Poisson discretizations of Wiener functionals and Malliavin operators with Wasserstein estimates

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Abstract

This article proposes a global, chaos-based procedure for the discretization of functionals of Brownian motion into functionals of a Poisson process with intensity $\lambda > 0$. Under this discretization we study the weak convergence, as the intensity of the underlying Poisson process goes to infinity, of Poisson functionals and their corresponding Malliavin-type derivatives to their Wiener counterparts. In addition, we derive a convergence rate of $O(\lambda^{-1/4})$ for the Poisson discretization of Wiener functionals by combining the multivariate Chen-Stein method with the Malliavin calculus. Our proposed sufficient condition for establishing the mentioned convergence rate involves the kernel functions in the Wiener chaos, yet we provide examples, especially the discretization of some common path dependent Wiener functionals, to which our results apply without committing the explicit computations of such kernels. To the best our knowledge, these are the first results in the literature on the universal convergence rate of a global discretization of general Wiener functionals.

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1 Introduction

The Malliavin calculus was originally designed as a tool to establish the regularity of the solution of certain parabolic partial differential equations via probabilistic arguments. Over the years, it has been developed into an indispensible tool in many research areas such as anticipating stochastic calculus and stochastic calculus for fractional Brownian motion, see for example [21]. Malliavin's approach relies on a heavy functional analytic apparatus, such as the Ornstein-Uhlenbeck operator and the definition of suitable Sobolev spaces, to which the diffusion processes belong.

A central tool for the unification of the Malliavin calculus in the Brownian and jump cases is the use of chaos expansions based on multiple stochastic integrals. When applied in the Poisson case, this approach is known to yield finite difference operators instead of derivation operators, as was noted in [13] and [23]; there also exists an alternative approaches to the Malliavin calculus for the standard Poisson process that use derivation operators, for instance, see e.g. [10], [25]. In the case of jump processes, the development of the stochastic calculus of variations was also initiated in [4], via the use of the Girsanov theorem; this approach was further developed in [3] with applications to the smoothness of the density of the solution to stochastic differential equations with jumps. This approach relies on the differentiation of quasi-invariance identities leading to integration by parts formulas for diffusion processes, which where obtained by Malliavin in an alternative way.

Still based on integration by parts, in the seminal paper [28], Stein proposed an alternative derivation of Berry-Essen's bounds for the error incurred in normal approximation, through the use of the Stein equation. This approach was further extended and enhanced by Chen [5] to obtain similar results for Poisson approximation. More recently, new developments combining the integration by parts in the Malliavin Calculus and in the Chen-Stein method have appeared [18], [22], [30], together with a growing number of applications in probability and statistics, including the discovery of a universal normality result, called the fourth moment theorem, for sequences of multiple stochastic integrals. Since then, popularity in combining the Malliavin calculus with Stein's method has been widely observed, and this has provided asymptotic statistical analysis tools for the normal approximation of Wiener chaoses, see [18], and [6]. Furthermore, [19] studied the approximation of Gamma distribution by a sequence of Wiener chaoses. [9] provided a technique to compare the tail of a given random variable to that of Pearson distributions. [15] considered the measurement of the distance between the law of a Malliavin differentiable random variable and a certain regular continuous probability distribution; however, the previous works did not provide a systematic study on approximating the distributions of any general Wiener functionals and by no means for the rates of these estimations. Our present work aims to fill this gap in the very first time.

On the other hand the study of the convergence of the Poisson Malliavin structure and operators to their Brownian counterparts has been left idle, though it seems quite natural question. See nevertheless the recent work [2] which considers the convergence of discretized Malliavin gradient and divergence using rescaled Bernoulli random walks. In this paper we address this issue under a suitable renormalization of the underlying compensated Poisson process.

Let $\{N_t\}_{t\geq 0}$ be the standard Poisson process with a unit intensity. It is well known that the distribution of Brownian motion can be approximated by that of a normalized sequence of renormalized compensated Poisson processes $\tilde{N}_t^\lambda := (N_{\lambda t} - \lambda t)\sqrt{\lambda}$, $t \in \mathbb{R}_+$, with intensity λ approaching infinity. In particular, we have

$$I_1^\lambda(\mathbb{1}_{[0,t)}) := \int_0^\infty \mathbb{1}_{[0,t)}(s) d\tilde{N}_s^\lambda \xrightarrow{d} B_t, \text{ as } \lambda \to \infty \ ,$$

where $\{B_t\}_{t\geq 0}$ is the standard Brownian motion. In the case of higher order functionals, the convergence in distribution of symmetric statistics to products of Hermite polynomials in Gaussian random variables has been treated in [27]. Such results have been extended to the convergence of symmetric statistics of series of multiple Wiener integrals in [8], see also the survey [29].

In this paper we consider a different type of convergence for series of multiple stochastic integrals. Namely, given a Wiener functional F written in the form of its chaos expansion:

$$F = \mathbb{E}[F] + \sum_{n=1}^{\infty} I_n(f_n),$$

where $I_n(f_n)$ is the multiple Wiener integral to be defined in (2.2) and the kernel function $f_n(t_1, ..., t_n)$ is a symmetric deterministic one in $L^2(dt_1 \times \cdots \times dt_n)$. We define the discretization $\overline{F}^\lambda$ of F at the level $\lambda > 0$ as

$$\overline{F}^\lambda = \mathbb{E}[F] + \sum_{n=1}^{\infty} I_n^\lambda(f_n),$$

where $I_n^\lambda(f_n)$ is the multiple Poisson stochastic integral to be defined in (2.3). In Theorem 3.2 and Proposition 4.2, we shall consider the convergence in distribution of discretized Malliavin gradient and divergence operators. As a consequence we obtain a discretization of integration by parts formulas on the Wiener space, by using Poisson functionals, and the related sensitivity analysis, see (3.9) below.

Next, by combining the multivariate Stein method with the Malliavin Calculus on the Poisson space, in Theorem 5.1 we establish the convergence in distribution of the discretized Poisson approximation $\overline{F}^\lambda$ to the Wiener functional F as λ tends to infinity, assuming summability and smoothness conditions on the symmetric kernel functions f_n, $n \geq 1$, and we provide examples of Wiener functionals satisfying the conditions of Theorem 5.1.

Furthermore, we identify the universal rate of convergence of $\overline{F}^\lambda$ to F to be of order $O(\lambda^{-1/4})$ in the Wasserstein-type distance $d(\cdot, \cdot)$ to be defined in (5.1), i.e.

$$d\big(\overline{F}^\lambda, F\big) = O(\lambda^{-1/4}), \qquad [\lambda \to \infty].$$

Since the multiple stochastic integral $I_n(f_n)$ may not follow a normal distribution, it could be challenging to derive its associated Stein equation. To circumvent this difficulty we use an off-diagonal discretization $f_n^{\alpha,\lambda}$ of f_n as defined in (5.12), which allows us to formulate a proxy multivariate Stein equation for $I_n(f_n^{\alpha,\lambda})$.

We illustrate the effectiveness of our approach via examples of error bounds obtained for path-dependent Wiener functionals such as solutions of stochastic differential equations.

Despite several studies on the approximation of the solutions of SDEs, see e.g. [7], [14] and [24], a systematic discretization for general Wiener functionals and the corresponding universal convergence rate have long been absent in the literature, and the goal of the present work is to fill this gap.

This paper is organized as follows. In Section 2 we start with preliminaries on the unifying framework used in this paper, namely the Wiener and Poisson multiple stochastic integrals and polynomials, and the tools of the corresponding Malliavin calculi. In Section 3 we deal with the weak convergence of Malliavin operators and we obtain a discretization of the Malliavin integration by parts formulas using Poisson finite difference operators. In Section 4, we also consider the convergence of derivation operators on the Poisson space. In Section 5 we deal with the rate of weak convergence from Poisson discretized functionals to Wiener functionals; in addition, two applications on the numerical approximation of path-dependent Wiener functionals are provided to illustrate the effectiveness of our proposed method. The proof of the main Theorem 5.1 relies on a sequence of lemmas which are given in Section 6.

2 Preliminaries and notations

Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which a standard Wiener process $(B_t)_{t\in\mathbb{R}_+}$ and a (not necessarily independent) standard Poisson process $(N_t)_{t\in\mathbb{R}_+}$ with unit intensity are defined. The renormalized compensated Poisson process with intensity $\lambda > 0$ is defined by $\tilde{N}_t^\lambda = (N_{\lambda t} - \lambda t)/\sqrt{\lambda}$.

2.1 Normal martingales, chaos and orthogonal polynomials

More thorough introduction on the topic in this section can be found in [26]. A real-valued square integrable martingale $\{M_t\}_{t\geq 0}$ such that

$$\mathbb{E}\left[(M_t - M_s)^2\right] = t - s \ , \text{ for all } 0 \leq s \leq t \ ,$$

is called a *normal martingale*. Clearly, both Wiener process $\{B_t\}_{t\geq 0}$ and renormalized compensated Poisson process $\{\tilde{N}_t^\lambda\}_{t\geq 0}$ are representative examples of normal martingales. For any (deterministic) symmetric function $f_n \in L^2(\mathbb{R}^n)$ with $n \geq 1$, and a normal martingale $(M_t)_{t\in\mathbb{R}_+}$, the multiple Wiener-Itô integral of degree n with respect to $(M_t)_{t\in\mathbb{R}_+}$, denoted by $I_n^M(f_n)$, is defined as:

$$I_n^M(f_n) := n! \int_0^\infty \int_0^{t_{n-1}} \cdots \int_0^{t_2} f_n(t_1, \dots, t_n) dM_{t_1} \cdots dM_{t_n},$$

satisfying the isometry property:

$$\mathbb{E}[(I_n^M(f_n))^2] = n!\|f_n\|^2_{L^2(\mathbb{R}_+^n)}, \qquad n \geq 1. \tag{2.1}$$

Furthermore, both the Wiener process $\{B_t\}_{t\geq 0}$ and the renormalized compensated Poisson process $\{\tilde{N}_t^\lambda\}_{t\geq 0}$ have the chaos representation property, in the sense that for any $F \in L^2$ it can be shown that there exists a sequence $(f_m)_{m\in\mathbb{N}}$ of symmetric functions $f_m \in L^2(\mathbb{R}_+^m)$ such that

$$F = \mathbb{E}[F] + \sum_{m=1}^{\infty} I_m^M(f_m).$$

In particular, if $M_t = B_t$, for simplicity, we denote

$$I_n(f_n) := I_n^M(f_n)\Big|_{M=B}\ , \tag{2.2}$$

as a Wiener chaos of degree n. On the other hand, if $M_t = \tilde{N}_t^\lambda$, we also denote

$$I_n^\lambda(f_n) := I_n^M(f_n)\Big|_{M=\tilde{N}^\lambda}\ , \tag{2.3}$$

as a Poisson chaos of degree n. For any $n \in \mathbb{N}$, define $E_n := \mathbb{Z}_+^n$ and let

$$E = \{(p_k)_{k\geq 1}\ :\ p_k \in \mathbb{Z}_+, \quad k \geq 1\}$$

be the set of sequences in $\mathbb{Z}_+$ in which each element has all components vanishing except for finitely many of them. For $p = (p_k)_{k\geq 1} \in E$, let

$$|p| := \sum_{i=1}^{\infty} p_i \qquad \text{and} \qquad p! := \prod_{i=1}^{\infty} p_i!.$$

Definition 2.1. *For $x \in \mathbb{R}^k$ (resp. $x \in \mathbb{R}^{\mathbb{N}}$) and $p \in E_k$ (resp. $p \in E$), the k-dimensional (resp. generalized) Hermite polynomial $H_p(x)$ is defined as*

$$H_p(x) := \prod_{k=1}^{k} H_{p_k}(x_k), \quad \textit{resp.} \quad \left(\prod_{k=1}^{\infty} H_{p_k}(x_k)\right)\ ,$$

where

$$H_n(y) := (-1)^n e^{y^2/2}\frac{d^n}{dy^n}e^{-y^2/2},\ y \in \mathbb{R},\ \forall n \geq 1,\ \textit{with } H_0(y) \equiv 1\ .$$

Definition 2.2. *The Charlier polynomial of degree $n \in \mathbb{N}$ with parameter $a \geq 0$ is defined via the following recursive relations:*

$$C_{n+1}(x,a) = (x - n - a)C_n(x,a) - naC_{n-1}(x,a) \ ,$$

$x \in \mathbb{R}$, $a \in \mathbb{R}_+$, $n \geq 1$, *with* $C_0(x,a) \equiv 1$ *and* $C_1(x,a) = x - a$.

We have the following relationship between Hermite (resp. Charlier) polynomials and iterated Wiener (resp. Poisson) integrals, cf. Theorem 7.2 in [12] (resp. Proposition 6.2.9 in [26]).

Proposition 2.3. *1. Let $h \in L^2(\mathbb{R}_+)$ such that $\|h\|_{L^2(\mathbb{R}_+)} = 1$, and define $h^{\otimes m}$ as the m-fold tensor product of h with itself, where $m \in \mathbb{N}$. Then we have*

$$H_m(I_1(h)) = I_m(h^{\otimes m}).$$

2. Let $A_1, \ldots, A_d$ be mutually disjoint intervals in $\mathbb{R}_+$ and $n = k_1 + \cdots + k_d$, then

$$I_n^\lambda\big(\mathbb{1}_{A_1}^{\otimes k_1} \circ \cdots \circ \mathbb{1}_{A_d}^{\otimes k_d}\big) = \lambda^{-n/2} \prod_{i=1}^{d} C_{k_i}\left(\int_0^\infty \mathbb{1}_{A_i} dN_{\lambda t}, \lambda\mu(A_i)\right),$$

where "$\circ$" denotes the symmetric tensor product defined by:

$$f_1 \circ \cdots \circ f_n := \frac{1}{n!} \sum_{\sigma \in \Sigma_n} f_1(t_{\sigma(1)}) \cdots f_n(t_{\sigma(n)}),$$

for $f_1, \ldots, f_n \in L^2(\mathbb{R}_+)$, and Σ_n denotes the set of all permutations over $\{1, \ldots, n\}$.

The following result states an asymptotic relation between Hermite and Charlier polynomials which will be used in the proof of Lemma 3.1.

Lemma 2.4. *For any $n \in \mathbb{N}$,*

$$a^{-n/2} C_n\left(a + x\sqrt{a}, a\right) = H_n(x) + O_n\left(a^{-1/2}\right),$$

where the convergence rate $O_n\left(a^{-1/2}\right)$ holds uniformly for all x in an arbitrary compact interval in $\mathbb{R}$.

