Quantum stochastic calculus applied to path spaces over Lie groups

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Abstract

Quantum stochastic calculus is applied to the proof of Skorokhod and Weitzenböck type identities for functionals of a Lie group-valued Brownian motion. In contrast to the case of $\mathbb{R}^d$-valued paths, the computations use all three basic quantum stochastic differentials.

Key words: Quantum stochastic calculus, Lie group-valued Brownian motion.
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1 Introduction

Quantum stochastic calculus [4], [7], and anticipating stochastic calculus [6] have been linked in [5], where the Skorokhod isometry is formulated and proved using the annihilation and creation processes. On the other hand, a Skorokhod type isometry has been constructed in [3] for functionals on the path space over Lie groups. This isometry yields in particular a Weitzenböck type identity in infinite-dimensional geometry. We refer to [1], [2] for the case of path spaces over Riemannian manifolds.

We will prove such a Skorokhod type isometry formula on the path space over a Lie group, using the conservation operator which is usually linked to stochastic calculus for jump processes. In this way we will recover the Weitzenböck formula established in [3]. This provides a link between the non-commutative settings of Lie groups and of quantum stochastic calculus.

This paper is organised as follows. In Sect. 2 we recall how the Skorokhod isometry can be derived from quantum stochastic calculus in the case of $\mathbb{R}^d$-valued Brownian motion. In Sect. 3 the gradient and divergence operators of stochastic analysis on path groups are introduced, and the Skorokhod type isometry of [3] is stated. The proof of this isometry is given in Sect. 4 via quantum stochastic calculus on the path space over a Lie group. Sect. 4 ends with a remark on the links between vanishing of torsion and quantum stochastic calculus.

2 Skorokhod isometry on the path space over $\mathbb{R}^d$

In this section we recall how the Skorokhod isometry is linked to quantum stochastic calculus. Let $(B(t))_{t\in\mathbb{R}_+}$ denote an $\mathbb{R}^d$-valued Brownian motion on the Wiener space W with Wiener measure μ. Let

$$\mathcal{S} = \{G = g(B(t_1), \dots, B(t_n)) \quad : \quad g \in \mathcal{C}_b^\infty((\mathbb{R}^d)^n),\ t_1, \dots, t_n > 0\},$$

and

$$\mathcal{U} = \left\{ \sum_{i=1}^n u_i G_i \quad : \quad G_i \in \mathcal{S},\ u_i \in L^2(\mathbb{R}_+; \mathbb{R}^d),\ i = 1, \dots, n,\ n \geq 1 \right\}.$$

Let $D : L^2(W) \to L^2(W \times \mathbb{R}_+; \mathbb{R}^d)$ be the closed operator given by

$$D_t G = \sum_{i=1}^n 1_{[0,t_i]}(t) \nabla_i g(B(t_1), , \dots, B(t_n)), \quad t \geq 0,$$

for $G = g(B(t_1), \dots, B(t_n)) \in \mathcal{S}$, and let δ denote its adjoint. Thus

$$E[G\delta(u)] = E[\langle DG, u \rangle], \quad G \in \mathrm{Dom}(D),\ u \in \mathrm{Dom}(\delta),$$

where $\mathrm{Dom}(D)$ and $\mathrm{Dom}(\delta)$ are the respective domains of D and δ. We let $\langle \cdot, \cdot \rangle$ denote the scalar product in both $L^2(\mathbb{R}_+; \mathbb{R}^d)$ and $L^2(W \times \mathbb{R}_+; \mathbb{R}^d)$, and let $(\cdot, \cdot)$ denote the scalar product on $\mathbb{R}^d$. Since D is a derivation, we have the divergence relation

$$\delta(uG) = G\delta(u) - \langle u, DG \rangle, \quad G \in \mathcal{S},\ u \in \mathcal{U}.$$

Given $u \in L^2(\mathbb{R}_+;\mathbb{R}^d)$, the quantum stochastic differentials $da^-(t)$ and $da^+(t)$ are defined from

$$a_u^- G = \int_0^\infty u(t) da^-(t) G = \langle DG, u \rangle, \quad G \in \mathrm{Dom}(D), \tag{2.1}$$

and

$$a_v^+ G = \int_0^\infty v(t) da^+(t) G = \delta(vG), \quad G \in \mathrm{Dom}(D). \tag{2.2}$$

