CLARK-OCONE FORMULA BY THE S-TRANSFORM ON THE POISSON WHITE NOISE SPACE

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Abstract. Given φ a square-integrable Poisson white noise functionals we show that the Segal-Bargmann (S-) transform $t \longmapsto S\varphi(P_t(g))$ is absolutely continuous with

$$\frac{d}{dt} S\varphi(P_t(g)) = S\big(\partial_t^* \mathbb{E}[\,\partial_t \varphi \,|\, \mathcal{F}_t\,]\big)(g)$$

for almost all $t \in \mathbb{R}$, where $P_t(g) = \mathbf{1}_{(-\infty,\, t]} \cdot g$ for g in the Schwartz space $\mathcal{S}$ on $\mathbb{R}$, and ∂_t means the Poisson white noise derivative. After integration with respect to t and applying the inverse S-transform, this identity recovers the Clark-Ocone formula for φ.

1. Introduction

The Clark-Ocone formula allows one to express a given random variable φ defined on the classical Wiener space as the sum of its expectation value and an Itô integral with respect to Brownian motion. This formula was first derived by J. M. C. Clark (1970) in [2], and later, generalized to weakly differentiable functionals by D. Ocone (1984) in [13]. Such a formula is a useful tool in mathematical finance, for example, to compute the replicating portfolio of a claim with given maturity (see [1]).

Recently, non-Gaussian processes and more general Lévy processes have been extensively studied and new applications have been found in financial mathematics, quantum probability, stochastic analysis, etc. (see [1, 10, 12, 14, 16]). It is natural to ask that if the Clark-Ocone formula for Gaussian functionals can be generalized to non-Gaussian Lévy white noise functionals. In Hida's white noise theory, the Segal-Bargmann transform (S-transform, for abbrev.) plays a key role. There the generalized white noise fuctionals have been defined and studied through their S-transforms (see [6]). In [9], Lee and Shih have shown that the S-transform allows one to connect the Clark-Ocone formula of Brownian functionals with the fundamental theorem of calculus for Segal-Bargmann analytic functions. Applying the integral representation of the S-transform of Lévy white noise functionals in the non-Gaussian case, Lee and Shih [7], have developed a theory of the Lévy

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white noise analysis for case of Lévy processes following Hida's idea and by means of the S-transform, cf. [10] (see also [12]).

In this paper we are concerned with the derivation of the Clark-Ocone formula for Poisson white noise functionals. The main result is obtained in Section 4, in which we apply the results of [10] to the derivation of the Clark-Ocone formula for square-integrable Poisson white noise functionals.

The main argument can be briefly described as follows. Given g an element of the Schwartz space $\mathcal{S}$ on $\mathbb{R}$ and $P_t(g) = \mathbf{1}_{(-\infty,\, t]} \cdot g$, we prove that the function $t \longmapsto S\varphi(P_t(g))$ is absolutely continuous, where $S\varphi$ is the Segal-Bargmann (S-) transform of the square-integrable Poisson white noise functional φ. As a consequence we obtain the following identity

$$S\varphi(g) = \mathbb{E}[\varphi] + \int_{-\infty}^{+\infty} \frac{d}{dt} S\varphi(P_t(g))\, dt, \qquad g \in \mathcal{S}, \tag{1.1}$$

by the fundamental theorem of calculus. Comparing with the Clark-Ocone formula (Theorem 4.2), we show that for almost all $t \in \mathbb{R}$ we have

$$\frac{d}{dt} S\varphi(P_t(g)) = S\big(\partial_t^* \mathbb{E}[\, \partial_t \varphi \,|\, \mathcal{F}_t \,]\big)(g), \qquad g \in \mathcal{S},$$

where ∂_t denotes the Poisson white noise derivative. This also shows that we can recover the Clark-Ocone formula for φ simply by taking the inverse S-transform for the equation (1.1).

We note that the results of this paper can be extended to compound Poisson or more general Lévy white noise functionals. Since the arguments are much more involved, they will be discussed in a forthcoming paper.

2. Elements of Poisson white noise analysis

In this section we briefly review some notions and basic results of Poisson white noise analysis. The interested reader is referred to [10] for details and generalization to Lévy white noise analysis. Let $X = \{X(t);\, t \in \mathbb{R}\}$ be a standard Poisson process with $X(0) = 0$ almost surely, and

$$\mathbb{E}[e^{\mathrm{i}\, r(X(t)-X(s))}] = \exp((t-s)(e^{\mathrm{i}\, r} - 1)), \qquad r \in \mathbb{R}, \quad 0 \le s \le t, \tag{2.1}$$

where $\mathrm{i} = \sqrt{-1}$. Let $\mathcal{S}$ be the Schwartz space with the dual $\mathcal{S}'$ of tempered distributions on $\mathbb{R}$. It is well-known that there exists a unique probability measure Λ on $(\mathcal{S}', \mathcal{B}(\mathcal{S}'))$ with the characteristic functional $\mathcal{C}$ on $\mathcal{S}$ given by

$$\mathcal{C}(\eta) \equiv \mathbb{E}[e^{\mathrm{i}\,\langle \cdot,\, \eta \rangle}] = \exp\left(\int_{-\infty}^{+\infty} \left(e^{\mathrm{i}\, \eta(t)} - 1\right) dt \right), \qquad \eta \in \mathcal{S},$$

where $\langle \cdot, \cdot \rangle$ is the $\mathcal{S}'$-$\mathcal{S}$ pairing and dt refers to the Lebesgue measure (see [7]). The probability measure Λ is then called the Poisson white noise measure and $(\mathcal{S}', \mathcal{B}(\mathcal{S}'), \Lambda)$ will serve as the underlying probability space.