Proof. By Theorem 1 in [17] we have

$$(2a)^{n/2}\,\tilde{C}_{[a+x\sqrt{2a}]}\,(n,a) = (-1)^n \tilde{H}_n(x) + O_n\left(a^{-1/2}\right),$$

for $a > 0$, where $\tilde{C}_n(x,a) := (-a)^{-n} C_n(x,a)$, $\tilde{H}_n(x) := 2^{n/2} H_n(\sqrt{2}x)$, $[a + x\sqrt{2a}]$ denotes the smallest integer larger than $a+x\sqrt{2a}$, and the convergence rate $O_n\left(a^{-1/2}\right)$ holds uniformly for all x in any compact interval. Our result then follows from the equation (10.25.7) in [11], which states that $\tilde{C}_n(x,a) = \tilde{C}_x(n,a)$. □

2.2 Malliavin calculus for normal martingales

Again, more details on the material introduced in this section can be found in [26]. In the sequel, we denote $\mathcal{C}^M$ to be the collection of cylindrical functionals:

$$\mathcal{C}^M := \left\{ F = f_n\left(I_1^M\left(\frac{\mathbb{1}_{A_1}}{\sqrt{\mu(A_1)}}\right), \ldots, I_1^M\left(\frac{\mathbb{1}_{A_n}}{\sqrt{\mu(A_n)}}\right)\right) \ : \ f_n \in \mathcal{P}(\mathbb{R}^n),\ n \in \mathbb{N}, \right.$$
$$\left. \{A_i\}_{1\le i\le n} \text{ are mutually disjoint intervals in } \mathbb{R}^+ \right\},$$

where $\mathcal{P}(\mathbb{R}^n)$ denotes the set of polynomial functions in n variables and μ is the Lebesgue measure on $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$. Let $L^2(\mathbb{R}_+)^{\circ n}$ denote the subspace of $L^2(\mathbb{R}_+)^{\otimes n} = L^2(\mathbb{R}_+^n)$ made of symmetric functions in n variables. Define

$$\mathcal{U}^M := \left\{ \sum_{k=1}^{n} g_k G_k \ : \ g_k \in L^2(\mathbb{R}_+),\ G_k \in \mathcal{C}^M,\ k = 1, \ldots, n,\ n \in \mathbb{N} \right\}.$$

Note that, the spaces $\mathcal{C}^M$ and $\mathcal{U}^M$ are respectively dense in $L^2(\Omega)$ and $L^2(\Omega \times \mathbb{R}_+)$. In addition, since the elements of $\mathcal{C}^M$ and those of $\mathcal{U}^M$ can be expressed as linear combinations of multiple Wiener-Itô integrals, the operators D^M and δ^M can be respectively defined on $\mathcal{C}^M$ and $\mathcal{U}^M$ as follows.

Definition 2.5. *a) The Malliavin derivative operator $D^M : \mathcal{C}^M \longrightarrow L^2(\Omega \times \mathbb{R}_+)$ is defined as:*

$$D_t^M I_m^M(f_m) = m I_{m-1}^M(f_m(*,t)), \quad f_m \in L^2(\mathbb{R}_+)^{\circ m}, \quad t \in \mathbb{R}_+, \quad m \in \mathbb{N}.$$

b) The operator $\delta^M : \mathcal{U}^M \to L^2(\Omega)$, the Skorokhod integral operator, is defined as:

$$\delta^M\left(I_n^M\left(f_{n+1}(*,\cdot)\right)\right) := I_{n+1}^M\big(\tilde{f}_{n+1}\big), \quad f_{n+1} \in L^2(\mathbb{R}_+)^{\circ n} \otimes L^2(\mathbb{R}_+),$$

where $\tilde{f}_{n+1}$ is the symmetrization of f_{n+1} in $n+1$ variables defined by:

$$\tilde{f}_{n+1}(t_1,\ldots,t_{n+1}) := \frac{1}{n+1}\sum_{k=1}^{n+1} f(t_1,\ldots,t_{k-1},t_{k+1},\ldots,t_{n+1},t_k).$$

The Malliavin operators D^M and δ^M are both closable, so that D^M can be extended in such a way that:

$$\mathrm{Dom}(D^M) := \left\{F = \mathbb{E}[F] + \sum_{n=1}^{\infty} I_n^M(f_n) \in L^2(\Omega),\ f_n \in L^2(\mathbb{R}_+)^{\circ n} \ :\ \sum_{n=1}^{\infty} n n! \|f_n\|^2_{L^2(\mathbb{R}_+^n)} < \infty\right\},$$

under the Sobolev norm $\|\cdot\|_{1,2}$ defined by

$$\|F\|_{1,2} := \|F\|_{L^2(\Omega)} + \|D^M F\|_{L^2(\Omega\times\mathbb{R}_+)}, \quad F \in \mathcal{C}^M\ ;$$

meanwhile δ^M can be extended to:

$$\mathrm{Dom}(\delta^M) := \Bigg\{u = \sum_{n=0}^{\infty} I_n^M(f_{n+1}(*,\cdot)) \in L^2(\Omega\times\mathbb{R}_+),\ f_{n+1} \in L^2(\mathbb{R}_+)^{\circ n} \otimes L^2(\mathbb{R}_+) \ :\ \sum_{n=1}^{\infty} (n+1)! \|\tilde{f}_{n+1}\|^2_{L^2(\mathbb{R}_+^{n+1})} < \infty\Bigg\}.$$

In addition, $\mathrm{Dom}(\delta^M)$ contains the space $\mathbb{L}_{1,2}$ of processes defined as

$$\mathbb{L}_{1,2} := \left\{u \in L^2(\Omega\times\mathbb{R}_+) \ :\ \|u\|_{1,2} := \|u\|_{L^2(\Omega\times\mathbb{R}_+)} + \|D^M u\|_{L^2(\Omega\times\mathbb{R}_+^2)} < \infty\right\},$$

cf. e.g. § 1.3.2 of [21]. Since we are only dealing with normal martingales, we can call back the duality (or integration by parts formula) relation of D^M and δ^M, where $\langle\cdot,\cdot\rangle$ is the inner product in $L^2(\mathbb{R}_+)$ defined as $\langle f, g\rangle := \int_0^\infty f(t)g(t)dt$, $f, g \in L^2(\mathbb{R}_+)$.

Proposition 2.6. *For any $F \in \mathrm{Dom}(D^M)$ and $u \in \mathrm{Dom}(\delta^M)$, we have*

$$\mathbb{E}\left[\langle D^M F, u\rangle\right] = \mathbb{E}\left[F\delta^M(u)\right]. \tag{2.4}$$

Particularly, if $M_t = B_t$ is the Wiener process, we denote

$$\mathcal{C} := \mathcal{C}^M\Big|_{M=B}, \quad \mathcal{U} := \mathcal{U}^M\Big|_{M=B}, \quad D_t(\cdot) := D_t^M(\cdot)\Big|_{M=B} \text{ and } \quad \delta(\cdot) := \delta^M(\cdot)\Big|_{M=B};$$

while if $M_t = \tilde{N}_t^\lambda$ is the renormalized compensated Poisson process, we denote

$$\mathcal{C}^\lambda := \mathcal{C}^M\Big|_{M=\tilde{N}^\lambda}, \quad \mathcal{U}^\lambda := \mathcal{U}^M\Big|_{M=\tilde{N}^\lambda}, \quad D_t^\lambda(\cdot) := D_t^M(\cdot)\Big|_{M=\tilde{N}^\lambda} \text{ and } \delta^\lambda(\cdot) := \delta^M(\cdot)\Big|_{M=\tilde{N}^\lambda}.$$

The following proposition provides the probabilistic interpretation of D^λ as a difference operator, also see Proposition 6.4.7 in [26] for more details.

Proposition 2.7. *For every $F \in \mathrm{Dom}(D^\lambda)$, we have*

$$D_t^\lambda F = \sqrt{\lambda}\left(F(N_\cdot + \mathbb{1}_{[t,\infty)}(\cdot)) - F(N_\cdot)\right), \qquad t \in \mathbb{R}_+. \tag{2.5}$$

3 Convergence of (annihilation) Malliavin operators

Given a square integrable Wiener functional F with the chaos expansion

$$F = \mathbb{E}[F] + \sum_{n=1}^{\infty} I_n(f_n), \tag{3.1}$$

where f_k is in the space $L^2(\mathbb{R}_+)^{\circ k}$ of symmetric square-integable functions such that

$$\sum_{n=1}^{\infty} n n! \|f_n\|^2_{L^2(\mathbb{R}_+^n)} < \infty,$$

we define the Poisson discretization $\overline{F}^\lambda$ of F at the level $\lambda > 0$ as

$$\overline{F}^\lambda := \mathbb{E}[F] + \sum_{n=1}^{\infty} I_n^\lambda(f_n). \tag{3.2}$$

We note that the mapping $F \mapsto \overline{F}^\lambda$ is an isometry on $L^2(\Omega)$ as we have

$$\|\overline{F}^\lambda\|^2_{L^2(\Omega)} = (\mathbb{E}[F])^2 + \sum_{n=1}^{\infty} n! \|f_n\|^2_{L^2(\mathbb{R}_+^n)} = \|F\|^2_{L^2(\Omega)}, \tag{3.3}$$

and we can further obtain

$$\|D^\lambda \overline{F}^\lambda\|^2_{L^2(\Omega\times\mathbb{R}_+)} = \sum_{k=1}^{\infty} nn!\|f_n\|^2_{L^2(\mathbb{R}^n_+)} = \|DF\|^2_{L^2(\Omega\times\mathbb{R}_+)},$$

from which it follows that

$$\overline{F}^\lambda \in \mathrm{Dom}(D^\lambda) \Longleftrightarrow F \in \mathrm{Dom}(D).$$

In this section, we shall establish the convergence in distribution of $\overline{F}^\lambda$ and $D_t^\lambda \overline{F}^\lambda$ respectively to F and $D_t F$ as λ tends to infinity.

Lemma 3.1. *a) For any $F \in \mathcal{C}$ with Poisson discretization $\overline{F}^\lambda$ and $t \in \mathbb{R}_+$, we have*

$$\overline{F}^\lambda \xrightarrow{d} F \quad \textit{and} \quad D_t^\lambda \overline{F}^\lambda \xrightarrow{d} D_t F, \qquad [\lambda \to \infty].$$

b) For any $u \in \mathcal{U}$ with Poisson discretization $\overline{u}^\lambda$ and a.e. $t \in \mathbb{R}_+$, we have

$$\overline{u}_t^\lambda \xrightarrow{d} u_t \quad \textit{and} \quad \delta^\lambda(\overline{u}^\lambda) \xrightarrow{d} \delta(u), \qquad [\lambda \to \infty].$$

Proof. Note that any function $f_n \in \mathcal{P}(\mathbb{R}^n)$ can be represented as

$$f_n(x_1, \dots, x_n) = \sum_{k=0}^{nq_n} \sum_{\substack{p\in E_n \\ |p|=k}} \frac{c_p(f_n)}{p!} \prod_{i=1}^{n} H_{p_i}(x_i),$$

where $\{c_p(f_n)\}_{p\in E_n}$ are constants depending only on f_n and q_n is the maximum power index of $f_n(x_1, \dots, x_n)$ in $(x_1, x_2, \dots, x_n)$. For any Wiener functional $F_n \in \mathcal{C}$, by Proposition 2.3, F_n admits the following Itô-Wiener chaotic decomposition:

$$\begin{aligned} F_n &= \sum_{k=0}^{nq_n} \sum_{\substack{p\in E_n \\ |p|=k}} \frac{c_p(f_n)}{p!} \prod_{i=1}^{n} H_{p_i}\left(I_1\left(\frac{\mathbb{1}_{A_i}}{\sqrt{\mu(A_i)}}\right)\right) \\ &= \sum_{k=0}^{nq_n} \sum_{\substack{p\in E_n \\ |p|=k}} \frac{c_p(f_n)}{p!} \prod_{i=1}^{n} I_{p_i}\left(\left(\frac{\mathbb{1}_{A_i}}{\sqrt{\mu(A_i)}}\right)^{\otimes p_i}\right) \\ &= \sum_{k=0}^{nq_n} I_k\left(g_k\right), \end{aligned} \tag{3.4}$$

where

$$g_k := \sum_{\substack{p\in E_n \\ |p|=k}} \frac{c_p(f_n)}{p!} \mu(A_1)^{-p_1/2} \cdots \mu(A_n)^{-p_n/2} \mathbb{1}_{A_1}^{\otimes p_1} \cdots \mathbb{1}_{A_n}^{\otimes p_n}, \tag{3.5}$$

cf. also Lemma 5.1.6 in [26]. From (3.4) and (3.5), we see that any $F \in \mathcal{C}$ admits the Itô-Wiener chaos decomposition

$$F = \mathbb{E}[F] + \sum_{k=1}^{nq_n} I_k(g_k),$$

for a given $n \geq 1$, where

$$g_k = \sum_{\substack{p \in E_n \\ |p|=k}} \frac{c_p(f_n)}{p!} \mu(A_1)^{-p_1/2} \cdots \mu(A_n)^{-p_n/2} \mathbb{1}_{A_1}^{\otimes p_1} \otimes \cdots \otimes \mathbb{1}_{A_n}^{\otimes p_n},$$

with $\mu(A_i) > 0$, $i = 1, \dots, n$. By the definition (3.2) of $\overline{F}^\lambda$ and Proposition 2.3-(2), we have

$$\begin{aligned}
\overline{F}^\lambda &= \mathbb{E}[F] + \sum_{k=1}^{nq_n} I_k^\lambda(g_k) \\
&= \mathbb{E}[F] + \sum_{k=1}^{nq_n} \lambda^{-k/2} \sum_{\substack{p \in E_n \\ |p|=k}} \frac{c_p(f_n)}{p!} \mu(A_1)^{-p_1/2} \cdots \mu(A_n)^{-p_n/2} \prod_{i=1}^n C_{p_i}\left(\int_0^\infty \mathbb{1}_{A_i} dN_{\lambda t}, \lambda \mu(A_i) \right) \\
&= J^\lambda(V_n^\lambda),
\end{aligned}$$

where

$$\begin{aligned}
J^\lambda(x_1, \dots, x_n) \;\; := \;\; & \sum_{k=0}^{nq_n} \lambda^{-k/2} \sum_{\substack{p \in E_n \\ |p|=k}} \frac{c_p(f_n)}{p!} \mu(A_1)^{-p_1/2} \cdots \mu(A_n)^{-p_n/2} \\
& \times \prod_{i=1}^n C_{p_i}\big(\lambda \mu(A_i) + \sqrt{\mu(A_i)\lambda} \cdot x_i, \lambda \mu(A_i)\big),
\end{aligned}$$

and

$$V^\lambda := \left((\mu(A_1))^{-1/2} \int_0^\infty \mathbb{1}_{A_1} d\tilde{N}_t^\lambda, \dots, (\mu(A_n))^{-1/2} \int_0^\infty \mathbb{1}_{A_n} d\tilde{N}_t^\lambda \right), \quad \lambda > 0.$$