They satisfy the Itô table

$\cdot$	dt	$da_v^-(t)$	$da_v^+(t)$
dt	0	0	0
$da_u^+(t)$	0	0	0
$da_u^-(t)$	0	0	$(u(t), v(t))dt$

with $da_u^-(t) = u(t)da^-(t)$ and $da_v^+(t) = v(t)da^+(t)$. Using the Itô table we have

$$a_u^- a_v^+ = \int_0^\infty \int_0^t da_v^+(s) da_u^-(t) + \int_0^\infty \int_0^t da_u^-(s) da_v^+(t) + \int_0^\infty u(t)v(t)dt$$

and

$$a_v^+ a_u^- = \int_0^\infty \int_0^t da_u^-(s) da_v^+(t) + \int_0^\infty \int_0^t da_v^+(s) da_u^-(t),$$

which implies the canonical commutation relation

$$a_u^- a_v^+ = \langle u, v \rangle + a_v^+ a_u^-. \tag{2.3}$$

This relation and its proof can be abbreviated as

$$d[a_u^-, a_v^+](t) = [da_u^-(t), da_v^+(t)] = (u(t), v(t))dt,$$

where $[\cdot,\cdot]$ denotes the commutator of operators. Relation (2.3) is easily translated back to the Skorokhod isometry:

$$\begin{aligned} E\left[\delta(uF)\delta(vG)\right] &= \langle a_u^+ F, a_v^+ G \rangle = \langle F, a_u^- a_v^+ G \rangle \\ &= \langle u \otimes F, v \otimes G \rangle + \langle F, a_v^+ a_u^- G \rangle \\ &= \langle u \otimes F, v \otimes G \rangle + \langle a_v^- F, a_u^- G \rangle \\ &= E[\langle u, v \rangle FG] + E\left[\int_0^\infty \int_0^\infty (u(t) \otimes D_s F, D_t G \otimes v(s)) ds dt\right], \end{aligned}$$

$F, G \in \mathcal{S}$, $u, v \in L^2(\mathbb{R}_+; \mathbb{R}^d)$, which implies

$$E[\delta(h)^2] = E[\|h\|^2_{L^2(\mathbb{R}_+;\mathbb{R}^d)}] + E\left[\int_0^\infty \int_0^\infty (D_s h(t), (D_t h(s))^*) ds dt\right], \quad h \in \mathcal{U},$$

where $(D_t h(s))^*$ denotes the adjoint of $D_t h(s)$ in $\mathbb{R}^d \otimes \mathbb{R}^d$. In this note we carry over this method to the proof of a Skorokhod type isometry on the path space over a Lie group, using the calculus of all the annihilation, creation and gauge (or conservation) process and the Itô table

$\cdot$	dt	$da_v^-(t)$	$da_v^+(t)$	$q(t)d\Lambda(t)$
dt	0	0	0	0
$da_u^+(t)$	0	0	0	0
$da_u^-(t)$	0	0	$(u(t), v(t))dt$	$q^*(t)u(t)da^-(t)$
$p(t)d\Lambda(t)$	0	0	$p(t)v(t)da^+(t)$	$p(t)q(t)d\Lambda(t)$

where $(q(t))_{t\in\mathbb{R}_+}$ is a (bounded) measurable operator process on $\mathbb{R}^d$ and $q(t)d\Lambda(t)$ is defined from

$$\int_0^\infty q(t)d\Lambda(t)F = \delta(q(\cdot)D.F),$$

for $F \in \mathrm{Dom}(D)$ such that $(q(t)D_t F)_{t\in\mathbb{R}_+} \in \mathrm{Dom}(\delta)$.

3 Skorokhod isometry on the path space over a Lie group

Let G be a compact connected d-dimensional Lie group with associated Lie algebra $\mathcal{G}$ identified to $\mathbb{R}^d$ and equipped with an Ad-invariant scalar product on $\mathbb{R}^d \simeq \mathcal{G}$, also denoted by $(\cdot,\cdot)$. The commutator in $\mathcal{G}$ is denoted by $[\cdot,\cdot]$. Let $\mathrm{ad}(u)v = [u, v]$, $u, v \in \mathcal{G}$, with $\mathrm{Ad}\, e^u = e^{\mathrm{ad} u}$, $u \in \mathcal{G}$.