For any $\eta \in \mathcal{S}$, the mean and variance of the random variables $\langle \cdot, \eta \rangle$ are given by

$$\mathbb{E}[\langle \cdot, \eta \rangle] = \int_{-\infty}^{+\infty} \eta(t)\, dt, \quad \text{and} \quad \mathbb{V}[\langle \cdot, \eta \rangle] = \int_{-\infty}^{+\infty} \eta(t)^2\, dt. \tag{2.2}$$

For each $\rho \in L^1 \cap L^2(\mathbb{R}, dt)$, choose a sequence $\{\eta_n\} \subset \mathcal{S}$ so that $\eta_n \to \rho$ in $L^1 \cap L^2(\mathbb{R}, dt)$ under the norm $|\cdot|_{L^1(\mathbb{R},dt)} + |\cdot|_{L^2(\mathbb{R},dt)}$. Then it follows from (2.2) that $\{\langle \cdot, \eta_n \rangle\}$ forms a Cauchy sequence in $L^2(\mathcal{S}', \Lambda)$. Denote by $\langle \cdot, \rho \rangle$ the L^2-limit of $\{\langle \cdot, \eta_n \rangle\}$. When $\rho = \mathbf{1}_{(s,\, t]}$, the indicator of $(s,\, t]$, the characteristic function of $\langle \cdot, \rho \rangle$ is exactly the same as in (2.1). So the Poisson process X on $(\mathcal{S}', \mathcal{B}(\mathcal{S}'), \Lambda)$ can be represented by

$$X(t\,;\, x) = \begin{cases} \langle x,\, 1_{[0,\, t]} \rangle, & \text{if } t \geq 0 \\ -\langle x,\, 1_{[t,\, 0]} \rangle, & \text{if } t < 0,\ x \in \mathcal{S}'. \end{cases}$$

Taking the time derivative formally, we get $\dot{X}(t;\, x) = x(t)$, $x \in \mathcal{S}'$. From this viewpoint, the elements of $\mathcal{S}'$ can be regarded as the sample paths of Poisson white noise, and thus members of $L^2(\mathcal{S}', \Lambda)$ are also called square-integrable Poisson white noise functionals.

For $\eta = \eta_1 + i\,\eta_2 \in L^1_c \cap L^2_c(\mathbb{R}, dt)$ with $\eta_1, \eta_2 \in L^1 \cap L^2(\mathbb{R}, dt)$, $\langle \cdot, \eta \rangle$ is defined to be $\langle \cdot, \eta_1 \rangle + i\, \langle \cdot, \eta_2 \rangle$, where V_c denotes the complexification of a real locally convex space V. Then the above identity in (2.2) also hold for complex-valued random variable $\langle \cdot, \eta \rangle$.

Fock-type decomposition. Let $\mathcal{B}_b(\mathbb{R})$ be the class of all bounded Borel subsets E of $\mathbb{R}$. For any $E \in \mathcal{B}_b(\mathbb{R})$, let $N(E\,;\, \cdot)$ be the integer-valued function on $(\mathcal{S}', \mathcal{B}(\mathcal{S}'))$ defined by

$$N(E\,;\, x) = \#\, \big|\{t \in E : X(t\,;\, x) - X(t^-\,;\, x) = 1\}\big|, \qquad x \in \mathcal{S}'.$$

Then $N(E\,;\cdot)$ is a random variable which has Poisson distribution and the Lebesgue measure *Leb* on $\mathbb{R}$ is the corresponding intensity measure. The system of $\{N(E\,;\, x) - Leb(E) : E \in \mathcal{B}_b(\mathbb{R})\,;\, x \in \mathcal{S}'\}$ forms an independent random measure with zero. We denote $N(E\,;\, x) - Leb(E)$ by $\tilde{N}(E\,;\, x)$. Then, for $f \in L^2_c(\mathbb{R})$, the stochastic integral $\int_{-\infty}^{\infty} f(t)\, d\tilde{N}(t)$ has mean zero and the variance $\int_{-\infty}^{\infty} |f(t)|^2\, dt$. We note that as $f \in L^1_c \cap L^2_c(\mathbb{R})$,

$$\int_{-\infty}^{\infty} f(t)\, d\tilde{N}(t;\, x) = \int_{-\infty}^{\infty} f(t)\, dX(t;\, x) - \int_{-\infty}^{\infty} f(t)\, dt$$

