# The Stein and Chen-Stein methods for functionals of non-symmetric Bernoulli processes

Nicolas Privault* Giovanni Luca Torrisi†

**Abstract**

Based on a new multiplication formula for discrete multiple stochastic integrals with respect to non-symmetric Bernoulli random walks, we extend the results of [14] on the Gaussian approximation of symmetric Rademacher sequences to the setting of possibly non-identically distributed independent Bernoulli sequences. We also provide Poisson approximation results for these sequences, by following the method of [15]. Our arguments use covariance identities obtained from the Clark-Ocone representation formula in addition to those usually based on the inverse of the Ornstein-Uhlenbeck operator.

*Keywords*: Bernoulli processes; Chen-Stein method; Stein method; Malliavin calculus; Clark-Ocone representation.
*Mathematics Subject Classification:* 60F05, 60G57, 60H07.

## 1 Introduction

Malliavin calculus and the Stein method were combined for the first time for Gaussian fields in the seminal paper [13], whose results have later been extended to other settings, including Poisson processes [15], [16]. In particular, the Stein method has been applied in [14] to Rademacher sequences $(X_n)_{n\in\mathbb{N}}$ of independent and identically distributed Bernoulli random variables with $P(X_1 = 1) = P(X_1 = -1) = 1/2$, in order to derive bounds on distances between the probability laws of functionals of $(X_n)_{n\in\mathbb{N}}$ and the law $\mathcal{N}(0,1)$ of a standard $\mathcal{N}(0,1)$ normal random variable $Z$. Those approaches exploit a covariance representation based on the number (or Ornstein-Uhlenbeck) operator $L$ and its inverse $L^{-1}$.

*Division of Mathematical Sciences, Nanyang Technological University, SPMS-MAS-05-43, 21 Nanyang Link Singapore 637371. e-mail: `nprivault@ntu.edu.sg`, `http://www.ntu.edu.sg/home/nprivault/`

†Istituto per le Applicazioni del Calcolo "Mauro Picone", CNR, Via dei Taurini 19, 00185 Roma, Italy. e-mail: `torrisi@iac.rm.cnr.it`, `http://www.iac.rm.cnr.it/~torrisi/`

From here onwards, we denote by $\mathcal{C}_b^2$ the set of all real-valued bounded functions with bounded derivatives up to the second order. In particular, for $h \in \mathcal{C}_b^2$, using a chain rule proved in the symmetric case, the bound

$$|\mathrm{E}[h(F)] - \mathrm{E}[h(Z)]| \leq A_1 \min\{4\|h\|_\infty, \|h''\|_\infty\} + \|h''\|_\infty A_2, \tag{1.1}$$

has been derived in [14] (see Theorem 3.1 therein) for centered functionals $F$ of a symmetric Bernoulli random walk $(X_n)_{n\in\mathbb{N}}$. Here, $(X_n)_{n\in\mathbb{N}}$ is built as the sequence of canonical projections on $\Omega := \{-1,1\}^{\mathbb{N}}$ and

$$A_1 = \mathrm{E}\left[\left|1 - \langle DF, -DL^{-1}F\rangle_{\ell^2(\mathbb{N})}\right|\right], \qquad A_2 = \frac{20}{3}\mathrm{E}\left[\langle |DL^{-1}F|, |DF|^3\rangle_{\ell^2(\mathbb{N})}\right],$$

where $\langle\cdot,\cdot\rangle_{\ell^2(\mathbb{N})}$ is the usual inner product on $\ell^2(\mathbb{N})$, and $D$ is the symmetric gradient defined as

$$D_k F(\omega) = \frac{1}{2}(F(\omega_+^k) - F(\omega_-^k)), \quad k \in \mathbb{N},$$

where, given $\omega = (\omega_0, \omega_1, \ldots) \in \Omega$, we let

$$\omega_+^k = (\omega_0, \ldots, \omega_{k-1}, +1, \omega_{k+1}, \ldots)$$

and

$$\omega_-^k = (\omega_0, \ldots, \omega_{k-1}, -1, \omega_{k+1}, \ldots).$$

The above bound (1.1) can be used to control the Wasserstein distance between $\mathcal{N}(0,1)$ and the law of $F$ as in Corollary 3.6 in [14]. In addition, the right-hand side of (1.1) yields explicit bounds in the case where $F$ is a single discrete stochastic integral (see Corollary 3.3 in [14]) or a multiple discrete stochastic integral (see Section 4 in [14]). In this latter case the derivation of explicit bounds is based on a multiplication formula proved in the symmetric case (see Proposition 2.9 in [14]).

In this paper we provide Gaussian and Poisson approximations for functionals of not-necessarily symmetric Bernoulli sequences via the Stein and Chen-Stein methods, respectively. See [11] for recent related results on Gaussian approximation, without relying on a multiplication formula for discrete multiple stochastic integrals.

The normal and Poisson approximations are based on suitable chain rules in Propositions 2.1 and 2.2 and on an extension to the non-symmetric case of the multiplication formula for discrete multiple stochastic integrals (see Proposition 5.1 and Section 9 for its proof). In addition to using the Ornstein-Uhlenbeck operator $L$ for covariance representations, we also derive error bounds for the normal and Poisson approximations using covariance representations based on the Clark-Ocone formula, following the argument implemented in [20]. Indeed the operator $L$ is of a more delicate use in applications to functionals whose multiple stochastic integral expansion is not explicitly known. In contrast with covariance identities based on the number operator, which rely on the divergence-gradient composition, the Clark-Ocone formula only requires the computation of a gradient and a conditional expectation.

A bound for the Wasserstein distance between a standard Gaussian random variable and a (standardized) function of a finite sequence of independent random variables has been obtained in [4], via the construction of an auxiliary random variable which allows one to approximate the Stein equation. Although the results in our paper are restricted to the Bernoulli case, they may be applied to functionals of an infinite sequence of Bernoulli distributed random variables. A comparison between a bound in our paper and that one in [4] is given at the end of the first example of Section 4.

As far as the Gaussian approximation is concerned, using a covariance representation based on the Clark-Ocone formula, in Theorem 3.2 below we find sufficient conditions on centered functionals $F$ of a not necessarily symmetric Bernoulli random walk so that

$$|\mathrm{E}[h(F)] - \mathrm{E}[h(Z)]| \leq B_1 \min\{4\|h\|_\infty, \|h''\|_\infty\} + \|h'\|_\infty B_2 + \|h''\|_\infty B_3 \qquad (1.2)$$

for any $h \in \mathcal{C}_b^2$, and for some positive constants $B_1, B_2, B_3 > 0$; similarly, using a covariance representation based on the Ornstein-Uhlenbeck operator, in Theorem 3.4 below we provide alternate sufficient conditions on centered functionals $F$ of a not necessarily symmetric Bernoulli random walk so that the bound (1.2) holds for different positive constants $C_1, C_2, C_3 > 0$, in place of $B_1, B_2, B_3$ respectively. In Theorem 3.6 below we show that the bound (1.2) can be used to control the Fortet-Mourier distance $d_{\mathrm{FM}}$ between $F$ and the standard $\mathcal{N}(0,1)$ normal random variable $Z$, i.e. we prove

$$d_{\mathrm{FM}}(F, Z) \leq \sqrt{2(B_1 + B_3)(5 + \mathrm{E}[|F|])} + B_2.$$

A similar bound holds, under alternate conditions on $F$, with the constant $B_i$ replaced by $C_i$ $(i = 1, 2, 3)$. Replacing the Stein method by the Chen-Stein method, we also show that this approach applies to the Poisson approximation in addition to the Gaussian approximation, and treat discrete multiple stochastic integrals as examples in both cases.

This paper is organized as follows. In Section 2 we recall some elements of stochastic analysis of Bernoulli processes, including chain rules for finite difference operators. In Section 3 we present the two different upper bounds for the quantity $|\mathrm{E}[h(F)]-\mathrm{E}[h(Z)]|$, $h \in \mathcal{C}_b^2$, described above and the related application to the Fortet-Mourier distance. Section 4 contains explicit first chaos bounds with application to determinantal processes, while Section 5 is concerned with bounds for the $n$th chaoses. The important case of quadratic functionals (second chaoses) is treated in a separate paragraph. In Section 6 we apply our arguments to the Poisson approximation and in Sections 7 and 8 we investigate the case of single and multiple discrete stochastic integrals. Finally, Section 9 deals with the new multiplication formula for discrete multiple stochastic integrals in the non-symmetric case, whose proof is modeled on normal martingales that are solution of a deterministic structure equation.