The Brownian motion $(\gamma(t))_{t\in\mathbb{R}_+}$ on G is constructed from $(B(t))_{t\in\mathbb{R}_+}$ via the Stratonovich differential equation

$$\begin{cases} d\gamma(t) = \gamma(t) \odot dB(t) \\ \gamma(0) = \mathrm{e}, \end{cases}$$

where e is the identity element in G. Let $\mathbb{P}(\mathsf{G})$ denote the space of continuous G-valued paths starting at e, with the image measure of the Wiener measure by $I : (B(t))_{t\in\mathbb{R}_+} \mapsto (\gamma(t))_{t\in\mathbb{R}_+}$. Let

$$\tilde{\mathcal{S}} = \{F = f(\gamma(t_1), \ldots, \gamma(t_n)) \quad : \quad f \in \mathcal{C}_b^\infty(\mathsf{G}^n)\},$$

and

$$\tilde{\mathcal{U}} = \left\{ \sum_{i=1}^n u_i F_i \quad : \quad F_i \in \tilde{\mathcal{S}}, \ u_i \in L^2(\mathbb{R}_+; \mathcal{G}), \ i = 1, \ldots, n, \ n \geq 1 \right\}.$$

Definition 3.1 *For* $F = f(\gamma(t_1), \ldots, \gamma(t_n)) \in \tilde{\mathcal{S}}$, $f \in \mathcal{C}_b^\infty(\mathsf{G}^n)$, *we let* $\tilde{D}F \in L^2(W \times \mathbb{R}_+; \mathcal{G})$ *be defined as*

$$\langle \tilde{D}F, v \rangle = \frac{d}{d\varepsilon} f \left(\gamma(t_1) e^{\varepsilon \int_0^{t_1} v(s)ds}, \ldots, \gamma(t_n) e^{\varepsilon \int_0^{t_n} v(s)ds} \right)_{|\varepsilon=0}, \quad v \in L^2(\mathbb{R}_+, \mathcal{G}).$$

In other terms, $\tilde{D}$ acts as a natural gradient on the cylindrical functionals on $\mathbb{P}(\mathsf{G})$ with

$$\tilde{D}_t F = \sum_{i=1}^n \partial_i f(\gamma(t_1), \ldots, \gamma(t_n)) 1_{[0,t_i]}(t), \quad t \geq 0.$$

Let $\tilde{\delta}$ denote the adjoint of $\tilde{D}$, that satisfies

$$E[F\tilde{\delta}(v)] = E[\langle \tilde{D}F, v \rangle], \quad F \in \tilde{\mathcal{S}}, \ v \in L^2(\mathbb{R}_+; \mathcal{G}), \tag{3.1}$$

(that $\tilde{\delta}$ exists and satisfies (3.1) can be seen as a consequence of Lemma 4.1 below). Given $v \in L^2(\mathbb{R}_+; \mathcal{G})$ we define

$$q_v(t) = \int_0^t \mathrm{ad}(v(s))ds, \quad t > 0.$$

Definition 3.2 *([3]) The covariant derivative of* $u \in L^2(\mathbb{R}_+; \mathcal{G})$ *in the direction* $v \in L^2(\mathbb{R}_+; \mathcal{G})$ *is the element* $\nabla_v u$ *of* $L^2(\mathbb{R}_+; \mathcal{G})$ *defined as follows:*

$$\nabla_v u(t) = q_v(t) u(t) = \int_0^t \mathrm{ad}(v(s)) u(t) ds, \quad t > 0.$$

In the following we will distinguish between ∇_v which acts on $L^2(\mathbb{R}_+; \mathcal{G})$ and $q_v(t)$ which acts on $\mathcal{G}$ and is needed in the quantum stochastic integrals to follow.
The operators $q_v(t)$ and ∇_v are antisymmetric on $\mathcal{G}$ and $L^2(\mathbb{R}_+; \mathcal{G})$ respectively, because $(\cdot, \cdot)$ is Ad-invariant.
The Skorokhod isometry on the path space over G holds for the covariant derivative ∇. The definition of ∇_v extends to $\tilde{\mathcal{U}}$, as

$$\nabla_v(uF)(t) = u(t) \langle \tilde{D}F, v \rangle + F q_v(t) u(t), \quad t > 0, \quad F \in \tilde{\mathcal{S}}, \ u \in L^2(\mathbb{R}_+; \mathcal{G}).$$

