

Convex comparison inequalities for non-Markovian stochastic integrals*

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Abstract

We derive convex comparison inequalities for stochastic integrals of the form $\int_0^T \sigma_t^* d\hat{B}_t$ and $\int_0^T \sigma_t dB_t$, where $0 \leq \sigma_t^* \leq \sigma_t$ are adapted processes with respect to the filtration generated by a standard Brownian motion $(B_t)_{t \in [0, T]}$, and $(\hat{B}_t)_{t \in [0, T]}$ is an independent Brownian motion. Our method uses forward-backward stochastic integration and the Malliavin calculus, and is also applied to jump-diffusion processes.

Key words: Convex concentration, Malliavin calculus, forward-backward stochastic calculus, jump diffusion processes.

Mathematics Subject Classification: 60E15, 60H05, 60G44, 60G55.

1 Introduction

Partial orderings of probability distributions via convex comparison inequalities have been introduced in economics as a risk management tool that yields finer information than mean-variance analysis, cf. e.g. [9]. Namely, a random variable X^* is said to be more concentrated than another random variable X if

$$E[\phi(X^*)] \leq E[\phi(X)], \quad (1.1)$$

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for all sufficiently integrable convex functions $\phi : \mathbb{R} \rightarrow \mathbb{R}$.

As is well known, when X^* and X are Gaussian random variables written as

$$X^* = x_0 + \int_0^T \sigma^*(t) dB_t, \quad X = x_0 + \int_0^T \sigma(t) dB_t,$$

where $\sigma(t)$ and $\sigma^*(t)$ are deterministic functions and $(B_t)_{t \in [0, T]}$ is a standard Brownian motion, (1.1) holds if and only if

$$\int_0^T |\sigma^*(t)|^2 dt \leq \int_0^T |\sigma(t)|^2 dt, \quad (1.2)$$

as can be shown using conditioning and the Jensen inequality, cf. e.g. [2], or by the Dubins-Schwarz theorem on time-changed Brownian motion.

In this paper we are interested in deriving sufficient conditions for the convex concentration inequality (1.1) in the case where σ_t and σ_t^* are random processes. When σ_t and σ_t^* become random, the above condition (1.2) alone cannot be expected to yield (1.1), nevertheless some results already exist in that direction. When $X^* = X_T^*$ is the terminal value of a diffusion process $(X_t^*)_{t \in [0, T]}$ solution of

$$X_t^* = x_0 + \int_0^t \sigma_s^*(X_s^*) dB_s, \quad t \in [0, T], \quad (1.3)$$

and $X = X_T$ is given by

$$X_T = x_0 + \int_0^T \sigma_t dB_t, \quad (1.4)$$

where $(\sigma_t)_{t \in [0, T]}$ is square-integrable and adapted with respect to the filtration $(\mathcal{F}_t)_{t \in [0, T]}$ generated by $(B_t)_{t \in [0, T]}$, it is known from [4] that (1.1) holds, i.e. we have

$$E[\phi(X_T^*)] \leq E[\phi(X_T)],$$

for convex $\phi : \mathbb{R} \rightarrow \mathbb{R}$, under the almost sure bound

$$|\sigma_t^*(X_t)| \leq |\sigma_t|, \quad t \in [0, T].$$

This result relies on the preservation of convexity by the Markov semigroup of (1.3), and this method has been extended to multidimensional jump-diffusion processes in

[3].

Our goal in this paper is to state a different extension of the above results to the case where $(\sigma_t^*)_{t \in [0, T]}$ does not have to be a diffusion coefficient. More precisely, $(\sigma_t^*)_{t \in [0, T]}$ will be an adapted process integrated against an independent Brownian motion $(\hat{B}_t)_{t \in [0, T]}$, i.e. (1.3) is replaced with

$$X_t^* = x_0 + \int_0^t \sigma_s^* d\hat{B}_s, \quad t \in [0, T],$$

while (1.4) still holds, i.e.

$$X_T = x_0 + \int_0^T \sigma_t dB_t,$$

where $(\sigma_t)_{t \in [0, T]}$ is (\mathcal{F}_t) -adapted as $(\sigma_t^*)_{t \in [0, T]}$.

This question has also been investigated in the multidimensional case as an application of forward-backward stochastic calculus in [2] Theorem 4.1, however the argument of [2] is valid only when $(\sigma_t^*)_{t \in [0, T]}$ is a deterministic function.

Clearly, the variance inequality

$$\text{Var}[X_T^*] = E \left[\int_0^T |\sigma_t^*|^2 dt \right] \leq E \left[\int_0^T |\sigma_t|^2 dt \right] = \text{Var}[X_T]$$

holds under the bound

$$|\sigma_t^*| \leq |\sigma_t|, \quad d\mathbb{P} - a.s., \quad t \in [0, T], \quad (1.5)$$

however this does not suffice to yield the convex concentration inequality (1.1) without additional assumptions, as noted from (1.8) below.

Our main tool will be Proposition 2.4 below, which states that

$$\begin{aligned} E [\phi(M_t + M_t^*)] &= E [\phi(M_s + M_s^*)] + \frac{1}{2} E \left[\int_s^t \phi''(M_u + M_u^*) (|\sigma_u|^2 - |\sigma_u^*|^2) du \right] \\ &\quad + \frac{1}{2} E \left[\int_s^t \sigma_u \phi^{(3)}(M_u + M_u^*) \int_u^T D_u^B |\sigma_v^*|^2 dv du \right], \end{aligned} \quad (1.6)$$

$0 \leq s \leq t$, for all $\phi \in \mathcal{C}^3(\mathbb{R})$ with $\phi^{(3)}$ bounded, where D^B denotes the Malliavin gradient on the Wiener space with respect to $(B_t)_{t \in \mathbb{R}_+}$, cf. (2.10) below. From (1.6) we show in Corollary 4.2 below that (1.1) holds for all convex functions ϕ with convex derivative, i.e.

$$E \left[\phi \left(\int_0^T \sigma_u^* d\hat{B}_u \right) \right] \leq E \left[\phi \left(\int_0^T \sigma_u dB_u \right) \right], \quad (1.7)$$

provided that, in addition to (1.5), the processes σ_t and σ_t^* satisfy the condition

$$\sigma_s D_s^B |\sigma_t^*|^2 \geq 0, \quad d\mathbb{P} - a.s., \quad 0 \leq s \leq t \leq T. \quad (1.8)$$

It can be checked from (1.6) that a condition of the form (1.8) can be necessary for (1.1) to hold when $\phi^{(3)}(x) > 0$, $x \in \mathbb{R}$, by taking for example $0 \leq \sigma_t = \sigma_t^*$, $t \in [0, T]$. Note also that (1.8) is invariant only under change of sign of σ_t^* as the law of $\int_0^T \sigma_t dB_t$ is not symmetric in general.

The pure jump and mixed jump-diffusion cases are treated in Corollary 4.3 and Theorem 4.1. Next, we consider some examples of applications of (1.7) in the continuous case.

Examples

As a first example of application of (1.7), when σ_t^* and σ_t have the form

$$\sigma_t^* = f^*(t, B_t) \quad \text{and} \quad \sigma_t = f(t, B_t), \quad t \in [0, T],$$

for $f, f^* \in \mathcal{C}^1(\mathbb{R}_+ \times \mathbb{R})$, we have

$$\sigma_s \sigma_t^* D_s^B \sigma_t^* = f(s, B_s) f^*(t, B_t) \frac{\partial f^*}{\partial x}(t, B_t) \mathbf{1}_{[0, t]}(s), \quad s, t \in [0, T],$$

and we find that

$$X^* = x_0 + \int_0^T f^*(t, B_t) d\hat{B}_t$$

is more concentrated than

$$X = x_0 + \int_0^T f(t, B_t) dB_t,$$

provided

$$0 \leq f^*(t, x) \leq f(t, x), \quad t \in [0, T], \quad x \in \mathbb{R},$$

and $x \rightarrow f^*(t, x)$ is non-decreasing for all $t \in [0, T]$.

