Probabilistic representations for the solutions of nonlinear PDEs with fractional Laplacians

Nicolas Privault

(joint work with Guillaume Penent)

This talk presents tree-based probabilistic algorithms for the existence of solutions of nonlinear fractional PDEs with their numerical implementation.

1. Parabolic case

Given $\eta : (0,\infty) \to [0,\infty)$ a Bernstein function, consider the (nonlocal) semilinear PDE

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) - \eta(-\Delta/2)u(t,x) + f\left(t,x,u(t,x),\dfrac{\partial u}{\partial x_1}(t,x),\ldots,\dfrac{\partial u}{\partial x_m}(t,x)\right) = 0, \\ u(T,x) = \phi(x), \qquad x = (x_1,\ldots,x_d) \in \mathbb{R}^d, \end{cases}$$

where $f(t,x,y,z_1,\ldots,z_m)$ is a polynomial nonlinearity given by

$$f(t,x,y,z_1,\ldots,z_m) = \sum_{l=(l_0,\ldots,l_m)\in\mathcal{L}_m} c_l(t,x) y^{l_0} z_1^{l_1}\cdots z_m^{l_m},$$

and $\mathcal{L}_m \subset \mathbb{N}^{m+1}$ is finite. By choosing $\eta(\lambda) := (2\lambda)^{\alpha/2}$, this setting includes the case of the standard fractional Laplacian Δ_α. We assume that the coefficients $c_l(t,x)$ are uniformly bounded and that the terminal condition ϕ is Lipschitz and bounded on $\mathbb{R}^d$.

Theorem 1.1. *([8]) Suppose that $\int_{\lambda_0}^\infty \frac{1}{\sqrt{\lambda}\eta(\lambda)} d\lambda < \infty$ for some $\lambda_0 > 0$. Then, there exists a small enough $T > 0$ such that the PDE*

$$u(t,x) = \int_{\mathbb{R}^d} \varphi(T-t,y-x)\phi(y)dy$$
$$+ \sum_{l=(l_0,\ldots,l_m)\in\mathcal{L}_m} \int_t^T \int_{\mathbb{R}^d} \varphi(s-t,y-x)c_l(s,y)u^{l_0}(s,y)\prod_{j=1}^m \left(\frac{\partial u}{\partial y_j}(s,y)\right)^{l_j} dyds,$$

admits an integral solution on $[0,T]$.

To prove the above result, for each $i = 0,1,\ldots,d$ we construct a sufficiently integrable functional $\mathcal{H}_\phi(\mathcal{T}_{t,x,i})$ of a random tree $\mathcal{T}_{t,x,i}$ driven by a subordinated Lévy process $(Z_t)_{t\in\mathbb{R}_+} := (B_{S_t})_{t\in\mathbb{R}_+}$, where $(B_t)_{t\in\mathbb{R}_+}$ is a multidimensional Brownian motion, such that we have the representations

$$u(t,x) := \mathbb{E}\big[\mathcal{H}_\phi(\mathcal{T}_{t,x,0})\big], \quad (t,x) \in [0,T]\times\mathbb{R}^d,$$

and

$$\frac{\partial u}{\partial x_i}(t,x) := \mathbb{E}\big[\mathcal{H}_\phi(\mathcal{T}_{t,x,i})\big], \quad (t,x)\in[0,T]\times\mathbb{R}^d, \quad i = 1,\ldots,d.$$

Dealing with gradient terms requires to perform an integration by parts, which is made possible using the Gaussian density of B_t in the subordination $Z_t := B_{S_t}$, as done in [7] in the case of stable processes with $\eta(\lambda) := (2\lambda)^{\alpha/2}$. Related local and global-in-time existence results have been obtained by deterministic arguments under more technical conditions in e.g. [5], [6]

Corollary 1.2. *([8]) Taking $\eta(\lambda) := (2\lambda)^{\alpha/2}$ with $\alpha \in (1,2)$, under the above assumptions there exists a small enough $T > 0$ such that the PDE*

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) - (-\Delta)^{\alpha/2} u(t,x) + f\left(t, x, u(t,x), \dfrac{\partial u}{\partial x_1}(t,x), \ldots, \dfrac{\partial u}{\partial x_m}(t,x)\right) = 0, \\ u(T,x) = \phi(x), \qquad x = (x_1, \ldots, x_d) \in \mathbb{R}^d, \end{cases}$$

with α-fractional Laplacian admits an integral solution on $[0,T]$.