Let $\nabla_s(uF)(t) \in \mathcal{G} \otimes \mathcal{G}$ be defined as

$$\langle e_i \otimes e_j, \nabla_s(uF)(t)\rangle = \langle u(t), e_j\rangle\langle e_i, \tilde{D}_s F\rangle + 1_{[0,t]}(s) F \langle e_j, \mathrm{ad}(e_i) u(t)\rangle, \quad i, j = 1, \dots, d.$$

In this context the following isometry has been proved in [3].

Theorem 3.3 *([3]) We have for $h \in \tilde{\mathcal{U}}$:*

$$E[\tilde{\delta}(h)^2] = E[\|h\|^2_{L^2(\mathbb{R}_+;\mathcal{G})}] + E\left[\int_0^\infty \int_0^\infty (\nabla_s h(t), (\nabla_t h(s))^*) dt ds\right]. \tag{3.2}$$

The proof in [3] is clear and self-contained, however its calculations involve a number of coincidences which are apparently not related to each other. In this paper we provide a short proof which offers some explanation for these. Let the analogs of (2.1)-(2.2) be defined as

$$\tilde{a}_u^- F = \int_0^\infty d\tilde{a}_u^-(t) F = \langle \tilde{D} F, u\rangle, \quad F \in \tilde{\mathcal{S}},$$

and

$$\tilde{a}_u^+ F = \int_0^\infty d\tilde{a}_u^+(t) F = \tilde{\delta}(uF), \quad F \in \tilde{\mathcal{S}},$$

$u \in L^2(\mathbb{R}_+; \mathcal{G})$, i.e. $d\tilde{a}_u^-(t) F = (u(t), \tilde{D}_t F) dt$.
Our proof relies on

a) the relation

$$d\tilde{a}_u^-(t) = da_u^-(t) + q_u(t) d\Lambda(t), \quad t > 0, \quad u \in L^2(\mathbb{R}_+; \mathcal{G}), \tag{3.3}$$

see Lemma 4.1 below,

b) the commutation relation between $\tilde{a}_u^-$ and $\tilde{a}_v^+$ which is analogous to (2.3) and is proved via quantum stochastic calculus in the following lemma.

Lemma 3.4 *We have on $\tilde{\mathcal{S}}$:*

$$\tilde{a}_u^- \tilde{a}_v^+ - \tilde{a}_v^+ \tilde{a}_u^- = \langle u, v\rangle + \tilde{a}^-_{\nabla_v u} + \tilde{a}^+_{\nabla_u v}, \tag{3.4}$$

$u, v \in L^2(\mathbb{R}_+; \mathcal{G})$.

Proof. Using the quantum Itô table, Relation (3.3), Lemma 4.2 below and the fact that $q_v^*(t) = -q_v(t)$, we have

$$
\begin{aligned}
& d\tilde{a}_u^-(t) \cdot d\tilde{a}_v^+(t) - d\tilde{a}_v^+(t) \cdot d\tilde{a}_u^-(t) \\
&= \big(da_u^-(t) + q_u(t)d\Lambda(t)\big) \cdot \big(da_v^+(t) - q_v(t)d\Lambda(t)\big) \\
&\quad - \big(da_v^+(t) - q_v(t)d\Lambda(t)\big) \cdot \big(da_u^-(t) + q_u(t)d\Lambda(t)\big) \\
&= (u(t), v(t))dt + q_v(t)u(t)da^-(t) - q_u(t)q_v(t)d\Lambda(t) \\
&\quad + q_u(t)v(t)da^+(t) + q_v(t)q_u(t)d\Lambda(t) \\
&= (u(t), v(t))dt + \nabla_v u(t)da^-(t) + q_{\nabla_v u}(t)d\Lambda(t) + \nabla_u v(t)da^+(t) - q_{\nabla_u v}(t)d\Lambda(t) \\
&= (u(t), v(t))dt + d\tilde{a}^-_{\nabla_v u}(t) + d\tilde{a}^+_{\nabla_u v}(t).
\end{aligned}
$$