As a second example, when

$$X_T^* = x_0 + \int_0^T \sigma_t^*(X_t) d\hat{B}_t, \quad \text{and} \quad X_T = x_0 + \int_0^T \sigma_t dB_t,$$

i.e. σ_t^* takes the form $\sigma_t^* = \sigma_t^*(X_t)$, where $x \mapsto \sigma_t^*(x)$ is Lipschitz uniformly in t , Condition (1.5) holds if

$$|\sigma_t^*(X_t)| \leq |\sigma_t|, \quad d\mathbb{P} - a.s., \quad t \in [0, T],$$

and (1.8) reads

$$\sigma_s \sigma_t^*(X_t) \sigma_t^{*'}(X_t) D_s^B X_t = \sigma_s \sigma_t^*(X_t) \sigma_t^{*'}(X_t) \left(\int_s^t D_s^B \sigma_r dB_r + \sigma_s \right) \geq 0, \quad (1.9)$$

$$0 \leq s \leq t \leq T.$$

When σ_t and σ_t^* both take the form $\sigma_t^* = \sigma_t^*(X_t)$ and $\sigma_t = \sigma_t(X_t)$, i.e.

$$X^* = x_0 + \int_0^T \sigma_t^*(X_t) d\hat{B}_t \quad \text{and} \quad X = x_0 + \int_0^T \sigma_t(X_t) dB_t,$$

Condition (1.5) reads

$$|\sigma_t^*(x)| \leq |\sigma_t(x)|, \quad x \in \mathbb{R}, \quad t \in [0, T],$$

and since

$$\begin{aligned} D_s^B \sigma_t^*(X_t) &= \sigma_t^{*'}(X_t) D_s^B X_t \\ &= \sigma_t^{*'}(X_t) \sigma_s(X_s) \exp \left(\int_0^t \sigma_u'(X_u) dB_u - \frac{1}{2} \int_0^t |\sigma_u'(X_u)|^2 du \right), \end{aligned}$$

$s, t \in [0, T]$, cf. e.g. Exercise 2.2.1 page 124 of [7], Condition (1.8) is equivalent to

$$(\sigma_s(X_s))^2 \sigma_t^*(X_t) \sigma_t^{*'}(X_t) \geq 0, \quad s, t \in [0, T], \quad (1.10)$$

hence (1.1) holds when $x \mapsto |\sigma_t^*(x)|$ is non-decreasing on the state space of X_t . This is the case in particular when X_T^* and X_T are represented as

$$X_T^* = x_0 + \int_0^T X_t \sigma^*(t) d\hat{B}_t \quad \text{and} \quad X_T = x_0 + \int_0^T X_t \sigma(t) dB_t,$$

where $x_0 > 0$ and $\sigma^*(t)$, $\sigma(t)$ are deterministic functions such that

$$|\sigma^*(t)| \leq |\sigma(t)|, \quad t \in [0, T].$$

The remaining of this paper is organized as follows. In Section 2 we state the main prerequisites on forward-backward stochastic calculus and on the Malliavin calculus, including Proposition 2.3 on increasing forward-backward martingales in the convex order. Next we derive convex concentration inequalities for forward-backward martingales in Section 3, and for stochastic integrals in Section 4.

2 Forward-backward integrals and the Malliavin calculus

In this section we present the forward-backward Itô formula and the associated identities in expectation that will be used to derive our main results. We denote by $(B_t)_{t \in [0, T]}$ and $\mu(dt, dx)$ a forward Brownian motion and jump measure on $[0, T] \times \mathbb{R}_+$, and by $(B_t^*)_{t \in [0, T]}$ and $\mu^*(dt, dx)$ a backward Brownian motion and jump measure on $[0, T] \times \mathbb{R}_+$, such that $\{B_t, \mu(dt, dx)\}$ is independent of $\{B_t^*, \mu^*(dt, dx)\}$. However, $(B_t)_{t \in [0, T]}$ may not be independent of $\mu(dt, dx)$, and $(B_t^*)_{t \in [0, T]}$ may not be independent of $\mu^*(dt, dx)$.

The forward and backward jump measures $\mu(dt, dx)$ and $\mu^*(dt, dx)$ on $[0, T] \times \mathbb{R}_+$ are assumed to have respective dual predictable projections of the form $dt\nu_t(dx)$ and $dt\nu_t^*(dx)$. We also let $(\mathcal{F}_t)_{t \in [0, T]}$ denote the forward filtration generated by $(B_t)_{t \in [0, T]}$ and $\mu(dt, dx)$, and we let $(\mathcal{F}_t^*)_{t \in [0, T]}$ denote the backward filtration generated by $(B_t^*)_{t \in [0, T]}$ and $\mu^*(dt, dx)$.

Our results rely on the Itô type change of variable formula (2.3) for the (\mathcal{F}_t) -forward, resp. $(\mathcal{F}_t^* \vee \mathcal{F}_T)$ -backward martingales

$$M_t = \int_0^t \sigma_s dB_s + \int_0^t \int_{-\infty}^{\infty} J_{s-,x}(\mu(ds, dx) - ds\nu_s(dx)), \quad t \in [0, T], \quad (2.1)$$

resp.

$$M_t^* = \int_t^T \sigma_s^* d^* B_s^* + \int_t^T \int_{-\infty}^{\infty} J_{s^+,x}^*(\mu^*(d^*s, dx) - ds^*\nu_s^*(dx)), \quad t \in [0, T], \quad (2.2)$$

in which $(\sigma_s)_{s \in [0, T]}$, $(J_{s,x})_{(s,x) \in [0, T] \times \mathbb{R}}$, $(\sigma_s^*)_{s \in [0, T]}$ and $(J_{s,x}^*)_{(s,x) \in [0, T] \times \mathbb{R}}$ are all square-integrable (\mathcal{F}_t) -forward adapted processes.

For all $\phi \in \mathcal{C}^2(\mathbb{R} \times \mathbb{R})$ we have the change of variable formula

$$\begin{aligned} \phi(M_t, M_t^*) &= \phi(M_s, M_s^*) + \int_{s^+}^t \frac{\partial \phi}{\partial x}(M_{u^-}, M_u^*) dM_u + \frac{1}{2} \int_s^t \frac{\partial^2 \phi}{\partial x^2}(M_u, M_u^*) |\sigma_u|^2 du \\ &+ \int_{s^+}^t \int_{-\infty}^{\infty} \left(\phi(M_{u^-} + J_{u^-,x}, M_u^*) - \phi(M_{u^-}, M_u^*) - J_{u^-,x} \frac{\partial \phi}{\partial x}(M_{u^-}, M_u^*) \right) \mu(du, dx) \\ &- \int_s^{t^-} \frac{\partial \phi}{\partial y}(M_u, M_{u^+}^*) d^* M_u^* - \frac{1}{2} \int_s^t \frac{\partial^2 \phi}{\partial y^2}(M_u, M_u^*) |\sigma_u^*|^2 du \\ &- \int_s^{t^-} \int_{-\infty}^{\infty} \left(\phi(M_u, M_{u^+}^* + J_{u^+,y}^*) - \phi(M_u, M_{u^+}^*) - J_{u^+,y}^* \frac{\partial \phi}{\partial y}(M_u, M_{u^+}^*) \right) \mu^*(d^*u, dy), \end{aligned} \quad (2.3)$$

$0 \leq s \leq t$, in which the forward integrals

$$\int_{s^+}^t \frac{\partial \phi}{\partial x}(M_{u^-}, M_u^*) dM_u \quad (2.4)$$

and

$$\int_{s^+}^t \int_{-\infty}^{\infty} \left(\phi(M_{u^-} + J_{u^-,x}, M_u^*) - \phi(M_{u^-}, M_u^*) - J_{u^-,x} \frac{\partial \phi}{\partial x}(M_{u^-}, M_u^*) \right) \mu(du, dx) \quad (2.5)$$

are anticipating and assumed to exist as limits in probability of Riemann sums, since in general M_u^* is only $(\mathcal{F}_u^* \vee \mathcal{F}_T)$ -measurable, and not \mathcal{F}_u -measurable, $0 \leq u \leq T$. In the sequel, sufficient conditions for the existence of the forward integrals will be provided using the Malliavin calculus.

