Skorohod and pathwise stochastic calculus with respect to an L^2 process

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Abstract

The purpose of this paper is to construct a stochastic calculus with respect to a class $\mathcal{V}$ of anticipating processes which is wider than the class of Skorohod integral processes. The main tool of this approach is the definition of a Skorohod type integral operator that acts with respect to $(X(t))_{t\in[0,1]} \in \mathcal{V}$. Under regularity assumptions on $(X(t))_{t\in[0,1]}$ we obtain an anticipating Itô formula, with sufficient conditions for the existence of quadratic variations and pathwise integrals with respect to $(X(t))_{t\in[0,1]}$.

Key words: Anticipating stochastic calculus, Skorohod integral, Stratonovich integral, forward integral.
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1 Introduction

The Itô formula, originally stated in the adapted case for Itô processes, has been extended to anticipating processes in mainly two directions. The pathwise approach using the forward, backward, and Stratonovich integrals is valid for a large class of processes, cf. [3], [12], [13], but such integrals do not retain an important property of the Itô integral which is useful in the applications of stochastic calculus, namely their expectation differs from zero in general. The Skorohod (or Hitsuda-Skorohod) integral operator δ, cf. [5], [14], conserves the latter property, moreover it can be used to give sufficient conditions for the existence of pathwise integrals, cf. [10]. Given a smooth non-adapted process u, the Skorohod integral process associated to u is defined as

$$X(t) = \delta(u(\cdot)1_{[0,t]}(\cdot)), \qquad t \in [0,1], \tag{1}$$

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cf. [6], [9], [10]. In the particular case where u is adapted, Skorohod integral processes include Itô processes of the form

$$X(t) = \int_0^t u(s)dB(s), \qquad t \in [0,1],$$

where $(B(t))_{t\in[0,1]}$ denotes Brownian motion. However, in the anticipating case Skorohod integral processes form a relatively restricted class of anticipating processes, cf. [2], due to the separation of the variables s and t in $h(s)1_{[0,t]}(s)$. The Itô calculus for such processes has been developed in [10], [15], and an extension of this calculus to processes given as $X(t) = \delta(u(\cdot)g(\cdot,t))$, where g is a deterministic function, has been been considered in [7]. In this paper we work with a general class of processes of the form

$$X(t) = \delta(u(\cdot,t)), \qquad t \in [0,1], \tag{2}$$

where $u(\cdot,t)$ belongs (for fixed t) to the domain of δ. This formulation is more general than (1) and it is not restrictive since every centered process $(X(t))_{t\in[0,1]} \in L^2([0,1]\times W)$ with multiple stochastic integral expansion

$$X(t) = \sum_{n=1}^{\infty} I_n(f_n(\cdot,t)), \quad t \in [0,1],$$

possesses such a representation, the process u being given by

$$u(s,t) = \sum_{n=0}^{\infty} I_n(f_{n+1}(\cdot,s,t)), \quad t \in [0,1].$$

This paper is organized as follows. Sect. 2 contains notation and preliminaries on the Skorohod integral operator δ. In Sect. 3 we state the definition and the basic properties of our generalized Skorohod integral operator δ^X that acts with respect to the process $(X(t))_{t\in[0,1]}$. We also introduce a gradient operator D^X which is the adjoint of δ^X. The extension of δ^X as a continuous operator on Sobolev spaces is studied in Sect. 4. In Sect. 5 we link δ^X to the forward, backward and Stratonovich anticipating integrals with respect to $(X(t))_{t\in[0,1]}$. In particular we obtain sufficient conditions for the existence of pathwise integrals with respect to $(X(t))_{t\in[0,1]}$ using the operators D_s^{X+} and D_s^{X-} defined from D^X by left and right limits. The pathwise quadratic covariation $[X,Y](t)$ of L^2 processes is considered in Sect. 6 where we obtain sufficient conditions for its existence, and the relation

$$d[X,X](t) = (D_t^{X+}X(t) - D_t^{X-}X(t))dt, \quad t \in [0,1].$$

In Sect. 7 we prove the following anticipating Itô formula for $(X(t))_{t\in[0,1]}$, using the extended Skorohod integral δ^X and its associated gradient operator (Th. 2):

$$\begin{aligned} f(X(t)) &= f(X(0)) + \delta^X(f'(X(\cdot))1_{[0,t]}(\cdot)) + \frac{1}{2}\int_0^t f''(X(s))d[X,X](s) \\ &\quad + \int_0^t f''(X(s))D_s^{X-}X(s)ds. \end{aligned}$$

This result is also stated in the multidimensional case, and can be formulated with pathwise anticipating integrals.

2 Notation

Let (W, H, μ) denote the classical Wiener space with Brownian motion $(B(t))_{t\in[0,1]}$, where $W = \mathcal{C}_0([0,1])$ is the space of continuous functions starting at 0 and

$$H = \left\{ \int_0^\cdot \dot{h}(s)ds \ : \ \dot{h} \in L^2([0,1]) \right\}$$

is the Cameron-Martin space, with norm $\|h\|_H = \|\dot{h}\|_{L^2([0,1])}$, where $\dot{h}$ denotes the derivative of the absolutely continuous function $h \in H$. Let $\mathcal{S}$ denote the set of smooth random variables defined as

$$\mathcal{S} = \left\{ f(B(t_1), \ldots, B(t_n)) \ : \ 0 \le t_i \le 1, \ i = 0, \ldots, n, \ f \in \mathcal{C}_b^1(\mathbb{R}^n) \right\},$$

which is dense in $L^2(W)$, and let

$$\mathcal{U} = \left\{ \sum_{i=0}^{i=n-1} F_i 1_{]t_i, t_{i+1}]} \ : \ F_1, \ldots, F_n \in \mathcal{S}, \ 0 \le t_1, \ldots, t_n \le 1 \right\},$$

which is also dense in $L^2([0,1] \times W)$. We denote by $D : L^2(W) \longrightarrow L^2([0,1] \times W)$ the closable gradient operator defined on $\mathcal{S}$ as

$$D_s f(B(t_1), \ldots, B(t_d)) = \sum_{k=1}^{k=d} 1_{[0,t_k]}(s) \partial_k f(B(t_1), \ldots, B(t_d)), \quad s \in [0,1].$$

Let $\mathbb{D}_{1,p}$, $p \in [1, \infty]$, denote the L^p domain of D, defined by the norm

$$\|F\|_{1,p} = \|F\|_{L^p(W)} + \|DF\|_{L^p(W, L^2([0,1]))}, \quad F \in \mathcal{S},$$

and let $\mathbb{L}^{1,p}$ be defined by the norm

$$\|u\|_{1,p} = \|u\|_{L^p(W, L^2([0,1]))} + \|Du\|_{L^p(W, L^2([0,1]\times[0,1]))}, \quad u \in \mathcal{U}.$$

Let δ denote the adjoint of D, which satisfies

$$E[F\delta(h)] = E[\langle DF, h\rangle_{L^2([0,1])}], \qquad F \in \mathrm{Dom}(D), \ h \in \mathrm{Dom}(\delta).$$

We recall the relation

$$\delta(Fu) = F\delta(u) - \langle DF, u\rangle_{L^2([0,1])}, \tag{3}$$

for $F \in \mathrm{Dom}(D)$ and $u \in \mathrm{Dom}(\delta)$ such that $F\delta(u) \in L^2(W)$, and the isometry

$$E[\delta(u)\delta(v)] = E\left[\int_0^1 u(s)v(s)ds\right] + E\left[\int_0^1 \int_0^1 D_s u(t) D_t v(s) ds dt\right], \qquad u, v \in \mathcal{U}. \tag{4}$$

Throughout this paper, π denotes a partition

$$\pi = \{0 = t_0 < t_1 < t_2 < \cdots < t_n = 1\},$$

of the interval $[0,1]$ and $|\pi| = \sup_{0 \le i \le n-1}(t_{i+1} - t_i)$. The notation $\lim_{|\pi|\to 0}$ denotes the limit over all sequences $\{\pi_n\}_{n\in\mathrm{N}}$ of partitions such that $|\pi_n| \to 0$ as $n \to \infty$.

3 Skorohod integral with respect to an L^2 process

We define an extended Skorohod integral operator $\delta^X : L^2([0,1] \times W) \longrightarrow L^2(W)$ that acts with respect to a given stochastic process $(X(t))_{t\in[0,1]}$ written as

$$X(t) = \delta(\dot{v}(\cdot, t)), \quad t \in [0,1].$$

Definition 1 *Let $(v(s,t))_{s,t\in[0,1]}$ be a two-parameter process such that*

a) $t \hookrightarrow v(s,t) \in H$ and $t \hookrightarrow v(t,s) \in H$, $ds \times dP$-a.e.,

b) $t \hookrightarrow \dot{v}(s,t) := \partial_s v(s,t)$ and $t \hookrightarrow \hat{v}(t,s) := \partial_s v(t,s)$ have finite variation, $ds \times dP$ a.e.,

c) $\dot{v}(\cdot,t) := \partial_s v(\cdot,t) \in Dom(\delta)$, $t \in [0,1]$.

We call v-process the process $(X(t))_{t\in[0,1]}$ defined as

$$X(t) = \delta(\dot{v}(\cdot,t)), \qquad t \in [0,1].$$

Under assumptions $a)$ and $b)$, v possesses the following important property which will be used throughout this paper:

$$\dot{v}(s,dt)ds = \hat{v}(ds,t)dt, \quad s,t \in [0,1], \quad a.s.$$

Brownian motion is obtained from the deterministic process $v(s,t) = s \wedge t$, $s,t \in [0,1]$. We define a Skorohod type integral with respect to $(X(t))_{t\in[0,1]}$.