$[\Lambda]$-almost everywhere $x \in \mathcal{S}'$, where the right-hand side integral with respect to X exists pathwise in the sense of the Lebesgue-Stieltjes integrals. In particular, for $b > a$,

$$X(b) - X(a) = (b - a) + \int_{-\infty}^{\infty} \mathbf{1}_{(a,b]}(t)\, d\tilde{N}(t).$$

Notice that, for any $E, F \in \mathcal{B}(\mathbb{R})$, $\mathbb{E}[\tilde{N}(E)\tilde{N}(F)] = Leb(E \cap F)$. Denote by $I_n(g)$, $g \in \hat{L}^2_c(\mathbb{R}^n)$, the closed subspace of $L^2_c(\mathbb{R}^n)$ consisting of all symmetric complex-valued functions in $L_c^2(\mathbb{R})$, the multiple stochastic integral of order n with respect to $\tilde{N}$

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} g(t_1, \ldots, t_n)\, d\tilde{N}(t_1) \cdots d\tilde{N}(t_n),$$

which satisfies the isometry

$$\left\| I_n(g) \right\|^2_{L^2(\mathcal{S}', \Lambda)} = n! \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left| g(t_1, \ldots, t_n) \right|^2 dt_1 \cdots dt_n.$$

The multiple stochastic integrals with respect to $\tilde{N}$ provide an orthogonal decomposition theorem for the square-integrable Poisson white noise functionals.

Theorem 2.1. (Itô [4]) *Let φ be given in $L^2(\mathcal{S}', \Lambda)$. Then*

(*i*) *there exist uniquely a series of kernel functions $\phi_n \in \hat{L_c^2}(\mathbb{R}^n)$, $n \in \mathbb{N} \cup \{0\}$, such that φ is equal to the orthogonal direct sum $\sum_{n=0}^{\infty} I_n(\phi_n)$. In notation, we write $\varphi \sim (\phi_n)$.*

(*ii*) *$L^2(\mathcal{S}', \Lambda)$ is isomorphic to the symmetric Fock space $\mathcal{F}_s(L_c^2(\mathbb{R}))$ over $L_c^2(\mathbb{R})$ by carrying $\varphi \sim (\, \phi_n\,)$ into $(\, \sqrt{0!}\, \phi_0, \sqrt{1!}\, \phi_1, \ldots, \sqrt{n!}\, \phi_n, \ldots)$.*

The Segal-Bargmann transform. For $\varphi \in L^2(\mathcal{S}', \Lambda)$, say $\varphi \sim (\phi_n)$, the Segal-Bargmann (or the S-) transform of φ is a complex-valued functional on $L_c^2(\mathbb{R})$ by

$$\begin{aligned} S\varphi(g) &\equiv \int_{\mathcal{S}'} \varphi(x)\, \mathcal{E}_X(g)(x)\, \Lambda(dx) \\ &= \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \phi_n(t_1, \ldots, t_n)\, g(t_1) \cdots g(t_n)\, dt_1 \cdots dt_n, \end{aligned}$$

where

$$\mathcal{E}_X(g) = \sum_{n=0}^{\infty} \frac{1}{n!}\, I_n(g^{\otimes n})$$

is the coherent state functional associated with g. The S-transform is a unitary operator mapping from $L^2(\mathcal{S}', \Lambda)$ onto the Bargmann-Segal-Dwyer space $\mathcal{F}^1(L_c^2(\mathbb{R}))$ over $L_c^2(\mathbb{R},)$. Moreover, we have

$$\|\varphi\|^2_{L^2(\mathcal{S}', \Lambda)} = \sum_{n=0}^{\infty} \frac{1}{n!} \, \|D^n S\varphi(0)\|^2_{\mathcal{L}_{(2)}^{(n)}(L_c^2(\mathbb{R}))},$$

where D is the Fréchet derivative of $S\varphi$ and $\|\cdot\|_{\mathcal{L}_{(2)}^{(n)}(H)}$ denotes the Hilbert-Schmidt operator norm of a n-linear functional on a Hilbert space H.

Theorem 2.2. (Lee-Shih [7]) *Let $\varphi \in L^2(\mathcal{S}', \Lambda)$. Then, for any $g \in L_c^1(\mathbb{R}) \cap L_c^{\infty}(\mathbb{R})$ we have*

$$\mathcal{E}_X(g) = \exp\left(- \int_{-\infty}^{\infty} g(t)\, dt \right) \prod_{t \in J_X(x)} (1 + g(t)),$$

where $J_X(x)$ is the set $\{t \in \mathbb{R} : X(t; x) - X(t^-; x) = 1\}$.

Remark 2.3. Let H be a complex Hilbert space. For $r > 0$, denote by $\mathcal{F}^r(H)$ the class of analytic functions on H with norm $\|\cdot\|_{\mathcal{F}^r(H)}$ such that

$$\|f\|^2_{\mathcal{F}^r(H)} = \sum_{n=0}^{\infty} \frac{r^n}{n!} \|D^n f(0)\|^2_{\mathcal{L}^{(n)}_{(2)}[H]} < +\infty,$$

called the Bargmann-Segal-Dwyer space.