Similarly, we have

$$\begin{aligned}
F &= \mathbb{E}[F] + \sum_{k=1}^{nq_n} I_k(g_k) \\
&= \sum_{k=0}^{nq_n} \sum_{\substack{p \in E_n \\ |p|=k}} \frac{c_p(f_n)}{p!} \mu(A_1)^{-p_1/2} \cdots \mu(A_n)^{-p_n/2} \prod_{i=1}^n I_{p_i}(\mathbb{1}_{A_i}^{\otimes p_i})
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=0}^{nq_n} \sum_{\substack{p\in E_n \\ |p|=k}} \frac{c_p(f_n)}{p!} \prod_{i=1}^{n} H_{p_i}\left(\int_0^\infty \frac{\mathbb{1}_{A_i}}{\sqrt{\mu(A_i)}} dB_t \right) \\
&= J(V),
\end{aligned}
$$

where

$$
J(x_1, \ldots, x_n) := \sum_{k=0}^{nq_n} \sum_{\substack{p\in E_n \\ |p|=k}} \frac{c_p(f_n)}{p!} \prod_{i=1}^{n} H_{p_i}(x_i),
$$

and

$$
V := \left((\mu(A_1))^{-1/2} \int_0^\infty \mathbb{1}_{A_1} dB_t, \ldots, (\mu(A_n))^{-1/2} \int_0^\infty \mathbb{1}_{A_n} dB_t \right).
$$

Since the sets $\{A_i\}_{1\le i\le n}$ are disjoint, $\left\{ \int_0^\infty \mathbb{1}_{A_i} d\tilde{N}_t^\lambda / \sqrt{\mu(A_i)} \right\}_{1\le i\le n}$ is a family of independent random variables, hence by the Donsker theorem we have*

$$
\lim_{\lambda\to\infty} V^\lambda = V,
$$

hence

$$
J(V^\lambda) \xrightarrow{d} J(V) = F, \qquad [\lambda \to \infty]. \tag{3.6}
$$

We next show that

$$
J^\lambda(V^\lambda) - J(V^\lambda) \xrightarrow{\mathbb{P}} 0. \tag{3.7}
$$

By Chebyshev's inequality and the fact that $\|V^\lambda\|_{L^2(\Omega)}^2 = \|V\|_{L^2(\Omega)}^2 = n$, then for any $\delta > 0$ and $K_\delta > \sqrt{n/\delta}$ we have the bound

$$
\mathbb{P}\left(|V^\lambda| > K_\delta \right) < \frac{\mathbb{E}[|V^\lambda|^2]}{K_\delta^2} = \frac{\mathbb{E}[|V|^2]}{K_\delta^2} < \delta, \qquad \lambda > 0.
$$

Accordingly, we see that J^λ converges locally uniformly to J as $\lambda \to \infty$; indeed, for any fixed $\epsilon > 0$, there exists a large enough constant λ_δ depending on δ, such that for all $\lambda > \lambda_\delta$,

$$
\mathbb{P}\left(\left| J^\lambda(V^\lambda) - J(V^\lambda) \right| > \varepsilon \right)
$$

*Independence of $\left\{ \int_0^\infty \frac{\mathbb{1}_{A_i}}{\sqrt{\mu(A_i)}} d\tilde{N}_t^\lambda \right\}_{1\le i\le n}$ is required, because in general, given two weakly convergent random sequences $X^\lambda \longrightarrow X$ and $Y^\lambda \longrightarrow Y$ we may not have $X^\lambda + Y^\lambda \longrightarrow X + Y$.

$$
\begin{aligned}
&\leq \mathbb{P}\big(\big|J^\lambda(V^\lambda) - J(V^\lambda)\big| > \varepsilon,\ |V^\lambda| \leq K_\delta\big) + \mathbb{P}\big(|V^\lambda| > K_\delta\big) \\
&\leq \limsup_{\lambda\to\infty} \left(\mathbb{P}\left(\sup_{|x|\leq K_\delta} \big|J^\lambda(x) - J(x)\big| > \varepsilon \right) + \mathbb{P}\left(|V^\lambda| > K_\delta\right) \right) \\
&\leq 0 + \delta = \delta,
\end{aligned}
$$

where we applied Lemma 2.4. Since δ can be arbitrarily chosen, we can claim that (3.7) holds. From the relation

$$
\overline{F}^\lambda = J^\lambda\left(V^\lambda\right) = J^\lambda(V^\lambda) - J(V^\lambda) + J(V^\lambda)
$$

and (3.6)-(3.7) we conclude that $\overline{F}^\lambda \xrightarrow{d} F$ by Slutsky's theorem. By using a similar argument and the relations $(D_t F)^\lambda = D_t^\lambda \overline{F}^\lambda$, and $(\delta(u))^\lambda = \delta^\lambda(u^\lambda)$, we also obtain $\overline{u}_t^\lambda \xrightarrow{d} u_t$, $D_t^\lambda \overline{F}^\lambda \xrightarrow{d} D_t F$ and $\delta^\lambda(\overline{u}^\lambda) \xrightarrow{d} \delta(u)$ as λ tends to infinity. □

Theorem 3.2. *a) For any $F \in \mathrm{Dom}(D)$ and $h \in L^2(\mathbb{R}_+)$, we have the weak convergences*

$$
\overline{F}^\lambda \xrightarrow{d} F \quad \textit{and} \quad \langle h, D^\lambda \overline{F}^\lambda \rangle \xrightarrow{d} \langle h, DF \rangle, \qquad [\lambda \to \infty].
$$

b) For any $u \in \mathbb{L}_{1,2}$, we have the weak convergence

$$
\delta^\lambda(\overline{u}^\lambda) \xrightarrow{d} \delta(u), \qquad [\lambda \to \infty].
$$

Proof. For any $F \in \mathrm{Dom}(D)$, there exists a sequence $(F_n)_{n\in\mathbb{N}} \in \mathcal{C}$ such that $\|F_n - F\|_{1,2} \to 0$ and $\|\overline{F}_n^\lambda - \overline{F}^\lambda\|_{1,2} \to 0$ as n tends to infinity, for any $\lambda > 0$. By using Lemma 3.1, for each $F_n \in \mathcal{C}$ we have

$$
\overline{F}_n^\lambda \xrightarrow{d} F_n \text{ as } \lambda \to \infty.
$$

Fix $\varepsilon > 0$, for any non-constant bounded and Lipschitz function $f : \mathbb{R} \longrightarrow \mathbb{R}$ there exists a large enough $\lambda \in \mathbb{R}_+$ and $n_0(\epsilon) \in \mathbb{N}$, such that

$$
\begin{cases}
\|F - F_{n_0(\epsilon)}\|_{L^2(\Omega)} \leq \dfrac{\varepsilon}{3\|f'\|_\infty}, & \text{(3.8a)} \\[2ex]
\|\overline{F}^\lambda - \overline{F}^\lambda_{n_0(\epsilon)}\|_{L^2(\Omega)} \leq \dfrac{\varepsilon}{3\|f'\|_\infty}, & \text{(3.8b)} \\[2ex]
\left|\mathbb{E}\left[f\left(\overline{F}^\lambda_{n_0(\epsilon)}\right) - f(F_{n_0(\epsilon)})\right]\right| < \dfrac{\varepsilon}{3}, & \text{(3.8c)}
\end{cases}
$$

where $\|f'\|_\infty := \sup_{x\in\mathbb{R}} |f'(x)|$, and (3.8c) follows from the limit $F^\lambda_{n_0(\epsilon)} \xrightarrow{d} F_{n_0(\epsilon)}$ as guaranteed by Lemma 3.1. By using (3.8a), (3.8b) and (3.8c), for large enough λ, we can deduce that

$$\begin{aligned}
&\left|\mathbb{E}\left[f(\overline{F}^\lambda) - f(F)\right]\right| \\
&\le \left|\mathbb{E}\left[f(\overline{F}^\lambda) - f(\overline{F}^\lambda_{n_0(\epsilon)})\right]\right| + \left|\mathbb{E}\left[f(\overline{F}^\lambda_{n_0(\epsilon)}) - f(F_{n_0(\epsilon)})\right]\right| + \left|\mathbb{E}[f(F_{n_0(\epsilon)}) - f(F)]\right| \\
&\le \left(\left\|\overline{F}^\lambda - \overline{F}^\lambda_{n_0(\epsilon)}\right\|_{L^2(\Omega)} + \|F - F_{n_0(\epsilon)}\|_{L^2(\Omega)}\right)\|f'\|_\infty + \left|\mathbb{E}\left[f(\overline{F}^\lambda_{n_0(\epsilon)}) - f(F_{n_0(\epsilon)})\right]\right| \\
&\le \frac{2}{3}\varepsilon + \frac{1}{3}\varepsilon = \varepsilon,
\end{aligned}$$

which justifies $\overline{F}^\lambda \xrightarrow{d} F$. Similarly, for $F \in \mathrm{Dom}(D)$ and $h \in L^2(\mathbb{R}_+)$ there exists a sequence $(F_n)_{n\in\mathbb{N}} \in \mathcal{C}$ such that $\|\langle h, D^\lambda \overline{F}^\lambda_n\rangle - \langle h, D^\lambda \overline{F}^\lambda\rangle\|_{L^2(\Omega)} \to 0$ as n tends to infinity, by the inequality

$$\|\langle h, D^\lambda \overline{F}^\lambda\rangle - \langle h, D^\lambda \overline{F}^\lambda_n\rangle\|_{L^2(\Omega)} \le \|h\|_{L^2(\mathbb{R}_+)}\|D^\lambda \overline{F}^\lambda - D^\lambda \overline{F}^\lambda_n\|_{L^2(\Omega\times\mathbb{R}_+)},$$

and a similar argument can be applied to conclude that $\langle h, D^\lambda \overline{F}^\lambda\rangle \xrightarrow{d} \langle h, DF\rangle$. In the case of $u \in \mathbb{L}_{1,2}$ we choose a sequence $(u_n)_{n\in\mathbb{N}} \in \mathcal{U}$ such that $\|u_n - u\|_{1,2} \to 0$ and $\|\overline{u}^\lambda_n - \overline{u}^\lambda\|_{1,2} \to 0$ as n tends to infinity, for any $\lambda > 0$, and we conclude similarly as above, since $\delta^\lambda(\overline{u}^\lambda_n) \xrightarrow{d} \delta(u_n)$ as λ tends to infinity. □

As an example, given $(F_\zeta)_{z\in\mathbb{R}}$ a family of regular enough random variables depending on a parameter ζ, consider the classical Malliavin calculus argument

$$\begin{aligned}
\frac{\partial}{\partial\zeta}\mathbb{E}[f(F_\zeta)] &= \mathbb{E}\left[f'(F_\zeta)\frac{\partial F_\zeta}{\partial\zeta}\right] \\
&= \mathbb{E}\left[\frac{\langle Df(F_\zeta), u\rangle}{\langle DF_\zeta, u\rangle}\frac{\partial F_\zeta}{\partial\zeta}\right] \\
&= \mathbb{E}\left[f(F_\zeta)\delta\left(\frac{u}{\langle DF_\zeta, u\rangle}\frac{\partial F_\zeta}{\partial\zeta}\right)\right], \qquad f \in \mathcal{C}^1_b(\mathbb{R}),
\end{aligned}$$

for the computation of the sensitivity $\partial\mathbb{E}[f(F_\zeta)]/\partial\zeta$, where $u = (u_t)_{t\in\mathbb{R}_+}$ is a process suitably chosen so that $u\frac{\partial F_\zeta}{\partial\zeta}/\langle DF_\zeta, u\rangle$ belongs to $\mathrm{Dom}(\delta)$. This identity can be discretized into its Poisson version

$$\frac{\partial}{\partial\zeta}\mathbb{E}[f(F_\zeta)] \simeq \mathbb{E}\left[f(\overline{F}^\lambda_\zeta)\delta^\lambda\left(\frac{\overline{u}^\lambda}{\langle D^\lambda \overline{F}^\lambda_\zeta, \overline{u}^\lambda\rangle}\overline{\left(\frac{\partial F_\zeta}{\partial\zeta}\right)}^\lambda\right)\right], \tag{3.9}$$

due to the weak convergences of $\overline{F}_\zeta^\lambda$, $D^\lambda \overline{F}_\zeta^\lambda$ and $\delta^\lambda(\overline{u}^\lambda)$ in Theorem 3.2. In addition, the rate of convergence of $(\overline{F}_\zeta^\lambda)_{\lambda>0}$ to F_ζ can be estimated using a Wasserstein-type distance according to Theorem 5.1 below.

4 Convergence of damped gradient operators

There are actually two Malliavin derivative operators in the Poisson space, the first one is the annihilation operator $D_t^\lambda(\cdot)$ introduced in the previous section, while another one is the damped gradient operator $\tilde{D}_t^\lambda(\cdot)$, which will be explained in this section; also see [26]. In particular, the relationship between the damped operator $\tilde{D}_t^\lambda$ defined in the Poisson space and the Malliavin derivative operator D_t defined in the Wiener space is illustrated as follows.

Consider the set $\mathcal{S}_\lambda$ of smooth Poisson functionals defined as

$$\mathcal{S}_\lambda := \left\{ F = f_n\left(T_1^\lambda, \ldots, T_n^\lambda\right) \ : \ f_n \in C_0^\infty\left(\mathbb{R}^n\right), \ n \in \mathbb{N} \right\},$$

where T_i^λ is the i^{th} jump time of $(N_{\lambda t})_{t\in\mathbb{R}_+}$.

Definition 4.1. *For every $F \in \mathcal{S}_\lambda$, the damped operator $\tilde{D}^\lambda : L^2(\Omega) \mapsto L^2(\Omega \times \mathbb{R}_+)$ is defined as*

$$\tilde{D}_t^\lambda F := -\frac{1}{\sqrt{\lambda}} \sum_{k=1}^n \mathbb{1}_{[0,T_k^\lambda]}(t) \partial_k f_n(T_1^\lambda, \ldots, T_n^\lambda), \ t \in \mathbb{R}_+.$$

The operator $\tilde{D}^\lambda$ is also closable and can be extended to its closed domain $\mathrm{Dom}(\tilde{D}^\lambda)$ under the Sobolev norm $\|\cdot\|_{1,2}^{\sim}$ defined by

$$\|F\|_{1,2}^{\sim} := \|F\|_{L^2(\Omega)} + \|\tilde{D}^\lambda F\|_{L^2(\Omega\times\mathbb{R}_+)}, \quad F \in \mathcal{S}_\lambda.$$

Consider the set $\mathcal{Q}$ of functionals defined as

$$\mathcal{Q} := \Bigg\{ F = \mathbb{E}[F] + \sum_{n=1}^\infty I_n^\lambda(h_n^{\otimes n}) \ : \ h_n \in C_0^1\left(\mathbb{R}_+\right), \\ \sum_{n=1}^\infty n^2 n! \left(\left\|h_{n+1}'\right\|_{L^2(\mathbb{R}_+)}^{2n+2} + \|h_n\|_{L^2(\mathbb{R}_+)}^{2n} \right) < \infty \Bigg\},$$

which is dense in the space of square-integrable Poisson functionals. In the next proposition, $F^\lambda = \sum_{n=0}^{\infty} I_n^\lambda \left(h_n^{\otimes n}\right)$ denotes the Poisson discretization of $F \in \mathcal{Q}$ of the form $F = \mathbb{E}[F] + \sum_{n=1}^{\infty} I_n(h_n^{\otimes n})$. Recall that by Theorem 3.2 we have the weak convergence $F^\lambda \longrightarrow F$.