The proof of the change of variable formula (2.3) follows the same lines as the proof of the forward-backward Itô formula of [5], to the exception that the forward integrals (2.4) and (2.5) with respect to dM_t and $\mu(dt, dx)$ are defined using limits in probability since they are anticipating. On the other hand, the backward integrals with respect to $d^*M_t^*$ and $\mu^*(d^*t, dx)$ are backward integrals defined in the adapted sense since $(B_t^*)_{t \in [0, T]}$ and $\mu^*(dt, dx)$ are independent of \mathcal{F}_T .

In the sequel the (\mathcal{F}_t) -forward martingale $(M_t)_{t \in [0, T]}$ of (2.1) and the (\mathcal{F}_t^*) -backward martingale $(M_t^*)_{t \in [0, T]}$ of (2.2) will be given by

$$M_t = \int_0^t \sigma_u dB_u + \int_0^t J_{u^-} (dN_u - \lambda_u du), \quad (2.6)$$

and

$$M_t^* = \int_t^T \sigma_u^* d^*B_u^* + \int_t^T J_{u^+}^* (d^*N_u^* - \lambda_u^* du), \quad (2.7)$$

where $(N_t)_{t \in [0, T]}$ and $(N_t^*)_{t \in [0, T]}$ are respectively a forward Poisson process with intensity $(\lambda_t)_{t \in [0, T]}$ and a backward Poisson process independent of $(N_t)_{t \in [0, T]}$ with intensity $(\lambda_t^*)_{t \in [0, T]}$, so that we have

$$\nu_t(dx) = \lambda_t \delta_{J_{t^-}}(dx) \quad \text{and} \quad \nu_t^*(dx) = \lambda_t^* \delta_{J_{t^+}^*}(dx),$$

where $(\sigma_t)_{t \in [0, T]}$, $(\sigma_t^*)_{t \in [0, T]}$, $(J_t)_{t \in [0, T]}$, $(J_t^*)_{t \in [0, T]}$ are square-integrable (\mathcal{F}_t) -adapted processes. The filtrations $(\mathcal{F}_t)_{t \in [0, T]}$ and $(\mathcal{F}_t^*)_{t \in [0, T]}$ are now given by

$$\mathcal{F}_t = \sigma(B_s, N_s : 0 \leq s \leq t) \quad \text{and} \quad \mathcal{F}_t^* = \sigma(B_s^*, N_s^* : t \leq s \leq T), \quad t \in [0, T]. \quad (2.8)$$

Recall also that $\{(B_t)_{t \in [0, T]}, (N_t)_{t \in [0, T]}\}$ is independent of $\{(B_t^*)_{t \in [0, T]}, (N_t^*)_{t \in [0, T]}\}$.

In this case the Itô formula (2.3) rewrites as

$$\begin{aligned} \phi(M_t + M_t^*) &= \phi(M_s + M_s^*) + \int_{s^+}^t \phi'(M_{u^-} + M_u^*) dM_u - \int_s^{t^-} \phi'(M_u + M_{u^+}^*) d^*M_u^* \\ &\quad + \int_{s^+}^t J_{u^-} \psi(M_{u^-} + M_u^*, J_{u^-}) dN_u - \int_s^{t^-} J_{u^+}^* \psi(M_u + M_{u^+}^*, J_{u^+}^*) d^*N_u^* \\ &\quad + \frac{1}{2} \int_s^t \phi''(M_u + M_u^*) (|\sigma_u|^2 - |\sigma_u^*|^2) du, \end{aligned} \quad (2.9)$$

$0 \leq s \leq t$, where

$$\psi(x, y) = \frac{\phi(x + y) - \phi(x) - y\phi'(x)}{y}, \quad x, y \in \mathbb{R}.$$

Next, we recall some results on the computation of the expectation of forward integrals for continuous and jump processes by the Malliavin calculus. We let D^B denote the Malliavin gradient with respect to the Brownian motion $(B_t)_{t \in [0, T]}$, defined from

$$D_t^B f(B_{t_1}, \dots, B_{t_n}) = \sum_{i=1}^n \mathbf{1}_{[0, t_i)}(t) \frac{\partial f}{\partial x_i}(B_{t_1}, \dots, B_{t_n}), \quad t \in [0, T], \quad (2.10)$$

$f \in \mathcal{C}_b^1(\mathbb{R}^n)$. We also define the partial finite difference operator D^N with respect to the Poisson process $(N_t)_{t \in [0, T]}$, as

$$D_t^N F(N.) = F(N. + \mathbf{1}_{[t, \infty)}(\cdot)) - F(N.), \quad t \in [0, T],$$

for any random variable $F : \Omega \rightarrow \mathbb{R}$, and consider the space $\mathbb{L}_{2,1}$ defined by the norm

$$\|u\|_{\mathbb{L}_{2,1}}^2 = \|u\|_{L^2(\Omega \times [0, T])}^2 + E \left[\int_0^T \int_0^T |D_s^B u_t|^2 ds dt \right] + E \left[\int_0^T \int_0^T |D_s^N u_t|^2 ds dt \right].$$

Let $(D^{B^-}u)_t$ denote the trace of $(D_t^{B^-}u_s)_{s, t \in [0, T]}$ defined by the left limit

$$\lim_{n \rightarrow \infty} \int_0^T \sup_{(s - \frac{1}{n}) \vee 0 \leq t < s} E [|D_s^B u_t - (D^{B^-}u)_s|^2] ds = 0, \quad (2.11)$$

cf. Relation (3.7) in Definition 3.1.1 of [7], for u a process in $\mathbb{L}_{2,1}$.

By Proposition 3.2.3 page 193 of [7], the expectation of the forward anticipating integral with respect to Brownian motion can be computed as in the next lemma.*

Lemma 2.1. *Assume that $(u_t)_{t \in [0, T]}$ is continuous in $\mathbb{L}_{2,1}$ and that the process $D^{B^-}u$ defined by (2.11) exists. Then the anticipating forward integral $\int_0^T u_t dB_t$ exists in $L^2(\Omega)$ and we have*

$$E \left[\int_0^T u_t dB_t \right] = E \left[\int_0^T (D^{B^-}u)_t dt \right].$$

*Note that a sign has to be changed in Proposition 3.2.3 page 193 of [7].

By Corollary 2.9 of [1] or Corollary 4.1 of [8], see also Proposition 3.1 of [6], we also have the following lemma, in which $(D^{N^-}u)_t$, $t \in [0, T]$ is defined by

$$\lim_{n \rightarrow \infty} \int_0^T \sup_{(s-\frac{1}{n}) \vee 0 \leq t < s} E [|D_s^N u_t - (D^{N^-}u)_s|^2] ds = 0,$$

as in (2.11).