Test and generalized Poisson white noise functionals. Let $A = -d^2/dt^2 + (1+t^2)$ denote the densely defined and self-adjoint operator on $L^2(\mathbb{R}, dt)$ and $\{e_n : n \in \mathbb{N}_0\}$ be eigenfunctions of A with corresponding eigenvalues $2n+2$, $n \in \mathbb{N} \cup \{0\}$. $\{e_n : n \in \mathbb{N}_0\}$ forms a complete orthonormal basis (CONS, in abbreviation) of $L^2(\mathbb{R}, dt)$. For any $p \in \mathbb{R}$ and $\eta \in L^2(\mathbb{R}, dt)$, define $|\eta|_p := |A^p \eta|_{L^2(\mathbb{R},dt)}$ and let $\mathcal{S}_p$ be the completion of the class $\{\eta \in L^2(\mathbb{R}, dt) : |\eta|_p < +\infty\}$ with respect to $|\cdot|_p$-norm. Then $\mathcal{S}_p$ is a real separable Hilbert space and we have the continuous inclusions:

$$\mathcal{S} = \varprojlim_{p>0} \mathcal{S}_p \subset \mathcal{S}_p \subset \mathcal{S}_q \subset L^2(\mathbb{R}, dt) \subset \mathcal{S}_{-q} \subset \mathcal{S}_{-p} \subset \mathcal{S}' = \varinjlim_{p>0} \mathcal{S}_{-p}, \quad p > q \geq 0.$$

We next proceed to construct the spaces of test and generalized functions as follows. For $p \in \mathbb{R}$ and $\varphi \in L^2(\mathcal{S}', \Lambda)$, define

$$\|\varphi\|^2_p = \sum_{n=0}^{\infty} \frac{1}{n!} \|D^n S\varphi(0)\|^2_{\mathcal{L}^{(n)}_{(2)}(\mathcal{S}_{-p})}$$

and let $\mathcal{L}_p$ be the completion of the class $\{\, \varphi \in L^2(\mathcal{S}', \Lambda) : \|\varphi\|_p < +\infty \,\}$ with respect to $\|\cdot\|_p$-norm. Then $\mathcal{L}_p$, $p \in \mathbb{R}$, is a Hilbert space with the inner product $\langle\langle \cdot, \cdot \rangle\rangle_p$ induced by $\|\cdot\|_p$-norm. For $p, q \in \mathbb{R}$ with $q \geq p$, $\mathcal{L}_q \subset \mathcal{L}_p$ and the embedding $\mathcal{L}_q \hookrightarrow \mathcal{L}_p$ is of Hilbert-Schmidt type, whenever $q - p > 1/2$. Let $\mathcal{L} = \varprojlim_{p>0} \mathcal{L}_p$. Then $\mathcal{L}$ is a nuclear space which will serve as the space of test functions and the dual space $\mathcal{L}'$ of $\mathcal{L}$ the space of generalized functions. The members of $\mathcal{L}'$ are called generalized Poisson white noise functionals. In this way, we obtain a Gel'fand triple $\mathcal{L} \subset L^2(\mathcal{S}', \Lambda) \subset \mathcal{L}'$ and have the continuous inclusion:

$$\mathcal{L} \subset \mathcal{L}_p \subset \mathcal{L}_q \subset L^2(\mathcal{S}', \Lambda) \subset \mathcal{L}'_q \subset \mathcal{L}'_p \subset \mathcal{L}' = \varinjlim_{p>0} \mathcal{L}'_p, \quad p \geq q > 0.$$

In what follows, the dual pairing of $\mathcal{L}'$ and $\mathcal{L}$ will be denoted by $\langle\langle \cdot, \cdot \rangle\rangle$.

For $g \in L^2_c(\mathbb{R})$, it is easy to see that $\|\mathcal{E}_X(g)\|_p = e^{(1/2)\,|g|^2_p}$ for any $p > 0$. Hence $\mathcal{E}_X(g) \in \mathcal{L}_p$ if and only if $g \in \mathcal{S}_{p,c}$ for $p > 0$. We define the S-transform for $F \in \mathcal{L}'$ by

$$SF(g) = \langle\langle F, \mathcal{E}_X(g) \rangle\rangle, \qquad g \in \mathcal{S}_c.$$

Annihilation and creation operators. Let $F \in \mathcal{L}_p$ and $\xi \in \mathcal{S}_{-p,c}$, $p \in \mathbb{R}$. The Gâteaux derivative $(d/dz)|_{z=0}\, SF(\cdot + z\,\xi)$ in the direction ξ is an analytic function on $\mathcal{S}_{-p,c}$. In fact, by using the Cauchy integral formula, one can show that $(d/dz)|_{z=0}\, SF(\cdot + z\,\xi\,) \in \mathcal{L}_{p-1}$. Define

$$\partial_\xi\, F = S^{-1}\left(\frac{d}{dz}\, SF(\cdot + z\,\xi\,)_{|\,z=0} \right).$$

Then, by the characterization theorem [11], we have $\partial_\xi F \in \mathcal{L}_{p-\frac{5}{2}}$. It is clear that ∂_ξ is continuous from $\mathcal{L}$ into itself. Its adjoint operator ∂_ξ^* is then defined from by

$$\langle\!\langle \partial_\xi^* F, \varphi \rangle\!\rangle := \langle\!\langle F, \partial_\xi \varphi \rangle\!\rangle \qquad \text{for } F \in \mathcal{L}' \text{ and } \varphi \in \mathcal{L}.$$

Here, ∂_ξ is called the annihilation operator and ∂_ξ^* is called the creation operator. For convenience, we denote the Poisson white noise derivative ∂_{δ_t} and its adjoint $\partial_{\delta_t}^*$ respectively by ∂_t and ∂_t^* for $t \in \mathbb{R}$, where δ_t is the Dirac measure concentrated on the point t.