Proposition 4.2. *For any $F \in \mathcal{Q}$ and $t \in \mathbb{R}_+$ we have the weak convergence*

$$\tilde{D}_t^\lambda F^\lambda \xrightarrow{d} D_t F, \qquad [\lambda \to \infty].$$

Proof. By e.g. § 4.8 and § 7.7 in [26], the annihilation operator D^λ and the operator $\tilde{D}^\lambda$ satisfy the relation

$$\tilde{D}_t^\lambda I_n^\lambda(h_n^{\otimes n}) = D_t^\lambda I_n^\lambda(h_n^{\otimes n}) - \frac{n}{\sqrt{\lambda}} I_n^\lambda \left(\left(h_n' \mathbb{1}_{[t,\infty)} \right) \circ h_n^{\otimes(n-1)} \right), \quad h_n \in C_0^1 \big(\mathbb{R}_+ \big).$$

Hence by the definition of F^λ we have

$$\tilde{D}^\lambda F^\lambda = D^\lambda F^\lambda - \sum_{n=1}^{\infty} \frac{n}{\sqrt{\lambda}} I_n^\lambda \big(\big(h_n' \mathbb{1}_{[t,\infty)} \big) \circ h_n^{\otimes(n-1)} \big).$$

Fix $m \in \mathbb{N}$, since $\|\tilde{f}_n\|_{L^2(\mathbb{R}_+^n)} \le \|f_n\|_{L^2(\mathbb{R}_+^n)}$ for any $f_n \in L^2(\mathbb{R}_+^n)$, we have

$$\begin{aligned} \mathbb{E}\left[\left(\sum_{n=1}^{m} \frac{n}{\sqrt{\lambda}} I_n^\lambda \big(\big(h_n' \mathbb{1}_{[t,\infty)} \big) \circ h_n^{\otimes(n-1)} \big) \right)^2 \right] &= \frac{1}{\lambda} \sum_{n=1}^{m} n^2 n! \left\| \big(h_n' \mathbb{1}_{[t,\infty)} \big) \circ h_n^{\otimes(n-1)} \right\|_{L^2(\mathbb{R}_+^n)}^2 \\ &\le \frac{1}{\lambda} \sum_{n=1}^{\infty} n n! \|h_n'\|_{L^2(\mathbb{R}_+)}^{2n} + \frac{1}{\lambda} \sum_{n=1}^{\infty} n^2 n! \|h_n\|_{L^2(\mathbb{R}_+)}^{2n}, \end{aligned} \tag{4.1}$$

where (4.1) follows from Young's inequality, so that the last term converges to zero as $\lambda \to \infty$ since $F^\lambda \in \mathcal{Q}$. Together with Theorem 3.2 we deduce that

$$\tilde{D}_t^\lambda F^\lambda \xrightarrow{d} D_t F, \qquad t \in \mathbb{R}_+,$$

as λ tends to infinity. □

For example, for the single Poisson integral

$$F^\lambda = \int_0^\infty h(t) d\tilde{N}_t^\lambda = \frac{1}{\sqrt{\lambda}} \sum_{k=1}^{\infty} h(T_k^\lambda) - \sqrt{\lambda} \int_0^\infty h(t) dt,$$

where $h \in C_0^1(\mathbb{R})$, we note that

$$\begin{aligned}\tilde{D}_t^\lambda F^\lambda &= -\frac{1}{\lambda}\sum_{T_k^\lambda > t} h'(T_k^\lambda) \\ &= -\frac{1}{\sqrt{\lambda}}\int_t^\infty h'(s)d\tilde{N}_s^\lambda - \int_t^\infty h'(s)ds \\ &= h(t) - \frac{1}{\sqrt{\lambda}}\int_t^\infty h'(s)d\tilde{N}_s^\lambda \ ,\end{aligned}$$

which converges weakly to $h(t) = D_t F$ as λ tends to infinity since

$$\mathbb{E}\left[\left(\frac{1}{\sqrt{\lambda}}\int_t^\infty h'(s)d\tilde{N}_s^\lambda\right)^2\right] = \frac{1}{\lambda}\int_t^\infty h'(s)^2 ds \xrightarrow{\lambda\to\infty} 0.$$

5 Convergence rate of discretized Wiener functionals

In this section we establish the convergence rate of the discretized Poisson functionals $\overline{F}^\lambda$ to the Wiener functional F when the intensity parameter λ tends to infinity. For measuring this closeness, we use the Wasserstein-type distance

$$d(F,G) := \sup_{g\in\mathcal{G}} |\mathbb{E}[g(F) - g(G)]|, \tag{5.1}$$

which is the distance $d_{\mathcal{H}_2}$ in [18] between two random variables F and G in $L^2(\Omega)$, where

$$\mathcal{G} := \big\{g \in C^2(\mathbb{R}) \ : \ \max(\|g\|_\infty, \|g'\|_\infty, \|g''\|_\infty) \le 1\big\}. \tag{5.2}$$

Although this distance differs from the common Wasserstein distance by the additional requirement that $\|g''\|_\infty \le 1$, it is clear that $d(F_n, G) \to 0$ implies $F_n \xrightarrow{d} G$ as n tends to infinity.

In the rest of this paper, we consider symmetric functions $f_n : [0,T]^n \to \mathbb{R}$ with a compact support in the hypercube $[0,T]^n$, $n \ge 1$, where $0 < T < \infty$ is fixed. Theorem 5.1 below provides the convergence rate of $O(\lambda^{-1/4})$ from Poisson discretizations to the corresponding Wiener functionals, up to a constant depending on the chaos expansion of the functional F. In the sequel, we let $H_{1,2}$ denote the standard Sobolev space on $\mathbb{R}_+^n$ with the norm

$$\|h\|_{H_{1,2}(\mathbb{R}_+^n)}^2 := \|h\|_{L^2(\mathbb{R}_+^n)}^2 + \|\nabla h\|_{L^2(\mathbb{R}_+^n;\mathbb{R}^n)}^2, \qquad h \in H_{1,2}(\mathbb{R}_+^n),$$

where ∇h denotes the weak gradient of h.

Theorem 5.1. *Let F be a Wiener functional with Wiener chaos expansion (3.1) such that for all $n \geq 1$ the symmetric function f_n belongs to $H_{1,2}(\mathbb{R}_+^n)$, which also satisfies*

$$\sum_{n=1}^{\infty} n \cdot n! \|\nabla f_n\|_{L^2(\mathbb{R}_+^n;\mathbb{R}^n)}^2 < \infty, \tag{5.3}$$

and

$$\int_0^T \mathbb{E}\left[(D_t D_t F)^2\right] dt = \sum_{n=2}^{\infty} n(n-1)n! \int_0^{\infty} \|f_n(\cdot, t, t)\|_{L^2(\mathbb{R}_+^{n-2})}^2 dt < \infty. \tag{5.4}$$

Then we have the universal rate of convergence:

$$d\big(\overline{F}^{\lambda}, F\big) = O(\lambda^{-1/4}), \qquad [\lambda \to \infty], \tag{5.5}$$

for the Poisson discretization $\overline{F}^{\lambda}$ to F.

Before proceeding to the proof of Theorem 5.1, we consider two examples of application.

Example - Time average of geometric Brownian motion

Given $\sigma : \mathbb{R}_+ \to \mathbb{R}_+$ a bounded deterministic function and $b \in \mathcal{C}_b^1$ such that

$$\|b\|_\infty, \|b'(t)\|_\infty, \|\sigma\|_\infty, \|\sigma'\|_\infty < C,$$

where $C > 0$ is a finite constant, consider the geometric Brownian motion

$$\begin{aligned} X_t &= X_0 \exp\left(\int_0^t b(s)ds + \int_0^t \sigma(s)dB_s - \frac{1}{2}\int_0^t \sigma^2(s)ds\right) \\ &= X_0 \exp\left(\int_0^t b(s)ds\right) \sum_{n=0}^{\infty} \frac{1}{n!} I_n(\sigma^{\circ n} \mathbb{1}_{[0,t]^n}). \end{aligned}$$

Given h a bounded deterministic function with $\|h\|_\infty < C$, the time integral $F := \int_0^T h(t)X_t dt$, admits the chaos expansion

$$F = \mathbb{E}[F] + \sum_{n=1}^{\infty} I_n(f_n),$$

with the kernels

$$f_n(t_1, \ldots, t_n) = \frac{1}{n!} \int_{t_n}^T h(s) e^{\int_0^s b(r)dr} ds \cdot \prod_{i=1}^n \sigma(t_i), \qquad 0 \leq t_1 \leq t_2 \leq \cdots \leq t_n.$$

We have

$$\partial_{t_i} f_n(t_1, \ldots, t_n) = \mathbb{1}_{\{1 \leq i < n\}} \frac{1}{n!} \sigma'(t_i) \int_{t_n}^T h(s) e^{\int_0^s b(r)dr} ds \cdot \prod_{\substack{i=1 \\ i \neq j}}^n \sigma(t_j)$$
$$+ \mathbb{1}_{\{i=n\}} \frac{1}{n!} \prod_{j=1}^{n-1} \sigma(t_j) \left(\sigma'(t_n) \int_{t_n}^T h(s) e^{\int_0^s b(r)dr} ds - \sigma(t_n) h(t_n) e^{\int_0^{t_n} b(r)dr} \right),$$

$i = 1, 2, \ldots, n$, and therefore

$$|f_n(t_1, \ldots, t_n)| \leq T \frac{c^{n+1}}{n!} e^{cT} \quad \textit{and} \quad |\partial_{t_i} f_n(t_1, \ldots, t_n)| \leq (T+1) \frac{c^{n+1}}{n!} e^{cT},$$

$i = 1, \ldots, n$, from which it follows that

$$\|f_n\|^2_{L^2([0,T]^n)} \leq T^{n+2} \frac{c^{2n+2}}{(n!)^2} e^{2cT} \quad \text{and} \quad \|\nabla f_n\|^2_{L^2([0,T]^n; \mathbb{R}^n)} \leq n(T+1)^2 \frac{c^{2n+2}}{(n!)^2} e^{2cT}.$$

On the other hand, we also check that

$$\sum_{n=1}^{\infty} n n! \|\nabla f_n\|^2_{L^2(\mathbb{R}^n)} < \infty,$$
$$\int_0^T \mathbb{E}[(D_t D_t F)^2] dt = \sum_{n=2}^{\infty} n! n(n-1) \int_{[0,T]^{n-1}} f_n^2(t_1, \ldots, t_{n-1} t_{n-1}) dt_1 \cdots dt_{n-1} < \infty,$$

which shows that the conditions of Theorem 5.1 are satisfied. In this case, the Poisson discretization $\overline{F}^\lambda$ of $F = \int_0^T h(t) X_t dt$ is given by

$$\overline{F}^\lambda = X_0 \int_0^T h(t) \exp\left(\int_0^t b(s) ds - \lambda \int_0^t \sigma^2(s) ds \right) \prod_{k=1}^{N_t^\lambda} (1 + \sigma(T_k)) dt,$$

where $(N_t^\lambda)_{t \in \mathbb{R}_+}$ is a standard Poisson process with intensity $\lambda > 0$, and sequence of jump times $(T_k)_{k \geq 1}$. The time average $F := \int_0^T h(t) X_t dt$ can be used for e.g. the pricing of Asian options.

Example - Stochastic differential equations

Consider the SDE

$$dX_t = b(X_t)dt + \sigma dB_t, \tag{5.6}$$

where $b \in \mathcal{C}^2(\mathbb{R})$ with $\|b'\|_\infty, \|b''\|_\infty < C < \infty$, and σ is a positive constant.

We note that

$$D_{t_1}X_t = \int_{t_1}^{t\vee t_1} b'(X_s)D_{t_1}X_s ds + \sigma, \quad \partial_{t_1}D_{t_1}X_t = \int_{t_1}^{t\vee t_1} b'(X_s)\partial_{t_1}D_{t_1}X_s ds - \sigma b'(X_{t_1}),$$

$0 \le t_1 \le t$, which implies, by using Gronwall's inequality:

$$\sup_{t_1\in[0,T]} |D_{t_1}X_T| < \sigma e^{CT}, \quad \sup_{t_1\in[0,T]} |\partial_{t_1}D_{t_1}X_T| < C\sigma e^{CT}, \tag{5.7}$$

for some constant $C > 0$. Furthermore we have

$$D_{t_1}D_{t_2}X_t = \int_{t_1\vee t_2}^{t} b'(X_s)D_{t_1}D_{t_2}X_s ds + \int_{t_1\vee t_2}^{t} b''(X_s)D_{t_1}X_s D_{t_2}X_s ds, \quad 0 \le t_1, t_2 \le t,$$

hence

a) for $0 \le t_1 \le t_2 \le t$,

$$\partial_{t_2}D_{t_1}D_{t_2}X_t = -\sigma b''(X_{t_2})D_{t_1}X_{t_2} + \int_{t_2}^{t} b'(X_s)(\partial_{t_2}D_{t_1}D_{t_2}X_s)ds + \int_{t_2}^{t} b''(X_s)(D_{t_1}X_s)(\partial_{t_2}D_{t_2}X_s)ds$$

b) for $0 \le t_2 \le t_1 \le t$,

$$\partial_{t_2}D_{t_1}D_{t_2}X_t = \int_{t_1}^{t} b'(X_s)(\partial_{t_2}D_{t_1}D_{t_2}X_s)ds + \int_{t_1}^{t} b''(X_s)(D_{t_1}X_s)(\partial_{t_2}D_{t_2}X_s)ds.$$

By using (5.7) and Gronwall's inequality, we find that

$$\sup_{t_1,t_2\in[0,T]} \mathbb{E}[|\partial_{t_2}D_{t_1}D_{t_2}X_T|^2] < \big(C\sigma^2 e^{CT} + C^2 T \cdot \sigma^2 e^{2CT}\big)e^{CT} < \infty. \tag{5.8}$$

Now, if

$$F := X_T = X_0 + \int_0^T b(X_s)ds + \sigma B_T$$

has the chaos expansion

$$X_T = \sum_{n=0}^{\infty} I_n(f_n),$$

we have

$$\partial_{t_2} D_{t_1} D_{t_2} X_T = \sum_{n=2}^{\infty} n(n-1) I_{n-2}(\partial_{t_2} f_n(*, t_1, t_2)),$$

hence by (5.8) we deduce that

$$\begin{aligned}
\sum_{n=1}^{\infty} nn! \|\nabla f_n\|^2_{L^2([0,T]^n;\mathbb{R}^n)} &= \sum_{n=2}^{\infty} nn! \int_0^T \cdots \int_0^T \|\nabla f_n(t_3, \ldots, t_n, t_1, t_2)\|^2_{\mathbb{R}^n} dt_1 \cdots dt_n \\
&\leq 2 \sum_{n=2}^{\infty} (n-1) n! \int_0^T \cdots \int_0^T \|\nabla f_n(t_3, \ldots, t_n, t_1, t_2)\|^2_{\mathbb{R}^n} dt_1 \cdots dt_n \\
&= 2 \sum_{n=2}^{\infty} n(n-1) n! \int_0^T \cdots \int_0^T |\partial_{t_2} f_n(t_3, \ldots, t_n, t_1, t_2)|^2 dt_1 \cdots dt_n \\
&= 2 \int_0^T \int_0^T \mathbb{E}\left[|\partial_{t_2} D_{t_1} D_{t_2} X_T|^2 \right] dt_1 dt_2 \\
&\leq 2 \big(C\sigma^2 e^{CT} + C^2 T \cdot \sigma^2 e^{2CT} \big) e^{CT} T^2,
\end{aligned}$$

where we applied the symmetric property of f_n for the second equality, i.e.

$$\partial_{t_2} f(t_2, t_1, \ldots, t_n) = \partial_{t_2} f(t_1, t_2, \ldots, t_n) = \cdots = \partial_{t_2} f(t_1, t_3, \ldots, t_n, t_2).$$

Similarly, by using Gronwall's inequality, we find

$$\mathbb{E}\left[|D_{t_1} D_{t_1} X_T|^2 \right] \leq CT\sigma^2 e^{3CT} < \infty, \qquad 0 \leq t_1 \leq T,$$

hence

$$\int_0^T \mathbb{E}[(D_t D_t F)^2] dt \leq CT < \infty,$$

showing that the conditions of Theorem 5.1 are satisfied. In this case, the Poisson discretization of X_t is given by solving the SDE

$$dX_t^\lambda = b(X_t^\lambda) dt + \frac{\sigma}{\sqrt{\lambda}} (dN_t^\lambda - \lambda dt), \text{ for } 0 \leq t \leq T,$$

where $(N_t^\lambda)_{t \in \mathbb{R}_+}$ is a standard Poisson process with intensity $\lambda > 0$. Diffusion equations with non-constant diffusion coefficient $dY_t = c(Y_t)dt + \sigma(Y_t)dB_t$ such that $1/\sigma(y)$

is locally integrable can also be considered by writing Y_t as a deterministic function $Y_t = \varphi(t, X_t)$ of X_t by the Lamperti transformation, depending on the regularity of $\varphi(t, x)$.