Definition 2 *We define the generalized Skorohod integral operator* $\delta^X : L^2([0,1] \times W) \longrightarrow L^2(W)$ *on* $\mathcal{U}$ *as*

$$\delta^X(u) = \delta\left(\int_0^1 u(s)\dot{v}(\cdot, ds)\right), \qquad u \in \mathcal{U}. \tag{5}$$

For $u \in \mathcal{U}$ of the form

$$u(t,\omega) = \sum_{i=0}^{n-1} F_i(\omega) 1_{]t_i,t_{i+1}]}(t), \quad t \in [0,1], \ \omega \in W, \tag{6}$$

the expression (5) is well-defined as:

$$\delta^X(u) = \sum_{i=0}^{n-1} \delta(F_i(\dot{v}(\cdot, t_{i+1}) - \dot{v}(\cdot, t_i))),$$

and $(X(t))_{t\in[0,1]}$ has the representation

$$X(t) = \delta^X(1_{[0,t]}), \quad t \in [0,1].$$

Note that in general $\delta^X(u)$ does not coincide with the sum

$$\sum_{i=0}^{n-1} F_i \diamond (X(t_{i+1}) - X(t_i)),$$

where $\diamond$ denotes the Wick product, except if $v(s,t)$ is deterministic. In the Brownian case ($v(s,t) = s \wedge t$, $s, t \in [0,1]$) we have

$$\dot{v}(s,t) = 1_{[0,t]}(s), \quad \hat{v}(s,t) = 1_{[t,\infty[}(s), \quad s,t \in [0,1],$$

and

$$\dot{v}(s,dt)ds = \varepsilon_s(dt)ds, \quad \hat{v}(ds,t)dt = \varepsilon_t(ds)dt, \tag{7}$$

where ε_x denotes the Dirac measure at $x \in \mathbb{R}$. Consequently, $\int_0^1 u(t)\dot{v}(s,dt) = u(s)$, $s \in [0,1]$, and

$$\delta^B(u) = \delta\left(\int_0^1 u(t)\dot{v}(\cdot, dt)\right) = \delta(u), \quad u \in \mathcal{U},$$

i.e. $\delta^B = \delta$ is the classical Skorohod integral. If $(X(t))_{t\in[0,1]}$ is a Skorohod integral process of the form

$$X(t) = \delta(h1_{[0,t]}), \quad t \in [0,1], \quad h \in \mathbb{L}_{1,4},$$

i.e. $v(s,t) = \int_0^{s\wedge t} h_\alpha d\alpha$, $s,t \in [0,1]$. We have $\dot{v}(s,t) = 1_{[0,t]}(s)h(s)$, and $\dot{v}(s,dt) = h(s)\varepsilon_s(dt)$, hence

$$\delta^X(u) = \delta\left(\int_0^1 u(s)\dot{v}(\cdot, ds)\right) = \delta(hu),$$

and this formula coincides with the definition of [6] of the Skorohod integral with respect to $(X(t))_{t\in[0,1]}$. In particular, if u is adapted then $(X(t))_{t\in[0,1]}$ is an Itô process with Itô differential $dX(t) = h(t)dB(t)$ and the operator δ^X coincides on adapted processes with the Itô integral with respect to $h(t)dB(t)$.
If $(X(t))_{t\in[0,1]}$ has absolutely continuous trajectories, e.g. $X(t) = \int_0^t \delta(a(\cdot, s))ds$ with $\dot{v}(s,t) = \int_0^t a(s,\alpha)d\alpha$, then

$$\delta^X(u) = \delta\left(\int_0^1 u(s)a(\cdot, s)ds\right),$$

i.e. δ^X defines a centered stochastic integral with respect to $(X(t))_{t\in[0,1]}$, which necessarily differs from pathwise integrals. We now define the adjoint D^X of δ^X.

Definition 3 *The $(X(t))_{t\in[0,1]}$-derivative operator $D^X : L^2(W) \longrightarrow L^2([0,1]\times W)$ is defined on $\mathcal{S}$ by*

$$D_s^X F = \int_0^1 D_\alpha F \hat{v}(d\alpha, s), \quad s \in [0,1], \ F \in \mathcal{S}.$$

For $F \in \mathcal{S}$ of the form $F = f(B(t_1), \ldots, B(t_n))$ we have

$$D_s^X F = \sum_{i=1}^{i=n} \partial_i f(B(t_1), \ldots, B(t_n))(\hat{v}(t_i, s) - \hat{v}(0, s)), \quad s \in [0,1].$$

The operator D^X is a derivation on $\mathcal{S}$, i.e. $D^X(FG) = FD^X G + GD^X F$, $F, G \in \mathcal{S}$.
In the Brownian case ($\hat{v}(s,t) = 1_{[0,s]}(t)$, $s,t \in [0,1]$), we have

$$D^B(s)F = \sum_{i=1}^{i=n} 1_{[0,t_i]}(s)\partial_i f(B(t_1), \ldots, B(t_n)), \quad s \in [0,1],$$

i.e. $D^B = D$ is the classical gradient on the Wiener space. If $(X(t))_{t\in[0,1]} = (\delta(h1_{[0,t]}))_{t\in[0,1]}$ is a Skorohod integral process, then $\hat{v}(s,t) = 1_{[0,s]}(t)h(t)$, and $\hat{v}(ds,t) = h(t)\varepsilon_t(ds)$, hence $D_t^X = h(t)D_t$, $t \in [0,1]$.

Proposition 1 *The operators D^X and δ^X are closable and the following duality relation holds:*

$$E[F\delta^X(u)] = E[\langle D^X F, u\rangle_{L^2([0,1])}], \quad F \in \mathcal{S}, \ u \in \mathcal{U}. \tag{8}$$

Proof: We have

$$E[F\delta^X(u)] \quad = \quad E\left[F\delta\left(\int_0^1 u(s)\dot{v}(\cdot, ds)\right)\right]$$

$$
\begin{aligned}
&= E\left[\int_0^1 D_\alpha F \int_0^1 u(s)\dot{v}(\alpha, ds) d\alpha\right] \\
&= E\left[\int_0^1 u(s) \int_0^1 D_\alpha F \hat{v}(d\alpha, s) ds\right] \\
&= E\left[\int_0^1 D_s^X F u(s) ds\right] = E\left[\langle D^X F, u\rangle_{L^2([0,1])}\right].
\end{aligned}
$$

The closability of δ^X and D^X follows from (8) and the fact that δ^X and D^X are densely defined, respectively on $\mathcal{U}$ and $\mathcal{S}$.

□

We may now extend δ^X to its closed domain $\mathrm{Dom}(\delta^X)$ which is the set of $u \in L^2([0,1] \times W)$ such that there is a constant $K > 0$ with

$$
|E[\langle D^X F, u\rangle_{L^2([0,1])}]| \leq K \|F\|^2_{L^2(W)}, \quad F \in \mathcal{S}.
$$

For $u \in \mathrm{Dom}(\delta^X)$, $\delta^X(u)$ is the unique random variable that satisfies

$$
E[\langle D^X F, u\rangle_{L^2([0,1])}] = E[F\delta^X(u)], \quad F \in \mathcal{S}.
$$

Proposition 2 *Let* $u \in \mathrm{Dom}(\delta^X)$ *and* $F \in \mathrm{Dom}(D^X)$ *such that*

$$
F\delta^X(u) - \langle u, D^X F\rangle_{L^2([0,1])} \in L^2(W).
$$

We have

$$
\delta^X(Fu) = F\delta^X(u) - \langle u, D^X F\rangle_{L^2([0,1])}. \tag{9}
$$

Proof: Let $u \in \mathcal{U}$ be a simple process of the form $u = 1_{]s,t]}G$. We have

$$
\begin{aligned}
\delta^X(Fu) &= \delta^X(FG1_{]s,t]}) = \delta\left(F \int_0^1 u(s)\dot{v}(\cdot, ds)\right) \\
&= \delta\left(FG(\dot{v}(\cdot, t) - \dot{v}(\cdot, s))\right) \\
&= F\delta\left(G(\dot{v}(\cdot, t) - \dot{v}(\cdot, s))\right) - \langle DF, G(\dot{v}(\cdot, t) - \dot{v}(\cdot, s))\rangle_{L^2([0,1])} \\
&= F\delta^X(u) - \int_0^1 \int_0^1 D_\alpha F u(\beta)\dot{v}(\alpha, d\beta) d\alpha \\
&= F\delta^X(u) - \int_0^1 \int_0^1 D_\alpha F u(\beta)\hat{v}(d\alpha, \beta) d\beta \\
&= F\delta^X(u) - \int_0^1 D_\alpha^X F u(\alpha) d\alpha, \quad F \in \mathcal{S}.
\end{aligned}
$$

Hence (9) holds for $F, G \in \mathcal{S}$ and $u \in \mathcal{U}$. The conclusion follows by closability of D^X and δ^X.

□

Relation (9) can also be obtained using the duality between D^X and δ^X and the derivation property of D^X, as

$$\begin{aligned}E[G\delta^X(uF)] &= E[\langle u, FD^XG\rangle_{L^2([0,1])}] = E[\langle u, D^X(FG) - GD^XF\rangle_{L^2([0,1])}] \\ &= E[G(F\delta^X(u) - \langle u, D^XF\rangle_{L^2([0,1])})], \quad F, G \in \mathcal{S},\ u \in \mathcal{U}.\end{aligned}$$

Finally we notice that D^X and δ^X have the locality property.

Proposition 3 *For $F \in \mathrm{Dom}(D^X)$ we have $D^XF = 0$ a.s. on the set $\{F = 0\}$. If $u \in \mathrm{Dom}(\delta^X)$ is such that $E[\|D^Xu\|^2_{L^2([0,1]^2)}] < \infty$ then $\delta^X(u) = 0$ a.s. on $\{\|u\|_{L^2([0,1])}\} = 0$.*

Proof: The proof of this proposition uses the duality between D^X and δ^X and is identical to the proof of the analog result in [1], [8], [10].

□

Consequently the operator δ^X may, as the usual Skorohod integral operator δ, be extended by a localisation argument.

4 The extended Skorohod integral as a continuous operator on Sobolev spaces

In this section we give a more precise description of the domain of δ^X via a bound in Sobolev norm. We assume that $(\dot{v}(s,t))_{s,t\in[0,1]}$ satisfies the following hypothesis.

