}u_1)+\langle f_1, DD_{t_1}F\rangle_{L^2(\mathrm{R}_+)}-\langle f_1, D_{t_1}DF\rangle_{L^2(\mathrm{R}_+)}+\langle\nabla_{t_1}f_1(\cdot), D_\cdot F\rangle_{L^2(\mathrm{R}_+)},\end{aligned}$$

for $u_1 = F\otimes f_1\in\mathcal{S}\otimes H$. By the definitions (2.12) and (3.2) of ∇ and d^n, and (2.17) or (2.18) applied to

$$u(s) = F\otimes f_n(s,t_1,\ldots,t_{j-1},t_{j+1},\ldots,t_n),$$

we get

$$\begin{aligned}\mathrm{d}_{t_n}^{n-1}\delta(F\otimes f_n(\cdot,t_1,\ldots,t_{n-1})) &= \sum_{j=1}^{n}(-1)^{j-1}\nabla_{t_j}\delta(F\otimes f_n(\cdot,t_1,\ldots,t_{j-1},t_{j+1},\ldots,t_n))\\
&= nF\otimes f_n(t_1,\ldots,t_n)+\sum_{j=1}^{n}\delta(\nabla_{t_j}(F\otimes f_n(t_1,\ldots,t_{j-1},\cdot,t_{j+1},\ldots,t_n)))\\
&\quad+\sum_{j=1}^{n}(-1)^{j-1}\langle f_n(\cdot,t_1,\ldots,t_{j-1},t_{j+1},\ldots,t_n), D_\cdot D_{t_j}F-D_{t_j}D_\cdot F\rangle_{L^2(\mathrm{R}_+)}\\
&\quad+\sum_{j=1}^{n}\langle\nabla_{t_j}^{(j)}f_n(t_1,\ldots,t_{j-1},\cdot,t_{j+1},\ldots,t_n), D_\cdot F\rangle_{L^2(\mathrm{R}_+)}\\
&= nF\otimes f_n(t_1,\ldots,t_n)+\sum_{j=1}^{n}\delta(\nabla_{t_j}(F\otimes f_n(t_1,\ldots,t_{j-1},\cdot,t_{j+1},\ldots,t_n)))\\
&\quad+\sum_{j=1}^{n}\int_0^\infty\langle\nabla_{t_j}^{(j)}g_n(t_1,\ldots,t_{j-1},t,t_{j+1},\ldots,t_n), D_tF\otimes f_n(t_1,\ldots,t_n)\rangle_{\hat{H}^{\wedge n}}dt.,\end{aligned}$$

by the vanishing of torsion (2.15) applied to $h(s) = f_n(t_1,\ldots,t_{j-1},s,t_{j+1},\ldots,t_n)$, using the operator $\nabla_{t_j}^{(j)}$ which applies to the $j-th$ component of $f_n(t_1,\ldots,t_{j-1},\cdot,t_{j+1},\ldots,t_n)$. We conclude by the definition

$$\mathrm{d}^{(n-1)*}(F\otimes f_n)(t_1,\ldots,t_{n-1}) = \delta(F\otimes f_n(\cdot,t_1,\ldots,t_{n-1}))-F\otimes\mathrm{trace}\nabla_\cdot f_n(\cdot,t_1,\ldots,t_{n-1})$$

of d^{n*}, cf. (4.2). □

For $n = 1$, Relation (4.8) and the fact that $\nabla_t f_1(t) = 0$ yield

$$\langle g_1(t_1), \mathrm{d}_{t_1}^0 \mathrm{d}^{0*} u_1(t_1)\rangle_{L^2(\mathrm{R}_+)}$$
$$= F\langle f(t_1), g_1(t_1)\rangle_{L^2(\mathrm{R}_+)} + \langle \delta(\nabla_{t_1}(F \otimes f_1)), g_1(t_1)\rangle_{L^2(\mathrm{R}_+)} + \langle \langle \nabla_{t_1} g_1(\cdot), D.F\rangle_{L^2(\mathrm{R}_+)}, f_1(t_1)\rangle_{L^2(\mathrm{R}_+)},$$

$F \in \mathcal{S}$, $f_1, g_1 \in H$, which coincides with the commutation relation (2.17) of Lemma 2.4

5 Weitzenböck identity for n-forms

In this section we will use the following dense subspace H_n of $\hat{H}^{\wedge n}$.

Definition 5.1. *Let H_n denote the space of continuously differentiable functions in $\mathcal{C}_c^1(\mathbb{R}_+^n; \mathbb{R})$ with compact support in $\mathbb{R}_+^n$, and vanish in a neighborhood of the diagonals $\{(t_1, \ldots, t_n) \ : \ t_i = t_j \text{ for some } i \neq j\}$ of $\mathbb{R}_+^n$.*

The proof of the Weitzenböck identity in Theorem 5.3 below relies on the following lemma which corresponds to Assumption B1 in [24].

Lemma 5.2. *For all $n \geq 1$ the covariant derivative operator satisfies*

$$\mathrm{d}_{t_n}^{n-1} \nabla_t f_n(t, t_1, \ldots, t_{n-1}) = \nabla_t \mathrm{d}_{t_n}^{n-1} f_n(t, t_1, \ldots, t_{n-1}), \tag{5.1}$$

$t_1, \ldots, t_n \in \mathbb{R}_+$, $t \in \mathbb{R}_+$, $f_n \in H_n$.

Proof. Since the functions in H_n vanish in a neighborhood of the diagonals, we check by explicit derivation that for any $f_n \in H_n$ we have

$$\begin{aligned}
&\mathrm{d}_{t_n}^{n-1} \nabla_t f_n(t, t_1, \ldots, t_{n-1}) \\
&= \sum_{j=1}^n (-1)^{j-1} \nabla_{t_j} \nabla_t f_n(t, t_1, \ldots, t_{j-1}, t_{j+1}, \ldots, t_n) \\
&= \sum_{j=1}^n (-1)^j \sum_{\substack{i=1 \\ i \neq j}}^n \nabla_{t_j} \left(\mathbf{1}_{[0,t_i]}(t) \frac{\partial f_n}{\partial t_i}(t, t_1, \ldots, t_{j-1}, t_{j+1}, \ldots, t_n) \right) \\
&= -\sum_{j=1}^n (-1)^j \sum_{\substack{i=1 \\ i \neq j}}^n \sum_{\substack{k=1 \\ k \neq j}}^n \mathbf{1}_{[0,t_i]}(t) \mathbf{1}_{[0,t_k]}(t_j) \frac{\partial^2 f_n}{\partial t_i \partial t_k}(t, t_1, \ldots, t_{j-1}, t_{j+1}, \ldots, t_n) \\
&\quad + \sum_{j=1}^n (-1)^j \sum_{\substack{i=1 \\ i \neq j}}^n \delta_t(t_i) \mathbf{1}_{[0,t_i]}(t_j) \frac{\partial f_n}{\partial t_i}(t, t_1, \ldots, t_{j-1}, t_{j+1}, \ldots, t_n)
\end{aligned}$$

$$= \sum_{j=1}^n (-1)^{j-1} \sum_{\substack{i=1\\ i\neq j}}^n \sum_{\substack{k=1\\ k\neq j}}^n \mathbf{1}_{[0,t_i]}(t)\mathbf{1}_{[0,t_k]}(t_j) \frac{\partial^2 f_n}{\partial t_i \partial t_k}(t, t_1, \ldots, t_{j-1}, t_{j+1}, \ldots, t_n),$$

due to the vanishing

$$\frac{\partial f_n}{\partial t_i}(t, t_1, \ldots, t_i, \ldots, t_{j-1}, t_{j+1}, \ldots, t_n) = 0$$

when $t = t_i$. On the other hand we have

$$\begin{aligned}
&\nabla_t \mathrm{d}_{t_n}^{n-1} f_n(t, t_1, \ldots, t_{n-1}) \\
&\quad = \sum_{j=1}^n (-1)^j \sum_{\substack{i=1\\ i\neq j}}^n \nabla_t \left(\mathbf{1}_{[0,t_i]}(t_j) \frac{\partial f_n}{\partial t_i}(t, t_1, \ldots, t_{j-1}, t_{j+1}, \ldots, t_n) \right) \\
&\quad = -\sum_{j=1}^n (-1)^j \sum_{\substack{i=1\\ i\neq j}}^n \sum_{\substack{k=1\\ k\neq j}}^n \mathbf{1}_{[0,t_i]}(t)\mathbf{1}_{[0,t_k]}(t_j) \frac{\partial^2 f_n}{\partial t_i \partial t_k}(t, t_1, \ldots, t_{j-1}, t_{j+1}, \ldots, t_n) \\
&\qquad - \sum_{j=1}^n (-1)^j \sum_{\substack{i=1\\ i\neq j}}^n \delta_{t_j}(t_i)\mathbf{1}_{[0,t_i]}(t) \frac{\partial f_n}{\partial t_i}(t, t_1, \ldots, t_{j-1}, t_{j+1}, \ldots, t_n) \\
&\qquad + \sum_{j=1}^n (-1)^j \sum_{\substack{i=1\\ i\neq j}}^n \delta_{t_i}(t_j)\mathbf{1}_{[0,t_i]}(t) \frac{\partial f_n}{\partial t_i}(t, t_1, \ldots, t_{j-1}, t_{j+1}, \ldots, t_n) \\
&\quad = -\sum_{j=1}^n (-1)^j \sum_{\substack{i=1\\ i\neq j}}^n \sum_{\substack{k=1\\ k\neq j}}^n \mathbf{1}_{[0,t_i]}(t)\mathbf{1}_{[0,t_k]}(t_j) \frac{\partial^2 f_n}{\partial t_i \partial t_k}(t, t_1, \ldots, t_{j-1}, t_{j+1}, \ldots, t_n).
\end{aligned}$$

□

Note that (5.1) differs from the vanishing of curvature condition which reads $\nabla_u \nabla_v - \nabla_v \nabla_u = \nabla_{\{u,v\}}$, where $\{u, v\}$ is the Lie bracket of two vector fields u, v, cf. Proposition 3.2 of [19] or Proposition 7.6.4 of [23].

