On tree-based methods for (partial) differential equations

NICOLAS PRIVAULT
(joint work with Guillaume Penent and Jiang Yu Nguwi)

Stochastic branching mechanisms have been used to represent the solutions of partial differential equations in [15], [7], [10], [8], and recently extended in [6] to the treatment of polynomial nonlinearities in first order gradient terms. This talk reviews an extension of such tree-based methods to functional nonlinearities with gradients of arbitrary orders.

Consider the ODE

$$u'(t) = f(u(t)), \qquad u(0) = u_0 \in \mathbb{R}^d, \quad t \in \mathbb{R}_+, \tag{1}$$

whose solution can be expanded as

$$u(t) = u_0 + tf(u_0) + \frac{t^2}{2} f' f(u_0) + \frac{t^3}{6} f' f' f(u_0) + \frac{t^3}{6} f''[f, f](u_0) + \cdots$$

which rewrites as the sum

$$u(t) = u_0 + \sum_{\mathcal{T}} \frac{t^{r(\mathcal{T})}}{\sigma(r(\mathcal{T}))\gamma(r(\mathcal{T}))} F(\mathcal{T})$$

over the family of Butcher trees $\mathcal{T}$, see [1], [2], Chapters 4-6 of [4], and [9], based on early work of [3]. In order to solve (1), we may also write

$$u(s) = u_0 + \int_0^s u'(r)dr = u_0 + \int_0^s f(u(r))dr,$$

and more generally we can expand the derivative $f^{(l)}(u(r))$ as

$$f^{(l)}(u(r)) = f^{(l)}(u_0) + \int_0^r f(u(r)) f^{(l+1)}(u(v))dv, \qquad l \geq 1.$$

We note that the above family of equations can be rewritten as

$$c(u)(t) = c(u)(0) + \sum_{Z \in \mathcal{M}(c)} \int_0^t \prod_{z \in Z} z(u)(s)ds \tag{2}$$

where c runs through a set $\mathcal{C} := \{\text{Id}, \ f^{(l)}, \ l \geq 0\}$, of functions called *codes* and $\mathcal{M}(c)$ is defined by letting $\mathcal{M}(\text{Id}) := \{f\}$ and $\mathcal{M}(g) := \{(f, g')\}$ for g a smooth function on $\mathbb{R}_+ \times \mathbb{R}$, see [12].

Next, consider a nonlinear PDE of the form

$$\begin{cases} \partial_t u(t, x) + \dfrac{1}{2} \Delta u(t, x) + f(u(t, x)) = 0 \\ u(T, x) = \phi(x), \qquad (t, x) \in [0, T] \times \mathbb{R}. \end{cases} \tag{3}$$

Letting $v(t,x) := g(u(t,x))$, we now have

$$\partial_t v(t,x) + \frac{1}{2}\Delta v(t,x) = g'(u(t,x))\left(\partial_t u(t,x) + \frac{1}{2}\Delta u(t,x)\right) + \frac{1}{2}(\partial_x u(t,x))^2 g''(u(t,x))$$

$$= -f(u(t,x))g'(u(t,x)) + \frac{1}{2}(\partial_x u(t,x))^2 g''(u(t,x)),$$

which shows that the functions u, $\partial_x u$, $af^{(k)} \circ u$ satisfy the integral equations

$$\begin{cases} u(t,x) = \displaystyle\int_{-\infty}^{\infty} \varphi(T-t,y-x)\phi(y)dy + \int_t^T \int_{-\infty}^{\infty} \varphi(s-t,y-x)f(u(s,y))dyds \\ af^{(k)}(u(t,x)) = \displaystyle\int_{-\infty}^{\infty} \varphi(T-t,y-x)af^{(k)}(\phi(y))dy \\ \displaystyle+\int_t^T \int_{-\infty}^{\infty} \varphi(s-t,y-x) \\ \qquad \times \left(af(u(s,y))f^{(k+1)}(u(s,y)) - \frac{a}{2}(\partial_x u(s,y))^2 f^{(k+2)}(u(s,y))\right) dyds \\ \partial_x u(t,x) = \displaystyle\int_{-\infty}^{\infty} \varphi(T-t,y-x)\partial_x\phi(y)dy \\ \quad \displaystyle+\int_t^T \int_{-\infty}^{\infty} \varphi(s-t,y-x)f'(u(s,y))\partial_x u(x,y)dyds, \end{cases}$$