Lemma 2.2. *Let $T > 0$ and assume that $u \in \mathbb{L}_{2,1}$ and $((D^{N^-}u)_t)_{t \in [0, T]} \in L^2(\Omega \times [0, T])$. Then we have*

$$E \left[\int_0^T u_t dN_t \right] = E \left[\int_0^T u_t \lambda_t dt \right] + E \left[\int_0^T \lambda_t (D^{N^-}u)_t dt \right].$$

The operators D^{B^*} , D^{B^*+} and D^{N^*} , D^{N^*+} are similarly defined for the backward Brownian motion $(B_t^*)_{t \in [0, T]}$ and Poisson process $(N_t^*)_{t \in [0, T]}$, from

$$D_t^{B^*} f(B_{t_1}^*, \dots, B_{t_n}^*) = \sum_{i=1}^n \mathbf{1}_{[t_i, T]}(t) \frac{\partial f}{\partial x_i}(B_{t_1}^*, \dots, B_{t_n}^*), \quad t \in [0, T],$$

$f \in C_b^1(\mathbb{R}^n)$, and

$$D_t^{N^*} F(N_\cdot) = F(N_\cdot + \mathbf{1}_{[0, t)}(\cdot)) - F(N_\cdot), \quad t \in [0, T],$$

and they satisfy the backward analogs of Lemmas 2.1 and 2.2, i.e.

$$E \left[\int_0^T u_t d^* B_t \right] = E \left[\int_0^T (D^{B^*+}u)_t dt \right], \quad (2.12)$$

and

$$E \left[\int_0^T u_t d^* N_t \right] = E \left[\int_0^T u_t \lambda_t^* dt \right] + E \left[\int_0^T \lambda_t^* (D^{N^*+}u)_t dt \right]. \quad (2.13)$$

The next proposition is the main result of this section.

Proposition 2.3. *Assume that the processes $(\sigma_t^*)_{t \in [0, T]}$ and $(J_t^*)_{t \in [0, T]}$ belong to the space $\mathbb{L}_{2,1}$. For all $\phi \in C^3(\mathbb{R})$ with $\phi^{(3)}$ bounded we have*

$$E [\phi(M_t + M_t^*)] = E [\phi(M_s + M_s^*)] + \frac{1}{2} E \left[\int_s^t \phi''(M_u + M_u^*) (|\sigma_u|^2 - |\sigma_u^*|^2) du \right]$$

$$\begin{aligned}
& +E \left[\int_s^t (J_u \lambda_u \psi(M_u + M_u^*, J_u) - J_u^* \lambda_u^* \psi(M_u + M_u^*, J_u^*)) du \right] \\
& + \frac{1}{2} E \left[\int_s^t \sigma_u \phi^{(3)}(M_u + M_u^*) \int_u^T D_u^B |\sigma_v|^2 dv du \right] \\
& + E \left[\int_s^t \sigma_u \int_u^T J_v^* \lambda_v^* (D_u^B J_v^*) \int_0^1 \phi^{(3)}(M_u + M_u^* + \tau J_v^*) d\tau dv du \right] \\
& + E \left[\int_s^t \lambda_u J_u \int_u^T (D_u^N \sigma_v^*) \int_0^1 (\sigma_v^* + \tau D_u^N \sigma_v^*) \phi^{(3)}(M_u + M_u^* + \tau D_u^N M_u^*) d\tau dv du \right] \\
& + E \left[\int_s^t \lambda_u J_u \int_u^T \lambda_v^* (D_u^N J_v^*) \int_0^1 (J_v^* + \tau D_u^N J_v^*) \right. \\
& \quad \left. \int_0^1 \phi^{(3)}(M_u + M_u^* + \tau D_u^N M_u^* + \rho(J_v^* + \tau D_u^N J_v^*)) d\rho d\tau dv du \right] \\
& + E \left[\int_s^t \lambda_u J_u \int_u^T \lambda_v^* (D_u^N J_v^*) \int_0^1 (J_v^* + \tau D_u^N J_v^*) \right. \\
& \quad \left. \int_0^1 \frac{\partial^2 \psi}{\partial x^2}(M_u + M_u^* + \tau D_u^N M_u^* + \rho(J_v^* + \tau D_u^N J_v^*), J_u) d\rho d\tau dv du \right] \\
& + E \left[\int_s^t \lambda_u J_u \int_u^T (D_u^N \sigma_v^*) \int_0^1 (\sigma_v^* + \tau D_u^N \sigma_v^*) \frac{\partial^2 \psi}{\partial x^2}(M_u + M_u^* + \tau D_u^N M_u^*, J_u) d\tau dv du \right],
\end{aligned}$$

$0 \leq s \leq t$, where

$$\psi(x, y) = \frac{\phi(x + y) - \phi(x) - y\phi'(x)}{y}, \quad x, y \in \mathbb{R}.$$

Proof. First we note that by the Itô formula (2.9) and the fact that the backward stochastic integral of a backward adapted process has zero expectation, we get

$$\begin{aligned}
E[\phi(M_t + M_t^*)] &= E[\phi(M_s + M_s^*)] + E \left[\int_{s^+}^t \phi'(M_{u^-} + M_u^*) dM_u \right] \\
&+ \frac{1}{2} E \left[\int_s^t \phi''(M_u + M_u^*) (|\sigma_u|^2 - |\sigma_u^*|^2) du \right] \\
&+ E \left[\int_{s^+}^t J_{u^-} \psi(M_{u^-} + M_u^*, J_{u^-}) dN_u \right] - E \left[\int_s^t J_u^* \psi(M_u + M_u^*, J_u^*) \lambda_u^* du \right].
\end{aligned}$$

Next we note that we have the relations $D_v^{B^*} M_u^* = \sigma_v^* \mathbf{1}_{[u, T]}(v)$, $D_v^{N^*} M_u^* = J_{v^+}^* \mathbf{1}_{[u, T]}(v)$, $(D^{B^-} \sigma)_u = 0$, $0 \leq u \leq v$, and

$$D_s^B M_u^* = \int_u^T D_s^B \sigma_v^* d^* B_v^* + \int_u^T D_s^B J_{v^+}^* (d^* N_v^* - \lambda_v^* dv), \quad 0 \leq u \leq s,$$

hence by the same argument as in Proposition 3.1.1 of [7], i.e. by the bound

$$\begin{aligned}
& \int_0^T \sup_{(s-\frac{1}{n}) \vee 0 \leq u < s} E \left[\left| D_s^B M_u^* - \int_s^T D_s^B \sigma_v^* d^* B_v^* - \int_s^T D_s^B J_{v^+}^* (d^* N_v^* - \lambda_v^* dv) \right|^2 \right] ds \\
&= \int_0^T \sup_{(s-\frac{1}{n}) \vee 0 \leq u < s} E \left[\left| \int_u^s D_s^B \sigma_v^* d^* B_v^* + \int_u^s D_s^B J_{v^+}^* (d^* N_v^* - \lambda_v^* dv) \right|^2 \right] ds \\
&\leq 2 \int_0^T E \left[\int_{(s-1/n) \vee 0}^s |D_s^B \sigma_v^*|^2 dv \right] ds + 2 \int_0^T E \left[\int_{(s-1/n) \vee 0}^s |D_s^B J_{v^+}^*|^2 \lambda_v^* dv \right] ds,
\end{aligned} \tag{2.14}$$

we have

$$(D^{B^-} M^*)_u = D_u^B M_u^* = \int_u^T D_u^B \sigma_v^* d^* B_v^* + \int_u^T D_u^B J_{v^+}^* (d^* N_v^* - \lambda_v^* dv).$$