# 2 Stochastic analysis of Bernoulli processes

In this section we provide some preliminaries. The reader is directed to [18] and references therein for more insight into the stochastic analysis of Bernoulli processes.

From now on we assume that the canonical projections $X_n : \Omega \to \{-1,1\}$, $\Omega = \{-1,1\}^{\mathbb{N}}$, are considered under the not necessarily symmetric measure $P$ given on cylinder sets by

$$P(\{\varepsilon_0,\dots,\varepsilon_n\}\times\{-1,1\}^{\mathbb{N}}) = \prod_{k=0}^{n} p_k^{(1+\varepsilon_k)/2} q_k^{(1-\varepsilon_k)/2}, \quad \varepsilon_k \in \{-1,1\},\ k=0,\dots,n.$$

Given $\omega = (\omega_0,\omega_1,\dots) \in \Omega$ and $\omega_+^k$, $\omega_-^k$ defined as above, for any $F : \Omega \to \mathbb{R}$ we consider the finite difference operator

$$D_kF(\omega) = \sqrt{p_kq_k}(F(\omega_+^k) - F(\omega_-^k)), \quad k \in \mathbb{N}$$

and, denoting by $\kappa$ the counting measure on $\mathbb{N}$, we consider the $L^2(\Omega\times\mathbb{N}) = L^2(\Omega\times\mathbb{N}, P\otimes\kappa)$-valued operator $D$ defined for any $F : \Omega \to \mathbb{R}$, by $DF = (D_kF)_{k\in\mathbb{N}}$. Given $n \geq 1$ we denote by $\ell^2(\mathbb{N})^{\otimes n} = \ell^2(\mathbb{N}^n)$ the class of functions on $\mathbb{N}^n$ that are square integrable with respect to $\kappa^{\otimes n}$, we denote by $\ell^2(\mathbb{N})^{\circ n}$ the subspace of $\ell^2(\mathbb{N})^{\otimes n}$ formed by functions that are symmetric in $n$ variables. The $L^2$ domain of $D$ is given by

$$\mathrm{Dom}(D) = \{F \in L^2(\Omega) : \ DF \in L^2(\Omega \times \mathbb{N})\} = \{F \in L^2(\Omega) : \ \mathrm{E}[\|DF\|^2_{\ell^2(\mathbb{N})}] < \infty\}.$$

We let $(Y_n)_{n\geq 0}$ denote the sequence of centered and normalized random variables defined by

$$Y_n = \frac{q_n - p_n + X_n}{2\sqrt{p_n q_n}},$$

which satisfies the discrete structure equation

$$Y_n^2 = 1 + \frac{q_n - p_n}{2\sqrt{p_n q_n}} Y_n. \tag{2.1}$$

Given $f_1 \in \ell^2(\mathbb{N})$ we define the first order discrete stochastic integral of $f_1$ as

$$J_1(f_1) = \sum_{k\geq 0} f_1(k) Y_k,$$

and we let

$$J_n(f_n) = \sum_{(i_1,\ldots,i_n)\in\Delta_n} f_n(i_1,\ldots,i_n) Y_{i_1} \ldots Y_{i_n}$$

denote the discrete multiple stochastic integral of order $n$ of $f_n$ in the subspace $\ell^2_{\mathfrak{s}}(\Delta_n)$ of $\ell^2(\mathbb{N})^{\circ n}$ composed of symmetric kernels that vanish on diagonals, i.e. on the complement of

$$\Delta_n = \{(k_1,\ldots,k_n) \in \mathbb{N}^n : \ k_i \neq k_j, \ 1 \leq i < j \leq n\}, \quad n \geq 1.$$

As a convention we identify $\ell^2(\mathbb{N}^0)$ to $\mathbb{R}$ and let $J_0(f_0) = f_0$, $f_0 \in \mathbb{R}$. Hereafter, we shall refer to the set of functionals of the form $J_n(f)$ as the $n$-chaos. The multiple stochastic integrals satisfy the isometry formula

$$\mathrm{E}[J_n(f_n) J_m(g_m)] = \mathbb{1}_{\{n=m\}} n! \langle f_n, g_m \rangle_{\ell^2_{\mathfrak{s}}(\Delta_n)},$$

$f_n \in \ell^2_{\mathfrak{s}}(\Delta_n)$, $g_m \in \ell^2_{\mathfrak{s}}(\Delta_m)$, cf. e.g. Proposition 1.3.2 of [19].

The finite difference operator acts on multiple stochastic integrals as follows:

$$D_k J_n(f_n) = n J_{n-1}(f_n(*,k)\mathbb{1}_{\Delta_n}(*,k)) = n J_{n-1}(f_n(*,k)), \tag{2.2}$$

$k \in \mathbb{N}$, $f_n \in \ell^2_{\mathfrak{s}}(\Delta_n)$. Due to the chaos representation property any square integrable $F$ may be represented as $F = \sum_{n\geq 0} J_n(f_n)$, $f_n \in \ell^2_{\mathfrak{s}}(\Delta_n)$, and so the $L^2$ domain of $D$ may be rewritten as

$$\mathrm{Dom}(D) = \left\{ F = \sum_{n\geq 0} J_n(f_n) \ : \ \sum_{n\geq 1} n\, n! \|f_n\|^2_{\ell^2(\mathbb{N})^{\otimes n}} < \infty \right\}.$$

Next we present a chain rule for the finite difference operator that extends Proposition 2.14 in [14] from the symmetric to the non-symmetric case. This chain rule will be used later on for the normal approximation. In the following we write $F_k^{\pm}$ in place of $F(\omega_{\pm}^k)$.

**Proposition 2.1** *Let* $F \in \mathrm{Dom}(D)$ *and* $f : \mathbb{R} \to \mathbb{R}$ *be thrice differentiable with bounded third derivative. Assume moreover that* $f(F) \in \mathrm{Dom}(D)$*. Then, for any integer* $k \geq 0$ *there exists a random variable* $R_k^F$ *such that*

$$D_k f(F) = f'(F) D_k F - \frac{|D_k F|^2}{4\sqrt{p_k q_k}} (f''(F_k^+) + f''(F_k^-)) X_k + R_k^F, \quad a.s. \tag{2.3}$$

*where*

$$|R_k^F| \leq \frac{5}{3!} \|f'''\|_\infty \frac{|D_k F|^3}{p_k q_k}, \quad a.s. \tag{2.4}$$

*Proof.* By a standard Taylor expansion we have

$$\begin{aligned} D_k f(F) &= \sqrt{p_k q_k}(f(F_k^+) - f(F_k^-)) = \sqrt{p_k q_k}(f(F_k^+) - f(F)) - \sqrt{p_k q_k}(f(F_k^-) - f(F)) \\ &= \sqrt{p_k q_k} f'(F)(F_k^+ - F) + \frac{\sqrt{p_k q_k}}{2} f''(F)(F_k^+ - F)^2 + R_k^+ \\ &\quad - \sqrt{p_k q_k} f'(F)(F_k^- - F) - \frac{\sqrt{p_k q_k}}{2} f''(F)(F_k^- - F)^2 + R_k^- \\ &= f'(F) D_k F + \frac{\sqrt{p_k q_k}}{2} f''(F)[(F_k^+ - F)^2 - (F_k^- - F)^2] + R_k^+ + R_k^-, \end{aligned} \tag{2.5}$$

where

$$|R_k^{\pm}| \leq \frac{\sqrt{p_k q_k}}{3!} \|f'''\|_\infty |F_k^{\pm} - F|^3. \tag{2.6}$$

By the mean value theorem we find

$$\begin{aligned} f''(F) &= \frac{f''(F_k^+) + f''(F_k^-)}{2} + \frac{f''(F) - f''(F_k^+) + f''(F) - f''(F_k^-)}{2} \\ &= \frac{f''(F_k^+) + f''(F_k^-)}{2} + R_k', \end{aligned}$$

where

$$|R_k'| \leq \frac{\|f'''\|_\infty}{2} (|F_k^+ - F| + |F_k^- - F|).$$

Substituting this into (2.5) we get

$$D_k f(F) = f'(F) D_k F + \frac{\sqrt{p_k q_k}}{4} (f''(F_k^+) + f''(F_k^-))[(F_k^+ - F)^2 - (F_k^- - F)^2] + R_k^+ + R_k^- + R_k^*, \tag{2.7}$$

where

$$|R_k^*| \leq \frac{\sqrt{p_k q_k}}{4} \|f'''\|_\infty (|F_k^+ - F| + |F_k^- - F|)(|F_k^+ - F|^2 - |F_k^- - F|^2)$$

$$\leq \frac{\sqrt{p_k q_k}}{4} \|f'''\|_\infty (|F_k^+ - F| + |F_k^- - F|)|F_k^+ - F|^2. \tag{2.8}$$