Proof of Theorem 5.1. For a given $\alpha \in (0,1)$, let

$$M_\lambda := \lceil \lambda^\alpha \rceil, \qquad T_\lambda := \frac{T}{M_\lambda} = \frac{T}{\lceil \lambda^\alpha \rceil}, \tag{5.9}$$

where $\lceil x \rceil$ denotes the smallest integer greater than x, and

$$A_k^\lambda := [t_{k-1}^\lambda, t_k^\lambda), \quad \text{where} \quad t_k^\lambda := kT_\lambda, \qquad k = 1, \ldots, M_\lambda. \tag{5.10}$$

Next, for $n \geq 1$, letting

$$c_{i_1,\ldots,i_n} := \frac{1}{(T_\lambda)^n} \int_{A_{i_1}^\lambda \times \cdots \times A_{i_n}^\lambda} f_n(t_1, \ldots, t_n) dt_1 \cdots dt_n, \tag{5.11}$$

$1 \leq i_1, \ldots, i_n \leq M_\lambda$, we define the off-diagonal discretization $f_n^{\alpha,\lambda}$ of f_n as

$$f_n^{\alpha,\lambda}(t_1, \ldots, t_n) := \sum_{\substack{i_1,\ldots,i_n=1 \\ i_1 \neq \cdots \neq i_n}}^{M_\lambda} c_{i_1,\ldots,i_n} \mathbb{1}_{A_{i_1}^\lambda \times \cdots \times A_{i_n}^\lambda}(t_1, \ldots, t_n), \tag{5.12}$$

where $i_1 \neq \cdots \neq i_n$ means $i_l \neq i_k$, $1 \leq l \neq k \leq n$, so that

$$\|f_n^{\alpha,\lambda}\|_{L^2([0,T]^n)} \leq \|f_n\|_{L^2([0,T]^n)} \quad \text{and} \quad \lim_{\lambda \to \infty} \|f_n^{\alpha,\lambda} - f_n\|_{L^2([0,T]^n)} = 0. \tag{5.13}$$

The multiple Wiener integral $I_n(f_n^{\alpha,\lambda})$ of $f_n^{\alpha,\lambda}$ can be written as the sum

$$I_n(f_n^{\alpha,\lambda}) = \sum_{\substack{i_1,\ldots,i_n=1 \\ i_1 \neq \cdots \neq i_n}}^{M_\lambda} c_{i_1,\ldots,i_n} I_n\big(\mathbb{1}_{A_{i_1}^\lambda \times \cdots \times A_{i_n}^\lambda}\big) = \sum_{\substack{i_1,\ldots,i_n=1 \\ i_1 \neq \cdots \neq i_n}}^{M_\lambda} c_{i_1,\ldots,i_n} \prod_{k=1}^n I_1\big(\mathbb{1}_{A_{i_k}^\lambda}\big) \tag{5.14}$$

of products of mutually independent first order Wiener integrals. Similarly, for the multiple Poisson integral $I_n^\lambda(f_n^{\alpha,\lambda})$, we have

$$I_n^\lambda(f_n^{\alpha,\lambda}) = \sum_{\substack{i_1,\ldots,i_n=1 \\ i_1 \neq \cdots \neq i_n}}^{M_\lambda} c_{i_1,\ldots,i_n} I_n^\lambda\big(\mathbb{1}_{A_{i_1}^\lambda \times \cdots \times A_{i_n}^\lambda}\big) = \sum_{\substack{i_1,\ldots,i_n=1 \\ i_1 \neq \cdots \neq i_n}}^{M_\lambda} c_{i_1,\ldots,i_n} \prod_{k=1}^n I_1^\lambda\big(\mathbb{1}_{A_{i_k}^\lambda}\big). \tag{5.15}$$

Letting, for any $m \geq 1$,

$$F_m := \mathbb{E}[F] + \sum_{n=1}^{m} I_n(f_n), F_m^{\alpha,\lambda} := \mathbb{E}[F] + \sum_{n=1}^{m} I_n(f_n^{\alpha,\lambda}) \text{ and } F^{\alpha,\lambda} := \mathbb{E}[F] + \sum_{n=1}^{\infty} I_n(f_n^{\alpha,\lambda}),$$

for any $g \in \mathcal{G}$, by a Taylor series expansion and the definition (5.2) of $\mathcal{G}$, we have

$$\begin{aligned}
\left|\mathbb{E}\left[g\left(F_m^{\alpha,\lambda}\right) - g\left(F_m\right)\right]\right| &= \left|\mathbb{E}\left[g\left(\sum_{n=1}^{m} I_n(f_n^{\alpha,\lambda})\right) - g\left(\sum_{n=1}^{m} I_n(f_n)\right)\right]\right| \\
&\leq \mathbb{E}\left[\left|g'\left(\sum_{n=1}^{m} I_n(f_n^{\alpha,\lambda})\right)\right| \left|\sum_{n=1}^{m}(I_n(f_n) - I_n(f_n^{\alpha,\lambda}))\right|\right] \\
&\quad + \frac{1}{2}\mathbb{E}\left[\left|g''(\xi)\right| \left|\sum_{n=1}^{m}(I_n(f_n) - I_n(f_n^{\alpha,\lambda}))\right|^2\right] \\
&\leq \mathbb{E}\left[\left|\sum_{n=1}^{m}(I_n(f_n) - I_n(f_n^{\alpha,\lambda}))\right|\right] + \frac{1}{2}\mathbb{E}\left[\left|\sum_{n=1}^{m}(I_n(f_n) - I_n(f_n^{\alpha,\lambda}))\right|^2\right] \\
&\leq \sqrt{\mathbb{E}\left[\left|\sum_{n=1}^{m} I_n(f_n - f_n^{\alpha,\lambda})\right|^2\right]} + \frac{1}{2}\mathbb{E}\left[\left|\sum_{n=1}^{m} I_n(f_n - f_n^{\alpha,\lambda})\right|^2\right] \\
&\leq \sqrt{\sum_{n=1}^{m} n!\|f_n - f_n^{\alpha,\lambda}\|^2_{L^2([0,T]^n)}} + \frac{1}{2}\sum_{n=1}^{m} n!\|f_n - f_n^{\alpha,\lambda}\|^2_{L^2([0,T]^n)},
\end{aligned}$$

where ξ lies on the line joining the random numbers $F_m^{\alpha,\lambda}$ and F_m. Hence we have, by considering a.s. convergent subsequences,

$$\begin{aligned}
d\big(F^{\alpha,\lambda}, F\big) &= \sup_{g \in \mathcal{G}} \left|\mathbb{E}\left[g\left(F^{\alpha,\lambda}\right) - g\left(F\right)\right]\right| \\
&= \sup_{g \in \mathcal{G}} \lim_{m \to \infty} \left|\mathbb{E}\left[g\left(F_m^{\alpha,\lambda}\right) - g\left(F_m\right)\right]\right| \\
&\leq \sqrt{\sum_{n=1}^{\infty} n!\|f_n - f_n^{\alpha,\lambda}\|^2_{L^2([0,T]^n)}} + \frac{1}{2}\sum_{n=1}^{\infty} n!\left\|f_n - f_n^{\alpha,\lambda}\right\|^2_{L^2([0,T]^n)},
\end{aligned}$$

and by Lemma 6.1 in Section 5, we get

$$d\big(F^{\alpha,\lambda}, F\big) = O\big(\lambda^{-\alpha/2}\big), \qquad [\lambda \to \infty].$$

Similarly, we can also obtain:

$$\overline{F^{\alpha,\lambda}}^{\lambda} := \mathbb{E}[F] + \sum_{n=1}^{\infty} I_n^{\lambda}(f_n^{\alpha,\lambda}), \qquad \lambda > 0,$$

by using the isometry (3.3) and Lemma 6.1, we can show that

$$d(\overline{F^{\alpha,\lambda}}^{\lambda}, \overline{F}^{\lambda}) = O\big(\lambda^{-\alpha/2}\big), \qquad [\lambda \to \infty],$$

In addition, by Lemma 6.2 in Section 5, we have

$$d(\overline{F^{\alpha,\lambda}}^{\lambda}, F^{\alpha,\lambda}) = O\big(\lambda^{(\alpha-1)/2}\big), \qquad [\lambda \to \infty],$$

hence by the triangle inequality we find

$$\begin{aligned} d\big(\overline{F}^{\lambda}, F\big) &\leq d\big(\overline{F}^{\lambda}, \overline{F^{\alpha,\lambda}}^{\lambda}\big) + d\big(\overline{F^{\alpha,\lambda}}^{\lambda}, F^{\alpha,\lambda}\big) + d\big(F^{\alpha,\lambda}, F\big) \\ &= O\big(\lambda^{-\alpha/2}\big) + O\big(\lambda^{(\alpha-1)/2}\big). \end{aligned}$$

We conclude by taking $\alpha = 1/2$, which yields the optimal rate. □

6 Key lemmas

In this section we prove the main Lemmas 6.1, 6.2, 6.3 and 6.4, which were used in the proof of Theorem 5.1. In Lemma 6.1, we first start by showing the convergence of $F^{\alpha,\lambda}$ to F (resp. $\overline{F^{\alpha,\lambda}}^{\lambda}$ to $\overline{F}^{\lambda}$) with a rate $O(\lambda^{-\alpha/2})$ under the $L^2(\Omega)$ norm.

Lemma 6.1. *Let $\alpha \in (0,1)$ and consider F a Wiener functional with Wiener chaos expansion (3.1) satisfying the conditions (5.3)-(5.4) of Theorem 5.1. Then we have*

$$\|F^{\alpha,\lambda} - F\|^2_{L^2(\Omega)} = \|\overline{F^{\alpha,\lambda}}^{\lambda} - \overline{F}^{\lambda}\|^2_{L^2(\Omega)} = \sum_{m=1}^{\infty} n! \|f_n^{\alpha,\lambda} - f_n\|^2_{L^2([0,T]^n)} = O\big(\lambda^{-\alpha}\big), \quad (6.1)$$

as λ tends to infinity.

Proof. Recall that

$$f_n^{\alpha,\lambda}(t_1,\dots,t_n) := \sum_{\substack{i_1,\dots,i_n=1 \\ i_1 \neq \cdots \neq i_n}}^{M_\lambda} c_{i_1,\dots,i_n} \mathbb{1}_{A^{\lambda}_{i_1} \times \cdots \times A^{\lambda}_{i_n}}(t_1,\dots,t_n),$$

where

$$c_{i_1,\dots,i_n} := \frac{1}{(T_\lambda)^n} \int_{A^{\lambda}_{i_1} \times \cdots \times A^{\lambda}_{i_n}} f_n(t_1,\dots,t_n) dt_1 \cdots dt_n, \quad 1 \leq i_1,\dots,i_n \leq M_\lambda.$$

Let now

$$\bar{f}_n^{\alpha,\lambda}(t_1,\ldots,t_n) := \sum_{i_1,\ldots,i_n=1}^{M_\lambda} c_{i_1,\ldots,i_n} \mathbb{1}_{A_{i_1}^\lambda \times\cdots\times A_{i_n}^\lambda}(t_1,\ldots,t_n),$$

with the triangle inequality

$$\|f_n - f_n^{\alpha,\lambda}\|^2_{L^2([0,T]^n)} \le 2\|f_n - \bar{f}_n^{\alpha,\lambda}\|^2_{L^2([0,T]^n)} + 2\|\bar{f}_n^{\alpha,\lambda} - f_n^{\alpha,\lambda}\|^2_{L^2([0,T]^n)}. \tag{6.2}$$

By definition, we have

$$\begin{aligned}\|f_n - \bar{f}_n^{\alpha,\lambda}\|^2_{L^2([0,T]^n)} &= \sum_{i_1,\ldots,i_n=1}^{M_\lambda} \int_{A_{i_1}^\lambda \times\cdots\times A_{i_n}^\lambda} (f_n - \bar{f}_n^{\alpha,\lambda})^2 dt_1\cdots dt_n \\ &= \sum_{i_1,\ldots,i_n=1}^{M_\lambda} \int_{A_{i_1}^\lambda \times\cdots\times A_{i_n}^\lambda} (f_n(t_1,\ldots,t_n) - c_{i_1,\ldots,i_n})^2 dt_1\cdots dt_n \\ &\le n\frac{(T_\lambda)^2}{\pi^2}\int_{[0,T]^n} \|\nabla f_n(t_1,\ldots,t_n)\|^2_{\mathbb{R}^n} dt_1\cdots dt_n, \end{aligned} \tag{6.3}$$

where the last inequality holds is due to the fact that the domain $A_{i_1}^\lambda \times\cdots\times A_{i_n}^\lambda$ is convex with diameter $\sqrt{n}T_\lambda$, and

$$\int_{A_{i_1}^\lambda \times\cdots\times A_{i_n}^\lambda} (f_n(t_1,\ldots,t_n) - c_{i_1,\ldots,i_n}) dt_1\cdots dt_n = 0.$$

Next, an application of the Poincaré inequality, cf. e.g. Theorem 3.2 of [1] shows the upper bound

$$\begin{aligned}&\int_{A_{i_1}^\lambda \times\cdots\times A_{i_n}^\lambda} (f_n(t_1,\ldots,t_n) - c_{i_1,\ldots,i_n})^2 dt_1\cdots dt_n \\ &\le \frac{n\cdot(T_\lambda)^2}{\pi^2}\int_{A_{i_1}^\lambda \times\cdots\times A_{i_n}^\lambda} \|\nabla f_n(t_1,\ldots,t_n)\|^2_{\mathbb{R}^n} dt_1\cdots dt_n,\end{aligned}$$