Definition 4 *We denote by $\mathcal{V}$ the set of two-parameter processes $(v(s,t))_{s,t\in[0,1]}$ satisfying Def. 1, such that $\dot{v}(\cdot,t) \in \mathbb{L}_{1,4}$, $t \in [0,1]$, and there is a constant $C_v > 0$ with*

i) $\|\dot{v}(\cdot,t) - \dot{v}(\cdot,s)\|_{1,4} \leq C_v\sqrt{t-s}$, $0 \leq s < t \leq 1$,

and for $0 \leq a < b < s < t \leq 1$:

ii) $\left\|\langle \dot{v}(\cdot,t) - \dot{v}(\cdot,s), \dot{v}(\cdot,b) - \dot{v}(\cdot,a)\rangle_{L^2([0,1])}\right\|_{L^2(W)} \leq C_v^2(t-s)(b-a)$,

iii) $E\left[\int_0^1 \left(\int_0^1 D_r(\dot{v}(u,t) - \dot{v}(u,s))(\dot{v}(r,b) - \dot{v}(r,a))dr\right)^2 du\right]^{1/2} \leq C_v^2(t-s)(b-a)$,

iv) $\left\|\int_0^1\int_0^1 D_r(\dot{v}(u,t) - \dot{v}(u,s))D_u(\dot{v}(r,b) - \dot{v}(r,a))dudr\right\|_{L^2(W)} \leq C_v^2(t-s)(b-a)$.

Assumption $c)$ is satisfied by Brownian motion since in this case $v(s,t) = s \wedge t$. Our main interest is in the case of random $v(s,t)$, but we mention that the first chaos process given by $\dot{v}(s,t) = (t-s)^\alpha 1_{[0,t]}(s)$, $\alpha > 1/2$, also satisfies $c)$, since $(t-r)^\alpha - (s-r)^\alpha \le \alpha(t-s)(s-r)^{\alpha-1}$, $0 \le r < s < t$, hence

$$\int_a^b (\dot{v}(r,t) - \dot{v}(r,s))\dot{v}(r,b)dr \le \alpha(t-s)(b-a)^{2\alpha}$$

and

$$\int_0^a (\dot{v}(r,t) - \dot{v}(r,s))(\dot{v}(r,b) - \dot{v}(r,a))dr \le \alpha^2(t-s)(b-a).$$

The following lemma allows to extend δ^X as a continuous operator from $L^2([0,1], I\!\!D_{1,4})$ into $L^2(W)$ by density of $\mathcal{U}$ in $L^2([0,1], I\!\!D_{1,4})$.

Lemma 1 *For any simple process $u \in \mathcal{U}$ we have*

$$\|\delta^X(u)\|_{L^2(W)} \le 2C_v \|u\|_{L^2([0,1], I\!\!D_{1,4})}.$$

Proof: We start by the following:

Lemma 2 *We have for $F \in I\!\!D_{1,4}$:*

$$\|\delta^X(1_{[s,t]}F)\|_{L^2(W)} \le 2C_v\sqrt{t-s}\|F\|_{1,4}, \quad 0 \le s \le t \le 1.$$

and for $F, G \in I\!\!D_{1,4}$:

$$\left| E\left[\delta^X(1_{[a,b]}F)\delta^X(1_{[s,t]}G)\right]\right| \le 2C_v^2\|F\|_{1,4}\|G\|_{1,4}(t-s)(b-a) \quad 0 \le a < b < s < t \le 1.$$

Proof: We have for $0 \le s < t \le 1$ and $0 \le a < b \le 1$:

$$\begin{aligned}
&|E\left[\delta(F(\dot{v}(\cdot,t) - \dot{v}(\cdot,s)))\delta(G(\dot{v}(\cdot,b) - \dot{v}(\cdot,a)))\right]| \\
&= \Big| E\left[FG\langle \dot{v}(\cdot,t) - \dot{v}(\cdot,s), \dot{v}(\cdot,b) - \dot{v}(\cdot,a)\rangle_{L^2([0,1])}\right] \\
&\quad + E\left[\int_0^1\int_0^1 D_r(F(\dot{v}(u,t) - \dot{v}(u,s)))D_u(G(\dot{v}(r,b) - \dot{v}(r,a)))dudr\right]\Big| \\
&= \Big| E\left[FG\langle \dot{v}(\cdot,t) - \dot{v}(\cdot,s), \dot{v}(\cdot,b) - \dot{v}(\cdot,a)\rangle_{L^2([0,1])}\right] \\
&\quad + E\left[\int_0^1\int_0^1 (\dot{v}(u,t) - \dot{v}(u,s))(\dot{v}(r,b) - \dot{v}(r,a))D_rFD_uGdudr\right] \\
&\quad + E\left[\int_0^1\int_0^1 F(\dot{v}(r,b) - \dot{v}(r,a))D_uGD_r(\dot{v}(u,t) - \dot{v}(u,s))dudr\right] \\
&\quad + E\left[\int_0^1\int_0^1 G(\dot{v}(u,t) - \dot{v}(u,s))D_rFD_u(\dot{v}(r,b) - \dot{v}(r,a))dudr\right]
\end{aligned}$$

$$+E\left[\int_0^1\int_0^1 FGD_r(\dot v(u,t)-\dot v(u,s))D_u(\dot v(r,b)-\dot v(r,a))dudr\right]\Bigg|$$

$$\leq \|F\|_{L^4(W)}\|G\|_{L^4(W)}\left\|\langle\dot v(\cdot,t)-\dot v(\cdot,s),\dot v(\cdot,b)-\dot v(\cdot,a)\rangle_{L^2([0,1])}\right\|_{L^2(W)}$$

$$+\left\|\|DG\|_{L^2([0,1])}\|\dot v(\cdot,t)-\dot v(\cdot,s)\|_{L^2([0,1])}\right\|_{L^2(W)}\left\|\|DF\|_{L^2([0,1])}\|\dot v(\cdot,b)-\dot v(\cdot,a)\|_{L^2([0,1])}\right\|_{L^2(W)}$$

$$+\|F\|_{L^4(W)}\|DG\|_{L^4(W,L^2([0,1]))}E\left[\int_0^1\left(\int_0^1 D_r(\dot v(u,t)-\dot v(u,s))(\dot v(r,b)-\dot v(r,a))dr\right)^2du\right]^{1/2}$$

$$+\|G\|_{L^4(W)}\|DF\|_{L^4(W,L^2([0,1]))}E\left[\int_0^1\left(\int_0^1 D_u(\dot v(r,b)-\dot v(r,a))(\dot v(u,t)-\dot v(u,s))du\right)^2dr\right]^{1/2}$$

$$+\|FG\|_{L^2(W)}E\left[\left(\int_0^1\int_0^1 D_r(\dot v(u,t)-\dot v(u,s))D_u(\dot v(r,b)-\dot v(r,a))dudr\right)^2\right]^{1/2}.$$

If $F=G$, $a=s$ and $b=t$, then $c-i)$ implies

$$E\left[\int_0^1\left(\int_0^1 D_r(\dot v(u,t)-\dot v(u,s))(\dot v(r,t)-\dot v(r,s))dr\right)^2du\right]^{1/2}\leq C_v^2(t-s),$$

and

$$E\left[\left(\int_0^1\int_0^1 D_r(\dot v(u,t)-\dot v(u,s))D_u(\dot v(r,b)-\dot v(r,a))dudr\right)^2\right]^{1/2}\leq C_v^2(t-s),$$

hence

$$\begin{aligned}
E\left[\delta(F(\dot v(\cdot,t)-\dot v(\cdot,s)))^2\right] &\leq C_v^2\|F\|^2_{L^4(W)}(t-s)+C_v^2\|DF\|^2_{L^4(W,L^2([0,1]))}(t-s)\\
&\quad +2C_v^2\|F\|_{L^4(W)}\|DF\|_{L^4(W,L^2([0,1]))}(t-s)\\
&\quad +C_v^2\|F\|^2_{L^4(W)}(t-s)\\
&\leq 2C_v^2\|F\|^2_{1,4}(t-s)
\end{aligned}$$

If $0<a<b<s<t$, then

$$\begin{aligned}
|E\left[\delta(F(\dot v(\cdot,t)-\dot v(\cdot,s)))\delta(G(\dot v(\cdot,b)-\dot v(\cdot,a)))\right]| &\leq C_v^2\|F\|_{L^4(W)}\|G\|_{L^4(W)}(t-s)(b-a)\\
&\quad +C_v^2\|DG\|_{L^4(W,L^2([0,1]))}\|DF\|_{L^4(W,L^2([0,1]))}(t-s)(b-a)\\
&\quad +C_v^2\|DF\|_{L^4(W,L^2([0,1]))}\|G\|_{L^4(W)}(t-s)(b-a)\\
&\quad +C_v^2\|DG\|_{L^4(W,L^2([0,1]))}\|F\|_{L^4(W)}(t-s)(b-a)\\
&\quad +C_v^2\|F\|_{L^4(W)}\|G\|_{L^4(W)}(t-s)(b-a)\\
&\leq 2C_v^2\|F\|_{1,4}\|G\|_{1,4}(t-s)(b-a).
\end{aligned}$$

Proof of Lemma 1. Let $u \in \mathcal{U}$ be of the form (6). We have

$$\begin{aligned}
\left\|\delta^X(u)\right\|^2_{L^2(W)} &= E\left[\left(\sum_{i=0}^{i=n-1} \delta(F_i(\dot{v}(\cdot,t_{i+1}) - \dot{v}(\cdot,t_i)))\right)^2\right] \\
&\leq 4C_v^2 \sum_{j=0}^{j=n-1}\sum_{i=0}^{i=j-1} \|F_i\|_{1,4}\|F_j\|_{1,4}(t_{i+1}-t_i)(t_{j+1}-t_j) \\
&\quad +2C_v^2 \sum_{i=0}^{i=n-1} \|F_i\|^2_{1,4}(t_{i+1}-t_i) \\
&\leq 2C_v^2\left(\sum_{i=0}^{i=n-1} \|F_i\|_{1,4}(t_{i+1}-t_i)\right)^2 + 2C_v^2\|u\|^2_{L^1([0,1],\mathbb{D}_{1,4})} \\
&\leq 2C_v^2\|u\|^2_{L^2([0,1],\mathbb{D}_{1,4})} + 2C_v^2\|u\|^2_{L^1([0,1],\mathbb{D}_{1,4})}. \leq 4C_v^2\|u\|^2_{L^2([0,1],\mathbb{D}_{1,4})}. \qquad \square
\end{aligned}$$