Note that

$$\begin{aligned} F_k^+ - F &= (F_k^+ - F)\mathbb{1}_{\{X_k=-1\}} + (F_k^+ - F)\mathbb{1}_{\{X_k=1\}} = (F_k^+ - F)\mathbb{1}_{\{X_k=-1\}} \\ &= (F_k^+ - F_k^-)\mathbb{1}_{\{X_k=-1\}}, \end{aligned} \tag{2.9}$$

and similarly,

$$F_k^- - F = -(F_k^+ - F_k^-)\mathbb{1}_{\{X_k=1\}}. \tag{2.10}$$

Therefore we have $|F_k^\pm - F| \leq |D_k F|/\sqrt{p_k q_k}$, and combining this with (2.6) and (2.8) we find

$$|R_k^\pm| \leq \frac{\|f'''\|_\infty}{3! p_k q_k} |D_k F|^3, \quad |R_k^*| \leq \frac{\|f'''\|_\infty}{2 p_k q_k} |D_k F|^3. \tag{2.11}$$

By (2.9) and (2.10) we also have

$$(F_k^+ - F)^2 = (F_k^+ - F_k^-)^2 \mathbb{1}_{\{X_k=-1\}} \quad \text{and} \quad (F_k^- - F)^2 = (F_k^+ - F_k^-)^2 \mathbb{1}_{\{X_k=1\}},$$

therefore

$$\begin{aligned} (F_k^+ - F)^2 - (F_k^- - F)^2 &= (F_k^+ - F_k^-)^2 (\mathbb{1}_{\{X_k=-1\}} - \mathbb{1}_{\{X_k=1\}}) \\ &= -(F_k^+ - F_k^-)^2 X_k = -\frac{|D_k F|^2}{p_k q_k} X_k. \end{aligned}$$

The claim follows substituting this expression into (2.7) and by using (2.11) to estimate the remainder. □

Now we present a chain rule for the finite difference operator, which is suitable for integer-valued functionals. This chain rule will be used later on for the Poisson approximation. Given a function $f : \mathbb{N} \to \mathbb{R}$ we define the operators

$$\Delta f(k) := f(k+1) - f(k), \quad \Delta^2 f := \Delta(\Delta f).$$

**Proposition 2.2** *Let $F \in \mathrm{Dom}(D)$ be an $\mathbb{N}$-valued random variable. Then, for any $f : \mathbb{N} \to \mathbb{R}$ so that $f(F) \in \mathrm{Dom}(D)$, we have*

$$D_k f(F) = \Delta f(F) D_k F + R_k^F, \tag{2.12}$$

*where*

$$|R_k^F| \leq \frac{\|\Delta^2 f\|_\infty}{2} \left( \left| \frac{D_k F}{\sqrt{p_k q_k}} \mathbb{1}_{\{X_k=-1\}} \left( \frac{D_k F}{\sqrt{p_k q_k}} - 1 \right) \right| + \left| \frac{D_k F}{\sqrt{p_k q_k}} \mathbb{1}_{\{X_k=1\}} \left( \frac{D_k F}{\sqrt{p_k q_k}} + 1 \right) \right| \right). \tag{2.13}$$

*Proof.* As shown in the proof of Theorem 3.1 in [15], for any $f : \mathbb{N} \to \mathbb{R}$ and any $k, a \in \mathbb{N}$,

$$|f(k) - f(a) - \Delta f(a)(k-a)| \leq \frac{\|\Delta^2 f\|_\infty}{2} |(k-a)(k-a-1)|. \tag{2.14}$$

Therefore, taking first $k = F_k^+$, $a = F$ and then $k = F_k^-$, $a = F$, we deduce

$$\begin{aligned} D_k f(F) &= \sqrt{p_k q_k}(f(F_k^+) - f(F)) - \sqrt{p_k q_k}(f(F_k^-) - f(F)) \\ &= \sqrt{p_k q_k} \Delta f(F)(F_k^+ - F) + R_k^{(1)} + \sqrt{p_k q_k} \Delta f(F)(F_k^- - F) + R_k^{(2)}, \end{aligned}$$

where by (2.14), setting $R_k^F := R_k^{(1)} + R_k^{(2)}$, we have

$$|R_k^F| \leq \frac{\|\Delta^2 f\|_\infty}{2} (|(F_k^+ - F)(F_k^+ - F - 1)| + |(F_k^- - F)(F_k^- - F - 1)|).$$

The claim follows from (2.9) and (2.10). □

Next we give two alternative covariance representation formulas. Let $(\mathcal{F}_n)_{n \geq -1}$ be the filtration defined by

$$\mathcal{F}_{-1} = \{\emptyset, \Omega\}, \qquad \mathcal{F}_n = \sigma\{X_0, \ldots, X_n\}, \quad n \geq 0.$$

By Proposition 1.10.1 of [19], for any $F, G \in \mathrm{Dom}(D)$ with $F$ centered we have the Clark-Ocone covariance representation formula

$$\mathrm{Cov}(F, G) = \mathrm{E}[FG] = \mathrm{E}\left[\sum_{k \geq 0} \mathrm{E}[D_k F \mid \mathcal{F}_{k-1}] D_k G\right]. \tag{2.15}$$

The second covariance representation formula involves the inverse of the Ornstein-Uhlenbeck operator. The domain $\mathrm{Dom}(L)$ of the Ornstein-Uhlenbeck operator $L : L^2(\Omega) \to L_0^2(\Omega)$, where $L_0^2(\Omega)$ denotes the subspace of $L^2(\Omega)$ composed of centered random variables, is given by

$$\mathrm{Dom}(L) = \left\{ F = \sum_{n \geq 0} J_n(f_n) \ : \ \sum_{n \geq 1} n^2 \, n! \|f_n\|^2_{\ell^2(\mathbb{N})^{\otimes n}} < \infty \right\}$$

and, for any $F \in \mathrm{Dom}(L)$,

$$LF = -\sum_{n=1}^{\infty} n J_n(f_n).$$

The inverse of $L$, denoted by $L^{-1}$, is defined on $L_0^2(\Omega)$ by

$$L^{-1} F = -\sum_{n=1}^{\infty} \frac{1}{n} J_n(f_n),$$

with the convention $L^{-1}F = L^{-1}(F - \mathrm{E}[F])$ in case $F$ is not centered, as in e.g. [15]. Using this convention, for any $F, G \in \mathrm{Dom}(D)$ we have

$$\mathrm{Cov}(F,G) = \mathrm{E}[G(F - \mathrm{E}[F])] = \mathrm{E}\left[\langle DG, -DL^{-1}F\rangle_{\ell^2(\mathbb{N})}\right], \tag{2.16}$$

cf. Lemma 2.12 of [14] in the symmetric case. For the sake of completeness, we provide an alternative expression for the covariance representation formula (2.16). Let $(P_t)_{t\geq 0}$ be the semigroup associated to the Ornstein-Uhlenbeck operator $L$ (we refer the reader to Section 10 of [18] for the details). Then

$$P_t J_n(f_n) = \mathrm{e}^{-nt} J_n(f_n), \quad n \geq 1,$$

and so for any $F = \sum_{n=0}^{\infty} J_n(f_n) \in \mathrm{Dom}(D)$ one has

$$\begin{aligned}
\int_0^\infty \mathrm{e}^{-t} P_t D_k F \,\mathrm{d}t &= \sum_{n=1}^{\infty} n \int_0^\infty \mathrm{e}^{-t} P_t J_{n-1}(f_n(*,k))\,\mathrm{d}t \\
&= \sum_{n\geq 1} n \int_0^\infty \mathrm{e}^{-t} \mathrm{e}^{-(n-1)t} J_{n-1}(f_n(*,k))\,\mathrm{d}t \\
&= \sum_{n=1}^{\infty} J_{n-1}(f_n(*,k)) \\
&= -D_k L^{-1} F.
\end{aligned}$$

Consequently, the covariance representation (2.16) may be rewritten as

$$\mathrm{Cov}(F,G) = \mathrm{E}[G(F - \mathrm{E}[F])] = \mathrm{E}\left[\sum_{k=0}^{\infty} \int_0^\infty \mathrm{e}^{-t} D_k G P_t D_k F \,\mathrm{d}t\right],$$

for any $F, G \in \mathrm{Dom}(D)$, cf. Proposition 1.10.2 of [19].