Similarly we have the relation

$$D_s^N M_u^* = \int_u^T D_s^N \sigma_v^* d^* B_v^* + \int_u^T D_s^N J_{v^+}^* (d^* N_v^* - \lambda_v^* dv), \quad 0 \leq u \leq s,$$

which shows that

$$(D^{N^-} M^*)_u = \int_u^T D_u^N \sigma_v^* d^* B_v^* + \int_u^T D_u^N J_{v^+}^* (d^* N_v^* - \lambda_v^* dv) = D_u^N M_u^*,$$

by the same bound as (2.14). Hence by Lemmas 2.1 and 2.2 and (2.12)-(2.13) the expectation of the forward integral can be computed as follows:

$$\begin{aligned}
& E \left[\int_s^t \sigma_u \phi'(M_u + M_u^*) dB_u \right] = E \left[\int_s^t (D^{B^-}(\sigma_u \phi'(M_u + M_u^*)))_u du \right] \\
&= E \left[\int_s^t \sigma_u (D^{B^-} \phi'(M_u + M_u^*))_u du \right] \\
&= E \left[\int_s^t \sigma_u \phi''(M_u + M_u^*) (D^{B^-} M^*)_u du \right] \\
&= E \left[\int_s^t \sigma_u \phi''(M_u + M_u^*) \int_u^T D_u^B \sigma_v^* d^* B_v^* du \right] \\
&\quad + E \left[\int_s^t \sigma_u \phi''(M_u + M_u^*) \int_u^T D_u^B J_{v^+}^* (d^* N_v^* - \lambda_v^* dv) du \right] \\
&= E \left[\int_s^T \int_s^{v \wedge t} \sigma_u \phi''(M_u + M_u^*) D_u^B \sigma_v^* du d^* B_v^* \right]
\end{aligned}$$

$$\begin{aligned}
& +E \left[\int_s^T \int_s^{v \wedge t} \sigma_u \phi''(M_u + M_u^*) D_u^B J_v^* du (d^* N_v^* - \lambda_v^* dv) \right] \\
= & E \left[\int_s^T D^{B^*+} \int_s^{v \wedge t} \sigma_u \phi''(M_u + M_u^*) D_u^B \sigma_v^* dudv \right] \\
& +E \left[\int_s^T D_v^{N^*+} \int_s^{v \wedge t} \sigma_u \phi''(M_u + M_u^*) D_u^B J_v^* du \lambda_v^* dv \right] \\
= & E \left[\int_s^T \int_s^{v \wedge t} \sigma_u D_v^{B^*} \phi''(M_u + M_u^*) D_u^B \sigma_v^* dudv \right] \\
& +E \left[\int_s^T \int_s^{v \wedge t} \sigma_u D_v^{N^*} \phi''(M_u + M_u^*) D_u^B J_v^* du \lambda_v^* dv \right] \\
= & E \left[\int_s^T \int_s^{v \wedge t} \sigma_u \phi^{(3)}(M_u + M_u^*) (D_v^{B^*} M_u^*) D_u^B \sigma_v^* dudv \right] \\
& +E \left[\int_s^T \int_s^{v \wedge t} \sigma_u (\phi''(M_u + M_u^* + D_v^{N^*} M_u^*) - \phi''(M_u + M_u^*)) D_u^B J_v^* du \lambda_v^* dv \right] \\
= & E \left[\int_s^t \sigma_u \phi^{(3)}(M_u + M_u^*) \int_u^T (D_v^{B^*} M_u^*) D_u^B \sigma_v^* dv du \right] \\
& +E \left[\int_s^t \sigma_u \int_u^T J_v^* \lambda_v^* (D_u^B J_v^*) \int_0^1 \phi^{(3)}(M_u + M_u^* + \tau J_v^*) d\tau dv du \right] \\
= & E \left[\int_s^t \sigma_u \phi^{(3)}(M_u + M_u^*) \int_u^T \sigma_v^* D_u^B \sigma_v^* dv du \right] \\
& +E \left[\int_s^t \sigma_u \int_u^T J_v^* \lambda_v^* (D_u^B J_v^*) \int_0^1 \phi^{(3)}(M_u + M_u^* + \tau J_v^*) d\tau dv du \right].
\end{aligned}$$

Similarly, by Lemma 2.2 we have, since $D_u^N J_{u-} = 0$,

$$\begin{aligned}
& E \left[\int_{s^+}^t J_{u-} \phi'(M_{u-} + M_u^*) (dN_u - \lambda_u du) \right] \\
= & E \left[\int_s^t (D^{N^-} (J \phi'(M + M^*)))_u \lambda_u du \right] \\
= & E \left[\int_s^t J_u (D^{N^-} \phi'(M + M^*))_u \lambda_u du \right] \\
= & E \left[\int_s^t J_u (\phi'(M_u + M_u^* + D_u^N M_u^*) - \phi'(M_u + M_u^*)) \lambda_u du \right] \\
= & E \left[\int_s^t J_u \lambda_u (D_u^N M_u^*) \int_0^1 \phi''(M_u + M_u^* + \tau D_u^N M_u^*) d\tau du \right] \\
= & E \left[\int_s^t J_u \lambda_u \int_u^T D_u^N \sigma_v^* d^* B_v^* \int_0^1 \phi''(M_u + M_u^* + \tau D_u^N M_u^*) d\tau du \right]
\end{aligned}$$

$$\begin{aligned}
& +E \left[\int_s^t J_u \lambda_u \int_u^T D_u^N J_{v^+}^* (d^* N_v^* - \lambda_v^* dv) \int_0^1 \phi''(M_u + M_u^* + \tau D_u^N M_u^*) d\tau du \right] \\
= & E \left[\int_s^T \int_s^{v \wedge t} J_u \lambda_u (D_u^N \sigma_v^*) \int_0^1 \phi''(M_u + M_u^* + \tau D_u^N M_u^*) d\tau du d^* B_v^* \right] \\
& +E \left[\int_s^T \int_s^{v \wedge t} J_u \lambda_u (D_u^N J_{v^+}^*) \int_0^1 \phi''(M_u + M_u^* + \tau D_u^N M_u^*) d\tau du (d^* N_v^* - \lambda_v^* dv) \right] \\
= & E \left[\int_s^T \int_s^{v \wedge t} J_u \lambda_u (D_u^N \sigma_v^*) \int_0^1 \phi^{(3)}(M_u + M_u^* + \tau D_u^N M_u^*) D_v^{B^*} (M_u^* + \tau D_u^N M_u^*) d\tau du dv \right] \\
& +E \left[\int_s^T \int_s^{v \wedge t} J_u \lambda_u (D_u^N J_v^*) \int_0^1 (\phi''(M_u + M_u^* + \tau D_u^N M_u^* + D_v^{N^*} (M_u^* + \tau D_u^N M_u^*)) \right. \\
& \quad \left. - \phi''(M_u + M_u^* + \tau D_u^N M_u^*)) d\tau du \lambda_v^* dv \right] \\
= & E \left[\int_s^t J_u \lambda_u \int_u^T (D_u^N \sigma_v^*) \int_0^1 \phi^{(3)}(M_u + M_u^* + \tau D_u^N M_u^*) (\sigma_v^* + \tau D_u^N \sigma_v^*) d\tau dv du \right] \\
& +E \left[\int_s^t J_u \lambda_u \int_u^T \lambda_v^* (D_u^N J_v^*) \int_0^1 (J_v^* + \tau D_u^N J_v^*) \right. \\
& \quad \left. \int_0^1 \phi^{(3)}(M_u + M_u^* + \tau D_u^N M_u^* + \rho (J_v^* + \tau D_u^N J_v^*)) d\rho d\tau dv du \right].
\end{aligned}$$