The next Theorem 5.3 is an extension to n-forms of Lemma 3.2 of [20] and Proposition 3.3 of [19], or Proposition 2 of [18], which are stated in the case of one-forms. The following Weitzenböck identity (5.3) is interpreted as

$$\Delta_n = n\mathrm{Id}_{\hat{H}^{\wedge n}} + \nabla^* \nabla$$

on $\mathcal{S} \otimes H_n$, $n \geq 1$, where ∇^* is the formal adjoint of ∇.

Theorem 5.3. *For any $n \geq 1$ we have the Weitzenböck identity*

$$\langle \Delta_n u_n, v_n \rangle_{L^2(\Omega, \hat{H}^{\otimes n})} = n \langle u_n, v_n \rangle_{L^2(\Omega, \hat{H}^{\otimes n})} + \langle \nabla u_n, \nabla v_n \rangle_{L^2(\Omega, \hat{H}^{\otimes (n+1)})}, \tag{5.2}$$

$u_n, v_n \in \mathcal{S} \otimes H_n$.

Proof. We will show that

$$n! \| \mathrm{d}^{(n-1)*} u_n \|^2_{L^2(\Omega, \hat{H}^{\wedge (n-1)})} + n! \| \mathrm{d}^n u_n \|^2_{L^2(\Omega, \hat{H}^{\wedge (n+1)})} = nn! \| u_n \|^2_{L^2(\Omega, \hat{H}^{\wedge n})} + \| \nabla u_n \|^2_{L^2(\Omega, \hat{H}^{\otimes (n+1)})},$$

i.e.

$$\| \mathrm{d}^{(n-1)*} u_n \|^2_{L^2(\Omega, \hat{H}^{\otimes (n-1)})} + \frac{1}{n+1} \| \mathrm{d}^n u_n \|^2_{L^2(\Omega, \hat{H}^{\otimes (n+1)})} = n \| u_n \|^2_{L^2(\Omega, \hat{H}^{\otimes n})} + \| \nabla u_n \|^2_{L^2(\Omega, \hat{H}^{\otimes (n+1)})}$$

for $u_n \in \mathcal{S} \otimes H_n$. The intertwining relation (4.8) of Lemma 4.3 reads, after application of Lemma 5.2 to get (5.4),

$$\begin{aligned}
\langle g_n(t_1, \ldots, t_n), \mathrm{d}^{n-1} \mathrm{d}^{(n-1)*} u_n(t_1, \ldots, t_n) \rangle_{\hat{H}^{\wedge n}} &= nF \langle f_n(t_1, \ldots, t_n), g_n(t_1, \ldots, t_n) \rangle_{\hat{H}^{\wedge n}} \\
&+ \sum_{j=1}^n \langle \delta(\nabla_{t_j}(F \otimes f_n(t_1, \ldots, t_{j-1}, \cdot, t_{j+1}, \ldots, t_n))), g_n(t_1, \ldots, t_n) \rangle_{\hat{H}^{\wedge n}} \\
&- \sum_{j=1}^n (-1)^{j-1} \Big\langle (D_{t_j} F) \otimes \int_0^\infty \nabla_s f_n(s, t_1, \ldots, t_{j-1}, t_{j+1}, \ldots, t_n) ds, g_n(t_1, \ldots, t_n) \Big\rangle_{\hat{H}^{\wedge n}} \qquad (5.3) \\
&- F \left\langle \int_0^\infty \nabla_t \mathrm{d}_{t_n}^{n-1} f_n(t, t_1, \ldots, t_{n-1}) dt, g_n(t_1, \ldots, t_n) \right\rangle_{\hat{H}^{\wedge n}} \qquad (5.4) \\
&+ \sum_{j=1}^n \int_0^\infty \langle \nabla_{t_j}^{(j)} g_n(t_1, \ldots, t_{j-1}, t, t_{j+1}, \ldots, t_n), (D_t F) \otimes f_n(t_1, \ldots, t_n) \rangle_{\hat{H}^{\wedge n}} dt.
\end{aligned}$$

Hence, using Lemma 3.1 from (5.3)-(5.4) to (5.5)-(5.6), and applying (2.13) from (5.6) to (5.9) and grouping (5.5) and (5.7) into (5.8) below, we find

$$\begin{aligned}
\langle \mathrm{d}_{t_n}^{n-1} \mathrm{d}^{(n-1)*} u_n(\cdot, t_1, \ldots, t_{n-1}), v_n(t_1, \ldots, t_n) \rangle_{L^2(\Omega, \hat{H}^{\wedge n})} &= n \langle u_n(t_1, \ldots, t_n), v_n(t_1, \ldots, t_n) \rangle_{L^2(\Omega, \hat{H}^{\wedge n})} \\
&+ \sum_{j=1}^n \langle \delta((D_{t_j} F) \otimes f_n(t_1, \ldots, t_{j-1}, \cdot, t_{j+1}, \ldots, t_n)), G \otimes g_n(t_1, \ldots, t_n) \rangle_{L^2(\Omega, \hat{H}^{\wedge n})} \\
&+ \sum_{j=1}^n \langle \delta(F \otimes \nabla_{t_j} f_n(t_1, \ldots, t_{j-1}, \cdot, t_{j+1}, \ldots, t_n)), G \otimes g_n(t_1, \ldots, t_n) \rangle_{L^2(\Omega, \hat{H}^{\wedge n})}
\end{aligned}$$

$$+\frac{1}{n}\sum_{j=1}^{n}(-1)^{n-j}\sum_{l=1}^{n-1}\int_0^\infty\int_0^\infty\langle(D_tF)\otimes f_n(t_1,\dots,t_{j-1},s,t_{j+1},\dots,t_n), \tag{5.5}$$

$$G\otimes\nabla_s^{(l)}g_n(t_1,\dots,t_{j-1},t_{j+1},\dots,t_n,t)\rangle_{L^2(\Omega,\hat{H}^{\wedge(n-1)})}dsdt$$

$$+\frac{1}{n}\sum_{l=1}^{n}\sum_{\substack{j=1\\j\neq l}}^{n}\int_0^\infty\int_0^\infty\langle F\otimes\nabla_s^{(j)}f_n(t_1,\dots,t_{l-1},t,t_{l+1},\dots,t_n), \tag{5.6}$$

$$G\otimes\nabla_t^{(l)}g_n(t_1,\dots,t_{j-1},s,t_{j+1},\dots,t_n)\rangle_{L^2(\Omega,\hat{H}^{\wedge(n-1)})}dtds$$

$$+\sum_{j=1}^{n}(-1)^{n-j}\int_0^\infty\Big\langle(D_tF)\otimes f_n(t_1,\dots,t_n),G\otimes\nabla_{t_j}^{(n)}g_n(t_1,\dots,t_{j-1},t_{j+1},\dots,t_n,t)\Big\rangle_{L^2(\Omega,\hat{H}^{\wedge n})}dt \tag{5.7}$$

$$=n\langle u_n(t_1,\dots,t_n),v_n(t_1,\dots,t_n)\rangle_{L^2(\Omega,\hat{H}^{\wedge n})}$$

$$+\frac{1}{n}\sum_{j=1}^{n}\int_0^\infty\int_0^\infty\langle(D_tF)\otimes f_n(t_1,\dots,t_{j-1},s,t_{j+1},\dots,t_n),$$

$$(D_sG)\otimes g_n(t_1,\dots,t_{j-1},t,t_{j+1},\dots,t_n)\rangle_{L^2(\Omega,\hat{H}^{\wedge(n-1)})}dtds$$

$$+\frac{1}{n}\sum_{j=1}^{n}\int_0^\infty\int_0^\infty\langle F\otimes\nabla_t f_n(t_1,\dots,t_{j-1},s,t_{j+1},\dots,t_n),$$

$$(D_sG)\otimes g_n(t_1,\dots,t_{j-1},t,t_{j+1},\dots,t_n)\rangle_{L^2(\Omega,\hat{H}^{\wedge(n-1)})}dtds$$

$$+\frac{1}{n}\sum_{j=1}^{n}(-1)^{n-j}\sum_{l=1}^{n}\int_0^\infty\int_0^\infty\langle(D_tF)\otimes f_n(t_1,\dots,t_{j-1},s,t_{j+1},\dots,t_n), \tag{5.8}$$