Proposition 2.4. ([10]) *Let $\varphi \sim (\phi_n) \in \mathcal{L}_p$ and $\xi \in \mathcal{S}_{-p,c}$ for $p \geq 0$. If $p - q \geq \ln 3/6 \ln 2$, then*

$$\partial_\xi \varphi = \langle \xi, \phi_1 \rangle + \sum_{n=2}^{\infty} n \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \langle \xi, \phi_n(\cdot, t_2, \ldots, t_n) \rangle d\tilde{N}(t_2) \cdots d\tilde{N}(t_n),$$

where $\langle \cdot, \cdot \rangle$ is the $\mathcal{S}_{-p,c}$-$\mathcal{S}_{p,c}$ pairing. Moreover,

$$\|\partial_\xi \varphi\|_q \leq 2\, |\xi|_{-p}\, \|\varphi\|_p.$$

From [10] we have the following proposition.

Proposition 2.5. ([10]) *Let $F \sim (f_n) \in \mathcal{L}_{-p}$ and $\xi \in \mathcal{S}'_c$. Then*

$$\partial_\xi^* F = \sum_{n=0}^{\infty} I_{n+1}(\xi \hat{\otimes} f_n) \qquad \text{in } \mathcal{L}'.$$

Moreover, for $q - p \geq \ln 3/6 \ln 2$,

$$\|\partial_\xi^* F\|_{-q} \leq 2\, |\xi|_{-q}\, \|F\|_{-p}.$$

3. Fundamental theorem of calculus on Bargmann-Segal-Dwyer spaces

First of all, we specify some notations as follows:

- For $t \in \mathbb{R}$ and $h \in L_c^2(\mathbb{R})$ we let $P_t(h) = h \cdot \mathbf{1}_{(-\infty,t]}$.
- For $g \in L^1 \cap L^\infty(\mathbb{R})$ and $x \in \mathcal{S}'$ we set

$$\gamma_1(g) = \exp\left(-\int_{-\infty}^{\infty} g(t)\, dt\right) \qquad \text{and} \qquad \gamma_2(g)(x) = \prod_{t \in J_X(x)} (1 + g(t)),$$

and $\gamma(g) = \gamma_1(g)\gamma_2(g)$.

Theorem 3.1. *For any $\varphi \in L^2(\mathcal{S}', \Lambda)$ and $g \in L^1 \cap L^\infty(\mathbb{R})$, the mapping*

$$t \in \mathbb{R} \longmapsto S\varphi(P_t(g))$$

is absolutely continuous. Moreover, for any $-\infty < a < b < +\infty$,

$$S\varphi(P_b(g)) - S\varphi(P_a(g)) = \int_a^b \frac{d}{dt} S\varphi(P_t(g))\, dt. \tag{3.1}$$

Proof. Let $g \in L^1 \cap L^\infty(\mathbb{R})$ and $\varphi \in L^2(\mathcal{S}', \Lambda)$ be fixed. We start by proving the absolute continuity of the mapping $t \in \mathbb{R} \longmapsto S\varphi(P_t(g))$. Let $\{[a_i, b_i] : i = 1, 2, 3, \ldots\}$ be any (finite or countably infinite) collection of nonoverlapping intervals. Then

$$\begin{aligned}\sum_i &|S\varphi(P_{b_i}(g)) - S\varphi(P_{a_i}(g))| \\ &= \sum_i \left| \int_{\mathcal{S}'} \varphi(x)(\gamma(P_{b_i}(g))(x) - \gamma(P_{a_i}(g))(x))\, \Lambda(dx) \right|. \qquad (3.2)\end{aligned}$$

We denote this sum by $\sum_N$. To estimate the sum $\sum_N$, we first note that

$$\sum\nolimits_N \leq \sum_i |\gamma_1(P_{b_i}(g)) - \gamma_1(P_{a_i}(g))| \int_{\mathcal{S}'} |\varphi(x)||\gamma_2(P_{b_i}(g))(x)|\, \Lambda(dx) \qquad (3.3)$$

$$+ \sum_i |\gamma_1(P_{a_i}(g))| \int_{\mathcal{S}'} |\varphi(x)||\gamma_2(P_{b_i}(g))(x) - \gamma_2(P_{a_i}(g))(x)|\, \Lambda(dx). \qquad (3.4)$$

