# Combinatorics, moments and quasi-invariance for Poisson random integrals

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## 1. Moment identities

Consider a Poisson point process with $\sigma$-finite diffuse measure $\sigma(dx)$ on a $\sigma$-compact metric space $X$. The underlying probability space $\Omega$ is a space of configurations whose elements $\omega \in \Omega$ are identified with the Radon point measures $\omega = \sum_{x\in\omega} \epsilon_x$, where $\epsilon_x$ denotes the Dirac measure at $x \in X$ and the Poisson probability measure with intensity $\sigma$ on $\Omega$ is denoted by $\pi_\sigma$. The isometry formula for the multiple compensated Poisson stochastic integrals $I_k(f_k)$ of symmetric square-integrable functions $f_k : X^k \to \mathbb{R}$ in $k$ variables shows that

$$\mathbb{E}\left[I_k(f^{\otimes k})F\right] = \mathbb{E}\left[\int_{X^k} f(x_1)\cdots f(x_k) D_{x_1}\cdots D_{x_k} F \sigma(dx_1)\cdots\sigma(dx_k)\right] \tag{1}$$

where $F : \Omega \to \mathbb{R}$ is a finite sum of multiple stochastic integrals and $D_x$ is the finite difference operator defined by

$$D_x F := \varepsilon_x^+ F(\omega) - F(\omega) = F(\omega\cup\{x\}) - F(\omega), \qquad \omega\in\Omega, \quad x \in X.$$

Next, using the relation

$$\mathcal{E}(g) := \exp\left(-\int_0^\infty g(x)dx\right)\prod_{x\in\omega}(1+g(x)) = \sum_{n=0}^\infty \frac{1}{n!} I_n(g^{\otimes n})$$

with $g = e^f - 1$ we find, by the Faà di Bruno formula applied to the exponential function,

$$\begin{aligned}
\sum_{n=0}^\infty \frac{1}{n!}\mathbb{E}\left[F\left(\int_X f d\omega\right)^n\right] &= \mathbb{E}[F e^{\int_X f d\omega}] = e^{\int_X (e^f-1)d\sigma}\mathbb{E}\left[F\mathcal{E}(e^f-1)\right] \\
&= \sum_{k=0}^\infty \frac{1}{k!}\int_{X^k}(e^{f(x_1)}-1)\cdots(e^{f(x_n)}-1)\mathbb{E}\left[\varepsilon_{\mathfrak{x}_k}^+ F\right]\sigma(dx_1)\cdots\sigma(dx_k) \\
&= \sum_{n=0}^\infty \frac{1}{n!}\sum_{P_1,\ldots,P_k\subset\{1,\ldots,n\}}\int_{X^k} f^{|P_1|}(x_1)\cdots f^{|P_k|}(x_k)\mathbb{E}\left[\varepsilon_{\mathfrak{x}_k}^+ F\right]\sigma(dx_1)\cdots\sigma(dx_k),
\end{aligned}$$

with $\varepsilon_{\mathfrak{x}_k}^+ = \varepsilon_{x_1}^+\cdots\varepsilon_{x_k}^+$, for $F : \Omega\to\mathbb{R}$ a bounded random variable, where the sum runs over all partitions $\{P_1,\ldots,P_k\}$ of $\{1,\ldots,n\}$, hence the relation

$$\mathbb{E}\left[F\left(\int_X f d\omega\right)^n\right] = \sum_{P_1,\ldots,P_k\subset\{1,\ldots,n\}}\int_{X^k} f^{|P_1|}(x_1)\cdots f^{|P_k|}(x_k)\mathbb{E}\left[\varepsilon_{\mathfrak{x}_k}^+ F\right]\sigma(dx_1)\cdots\sigma(dx_k),$$