On the other hand, still applying Lemma 2.2, we have

$$\begin{aligned}
& E \left[\int_s^t J_{u^-} \psi(M_{u^-} + M_u^*, J_u) (dN_u - \lambda_u du) \right] \\
= & E \left[\int_s^t J_u D_u^N \psi(M_u + M_u^*, J_u) \lambda_u du \right] \\
= & E \left[\int_s^t J_u (\psi(M_u + M_u^* + D_u^N M_u^*, J_u) - \psi(M_u + M_u^*, J_u)) \lambda_u du \right] \\
= & E \left[\int_s^t J_u (D_u^N M_u^*) \int_0^1 \frac{\partial \psi}{\partial x}(M_u + M_u^* + \tau D_u^N M_u^*, J_u) d\tau \lambda_u du \right] \\
= & E \left[\int_s^t J_u \int_u^T D_u^N J_v^* (d^* N_v^* - \lambda_v^* dv) \int_0^1 \frac{\partial \psi}{\partial x}(M_u + M_u^* + \tau D_u^N M_u^*, J_u) d\tau \lambda_u du \right] \\
& +E \left[\int_s^t J_u \int_u^T D_u^N \sigma_v^* d^* B_v^* \int_0^1 \frac{\partial \psi}{\partial x}(M_u + M_u^* + \tau D_u^N M_u^*, J_u) d\tau \lambda_u du \right] \\
= & \int_s^t \int_0^1 E \left[J_u \frac{\partial \psi}{\partial x}(M_u + M_u^* + \tau D_u^N M_u^*, J_u) \int_u^T D_u^N J_v^* (d^* N_v^* - \lambda_v^* dv) \right] d\tau \lambda_u du \\
& + \int_s^t \int_0^1 E \left[J_u \frac{\partial \psi}{\partial x}(M_u + M_u^* + \tau D_u^N M_u^*, J_u) \int_u^T D_u^N \sigma_v^* d^* B_v^* \right] d\tau \lambda_u du \\
= & \int_s^t \int_0^1 E \left[J_u \int_u^T \lambda_v^* (D_u^N J_v^*) D_v^{N^*} \frac{\partial \psi}{\partial x}(M_u + M_u^* + \tau D_u^N M_u^*, J_u) dv \right] d\tau \lambda_u du
\end{aligned}$$

$$\begin{aligned}
& + \int_s^t \int_0^1 E \left[J_u \int_u^T (D_u^N \sigma_v^*) D_v^{B^*} \frac{\partial \psi}{\partial x} (M_u + M_u^* + \tau D_u^N M_u^*, J_u) dv \right] d\tau \lambda_u du \\
= & E \left[\int_s^t \lambda_u J_u \int_0^1 \int_u^T \lambda_v^* (D_u^N J_v^*) D_v^{N^*} \frac{\partial \psi}{\partial x} (M_u + M_u^* + \tau D_u^N M_u^*, J_u) dv d\tau du \right] \\
& + E \left[\int_s^t \lambda_u J_u \int_0^1 \int_u^T (D_u^N \sigma_v^*) D_v^{B^*} \frac{\partial \psi}{\partial x} (M_u + M_u^* + \tau D_u^N M_u^*, J_u) dv d\tau du \right] \\
= & E \left[\int_s^t \lambda_u J_u \int_0^1 \int_u^T (D_u^N J_v^*) \left(\frac{\partial \psi}{\partial x} (M_u + M_u^* + \tau D_u^N M_u^* + D_v^{N^*} (M_u^* + \tau D_u^N M_u^*), J_u) \right. \right. \\
& \quad \left. \left. - \frac{\partial \psi}{\partial x} (M_u + M_u^* + \tau D_u^N M_u^*, J_u) \right) \lambda_v^* dv d\tau du \right] \\
& + E \left[\int_s^t \lambda_u J_u \int_0^1 \int_u^T (D_u^N \sigma_v^*) \frac{\partial^2 \psi}{\partial x^2} (M_u + M_u^* + \tau D_u^N M_u^*, J_u) D_v^{B^*} (M_u^* + \tau D_u^N M_u^*) dv d\tau du \right] \\
= & E \left[\int_s^t \lambda_u J_u \int_0^1 \int_u^T (D_u^N J_v^*) D_v^{N^*} (M_u^* + \tau D_u^N M_u^*) \right. \\
& \quad \left. \int_0^1 \frac{\partial^2 \psi}{\partial x^2} (M_u + M_u^* + \tau D_u^N M_u^* + \rho D_v^{N^*} (M_u^* + \tau D_u^N M_u^*), J_u) d\rho \lambda_v^* dv d\tau du \right] \\
& + E \left[\int_s^t \lambda_u J_u \int_0^1 \int_u^T (D_u^N \sigma_v^*) \frac{\partial^2 \psi}{\partial x^2} (M_u + M_u^* + \tau D_u^N M_u^*, J_u) D_v^{B^*} (M_u^* + \tau D_u^N M_u^*) dv d\tau du \right] \\
= & E \left[\int_s^t \lambda_u J_u \int_u^T \lambda_v^* D_u^N J_v^* \right. \\
& \quad \left. \int_0^1 (J_v^* + \tau D_u^N J_v^*) \int_0^1 \frac{\partial^2 \psi}{\partial x^2} (M_u + M_u^* + \tau D_u^N M_u^* + \rho (J_v^* + \tau D_u^N J_v^*), J_u) d\rho d\tau dv du \right] \\
& + E \left[\int_s^t \lambda_u J_u \int_0^1 \int_u^T (D_u^N \sigma_v^*) \frac{\partial^2 \psi}{\partial x^2} (M_u + M_u^* + \tau D_u^N M_u^*, J_u) (\sigma_v^* + \tau D_u^N \sigma_v^*) dv d\tau du \right].
\end{aligned}$$

□

Next we consider the application of Proposition 2.3 to the particular cases of continuous and pure jump processes, with respectively $J_u = J_u^* = 0$ and $\sigma_u = \sigma_u^* = 0$, $u \in [0, T]$.

Continuous case

In the next proposition we assume that

$$J_u = J_u^* = 0 \quad \text{or} \quad \lambda_u = \lambda_u^* = 0, \quad u \in [0, T]. \quad (2.15)$$

Proposition 2.4. *Assume that the process $(\sigma_t^*)_{t \in [0, T]}$ belongs to the space $\mathbb{L}_{2,1}$. Under Condition (2.15), for all $\phi \in \mathcal{C}^3(\mathbb{R})$ with $\phi^{(3)}$ bounded we have*

$$\begin{aligned} E [\phi(M_t + M_t^*)] &= E [\phi(M_s + M_s^*)] + \frac{1}{2} E \left[\int_s^t \phi''(M_u + M_u^*) (|\sigma_u|^2 - |\sigma_u^*|^2) du \right] \\ &\quad + \frac{1}{2} E \left[\int_s^t \sigma_u \phi^{(3)}(M_u + M_u^*) \int_u^T D_u^B |\sigma_v^*|^2 dv du \right], \end{aligned}$$

$$0 \leq s \leq t.$$

Pure jump case

In the next proposition we assume that

$$\sigma_u = \sigma_u^* = 0, \quad u \in [0, T]. \quad (2.16)$$

Proposition 2.5. *Assume that the process $(J_t^*)_{t \in [0, T]}$ belongs to the space $\mathbb{L}_{2,1}$. Under Condition (2.16), for all $\phi \in \mathcal{C}^3(\mathbb{R})$ with $\phi^{(3)}$ bounded we have*

$$\begin{aligned} E [\phi(M_t + M_t^*)] &= E [\phi(M_s + M_s^*)] \\ &\quad + E \left[\int_s^t (\lambda_u J_u \psi(M_u + M_u^*, J_u) - \lambda_u^* J_u^* \psi(M_u + M_u^*, J_u^*)) du \right] \\ &\quad + E \left[\int_s^t \lambda_u J_u \int_u^T \lambda_v^* (D_u^N J_v^*) \int_0^1 (J_v^* + \tau D_u^N J_v^*) \right. \\ &\quad \quad \left. \int_0^1 \phi^{(3)}(M_u + M_u^* + \tau D_u^N M_u^* + \rho(J_v^* + \tau D_u^N J_v^*)) d\rho d\tau dv du \right] \\ &\quad + E \left[\int_s^t \lambda_u J_u \int_u^T \lambda_v^* (D_u^N J_v^*) \int_0^1 (J_v^* + \tau D_u^N J_v^*) \right. \\ &\quad \quad \left. \int_0^1 \frac{\partial^2 \psi}{\partial x^2}(M_u + M_u^* + \tau D_u^N M_u^* + \rho(J_v^* + \tau D_u^N J_v^*), J_u) d\rho d\tau dv du \right], \end{aligned}$$

$$0 \leq s \leq t.$$

3 Convex increasing forward-backward martingales

In this section we derive sufficient conditions for the forward-backward martingale $(M_t + M_t^*)_{t \in [0, T]}$ defined from (2.6) and (2.7) to be non-decreasing in the convex order (1.1). The forward and backward filtrations $(\mathcal{F}_t)_{t \in [0, T]}$ and $(\mathcal{F}_t^*)_{t \in [0, T]}$ are given by (2.8).