Let $\sum_N(1)$ and $\sum_N(2)$ stand separately for the sums in the right hand side of (3.3) and (3.4). Since, for any $a < b$,

$$|\gamma_1(P_b(g)) - \gamma_1(P_a(g))| \leq \int_a^b |g(t)|\, dt \cdot \exp\left(2 \int_{-\infty}^{\infty} |g(t)|\, dt\right)$$

and

$$|\gamma_2(P_b(g))(x)| \leq \exp\left(\int_{-\infty}^{\infty} |g(t)|\, dt\right) \cdot |\mathcal{E}_X(P_b(g))(x)|, \qquad x \in \mathcal{S}',$$

we see, by the Cauchy-Schwarz inequality, that

$$\sum\nolimits_N(1) \leq C(\varphi, g) \int_{\cup_i [a_i,\, b_i]} |g(t)|\, dt, \qquad (3.5)$$

where $C(\varphi, g)$ is a constant, depending only on φ and g, given by

$$C(\varphi, g) = \|\varphi\|_{L^2(\mathcal{S}', \Lambda)} \cdot \exp\left(3 \int_{\mathbb{R}} |g(t)|\, dt + \frac{1}{2} \int_{\mathbb{R}} |g(t)|^2\, dt\right).$$

For the estimation of $\sum_N(2)$, we need the following inequality:

Lemma 3.2. *Let E, F be two countable subsets of complex numbers such that $E \subset F$ and $\sum_{z \in F} |z| < +\infty$. Then*

$$\left| \prod_{z \in F} (1 + z) - \prod_{z \in E} (1 + z) \right| \leq \prod_{z \in F} (1 + |z|) - \prod_{z \in E} (1 + |z|).$$

By Lemma 3.2, we see that for any $a < b$,

$$\begin{aligned}&|\gamma_2(P_b(g))(x) - \gamma_2(P_a(g))(x)| \\ &\quad \leq \exp\left(\int_{-\infty}^{b} \ln(1 + |g(t)|)\, dX(t;\, x)\right) - \exp\left(\int_{-\infty}^{a} \ln(1 + |g(t)|)\, dX(t;\, x)\right) \\ &\quad \leq \int_a^b \ln(1 + |g(t)|)\, dX(t;\, x) \cdot \exp\left(2 \int_{-\infty}^{\infty} \ln(1 + |g(t)|)\, dX(t;\, x)\right). \qquad (3.6)\end{aligned}$$

It follows from (3.6) that

$$\begin{aligned}
\sum\nolimits_N(2) \le\ & e^{\int_{-\infty}^{\infty}|g(t)|\,dt} \int_{\mathcal{S}'} |\varphi(x)| \left(\int_{\cup_i [a_i,b_i]} \ln(1+|g(t)|)\, dX(t;\, x) \right. \\
& \left. \times \exp\left(2\int_{-\infty}^{\infty} \ln(1+|g(t)|)\, dX(t;\, x) \right) \right) \Lambda(dx) \\
\le\ & e^{\int_{-\infty}^{\infty}|g(t)|\,dt} \cdot \|\varphi\|_{L^2(\mathcal{S}',\Lambda)} \\
& \times \left\{ \int_{\mathcal{S}'} \left[\left(\int_{\cup_i [a_i,b_i]} \ln(1+|g(t)|)\, dX(t;\, x) \right)^2 \right. \right. \\
& \left. \left. \times \exp\left(4\int_{-\infty}^{\infty} \ln(1+|g(t)|)\, dX(t;\, x) \right) \right] \Lambda(dx) \right\}^{1/2} \\
\le\ & D(\varphi, g) \cdot \left\{ \int_{\mathcal{S}'} \left(\int_{\cup_i [a_i,b_i]} \ln(1+|g(t)|)\, dX(t;\, x) \right. \right. \\
& \left. \left. \times \exp\left(5\int_{-\infty}^{\infty} \ln(1+|g(t)|)\, dX(t;\, x) \right) \right) \Lambda(dx) \right\}^{1/2}, \qquad (3.7)
\end{aligned}$$

where

$$D(\varphi, g) = e^{\int_{-\infty}^{\infty}|g(t)|\,dt} \cdot \|\varphi\|_{L^2(\mathcal{S}',\Lambda)}$$

is a constant depending only only on φ and g.

Note that for any two non-negative Borel measurable functions $p(t), q(t)$, $t \in \mathbb{R}$, we have

$$\begin{aligned}
& \int_{\mathcal{S}'} \left(\int_{-\infty}^{\infty} p(t)\, dX(t;\, x) \right) \cdot \exp\left(\int_{-\infty}^{\infty} q(t)\, dX(t;\, x) \right) \Lambda(dx) \\
&= \left(\int_{-\infty}^{\infty} p(t)\, e^{q(t)}\, dt \right) \cdot \exp\left(\int_{-\infty}^{\infty} (e^{q(t)} - 1)\, dt \right).
\end{aligned}$$

By applying this formula to (3.7), we see that

$$\begin{aligned}
\sum\nolimits_N(2) \le\ & D(\varphi, g) \times \left(\int_{\cup_i [a_i,b_i]} (1+|g(t)|)^5 \ \ln(1+|g(t)|)\, dt \right)^{1/2} \\
& \times \exp\left(\frac{1}{2} \int_{-\infty}^{\infty} \left((1+|g(t)|)^5 - 1 \right)\ dt \right),
\end{aligned}$$

which tends to zero whenever $\sum(b_i - a_i)$ tends to zero. Combining this bound with (3.5) yields the absolute continuity property and (3.1). □

4. Clark-Ocone formula for Poisson white noise functionals

In [2], Clark proved that a sufficiently well-behaved Fréchet-differentiable Brownian functional can be represented as a stochastic integral in which the integrand has the form of conditional expectations of the differential. We note that this result can be extended to all square integrable Poisson white noise functionals.