which extends as the moment identity

$$\mathbb{E}\left[\left(\int_X u_x(\omega)\omega(dx)\right)^n\right] = \sum_{P_1,\ldots,P_k} \mathbb{E}\left[\int_{X^k} \varepsilon^+_{\mathfrak{x}_k}(u_{x_1}^{|P_1|}\cdots u_{x_k}^{|P_k|})\sigma(dx_1)\cdots\sigma(dx_k)\right], \tag{2}$$

for $u(x,\omega)$ a sufficiently integrable random process on $X \times \Omega$, cf. Prop. 3.1 of [6]. From the relation

$$\varepsilon^+_{\mathfrak{x}_k}(u_{x_1}\cdots u_{x_k}) = \varepsilon^+_{x_1,\ldots,x_k}(u_{x_1}\cdots u_{x_k}) = \sum_{\Theta\subset\{1,\ldots,k\}} D_\Theta(u_{x_1}\cdots u_{x_k}),$$

where $D_\Theta = D_{x_1}\cdots D_{x_l}$ when $\Theta = \{1,\ldots,l\}$, we deduce that

$$\begin{aligned}\mathbb{E}\left[\left(\int_X u_x(\omega)\omega(dx)\right)^n\right] &= \sum_{P_1,\ldots,P_k} \mathbb{E}\left[\int_{X^k} \varepsilon^+_{\mathfrak{x}_k}(u_{x_1}^{|P_1|}\cdots u_{x_k}^{|P_k|})\sigma(dx_1)\cdots\sigma(dx_k)\right] \\ &= \sum_{P_1,\ldots,P_k}\sum_{\Theta\subset\{1,\ldots,k\}} \mathbb{E}\left[\int_{X^k} D_\Theta(u_{x_1}^{|P_1|}\cdots u_{x_k}^{|P_k|})\sigma(dx_1)\cdots\sigma(dx_k)\right].\end{aligned}$$

Under the cyclic condition $D_{x_1}u_{x_2}\cdots D_{x_k}u_{x_1} = 0$, we get $D_{x_1}\cdots D_{x_k}(u_{x_1}\cdots u_{x_k}) = 0$, $x_1,\ldots,x_k \in X$, $\omega \in \Omega$, cf. [3], [5], and provided in addition that the moment $\int_X u^k(s)\sigma(ds)$ is deterministic, $k \geq 1$, a decreasing induction shows that

$$\mathbb{E}\left[\left(\int_X u_x(\omega)\omega(dx)\right)^n\right] = \sum_{P_1,\ldots,P_k}\int_{X^k} u_{x_1}^{|P_1|}\cdots u_{x_k}^{|P_k|}\sigma(dx_1)\cdots\sigma(dx_k), \qquad n \geq 1,$$

i.e. $\int_X u_x(\omega)\omega(dx)$ has a compound Poisson distribution. See [7] for related consequence for the mixing of random transformations of Poisson measures. Such results have been recently extended to point processes with Papangelou intensities in [1].

## 2. Quasi-invariance

Formula (1) can be extended to indicator functions $\mathbf{1}_{A(\omega)}$ over random sets $A(\omega)$, as

$$\begin{aligned}\mathbb{E}\left[FI_n(\mathbf{1}_A^{\otimes n})\right] &= \mathbb{E}\left[FC_n(\omega(A),\sigma(A))\right] \\ &= \mathbb{E}\left[\int_{X^n} D_{x_1}\cdots D_{x_n}\left(F\prod_{p=1}^n \mathbf{1}_A(x_p)\right)\sigma(dx_1)\cdots\sigma(dx_n)\right],\end{aligned} \tag{3}$$

via a pathwise extension of the multiple stochastic integral, by application of Stirling inversion to (2) and to the Charlier polynomial $C_n(x,\lambda)$ of order $n \in \mathbb{N}$ with parameter $\lambda > 0$, cf. [4]. As a consequence, if $\tau : \Omega \times X \to Y$ satisfies the cyclic condition $D_{t_1}\tau(\omega,t_2)\cdots D_{t_k}\tau(\omega,t_1) = 0$, $t_1,\ldots,t_k \in X$, $\omega \in \Omega$, for all $k \geq 1$, and $g : Y \to \mathbb{R}$ is sufficiently integrable we get

$$\mathbb{E}\left[e^{-\int_X g(\tau(\omega,x))\sigma(dx)}\prod_{x\in\omega}(1+g(\tau(\omega,x)))\right] = 1.$$