5 Existence of pathwise integrals

In this section we give conditions for the existence in $L^1(W)$ of pathwise integrals with respect to $(X(t))_{t\in[0,1]}$, and obtain their expressions with δ^X and D^X. For any $u \in L^2([0,1]\times W)$ and any partition π of [0,1] we define the processes

$$\begin{aligned}
u_t^{\pi\circ} &= \sum_{i=0}^{n-1} \frac{1}{t_{i+1}-t_i}\left(\int_{t_i}^{t_{i+1}} u(s)ds\right) 1_{]t_i,t_{i+1}]}(t), \\
u_t^{\pi-} &= \sum_{i=1}^{n} \frac{1}{t_i-t_{i-1}}\left(\int_{t_{i-1}}^{t_i} u(s)ds\right) 1_{]t_i,t_{i+1}]}(t), \\
u_t^{\pi+} &= \sum_{i=0}^{n-2} \frac{1}{t_{i+2}-t_{i+1}}\left(\int_{t_{i+1}}^{t_{i+2}} u(s)ds\right) 1_{]t_i,t_{i+1}]}(t), \quad t \in [0,1].
\end{aligned}$$

We know, cf. [9], that $(u^{\pi\circ})_\pi$, $(u^{\pi-})_\pi$ and $(u^{\pi+})_\pi$ converge to u in $L^2([0,1]\times W)$ as $|\pi| \to 0$, and if $u \in \mathbb{L}_{1,4}$, then the convergence holds in $\mathbb{L}_{1,4}$. The Riemann sums associated to $u^{\pi\circ}$, $u^{\pi-}$ and $u^{\pi+}$ are

$$\begin{aligned}
S^{\pi\circ}(u) &= \sum_{i=0}^{n-1} \frac{1}{t_{i+1}-t_i}\left(\int_{t_i}^{t_{i+1}} u(s)ds\right)(X(t_{i+1})-X(t_i)), \\
S^{\pi-}(u) &= \sum_{i=1}^{n} \frac{1}{t_i-t_{i-1}}\left(\int_{t_{i-1}}^{t_i} u(s)ds\right)(X(t_{i+1})-X(t_i)), \\
S^{\pi+}(u) &= \sum_{i=0}^{n-2} \frac{1}{t_{i+2}-t_{i+1}}\left(\int_{t_{i+1}}^{t_{i+2}} u(s)ds\right)(X(t_{i+1})-X(t_i)).
\end{aligned}$$

The following is a definition of pathwise type anticipating integrals with respect to $(X(t))_{t\in[0,1]}$, cf. [9] for the Stratonovich case. Another family of approximating sequences is used in [12].

Definition 5 *The process u is Stratonovich, resp. forward, backward integrable with respect to $(X(t))_{t\in[0,1]}$ if the family $\{S^{\pi\circ}(u)\}_\pi$, resp. $\{S^{\pi-}(u)\}_\pi$, $\{S^{\pi+}(u)\}_\pi$ converges in probability as $|\pi| \to 0$ to a limit denoted by $\int_0^1 u(s)d^\circ X(s)$, resp. $\int_0^1 u(s)d^- X(s)$, $\int_0^1 u(s)d^+ X(s)$.*

Let $\mathbb{L}_C$ be the subspace of $L^2([0,1], \mathbb{D}_{1,4})$ such that for $u \in \mathbb{L}_C$ the limits

$$D_t^+ u(t) = \lim_{\varepsilon\downarrow 0} D_t u(t+\varepsilon), \qquad D_t^- u(t) = \lim_{\varepsilon\downarrow 0} D_t u(t-\varepsilon),$$

exist in $L^2(W)$, uniformly in $t \in [0,1]$. For $u \in \mathbb{L}_C$ we let

$$\nabla_\alpha u(\beta) = D_\alpha^+ u(\beta) + D_\alpha^- u(\beta), \quad \alpha, \beta \in [0,1].$$

We now define a subspace $\mathbb{L}_v$ of $\mathbb{L}_C$ whose elements will be pathwise integrable with respect to $(X(t))_{t\in[0,1]}$. The total variation of the finite measure $\hat{v}(d\alpha, s)$ is denoted by $|\hat{v}|(d\alpha, s)$, $ds \times dP$ a.e.

Definition 6 *Let $v \in \mathcal{V}$. We denote by $\mathbb{L}_v$ the set of processes $u \in \mathbb{L}_C$ such that there is a version of Du with*

$$\lim_{\varepsilon\to 0} \sup_{0\le\beta<s<\beta+\varepsilon\le 1} \int_0^1 |D_\alpha u(s) - D_\alpha^+ u(\beta)||\hat{v}|(d\alpha, \beta) = 0 \quad \text{in } L^2(W),$$
$$\lim_{\varepsilon\to 0} \sup_{0\le\beta-\varepsilon<s<\beta\le 1} \int_0^1 |D_\alpha u(s) - D_\alpha^- u(\beta)||\hat{v}|(d\alpha, \beta) = 0 \quad \text{in } L^2(W).$$

We also define the operators ∇_s^X, D_s^{X+} and D_s^{X-}, $s \in [0,1]$, as

$$D_s^{X+} u(s) = \int_0^1 D_\alpha^+ u(s)\hat{v}(d\alpha, s), \quad D_s^{X-} u(s) = \int_0^1 D_\alpha^- u(s)\hat{v}(d\alpha, s),$$

and

$$\nabla_s^X u(s) = \int_0^1 \nabla_\alpha u(s)\hat{v}(d\alpha, s) = D_s^{X+} u(s) + D_s^{X-} u(s), \quad u \in \mathbb{L}_v.$$

From (7), we have in the Brownian case:

$$\nabla_s^B = \nabla_s, \quad D_s^{B-} = D_s^-, \text{ and } D_s^{B+} = D_s^+, \quad s \in [0,1]. \tag{10}$$

Remark 1 *We can also define $\mathbb{L}_v$ as the set of processes $u \in L^2([0,1], \mathbb{D}_{1,4})$ such that there is a version of $D^X u$ such that $s \hookrightarrow D_{t\vee s}^X u(t\wedge s)$ and $s \hookrightarrow D_{t\wedge s}^X u(t \vee s)$ are continuous from $[0,1]$ into $L^2(W)$, uniformly in $t \in [0,1]$. Then one can define*

$$D_t^{X+} = \lim_{\varepsilon\downarrow 0} D_t^X u(t+\varepsilon), \qquad D_t^{X-} = \lim_{\varepsilon\downarrow 0} D_t^X u(t-\varepsilon).$$

In this case the proofs of Prop. 4 and Th. 2 below would stay closer to [9] but the representation of D^{X+}, D^{X-} in terms of D^+, D^- would not necessarily hold.

The following proposition gives sufficient conditions for the existence of the Stratonovich, forward and backward integrals with respect to $(X(t))_{t\in[0,1]}$, and links these integrals to the Skorohod integral operator δ^X.

Proposition 4 *Let $u \in \mathbb{L}_v$. Then u is forward, backward and Stratonovich integrable, and the following relations hold in $L^1(W)$:*

$$\delta^X(u) = \int_0^1 u(s)d^\circ X(s) - \frac{1}{2}\int_0^1 \nabla_s^X u(s)ds, \tag{11}$$

$$\delta^X(u) = \int_0^1 u(s)d^- X(s) - \int_0^1 D_s^{X-} u(s)ds, \tag{12}$$

$$\delta^X(u) = \int_0^1 u(s)d^+ X(s) - \int_0^1 D_s^{X+} u(s)ds. \tag{13}$$

Proof: For simplicity of notation we assume that the measure $\hat{v}(d\alpha, s)$ is positive, $ds \times dP$ a.e. The general case follows easily from this particular case. We will prove

$$\delta^X(u) = \int_0^1 u(s)d^\circ X(s) - \frac{1}{2}\int_0^1\int_0^1 \nabla_\alpha u(s)\hat{v}(d\alpha, s)ds,$$

$$\delta^X(u) = \int_0^1 u(s)d^- X(s) - \int_0^1\int_0^1 D_\alpha^- u(s)\hat{v}(d\alpha, s)ds,$$

$$\delta^X(u) = \int_0^1 u(s)d^+ X(s) - \int_0^1\int_0^1 D_\alpha^+ u(s)\hat{v}(d\alpha, s)ds.$$

From Prop. 2 we have

$$\begin{aligned}\delta^X(u^{\pi\circ}) &= S^{\pi\circ}(u) - \sum_{i=0}^{n-1} \frac{1}{t_{i+1}-t_i}\int_{t_i}^{t_{i+1}} D_\alpha^X \int_{t_i}^{t_{i+1}} u(s)dsd\alpha \\ &= S^{\pi\circ}(u) - \sum_{i=0}^{n-1} \frac{1}{t_{i+1}-t_i}\int_{t_i}^{t_{i+1}}\int_{t_i}^{t_{i+1}}\int_0^1 D_\alpha u(s)\hat{v}(d\alpha,\beta)dsd\beta.\end{aligned}$$

We will show the convergence in $L^1(W)$ of the last term to

$$\frac{1}{2}\int_0^1\int_0^1 \nabla_\alpha u(s)\dot{v}(\alpha, ds)d\alpha = \frac{1}{2}\int_0^1\int_0^1 \nabla_\alpha u(s)\hat{v}(d\alpha, s)ds.$$

We have

$$\begin{aligned}&\left|\sum_{i=0}^{n-1} \frac{1}{t_{i+1}-t_i}\int_{t_i}^{t_{i+1}}\int_\beta^{t_{i+1}}\int_0^1 (D_\alpha u(s) - \frac{1}{2}D_\alpha^+ u(\beta))\hat{v}(d\alpha,\beta)dsd\beta\right| \\ &\leq \sum_{i=0}^{n-1} \frac{1}{t_{i+1}-t_i}\int_{t_i}^{t_{i+1}}\int_\beta^{t_{i+1}}\int_0^1 |D_\alpha u(s) - D_\alpha^+ u(\beta)|\hat{v}(d\alpha,\beta)dsd\beta \\ &\quad + \left|\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}}\left(\frac{t_{i+1}-\beta}{t_{i+1}-t_i} - \frac{1}{2}\right)\int_0^1 D_\alpha^+ u(\beta)\hat{v}(d\alpha,\beta)d\beta\right|,\end{aligned}$$

which converges to 0. (The weak convergence in $L^2([0,1])$ of $\beta \hookrightarrow \sum_{i=0}^{n-1} \frac{t_{i+1}-\beta}{t_{i+1}-t_i}$ to $\frac{1}{2}$ as $|\pi| \to 0$ is used as in [9]). Hence the convergence of

$$\sum_{i=0}^{n-1} \frac{1}{t_{i+1}-t_i} \int_{t_i}^{t_{i+1}} \int_{\beta}^{t_{i+1}} \int_0^1 D_\alpha u(s)\hat{v}(d\alpha,\beta)dsd\beta$$

to

$$\frac{1}{2}\int_0^1 \int_0^1 D_\alpha^+ u(\beta)\hat{v}(d\alpha,\beta)d\beta.$$