$$G\otimes\nabla_s^{(l)}g_n(t_1,\dots,t_{j-1},t_{j+1},\dots,t_n,t)\rangle_{L^2(\Omega,\hat{H}^{\wedge(n-1)})}dsdt$$

$$+\frac{1}{n}\sum_{j=1}^{n}(-1)^{n-j}\sum_{l=1}^{n}\int_0^\infty\int_0^\infty\langle F\otimes\nabla_s^{(j)}f_n(t_1,\dots,t_{l-1},t,t_{l+1},\dots,t_n), \tag{5.9}$$

$$G\otimes\nabla_t^{(l)}g_n(t_1,\dots,t_{j-1},t_{j+1},\dots,t_n,s)\rangle_{L^2(\Omega,\hat{H}^{\wedge(n-1)})}dtds$$

$$=n\langle u_n(t_1,\dots,t_n),v_n(t_1,\dots,t_n)\rangle_{L^2(\Omega,\hat{H}^{\wedge n})}$$

$$+\frac{1}{n!}\Big\langle(D_{t_{n+1}}F)\otimes f_n(t_1,\dots,t_n)+F\otimes\nabla_{t_{n+1}}f_n(t_1,\dots,t_n),$$

$$\sum_{j=1}^{n}(-1)^{n-j}(D_{t_j}G)\otimes g_n(t_1,\dots,t_{j-1},t_{j+1},\dots,t_n,t_{n+1})$$

$$+\sum_{j=1}^{n}(-1)^{n-j}\sum_{l=1}^{n}G\otimes\nabla_{t_j}^{(l)}g_n(t_1,\dots,t_{j-1},t_{j+1},\dots,t_n,t_{n+1})\Big\rangle_{L^2(\Omega,\hat{H}^{\otimes(n+1)})}$$

$$=n\langle u_n(t_1,\dots,t_n),v_n(t_1,\dots,t_n)\rangle_{L^2(\Omega,\hat{H}^{\wedge n})}$$

$$
\begin{aligned}
&- \frac{(-1)^n}{n!} \Big\langle (D_{t_{n+1}} F) \otimes f_n(t_1, \dots, t_n) + F \otimes \nabla_{t_{n+1}} f_n(t_1, \dots, t_n), \\
&\sum_{j=1}^{n+1} (-1)^{j-1} (D_{t_j} G) \otimes g_n(t_1, \dots, t_{j-1}, t_{j+1}, \dots, t_{n+1}) \\
&+ \sum_{j=1}^{n+1} (-1)^{j-1} \sum_{l=1}^{n} G \otimes \nabla_{t_j}^{(l)} g_n(t_1, \dots, t_{j-1}, t_{j+1}, \dots, t_{n+1}) \Big\rangle_{L^2(\Omega, \hat{H}^{\otimes(n+1)})} \\
&+ \frac{1}{n!} \Big\langle (D_{t_{n+1}} F) \otimes f_n(t_1, \dots, t_n) + F \otimes \nabla_{t_{n+1}} f_n(t_1, \dots, t_n), \\
&(D_{t_{n+1}} G) \otimes g_n(t_1, \dots, t_n) + G \otimes \sum_{l=1}^{n} \nabla_{t_{n+1}}^{(l)} g_n(t_1, \dots, t_n) \Big\rangle_{L^2(\Omega, \hat{H}^{\otimes(n+1)})} \\
&= n \langle u_n(t_1, \dots, t_n), v_n(t_1, \dots, t_n) \rangle_{L^2(\Omega, \hat{H}^{\wedge n})} \\
&- \frac{(-1)^n}{n!} \Big\langle (D_{t_{n+1}} F) \otimes f_n(t_1, \dots, t_n) + F \otimes \nabla_{t_{n+1}} f_n(t_1, \dots, t_n), \\
&\sum_{j=1}^{n+1} (-1)^{j-1} (D_{t_j} G) \otimes g_n(t_1, \dots, t_{j-1}, t_{j+1}, \dots, t_{n+1}) \\
&+ \sum_{j=1}^{n+1} (-1)^{j-1} G \otimes \nabla_{t_j} g_n(t_1, \dots, t_{j-1}, t_{j+1}, \dots, t_{n+1}) \Big\rangle_{L^2(\Omega, \hat{H}^{\otimes(n+1)})} \\
&+ \frac{1}{n!} \big\langle (D_{t_{n+1}} F) \otimes f_n(t_1, \dots, t_n) + F \otimes \nabla_{t_{n+1}} f_n(t_1, \dots, t_n), \\
&(D_{t_{n+1}} G) \otimes g_n(t_1, \dots, t_n) + G \otimes \nabla_{t_{n+1}} g_n(t_1, \dots, t_n) \big\rangle_{L^2(\Omega, \hat{H}^{\otimes(n+1)})} \\
&= n \langle u_n(t_1, \dots, t_n), v_n(t_1, \dots, t_n) \rangle_{L^2(\Omega, \hat{H}^{\wedge n})} \\
&- \frac{1}{(n+1)!} \Big\langle \sum_{j=1}^{n+1} (-1)^{j-1} (D_{t_j} F) \otimes f_n(t_1, \dots, t_{j-1}, t_{j+1}, \dots, t_{n+1}) \\
&+ \sum_{j=1}^{n+1} (-1)^{j-1} F \otimes \nabla_{t_j} f_n(t_1, \dots, t_{j-1}, t_{j+1}, \dots, t_{n+1}), \\
&\sum_{j=1}^{n+1} (-1)^{j-1} (D_{t_j} G) \otimes g_n(t_1, \dots, t_{j-1}, t_{j+1}, \dots, t_{n+1}) \\
&+ \sum_{j=1}^{n+1} (-1)^{j-1} G \otimes \nabla_{t_j} g_n(t_1, \dots, t_{j-1}, t_{j+1}, \dots, t_{n+1}) \Big\rangle_{L^2(\Omega, \hat{H}^{\otimes(n+1)})} \\
&+ \frac{1}{n!} \big\langle (D_{t_{n+1}} F) \otimes f_n(t_1, \dots, t_n) + F \otimes \nabla_{t_{n+1}} f_n(t_1, \dots, t_n), \\
&(D_{t_{n+1}} G) \otimes g_n(t_1, \dots, t_n) + G \otimes \nabla_{t_{n+1}} g_n(t_1, \dots, t_n) \big\rangle_{L^2(\Omega, \hat{H}^{\otimes(n+1)})}
\end{aligned}
$$

$$= n\langle u_n(t_1,\dots,t_n), v_n(t_1,\dots,t_n)\rangle_{L^2(\Omega,\hat{H}^{\wedge n})} - \langle \mathrm{d}^n u_n, \mathrm{d}^n v_n\rangle_{L^2(\Omega,\hat{H}^{\wedge(n+1)})}$$
$$+ \frac{1}{n!}\langle \nabla_{t_{n+1}} u_n(t_1,\dots,t_n), \nabla_{t_{n+1}} v_n(t_1,\dots,t_n)\rangle_{L^2(\Omega,\hat{H}^{\wedge(n+1)})}.$$

Hence we have

$$\langle \mathrm{d}^{(n-1)*} u_n, \mathrm{d}^{(n-1)*} v_n\rangle_{L^2(\Omega,\hat{H}^{\wedge n})} + \langle \mathrm{d}^n u_n, \mathrm{d}^n v_n\rangle_{L^2(\Omega,\hat{H}^{\wedge(n+1)})}$$
$$= n\langle u_n, v_n\rangle_{L^2(\Omega,\hat{H}^{\wedge n})} + \frac{1}{n!}\langle \nabla u_n, \nabla v_n\rangle_{L^2(\Omega,\hat{H}^{\otimes(n+1)})},$$

i.e.

$$\langle \mathrm{d}^{(n-1)*} u_n, \mathrm{d}^{(n-1)*} v_n\rangle_{L^2(\Omega,\hat{H}^{\otimes n})} + \frac{1}{n+1}\langle \mathrm{d}^n u_n, \mathrm{d}^n v_n\rangle_{L^2(\Omega,\hat{H}^{\otimes(n+1)})}$$
$$= n\langle u_n, v_n\rangle_{L^2(\Omega,\hat{H}^{\otimes n})} + \langle \nabla u_n, \nabla v_n\rangle_{L^2(\Omega,\hat{H}^{\otimes(n+1)})},$$

and applying the duality

$$\langle \nabla u_n, \nabla v_n\rangle_{L^2(\Omega,\hat{H}^{\otimes(n+1)})} = \langle \nabla^*\nabla u_n, v_n\rangle_{L^2(\Omega,\hat{H}^{\wedge n})}, \qquad u_n, v_n \in \mathcal{S}\otimes\hat{H}^{\wedge n},$$

we get

$$\mathrm{d}^{n-1}\mathrm{d}^{(n-1)*} + \mathrm{d}^{n*}\mathrm{d}^n = n\mathrm{I}_{\hat{H}^{\wedge n}} + \nabla^*\nabla.$$

□

Relation (5.2) shows in particular that the Bochner Laplacian $L = -\nabla^*\nabla$ and the Hodge Laplacian Δ_n have same closed domain $\mathrm{Dom}(\Delta_n)$ on the random n-forms. In addition, Theorem 5.3 shows the following.