$1 \le i_1,\ldots,i_n \le M_\lambda$. For the second term in (6.2), we have

$$\begin{aligned}&\|\bar{f}_n^{\alpha,\lambda} - f_n^{\alpha,\lambda}\|^2_{L^2([0,T]^n)} \\ &= \sum_{\substack{i_1,\ldots,i_n=1 \\ i_l=i_k \text{ for some } i\neq k}}^{M_\lambda} \int_{A_{i_1}^\lambda \times\cdots\times A_{i_n}^\lambda} c^2_{i_1,\ldots,i_n} dt_1\cdots dt_n \\ &= (T_\lambda)^n \sum_{\substack{i_1,\ldots,i_n=1 \\ i_l=i_k \text{ for some } i\neq k}}^{M_\lambda} \left(\frac{1}{(T_\lambda)^n}\int_{A_{i_1}^\lambda \times\cdots\times A_{i_n}^\lambda} f_n(t_1,\ldots,t_n) dt_1\cdots dt_n\right)^2\end{aligned}$$

$$\leq \sum_{\substack{i_1,\ldots,i_n=1 \\ i_l=i_k \text{ for some } i\neq k}}^{M_\lambda} \int_{A^\lambda_{i_1}\times\cdots\times A^\lambda_{i_n}} f_n^2(t_1,\ldots,t_n)dt_1\cdots dt_n.$$

Therefore we deduce that

$$\begin{aligned}
&\sum_{n=1}^\infty n!\|\bar{f}_n^{\alpha,\lambda}-f_n^{\alpha,\lambda}\|^2_{L^2([0,T]^n)} \\
&\leq \sum_{n=1}^\infty n! \sum_{\substack{i_1,\ldots,i_n=1 \\ i_l=i_k \text{ for some } i\neq k}}^{M_\lambda} \int_{A^\lambda_{i_1}\times\cdots\times A^\lambda_{i_n}} f_n^2(t_1,\ldots,t_n)dt_1\cdots dt_n \\
&= \sum_{n=1}^\infty n!\binom{n}{2} \sum_{i_{n-1}=1}^{M_\lambda} \int_{[0,T]^{n-2}\times A^\lambda_{i_{n-1}}\times A^\lambda_{i_{n-1}}} f_n^2(t_1,\ldots,t_{n-1},t_n)dt_1\cdots dt_{n-2}dt_{n-1}dt_n \\
&= \sum_{n=1}^\infty n!\binom{n}{2} \sum_{i_{n-1}=1}^{M_\lambda} \int_{A^\lambda_{i_{n-1}}\times A^\lambda_{i_{n-1}}} \tilde{f}_n(t_{n-1},t_n)dt_{n-1}dt_n,
\end{aligned}$$

where the first equality follows by symmetry of f_n, and

$$\tilde{f}_n(x,y) := \int_{[0,T]^{n-2}} f_n^2(t_1,\ldots,t_{n-2},x,y)dt_1\cdots dt_{n-2},\ \ x,y\in\mathbb{R}.$$

By the fundamental theorem of calculus, we have

$$\begin{aligned}
&\int_{A^\lambda_{i_{n-1}}\times A^\lambda_{i_{n-1}}} \tilde{f}_n(t_{n-1},t_n)dt_{n-1}dt_n \qquad (6.4)\\
&= \int_{A^\lambda_{i_{n-1}}}\left(\int_{A^\lambda_{i_{n-1}}} \tilde{f}_n(t_{n-1},t_{n-1})dt_{n-1}\right)dt_n + \int_{A^\lambda_{i_{n-1}}\times A^\lambda_{i_{n-1}}}\left(\int_{t_{n-1}}^{t_n} \partial_s\tilde{f}_n(t_{n-1},s)ds\right)dt_{n-1}dt_n \\
&= T_\lambda\cdot\int_{[0,T]^{n-2}\times A^\lambda_{i_{n-1}}} f_n^2(t_1,\ldots,t_{n-2},t_{n-1},t_{n-1})dt_1\cdots dt_{n-1} \\
&\quad + 2\int_{A^\lambda_{i_{n-1}}\times A^\lambda_{i_{n-1}}}\left(\int_{t_{n-1}}^{t_n}\int_{[0,T]^{n-2}} f_n(t_1,\ldots,t_{n-2},t_{n-1},s)\right. \\
&\qquad\qquad \left.\cdot\,\partial_s f_n(t_1,\ldots,t_{n-2},t_{n-1},s)dt_1\cdots dt_{n-2}ds\right)dt_{n-1}dt_n.
\end{aligned}$$

Next, using Young's inequality $2ab\leq a^2/2+4b^2$, we can deduce that

$$\begin{aligned}
&2\int_{A^\lambda_{i_{n-1}}\times A^\lambda_{i_{n-1}}}\left(\int_{t_{n-1}}^{t_n}\int_{[0,T]^{n-2}} f_n(t_1,\ldots,t_{n-2},t_{n-1},s)\right. \\
&\qquad\qquad \left.\times\,\partial_s f_n(t_1,\ldots,t_{n-2},t_{n-1},s)dt_1\cdots dt_{n-2}ds\right)dt_{n-1}dt_n
\end{aligned}$$

$$\leq \frac{T_\lambda}{2} \int_{A^\lambda_{i_{n-1}} \times A^\lambda_{i_{n-1}}} \tilde{f}_n(t_{n-1}, t_n) dt_{n-1} dt_n$$

$$+ 4T_\lambda \int_{[0,T]^{n-2} \times A^\lambda_{i_{n-1}} \times A^\lambda_{i_{n-1}}} |\partial_{t_n} f_n(t_1, \ldots, t_{n-1}, t_n)|^2 dt_1 \cdots dt_{n-1} dt_n.$$

Substituting the last inequality back into (6.4), after rearrangement, we further deduce that

$$\begin{aligned}
&\int_{A^\lambda_{i_{n-1}} \times A^\lambda_{i_{n-1}}} \tilde{f}_n(t_{n-1}, t_n) dt_{n-1} dt_n \\
&\leq 2 \cdot T_\lambda \cdot \int_{[0,T]^{n-2} \times A^\lambda_{i_{n-1}}} f_n^2(t_1, \ldots, t_{n-2}, t_{n-1}, t_{n-1}) dt_1 \cdots dt_{n-1} \\
&\quad + 8 \cdot T_\lambda \int_{[0,T]^{n-2} \times A^\lambda_{i_{n-1}} \times A^\lambda_{i_{n-1}}} |\partial_{t_n} f_n(t_1, \ldots, t_{n-1}, t_n)|^2 dt_1 \cdots dt_{n-1} dt_n.
\end{aligned}$$

Then

$$\begin{aligned}
&\sum_{i_{n-1}=1}^{M_\lambda} \int_{A^\lambda_{i_{n-1}} \times A^\lambda_{i_{n-1}}} \tilde{f}_n(t_{n-1}, t_n) dt_{n-1} dt_n \\
&\leq 2T_\lambda \int_{[0,T]^{n-1}} f_n^2(t_1, \ldots, t_{n-2}, t_{n-1}, t_{n-1}) dt_1 \cdots dt_{n-1} \\
&\quad + 8 \frac{T_\lambda}{n} \int_{[0,T]^n} \|\nabla f_n(t_1, \ldots, t_n)\|^2 dt_1 \cdots dt_n,
\end{aligned}$$

where the last inequality follows from the symmetric property of f_n. Now we can obtain

$$\begin{aligned}
&\sum_{n=1}^\infty n! \|\bar{f}_n^{\alpha,\lambda} - f_n^{\alpha,\lambda}\|_{L^2([0,T]^n)} \\
&\leq T_\lambda \sum_{n=1}^\infty (n-1)! n^2 (n-1) \int_{[0,T]^{n-1}} f_n^2(t_1, \ldots, t_{n-2}, t_{n-1}, t_{n-1}) dt_1 \cdots dt_{n-1} \\
&\quad + 4T_\lambda \sum_{n=1}^\infty (n-1) \cdot n! \int_{[0,T]^n} \|\nabla f_n(t_1, \ldots, t_n)\|^2 dt_1 \cdots dt_n \\
&< T_\lambda \int_0^T \mathbb{E}[(D_t D_t F)^2] dt \\
&\quad + 4T_\lambda \sum_{n=1}^\infty n \cdot n! \int_{[0,T]^n} \|\nabla f_n(t_1, \ldots, t_n)\|^2 dt_1 \cdots dt_n. \qquad (6.5)
\end{aligned}$$

Combining (6.2), (6.3) and (6.5), we conclude that

$$\sum_{n=1}^{\infty} n! \|f_n - f_n^{\alpha,\lambda}\|^2_{L^2([0,T]^n)}$$
$$\leq \frac{2}{\pi^2}(T_\lambda)^2 \sum_{n=1}^{\infty} n \cdot n! \int_{[0,T]^n} \|\nabla f_n(t_1, \ldots, t_n)\|^2 dt_1 \cdots dt_n$$

$$+ 2T_\lambda \int_0^T \mathbb{E}[(D_t D_t F)^2]dt + 8T_\lambda \sum_{n=1}^{\infty} n \cdot n! \int_{[0,T]^n} \|\nabla f_n(t_1, \ldots, t_n)\|^2 dt_1 \cdots dt_n < \tilde{C} T_\lambda,$$

where

$$\tilde{C} := 9 \sum_{n=1}^{\infty} n \cdot n! \int_{[0,T]^n} \|\nabla f_n\|^2 + 2 \int_0^T \mathbb{E}[(D_t D_t F)^2]dt < \infty$$

is independent of λ and partition, and it yields (6.1). □

The proof of Lemma 6.2 relies on the multivariate Stein method and the Malliavin calculus on the Poisson space.

Lemma 6.2. *For any $\alpha > 0$ we have*

$$d\big(\overline{F^{\alpha,\lambda}}^{\lambda}, F^{\alpha,\lambda}\big) = O\big(\lambda^{(\alpha-1)/2}\big), \qquad [\lambda \to \infty]. \tag{6.6}$$

Proof. Let

$$V^\lambda = \big(V_1^\lambda, \ldots, V_{M_\lambda}^\lambda\big)^\top := \big(I_1^\lambda(\mathbb{1}_{A_1^\lambda}), I_1^\lambda(\mathbb{1}_{A_2^\lambda}), \ldots, I_1^\lambda(\mathbb{1}_{A_{M_\lambda}^\lambda})\big)^\top$$

and

$$U^\lambda = \big(U_1^\lambda, \ldots, U_{M_\lambda}^\lambda\big)^\top := \big(I_1(\mathbb{1}_{A_1^\lambda}), I_1(\mathbb{1}_{A_2^\lambda}), \ldots, I_1(\mathbb{1}_{A_{M_\lambda}^\lambda})\big)^\top,$$

where $A_1^\lambda, \ldots, A_{M_\lambda}^\lambda$ are defined in (5.10). We note that

$$\begin{aligned}
I_n^\lambda(f_n^{\alpha,\lambda}) &= \sum_{\substack{i_1,\ldots,i_n=1 \\ i_1 \neq \cdots \neq i_n}}^{M_\lambda} c_{i_1,\ldots,i_n} I_n^\lambda\big(\mathbb{1}_{A_{i_1}^\lambda \times \cdots \times A_{i_n}^\lambda}\big) \\
&= \sum_{\substack{i_1,\ldots,i_n=1 \\ i_1 \neq \cdots \neq i_n}}^{M_\lambda} c_{i_1,\ldots,i_n} \prod_{k=1}^{n} I_1^\lambda(\mathbb{1}_{A_{i_k}^\lambda}), \\
&= \sum_{\substack{i_1,\ldots,i_n=1 \\ i_1 \neq \cdots \neq i_n}}^{M_\lambda} c_{i_1,\ldots,i_n} \Big(I_1^\lambda(\mathbb{1}_{A_1^\lambda})\Big)^{\sum_{k=1}^n \mathbb{1}_{\{i_k=1\}}} \cdots \Big(I_1^\lambda(\mathbb{1}_{A_{M_\lambda}^\lambda})\Big)^{\sum_{k=1}^n \mathbb{1}_{\{i_k=M_\lambda\}}}
\end{aligned}$$

$$= \quad h_n^{\alpha,\lambda}(V^\lambda),$$

where $c_{i_1,i_2,\ldots,i_n}$ is defined in (5.11) and

$$h_n^{\alpha,\lambda}(x_1,\ldots,x_{M_\lambda}) := \sum_{\substack{i_1,\ldots,i_n=1\\ i_1\neq\cdots\neq i_n}}^{M_\lambda} c_{i_1,\ldots,i_n} x_1^{\sum_{k=1}^n \mathbb{1}_{\{i_k=1\}}} x_2^{\sum_{k=1}^n \mathbb{1}_{\{i_k=2\}}} \cdots x_{M_\lambda}^{\sum_{k=1}^n \mathbb{1}_{\{i_k=M_\lambda\}}}. \quad (6.7)$$

Similarly, we can also obtain:

$$I_n(f_n^{\alpha,\lambda}) = h_n^{\alpha,\lambda}(U^\lambda).$$

Hence, letting

$$\overline{F_m^{\alpha,\lambda}}^\lambda := \mathbb{E}[F] + \sum_{n=1}^m I_n^\lambda(f_n^{\alpha,\lambda}) \quad \text{and} \quad F_m^{\alpha,\lambda} := \mathbb{E}[F] + \sum_{n=1}^m I_n(f_n^{\alpha,\lambda}),$$

we have

$$\left|\mathbb{E}\left[g\Big(\overline{F_m^{\alpha,\lambda}}^\lambda\Big) - g\left(F_m^{\alpha,\lambda}\right)\right]\right| = \left|\mathbb{E}\left[g^{\alpha,\lambda}\left(V^\lambda\right) - g^{\alpha,\lambda}\left(U^\lambda\right)\right]\right|, \quad (6.8)$$

where

$$g^{\alpha,\lambda}(x) := g\left(\sum_{n=1}^m h_n^{\alpha,\lambda}(x)\right), \qquad x \in \mathbb{R}^{M_\lambda}.$$

We shall estimate (6.8) by the multiple Stein method combined with the Malliavin calculus. We use the representation (5.14) of $I_n(f_n^{\alpha,\lambda})$ as a series of products of mutually independent first order Wiener chaos random variables whose joint distribution is multivariate Gaussian, which allows us to apply the multivariate Stein method in order to quantify the Wasserstein-type distance between $\overline{F}_m^{\alpha,\lambda}$ and $F_m^{\alpha,\lambda}$ for all $m \geq 1$, therefore allowing us to bound $d\big(\overline{F^{\alpha,\lambda}}^\lambda, F^{\alpha,\lambda}\big)$.