Lemma 4.1. [8] *Let $F \sim (f_n) \in L^2(\mathcal{S}', \Lambda)$. The conditional expectation $\mathbb{E}[F \,|\, \mathcal{F}_t]$ of F relative to $\mathcal{F}_t$, $t \in \mathbb{R}$, is given by*

$$\mathbb{E}[F \,|\, \mathcal{F}_t] = \sum_{n=0}^{\infty} I_n(\mathbf{1}_{(-\infty,t]^n} f_n),$$

where E^n means the product $E \times \cdots \times E$ for any $E \in \mathcal{B}(\mathbb{R})$.

Let $f_n \in \hat{L}^2_c(\mathbb{R}^n)$ and $\psi \sim (\psi_n) \in \mathcal{L}$. We observe that for almost all $t \in \mathbb{R}$, $\partial_t I_n(f_n) \in L^2(\mathcal{S}', \Lambda)$, and thus, for $p > 1$, by Lemma 4.1,

$$\begin{aligned} |\langle\!\langle \mathbb{E}[\partial_t I_n(f_n) \,|\, \mathcal{F}_t], \psi \rangle\!\rangle|^2 &\le \| \mathbb{E}[\partial_t I_n(f_n) \,|\, \mathcal{F}_t] \|_{-p}^2 \|\psi\|_p^2 \\ &= (n-1)!\, n^2\, |\mathbf{1}_{(-\infty,t]^{n-1}} \tilde{f}_n(t)|_{-p}^2\, \|\psi\|_p^2 \\ &\quad \text{(by Proposition 2.4 and Lemma 4.1)} \\ &\le 2^{-2(p-1)(n-1)}\, n!\, |\tilde{f}_n(t)|_0^2\, \|\psi\|_p^2, \end{aligned} \tag{4.1}$$

where $\tilde{f}_n(t)$ is the function on $\mathbb{R}^{n-1}$ to $\mathbb{C}$ defined by

$$\tilde{f}_n(t)(t_1, \ldots, t_{n-1}) = f_n(t_1, \ldots, t_{n-1}, t).$$

So, for a fixed $F \sim (f_n) \in L^2(\mathcal{S}', \Lambda)$, it follows from (4.1) that

$$\int_{-\infty}^{\infty} \limsup_{m,m' \to \infty} \left\| \sum_{n=m}^{m'} \mathbb{E}[\partial_t I_n(f_n) \,|\, \mathcal{F}_t] \right\|_{-p}^2 dt \le \lim_{m\to\infty} \sum_{n=m}^{\infty} n! \cdot |f_n|_0^2 = 0.$$

In other words, for almost all $t \in \mathbb{R}$, $\left\{ \sum_{n=0}^{m} \mathbb{E}[\partial_t I_n(f_n) | \mathcal{F}_t] \right\}_m$ is a Cauchy sequence in $\mathcal{L}_{-p}$, $p > 1$. Consequently we can define

$$\mathbb{E}[\partial_t F | \mathcal{F}_t] := \text{ the } \mathcal{L}_{-p}\text{-limit of } \left\{ \sum_{n=0}^{m} \mathbb{E}[\partial_t I_n(f_n) \,|\, \mathcal{F}_t] \right\} \text{ as } m \to \infty.$$

Theorem 4.2. (Clark-Ocone formula for Poisson white noise functionals) *Let $F \sim (f_n)$ be in $L^2(\mathcal{S}', \Lambda)$. Then*

$$F = \mathbb{E}[F] + \int_{-\infty}^{\infty} \partial_t^* \mathbb{E}[\partial_t F \,|\, \mathcal{F}_t]\, dt \text{ in } \mathcal{L}_{-p},\ p > 5/4,$$

where the above integral is regarded as a Bochner integral.