2. Elliptic case

We consider the class of semilinear elliptic PDEs on the open ball $B(0,R)$ of radius $R > 0$ in $\mathbb{R}^d$, of the form

$$\begin{cases} \Delta_\alpha u(x) + f\big(x, u(x), \nabla u(x)\big) = 0, \qquad x \in B(0,R), \\ u(x) = \phi(x), \qquad x \in \mathbb{R}^d \backslash B(0,R), \end{cases} \tag{1}$$

where $\phi : \mathbb{R}^d \to \mathbb{R}$ is a bounded Lipschitz function on $\mathbb{R}^d \setminus B(0,R)$, Δ_α denotes the fractional Laplacian with parameter $\alpha \in (0,2)$, and $f(x,y,z)$ is a polynomial nonlinearity term. Next, we provide existence results and probabilistic representations for the solution of (1) under boundedness and smoothness conditions on polynomial coefficients. Our approach allows us to take into account gradient nonlinearities, which has not been done by deterministic finite difference methods, see e.g. [4].

Theorem 2.1. *([9], [10]) Let $d \geq 2$ and $\alpha \in (1,2)$, assume that the boundary condition ϕ belongs to $H^\alpha(\mathbb{R}^d)$ and is bounded on $\mathbb{R}^d \setminus B(0,R)$. Under the above assumptions, the semilinear elliptic PDE*

$$\begin{cases} \Delta_\alpha u(x) + f\big(x, u(x), \nabla u(x)\big) = 0, \qquad x \in B(0,R), \\ u(x) = \phi(x), \qquad x \in \mathbb{R}^d \backslash B(0,R), \end{cases}$$

admits a viscosity solution in $\mathcal{C}^1(B(0,R)) \cap \mathcal{C}^0(\overline{B}(0,R))$ provided that R and $|c_l|_\infty$, $l \in \mathcal{L}_m$, are sufficiently small.

Existence of solutions are obtained through a probabilistic representation of the form $u(x) := \mathbb{E}[\mathcal{H}_\phi(\mathcal{T}_{x,0})]$, $x \in B(0,R)$, where $\mathcal{H}_\phi(\mathcal{T}_{x,0})$ is a functional of a random branching tree $\mathcal{T}_{x,0}$. For each $i = 0, 1, \ldots, d$ we construct a sufficiently integrable functional $\mathcal{H}_\phi(\mathcal{T}_{x,i})$ of a random tree $\mathcal{T}_{x,i}$ such that we have the representation

$$u(x) = \mathbb{E}\big[\mathcal{H}_\phi(\mathcal{T}_{x,0})\big], \qquad x \in \mathbb{R}^d. \tag{2}$$

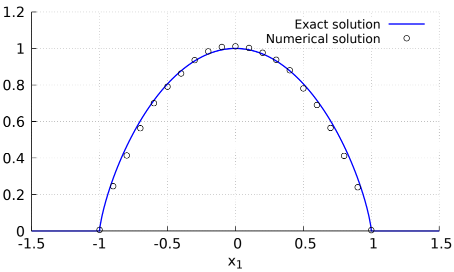
The main difficulty in the proof is to show the uniform integrability required on $\mathcal{H}(\mathcal{T}_{x,i})$ for $\mathbb{E}[\mathcal{H}(\mathcal{T}_{x,i})]$ to be continuous in $x \in \mathbb{R}^d$ is satisfied for $\alpha \in (1,2)$, as required in the framework of viscosity solutions. For this, we extend arguments of

[1] from the standard Laplacian Δ and Brownian motion to the fractional Laplacian $\Delta_\alpha := -(-\Delta)^{\alpha/2}$ and its associated stable process, and use bounds on the fractional Green and Poisson kernel and stable process hitting times from [3], [2].