Given that $U^\lambda \sim \mathcal{N}(0, T_\lambda I_d)$ is Gaussian with diagonal covariance matrix $T_\lambda I_d$, where I_d denotes the identity matrix on $\mathbb{R}^{M_\lambda}$ and M_λ is defined in (5.9), by Lemma 3.3 in [20] the function

$$\tilde{g}_{h,\lambda}(x) := \frac{1}{2}\int_0^1 \mathbb{E}\left[g^{\alpha,\lambda}\left(x\sqrt{s} + U^\lambda\sqrt{1-s}\right) - g^{\alpha,\lambda}(U^\lambda)\right]\frac{ds}{s}, \quad x \in \mathbb{R}^{M_\lambda}, \quad (6.9)$$

satisfies the multivariate Stein equation

$$\mathbb{E}\left[g^{\alpha,\lambda}\left(V^\lambda\right) - g^{\alpha,\lambda}\left(U^\lambda\right)\right] = \mathbb{E}\left[T_\lambda \operatorname{Tr}\left(\operatorname{Hess} \tilde{g}_{h,\lambda}(V^\lambda)\right) - \langle V^\lambda, \nabla \tilde{g}_{h,\lambda}(V^\lambda)\rangle_{\mathbb{R}^{M_\lambda}}\right]$$

$$= \mathbb{E}\left[T_\lambda \operatorname{Tr}\left(\text{Hess } \tilde{g}_{h,\lambda}(V^\lambda)\right) - \sum_{j=1}^{M_\lambda} \left\langle L^\lambda (L^\lambda)^{-1} V_j^\lambda, \frac{\partial \tilde{g}_{h,\lambda}}{\partial x_j}(V^\lambda) \right\rangle_{\mathbb{R}^{M_\lambda}}\right]$$

$$= \mathbb{E}\left[T_\lambda \operatorname{Tr}\left(\text{Hess } \tilde{g}_{h,\lambda}(V^\lambda)\right)\right] - \sum_{j=1}^{M_\lambda} \mathbb{E}\left[\left\langle D^\lambda \left(L^\lambda\right)^{-1} V_j^\lambda, D^\lambda \left(\frac{\partial \tilde{g}_{h,\lambda}}{\partial x_j}\left(V^\lambda\right)\right)\right\rangle_{\mathbb{R}^{M_\lambda}}\right], \tag{6.10}$$

where V_j^λ is the j^{th} component of V^λ, $L^\lambda := \delta^\lambda D^\lambda$ is the Ornstein-Uhlenbeck operator on the Poisson space, and the last equality follows by the duality (2.4) between D^λ and δ^λ.

Next, by the finite difference property (2.5) in Proposition 2.9 of the Poisson Malliavin derivative operator and the mean value theorem we find that

$$D_t^\lambda \left(\frac{\partial \tilde{g}_{h,\lambda}}{\partial x_j}\left(V^\lambda\right)\right) = \sqrt{\lambda}\left(\frac{\partial \tilde{g}_{h,\lambda}}{\partial x_j}\left(V^\lambda(N_\cdot + \mathbb{1}_{[t,\infty)}(\cdot))\right) - \frac{\partial \tilde{g}_{h,\lambda}}{\partial x_j}\left(V^\lambda\right)\right)$$

$$= \sum_{i=1}^{M_\lambda} \frac{\partial^2 \tilde{g}_{h,\lambda}}{\partial x_i \partial x_j}\left(V^\lambda\right) D_t^\lambda V_i^\lambda + \frac{1}{2\sqrt{\lambda}} (D_t^\lambda V^\lambda)^\top \left(\text{Hess } \frac{\partial \tilde{g}_{h,\lambda}}{\partial x_j}(\Psi_t^\lambda)\right) D_t^\lambda V^\lambda, \tag{6.11}$$

for every $t \in [0,T]$, where Ψ_t^λ lies on the line joining $V^\lambda(N_\cdot)$ and $V^\lambda\left(N_\cdot + \mathbb{1}_{[t,\infty)}(\cdot)\right)$. Given that

$$(L^\lambda)^{-1} V_j^\lambda = I_1^\lambda(\mathbb{1}_{A_j^\lambda}) = V_j^\lambda, \qquad j = 1, \ldots, M_\lambda, \tag{6.12}$$

cf. also Section 4.4 in [26], applying (6.11) to (6.10) we get

$$\mathbb{E}\left[g^{\alpha,\lambda}(V^\lambda) - g^{\alpha,\lambda}(U^\lambda)\right]$$

$$= \mathbb{E}\left[T_\lambda \operatorname{Tr}\left(\text{Hess } \tilde{g}_{h,\lambda}(V^\lambda)\right)\right]$$

$$- \sum_{j=1}^{M_\lambda} \sum_{i=1}^{M_\lambda} \mathbb{E}\left[\left\langle D_\cdot^\lambda \left(L^\lambda\right)^{-1} V_j^\lambda, \frac{\partial^2 \tilde{g}_{h,\lambda}}{\partial x_i \partial x_j}\left(V^\lambda\right) D_\cdot^\lambda V_i^\lambda \right\rangle_{\mathbb{R}^{M_\lambda}}\right] \tag{6.13}$$

$$- \frac{1}{2\sqrt{\lambda}} \sum_{j=1}^{M_\lambda} \mathbb{E}\left[\left\langle D_\cdot^\lambda \left(L^\lambda\right)^{-1} V_j^\lambda, (D_\cdot^\lambda V^\lambda)^\top \left(\text{Hess } \frac{\partial \tilde{g}_{h,\lambda}}{\partial x_j}(\Psi_\cdot^\lambda)\right) D_\cdot^\lambda V^\lambda \right\rangle_{\mathbb{R}^{M_\lambda}}\right] \tag{6.14}$$

$$= -\frac{1}{2\sqrt{\lambda}} \sum_{k=1}^{M_\lambda} \mathbb{E}\left[\left\langle \mathbb{1}_{A_k^\lambda}(\cdot), \mathbb{1}_{A_k^\lambda}(\cdot) \left(\frac{\partial^3 \tilde{g}_{h,\lambda}}{\partial x_k^3}(\Psi_\cdot^\lambda)\right)\right\rangle_{\mathbb{R}}\right]$$

$$= -\frac{1}{2\sqrt{\lambda}} \sum_{k=1}^{M_\lambda} \int_{t_{k-1}^\lambda}^{t_k^\lambda} \mathbb{E}\left[\frac{\partial^3 \tilde{g}_{h,\lambda}}{\partial x_k^3}\left(\Psi_t^\lambda\right)\right] dt, \tag{6.15}$$

where the second equality follows by (6.12), while (6.14) becomes the expression in the next line by using (6.12) again, and (6.13) vanishes as

$$\begin{aligned}
& T_\lambda \mathbb{E}\left[\operatorname{Tr}\left(\text{Hess } \tilde{g}_{h,\lambda}(V^\lambda)\right)\right] \\
& - \sum_{j=1}^{M_\lambda}\sum_{i=1}^{M_\lambda} \mathbb{E}\left[\left\langle D_\cdot^\lambda \left(L^\lambda\right)^{-1} V_j^\lambda, \frac{\partial^2 \tilde{g}_{h,\lambda}}{\partial x_i \partial x_j}\left(V^\lambda\right) D_\cdot^\lambda V_i^\lambda \right\rangle_{\mathbb{R}^{M_\lambda}}\right] \\
&= T_\lambda \sum_{i=1}^{M_\lambda} \mathbb{E}\left[\frac{\partial^2 \tilde{g}_{h,\lambda}}{\partial x_i^2}\left(V^\lambda\right)\right] - \sum_{j=1}^{M_\lambda}\sum_{i=1}^{M_\lambda} \mathbb{E}\left[\left\langle \mathbb{1}_{A_j^\lambda}(\cdot), \frac{\partial^2 \tilde{g}_{h,\lambda}}{\partial x_i \partial x_j}\left(V^\lambda\right) \mathbb{1}_{A_i^\lambda}(\cdot)\right\rangle_{\mathbb{R}^{M_\lambda}}\right] \\
&= T_\lambda \sum_{i=1}^{M_\lambda} \mathbb{E}\left[\frac{\partial^2 \tilde{g}_{h,\lambda}}{\partial x_i^2}\left(V^\lambda\right)\right] - T_\lambda \sum_{i=1}^{M_\lambda} \mathbb{E}\left[\frac{\partial^2 \tilde{g}_{h,\lambda}}{\partial x_i^2}\left(V^\lambda\right)\right] \\
&= 0.
\end{aligned}$$

By applying Lemma 6.4 below to (6.15) we conclude that

$$\begin{aligned}
&\left|\mathbb{E}\left[g\left(\overline{F_m^{\alpha,\lambda}}^\lambda\right) - g\left(F_m^{\alpha,\lambda}\right)\right]\right| = \left|\mathbb{E}\left[g^{\alpha,\lambda}(V^\lambda) - g^{\alpha,\lambda}(U^\lambda)\right]\right| \\
&\leq \frac{\pi}{4}\sqrt{\frac{\lceil\lambda^\alpha\rceil}{\lambda T}} \|g''\|_\infty \sum_{k=1}^{M_\lambda} \int_{t_{k-1}^\lambda}^{t_k^\lambda} \sum_{n=1}^m n n! \|f_n(\cdot,t)\|^2_{L^2(\mathbb{R}_+^{n-1})} dt \\
&= \frac{\pi}{4}\sqrt{\frac{\lceil\lambda^\alpha\rceil}{\lambda T}} \|g''\|_\infty \sum_{n=1}^m n n! \|f_n\|^2_{L^2([0,T]^n)}, \quad g \in \mathcal{U},
\end{aligned}$$

from which (6.6) follows as

$$\begin{aligned}
d\left(\overline{F^{\alpha,\lambda}}^\lambda, F^{\alpha,\lambda}\right) &= \sup_{g\in\mathcal{G}} \left|\mathbb{E}\left[g\left(\overline{F^{\alpha,\lambda}}^\lambda\right) - g\left(F^{\alpha,\lambda}\right)\right]\right| \\
&= \sup_{g\in\mathcal{G}} \lim_{m\to\infty} \left|\mathbb{E}\left[g\left(\overline{F_m^{\alpha,\lambda}}^\lambda\right) - g\left(F_m^{\alpha,\lambda}\right)\right]\right| \\
&\leq \frac{\pi}{4}\sqrt{\frac{\lceil\lambda^\alpha\rceil}{\lambda T}} \sum_{n=1}^\infty n n! \|f_n\|^2_{L^2([0,T]^n)}.
\end{aligned}$$

□

The proof of Lemma 6.4 below relies on the next Lemma 6.3. Define an *i.i.d.* copy of V^λ as

$$\hat{V}^\lambda := \left(\int_0^\infty \mathbb{1}_{A_1^\lambda}(s) d\hat{N}_s^\lambda, \ldots, \int_0^\infty \mathbb{1}_{A_{M_\lambda}^\lambda}(s) d\hat{N}_s^\lambda\right)^\top,$$

where $\hat{N}^\lambda_s := (N'_{\lambda s} - \lambda s)/\sqrt{\lambda}$ is a renormalized compensated standard Poisson process with intensity λ, independent of both $(B_t)_{t\in\mathbb{R}_+}$ and $(\tilde{N}^\lambda_t)_{t\in\mathbb{R}_+}$.

Lemma 6.3. *Let $\hat{\Psi}^\lambda_t$ denote an i.i.d. copy of Ψ^λ_t as defined in the proof of Lemma 6.2, lying on the line joining $\hat{V}^\lambda(\hat{N}^\lambda_\cdot)$ to $\hat{V}^\lambda\big(\hat{N}^\lambda_\cdot + \mathbb{1}_{[t,\infty)}\big)$. We have the relation*

$$\frac{\partial h^{\alpha,\lambda}_n}{\partial x_k}(\hat{\Psi}^\lambda\sqrt{s} + U^\lambda\sqrt{1-s}) = nI^{\lambda,s}_{n-1}(f^{\alpha,\lambda}_n(\cdot,t)), \qquad t \in A^\lambda_k, \tag{6.16}$$

where $I^{\lambda,s}_n(\cdot) : L^2(\mathbb{R})^{\circ n} \to \mathbb{R}$ is the operator defined as

$$\begin{aligned} I^{\lambda,s}_n(f_n) \;&:=\; n!\sqrt{s}\int_0^\infty \cdots \int_0^{s_2^-} f_n(s_1,\ldots,s_n)d\hat{N}^\lambda_{s_1}\cdots d\hat{N}^\lambda_{s_n} \\ &\qquad + n!\sqrt{1-s}\int_0^\infty \cdots \int_0^{s_2} f_n(s_1,\ldots,s_n)dB_{s_1}\cdots dB_{s_n}, \end{aligned} \tag{6.17}$$

with the isometry property

$$\|I^{\lambda,s}_n(f_n)\|^2_{L^2(\Omega)} = n!\|f_n\|^2_{L^2([0,T]^n)}, \qquad f_n \in L^2(\mathbb{R})^{\circ n}. \tag{6.18}$$

Proof. Given that

$$\begin{aligned} &\hat{V}^\lambda(N_\cdot + \mathbb{1}_{[t,\infty)}(\cdot)) \\ &= \left(\int_0^\infty \mathbb{1}_{A^\lambda_1}(s)d\tilde{N}^\lambda_s + \frac{\mathbb{1}_{A^\lambda_1}(t)}{\sqrt{\lambda}},\ldots,\int_0^\infty \mathbb{1}_{A^\lambda_{M_\lambda}}(s)d\tilde{N}^\lambda_s + \frac{\mathbb{1}_{A^\lambda_{M_\lambda}}(t)}{\sqrt{\lambda}}\right), \end{aligned}$$

for any $j = 1,\ldots,M_\lambda$ with $j \neq k$ and $t \in A^\lambda_k$ we have

$$\hat{V}^\lambda_j(N_\cdot + \mathbb{1}_{[t,\infty)}(\cdot)) = \int_0^\infty \mathbb{1}_{A^\lambda_j}(s)d\tilde{N}^\lambda_s + \frac{\mathbb{1}_{A^\lambda_j}(t)}{\sqrt{\lambda}} = \int_0^\infty \mathbb{1}_{A^\lambda_j}(s)d\tilde{N}^\lambda_s = \hat{V}^\lambda_j.$$

Consequently, since $\hat{\Psi}^\lambda_t$ lies on the line joining $\hat{V}^\lambda(N_\cdot)$ to $\hat{V}^\lambda\left(N_\cdot + \mathbb{1}_{[t,\infty)}(\cdot)\right)$, we have

$$\hat{\Psi}^\lambda_{t,j} = \int_0^\infty \mathbb{1}_{A^\lambda_j}(s)d\tilde{N}^\lambda_s, \quad t \in A^\lambda_k, \quad j = 1,\ldots,M_\lambda, \quad j \neq k, \tag{6.19}$$

for every component of $\hat{\Psi}^\lambda = (\hat{\Psi}^\lambda_1,\ldots,\hat{\Psi}^\lambda_{M_\lambda})$. Given that the sum in (6.7) is over distinct i_k's, the power $\sum_{l=1}^n \mathbb{1}_{\{i_l=k\}}$ of x_k in $h^{\alpha,\lambda}_n(x)$ is either 1 or 0, cf. (6.7). Therefore we have

$$\frac{\partial h^{\alpha,\lambda}_n}{\partial x_k}(x) = \sum_{\substack{i_1,\ldots,i_n=1 \\ i_1\neq\cdots\neq i_n}}^{M_\lambda} c_{i_1,\ldots,i_n}\left(\sum_{l=1}^n \mathbb{1}_{\{i_l=k\}}\right) \tag{6.20}$$

$$\times\, x_1^{\sum_{l=1}^n \mathbb{1}_{\{i_l=1\}}} \cdots x_{k-1}^{\sum_{l=1}^n \mathbb{1}_{\{i_l=k-1\}}} x_{k+1}^{\sum_{l=1}^n \mathbb{1}_{\{i_l=k+1\}}} \cdots x_{M_\lambda}^{\sum_{l=1}^n \mathbb{1}_{\{i_l=M_\lambda\}}}\,.$$