Similarly we prove that

$$\left| \sum_{i=0}^{n-1} \frac{1}{t_{i+1}-t_i} \int_{t_i}^{t_{i+1}} \int_{t_i}^{\beta} \int_0^1 (D_\alpha u(s) - \frac{1}{2} D_\alpha^- u(\beta))\hat{v}(d\alpha,\beta)dsd\beta \right|$$

converges to 0 in $L^1(W)$ as $|\pi| \to 0$. Concerning (12) we use the relation

$$\begin{aligned} \delta^X(u^{\pi-}) &= S^{\pi-}(u) - \sum_{i=1}^{n-1} \frac{1}{t_i - t_{i-1}} \int_{t_{i-1}}^{t_i} \int_0^1 D_\alpha u(s)(\dot{v}(\alpha,t_{i+1}) - \dot{v}(\alpha,t_i))d\alpha ds \\ &= S^{\pi-}(u) - \sum_{i=0}^{n-1} \frac{1}{t_i - t_{i-1}} \int_{t_i}^{t_{i+1}} \int_{t_{i-1}}^{t_i} \int_0^1 D_\alpha u(s)\hat{v}(d\alpha,\beta)dsd\beta. \end{aligned}$$

The convergence in $L^1(W)$ of the last term to

$$\int_0^1 \int_0^1 D_\alpha^- u(\beta)\hat{v}(d\alpha,\beta)d\beta$$

follows then from the bound

$$\begin{aligned} &\left| \sum_{i=0}^{n-1} \frac{1}{t_{i-1}-t_i} \int_{t_i}^{t_{i+1}} \int_{t_{i-1}}^{t_i} \int_0^1 (D_\alpha u(s) - \frac{1}{2} D_\alpha^- u(\beta))\hat{v}(d\alpha,\beta)dsd\beta \right| \\ &\leq \int_0^1 \sup_{s\in[\beta,\beta+|\pi|]} \int_0^1 |D_\alpha u(s) - D_\alpha^- u(\beta)|\hat{v}(d\alpha,\beta)d\beta. \end{aligned}$$

Relation (13) is proved analogously.

□

6 Quadratic variation of L^2 processes

In this section we obtain sufficient conditions for the existence of the pathwise quadratic variation of the v-process $(X(t))_{t\in[0,1]}$, $v \in \mathcal{V}$. It is known that the quadratic variation of the Skorohod integral process $(X(t))_{t\in[0,1]} = (\delta(h1_{[0,t]}))_{t\in[0,1]}$ is

$$d[X,X](t) = h(t)^2 dt.$$

Moreover, $D_t^+ X(t) = h(t) + \delta(D_t h 1_{[0,t]})$ and $D_t^- X(t) = \delta(D_t h 1_{[0,t]})$, $t \in [0,1]$, and this suggests the identity

$$d[X,X](t) = h(t)(D^+ X(t) - D^- X(t)), \quad t \in [0,1],$$

which can also be written as

$$d[X,X](t) = D^{X+} X(t) - D^{X-} X(t), \quad t \in [0,1],$$

since $\hat{v}(s,t) = h(s) 1_{[t,\infty[}(s)$, $\hat{v}(ds,t) = h(t)\varepsilon_t(ds)$, $D_t^{X+} X(t) = h(t) D_t^+ X(t) = h(t)^2 + h(t)\delta(D_t h 1_{[0,t]})$ and $D_t^{X-} X(t) = h(t) D_t^- X(t) = h(t)\delta(D_t h 1_{[0,t]})$, $t \in [0,1]$. We give a more general meaning to this formula, extending it to $(X(t))_{t\in[0,1]} \in \mathbb{L}_v$ by use of the operators D_s^{X+} and D_s^{X-}.

Definition 7 *Let $v \in \mathcal{V}$, let $(X(t))_{t\in[0,1]}$ be a v-process, and let $(Y(t))_{t\in[0,1]} \in \mathbb{L}_v$. We define the absolutely continuous process $([X,Y](t))_{t\in[0,1]}$ by*

$$d[X,Y](t) = \int_0^1 (D_\alpha^+ Y(t) - D_\alpha^- Y(t))\hat{v}(d\alpha, t)dt, \quad t \in [0,1].$$

The bracket considered here corresponds to the pathwise quadratic variation of the process, as in e.g. [12]. In this respect it differs from other brackets that may be not symmetric or not positive and act as a correcting term for the Skorohod integral term of the Itô formula, cf. [4], [11].

Theorem 1 *Let $v \in \mathcal{V}$, let $(X(t))_{t\in[0,1]}$ be a v-process, and let $(Y(t))_{t\in[0,1]} \in \mathbb{L}_v$ be continuous in $\mathbb{D}_{1,4}$. The sequence $(V^\pi)_\pi$ defined as*

$$V^\pi = \sum_{i=0}^{n-1} (X_{t_{i+1}} - X_{t_i})(Y(t_{i+1}) - Y(t_i))$$

converges in $L^1(W)$ to

$$\int_0^1 (D_s^{X+} Y(s) - D_s^{X-} Y(s))ds = [X,Y](1).$$

Proof: We have from Prop. 2:

$$\begin{aligned} V^\pi &= \sum_{i=0}^{n-1} (Y(t_{i+1}) - Y(t_i))(X_{t_{i+1}} - X_{t_i}) = \sum_{i=0}^{n-1} (Y(t_{i+1}) - Y(t_i))\delta^X(1_{]t_i,t_{i+1}]}) \\ &= \sum_{i=0}^{i=n-1} \delta^X((Y(t_{i+1}) - Y(t_i))1_{]t_i,t_{i+1}]}) - \int_{t_i}^{t_{i+1}} D_s^X Y(t_{i+1})ds + \int_{t_i}^{t_{i+1}} D_s^X Y(t_i)ds \end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=0}^{n-1} \delta^X((Y(t_{i+1}) - Y(t_i))1_{]t_i,t_{i+1}]}) \\
&\quad + \int_{t_i}^{t_{i+1}} \int_0^1 D_\alpha Y(t_i)\hat{v}(d\alpha,\beta)d\beta - \int_{t_i}^{t_{i+1}} \int_0^1 D_\alpha Y(t_{i+1})\hat{v}(d\alpha,\beta)d\beta \\
&= J_1 + J_2 + J_3.
\end{aligned}
$$

For the term J_1 we have the inequalities

$$
\begin{aligned}
\|J_1\|^2_{L^2(W)} &= \left\| \sum_{i=0}^{n-1} \delta^X((Y(t_{i+1}) - Y(t_i))1_{]t_i,t_{i+1}]}) \right\|^2_{L^2(W)} \\
&\leq 4C_v^2 \left\| \sum_{i=0}^{n-1} (Y(t_{i+1}) - Y(t_i))1_{]t_i,t_{i+1}]} \right\|^2_{L^2([0,1],\mathbb{D}_{1,4})} \\
&= 4C_v^2 \sum_{i=0}^{n-1} \left\| (Y(t_{i+1}) - Y(t_i))1_{]t_i,t_{i+1}]} \right\|^2_{L^2([0,1],\mathbb{D}_{1,4})} \\
&= 4C_v^2 \sum_{i=0}^{n-1} (t_{i+1} - t_i)\|(Y(t_{i+1}) - Y(t_i))\|^2_{1,4} \\
&\leq 4C_v^2 \sup_{|s-t|<|\pi|} \|Y(t) - Y(s)\|_{1,4},
\end{aligned}
$$

hence $J_1 \to 0$ in $L^2(W)$. The convergence of J_2 to $\int_0^1 \int_0^1 D_\alpha^+ Y(\beta)\hat{v}(d\alpha,\beta)d\beta$ follows from

$$
\sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \int_0^1 |D_\alpha Y(t_i) - D_\alpha^+ Y(\beta)|\hat{v}(d\alpha,\beta)d\beta \leq \int_0^1 \sup_{\beta\in[s,s+|\pi|]} |D_\alpha Y(s) - D_\alpha^+ Y(\beta)|\hat{v}(d\alpha,\beta)d\beta.
$$

Similarly, J_3 converges to $\int_0^1 \int_0^1 D_\alpha^- Y(s)\hat{v}(d\alpha,\beta)d\beta$. Finally we use the relation

$$
\int_0^t (D_s^{X+} Y(s) - D_s^{X-} Y(s))ds = \int_0^t \int_0^1 (D_\alpha^+ Y(s) - D_\alpha^- Y(s))\hat{v}(d\alpha,s)ds. \quad \square
$$

The bracket $[X,Y](t)$ being symmetric by definition, this leads to the relation

$$
(D_t^{Y+} X(t) - D_t^{Y-} X(t)) = (D_t^{X+} Y(t) - D_t^{X-} Y(t)), \quad t \in [0,1].
$$

In the case of a quadratic variation $[X,X](t)$ this result also implies that

$$
D_t^{X+} X(t) \geq D_t^{X-} X(t), \qquad t \in [0,1].
$$

7 Anticipating Itô formula

In this section we prove the anticipating Itô formula

$$
\begin{aligned}
f(X(t)) &= f(X(0)) + \delta^X(f'(X(\cdot))1_{[0,t]}(\cdot)) \\
&\quad + \frac{1}{2}\int_0^t f''(X(s))d[X,X](s) + \int_0^t f''(X(s))D_s^{X-} X(s)ds.
\end{aligned}
$$

for a v-process $(X(t))_{t\in[0,1]}$ satisfying certain regularity conditions. This formula uses the operator δ^X and it is linked to pathwise approaches in Cor. 1. We first notice that even if $(X(t))_{t\in[0,1]}$ has absolutely continuous trajectories, δ^X can be used to write a change of variable formula that contains a zero expectation term. Let $a(\cdot,s)\in \mathbb{L}_{1,4}$, $s\in[0,1]$, and assume that $X(t)$ has the form

$$X(t)=\int_0^t \delta(a(\cdot,s))ds,\quad t\in[0,1].$$

We have

$$\begin{aligned} f(X(t)) &= f(X(0))+\int_0^t f'(X(s))\delta(a(\cdot,s))ds \qquad (14)\\ &= f(X(0))+\delta\left(\int_0^t f'(X(s))a(\cdot,s)ds\right)+\int_0^t f''(X(s))\langle D.X(s),a(\cdot,s)\rangle_{L^2([0,1])}ds\\ &= f(X(0))+\delta^X(1_{[0,t]}(\cdot)f'(X(\cdot)))+\int_0^t f''(X(s))D_s^X X(s)ds,\quad f\in\mathcal{C}_b^2(\mathbb{R}), \end{aligned}$$

which allows to identify a zero expectation component given by δ^X in the process $t\hookrightarrow f(X(t))$. The processes considered in the next theorem do not necessarily have absolutely continuous trajectories.