Theorem 3.1. Assume that the processes $(\sigma_t^*)_{t \in [0, T]}$ and $(J_t^*)_{t \in [0, T]}$ belong to the space $\mathbb{L}_{2,1}$, and

$$i) |\sigma_u^*| \leq |\sigma_u|, \quad d\mathbb{P}du\text{-a.e.},$$

$$ii) 0 \leq J_u^* \leq J_u, \quad d\mathbb{P}du\text{-a.e.},$$

$$iii) 0 \leq \lambda_u^* J_u^* \leq \lambda_u J_u, \quad d\mathbb{P}du\text{-a.e.},$$

$$iv) (\sigma_v^* + \tau D_u^N \sigma_v^*) D_u^N \sigma_v^* \geq 0, \quad d\mathbb{P}dudv\text{-a.e.}, \quad 0 \leq u \leq v, \quad \tau \in [0, 1],$$

$$v) \sigma_u \sigma_v^* D_u^B \sigma_v^* \geq 0, \quad \sigma_u D_u^B J_v^* \geq 0, \quad D_u^N J_v^* \geq 0, \quad d\mathbb{P}dudv\text{-a.e.}, \quad 0 \leq u \leq v.$$

Then we have

$$E[\phi(M_s + M_s^*)] \leq E[\phi(M_t + M_t^*)], \quad 0 \leq s \leq t, \quad (3.1)$$

for all convex functions $\phi \in \mathcal{C}^2(\mathbb{R})$ such that ϕ' and ϕ'' are convex.

Proof. We start by assuming that $\phi \in \mathcal{C}^3(\mathbb{R})$. Clearly, the terms

$$\begin{aligned} & \frac{1}{2} E \left[\int_s^t \phi''(M_u + M_u^*) (|\sigma_u|^2 - |\sigma_u^*|^2) du \right] \\ & + E \left[\int_s^t (J_u \lambda_u \psi(M_u + M_u^*, J_u) - J_u^* \lambda_u^* \psi(M_u + M_u^*, J_u^*)) du \right] \end{aligned}$$

in Proposition 2.3 are non-negative due to Conditions (i) – (iii) and the convexity of ϕ , which show that the function

$$\psi(x, y) = \frac{\phi(x + y) - \phi(x) - y\phi'(x)}{y} = y \int_0^1 (1 - \tau) \phi''(x + \tau y) d\tau \geq 0, \quad (3.2)$$

is non-negative in $(x, y) \in \mathbb{R} \times \mathbb{R}_+$, and non-decreasing in $y > 0$ for all fixed $x \in \mathbb{R}$.

Next, by Conditions (iv) and (v) we have

$$\begin{aligned} & E \left[\int_s^t \sigma_u \phi^{(3)}(M_u + M_u^*) \int_u^T D_u^B |\sigma_v^*|^2 dv du \right] \geq 0, \\ & E \left[\int_s^t \sigma_u \int_u^T J_v^* \lambda_v^* (D_u^B J_v^*) \int_0^1 \phi^{(3)}(M_u + M_u^* + \tau J_v) d\tau dv du \right] \geq 0, \\ & E \left[\int_s^t J_u \lambda_u \int_u^T (D_u^N \sigma_v^*) \int_0^1 \phi^{(3)}(M_u + M_u^* + \tau D_u^N M_u^*) (\sigma_v^* + \tau D_u^N \sigma_v^*) d\tau dv du \right] \geq 0, \end{aligned}$$

and

$$E \left[\int_s^t \lambda_u J_u \int_0^1 \int_u^T (D_u^N \sigma_v^*) \frac{\partial^2 \psi}{\partial x^2} (M_u + M_u^* + \tau D_u^N M_u^*, J_u) (\sigma_v^* + \tau D_u^N \sigma_v^*) dv d\tau du \right] \geq 0,$$

since

$$\frac{\partial^2 \psi}{\partial x^2} (x, y) = \frac{\phi''(x+y) - \phi''(x) - y\phi^{(3)}(x)}{y} \geq 0,$$

$(x, y) \in \mathbb{R} \times \mathbb{R}_+$, by (3.2) and the convexity of ϕ'' , and

$$E \left[\int_s^t \lambda_u J_u \int_u^T \lambda_v^* (D_u^N J_v^*) \int_0^1 (J_v^* + \tau D_u^N J_v^*) \int_0^1 \left(\phi^{(3)}(X_{u,v}) + \frac{\partial^2 \psi}{\partial x^2} (X_{u,v}, J_u) \right) d\rho d\tau dv du \right] \geq 0,$$

where

$$X_{u,v} = M_u + M_u^* + \tau D_u^N M_u^* + \rho(J_v^* + \tau D_u^N J_v^*),$$

since

$$\begin{aligned} J_v^* + \tau D_u^N J_v^* &= J_v^* + \tau (J_v^*(B, N + \mathbf{1}_{[u, +\infty)}(\cdot)) - J_v^*) \\ &= (1 - \tau) J_v^* + \tau J_v^*(B, N + \mathbf{1}_{[u, +\infty)}(\cdot)) \\ &\geq 0, \end{aligned} \tag{3.3}$$

$dudvd\mathbb{P}$ -a.e., $\tau \in [0, 1]$. The conclusion then follows from Proposition 2.3 when $\phi \in \mathcal{C}^3(\mathbb{R})$, and (3.1) finally extends to functions $\phi \in \mathcal{C}^2(\mathbb{R})$. \square

Next we consider the particular cases of continuous and pure jump processes, in which some of the above assumptions can be relaxed.

Continuous case

Theorem 3.2. *Assume that the process $(\sigma_t^*)_{t \in [0, T]}$ belongs to the space $\mathbb{L}_{2,1}$. Under Condition (2.15), assume $|\sigma_u^*| \leq |\sigma_u|$, $d\mathbb{P}du$ -a.e., and*

$$\sigma_u D_u^B |\sigma_v^*|^2 \geq 0, \quad d\mathbb{P}dudv - a.e., \quad 0 \leq u \leq v. \tag{3.4}$$

Then for all $0 \leq s \leq t \leq T$, we have

$$E[\phi(M_s + M_s^*)] \leq E[\phi(M_t + M_t^*)], \quad 0 \leq s \leq t \leq T, \tag{3.5}$$

for all convex functions $\phi \in \mathcal{C}^1(\mathbb{R})$ with convex derivative ϕ' .

Replacing (3.4) with

$$\sigma_u D_u^B |\sigma_v^*|^2 \leq 0, \quad d\mathbb{P}dudv - a.e.,$$

we find that (3.5) holds true for all convex function $\phi \in \mathcal{C}^1(\mathbb{R})$ with concave derivative ϕ' .