Corollary 5.4. *Let $n \geq 1$. All eigenvalues λ_n of the Hodge Laplacian Δ_n on the n-forms satisfy $\lambda_n \geq n$.*

Proof. Relation (5.2) rewrites as

$$L = n\mathrm{Id}_{\hat{H}^{\wedge n}} - \Delta_n,$$

so the Bochner Laplacian $L = -\nabla^*\nabla$ and the Hodge Laplacian Δ_n share the same eigenvectors. Next, for any $w_n \in \mathcal{S}\otimes H_n$ we have

$$0 \;\leq\; \langle \nabla w_n, \nabla w_n\rangle_{L^2(\Omega,\hat{H}^{\wedge n})}$$

$$
\begin{aligned}
&= -\langle Lw_n, w_n\rangle_{L^2(\Omega,\hat{H}^{\wedge n})} \\
&= \langle(\Delta_n - n\mathrm{Id}_{\hat{H}^{\wedge n}})w_n, w_n\rangle_{L^2(\Omega,\hat{H}^{\wedge n})} \\
&= \langle\Delta_n w_n, w_n\rangle_{L^2(\Omega,\hat{H}^{\wedge n})} - n\langle w_n, w_n\rangle_{L^2(\Omega,\hat{H}^{\wedge n})}, \qquad n \geq 1,
\end{aligned}
$$

which shows by closability of Δ_n that

$$
n\langle w_n, w_n\rangle_{L^2(\Omega,\hat{H}^{\wedge n})} \leq \langle\Delta_n w_n, w_n\rangle_{L^2(\Omega,\hat{H}^{\wedge n})}, \tag{5.10}
$$

for any $w_n \in \mathrm{Dom}(\Delta_n)$. In particular, if $w_n \in \mathrm{Dom}(\Delta_n)$ is an eigenvector of Δ_n with eigenvalue λ_n, then (5.10) shows that

$$
\lambda_n\langle w_n, w_n\rangle_{L^2(\Omega,\hat{H}^{\wedge n})} = \langle\Delta_n w_n, w_n\rangle_{L^2(\Omega,\hat{H}^{\wedge n})} \geq n\langle w_n, w_n\rangle_{L^2(\Omega,\hat{H}^{\wedge n})},
$$

hence $\lambda_n \geq n$. □

The next corollary is a consequence of Proposition 4.2 and Corollary 5.4.

Corollary 5.5. *We have* $\mathrm{Ker}\,\Delta_n = \{0\}$, $n \geq 1$, *and the de Rham-Hodge-Kodaira decomposition* (4.7) *reads*

$$
L^2(\Omega, \hat{H}^{\wedge n}) = \mathrm{Im}\,\mathrm{d}^{n-1} \oplus \mathrm{Im}\,\mathrm{d}^{n*}, \qquad n \geq 1.
$$

Proof. Corollary 5.4 shows that any harmonic form for the de Rham Laplacian Δ_n has to vanish, i.e. the space $\mathrm{Ker}\,\Delta_n$ of harmonic forms for the Δ_n is equal to $\{0\}$. We conclude by (4.7). □

6 Clark-Ocone representation formula for n-forms

Recall that the operator D satisfies the Clark-Ocone formula

$$
F = E[F \mid \mathcal{F}_t] + \int_t^\infty E[D_r F \mid \mathcal{F}_r] d(N_r - r), \qquad t \in \mathbb{R}_+, \tag{6.1}
$$

for $F \in \mathrm{Dom}(D)$, cf. e.g. Theorem 1 of [16] or Proposition 3.2.3 page 115 of [23], and Assumption C1 in [24]. Here, the stochastic integral with respect to the compensated Poisson process $(N_r - r)_{r\in\mathrm{R}_+}$ is the Itô integral defined in the L^2 sense.

Under the duality relation (2.7), the formula (6.1) is equivalent to stating that $(N_t - t)_{t\in\mathbb{R}_+}$ has the predictable representation property and δ coincides with the stochastic integral with respect to $(N_t - t)_{t\in\mathbb{R}_+}$ on the square-integrable predictable processes, cf. Corollary 3.2.8 and Propositions 3.3.1 and 3.3.2 of [23]. Also it is sufficient to assume that (6.1) holds for $t = 0$, cf. Proposition 3.2.3 of [23].

Note that the Clark-Ocone formula also holds in the different framework of finite difference operators on the Poisson space cf. Proposition 7.2.7 page 259 of [23] and Assumption C2 in [24].

Lemma 6.1. *The operators ∇ and δ satisfy the relations*

$$\nabla_s E[F \mid \mathcal{F}_t] = \mathbf{1}_{[0,t]}(s) E[D_s F \mid \mathcal{F}_t], \qquad s, t \in \mathbb{R}_+, \tag{6.2}$$

and

$$\delta(\mathbf{1}_{[0,t]}(\cdot) E[u_\cdot \mid \mathcal{F}_t]) = E[\delta(u) \mid \mathcal{F}_t], \qquad t \in \mathbb{R}_+, \tag{6.3}$$

for $F \in \mathcal{S}$ and $u \in \mathcal{S} \otimes H$, respectively.

Proof. Regarding (6.2) it suffices to consider $s \in [0, t]$ since both sides vanish when $s > t$. By Proposition 2.3.6 of [23], for any $f \in L^1(\mathbb{R}_+^n)$ we have

$$\begin{aligned}
&E[f(T_1, \ldots, T_n) \mid \mathcal{F}_t] \\
&= \int_t^\infty \mathrm{e}^{-(s_n - t)} \int_t^{s_n} \cdots \int_t^{s_{N_t+2}} f(T_1, \ldots, T_{N_t}, s_{N_t+1}, \ldots, s_n) ds_{N_t+1} \cdots ds_n \\
&= \sum_{k=0}^\infty \mathbf{1}_{\{N_t = k\}} \int_t^\infty \mathrm{e}^{-(s_n - t)} \int_t^{s_n} \cdots \int_t^{s_{k+2}} f(T_1, \ldots, T_k, s_{k+1}, \ldots, s_n) ds_{k+1} \cdots ds_n \\
&= \sum_{k=0}^\infty \mathbf{1}_{[T_k, T_{k+1})}(t) \int_0^\infty \mathrm{e}^{-s_n} \int_0^{s_n} \cdots \int_0^{s_{k+2}} f(T_1, \ldots, T_k, s_{k+1} + t, \ldots, s_n + t) ds_{k+1} \cdots ds_n.
\end{aligned}$$

Hence for $F \in \mathcal{S}$ of the form (2.2), by the definition (2.12) of ∇_s we have

$$\begin{aligned}
&\nabla_s E[F \mid \mathcal{F}_t] \\
&= -\sum_{k=0}^\infty \mathbf{1}_{[T_k, T_{k+1})}(t) \sum_{i=1}^k \mathbf{1}_{[0,T_i]}(s) \int_0^\infty \mathrm{e}^{-s_n} \int_0^{s_n} \cdots \int_0^{s_{k+2}} \partial_i f(T_1, \ldots, T_k, s_{k+1} + t, \ldots, s_n + t) ds_{k+1} \cdots ds_n \\
&+ \sum_{k=0}^\infty \delta_t(T_k) \mathbf{1}_{[0,T_k]}(s) \int_0^\infty \mathrm{e}^{-s_n} \int_0^{s_n} \cdots \int_0^{s_{k+2}} f(T_1, \ldots, T_k, s_{k+1} + t, \ldots, s_n + t) ds_{k+1} \cdots ds_n
\end{aligned}$$

$$
\begin{aligned}
&-\sum_{k=0}^{\infty}\delta_t(T_{k+1})\mathbf{1}_{[0,T_{k+1}]}(s)\int_0^{\infty}\mathrm{e}^{-s_n}\int_0^{s_n}\cdots\int_0^{s_{k+2}} f(T_1,\ldots,T_k,s_{k+1}+t,\ldots,s_n+t)ds_{k+1}\cdots ds_n\\
&-\sum_{k=0}^{\infty}\mathbf{1}_{[0,t]}(s)(\delta_{T_k}(t)-\delta_{T_{k+1}}(t))\int_0^{\infty}\mathrm{e}^{-s_n}\int_0^{s_n}\cdots\int_0^{s_{k+2}} f(T_1,\ldots,T_k,s_{k+1}+t,\ldots,s_n+t)ds_{k+1}\cdots ds_n\\
&-\sum_{k=0}^{\infty}\mathbf{1}_{[0,t]}(s)\mathbf{1}_{[T_k,T_{k+1})}(t)\sum_{i=k+1}^{n}\int_0^{\infty}\mathrm{e}^{-s_n}\int_0^{s_n}\cdots\int_0^{s_{k+2}} \partial_i f(T_1,\ldots,T_k,s_{k+1}+t,\ldots,s_n+t)ds_{k+1}\cdots ds_n\\
&=-\mathbf{1}_{[0,t]}(s)\sum_{k=0}^{\infty}\mathbf{1}_{[T_k,T_{k+1})}(t)\sum_{i=1}^{k}\mathbf{1}_{[0,T_i]}(s)\int_0^{\infty}\mathrm{e}^{-s_n}\int_0^{s_n}\cdots\int_0^{s_{k+2}} \partial_i f(T_1,\ldots,T_k,s_{k+1}+t,\ldots,s_n+t)ds_{k+1}\cdots ds_n\\
&-\mathbf{1}_{[0,t]}(s)\sum_{k=0}^{\infty}\mathbf{1}_{[T_k,T_{k+1})}(t)\mathbf{1}_{[0,s_i+t]}(s)\sum_{i=k+1}^{n}\int_0^{\infty}\mathrm{e}^{-s_n}\int_0^{s_n}\cdots\int_0^{s_{k+2}} \partial_i f(T_1,\ldots,T_k,s_{k+1}+t,\ldots,s_n+t)ds_{k+1}\cdots ds_n\\
&=-\mathbf{1}_{[0,t]}(s)\int_0^{\infty}\mathrm{e}^{-s_n}\int_0^{s_n}\cdots\int_0^{s_{N_t+2}}\sum_{i=1}^{N_t}\mathbf{1}_{[0,T_i]}(s)\partial_i f(T_1,\ldots,T_{N_t},s_{N_t+1}+t,\ldots,s_n+t)ds_{N_t+1}\cdots ds_n\\
&-\mathbf{1}_{[0,t]}(s)\int_0^{\infty}\mathrm{e}^{-s_n}\int_0^{s_n}\cdots\int_0^{s_{N_t+2}}\sum_{i=N_t+1}^{n}\mathbf{1}_{[0,s_i+t]}(s)\partial_i f(T_1,\ldots,T_{N_t},s_{N_t+1}+t,\ldots,s_n+t)ds_{N_t+1}\cdots ds_n\\
&=\mathbf{1}_{[0,t]}(s)E[D_sF\mid\mathcal{F}_t],\qquad s,t\in\mathbf{R}_+.
\end{aligned}
$$