Theorem 2 *Let $v\in\mathcal{V}$. We assume that $(X(t))_{t\in[0,1]}\in\mathbb{L}_v$ has a continuous version and is continuous in $\mathbb{D}_{1,4}$ with $\sup_{t\in[0,1]}\|DX(t)\|_{L^4(W,L^\infty([0,1]))}<\infty$, and $\|\dot v(\cdot,t)-\dot v(\cdot,s)\|_{L^4(W,L^1([0,1]))}\le C_v(t-s)$. Then for $f\in\mathcal{C}_b^2(\mathbb{R})$,*

$$f(X(t))=f(X(0))+\delta^X(f'(X(\cdot))1_{[0,t]}(\cdot))+\frac{1}{2}\int_0^t f''(X(s))\nabla_s^X X(s)ds.$$

Before proving Th. 2 we make the following remarks.

- The v-processes considered in Th. 2 are centered by definition, but the formula is easily extended with an additional deterministic drift. On the other hand, the initial value $X(0)$ of $(X(t))_{t\in[0,1]}$ may be random.

- Most versions of the anticipating Itô formula for Skorohod integral processes, cf. [9], consider an additional random absolutely continuous drift term. This is not needed in Th. 2 because the class of processes it applies to is sufficiently large to contain such drifts, e.g. as in Relation (14).

- If we assume that $(D_s^X X(t))_{s,t\in[0,1]}\in L^2([0,1]^2\times W)$ then using Prop. 3 and a classical locality argument, Th. 2 can be extended to $f\in\mathcal{C}^2(\mathbb{R})$.

Proof of Th. 1: We will prove the formula

$$f(X(t)) = f(X(0)) + \delta^X(f'(X(\cdot))1_{[0,t]}(\cdot)) + \frac{1}{2}\int_0^t f''(X(s))\int_0^1 \nabla_\alpha X(s)\hat{v}(d\alpha, s)ds.$$

Following the method of [10] we write Taylor's formula

$$\begin{aligned} f(X(t)) &= f(X(0)) + \sum_{i=0}^{i=n-1} f'(X(t_i))(X(t_{i+1}) - X(t_i)) \\ &\quad + \frac{1}{2}\sum_{i=0}^{i=n-1} f''(\bar{X}(t_i))(X(t_{i+1}) - X(t_i))^2, \end{aligned}$$

where $\bar{X}(t_i)$ denotes a random point located between $X(t_i)$ and $X(t_{i+1})$. We have

$$\begin{aligned} f(X(t)) &= f(X(0)) + \sum_{i=0}^{i=n-1} \delta(f'(X(t_i))(\dot{v}(\cdot, t_{i+1}) - \dot{v}(\cdot, t_i))) \\ &\quad + \sum_{i=0}^{i=n-1} f''(X(t_i))\langle DX(t_i), \dot{v}(\cdot, t_{i+1}) - \dot{v}(\cdot, t_i)\rangle_{L^2([0,1])} \\ &\quad + \frac{1}{2}\sum_{i=0}^{i=n-1} f''(\bar{X}(t_i))\delta((X(t_{i+1}) - X(t_i))(\dot{v}(\cdot, t_{i+1}) - \dot{v}(\cdot, t_i))) \\ &\quad + \frac{1}{2}\sum_{i=0}^{i=n-1} f''(\bar{X}(t_i))\langle DX(t_{i+1}) - DX(t_i), \dot{v}(\cdot, t_{i+1}) - \dot{v}(\cdot, t_i)\rangle_{L^2([0,1])} \\ &= f(X(0)) + \sum_{i=0}^{i=n-1} \delta(f'(X(t_i))(\dot{v}(\cdot, t_{i+1}) - \dot{v}(\cdot, t_i))) \\ &\quad + \sum_{i=0}^{i=n-1} (f''(X(t_i)) - f''(\bar{X}(t_i)))\langle DX(t_i), \dot{v}(\cdot, t_{i+1}) - \dot{v}(\cdot, t_i)\rangle_{L^2([0,1])} \\ &\quad + \frac{1}{2}\sum_{i=0}^{i=n-1} f''(\bar{X}(t_i))\langle DX(t_{i+1}), \dot{v}(\cdot, t_{i+1}) - \dot{v}(\cdot, t_i)\rangle_{L^2([0,1])} \\ &\quad + \frac{1}{2}\sum_{i=0}^{i=n-1} f''(\bar{X}(t_i))\langle DX(t_i), \dot{v}(\cdot, t_{i+1}) - \dot{v}(\cdot, t_i)\rangle_{L^2([0,1])} \\ &\quad + \frac{1}{2}\sum_{i=0}^{i=n-1} f''(\bar{X}(t_i))\delta((X(t_{i+1}) - X(t_i))(\dot{v}(\cdot, t_{i+1}) - \dot{v}(\cdot, t_i))) \\ &= f(X(0)) + J_1 + J_2 + J_3 + J_4 + J_5. \end{aligned}$$

We start by proving that J_2 and J_5 converge to 0 in $L^1(W)$ as $|\pi|$ tends to zero. We have, using the a.s. continuity of the trajectories of $(X(t))_{t\in[0,1]}$:

$$E[|J_2|] = E\left[\left|\sum_{i=0}^{i=n-1} (f''(X(t_i)) - f''(\bar{X}(t_i)))\int_0^1 D_\alpha X(t_i)(\dot{v}(\alpha, t_{i+1}) - \dot{v}(\alpha, t_i))d\alpha\right|\right]$$

$$
\begin{aligned}
&\leq E\left[\sup_{|a-b|<|\pi|}|f''(X(a))-f''(X(b))|\right.\\
&\quad\left.\times\sum_{i=0}^{i=n-1}\int_0^1|D_\alpha X(t_i)(\dot{v}(\alpha,t_{i+1})-\dot{v}(\alpha,t_i))|d\alpha\right]\\
&\leq \left\|\sup_{|a-b|<|\pi|}|f''(X(a))-f''(X(b))|\right\|_{L^2(W)}\\
&\quad\times\sum_{i=0}^{i=n-1}\left\|\int_0^1|D_\alpha X(t_i)(\dot{v}(\alpha,t_{i+1})-\dot{v}(\alpha,t_i))|d\alpha\right\|_{L^2(W)}\\
&\leq \left\|\sup_{|a-b|<|\pi|}|f''(X(a))-f''(X(b))|\right\|_{L^2(W)}\\
&\quad\times\sup_{t\in[0,1]}\|DX(t_i)\|_{L^4(W,L^\infty([0,1]))}\sum_{i=0}^{i=n-1}\|\dot{v}(\cdot,t_{i+1})-\dot{v}(\cdot,t_i)\|_{L^4(W,L^1([0,1]))}\\
&\leq C_v\left\|\sup_{|a-b|<|\pi|}|f''(X(a))-f''(X(b))|\right\|_{L^2(W)}\sup_{t\in[0,1]}\|DX(t)\|_{L^4(W,L^\infty([0,1]))},
\end{aligned}
$$

hence J_2 converges to 0 in $L^1(W)$. Concerning J_5 we have from Lemma 2:

$$
\begin{aligned}
\|J_5\|^2_{L^2(W)} &= \frac{1}{4}\left\|\sum_{i=0}^{i=n-1}f''(\bar{X}(t_i))\delta^X((X(t_{i+1})-X(t_i))1_{]t_i,t_{i+1}]})\right\|^2_{L^2(W)}\\
&\leq \frac{1}{4}\|f''\|^2_\infty\sum_{i,j=0}^{i=n-1}\left|E\left[\delta^X((X(t_{i+1})-X(t_i))1_{]t_i,t_{i+1}]})\delta^X((X(t_{j+1})-X(t_j))1_{]t_j,t_{j+1}]})\right]\right|\\
&\leq C_v^2\|f''\|^2_\infty\sum_{i=0}^{i=n-1}\sum_{j=0}^{j=i-1}\|X(t_{i+1})-X(t_i)\|_{1,4}\|X(t_{j+1})-X(t_j)\|_{1,4}(t_{i+1}-t_i)(t_{j+1}-t_j)\\
&\quad+C_v^2\|f''\|^2_\infty\sum_{i=0}^{i=n-1}\|X(t_{i+1})-X(t_i)\|^2_{1,4}(t_{i+1}-t_i)\\
&\leq C_v^2\|f''\|^2_\infty\sup_{|s-t|<|\pi|}\|X(s)-X(t)\|^2_{1,4},
\end{aligned}
$$

from Lemma 1, hence the convergence of J_5 to 0. Next we show that J_3 converges to

$$
\frac{1}{2}\int_0^t f''(X(s))\int_0^1 D_\alpha^- X(s)\hat{v}(d\alpha,s)ds.
$$

We have

$$
\begin{aligned}
&\sum_{i=0}^{i=n-1}\int_{t_i}^{t_{i+1}}\int_0^1|f''(X(t_i))D_\alpha X(t_i)-f''(X(s))D_\alpha^- X(s)|\hat{v}(d\alpha,s)ds\\
&\quad\leq \sum_{i=0}^{i=n-1}\int_{t_i}^{t_{i+1}}\int_0^1|f''(X(t_i))(D_\alpha X(t_i)-D_\alpha^- X(s))|\hat{v}(d\alpha,s)ds
\end{aligned}
$$

$$
\begin{aligned}
& + \sum_{i=0}^{i=n-1} \int_{t_i}^{t_{i+1}} \int_0^1 |D_\alpha^- X(s)(f''(X(t_i)) - f''(X(s)))| \hat{v}(d\alpha, s) ds \\
\leq \;& \sum_{i=0}^{i=n-1} |f''(X(t_i))| \int_{t_i}^{t_{i+1}} \int_0^1 \sup_{\{a<s,\ |s-a|<|\pi|\}} |D_\alpha X(a) - D_\alpha^- X(s)| \hat{v}(d\alpha, s) ds \\
& + \sup_{|a-b|<|\pi|} |f''(X(a)) - f''(X(b))| \int_0^1 \int_0^1 |D_\alpha^- X(s)| \hat{v}(d\alpha, s) ds \\
\leq \;& \|f''\|_\infty \int_0^1 \sup_{\{a<s,\ |a-s|<|\pi|\}} \int_0^1 |D_\alpha X(a) - D_\alpha^- X(s)| \hat{v}(d\alpha, s) ds \\
& + \sup_{|a-b|<|\pi|} |f''(X(a)) - f''(X(b))| \int_0^1 \int_0^1 |D_\alpha^- X(s)| \hat{v}(d\alpha, s) ds.
\end{aligned}
$$