# 3 Normal approximation of Bernoulli functionals

In this section we present two different upper bounds for the quantity $|\mathrm{E}[h(F)] - \mathrm{E}[h(Z)]|$, $h \in \mathcal{C}_b^2$. The first one is obtained by using the covariance representation formula (2.15), while the second one, obtained by using the covariance representation formula (2.16), is a strict extension of the bound given in Theorem 3.1 of [14].

Before proceeding further we recall some necessary background on the Stein method for the normal approximation and refer to [1], [10], [24], [25] and to [14] for more insight into this technique.

**Stein's method for normal approximation**

Let $Z$ be a standard $\mathcal{N}(0,1)$ normal random variable and consider the so-called Stein's equation associated with $h : \mathbb{R} \to \mathbb{R}$:

$$h(x) - \mathrm{E}[h(Z)] = f'(x) - xf(x), \quad x \in \mathbb{R}.$$

We refer to part $(ii)$ of Lemma 2.15 in [14] for the following lemma. More precisely, the estimates on the first and second derivative are proved in Lemma II.3 of [25], the estimate of the third derivative is proved in Theorem 1.1 of [6] and the alternative estimate on the first derivative may be found in [1] and [10].

**Lemma 3.1** *If $h \in \mathcal{C}_b^2$, then the Stein equation has a solution $f_h$ which is thrice differentiable and such that $\|f_h'\|_\infty \leq 4\|h\|_\infty$, $\|f_h''\|_\infty \leq 2\|h'\|_\infty$ and $\|f_h'''\|_\infty \leq 2\|h''\|_\infty$. We also have $\|f_h'\|_\infty \leq \|h''\|_\infty$.*

Combining the Stein equation with this lemma, for a generic square integrable and centered random variable $F$ we have

$$|\mathrm{E}[h(F)] - \mathrm{E}[h(Z)]| = |\mathrm{E}[f_h'(F) - Ff_h(F)]|. \tag{3.1}$$

Let $(F_n)_{n\geq 1}$ be a sequence of square integrable and centered random variables. If

$$|\mathrm{E}[h(F_n)] - \mathrm{E}[h(Z)]| \to 0, \qquad h \in \mathcal{C}_b^2$$

then $(F_n)_{n\geq 1}$ converges to $Z$ in distribution as $n$ tends to infinity, and so an upper bound for the right-hand side of (3.1) may provide informations about this normal approximation.

The results of Sections 3.1 and 3.2 below are given in terms of bounds for $|\mathrm{E}[h(F)]-\mathrm{E}[h(Z)]|$, for test functions in $\mathcal{C}_b^2$, and they are applied in Section 3.3 to derive bounds for the Fortet-Mourier distance between the laws of two random variables $X$ and $Y$, which is defined by

$$d_{\mathrm{FM}}(X, Y) = \sup_{h \in \mathcal{FM}} |\mathrm{E}[h(X)] - \mathrm{E}[h(Y)]|, \tag{3.2}$$

where $\mathcal{FM}$ is the class of functions $h$ such that $\|h\|_{BL} = \|h\|_L + \|h\|_\infty \leq 1$, where $\|\cdot\|_L$ denotes the standard Lipschitz semi-norm. Clearly, any $h \in \mathcal{FM}$ is Lipschitz with Lipschitz constant less than or equal to 1 and so it is Lebesgue a.e. differentiable and $\|h'\|_\infty \leq 1$. One can also shows that $d_{\mathrm{FM}}$ metrizes the convergence in distribution, see e.g. Chapter 11 in [7].

### 3.1 Clark-Ocone bound

**Theorem 3.2** *Let $F \in \mathrm{Dom}(D)$ be a centered random variable and assume that*

$$\begin{aligned} B_1 : &= \mathrm{E}\left[\left|1 - \sum_{k\geq 0} \mathrm{E}[D_k F \mid \mathcal{F}_{k-1}] D_k F\right|\right], \\ B_2 : &= \sum_{k\geq 0} \frac{|1-2p_k|}{\sqrt{p_k q_k}} \mathrm{E}[|\mathrm{E}[D_k F \mid \mathcal{F}_{k-1}]||D_k F|^2], \qquad (3.3) \\ B_3 : &= \frac{5}{3}\sum_{k\geq 0} \frac{1}{p_k q_k} \mathrm{E}[|\mathrm{E}[D_k F \mid \mathcal{F}_{k-1}]||D_k F|^3] \qquad (3.4) \end{aligned}$$

*are finite. Then we have*

$$|\mathrm{E}[h(F)] - \mathrm{E}[h(Z)]| \leq B_1 \min\{4\|h\|_\infty, \|h''\|_\infty\} + \|h'\|_\infty B_2 + \|h''\|_\infty B_3 \qquad (3.5)$$

*for all $h \in \mathcal{C}_b^2$.*

*Proof.* Since the first derivative of $f_h$ is bounded we have that $f_h$ is Lipschitz. So $f_h(F) \in L^2(\Omega)$ and

$$|D_k f_h(F)| = \sqrt{p_k q_k}|f_h(F_k^+) - f_h(F_k^-)| \leq \|f_h'\|_\infty |D_k F|.$$

Consequently we have

$$\mathrm{E}[\|Df_h(F)\|^2_{\ell^2(\mathbb{N})}] \leq \|f_h'\|_\infty \mathrm{E}[\|DF\|^2_{\ell^2(\mathbb{N})}]$$

and $f_h(F) \in \mathrm{Dom}(D)$. Since $F$ is centered, by the covariance representation (2.15) and the chain rule of Proposition 2.1 we have

$$\begin{aligned} \mathrm{E}[F f_h(F)] &= \mathrm{E}\left[\sum_{k\geq 0} \mathrm{E}[D_k F \mid \mathcal{F}_{k-1}] D_k f_h(F)\right] \\ &= \mathrm{E}\left[\sum_{k\geq 0} \mathrm{E}[D_k F \mid \mathcal{F}_{k-1}] D_k F f_h'(F)\right] - \mathrm{E}\left[\sum_{k\geq 0} \mathrm{E}[D_k F \mid \mathcal{F}_{k-1}] \frac{|D_k F|^2}{4\sqrt{p_k q_k}}(f_h''(F_k^+) + f_h''(F_k^-))X_k\right] \\ &\quad + \mathrm{E}\left[\sum_{k\geq 0} \mathrm{E}[D_k F \mid \mathcal{F}_{k-1}] R_k^F(h)\right]. \qquad (3.6) \end{aligned}$$

Note that the three expectations in (3.6) are finite. The first one since $DF \in L^2(\Omega \times \mathbb{N})$ and $f_h'$ is bounded, indeed by Jensen's inequality

$$\mathrm{E}\left[\left|\sum_{k\geq 0} \mathrm{E}[D_k F \mid \mathcal{F}_{k-1}] D_k F f_h'(F)\right|\right] \leq 4\|h\|_\infty \mathrm{E}\left[\sum_{k\geq 0} \mathrm{E}[|D_k F| \mid \mathcal{F}_{k-1}]|D_k F|\right]$$

$$= 4\|h\|_\infty \mathrm{E}\left[\sum_{k\geq 0} \mathrm{E}[\mathrm{E}[|D_kF| \mid \mathcal{F}_{k-1}]|D_kF| \mid \mathcal{F}_{k-1}]\right]$$
$$= 4\|h\|_\infty \mathrm{E}\left[\sum_{k\geq 0} \mathrm{E}[\mathrm{E}[|D_kF| \mid \mathcal{F}_{k-1}]^2]\right]$$
$$\leq 4\|h\|_\infty \mathrm{E}\left[\sum_{k\geq 0} \mathrm{E}[\mathrm{E}[|D_kF|^2 \mid \mathcal{F}_{k-1}]\right]$$
$$= 4\|h\|_\infty \sum_{k\geq 0} \mathrm{E}[|D_kF|^2] < \infty;$$

the second relation follows from the boundedness of $f_h''$ and (3.3), while the third one follows from (2.4) and (3.4). The random variables $\mathrm{E}[D_kF \mid \mathcal{F}_{k-1}]$, $D_kF$, $F_k^\pm$ are independent of $X_k$ (the first one because it is $\mathcal{F}_{k-1}$-measurable and the random variables $(X_k)_{k\in\mathbb{N}}$ are independent, the others by their definition). Therefore, the equality (3.6) reduces to

$$\mathrm{E}[Ff_h(F)] = \mathrm{E}\left[f_h'(F)\sum_{k\geq 0}\mathrm{E}[D_kF \mid \mathcal{F}_{k-1}]D_kF\right] \tag{3.7}$$
$$+ \sum_{k\geq 0} \frac{1-2p_k}{4\sqrt{p_kq_k}}\mathrm{E}[\mathrm{E}[D_kF \mid \mathcal{F}_{k-1}]|D_kF|^2(f_h''(F_k^+) + f_h''(F_k^-))]$$
$$+ \mathrm{E}\left[\sum_{k\geq 0}\mathrm{E}[D_kF \mid \mathcal{F}_{k-1}]R_k^F(h)\right]. \tag{3.8}$$