□

This commutation relation can be interpreted to give a proof of the Skorokhod isometry (3.2):

Proof of Th. 3.3. Applying Lemma 3.4 we have

$$
\begin{aligned}
& E\left[\tilde{\delta}(uF)\tilde{\delta}(vG)\right] = \langle \tilde{a}_u^+ F, \tilde{a}_v^+ G\rangle = \langle F, \tilde{a}_u^- \tilde{a}_v^+ G\rangle \\
&= \langle u \otimes F, v \otimes G\rangle + \langle \tilde{a}_v^- F, \tilde{a}_u^- G\rangle + \langle F\nabla_v u, \tilde{D}G\rangle + \langle \tilde{D}F, G\nabla_u v\rangle \\
&= E[\langle u, v\rangle FG] + E\left[\int_0^\infty (F\nabla_s u(t) + \tilde{D}_s F \otimes u(t), G(\nabla_t v(s))^* + v(s) \otimes \tilde{D}_t G) ds dt\right] \\
&= E[\langle u, v\rangle FG] + E\left[\int_0^\infty \langle \nabla_s (uF)(t), (\nabla_t (vG)(s))^*\rangle ds dt\right],
\end{aligned}
$$

$F, G \in \tilde{\mathcal{S}},\ u, v \in L^2(\mathbb{R}_+; \mathcal{G})$. □

We mention that a consequence of Th. 3.3 is the following Weitzenböck type identity, cf. [3], which extends the Shigekawa identity [8] to path spaces over Lie groups:

Theorem 3.5 *([3]) We have for $u \in \tilde{\mathcal{U}}$:*

$$
E[\tilde{\delta}(u)^2] + E\left[\|du\|^2_{L^2(\mathbb{R}_+;\mathcal{G})\wedge L^2(\mathbb{R}_+;\mathcal{G})}\right] = E[\|u\|^2_{L^2(\mathbb{R}_+)}] + E\left[\|\nabla u\|^2_{L^2(\mathbb{R}_+;\mathcal{G})\otimes L^2(\mathbb{R}_+;\mathcal{G})}\right]. \tag{3.5}
$$

The next section is devoted to two lemmas that are used to prove (3.3).

4 Quantum stochastic differentials on path space

The following expression for $\tilde{D}$ using quantum stochastic integrals can be viewed as an intertwining formula between $\tilde{D}$, D and I.

Lemma 4.1 *We have for $v \in L^2(\mathbb{R}_+;\mathcal{G})$:*

$$d\tilde{a}_v^-(t) = da_v^-(t) + q_v(t)d\Lambda(t), \qquad t > 0.$$

Proof. The process $t \mapsto \gamma(t)e^{\int_0^t v(s)ds}$ satisfies the following stochastic differential equation in the Stratonovich sense:

$$\begin{aligned} d\left(\gamma(t)e^{\int_0^t v(s)ds}\right) &= \gamma(t)e^{\int_0^t v(s)ds}\left(\odot \mathrm{Ad}e^{-\int_0^t v(s)ds}dB(t) + v(t)dt\right) \\ &= \gamma(t)e^{\int_0^t v(s)ds}\left(\odot e^{-q_v(t)}dB(t) + v(t)dt\right), \quad t > 0. \end{aligned}$$

Let $I_1(u)$ denote the Wiener integral of $u \in L^2(\mathbb{R}_+;\mathbb{R}^d)$ with respect to $(B(t))_{t\in\mathbb{R}_+}$, and let $G = g\left(I_1(u_1),\ldots,I_1(u_n)\right) \in \mathcal{S}$, and $F = f(\gamma(t_1),\ldots,\gamma(t_n)) \in \tilde{\mathcal{S}}$. Since $\exp(q_v(t)) : \mathcal{G} \longrightarrow \mathcal{G}$ is isometric, we have from the Girsanov theorem:

$$E\left[f\left(\gamma(t_1)e^{\int_0^{t_1} v(s)ds},\ldots,\gamma(t_n)e^{\int_0^{t_n} v(s)ds}\right)G\right] = E\left[Fe^{I_1(v)-\frac{1}{2}\|v\|^2}\Theta_v G\right],$$

where

$$\Theta_v G = g\left(\int_0^\infty u_1(s)e^{q_v(s)}dB(s) - \langle u_1, v\rangle,\ldots,\int_0^\infty u_n(s)e^{q_v(s)}dB(s) - \langle u_n, v\rangle\right).$$