Pure jump case

Theorem 3.3. *Assume that the process $(J_t^*)_{t \in [0, T]}$ belongs to the space $\mathbb{L}_{2,1}$. Under Condition (2.16), assume that the following conditions are satisfied:*

- i) $0 \leq J_u^* \leq J_u$, $d\mathbb{P}du$ -a.e.,
- ii) $0 \leq \lambda_u^* J_u^* \leq \lambda_u J_u$, $d\mathbb{P}du$ -a.e.,
- iii) $D_u^N J_v^* \geq 0$, $d\mathbb{P}dudv$ -a.e., $0 \leq u \leq v$,

then we have

$$E[\phi(M_s + M_s^*)] \leq E[\phi(M_t + M_t^*)], \quad 0 \leq s \leq t \leq T, \quad (3.6)$$

for all convex functions $\phi \in \mathcal{C}^2(\mathbb{R})$ such that ϕ' and ϕ'' are convex.

4 Convex ordering for stochastic integrals

In this section we apply the results of Section 3 to the derivation of convex concentration inequalities for random variables represented as stochastic integrals, as

$$\int_0^T \sigma_u dB_u + \int_0^T J_{u-} (dN_u - \lambda_u du),$$

and

$$\int_0^T \sigma_u^* d\hat{B}_u + \int_0^T J_{u-}^* (d\hat{N}_u - \lambda_u^* du),$$

where $(\sigma_t)_{t \in [0, T]}$, $(\sigma_t^*)_{t \in [0, T]}$, $(J_t)_{t \in [0, T]}$, $(J_t^*)_{t \in [0, T]}$ are square-integrable and (\mathcal{F}_t) -adapted, $(\hat{B}_t)_{t \in [0, T]}$ is a standard (forward) Brownian motion and $(\hat{N}_t)_{t \in [0, T]}$ is a standard (forward) Poisson process with intensity $(\lambda_t^*)_{t \in [0, T]}$, mutually independent and independent of $(B_t)_{t \in [0, T]}$ and $(N_t)_{t \in [0, T]}$.

Theorem 4.1. *Assume that the processes $(\sigma_t^*)_{t \in [0, T]}$ and $(J_t^*)_{t \in [0, T]}$ belong to the space $\mathbb{L}_{2,1}$, and*

$$i) |\sigma_u^*| \leq |\sigma_u|, \quad d\mathbb{P}du\text{-a.e.},$$

$$ii) 0 \leq J_u^* \leq J_u, \quad d\mathbb{P}du\text{-a.e.},$$

$$iii) \lambda_u^* J_u^* \leq \lambda_u J_u, \quad d\mathbb{P}du\text{-a.e.},$$

$$iv) (\sigma_v^* + \tau D_u^N \sigma_v^*) D_u^N \sigma_v^* \geq 0, \quad d\mathbb{P}dudv\text{-a.e.}, \quad 0 \leq u \leq v, \quad \tau \in [0, 1],$$

$$v) \sigma_u \sigma_v^* D_u^B \sigma_v^* \geq 0, \quad \sigma_u D_u^B J_v^* \geq 0, \quad D_u^N J_v^* \geq 0, \quad d\mathbb{P}dudv\text{-a.e.}, \quad 0 \leq u \leq v.$$

Then we have

$$E \left[\phi \left(\int_0^T \sigma_u^* d\hat{B}_u + \int_0^T J_{u-}^* (d\hat{N}_u - \lambda_u^* du) \right) \right] \leq E \left[\phi \left(\int_0^T \sigma_u dB_u + \int_0^T J_{u-} (dN_u - \lambda_u du) \right) \right], \quad (4.1)$$

for all convex functions $\phi \in \mathcal{C}^3(\mathbb{R})$ such that ϕ' and ϕ'' are convex.

Proof. Consider the (forward) martingale

$$\hat{M}_t = \int_0^t \sigma_u^* d\hat{B}_u + \int_0^t J_{u-}^* (d\hat{N}_u - \lambda_u^* du), \quad t \in [0, T],$$

where the processes $(\sigma_t^*)_{t \in [0, T]}$ and $(\hat{J}_t^*)_{t \in [0, T]}$ are (\mathcal{F}_t) -adapted. Defining the backward Brownian motion $(B_t^*)_{t \in [0, T]}$ and Poisson process $(N_t^*)_{t \in [0, T]}$ by

$$B_t^* = \hat{B}_T - \hat{B}_t, \quad N_t^* = \hat{N}_T - \hat{N}_t, \quad t \in [0, T], \quad (4.2)$$

we have the identity in law

$$\begin{aligned} \hat{M}_T &:= \int_0^T \sigma_u^* d\hat{B}_u + \int_0^T J_{u-}^* (d\hat{N}_u - \lambda_u^* du) \\ &= \int_0^T \sigma_u^* d^* B_u^* + \int_0^T J_{u+}^* (d^* N_u^* - \lambda_u^* du) \\ &= M_0^*, \end{aligned}$$

which holds since the integrands $(\sigma_t^*)_{t \in [0, T]}$ and $(J_t^*)_{t \in [0, T]}$ are independent of the integrators $(\hat{B}_t)_{t \in [0, T]}$, $(\hat{N}_t)_{t \in [0, T]}$, and by the definition (4.2) of $(B_t^*)_{t \in [0, T]}$ and $(N_t^*)_{t \in [0, T]}$.

We conclude by Theorem 3.1 which shows that

$$E[\phi(\hat{M}_T)] = E[\phi(M_0 + M_0^*)] \leq E[\phi(M_T + M_T^*)] = E[\phi(M_T)],$$

$t \in [0, T]$. □

Note that if $\sigma_v^* \geq 0$ then Condition (iv) above can be replaced with

$$iv') D_u^N \sigma_v^* \geq 0, \text{ d}\mathbb{P}dudv\text{-a.e.},$$

by the same argument as in (3.3) above.

The following results can be proved in a similar way in the continuous and pure jump cases.

Corollary 4.2. *Assume that the process $(\sigma_t^*)_{t \in [0, T]}$ belongs to the space $\mathbb{L}_{2,1}$. Under Condition (2.15), assume that $|\sigma_u^*| \leq |\sigma_u|$, $d\mathbb{P}dudv\text{-a.e.}$ and*

$$\sigma_u D_u^B |\sigma_v^*|^2 \geq 0, \quad d\mathbb{P}dudv - \text{a.e.}, \quad 0 \leq u \leq v. \quad (4.3)$$

Then we have

$$E \left[\phi \left(\int_0^T \sigma_u^* d\hat{B}_u \right) \right] \leq E \left[\phi \left(\int_0^T \sigma_u dB_u \right) \right], \quad (4.4)$$

for all convex functions $\phi \in \mathcal{C}^1(\mathbb{R})$ with convex derivative ϕ' .

The next corollary is stated in the pure jump case.

Corollary 4.3. *Assume that the process $(J_t^*)_{t \in [0, T]}$ belongs to the space $\mathbb{L}_{2,1}$. Under Condition (2.16), assume that the following conditions are satisfied*

$$i) 0 \leq J_u^* \leq J_u, \text{ d}\mathbb{P}du\text{-a.e.},$$

$$ii) 0 \leq \lambda_u^* J_u^* \leq \lambda_u J_u, \text{ d}\mathbb{P}du\text{-a.e.},$$

$$iii) D_u^N J_v^* \geq 0, \text{ d}\mathbb{P}dudv\text{-a.e.}, \quad 0 \leq u \leq v.$$

Then for all $0 \leq s \leq t \leq T$, we have

$$E \left[\phi \left(\int_0^T J_{u-}^* (d\hat{N}_u - \lambda_u^* du) \right) \right] \leq E \left[\phi \left(\int_0^T J_{u-} (dN_u - \lambda_u du) \right) \right], \quad (4.5)$$

for all convex functions $\phi \in \mathcal{C}^2(\mathbb{R})$ such that ϕ' and ϕ'' are convex.

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