Inserting this expression into the right-hand side of (3.1) we deduce

$$|\mathrm{E}[h(F)] - \mathrm{E}[h(Z)]| \leq B_1 \min\{4\|h\|_\infty, \|h''\|_\infty\} + \|h'\|_\infty B_2 \tag{3.9}$$
$$+\mathrm{E}\left[\sum_{k\geq 0}|\mathrm{E}[D_kF \mid \mathcal{F}_{k-1}]||R_k^F(h)|\right], \tag{3.10}$$

where to get the term (3.9) we used the inequalities $\|f_h'\|_\infty \leq \min\{4\|h\|_\infty, \|h''\|_\infty\}$ and $\|f_h''\|_\infty \leq 2\|h'\|_\infty$ (see Lemma 3.1). Using (2.4) one may easily see that the term in (3.10) is bounded above by $\|h''\|_\infty B_3$. The proof is complete. □

**Corollary 3.3** *Let $F \in \mathrm{Dom}(D)$ be a centered random variable and assume that*

$$B_1 := |1 - \|F\|_{L^2(\Omega)}^2| + \|\langle D_\cdot F, \mathrm{E}[D_\cdot F \mid \mathcal{F}_{\cdot-1}]\rangle_{\ell^2(\mathbb{N})} - \mathrm{E}[\langle D_\cdot F, \mathrm{E}[D_\cdot F \mid \mathcal{F}_{\cdot-1}]\rangle_{\ell^2(\mathbb{N})}]\|_{L^2(\Omega)},$$

$$B_2 := \sum_{k\geq 0}\frac{|1-2p_k|}{\sqrt{p_kq_k}}\|D_kF\|_{L^2(\Omega)}\sqrt{\mathrm{E}[|D_kF|^4]}, \tag{3.11}$$

$$B_3 : \quad = \quad \frac{5}{3} \sum_{k \geq 0} \frac{1}{p_k q_k} \mathrm{E}[|D_k F|^4]$$

*are finite. Then* (3.5) *holds for all* $h \in \mathcal{C}_b^2$.

*Proof.* By the Cauchy-Schwarz and the triangular inequalities we have

$$\begin{aligned}
\mathrm{E}\left[ \left| 1 - \sum_{k \geq 0} \mathrm{E}[D_k F \mid \mathcal{F}_{k-1}] D_k F \right| \right] &\leq \left\| 1 - \sum_{k \geq 0} \mathrm{E}[D_k F \mid \mathcal{F}_{k-1}] D_k F \right\|_{L^2(\Omega)} \\
&\leq |1 - \|F\|_{L^2(\Omega)}^2| + \|\langle D_\cdot F, \mathrm{E}[D_\cdot F \mid \mathcal{F}_{\cdot-1}] \rangle_{\ell^2(\mathbb{N})} - \|F\|_{L^2(\Omega)}^2\|_{L^2(\Omega)}.
\end{aligned}$$

By the Clark-Ocone formula (2.15) we have

$$\begin{aligned}
\mathrm{E}[\langle D_\cdot F, \mathrm{E}[D_\cdot F \mid \mathcal{F}_{\cdot-1}] \rangle_{\ell^2(\mathbb{N})}] &= \mathrm{E}\left[ \sum_{k=0}^{\infty} D_k F \, \mathrm{E}[D_k F \mid \mathcal{F}_{k-1}] \right] \\
&= \|F\|_{L^2(\Omega)}^2.
\end{aligned}$$

Therefore

$$\mathrm{E}\left[ \left| 1 - \sum_{k \geq 0} \mathrm{E}[D_k F \mid \mathcal{F}_{k-1}] D_k F \right| \right] \leq B_1.$$

By the Cauchy-Schwarz and Jensen inequalities we have

$$\begin{aligned}
\mathrm{E}[|\mathrm{E}[D_k F \mid \mathcal{F}_{k-1}]| |D_k F|^2] &\leq \sqrt{\mathrm{E}[|\mathrm{E}[D_k F \mid \mathcal{F}_{k-1}]|^2]} \sqrt{\mathrm{E}[|D_k F|^4]} \\
&\leq \sqrt{\mathrm{E}[\mathrm{E}[|D_k F|^2 \mid \mathcal{F}_{k-1}]]} \sqrt{\mathrm{E}[|D_k F|^4]} \\
&= \sqrt{\mathrm{E}[|D_k F|^2]} \sqrt{\mathrm{E}[|D_k F|^4]},
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{E}[|\mathrm{E}[D_k F \mid \mathcal{F}_{k-1}]| |D_k F|^3] &= \sqrt{\mathrm{E}[|\mathrm{E}[D_k F \mid \mathcal{F}_{k-1}]|^2 |D_k F|^2]} \sqrt{\mathrm{E}[|D_k F|^4]} \\
&\leq \sqrt{\mathrm{E}[\mathrm{E}[|D_k F|^2 \mid \mathcal{F}_{k-1}] |D_k F|^2]} \sqrt{\mathrm{E}[|D_k F|^4]} \\
&\leq \sqrt{\mathrm{E}[|D_k F|^4]} \times \sqrt{\mathrm{E}[|D_k F|^4]} = \mathrm{E}[|D_k F|^4].
\end{aligned}$$

The claim follows from Theorem 3.2. □

## 3.2 Semigroup bound

**Theorem 3.4** *Let* $F \in \mathrm{Dom}(D)$ *be a centered random variable and let*

$$C_1 : \quad = \quad \mathrm{E}\left[ \left| 1 - \langle DF, -DL^{-1}F \rangle_{\ell^2(\mathbb{N})} \right| \right],$$

$$C_2 : \quad = \quad \sum_{k\geq 0} \frac{|1-2p_k|}{\sqrt{p_k q_k}} \mathrm{E}[|D_k L^{-1} F||D_k F|^2], \tag{3.12}$$

$$C_3 : \quad = \quad \frac{5}{3} \sum_{k\geq 0} \frac{1}{p_k q_k} \mathrm{E}[|D_k L^{-1} F||D_k F|^3] \tag{3.13}$$

*be finite. Then for all* $h \in \mathcal{C}_b^2$ *we have*

$$|\mathrm{E}[h(F)] - \mathrm{E}[h(Z)]| \leq C_1 \min\{4\|h\|_\infty, \|h''\|_\infty\} + \|h'\|_\infty C_2 + \|h''\|_\infty C_3. \tag{3.14}$$

*Proof.* Although the proof is similar to that of Theorem 3.2, we give the details since some points need a different justification. As in the proof of Theorem 3.2 one has $f_h(F) \in \mathrm{Dom}(D)$. Since $F$ is centered, by the covariance representation (2.16) and the chain rule of Proposition 2.1 we have

$$\begin{aligned}
\mathrm{E}[F f_h(F)] &= -\mathrm{E}\left[\sum_{k\geq 0} D_k f_h(F) D_k L^{-1} F\right] \\
&= -\mathrm{E}\left[\sum_{k\geq 0} D_k F f_h'(F) D_k L^{-1} F\right] + \mathrm{E}\left[\sum_{k\geq 0} X_k \frac{|D_k F|^2}{4\sqrt{p_k q_k}} (f_h''(F_k^+) + f_h''(F_k^-)) D_k L^{-1} F\right] \\
&\quad -\mathrm{E}\left[\sum_{k\geq 0} D_k L^{-1} F R_k^F(h)\right].
\end{aligned} \tag{3.15}$$