Concerning (6.3), for all $\mathcal{F}_t$-measurable random variables $F\in\mathcal{S}$ we note that $D_sF=\mathbf{1}_{[0,t]}(s)D_sF$, $s\in\mathbb{R}_+$, cf. e.g. Lemma 7.2.3 of [23], hence we have

$$
E[F\delta(u)]=E[\langle DF,u\rangle_{L^2(\mathrm{R}_+)}]=E[\langle\mathbf{1}_{[0,t]}(\cdot)D.F,u(\cdot)\rangle_{L^2(\mathrm{R}_+)}]=E[F\delta(\mathbf{1}_{[0,t]}(\cdot)u(\cdot))],
$$

which leads to (6.3) by a density argument and a classical characterization of conditional expectations. □

In the appendix Section 7 we present an alternative proof of Lemma 6.1 based on chaos expansions. We note that (6.2) implies the following Corollary 6.2 which will be used in the proof of Theorem 6.3 below.

Corollary 6.2. *For any n-form $u_n\in\mathcal{S}\otimes\hat{H}^{\wedge n}$ we have*

$$
\nabla_sE[u_n(t_1,\ldots,t_n)\mid\mathcal{F}_{t_1}]=\mathbf{1}_{[0,t_1]}(s)E[\nabla_su_n(t_1,\ldots,t_n)\mid\mathcal{F}_{t_1}],\qquad s\in\mathbb{R}_+,
$$

whenever $0\leq t_i\leq t_1$, $i=2,\ldots,n$.

Proof. Consider $u_n\in\mathcal{S}\otimes\hat{H}^{\wedge n}$ of the form $u_n(t_1,\ldots,t_n)=F\otimes f_n(t_1,\ldots,t_n)$, $t_1,\ldots,t_n\in\mathbb{R}_+$, $f_n\in\hat{H}^{\wedge n}$. Since $\nabla_sf_n(t_1,\ldots,t_n)$ is deterministic we find, from (2.13) and Lemma 6.1,

$$
\nabla_s\mathbb{E}[u_n(t_1,\ldots,t_n)\mid\mathcal{F}_{t_1}]=\nabla_s(f_n(t_1,\ldots,t_n)\mathbb{E}[F\mid\mathcal{F}_{t_1}])
$$

$$
\begin{aligned}
&= \mathbb{E}[F \mid \mathcal{F}_{t_1}]\nabla_s f_n(t_1,\ldots,t_n) + f_n(t_1,\ldots,t_n)\nabla_s \mathbb{E}[F \mid \mathcal{F}_{t_1}] \\
&= \mathbb{E}[F\nabla_s f_n(t_1,\ldots,t_n) \mid \mathcal{F}_{t_1}] + \mathbf{1}_{[0,t_1]}(s) f_n(t_1,\ldots,t_n)\mathbb{E}[D_s F \mid \mathcal{F}_{t_1}] \\
&= \sum_{j=1}^{n} \mathbf{1}_{[0,t_j]}(s)\mathbb{E}[F\nabla_s^{(j)} f_n(t_1,\ldots,t_n) \mid \mathcal{F}_{t_1}] + \mathbf{1}_{[0,t_1]}(s) f_n(t_1,\ldots,t_n)\mathbb{E}[D_s F \mid \mathcal{F}_{t_1}] \\
&= \mathbf{1}_{[0,t_1]}(s)\sum_{j=1}^{n} \mathbf{1}_{[0,t_j]}(s)\mathbb{E}[F\nabla_s^{(j)} f_n(t_1,\ldots,t_n) \mid \mathcal{F}_{t_1}] + \mathbf{1}_{[0,t_1]}(s) f_n(t_1,\ldots,t_n)\mathbb{E}[D_s F \mid \mathcal{F}_{t_1}] \\
&= \mathbf{1}_{[0,t_1]}(s)\mathbb{E}[F\nabla_s f_n(t_1,\ldots,t_n) \mid \mathcal{F}_{t_1}] + \mathbf{1}_{[0,t_1]}(s) f_n(t_1,\ldots,t_n)\mathbb{E}[D_s F \mid \mathcal{F}_{t_1}] \\
&= \mathbf{1}_{[0,t_1]}(s)\mathbb{E}[\nabla_s(F \otimes f_n(t_1,\ldots,t_n)) \mid \mathcal{F}_{t_1}] \\
&= \mathbf{1}_{[0,t_1]}(s)\mathbb{E}[\nabla_s u_n(t_1,\ldots,t_n) \mid \mathcal{F}_{t_1}].
\end{aligned}
$$

□

Clark-Ocone formula for n-forms

The following result is a consequence of Proposition 2.3, Lemmas 2.4 and 6.1, and Relation (6.1). In the sequel, $t_1 \vee \cdots \vee t_n$ denotes $\max(t_1,\ldots,t_n)$.

Theorem 6.3. *For all $u_n \in \mathrm{Dom}(\mathrm{d}^n)$, we have, for a.e. $t_1,\ldots,t_n \in \mathbb{R}_+$,*

$$
\begin{aligned}
u_n(t_1,\ldots,t_n) &= \mathrm{d}_{t_n}^{n-1} \int_{t_1\vee\cdots\vee t_{n-1}}^{\infty} E[u_n(r,t_1,\ldots,t_{n-1}) \mid \mathcal{F}_r] d(N_r - r) \\
&\quad + \int_{t_1\vee\cdots\vee t_n}^{\infty} E[\mathrm{d}_{t_n}^n u_n(t_0,t_1,\ldots,t_{n-1}) \mid \mathcal{F}_{t_0}] d(N_{t_0} - t_0).
\end{aligned}
$$

Proof. We start by proving the formula for $u_n \in \mathcal{S} \otimes \hat{H}^{\wedge n}$. By (2.13) and the Clark-Ocone formula (6.1) we have

$$
\begin{aligned}
u_n(t_1,\ldots,t_n) &= E[u_n(t_1,\ldots,t_n) \mid \mathcal{F}_{t_1\vee\cdots\vee t_n}] + \int_{t_1\vee\cdots\vee t_n}^{\infty} E[D_r u_n(t_1,\ldots,t_n) \mid \mathcal{F}_r] d(N_r - r) \\
&= E[u_n(t_1,\ldots,t_n) \mid \mathcal{F}_{t_1\vee\cdots\vee t_n}] + \int_{t_1\vee\cdots\vee t_n}^{\infty} E[\nabla_r u_n(t_1,\ldots,t_n) \mid \mathcal{F}_r] d(N_r - r),
\end{aligned}
\tag{6.4}
$$