Proof. Letting

$$\Delta_n(t) = \{(t_1, \ldots, t_{n-1}) \in \mathbb{R}^{n-1};\, t_j < t,\ \forall\, j = 1, 2, \ldots, n-1\},$$

we have

$$\begin{aligned} F &= \mathbb{E}[F] + \sum_{n=1}^{\infty} I_n(\, f_n\,) \\ &= \mathbb{E}[F] + \sum_{n=1}^{\infty} n \int_{-\infty}^{\infty} \Big(\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \mathbf{1}_{\Delta_n(t)}(t_1, \ldots, t_{n-1}) \\ &\qquad \times f_n(t_1, \ldots, t_{n-1}, t)\, d\tilde{N}(t_1) \cdots d\tilde{N}(t_{n-1}) \Big)\, d\tilde{N}(t), \\ &= \mathbb{E}[F] + \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} \mathbb{E}[\, \partial_t I_n(\, f_n\,) \,|\, \mathcal{F}_t\,]\, d\tilde{N}(t), \end{aligned}$$

cf. also Proposition 12 of [15] and Proposition 4.2.3 of [16]. Then, by applying [10, Theorem 7.2],

$$F = \mathbb{E}[F] + \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} \partial_t^* \mathbb{E}[\, \partial_t I_n(\, f_n\,) \,|\, \mathcal{F}_t\,]\, dt \text{ in } \mathcal{L}'. \tag{4.2}$$

Moreover, since for $p > 5/4$,

$$\begin{aligned} &\sum_{n=1}^{\infty} \int_{-\infty}^{\infty} \|\, \partial_t^* \mathbb{E}[\, \partial_t I_n(\, f_n\,) \,|\, \mathcal{F}_t\,] \,\|_{-p}\, dt \\ &\qquad \leq 2 \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} |\delta_t|_{-p+1/4} \,\|\, \mathbb{E}[\, \partial_t I_n(\, f_n\,) \,|\, \mathcal{F}_t\,] \|_{-p+1/4}\, dt \\ &\text{(by Proposition 2.5)} \\ &\qquad \leq 2 \sum_{n=1}^{\infty} 2^{-(p-5/4)(n-1)} (n!)^{1/2} \int_{-\infty}^{\infty} |\delta_t|_{-p+1/4}\, |\tilde{f}_n(t)|_0\, dt \\ &\text{(by 4.1)} \\ &\qquad \leq 2 \left(\int_{-\infty}^{\infty} |\delta_t|_{-1}^2\, dt \right)^{1/2} \left(\sum_{n=1}^{\infty} 2^{-2(p-5/4)(n-1)} \right)^{1/2} \|\, F\,\|_0 < +\infty, \end{aligned}$$

we see that the identity (4.2) becomes that

$$\begin{aligned} F &= \mathbb{E}[F] + \int_{-\infty}^{\infty} \sum_{n=1}^{\infty} \partial_t^* \mathbb{E}[\, \partial_t I_n(\, f_n\,) \,|\, \mathcal{F}_t\,]\, dt \\ &= \mathbb{E}[F] + \int_{-\infty}^{\infty} \partial_t^* \mathbb{E}[\, \partial_t F \,|\, \mathcal{F}_t\,]\, dt \text{ in } \mathcal{L}_{-p}, \end{aligned}$$

where the above integrals are Bochner integrals. □

Combining Theorem 4.2 with Theorem 3.1 yields that, for any $\varphi \in L^2(\mathcal{S}', \Lambda)$,

$$\begin{aligned}
&\int_a^b \frac{d}{dt} S\varphi(P_t(g))\, dt \\
&\quad = S\left(\int_{-\infty}^{\infty} \partial_t^* \mathbb{E}[\,\partial_t F \,|\, \mathcal{F}_t\,]\, dt\right)(P_b(g)) - S\left(\int_{-\infty}^{\infty} \partial_t^* \mathbb{E}[\,\partial_t F \,|\, \mathcal{F}_t\,]\, dt\right)(P_a(g)) \\
&\quad = \int_{-\infty}^{\infty} S\big(\partial_t^* \mathbb{E}[\,\partial_t F \,|\, \mathcal{F}_t\,]\big)(P_b(g))\, dt - \int_{-\infty}^{\infty} S\big(\partial_t^* \mathbb{E}[\,\partial_t F \,|\, \mathcal{F}_t\,]\big)(P_a(g))\, dt \\
&\quad = \int_{-\infty}^{b} S\big(\partial_t^* \mathbb{E}[\,\partial_t F \,|\, \mathcal{F}_t\,]\big)(g)\, dt - \int_{-\infty}^{a} S\big(\partial_t^* \mathbb{E}[\,\partial_t F \,|\, \mathcal{F}_t\,]\big)(g)\, dt \\
&\quad = \int_a^b S\big(\partial_t^* \mathbb{E}[\,\partial_t F \,|\, \mathcal{F}_t\,]\big)(g)\, dt, \qquad \forall\ a < b,\ g \in L^1 \cap L^\infty(\mathbb{R}).
\end{aligned}$$

Therefore, by applying Lebesgue's differentiation theorem, we arrive at the following result, which can be read as the S-transform version of the Clark-Ocone formula.

Corollary 4.3. *Let $\varphi \in L^2(\mathcal{S}', \Lambda)$. Then, for any $g \in L^1 \cap L^\infty(\mathbb{R})$,*

$$\frac{d}{dt} S\varphi(P_t(g)) = S\big(\partial_t^* \mathbb{E}[\,\partial_t \varphi \,|\, \mathcal{F}_t\,]\big)(g), \qquad [Leb] - a.e.\ t \in \mathbb{R}.$$

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