Note that the three expectations in (3.15) are finite. The first one since $DF \in L^2(\Omega \times \mathbb{N})$ and $f_h'$ is bounded, indeed

$$\begin{aligned}
\mathrm{E}\left[\left|\sum_{k\geq 0} -D_k L^{-1} F D_k F f_h'(F)\right|\right] &\leq 4\|h\|_\infty \mathrm{E}\left[\sum_{k\geq 0} |D_k L^{-1} F||D_k F|\right] \\
&\leq 4\|h\|_\infty \left(\mathrm{E}\sum_{k\geq 0} |D_k L^{-1} F|^2\right)^{1/2} \left(\mathrm{E}\sum_{k\geq 0} |D_k F|^2\right)^{1/2} \\
&= 4\|h\|_\infty \left(\mathrm{E}\|DL^{-1}F\|_{\ell^2(\mathbb{N})}^2\right)^{1/2} \left(\mathrm{E}\|DF\|_{\ell^2(\mathbb{N})}^2\right)^{1/2} \\
&\leq 4\|h\|_\infty \mathrm{E}[\|DF\|_{\ell^2(\mathbb{N})}^2] < \infty,
\end{aligned}$$

where for the latter inequality we used the relation

$$\mathrm{E}[\|DL^{-1}F\|_{\ell^2(\mathbb{N})}^2] \leq \mathrm{E}[\|DF\|_{\ell^2(\mathbb{N})}^2]$$

(see Lemma 2.13(3) in [14]); the second one due to the boundedness of $f_h''$ and (3.12); the third one due to (2.4) and (3.13). By Lemma 2.13 in [14] we have that the random variables

$D_k L^{-1}F$, $D_kF$ and $F_k^{\pm}$ are independent of $X_k$. Therefore, the equality (3.15) reduces to

$$\mathrm{E}[Ff_h(F)] = -\mathrm{E}\left[\sum_{k\geq 0} f'_h(F)D_kFD_kL^{-1}F\right] + \sum_{k\geq 0}\frac{1-2p_k}{4\sqrt{p_kq_k}}\mathrm{E}[|D_kF|^2(f''(F_k^+)+f''(F_k^-))D_kL^{-1}F]$$
$$-\mathrm{E}\left[\sum_{k\geq 0} R_k^F(h)D_kL^{-1}F\right]. \qquad (3.16)$$

Inserting this expression into the right-hand side of (3.1) we deduce

$$|\mathrm{E}[h(F)] - \mathrm{E}[h(Z)]| \leq C_1\min\{4\|h\|_\infty, \|h''\|_\infty\} + \|h'\|_\infty C_2 \qquad (3.17)$$
$$+\mathrm{E}\left[\sum_{k\geq 0}|D_kL^{-1}F||R_k^F(h)|\right], \qquad (3.18)$$

where to get the term (3.17) we used the inequalities $\|f'_h\|_\infty \leq \min\{4\|h\|_\infty, \|h''\|_\infty\}$ and $\|f''_h\|_\infty \leq 2\|h'\|_\infty$ (see Lemma 3.1). Using (2.4) one may easily see that the term in (3.18) is bounded above by $\|h''\|_\infty C_3$. The proof is complete. □

Note that, formally, the upper bound (3.5) may be obtained by (3.14) substituting the term $-D_kL^{-1}F$ in the definitions of $C_1$, $C_2$, $C_3$, with $\mathrm{E}[D_kF \mid \mathcal{F}_{k-1}]$, and vice versa.

**Corollary 3.5** *Let $F \in \mathrm{Dom}(D)$ be a centered random variable and let*

$$C_1 : \; = \; |1-\|F\|^2_{L^2(\Omega)}| + \|\langle D.F, -D.L^{-1}F\rangle_{\ell^2(\mathbb{N})} - \mathrm{E}[\langle D.F, -D.L^{-1}F\rangle_{\ell^2(\mathbb{N})}]\|_{L^2(\Omega)},$$

$$C_2 : \; = \; B_2, \quad \text{where } B_2 \text{ is defined by (3.11)}$$

*and $C_3$ defined by* (3.13) *be finite. Then* (3.14) *holds for all $h \in \mathcal{C}_b^2$.*

*Proof.* By the Cauchy-Schwarz and the triangular inequalities we have

$$\mathrm{E}\left[\left|1-\langle D.F, -D.L^{-1}F\rangle_{\ell^2(\mathbb{N})}\right|\right] \leq \left\|1-\langle D.F, -D.L^{-1}F\rangle_{\ell^2(\mathbb{N})}\right\|_{L^2(\Omega)}$$
$$\leq \; |1-\|F\|^2_{L^2(\Omega)}| + \|\langle D.F, -D.L^{-1}F\rangle_{\ell^2(\mathbb{N})} - \|F\|^2_{L^2(\Omega)}\|_{L^2(\Omega)}.$$

By the covariance representation formula (2.16) we have

$$\mathrm{E}[\langle D.F, -D.L^{-1}F\rangle_{\ell^2(\mathbb{N})}] = \|F\|^2_{L^2(\Omega)}.$$

Therefore

$$\mathrm{E}\left[\left|1-\langle D.F, -D.L^{-1}F\rangle_{\ell^2(\mathbb{N})}\right|\right] \leq C_1.$$

Let $F \in \mathrm{Dom}(D)$ be of the form

$$F = \sum_{n\geq 0} J_n(f_n), \quad f_n \in \ell^2_{\mathfrak{s}}(\Delta_n).$$

Then

$$-D_k L^{-1} F = \sum_{n\geq 1} J_{n-1}(f_n(*,k)) \quad \text{and} \quad D_k F = \sum_{n\geq 1} n J_{n-1}(f_n(*,k)).$$

So, by the isometry formula, we have

$$\begin{aligned} \mathrm{E}[|D_k L^{-1} F|^2] &= \mathrm{E}\left[\Big|\sum_{n\geq 1} J_{n-1}(f_n(*,k))\Big|^2\right] \\ &= \sum_{n\geq 1} \mathrm{E}[|J_{n-1}(f_n(*,k))|^2] \\ &= \sum_{n\geq 1} (n-1)! \|f_n(*,k)\|^2_{\ell^2(\mathbb{N})^{\otimes(n-1)}} \end{aligned}$$

and

$$\begin{aligned} \mathrm{E}[|D_k F|^2] &= \mathrm{E}\left[\Big|\sum_{n\geq 1} n J_{n-1}(f_n(*,k))\Big|^2\right] \\ &= \sum_{n\geq 1} n^2 \mathrm{E}[|J_{n-1}(f_n(*,k))|^2] \\ &= \sum_{n\geq 1} n^2 (n-1)! \|f_n(*,k)\|^2_{\ell^2(\mathbb{N})^{\otimes(n-1)}}. \end{aligned}$$

So

$$\mathrm{E}[|D_k L^{-1} F|^2] \leq \mathrm{E}[|D_k F|^2] \tag{3.19}$$

and by the Cauchy-Schwarz inequality, we deduce

$$\begin{aligned} \mathrm{E}[|D_k L^{-1} F||D_k F|^2] &\leq \sqrt{\mathrm{E}[|D_k L^{-1} F|^2]}\sqrt{\mathrm{E}[|D_k F|^4]} \\ &\leq \sqrt{\mathrm{E}[|D_k F|^2]}\sqrt{\mathrm{E}[|D_k F|^4]}. \end{aligned}$$

The claim follows from Theorem 3.4. □

## 3.3 Fortet-Mourier distance

In this section we provide bounds in the Fortet-Mourier distance (3.2).