$t_1,\ldots,t_n \in \mathbb{R}_+$. Next we have, using (2.19),

$$
\mathrm{d}_{t_n}^{n-1} \int_{t_1\vee\cdots\vee t_{n-1}}^{\infty} E[u_n(r,t_1,\ldots,t_{n-1}) \mid \mathcal{F}_r] d(N_r - r)
$$

$$
\begin{aligned}
&= \sum_{j=1}^{n}(-1)^{j-1}\nabla_{t_j}\int_{t_1\vee\cdots\vee t_{j-1}\vee t_{j+1}\vee\cdots\vee t_n}^{\infty} E[u_n(r,t_1,\ldots,t_{j-1},t_{j+1},\ldots,t_n)\mid\mathcal{F}_r]d(N_r-r)\\
&= \sum_{j=1}^{n}(-1)^{j-1}\mathbf{1}_{[t_1\vee\cdots\vee t_n,\infty)}(t_j)E[u_n(t_j,t_1,\ldots,t_{j-1},t_{j+1},\ldots,t_n)\mid\mathcal{F}_{t_j}]\\
&+\sum_{j=1}^{n}(-1)^{j-1}\int_{t_1\vee\cdots\vee t_{j-1}\vee t_{j+1}\cdots\vee t_n}^{\infty}\nabla_{t_j}E[u_n(t_0,t_1,\ldots,t_{j-1},t_{j+1},\ldots,t_n)\mid\mathcal{F}_{t_0}]d(N_{t_0}-t_0)\\
&= \sum_{j=1}^{n}\mathbf{1}_{[t_1\vee\cdots\vee t_n,\infty)}(t_j)E[u_n(t_1,\ldots,t_n)\mid\mathcal{F}_{t_j}]\\
&+\sum_{j=1}^{n}(-1)^{j-1}\int_{t_1\vee\cdots\vee t_{j-1}\vee t_{j+1}\cdots\vee t_n}^{\infty}\nabla_{t_j}E[u_n(t_0,t_1,\ldots,t_{j-1},t_{j+1},\ldots,t_n)\mid\mathcal{F}_{t_0}]d(N_{t_0}-t_0)\\
&= E[u_n(t_1,\ldots,t_n)\mid\mathcal{F}_{t_1\vee\cdots\vee t_n}]\\
&+\sum_{j=1}^{n}(-1)^{j-1}\int_{t_1\vee\cdots\vee t_n}^{\infty}E[\nabla_{t_j}u_n(t_0,t_1,\ldots,t_{j-1},t_{j+1},\ldots,t_n)\mid\mathcal{F}_{t_0}]d(N_{t_0}-t_0), \qquad (6.5)
\end{aligned}
$$

where we applied Corollary 6.2 to get (6.5), hence by taking the difference of (6.4) and (6.5) we get

$$
\begin{aligned}
u_n(t_1,\ldots,t_n) &= \mathrm{d}_{t_n}^{n-1}\int_{t_1\vee\cdots\vee t_{n-1}}^{\infty}E[u_n(r,t_1,\ldots,t_{n-1})\mid\mathcal{F}_r]d(N_r-r)\\
&\quad-\sum_{j=1}^{n}(-1)^{j-1}\int_{t_1\vee\cdots\vee t_n}^{\infty}E[\nabla_{t_j}u_n(t_0,t_1,\ldots,t_{j-1},t_{j+1},\ldots,t_n)\mid\mathcal{F}_{t_0}]d(N_{t_0}-t_0)\\
&\quad+\int_{t_1\vee\cdots\vee t_n}^{\infty}E[\nabla_r u_n(t_1,\ldots,t_n)\mid\mathcal{F}_r]d(N_r-r)\\
&= \mathrm{d}_{t_n}^{n-1}\int_{t_1\vee\cdots\vee t_{n-1}}^{\infty}E[u_n(r,t_1,\ldots,t_{n-1})\mid\mathcal{F}_r]d(N_r-r)\\
&\quad+\int_{t_1\vee\cdots\vee t_n}^{\infty}E[\mathrm{d}_{t_n}^{n}u_n(t_0,t_1,\ldots,t_{n-1})\mid\mathcal{F}_{t_0}]d(N_{t_0}-t_0),
\end{aligned}
$$

$t_1,\ldots,t_n\in\mathbb{R}_+$. We conclude the proof by the denseness of $\mathcal{S}$ in $L^2(\Omega)$, using the closability of the operator d^n. □

By Theorem 6.3 we have the following corollaries.

Corollary 6.4. *For any closed form $u_n\in\mathrm{Dom}(\mathrm{d}^n)$ we have*

$$
u_n(t_1,\ldots,t_n)=\mathrm{d}_{t_n}^{n-1}\int_{t_1\vee\cdots\vee t_{n-1}}^{\infty}E[u_n(r,t_1,\ldots,t_{n-1})\mid\mathcal{F}_r]d(N_r-r),
$$

$t_1,\ldots,t_n\in\mathbb{R}_+$.

In the next Corollary 6.5 the Clark-Ocone formula allows us to recover Corollary 5.5 by a different method and to derive the exactness of the sequence (6.6) as in Theorem 3.2 of [25].

Corollary 6.5. *The range of the exterior derivative* d *and its adjoint* d^* *are closed and we have* $\mathrm{Im}\,\mathrm{d}^n = \mathrm{Ker}\,\mathrm{d}^{n+1}$, $n \in \mathbb{N}$. *As a consequence, the de Rham-Hodge-Kodaira decomposition* (4.7) *reads*

$$L^2(\Omega; \hat{H}^{\wedge n}) = \mathrm{Im}\,\mathrm{d}^{n-1} \oplus \mathrm{Im}\,\mathrm{d}^{n*}, \qquad n \geq 1,$$

and the following sequence is exact:

$$\mathrm{Dom}\,(\mathrm{d}^n) \xrightarrow{\mathrm{d}^n} \mathrm{Im}\,(\mathrm{d}^n) = \mathrm{Ker}\,(\mathrm{d}^{n+1}) \xrightarrow{\mathrm{d}^{n+1}} \mathrm{Im}\,(\mathrm{d}^{n+1}), \qquad n \in \mathbb{N}. \tag{6.6}$$

Proof. By Corollary 6.4 we have $\mathrm{Im}\,\mathrm{d}^n \supset \mathrm{Ker}\,\mathrm{d}^{n+1}$, which shows by (4.4) that $\mathrm{Im}\,\mathrm{d}^n = \mathrm{Ker}\,\mathrm{d}^{n+1}$, $n \in \mathbb{N}$. □

In this way we recover the fact that the Hodge Laplacian Δ_n has a closed range as well, so it has a spectral gap, cf. Theorem 6.6 and Corollary 6.7 of [9]. However this does not yield an explicit Poincaré inequality and lower bound for the spectral gap, unlike for the classical Clark-Ocone formula cf. e.g. Proposition 3.2.7 of [23].

By duality of Corollary 6.5 we also find that $\mathrm{Im}\,\mathrm{d}^{(n+1)*} = \mathrm{Ker}\,\mathrm{d}^{n*}$, and the following sequence is also exact:

$$\mathrm{Im}\,(\mathrm{d}^{n*}) \xleftarrow{\mathrm{d}^{n*}} \mathrm{Ker}\,(\mathrm{d}^{n*}) = \mathrm{Im}\,(\mathrm{d}^{(n+1)*}) \xleftarrow{\mathrm{d}^{(n+1)*}} \mathrm{Dom}\,(\mathrm{d}^{(n+1)*}), \qquad n \in \mathbb{N}.$$

7 Appendix

An alternative derivation of (3.3)

Here we show that (3.3) can be recovered by an explicit computation and a symmetry argument which involve the cancellation of distribution terms, using calculus in distribution sense for the operator D on the Poisson space, cf. [21], [22]. Given a smooth n-form $u_n(T_1, \ldots, T_m, t_1, \ldots, t_n)$ in $\mathcal{S} \otimes \hat{H}^{\wedge n}$ we have

$$\nabla_s u_n(t_1, \ldots, t_n) \;=\; D_s u_n(t_1, \ldots, t_n) - \sum_{j=1}^{n} \mathbf{1}_{[0,t_j]}(s) \frac{\partial u_n}{\partial t_j}(t_1, \ldots, t_n),$$

where for simplicity of notation we omitted the dependence of $u_n(T_1,\dots,T_m,t_1,\dots,t_n)$ in the jump times $T_1,\dots,T_m$. Denoting by $\partial_k u_n(t_1,\dots,t_n)$ the partial derivative of $u_n(T_1,\dots,T_m,t_1,\dots,t_n)$ in (2.3) with respect to the variable T_k, we get

$$\begin{aligned}
\mathrm{d}^n_{t_{n+1}} u_n(t_1,\dots,t_n) &= \sum_{j=1}^{n+1}(-1)^{j-1}\nabla_{t_j} u_n(t_1,\dots,t_{j-1},t_{j+1},\dots,t_{n+1}) \\
&= \sum_{j=1}^{n+1}(-1)^{j-1} D_{t_j} u_n(t_1,\dots,t_{j-1},t_{j+1},\dots,t_{n+1}) \\
&\quad -\sum_{j=1}^{n+1}(-1)^{j-1}\sum_{\substack{l=1\\ l\neq j}}^{n+1}\mathbf{1}_{[0,t_l]}(t_j)\frac{\partial u_n}{\partial t_l}u_n(t_1,\dots,t_{j-1},t_{j+1},\dots,t_{n+1}) \\
&= -\sum_{j=1}^{n+1}(-1)^{j-1}\sum_{k=1}^{m}\mathbf{1}_{[0,T_k]}(t_j)\partial_k u_n(t_1,\dots,t_{j-1},t_{j+1},\dots,t_{n+1}) \\
&\quad -\sum_{j=1}^{n+1}(-1)^{j-1}\sum_{\substack{l=1\\ l\neq j}}^{n+1}\mathbf{1}_{[0,t_l]}(t_j)\frac{\partial u_n}{\partial t_l}u_n(t_1,\dots,t_{j-1},t_{j+1},\dots,t_{n+1}).
\end{aligned}$$