Denoting by $\tau_* : \Omega \to \Omega$ the mapping defined by shifting configuration points according to $\tau$, this implies the non-adapted Girsanov identity

$$\mathbb{E}\left[F(\tau_*(\omega))e^{-\int_X \phi(\omega,x)\sigma(dx)}\prod_{x\in\omega}(1+\phi(\omega,x))\right] = \mathbb{E}[F], \qquad F \in L^1(\Omega),$$

provided $\tau(\omega,\cdot) : X \to X$ is invertible on $X$ for all $\omega \in \Omega$, and the density

$$\phi(\omega,x) := \frac{d\tau_*^{-1}(\omega,\cdot)\sigma}{d\sigma}(x) - 1, \qquad x \in X,$$

exists for all $\omega \in \Omega$. If $\tau_* : \Omega \to \Omega$ is invertible then the random transformation $\tau_*^{-1} : \Omega \to \Omega$ is absolutely continuous with respect to $\pi_\sigma$, with density

$$\frac{d\tau_*^{-1}\pi_\sigma}{d\pi_\sigma} = e^{-\int_X \phi(\omega,x)\sigma(dx)}\prod_{x\in\omega}(1+\phi(\omega,x)). \tag{4}$$

## 3. Examples and stopping sets

Examples can be constructed when $A(\omega)$ is a stopping set, i.e. a random set such that $\{A \subset U\} \in \mathcal{F}_K$ for all $U \subset K$, where $\mathcal{F}_K$ denotes the $\sigma$-algebra generated by points inside $K$, cf. [8] and Def. 2.27 in [2]. In this case (3) shows that $\mathbb{E}[I_n(\mathbf{1}_A^{\otimes n})] = 0$. Examples of transformations $\tau(\omega,x)$ can be defined by leaving $A(\omega)$ invariant and by shifting $x \mapsto \tau(\omega,x)$ depending only on those points of $\omega$ that belong to $A(\omega)$. Specific examples include $A$ the smallest ball containing the $n$ points closest to the origin when $X = \mathbb{R}^d$, or $A = [0,T_n]$ when $X = \mathbb{R}_+$ and $T_n$ is the $n$th Poisson jump time, and $A$ the complement of the open convex hull of the points of $\omega$ that belong to the unit ball.

## References


[1] L. Decreusefond and I. Flint. Moment formulae for general point processes. Preprint arXiv:1211.4811, 2012.

[2] I. Molchanov. *Theory of random sets*. Probability and its Applications (New York). Springer-Verlag London Ltd., London, 2005.

[3] N. Privault. Moment identities for Poisson-Skorohod integrals and application to measure invariance. *C. R. Math. Acad. Sci. Paris*, 347:1071–1074, 2009.

[4] N. Privault. Girsanov identities for Poisson measures under quasi-nilpotent transformations. *Ann. Probab.*, 40(3):1009–1040, 2012.

[5] N. Privault. Invariance of Poisson measures under random transformations. *Ann. Inst. H. Poincaré Probab. Statist.*, 48(4):947–972, 2012.

[6] N. Privault. Moments of Poisson stochastic integrals with random integrands. *Probability and Mathematical Statistics*, 32(2):227–239, 2012.

[7] N. Privault. Mixing of Poisson random measures under interacting transformations. arXiv:1301.0672, 2013.

[8] S. Zuyev. Stopping sets: gamma-type results and hitting properties. *Adv. in Appl. Probab.*, 31(2):355–366, 1999.