**Theorem 3.6** *Let $F \in \mathrm{Dom}(D)$ be centered. We have:*
*(i) If* (3.5) *holds for any $h \in \mathcal{C}_b^2$ and $B_1 + B_3 \le (5 + \mathrm{E}[|F|])/4$, then*

$$d_{\mathrm{FM}}(F, Z) \le \sqrt{2(B_1 + B_3)(5 + \mathrm{E}[|F|])} + B_2. \tag{3.20}$$

*(ii) If* (3.14) *holds for any $h \in \mathcal{C}_b^2$ and $C_1 + C_3 \le (5 + \mathrm{E}[|F|])/4$, then*

$$d_{\mathrm{FM}}(F, Z) \le \sqrt{2(C_1 + C_3)(5 + \mathrm{E}[|F|])} + C_2. \tag{3.21}$$

*Proof.* We only give the details for the proof of (3.20). The inequality (3.21) can be proved similarly. Take $h \in \mathcal{FM}$ and define

$$h_t(x) = \int_{\mathbb{R}} h(\sqrt{t}y + \sqrt{1-t}x)\phi(y)\,\mathrm{d}y, \qquad t \in [0,1],$$

where $\phi$ is the density of the standard $\mathcal{N}(0,1)$ normal random variable $Z$. As in the proof of Corollary 3.6 in [14], for $0 < t \le 1/2$, one has $h_t \in \mathcal{C}_b^2$ and the bounds

$$\|h_t''\|_\infty \le 1/\sqrt{t}, \tag{3.22}$$

and

$$|\mathrm{E}[h(F)] - \mathrm{E}[h_t(F)]| \le \sqrt{t}\left(1 + \frac{\mathrm{E}[|F|]}{2}\right), \qquad |\mathrm{E}[h(Z)] - \mathrm{E}[h_t(Z)]| \le \frac{3}{2}\sqrt{t}.$$

So

$$\begin{aligned}
|\mathrm{E}[h(F)] - \mathrm{E}[h(Z)]| &= |(\mathrm{E}[h(F)] - \mathrm{E}[h_t(F)]) + (\mathrm{E}[h_t(F)] - \mathrm{E}[h_t(Z)]) + (\mathrm{E}[h_t(Z)] - \mathrm{E}[h(Z)])| \\
&\le |\mathrm{E}[h(F)] - \mathrm{E}[h_t(F)]| + |\mathrm{E}[h_t(F)] - \mathrm{E}[h_t(Z)]| + |\mathrm{E}[h_t(Z)] - \mathrm{E}[h(Z)]| \\
&\le \sqrt{t}\left(1 + \frac{\mathrm{E}[|F|]}{2}\right) + B_1 \min\{4\|h_t\|_\infty, \|h_t''\|_\infty\} + \|h_t'\|_\infty B_2 + \|h_t''\|_\infty B_3 + \frac{3}{2}\sqrt{t} \\
&\le \sqrt{t}\left(\frac{5 + \mathrm{E}[|F|]}{2}\right) + \frac{B_1 + B_3}{\sqrt{t}} + B_2,
\end{aligned} \tag{3.23}$$

where in the latter inequality we used (3.22) and that $\|h_t'\|_\infty \le 1$, for all $t$. Minimizing in $t \in (0, 1/2]$ the term in (3.23), we have that the optimal is attained at $t^* = 2(B_1 + B_3)/(5 + \mathrm{E}[|F|]) \in (0, 1/2]$. The conclusion follows substituting $t^*$ in (3.23) and then taking the supremum over all the $h \in \mathcal{FM}$. □

# 4 First chaos bound for the normal approximation

In this section we specialize the results of Section 3 to first order discrete stochastic integrals. As we shall see, the bounds (3.5) and (3.14) (and the corresponding assumptions) coincide on the first chaos, although they differ on $n$-chaoses, $n \ge 2$.

**Corollary 4.1** *Assume that* $\alpha = (\alpha_k)_{k\geq 0}$ *is in* $\ell^2(\mathbb{N})$,

$$\sum_{k\geq 0} \frac{|1-2p_k|}{\sqrt{p_k q_k}} |\alpha_k|^3 < \infty \quad \textit{and} \quad \sum_{k\geq 0} \frac{1}{p_k q_k} |\alpha_k|^4 < \infty. \tag{4.1}$$

*Then for the first chaos*

$$F = J_1(\alpha) = \sum_{k\geq 0} \alpha_k Y_k$$

*the bound* (3.5) *(which in this case coincides with the bound* (3.14)*) holds with*

$$B_1 = C_1 = \left|1 - \sum_{k\geq 0} |\alpha_k|^2\right|, \qquad B_2 = C_2 = \sum_{k\geq 0} \frac{|1-2p_k|}{\sqrt{p_k q_k}} |\alpha_k|^3,$$

*and*

$$B_3 = C_3 = \frac{5}{3} \sum_{k\geq 0} \frac{1}{p_k q_k} |\alpha_k|^4.$$

*Proof.* Since $\alpha \in \ell^2(\mathbb{N})$ we have that $F \in L^2(\Omega)$. Moreover $F$ is centered, and since

$$D_k F = \alpha_k \sqrt{p_k q_k} \left( \frac{q_k - p_k + 1}{2\sqrt{p_k q_k}} - \frac{q_k - p_k - 1}{2\sqrt{p_k q_k}} \right) = \alpha_k,$$

we have $F \in \mathrm{Dom}(D)$. The finiteness of the corresponding quantities $B_1$, $B_2$ and $B_3$ is guaranteed by $\alpha \in \ell^2(\mathbb{N})$ and (4.1). The claim follows from e.g. Theorem 3.2. □

**Example**

Consider the sequence of functionals $(F_n)_{n\geq 1}$ defined by

$$F_n = \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} Y_k.$$

Setting

$$\alpha_k = \frac{1}{\sqrt{n}}, \quad k = 0, \ldots, n-1, \text{ and } \quad \alpha_k = 0, \quad k \geq n,$$

we have $B_1 = 0$ and

$$B_2 = \frac{1}{n^{3/2}} \sum_{k=0}^{n-1} \frac{|1-2p_k|}{\sqrt{p_k q_k}} \quad \text{and} \quad B_3 = \frac{5}{3n^2} \sum_{k=0}^{n-1} \frac{1}{p_k q_k}.$$

In the symmetric case $p_k = q_k = 1/2$ we find $B_2 = 0$ and the bound is of order $1/n$, implying a faster rate than in the classical Berry-Esséen estimate (however here we are using $\mathcal{C}_b^2$ test functions; cf. the comment after Corollary 3.3 in [14]).

In the non-symmetric case $p_k = p$ and $q_k = q$, $p \neq q$, the bound is of order $n^{-1/2}$ as in the classical Berry-Esséen estimate. Indeed we have

$$B_2 = B_2^{(n)} = \frac{1}{\sqrt{n}} \frac{|1-2p|}{\sqrt{p(1-p)}} \quad \text{and} \quad B_3 = B_3^{(n)} = \frac{5}{3n} \frac{1}{p(1-p)}$$

hence the inequality $B_1 + B_3 \leq (5 + \mathrm{E}[|F_n|])/4$ of Theorem 3.6 reads

$$\frac{5}{3p(1-p)} \frac{1}{\sqrt{n}} \leq \frac{5}{4}\sqrt{n} + 4^{-1}\mathrm{E}\left[\left|n\left(\frac{1-2p}{2\sqrt{p(1-p)}}\right) + \sum_{k=0}^{n-1} X_k\right|\right],$$

which holds if e.g. $n \geq \frac{4}{3p(1-p)}$. Consequently, by (3.20) it follows that for any $n \geq \frac{4}{3p(1-p)}$ we have

$$\begin{aligned} d_{\mathrm{FM}}(F_n, Z) &\leq \sqrt{2B_3^{(n)}(5 + \mathrm{E}[|F_n|]) + B_2^{(n)}} \\ &= \sqrt{\frac{50}{3p(1-p)} \frac{1}{n} + \frac{10}{3p(1-p)} \frac{1}{n} \mathrm{E}\left[\left|\frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} \left(\frac{1-2p+X_k}{2\sqrt{p(1-p)}}\right)\right|\right]} + \frac{|1-2p|}{\sqrt{p(1-p)}} \frac{1}{\sqrt{n}} \\ &\leq \sqrt{\frac{50}{3p(1-p)} \frac{1}{n} + \frac{10}{3p(1-p)} \frac{1}{n} \mathrm{E}\left[\left|\frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} \left(\frac{1-2p+X_k}{2\sqrt{p(1-p)}}\right)\right|^2\right]^{1/2}} + \frac{|1-2p|}{\sqrt{p(1-p)}} \frac{1}{\sqrt{n}}. \end{aligned}$$