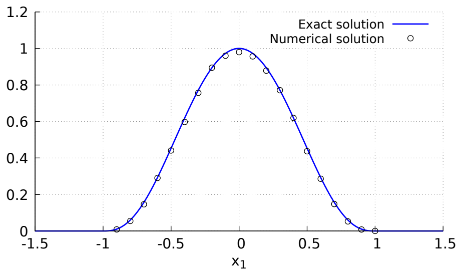
As an example, consider the elliptic PDE with nonlinear gradient term

$$\Delta_s u(x) + \Psi_{k,\alpha}(x) + (2k+\alpha)^2|x|^4(1-|x|^2)^{2k+\alpha} + ((1-|x|^2)x\cdot\nabla u(x))^2 = 0, \tag{3}$$

$x \in B(0,1)$, with $u(x) = 0$ for $x \in \mathbb{R}^d \setminus B(0,R)$, and explicit solution $u(x) = \Phi_{k,\alpha}(x) = (1-|x|^2)_+^{k+\alpha/2}$, $x \in \mathbb{R}^d$. Numerical estimates of (2) by the Monte Carlo method are presented in the figure below.



(A) Numerical solution of (3) with $k = 0$.



(B) Numerical solution of (3) with $k = 2$.

FIGURE 1. Numerical solutions with $d = 10$ and $\alpha = 1.75$.

REFERENCES

[1] A. Agarwal and J. Claisse. Branching diffusion representation of semi-linear elliptic PDEs and estimation using Monte Carlo method. *Stochastic Processes and their Applications*, 130(8):5006–5036, 2020.

[2] K. Bogdan, T. Grzywny, and M. Ryznar. Barriers, exit time and survival probability for unimodal Lévy processes. *Probab. Theory Related Fields*, 162(1-2):155–198, 2015.

[3] K. Bogdan, T. Kulczycki, and A. Nowak. Gradient estimates for harmonic and q-harmonic functions of symmetric stable processes. *Illinois J. Math.*, 46(2):541–556, 2002.

[4] Y. Huang and A. Oberman. Numerical methods for the fractional Laplacian: a finite difference-quadrature approach. *SIAM J. Numer. Anal.*, 52(6):3056–3084, 2014.

[5] K. Ishige, T. Kawakami, and K. Kobayashi. Global solutions for a nonlinear integral equation with a generalized heat kernel. *Discrete Contin. Dyn. Syst. Ser. S*, 7(4):767–783, 2014.

[6] K. Ishige, T. Kawakami, and K. Okabe. Existence of solutions to nonlinear parabolic equations via majorant integral kernel. arXiv:2101.06581, 2021.

[7] R. Kawai and A. Takeuchi. Computation of Greeks for asset price dynamics driven by stable and tempered stable processes. *Quantitative Finance*, 13(8):1303–1316, 2013.

[8] G. Penent and N. Privault. Existence and probabilistic representation of the solutions of semilinear parabolic PDEs with fractional Laplacians. To appear in Stochastics and Partial Differential Equations: Analysis and Computations, 30 pages, 2021.

[9] G. Penent and N. Privault. Existence of solutions for nonlinear elliptic PDEs with fractional Laplacians on open balls. Preprint arXiv:2110.09941, 21 pages, 2021.

[10] G. Penent and N. Privault. Semilinear fractional elliptic PDEs with gradient nonlinearities on open balls: existence of solutions and probabilistic representation. Preprint, 23 pages, 2021.