By (6.19), (6.20), and the definition (6.17) of $I_n^{\lambda,s}(f_n)$ we find

$$\frac{\partial h_n^{\alpha,\lambda}}{\partial x_k}(\hat{\Psi}^\lambda\sqrt{s}+U^\lambda\sqrt{1-s}) = \frac{\partial h_n^{\alpha,\lambda}}{\partial x_k}(\hat{V}^\lambda\sqrt{s}+U^\lambda\sqrt{1-s})$$

$$= \sum_{\substack{i_1,\dots,i_n=1\\ i_1\neq\cdots\neq i_n}}^{M_\lambda} c_{i_1,\dots,i_n}\sum_{l=1}^n \mathbb{1}_{\{i_l=k\}}\big(I_1^{\lambda,s}(\mathbb{1}_{A_1^\lambda})\big)^{\sum_{l=1}^n \mathbb{1}_{\{i_l=1\}}}\cdots\big(I_1^{\lambda,s}(\mathbb{1}_{A_{k-1}^\lambda})\big)^{\sum_{l=1}^n \mathbb{1}_{\{i_l=k-1\}}}$$

$$\times\big(I_1^{\lambda,s}(\mathbb{1}_{A_{k+1}^\lambda})\big)^{\sum_{l=1}^n \mathbb{1}_{\{i_l=k+1\}}}\cdots\big(I_1^{\lambda,s}(\mathbb{1}_{A_{M_\lambda}^\lambda})\big)^{\sum_{l=1}^n \mathbb{1}_{\{i_l=M_\lambda\}}}$$

$$= \sum_{\substack{i_1,\dots,i_n=1\\ i_1\neq\cdots\neq i_n}}^{M_\lambda} c_{i_1,\dots,i_n}\sum_{l=1}^n \mathbb{1}_{\{i_l=k\}}\big(I_1^{\lambda,s}(\mathbb{1}_{A_{i_1}^\lambda})\big)^{\mathbb{1}_{\{i_1=1\}}}\cdots\big(I_1^{\lambda,s}(\mathbb{1}_{A_{i_n}^\lambda})\big)^{\mathbb{1}_{\{i_n=1\}}}$$

$$\times\big(I_1^{\lambda,s}(\mathbb{1}_{A_{i_1}^\lambda})\big)^{\mathbb{1}_{\{i_1=2\}}}\cdots\big(I_1^{\lambda,s}(\mathbb{1}_{A_{i_n}^\lambda})\big)^{\mathbb{1}_{\{i_n=2\}}}$$

$$\cdots$$

$$\times\big(I_1^{\lambda,s}(\mathbb{1}_{A_{i_1}^\lambda})\big)^{\mathbb{1}_{\{i_1=k-1\}}}\cdots\big(I_1^{\lambda,s}(\mathbb{1}_{A_{i_n}^\lambda})\big)^{\mathbb{1}_{\{i_n=k-1\}}}$$

$$\times\big(I_1^{\lambda,s}(\mathbb{1}_{A_{i_1}^\lambda})\big)^{\mathbb{1}_{\{i_1=k+1\}}}\cdots\big(I_1^{\lambda,s}(\mathbb{1}_{A_{i_n}^\lambda})\big)^{\mathbb{1}_{\{i_n=k+1\}}}$$

$$\cdots$$

$$\times\big(I_1^{\lambda,s}(\mathbb{1}_{A_{i_1}^\lambda})\big)^{\mathbb{1}_{\{i_1=M_\lambda\}}}\cdots\big(I_1^{\lambda,s}(\mathbb{1}_{A_{i_n}^\lambda})\big)^{\mathbb{1}_{\{i_n=M_\lambda\}}}$$

$$= \sum_{\substack{i_1,\dots,i_n=1\\ i_1\neq\cdots\neq i_n}}^{M_\lambda} c_{i_1,\dots,i_n}\sum_{l=1}^n \mathbb{1}_{\{i_l=k\}}\big(I_1^{\lambda,s}(\mathbb{1}_{A_{i_1}^\lambda})\big)^{\mathbb{1}_{\{i_1\neq k\})}}\cdots\big(I_1^{\lambda,s}(\mathbb{1}_{A_{i_n}^\lambda})\big)^{\mathbb{1}_{\{i_n\neq k\}}}$$

$$= \sum_{\substack{i_2,i_3,\dots,i_n=1\\ i_1\neq\cdots\neq i_n}}^{M_\lambda} c_{k,i_2\dots,i_n} I_1^{\lambda,s}(\mathbb{1}_{A_{i_2}^\lambda}) I_1^{\lambda,s}(\mathbb{1}_{A_{i_3}^\lambda})\cdots I_1^{\lambda,s}(\mathbb{1}_{A_{i_n}^\lambda}) \tag{6.21}$$

$$+ \sum_{\substack{i_1,i_3\dots,i_n=1\\ i_1\neq\cdots\neq i_n}}^{M_\lambda} c_{i_1,k,\dots,i_n} I_1^{\lambda,s}(\mathbb{1}_{A_{i_1}^\lambda}) I_1^{\lambda,s}(\mathbb{1}_{A_{i_3}^\lambda})\cdots I_1^{\lambda,s}(\mathbb{1}_{A_{i_n}^\lambda})$$

$$+\cdots$$

$$+ \sum_{\substack{i_1,i_2\dots,i_{n-1}=1\\ i_1\neq\cdots\neq i_n}}^{M_\lambda} c_{i_1,\dots,i_{n-1},k} I_1^{\lambda,s}(\mathbb{1}_{A_{i_1}^\lambda})\cdots I_1^{\lambda,s}(\mathbb{1}_{A_{i_{n-1}}^\lambda})$$

$$= n\sum_{\substack{i_1,i_2\dots,i_{n-1}=1\\ i_1\neq\cdots\neq i_n}}^{M_\lambda} c_{i_1,\dots,i_{n-1},k} I_1^{\lambda,s}(\mathbb{1}_{A_{i_1}^\lambda})\cdots I_1^{\lambda,s}(\mathbb{1}_{A_{i_{n-1}}^\lambda}) \tag{6.22}$$

$$= nI^{\lambda,s}_{n-1}(f^{\alpha,\lambda}_n(\cdot,t)), \quad t \in A^\lambda_k,$$

where (6.21) holds because $c_{k,i_2,\ldots,i_n} = 0$ if $i_j = k$ for some $j \in \{2,\ldots,n\}$, and (6.22) follows from the symmetry relation

$$c_{i_1,\ldots,i_{m-1},k,i_{m+1},\ldots,i_n} = c_{i_1,\ldots,i_{n-1},k}, \qquad m = 1,\ldots,n.$$

$\square$

Lemma 6.4. *For all $\alpha \in (0,1)$ and $k = 1,\ldots,M_\lambda$, we have*

$$\left| \int_{t^\lambda_{k-1}}^{t^\lambda_k} \mathbb{E}\left[\frac{\partial^3 \tilde{g}_{h,\lambda}}{\partial x_k^3}(\Psi^\lambda_t)\right] dt \right| \le \frac{\pi}{2}\sqrt{\frac{\lceil \lambda^\alpha \rceil}{T}} \|g''\|_\infty \sum_{n=1}^m nn! \int_{t^\lambda_{k-1}}^{t^\lambda_k} \|f_n(\cdot,t)\|^2_{L^2(\mathbb{R}^{n-1}_+)} dt,$$

where $\tilde{g}_{h,\lambda}$ is defined in (6.9).

Proof. For any $g \in \mathcal{G}$, since $\|g''\|_\infty \le 1$ and $U^\lambda \sim \mathcal{N}(0, T_\lambda I_d)$, we have

$$\begin{aligned}
\tilde{g}_{h,\lambda}(x) &= \frac{1}{2}\int_0^1 \mathbb{E}\left[g^{\alpha,\lambda}(x\sqrt{s} + U^\lambda\sqrt{1-s}) - g(U^\lambda)\right]\frac{ds}{s} \\
&= \frac{1}{2(2\pi T_\lambda)^{M_\lambda/2}} \int_0^1 \int_{\mathbb{R}^{M_\lambda}} g^{\alpha,\lambda}(z) \exp\left(-\frac{\|x\sqrt{s} - z\|^2}{2(1-s)T_\lambda}\right) \frac{dzds}{s\sqrt{1-s}} \\
&\quad - \frac{1}{2(2\pi T_\lambda)^{M_\lambda/2}} \int_0^1 \int_{\mathbb{R}^{M_\lambda}} g^{\alpha,\lambda}(y) \exp\left(-\frac{\|y\|^2}{2T_\lambda}\right) dy \frac{ds}{s}. \qquad (6.23)
\end{aligned}$$

By differentiation of (6.23) with respect to x_k, we obtain

$$\begin{aligned}
&\frac{\partial \tilde{g}_{h,\lambda}}{\partial x_k}(x) \\
&= \frac{-1}{2(2\pi T_\lambda)^{M_\lambda/2} \cdot T_\lambda} \int_0^1 \int_{\mathbb{R}^{M_\lambda}} (x_k\sqrt{s} - z_k) g^{\alpha,\lambda}(z) \exp\left(-\frac{\|x\sqrt{s} - z\|^2}{2(1-s)T_\lambda}\right) \frac{dzds}{s^{1/2}(1-s)^{3/2}} \\
&= \frac{1}{2(2\pi T_\lambda)^{M_\lambda/2} \cdot T_\lambda} \int_0^1 \int_{\mathbb{R}^{M_\lambda}} y_k g^{\alpha,\lambda}(x\sqrt{s} + y\sqrt{1-s}) \exp\left(-\frac{\|y\|^2}{2T_\lambda}\right) \frac{dyds}{\sqrt{s(1-s)}} \\
&= \frac{1}{2T_\lambda} \int_0^1 \mathbb{E}\left[g^{\alpha,\lambda}(x\sqrt{s} + U^\lambda\sqrt{1-s})U^\lambda_k\right] \frac{ds}{\sqrt{s(1-s)}},
\end{aligned}$$

and

$$\frac{\partial^3 \tilde{g}_{h,\lambda}}{\partial x_k^3}(x) = \frac{1}{2T_\lambda} \int_0^1 \mathbb{E}\left[\frac{\partial^2 g^{\alpha,\lambda}}{\partial x_k^2}(x\sqrt{s} + U^\lambda\sqrt{1-s})U^\lambda_k\right] \frac{ds}{\sqrt{s(1-s)}}, \qquad (6.24)$$

$x \in \mathbb{R}^{M_\lambda}$, $k = 1, \dots, M_\lambda$. Next, letting

$$h^{\alpha,\lambda}(x) := \sum_{n=1}^{m} h_n^{\alpha,\lambda}(x), \qquad x \in \mathbb{R}^{M_\lambda},$$

we have

$$\frac{\partial g^{\alpha,\lambda}}{\partial x_k}(x) = \frac{\partial g(h^{\alpha,\lambda})}{\partial x_k}(x) = g'(h^{\alpha,\lambda}(x))\frac{\partial h^{\alpha,\lambda}}{\partial x_k}(x),$$

and

$$\begin{aligned}\frac{\partial^2 g^{\alpha,\lambda}}{\partial x_k^2}(x) &= g''(h^{\alpha,\lambda}(x))\left(\frac{\partial h^{\alpha,\lambda}}{\partial x_k}(x)\right)^2 + g'(h^{\alpha,\lambda}(x))\frac{\partial^2 h^{\alpha,\lambda}}{\partial x_k^2}(x) \\ &= g''(h^{\alpha,\lambda}(x))\left(\frac{\partial h^{\alpha,\lambda}}{\partial x_k}(x)\right)^2, \end{aligned} \tag{6.25}$$

since from (6.20) we have $\frac{\partial^2 h^{\alpha,\lambda}}{\partial x_k^2}(x) = 0$ as $\frac{\partial h^{\alpha,\lambda}}{\partial x_k}$ does not depend on x_k. Substituting (6.25) into (6.24) and using (6.16), we deduce that for all $k = 1, \dots, M_\lambda$ we have

$$\begin{aligned}&\left|\int_{t_{k-1}^\lambda}^{t_k^\lambda} \mathbb{E}\left[\frac{\partial^3 \tilde{g}_{h,\lambda}}{\partial x_k^3}(\Psi_t^\lambda)\right] dt\right| \\ &\leq \frac{1}{2T_\lambda}\|g''\|_\infty \int_{t_{k-1}^\lambda}^{t_k^\lambda}\int_0^1 \mathbb{E}\left[\left(\frac{\partial h^{\alpha,\lambda}}{\partial x_k}(\hat{\Psi}^\lambda(t)\sqrt{s} + U^\lambda\sqrt{1-s})\right)^2 \left|\int_0^\infty \mathbb{1}_{A_k^\lambda}(s)dB_s\right|\right]\sqrt{\frac{s}{1-s}}dsdt \\ &\leq \frac{1}{2T_\lambda}\|g''\|_\infty \int_{t_{k-1}^\lambda}^{t_k^\lambda}\int_0^1 \mathbb{E}\left[\left|\sum_{n=1}^m nI_{n-1}^{\lambda,s}(f_n^{\alpha,\lambda}(\cdot;t))\right|^2\right]\sqrt{\mathbb{E}\left[\left|\int_0^\infty \mathbb{1}_{A_k^\lambda}(s)dB_s\right|^2\right]}\sqrt{\frac{s}{1-s}}dsdt \qquad (6.26) \\ &\leq \frac{\pi}{2\sqrt{T_\lambda}}\|g''\|_\infty \sum_{n=1}^m nn! \int_{t_{k-1}^\lambda}^{t_k^\lambda} \|f_n(\cdot,t)\|^2_{L^2(\mathbb{R}_+^{n-1})}dt,\end{aligned}$$

where we used (5.13) and the identity $\int_0^1 \sqrt{\frac{s}{1-s}}ds = \pi$, and the second inequality in (6.26) holds by recalling that $\frac{\partial h^{\alpha,\lambda}}{\partial x_k}(\hat{\Psi}^\lambda(t)\sqrt{s} + U^\lambda\sqrt{1-s})$ is independent of $\int_0^\infty \mathbb{1}_{A_k^\lambda}(s)dB_s$ since (6.22) contains no term in the form $I_1^{\lambda,s}(\mathbb{1}_{A_k^\lambda})$. □

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