From the derivation property of D and the divergence relation $\delta(vG) = G\delta(v) - \langle v, DG\rangle$ we have

$$\begin{aligned} &\frac{d}{d\varepsilon}\Theta_{\varepsilon v}G_{|\varepsilon=0} = \sum_{i=1}^n \partial_i g\left(I_1(u_1),\ldots,I_1(u_n)\right)\frac{d}{d\varepsilon}\Theta_{\varepsilon v}I_1(u_i)_{|\varepsilon=0} \\ &= \sum_{i=1}^n \partial_i g\left(I_1(u_1),\ldots,I_1(u_n)\right)\left(-\langle v, u_i\rangle + \int_0^\infty q_v^*(s)u_i(s)dB(s)\right) \\ &= -\sum_{i=1}^n \partial_i g\left(I_1(u_1),\ldots,I_1(u_n)\right)\left(\langle v, DI_1(u_i)\rangle + \delta(\nabla_v DI_1(u_i))\right) \\ &= -\langle v, Dg\left(I_1(u_1),\ldots,I_1(u_n)\right)\rangle - \sum_{i=1}^n \partial_i g\left(I_1(u_1),\ldots,I_1(u_n)\right)\delta(\nabla_v DI_1(u_i)) \end{aligned}$$

$$
\begin{aligned}
&= -\langle v, DG\rangle - \sum_{i=1}^n \delta(\partial_i g\,(I_1(u_1),\ldots,I_1(u_n))\,\nabla_v DI_1(u_i)) \\
&\quad -\frac{1}{2}\sum_{i,j=1}^n \partial_j\partial_i g\,(I_1(u_1),\ldots,I_1(u_n))\,(\langle u_j,\nabla_v u_i\rangle + \langle u_i,\nabla_v u_j\rangle) \\
&= -\langle v, DG\rangle - \delta(\nabla_v DG),
\end{aligned}
$$

since ∇_v is antisymmetric. Hence

$$
\begin{aligned}
E[\langle \tilde{D}F, v\rangle G] &= \frac{d}{d\varepsilon} E\left[f\left(\gamma(t_1)e^{\varepsilon\int_0^{t_1} h(s)ds},\ldots,\gamma(t_n)e^{\varepsilon\int_0^{t_n} h(s)ds}\right) G\right]_{|\varepsilon=0} \\
&= \frac{d}{d\varepsilon} E\left[F e^{\varepsilon I_1(v)-\frac{1}{2}\varepsilon^2\|v\|^2}\Theta_{\varepsilon v} G\right]_{|\varepsilon=0} \\
&= E\left[F(GI_1(v) - \langle v, DG\rangle - \delta(\nabla_v DG))\right].
\end{aligned}
$$

Using the identity $\delta(v) = I_1(v)$ we have

$$
\begin{aligned}
E[\langle \tilde{D}F, v\rangle G] &= E\left[F(G\delta(v) - \langle v, DG\rangle - \delta(\nabla_v DG))\right] \\
&= E\left[F(\delta(vG) - \delta(\nabla_v DG))\right] \\
&= E\left[F\left(a_v^+ - \int_0^\infty q_v(t)d\Lambda(t)\right)G\right] \\
&= E\left[G\left(a_v^- + \int_0^\infty q_v(t)d\Lambda(t)\right)F\right].
\end{aligned}
$$

□

It follows from the proof of Lemma 4.1 that $\tilde{D}$ admits an adjoint $\tilde{\delta}$ that satisfies

$$
\tilde{\delta}(uF) = a_u^+ F - \int_0^\infty q_u(t)d\Lambda(t)F, \quad F \in \tilde{\mathcal{S}},
$$

and

$$
E[F\tilde{\delta}(u)] = E[\langle \tilde{D}F, u\rangle], \quad F \in \tilde{\mathcal{S}},\ u \in \tilde{\mathcal{U}}.
$$