The convergence of J_4 to $\frac{1}{2}\int_0^t \int_0^1 f''(X(s)) D_\alpha^+ X(s) \hat{v}(d\alpha, s) ds$ in $L^2(W)$ is proved similarly. Concerning J_1, we will show the convergence of

$$
\sum_{i=0}^{i=n-1} (f'(X(t_i)) - f'(X(\cdot))) 1_{]t_i, t_{i+1}]}(\cdot)
$$

to 0 in $L^2([0,1], I\!D_{1,4})$. The convergence in $L^2([0,1], L^4(W))$ clearly holds. Regarding the convergence in $L^2([0,1], I\!D_{1,4})$ we have

$$
\begin{aligned}
& \int_0^1 \left| \sum_{i=0}^{i=n-1} D_\alpha (f'(X(t_i)) - f'(X(t))) 1_{]t_i, t_{i+1}]}(t) \right|^2 d\alpha \\
= \;& \int_0^1 \left| \sum_{i=0}^{i=n-1} (f''(X(t_i)) D_\alpha X(t_i) - f''(X(t)) D_\alpha X(t)) 1_{]t_i, t_{i+1}]}(t) \right|^2 d\alpha \\
\leq \;& 2 \int_0^1 \left| \sum_{i=0}^{i=n-1} (f''(X(t_i)) - f''(X(t))) D_\alpha X(t_i) 1_{]t_i, t_{i+1}]}(t) \right|^2 d\alpha \\
& + 2 \int_0^1 \left| \sum_{i=0}^{i=n-1} f''(X(t)) (D_\alpha X(t_i) - D_\alpha X(t)) 1_{]t_i, t_{i+1}]}(t) \right|^2 d\alpha \\
\leq \;& 2 \sup_{|a-b|<|\pi|} |f''(X(a)) - f''(X(b))|^2 \left(\sum_{i=0}^{i=n-1} \|DX(t_i)\|_{L^2([0,1])} 1_{]t_i, t_{i+1}]}(t) \right)^2 \\
& + 2 \sup_{a \in [0,1]} |f''(X(a))|^2 \left(\sum_{i=0}^{i=n-1} \|DX(t_i) - DX(t)\|_{L^2([0,1])} 1_{]t_i, t_{i+1}]}(t) \right)^2 .
\end{aligned}
$$

Hence

$$
E\left[\int_0^1 \int_0^1 \left| \sum_{i=0}^{i=n-1} D_\alpha (f'(X(t_i)) - f'(X(t))) 1_{]t_i, t_{i+1}]}(t) \right|^2 d\alpha dr \right]
$$

$$\leq 2E\left[\sup_{|a-b|<|\pi|}|f''(X(a))-f''(X(b))|^2\sum_{i=0}^{i=n-1}\|DX(t_i)\|^2_{L^2([0,1])}(t_{i+1}-t_i)\right.$$
$$\left.+2\sup_{a\in[0,1]}|f''(X(a))|^2\sum_{i=0}^{i=n-1}\|DX(t_i)-DX(t)\|^2_{L^2([0,1])}(t_{i+1}-t_i)\right]$$
$$\leq 2\left\|\sup_{|a-b|<|\pi|}|f''(X(a))-f''(X(b))|^2\right\|_{L^2(W)}\sup_{t\in[0,1]}\|DX(t)\|^2_{L^4(W,L^2([0,1]))}$$
$$+2\left\|\sup_{a\in[0,1]}|f''(X(a))|^2\right\|\sup_{|a-b|<|\pi|}\|DX(a)-DX(b)\|^2_{L^4(W,L^2([0,1]))}.$$

Both terms converge to zero in $L^2([0,1]\times[0,1],L^2(W))$ since $\sup_{s\in[0,1]}\|X(s)\|_{1,4}<\infty$ and $(X(t))_{t\in[0,1]}$ is continuous in $I\!D_{1,4}$.

□

In the Brownian case, $\nabla^B_s B(s)=\nabla_s B(s)$, $s\in[0,1]$, from (10), hence we obtain the classical Itô formula for Brownian motion:

$$\begin{aligned} f(B(t)) &= f(0)+\delta^B(1_{[0,t]}(\cdot)f'(B(\cdot)))+\frac{1}{2}\int_0^t f''(B(s))\nabla^B_s B(s)ds \\ &= f(0)+\int_0^t f'(B(s))dB(s)+\frac{1}{2}\int_0^t f''(B(s))ds. \end{aligned}$$

Th. 2 can be rewritten using the quadratic variation of $(X(t))_{t\in[0,1]}$ and pathwise integrals as in the following Corollary which is a consequence of Th. 1.

Corollary 1 *Let $v\in\mathcal{V}$. We assume that $(X(t))_{t\in[0,1]}\in I\!L_v$ has a continuous version and is continuous in $I\!D_{1,4}$ with $\sup_{t\in[0,1]}\|DX(t)\|_{L^4(W,L^\infty([0,1]))}<\infty$, and $\|\dot v(\cdot,t)-\dot v(\cdot,s)\|_{L^4(W,L^1([0,1]))}\leq C_v(t-s)$. Then for $f\in\mathcal{C}^2_b(\mathbb{R})$,*

$$\begin{aligned} f(X(t)) &= f(X(0))+\delta^X(f'(X(\cdot))1_{[0,t]}(\cdot)) \\ &\quad +\frac{1}{2}\int_0^t f''(X(s))d[X,X](s)+\int_0^t f''(X(s))D^{X-}_s X(s)ds. \end{aligned}$$

As a corollary we obtain the pathwise extensions of the Itô formula, cf. [13], which are linked by Cor. 1 to the generalized Skorohod integral δ^X:

$$\begin{aligned} f(X(t)) &= f(X(0))+\int_0^t f'(X(s))d^\circ X(s), \\ f(X(t)) &= f(X(0))+\int_0^t f'(X(s))d^- X(s)+\frac{1}{2}\int_0^t f''(X(s))d[X,X](s), \\ f(X(t)) &= f(X(0))+\int_0^t f'(X(s))d^+ X(s)-\frac{1}{2}\int_0^t f''(X(s))d[X,X](s). \end{aligned}$$

We now state a multidimensional version of our formula.

Theorem 3 *Let $n \geq 1$. Let $v^1, \ldots, v^n \in \mathcal{V}$ and let $X^i \in \cap_{j=1}^{j=n} \mathbb{L}_{v^j}$ be a v_i-process, continuous in $\mathbb{D}_{1,4}$ and having a continuous version, $i = 1, \ldots, n$. Then for $f \in \mathcal{C}_b^2(\mathbb{R}^n)$,*

$$\begin{aligned} f(X^1(t), \ldots, X^n(t)) &= f(X_0^1, \ldots, X_0^n) + \sum_{i=1}^{i=n} \delta^{X^i}(\partial_i f(X^1(\cdot), \ldots, X^n(\cdot)) 1_{[0,t]}(\cdot)) \\ &+ \frac{1}{2} \sum_{i,j=1}^{n} \int_0^t \partial_i \partial_j f(X^1(s), \ldots, X^n(s)) d[X^i, X^j](s) \\ &+ \sum_{i,j=1}^{n} \int_0^t \partial_i \partial_j f(X^1(s), \ldots, X^n(s)) D_s^{X^i -} X^j(s) ds. \end{aligned} \quad (15)$$

Proof: We do the computation in the case $n = 2$, the limiting arguments being the same as in the proof of Th. 2. We have

$$\begin{aligned} & f(X^1(t), X^2(t)) \\ &= f(X^1(t_0), X^2(t_0)) + (f(X^1(t), X^2(t_0)) - f(X^1(t_0), X^2(t_0))) \\ &\quad + (f(X^1(t), X^2(t)) - f(X^1(t), X^2(t_0))) \\ &= f(X^1(t_0), X^2(t_0)) + \partial_1 f(X^1(t_0), X^2(t_0))(X^1(t) - X^1(t_0)) \\ &\quad + \partial_2 f(X^1(t), X^2(t_0))(X^2(t) - X^2(t_0)) \\ &\quad + \frac{1}{2} \partial_1 \partial_1 f(\bar{X}_{t_0}^1, X^2(t_0))(X^1(t) - X^1(t_0))^2 + \frac{1}{2} \partial_2 \partial_2 f(X^1(t), \bar{X}^2(t_0))(X^2(t) - X^2(t_0))^2 \\ &= f(X^1(t_0), X^2(t_0)) + \delta^{X^1}(\partial_1 f(X^1(t_0), X^2(t_0)) 1_{]t_0,t]}) + \delta^{X^2}(\partial_2 f(X^1(t), X^2(t_0)) 1_{]t_0,t]}) \\ &\quad + \frac{1}{2} \partial_1 \partial_1 f(\bar{X}_{t_0}^1, X^2(t_0))(X^1(t) - X^1(t_0))^2 + \frac{1}{2} \partial_2 \partial_2 f(X^1(t), \bar{X}^2(t_0))(X^2(t) - X^2(t_0))^2 \\ &\quad + \int_{t_0}^t D_s^{X^1} \partial_1 f(X^1(t_0), X^2(t_0)) ds + \int_{t_0}^t D_s^{X^2} \partial_2 f(X^1(t), X^2(t_0)) ds \\ &= f(X^1(t_0), X^2(t_0)) + \delta^{X^1}(\partial_1 f(X^1(t_0), X^2(t_0)) 1_{]t_0,t]}) + \delta^{X^2}(\partial_2 f(X^1(t), X^2(t_0)) 1_{]t_0,t]}) \\ &\quad + \frac{1}{2} \partial_1 \partial_1 f(\bar{X}_{t_0}^1, X^2(t_0))(X^1(t) - X^1(t_0))^2 + \frac{1}{2} \partial_2 \partial_2 f(X^1(t), \bar{X}^2(t_0))(X^2(t) - X^2(t_0))^2 \\ &\quad + \int_{t_0}^t \partial_1 \partial_1 f(X^1(t_0), X^2(t_0)) D_s^{X^1} X^1(t_0) ds + \int_{t_0}^t \partial_2 \partial_2 f(X^1(t), X^2(t_0)) D_s^{X^2} X^2(t_0) ds \\ &\quad + \int_{t_0}^t \partial_1 \partial_2 f(X^1(t_0), X^2(t_0)) D_s^{X^1} X^2(t_0) ds + \int_{t_0}^t \partial_2 \partial_1 f(X^1(t), X^2(t_0)) D_s^{X^2} X^1(t) ds, \end{aligned}$$