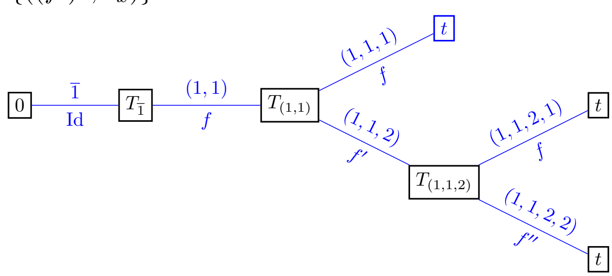
$a \neq 0$, $k \in \mathbb{N}$. We note that the above set of equations admits a formulation identical to (2) provided that we use the codes

$$\mathcal{C} := \left\{\mathrm{Id},\ \partial_x,\ af^{(k)},\ a \neq 0,\ k \in \mathbb{N}\right\}$$

and the mechanism defined as

$$\mathcal{M}(\mathrm{Id}) := \{f^*\}, \quad \mathcal{M}(g^*) := \left\{(f^*, (g')^*), \left(\partial_x, \partial_x, -\frac{1}{2}(g'')^*\right)\right\},$$

and $\mathcal{M}(\partial_x) := \{((f')^*, \partial_x)\}$.



We consider a random coding tree $\mathcal{T}_{t,x,c}$ illustrated by the above sample, started at (t,x) with a code $c \in \mathcal{C}$ and partitioned as $\mathcal{K}^\partial \cup \mathcal{K}^\circ$, where $\mathcal{K}^\circ$ denotes the set of leaves. In the next result, we use the random functional

$$\mathcal{H}(\mathcal{T}_{t,x,c}) := \prod_{\bar{k} \in \mathcal{K}^\circ} \frac{1}{q_{c_{\bar{k}}} \rho(\tau_{\bar{k}})} \prod_{\bar{k} \in \mathcal{K}^\partial} \frac{c_{\bar{k}}(u)\big(T, X^{\bar{k}}_{T_{\bar{k}}}\big)}{\overline{F}(T - T_{\bar{k}-})}.$$

of the random coding tree $\mathcal{T}_{t,x,c}$, in which branching at a node $\overline{k}$ occurs at the random time $T_{\overline{k}}$, the interjump time $\tau_{\overline{k}} = T_{\overline{k}} - T_{\overline{k}-}$ has tail CDF $\overline{F}$ and PDF ρ, and $\big(X_t^{\overline{k}}\big)_{t \geq T_{\overline{k}-}}$ is an independent Brownian motion started at time $T_{\overline{k}-}$.

Theorem 1. ([13]) *Assume that the integral solution of the system* (2) *is unique and that there exists a constant* $K > 0$ *such that:*

$$|f^{(k)} \circ \phi|_\infty \leq K, \quad k \geq 0, \quad |\phi|_\infty \leq K, \quad |\phi'|_\infty \leq K.$$

Then, there exists $T > 0$ *such that the solution of* (3) *admits the probabilistic representation*

$$u(t,x) = \mathbb{E}\left[\mathcal{H}(\mathcal{T}_{t,x,\mathrm{Id}})\right], \qquad (t,x) \in [0,T] \times \mathbb{R}.$$

The above method also extends to fully nonlinear PDEs of the form

$$\begin{cases} \partial_t u(t,x) + \dfrac{1}{2}\Delta u(t,x) + f\big(u(t,x), \nabla u(t,x), \ldots, \nabla^n u(t,x)\big) = 0, \\ u(T,x) = \phi(x), \qquad (t,x) = (t, x_1, \ldots, x_d) \in [0,T] \times \mathbb{R}^d, \end{cases}$$

$d \geq 1$, see [13], [11]. As an example, we consider a cosine nonlinearity with a gradient of order four, for which our method appears more accurate than the deep Galerkin method [14]. Related comparisons can be found in [11] with respect to the deep BSDE method [5].

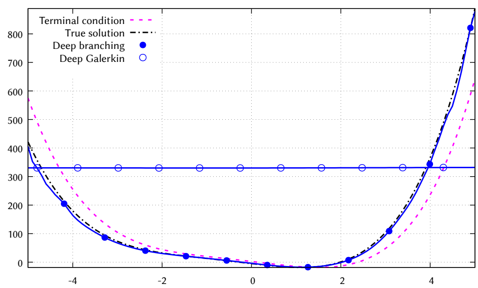


Figure 1. Comparison graphs in dimension $d = 5$.

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