Letting

$$
\tilde{a}_u^+ F = \int_0^\infty d\tilde{a}_u^+(t)F = \tilde{\delta}(uF),
$$

we have

$$
d\tilde{a}_u^+(t) = da_u^+(t) - q_u(t)d\Lambda(t).
$$

The following Lemma shows that

$$
[\nabla_v, \nabla_u] = \nabla_{\nabla_v u} - \nabla_{\nabla_u v}.
$$

This means that the Lie bracket $\{u, v\}$ associated to the gradient ∇ on $L^2(\mathbb{R}_+; \mathcal{G})$ via $[\nabla_u, \nabla_v] = \nabla_{\{u,v\}}$ satisfies $\{u, v\} = \nabla_u v - \nabla_v u$, i.e. the connection defined by ∇ on $L^2(\mathbb{R}_+; \mathcal{G})$ has vanishing torsion.

Lemma 4.2 *We have*

$$[q_u(t), q_v(t)] = q_{\nabla_u v}(t) - q_{\nabla_v u}(t), \quad t > 0, \quad u, v \in L^2(\mathbb{R}_+; \mathcal{G}).$$

Proof. The Jacobi identity on $\mathcal{G}$ shows that

$$\begin{aligned}
[q_u(t), q_v(t)] &= \left[\int_0^t \mathrm{ad}(u(s))ds, \int_0^t \mathrm{ad}(v(s))ds\right] = \mathrm{ad}\left(\left[\int_0^t u(s)ds, \int_0^t v(s)ds\right]\right) \\
&= \int_0^t \int_0^s \mathrm{ad}([u(\tau), v(s)])d\tau ds - \int_0^t \int_0^s \mathrm{ad}([v(\tau), u(s)])d\tau ds \\
&= \int_0^t \mathrm{ad}(q_u(s)v(s))ds - \int_0^t \mathrm{ad}(q_v(s)u(s))ds \\
&= q_{\nabla_u v}(t) - q_{\nabla_v u}(t).
\end{aligned}$$

□

The Lie derivative on $\mathbb{P}(\mathsf{G})$ in the direction $u \in L^2(\mathbb{R}_+; \mathcal{G})$, introduced in [3], can be written $\tilde{a}_u^-$ in our context. Finally we show that the vanishing of torsion discovered in [3] can be obtained via quantum stochastic calculus. Precisely, the Lie bracket $\{u, v\}$ associated to $\tilde{a}_v^-$ via $[\tilde{a}_u^-, \tilde{a}_v^-] = \tilde{a}_{\{u,v\}}^-$ satisfies $\{u, v\} = \nabla_u v - \nabla_v u$, i.e. the connection defined by ∇ on $\mathbb{P}(\mathsf{G})$ also has a vanishing torsion.

Proposition 4.3 *We have on $\tilde{\mathcal{S}}$:*

$$\tilde{a}_u^- \tilde{a}_v^- - \tilde{a}_v^- \tilde{a}_u^- = \tilde{a}_{\nabla_v u}^- - \tilde{a}_{\nabla_u v}^-.$$

$u, v \in L^2(\mathbb{R}_+; \mathcal{G})$.

Proof. Using Lemma 4.2, the quantum Itô table implies

$$\begin{aligned}
& d\tilde{a}_u^-(t) \cdot d\tilde{a}_v^-(t) - d\tilde{a}_v^-(t) \cdot d\tilde{a}_u^-(t) \\
&= \big(da_u^-(t) + q_u(t)d\Lambda(t)\big) \cdot \big(da_v^-(t) + q_v(t)d\Lambda(t)\big) \\
&\quad - \big(da_v^-(t) + q_v(t)d\Lambda(t)\big) \cdot \big(da_u^-(t) + q_u(t)d\Lambda(t)\big) \\
&= q_u(t)v(t)da^-(t) + q_u(t)q_v(t)d\Lambda(t) - q_v(t)u(t)da^-(t) - q_v(t)q_u(t)d\Lambda(t)
\end{aligned}$$

$$
\begin{aligned}
&= \nabla_u v(t) da^-(t) - q_{\nabla_u v}(t) d\Lambda(t) - (\nabla_v u(t) da^-(t) - q_{\nabla_v u}(t) d\Lambda(t)) \\
&= d\tilde{a}^-_{\nabla_u v}(t) - d\tilde{a}^-_{\nabla_v u}(t),
\end{aligned}
$$

from Lemma 4.1. □

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