where $\bar{X}_{t_0}^1 \in [X^1(t_0), X^1(t)]$ and $\bar{X}^2(t_0) \in [X^2(t_0), X^2(t)]$. As $|\pi| \to 0$ we have for general $n \geq 1$

$$f(X^1(t), \ldots, X^n(t)) = f(X_0^1, \ldots, X_0^n) + \sum_{i=1}^{i=n} \delta^{X^i}(\partial_i f(X^1(\cdot), \ldots, X^n(\cdot)) 1_{[0,t]}(\cdot))$$

$$+\frac{1}{2}\sum_{i=1}^{n}\int_0^t \partial_i\partial_i f(X^1(s),\ldots,X^n(s))d[X^i,X^i](s)$$
$$+\sum_{i=1}^{n}\int_0^t \partial_i\partial_i f(X^1(s),\ldots,X^n(s))D_s^{X^i-}X^i(s)ds$$
$$+\sum_{i=2}^{i=n}\sum_{j=1}^{j=i-1}\int_0^t \partial_i\partial_j f(X^1(s),\ldots,X^n(s))D_s^{X^i-}X^j(s)ds$$
$$+\sum_{i=1}^{i=n-1}\sum_{j=i+1}^{j=n}\int_0^t \partial_i\partial_j f(X^1(s),\ldots,X^n(s))D_s^{X^i+}X^j(s)ds.$$

Using Th. 1 and Def. 7:

$$D_t^{X^i+}X^j(t) = d[X^i,X^j](t) + D_t^{X^i-}X^j(t)dt,$$

we obtain

$$\begin{aligned}
f(X^1(t),\ldots,X^n(t)) &= f(X_0^1,\ldots,X_0^n) + \sum_{i=1}^{i=n}\delta^{X^i}(\partial_i f(X^1(\cdot),\ldots,X^n(\cdot))1_{[0,t]}(\cdot)) \\
&+\frac{1}{2}\sum_{i=1}^{n}\int_0^t \partial_i\partial_i f(X^1(s),\ldots,X^n(s))d[X^i,X^i](s) \\
&+\sum_{i,j=1}^{n}\int_0^t \partial_i\partial_j f(X^1(s),\ldots,X^n(s))D_s^{X^i-}X^j(s)ds \\
&+\sum_{i=1}^{i=n-1}\sum_{j=i+1}^{j=n}\int_0^t \partial_i\partial_j f(X^1(s),\ldots,X^n(s))d[X^i,X^j](s)ds,
\end{aligned}$$

hence (15), from the symmetry of the bracket $[X^i,X^j](t)$.

□

We list below different versions of this formula using pathwise integrals and Prop. 4.

$$\begin{aligned}
f(X^1(t),\ldots,X^n(t)) &= f(X_0^1,\ldots,X_0^n) + \sum_{i=1}^{i=n}\delta^{X^i}(\partial_i f(X^1(\cdot),\ldots,X^n(\cdot))1_{[0,t]}(\cdot)) \\
&+\frac{1}{2}\sum_{i,j=1}^{n}\int_0^t \partial_i\partial_j f(X^1(s),\ldots,X^n(s))\nabla^{X^i}X^j ds,
\end{aligned}$$

$$\begin{aligned}
f(X^1(t),\ldots,X^n(t)) &= f(X_0^1,\ldots,X_0^n) + \sum_{i=1}^{i=n}\int_0^t \partial_i f(X^1(s),\ldots,X^n(s))d^-X^i(s) \\
&+\frac{1}{2}\sum_{i,j=1}^{n}\int_0^t \partial_i\partial_j f(X^1(s),\ldots,X^n(s))d[X^i,X^j](s),
\end{aligned}$$

$$f(X^1(t),\dots,X^n(t)) = f(X_0^1,\dots,X_0^n) + \sum_{i=1}^{i=n} \int_0^t \partial_i f(X^1(s),\dots,X^n(s)) d^+ X^i(s)$$
$$-\frac{1}{2}\sum_{i,j=1}^{n} \int_0^t \partial_i\partial_j f(X^1(s),\dots,X^n(s)) d[X^i,X^j](s),$$

$$f(X^1(t),\dots,X^n(t)) = f(X_0^1,\dots,X_0^n) + \sum_{i=1}^{i=n} \int_0^t \partial_i f(X^1(s),\dots,X^n(s)) d^\circ X^i(s).$$

Finally we show how the Itô formula for Skorohod integral processes of [10], [15], can be written as a particular case of our result. Let us first consider a Skorohod integral process

$$X(t) = \delta(h1_{[0,t]}), \quad t \in [0,1],$$

i.e. $v(s,t) = \int_0^{s\wedge t} h_\alpha d\alpha$, $s,t \in [0,1]$, with $h \in \mathbb{L}_{1,4}$. In this case we have $\nabla_s^X = h(s)\nabla_s$, $D_s^{X-} = h(s)D_s^-$, $D_s^{X+} = h(s)D_s^+$, and we obtain the Itô formula of [10] for Skorohod integral processes:

$$f(X(t)) = f(X(0)) + \delta(f'(X(\cdot))h(\cdot)1_{[0,t]}(\cdot)) + \frac{1}{2}\int_0^t h_s f''(X(s))\nabla_s X(s) ds.$$

The continuity condition on $(X(t))_{t\in[0,1]}$ in $\mathbb{D}_{1,4}$ is satisfied provided

$$\sup_{\alpha\in[0,1]} \|h_\alpha\|_{L^4(W)} + \int_0^1 \sup_{\alpha\in[0,1]} \|D_\beta h_\alpha\|^2_{L^4(W)} d\beta < \infty. \tag{16}$$

We then consider an additional absolutely continuous drift.

Proposition 5 *Let $h \in \mathbb{L}_{1,4}$ satisfy (16) and let $a(\cdot,t) \in \mathbb{L}_{1,4}$, $t \in [0,1]$. Let $X(t) = \delta(h1_{[0,t]})$ and $Y(t) = \int_0^t V(s)ds$ with $V(s) = \delta(a(\cdot,s))$, $s \in [0,1]$. Then*

$$f(X(t),Y(t)) = f(0,0) + \delta(\partial_1 f(X(\cdot),Y(\cdot))1_{[0,t]}(\cdot)) + \int_0^t \partial_2 f(X(s),Y(s))V(s)ds$$
$$+\frac{1}{2}\int_0^t \partial_1\partial_1 f(X(s),Y(s))h^2(s)ds$$
$$+\int_0^t \partial_1\partial_2 f(X(s),Y(s))h(s)D_sY(s)ds$$
$$+\int_0^t h(s)\partial_1\partial_1 f(X(s),Y(s))\delta(D_s h1_{[0,s]})ds,$$

$f \in \mathcal{C}_b^2(\mathbb{R})$.

Proof: We have $\delta^X(u) = \delta(uh)$, $\delta^Y(u) = \delta\left(\int_0^1 u(s)a(\cdot,s)ds\right)$, and $D_t^{X+} = h(t)D_t^+$, hence

$$\begin{aligned}
f(X(t),Y(t)) &= f(0,0) + \delta(\partial_1 f(X(\cdot),Y(\cdot))1_{[0,t]}(\cdot)) + \delta^Y(1_{[0,t]}\partial_2 f(X(s),Y(s))) \\
&\quad + \frac{1}{2}\int_0^t \partial_1\partial_1 f(X(s),Y(s))h^2(s)ds + \sum_{i,j=1}\int_0^t \partial_i\partial_j f(X(s),Y(s))D_s^{X^i+}X^j(s)ds \\
&= f(0,0) + \delta(\partial_1 f(X(s),Y(s))1_{[0,t]}) + \int_0^t \partial_2 f(X(s),Y(s))V(s)ds \\
&\quad + \frac{1}{2}\int_0^t \partial_1\partial_1 f(X(s),Y(s))h^2(s)ds \\
&\quad + \int_0^t \partial_1\partial_2 f(X(s),Y(s))D_s^{X-}Y(s)ds + \int_0^t \partial_1\partial_1 f(X(s),Y(s))D_s^{X-}X(s)ds \\
&= f(0,0) + \delta(\partial_1 f(X(\cdot),Y(\cdot))1_{[0,t]}(\cdot)) + \int_0^t \partial_2 f(X(s),Y(s))V(s)ds \\
&\quad + \frac{1}{2}\int_0^t \partial_1\partial_1 f(X(s),Y(s))h^2(s)ds \\
&\quad + \int_0^t \partial_1\partial_2 f(X(s),Y(s))h(s)D_sY(s)ds + \int_0^t h(s)\partial_1\partial_1 f(X(s),Y(s))\delta(D_sh1_{[0,s]})ds
\end{aligned}$$

since $D_s^{X-}X(s) = h(s)\delta(D_sh1_{[0,s]})$ and $D_s^{X-}Y(s) = h(s)D_sY(s)$, $s \in [0,1]$.

□

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