Let us first check the statement for $n=0$ when u_0 simply depends only on the k-th Poisson jump time T_k defined in (2.1). We have

$$\begin{aligned}
&\mathrm{d}^1_{t_2}\mathrm{d}^0_{t_1}u_0 = \mathrm{d}^1_{t_2}D_{t_1}u_0 = \nabla_{t_1}D_{t_2}u_0 - \nabla_{t_2}D_{t_1}u_0 \\
&= -\nabla_{t_1}(\mathbf{1}_{[0,T_k]}(t_2)\partial_k u_0) + \nabla_{t_2}(\mathbf{1}_{[0,T_k]}(t_1)\partial_k u_0) \\
&= -\mathbf{1}_{[0,T_k]}(t_2)D_{t_1}\partial_k u_0 - \partial_k u_0\nabla_{t_1}\mathbf{1}_{[0,T_k]}(t_2) + \mathbf{1}_{[0,T_k]}(t_1)D_{t_2}\partial_k u_0 + \partial_k u_0\nabla_{t_2}\mathbf{1}_{[0,T_k]}(t_1) \\
&= -\mathbf{1}_{[0,T_k]}(t_2)D_{t_1}\partial_k u_0 + \partial_k u_0\mathbf{1}_{[0,T_k]}(t_1)\delta_{t_2}(T_k) - \partial_k u_0\mathbf{1}_{[0,t_2]}(t_1)\delta_{T_k}(t_2) \\
&+ \mathbf{1}_{[0,T_k]}(t_1)D_{t_2}\partial_k u_0 - \partial_k u_0\mathbf{1}_{[0,T_k]}(t_2)\delta_{t_1}(T_k) + \partial_k u_0\mathbf{1}_{[0,t_1]}(t_2)\delta_{T_k}(t_1) \\
&= \partial_k u_0\mathbf{1}_{[0,T_k]}(t_1)\delta_{t_2}(T_k) - \partial_k u_0\mathbf{1}_{[0,t_2]}(t_1)\delta_{T_k}(t_2) - \partial_k u_0\mathbf{1}_{[0,T_k]}(t_2)\delta_{t_1}(T_k) + \partial_k u_0\mathbf{1}_{[0,t_1]}(t_2)\delta_{T_k}(t_1) \\
&= \partial_k u_0\mathbf{1}_{[0,t_2]}(t_1)\delta_{t_2}(T_k) - \partial_k u_0\mathbf{1}_{[0,t_2]}(t_1)\delta_{T_k}(t_2) - \partial_k u_0\mathbf{1}_{[0,t_1]}(t_2)\delta_{t_1}(T_k) + \partial_k u_0\mathbf{1}_{[0,t_1]}(t_2)\delta_{T_k}(t_1) \\
&= 0.
\end{aligned}$$

More generally for $n\geq 1$ we find

$$\begin{aligned}
\mathrm{d}^{n+1}_{t_{n+2}}\mathrm{d}^n_{t_{n+1}}u_n(t_1,\dots,t_n) &= \sum_{1\leq i\leq j\leq n+1}(-1)^{i-1}(-1)^{j-1}\nabla_{t_i}\nabla_{t_{j+1}}u_n(t_1,\dots,t_{i-1},t_{i+1},\dots,t_j,t_{j+2},\dots,t_{n+2}) \\
&+ \sum_{1\leq j<i\leq n+2}(-1)^{i-1}(-1)^{j-1}\nabla_{t_i}\nabla_{t_j}u_n(t_1,\dots,t_{j-1},t_{j+1},\dots,t_{i-1},t_{i+1},\dots,t_{n+2})
\end{aligned}$$

$$
\begin{aligned}
&= -\sum_{1\le i<j\le n+2}(-1)^{i-1}(-1)^{j-1}\nabla_{t_i}\nabla_{t_j}u_n(t_1,\dots,t_{i-1},t_{i+1},\dots,t_{j-1},t_{j+1},\dots,t_{n+2})\\
&\quad+\sum_{1\le j<i\le n+2}(-1)^{i-1}(-1)^{j-1}\nabla_{t_i}\nabla_{t_j}u_n(t_1,\dots,t_{j-1},t_{j+1},\dots,t_{i-1},t_{i+1},\dots,t_{n+2})
\end{aligned}
$$

Next we focus on the first term above. We have

$$
\begin{aligned}
&\sum_{1\le i<j\le n+2}(-1)^{i-1}(-1)^{j-1}\nabla_{t_i}\nabla_{t_j}u_n(t_1,\dots,t_{i-1},t_{i+1},\dots,t_{j-1},t_{j+1},\dots,t_{n+2})\\
&=\sum_{1\le i<j\le n+2}(-1)^{j-1}(-1)^{i-1}\sum_{k,l=1}^{m}\mathbf{1}_{[0,T_l]}(t_i)\partial_l\left(\mathbf{1}_{[0,T_k]}(t_j)\partial_k u_n(t_1,\dots,t_{i-1},t_{i+1},\dots,t_{j-1},t_{j+1},\dots,t_{n+2})\right)\\
&+\sum_{1\le i<j\le n+2}(-1)^{i-1}(-1)^{j-1}\sum_{k=1}^{m}\sum_{\substack{l=1\\l\neq i}}^{n+2}\mathbf{1}_{[0,t_l]}(t_i)\frac{\partial}{\partial t_l}\left(\mathbf{1}_{[0,T_k]}(t_j)\partial_k u_n(t_1,\dots,t_{i-1},t_{i+1},\dots,t_{j-1},t_{j+1},\dots,t_{n+2})\right)\\
&+\sum_{1\le i<j\le n+2}(-1)^{i-1}(-1)^{j-1}\sum_{k=1}^{m}\sum_{\substack{l=1\\l\neq j}}^{n+2}\mathbf{1}_{[0,T_k]}(t_i)\mathbf{1}_{[0,t_l]}(t_j)\partial_k\frac{\partial u_n}{\partial t_l}(t_1,\dots,t_{i-1},t_{i+1},\dots,t_{j-1},t_{j+1},\dots,t_{n+2})\\
&+\sum_{1\le i<j\le n+2}(-1)^{i-1}(-1)^{j-1}\sum_{\substack{k,l=1\\k\neq i,\ l\neq i,\ l\neq j}}^{n+2}\mathbf{1}_{[0,t_k]}(t_i)\frac{\partial}{\partial t_k}\left(\mathbf{1}_{[0,t_l]}(t_j)\frac{\partial u_n}{\partial t_l}(t_1,\dots,t_{i-1},t_{i+1},\dots,t_{j-1},t_{j+1},\dots,t_{n+2})\right)\\
&=\sum_{1\le i<j\le n+2}(-1)^{j-1}(-1)^{i-1}\sum_{k,l=1}^{m}\mathbf{1}_{[0,T_l]}(t_i)\mathbf{1}_{[0,T_k]}(t_j)\partial_l\partial_k u_n(t_1,\dots,t_{i-1},t_{i+1},\dots,t_{j-1},t_{j+1},\dots,t_{n+2})\\
&+\sum_{1\le i<j\le n+2}(-1)^{j-1}(-1)^{i-1}\sum_{k=1}^{m}\mathbf{1}_{[0,T_k]}(t_i)\delta_{t_j}(T_k)\partial_k u_n(t_1,\dots,t_{i-1},t_{i+1},\dots,t_{j-1},t_{j+1},\dots,t_{n+2})\\
&+\sum_{1\le i<j\le n+2}(-1)^{i-1}(-1)^{j-1}\sum_{k=1}^{m}\sum_{\substack{l=1\\l\neq i}}^{n+2}\mathbf{1}_{[0,t_l]}(t_i)\mathbf{1}_{[0,T_k]}(t_j)\frac{\partial}{\partial t_l}\partial_k u_n(t_1,\dots,t_{i-1},t_{i+1},\dots,t_{j-1},t_{j+1},\dots,t_{n+2})\\
&-\sum_{1\le i<j\le n+2}(-1)^{i-1}(-1)^{j-1}\sum_{k=1}^{m}\mathbf{1}_{[0,t_j]}(t_i)\delta_{T_k}(t_j)\partial_k u_n(t_1,\dots,t_{i-1},t_{i+1},\dots,t_{j-1},t_{j+1},\dots,t_{n+2})\\
&+\sum_{1\le i<j\le n+2}(-1)^{i-1}(-1)^{j-1}\sum_{k=1}^{m}\sum_{\substack{l=1\\l\neq j}}^{n+2}\mathbf{1}_{[0,T_k]}(t_i)\mathbf{1}_{[0,t_l]}(t_j)\partial_k\frac{\partial u_n}{\partial t_l}(t_1,\dots,t_{i-1},t_{i+1},\dots,t_{j-1},t_{j+1},\dots,t_{n+2})\\
&+\sum_{1\le i<j\le n+2}(-1)^{i-1}(-1)^{j-1}\sum_{\substack{k,l=1\\k\neq i,\ l\neq j}}^{n+2}\mathbf{1}_{[0,t_k]}(t_i)\mathbf{1}_{[0,t_l]}(t_j)\frac{\partial}{\partial t_k}\frac{\partial u_n}{\partial t_l}(t_1,\dots,t_{i-1},t_{i+1},\dots,t_{j-1},t_{j+1},\dots,t_{n+2})\\
&-\sum_{1\le i<j\le n+2}(-1)^{i-1}(-1)^{j-1}\sum_{\substack{l=1\\l\neq i,l\neq j}}^{n+2}\mathbf{1}_{[0,t_j]}(t_i)\delta_{t_l}(t_j)\frac{\partial u_n}{\partial t_l}(t_1,\dots,t_{i-1},t_{i+1},\dots,t_{j-1},t_{j+1},\dots,t_{n+2})\\
&+\sum_{1\le i<j\le n+2}(-1)^{i-1}(-1)^{j-1}\sum_{\substack{l=1\\l\neq i,\ l\neq j}}^{n+2}\mathbf{1}_{[0,t_l]}(t_i)\delta_{t_j}(t_l)\frac{\partial u_n}{\partial t_l}(t_1,\dots,t_{i-1},t_{i+1},\dots,t_{j-1},t_{j+1},\dots,t_{n+2})\\
&=\sum_{1\le i<j\le n+2}(-1)^{j-1}(-1)^{i-1}\sum_{k,l=1}^{m}\mathbf{1}_{[0,T_l]}(t_i)\mathbf{1}_{[0,T_k]}(t_j)\partial_l\partial_k u_n(t_1,\dots,t_{i-1},t_{i+1},\dots,t_{j-1},t_{j+1},\dots,t_{n+2})
\end{aligned}
$$

$$+\sum_{1\le i<j\le n+2}(-1)^{i-1}(-1)^{j-1}\sum_{k=1}^{m}\sum_{\substack{l=1\\ l\neq i}}^{n+2}\mathbf{1}_{[0,t_l]}(t_i)\mathbf{1}_{[0,T_k]}(t_j)\frac{\partial}{\partial t_l}\partial_k u_n(t_1,\ldots,t_{i-1},t_{i+1},\ldots,t_{j-1},t_{j+1},\ldots,t_{n+2})$$

$$+\sum_{1\le i<j\le n+2}(-1)^{i-1}(-1)^{j-1}\sum_{k=1}^{m}\sum_{\substack{l=1\\ l\neq j}}^{n+2}\mathbf{1}_{[0,T_k]}(t_i)\mathbf{1}_{[0,t_l]}(t_j)\partial_k\frac{\partial u_n}{\partial t_l}(t_1,\ldots,t_{i-1},t_{i+1},\ldots,t_{j-1},t_{j+1},\ldots,t_{n+2})$$

$$+\sum_{1\le i<j\le n+2}(-1)^{i-1}(-1)^{j-1}\sum_{\substack{k,l=1\\ k\neq i,\ l\neq j}}^{n+2}\mathbf{1}_{[0,t_k]}(t_i)\mathbf{1}_{[0,t_l]}(t_j)\frac{\partial}{\partial t_k}\frac{\partial u_n}{\partial t_l}(t_1,\ldots,t_{i-1},t_{i+1},\ldots,t_{j-1},t_{j+1},\ldots,t_{n+2}).$$

A similar computation by exchanging i and j shows that

$$\sum_{1\le j<i\le n+2}(-1)^{i-1}(-1)^{j-1}\nabla_{t_i}\nabla_{t_j}u_n(t_1,\ldots,t_{j-1},t_{j+1},\ldots,t_{i-1},t_{i+1},\ldots,t_{n+2}),$$

is equal to the above summation by exchanging the indexes i and j, which allows us to conclude to the cancellation of the two terms.

A Poisson chaos-based proof of Relation (6.2)

Here we provide an alternative, chaos-based proof for (6.2) by considering F of the form

$$F=I_n(f^{\otimes n}),\qquad f\in\mathcal{C}_c^1(\mathbb{R}_+;\mathbb{R}),\quad n\geq 1,$$

with

$$E[F\mid\mathcal{F}_t]=I_n((f\mathbf{1}_{[0,t]})^{\otimes n}),\quad t\in\mathbb{R}_+,$$

and using the following limiting argument. In order to apply the relation

$$D_sI_n(f^{\otimes n})=nf(s)I_{n-1}(f^{\otimes(n-1)})-nI_n(f^{\otimes(n-1)}\circ(f'\mathbf{1}_{[s,\infty)})),\tag{7.1}$$

cf. Proposition 8 of [17] or Proposition 7.7.2 page 279 of [23], we first need to apply a regularization argument to $f\mathbf{1}_{[0,r]}$. Let $\phi\in\mathcal{C}^1(\mathbb{R};\mathbb{R})$, $\phi\geq 0$, with support contained in $(-1,1)$, such that $\int_{-1}^1\phi(r)dr=1$, and let

$$\phi^\varepsilon(r)=\varepsilon^{-1}\phi(\varepsilon^{-1}r),\qquad r\in\mathbb{R},\quad \varepsilon>0,$$

and

$$\psi_t^\varepsilon(r)=\int_{-\infty}^t\phi^\varepsilon(r-s)ds=\int_{-\infty}^0\phi^\varepsilon(r-s-t)ds,$$

$s \in \mathbb{R}$, $\varepsilon > 0$, denote the regularization of the indicator function $\mathbf{1}_{(-\infty,t]}$. For any $\varepsilon > 0$ we consider the regularized conditional expectation operator

$$P^\varepsilon : L^2(\Omega) \longrightarrow L^2(\Omega)$$

which is continuous on $L^2(\Omega)$ and defined by

$$P_r^\varepsilon I_n(f^{\otimes n}) := I_n((f\psi_r^\varepsilon)^{\otimes n}), \qquad r \in \mathbb{R}_+,$$

which converges in $L^2(\Omega)$ to the conditional expectation

$$E[I_n(f^{\otimes n}) \mid \mathcal{F}_r] = I_n(f^{\otimes n}\mathbf{1}_{[0,r]}^{\otimes n})$$

as ε tends to 0. Next, by applying (7.1) we find

$$\begin{aligned}
\nabla_s P_t^\varepsilon F &= \nabla_s I_n((f\psi_t^\varepsilon)^{\otimes n}) \\
&= D_s I_n((f\psi_t^\varepsilon)^{\otimes n}) - \mathbf{1}_{[0,t]}(s) I_n\left(f^{\otimes n}\frac{\partial}{\partial t}(\psi_t^\varepsilon)^{\otimes n}\right) \\
&= nf(s)\psi_t^\varepsilon(s) I_{n-1}((f\psi_t^\varepsilon)^{\otimes(n-1)}) - n\mathbf{1}_{[0,t]}(s) I_n\left((f\psi_t^\varepsilon)^{\otimes(n-1)} \circ ((f\psi_t^\varepsilon)'\mathbf{1}_{[s,\infty)})\right) \\
&\quad -\mathbf{1}_{[0,t]}(s) I_n\left(f^{\otimes n}\frac{\partial}{\partial t}(\psi_t^\varepsilon)^{\otimes n}\right) \\
&= nf(s)\psi_t^\varepsilon(s) I_{n-1}((f\psi_t^\varepsilon)^{\otimes(n-1)}) - n\mathbf{1}_{[0,t]}(s) I_n\left((f\psi_t^\varepsilon)^{\otimes(n-1)} \circ (f'\psi_t^\varepsilon\mathbf{1}_{[s,\infty)})\right) \\
&\quad -n\mathbf{1}_{[0,t]}(s) I_n\left((f\psi_t^\varepsilon)^{\circ(n-1)} \circ (f(\psi_t^\varepsilon)'\mathbf{1}_{[s,\infty)})\right) - n\mathbf{1}_{[0,t]}(s) I_n\left((f\psi_t^\varepsilon)^{\circ(n-1)} \circ \left(f\frac{\partial}{\partial t}\psi_t^\varepsilon\right)\right) \\
&= nf(s)\psi_t^\varepsilon(s) I_{n-1}((f\psi_t^\varepsilon)^{\otimes(n-1)}) - n\mathbf{1}_{[0,t]}(s) I_n\left((f\psi_r^\varepsilon)^{\otimes(n-1)} \circ (f'\psi_t^\varepsilon\mathbf{1}_{[s,\infty)})\right) \\
&= nf(s)\psi_t^\varepsilon(s) P_t^\varepsilon I_{n-1}(f^{\otimes(n-1)}) - n\mathbf{1}_{[0,t]}(s) P_t^\varepsilon I_n(f^{\otimes(n-1)} \circ (f'\mathbf{1}_{[s,\infty)})),
\end{aligned}$$

where we used the relation

$$(\psi_t^\varepsilon)'(r) = \frac{\partial}{\partial r}\psi_t^\varepsilon(r) = -\frac{\partial}{\partial t}\psi_t^\varepsilon(r), \qquad r \in \mathbb{R}.$$

As ε tends to 0 we find

$$\begin{aligned}
&\lim_{\varepsilon\to 0} \nabla_s P_t^\varepsilon F \\
&\quad = nf(s)\mathbf{1}_{[0,t]}(s) I_{n-1}(f^{\otimes(n-1)}\mathbf{1}_{[0,t]^{n-1}}) - n\mathbf{1}_{[0,t]}(s) I_n(f^{\otimes(n-1)} \circ (f'\mathbf{1}_{[s,\infty)})\mathbf{1}_{[0,t]^n}) \\
&\quad = \mathbf{1}_{[0,t]}(s) E[D_s F \mid \mathcal{F}_t].
\end{aligned}$$

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