A straightforward computation gives

$$\mathrm{E}\left[\left|\frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} \left(\frac{1-2p+X_k}{2\sqrt{p(1-p)}}\right)\right|^2\right] = 1,$$

hence

$$d_{\mathrm{FM}}(F_n, Z) \leq \frac{1}{\sqrt{n}} K_1(p), \qquad n \geq \frac{4}{3p(1-p)} \tag{4.2}$$

where

$$K_1(p) := \frac{2\sqrt{5} + |1-2p|}{\sqrt{p(1-p)}}$$

In the general case, if

$$a_n := \frac{1}{n^{3/2}} \sum_{k=0}^{n-1} \frac{|1-2p_k|}{\sqrt{p_k q_k}} \to 0 \quad \text{and} \quad b_n := \frac{1}{n^2} \sum_{k=0}^{n-1} \frac{1}{p_k q_k} \to 0, \quad \text{as } n \to \infty, \tag{4.3}$$

then $F_n \to Z$ in distribution, and the rate depends on the rate of convergence to zero of the sequences $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$. For instance, if $p_k = (k+2)^{-\alpha}$, $0 < \alpha < 1$, $k \geq 0$, we have $p_k q_k \geq (n+1)^{-\alpha}(1 - (1/2)^\alpha)$, $k = 0, \ldots, n-1$. Consequently we have

$$\frac{1}{n^{3/2}} \sum_{k=0}^{n-1} \frac{|1-2p_k|}{\sqrt{p_k q_k}} \leq (1 - (1/2)^\alpha)^{-1/2} \frac{(n+1)^{\alpha/2}}{n^{3/2}} \sum_{k=0}^{n-1} |1-2p_k|$$

$$\leq \frac{1+2^{-(\alpha-1)}}{(1-(1/2)^\alpha)^{1/2}} \frac{(n+1)^{\alpha/2}}{n^{1/2}},$$

and

$$\frac{1}{n^2}\sum_{k=0}^{n-1}\frac{1}{p_k q_k} \leq (1-(1/2)^\alpha)^{-1} n^{-1}(n+1)^\alpha,$$

which yields a bound of order $n^{-(1-\alpha)/2}$.

Finally we note that the bound (4.2) in the non-symmetric case $p_k = p$ and $q_k = q$, $p \neq q$, is consistent with the bound on the Wasserstein distance between $F_n$ and $Z$ provided by Theorem 2.2 in [4]. Indeed, letting $d_W$ denote the Wasserstein distance and $Y_1'$ an independent copy of $Y_1$, a simple computation shows that

$$\begin{aligned} d_{\mathrm{FM}}(F_n, Z) &\leq d_W(F_n, Z) && (4.4)\\ &\leq \frac{1}{2\sqrt{n}}\left(\sqrt{\mathrm{E}[|Y_1 - Y_1'|^4] - (\mathrm{E}[|Y_1 - Y_1'|^2])^2} + \mathrm{E}[|Y_1|^3]\right)\\ &= \frac{1}{2\sqrt{n}}\left(\sqrt{\mathrm{E}[|Y_1 - Y_1'|^4] - 4} + \mathrm{E}[|Y_1|^3]\right)\\ &= \frac{1}{\sqrt{n}} K_2(p), && (4.5) \end{aligned}$$

where

$$K_2(p) := \frac{1}{2}\sqrt{\frac{1}{p^2(1-p)} - 4} + \frac{1+2|1-2p|}{4\sqrt{p(1-p)}}$$

since we have

$$\mathrm{E}[|Y_1|^3] \leq \frac{1+2|1-2p|}{2\sqrt{p(1-p)}} \quad \text{and} \quad \mathrm{E}[|Y_1 - Y_1'|^4] = \frac{1}{p^2(1-p)}.$$

We note that when e.g. $p$ is small it holds $K_2(p) > K_1(p)$.

## Application to determinantal processes

Let $E$ be a locally compact Hausdorff space with countable basis and $\mathcal{B}(E)$ the Borel $\sigma$-field. We fix a Radon measure $\lambda$ on $(E, \mathcal{B}(E))$. The configuration space $\Gamma_E$ is the family of non-negative $\mathbb{N}$-valued Radon measures on $E$. We equip $\Gamma_E$ with the topology which is generated by the functions $\Gamma_E \ni \xi \longmapsto \xi(A) \in \mathbb{N}$, $A \in \mathcal{B}(E)$, where $\xi(A)$ denotes the number of points of $\xi$ in $A$. The existence and uniqueness of a determinantal process with Hermitian kernel $K$ is due to Macchi [12] and Soshnikov [22] and can be summarized as follows (we refer the reader to [3] for notions of functional analysis).

**Theorem 4.2** *Let $\mathcal{K}$ be a self-adjoint integral operator on $L^2(E,\lambda)$ with kernel $K$. Suppose that the spectrum of $\mathcal{K}$ is contained in $[0,1]$ and that $\mathcal{K}$ is locally of trace-class, i.e. for any relatively compact $\Lambda \subset E$, $\mathcal{K}_\Lambda = P_\Lambda \mathcal{K} P_\Lambda$ is of trace-class (here $P_\Lambda f = f\mathbb{1}_\Lambda$ is the orthogonal projection.) Then there exists a unique probability measure $\mu_K$ on $\Gamma_E$ with $n$-th correlation measure*

$$\lambda_n(\mathrm{d}x_1,\ldots,\mathrm{d}x_n) = \det(K(x_i,x_j))_{1\le i,j\le n}\lambda(\mathrm{d}x_1)\ldots\lambda(\mathrm{d}x_n),$$

*where $\det(K(x_i,x_j))_{1\le i,j\le n}$ is the determinant of the $n\times n$ matrix with $ij$-entry $K(x_i,x_j)$.*

The probability measure $\mu_K$ is called determinantal process with kernel $K$.

Given a relatively compact set $\Lambda \subset E$, we focus on the random variable $\xi(\Lambda)$ and recall the following basic result (see e.g. Proposition 2.2 in [21]).

**Theorem 4.3** *Let $\mathcal{K}$ be as in the statement of Theorem 4.2 and denote by $\kappa_k \in [0,1]$, $k\ge 0$, the eigenvalues of $\mathcal{K}_\Lambda$. Under $\mu_K$ the random variable $\xi(\Lambda)$ has the same distribution of $\sum_{k\ge 0} Z_k$, where $Z_0, Z_1, \ldots$ are independent random variables such that $Z_k$ obeys the Bernoulli distribution with mean $\kappa_k$, i.e.*

$$Z_n = \frac{X_n+1}{2} \in \{0,1\}, \qquad n\in\mathbb{N}$$

*where the $X$'s take values on $\{-1,1\}$ and are independent with $P(X_n=1)=\kappa_n$.*

The central limit theorem for the number of points on a relatively compact set of a determinantal process may be obtained in different manners, see [5], [21] and [23]. In the following we provide an alternate derivation which gives the rate of the normal approximation.

**Corollary 4.4** *Let $\mathcal{K}$ be as in the statement of Theorem 4.2 and $(\Lambda_n)_{n\ge 0}\subset E$ be an increasing sequence of relatively compact sets such that*

$$\mathrm{Var}_{\mu_K}(\xi(\Lambda_n)) = \sum_{k\ge 0} \kappa_k^{(n)}(1-\kappa_k^{(n)}) \to \infty, \quad \text{as } n\to\infty$$

*where $\kappa_k^{(n)} \in [0,1]$, $k\ge 0$, are the eigenvalues of $\mathcal{K}_{\Lambda_n}$, $n\ge 0$. Setting*

$$F_n = \frac{\xi(\Lambda_n) - \mathrm{E}_{\mu_K}[\xi(\Lambda_n)]}{\sqrt{\mathrm{Var}_{\mu_K}(\xi(\Lambda_n))}},$$

*for any $h\in\mathcal{C}_b^2$, we have*

$$|\mathrm{E}_{\mu_K}[h(F_n)] - \mathrm{E}[h(Z)]| \le \|h'\|_\infty B_2^{(n)} + \|h''\|_\infty B_3^{(n)}, \quad n\ge 0$$

*where*

$$B_2^{(n)} = \frac{\sum_{k\geq 0}\kappa_k^{(n)}(1-\kappa_k^{(n)})|1-2\kappa_k^{(n)}|}{\left(\sum_{k\geq 0}\kappa_k^{(n)}(1-\kappa_k^{(n)})\right)^{3/2}} \leq \frac{1}{\left(\sum_{k\geq 0}\kappa_k^{(n)}(1-\kappa_k^{(n)})\right)^{1/2}}$$

*and*

$$B_3^{(n)} = \frac{5}{3}\frac{1}{\sum_{k\geq 0}\kappa_k^{(n)}(1-\kappa_k^{(n)})}.$$

*So we have a bound of order* $[\mathrm{Var}_{\mu_K}(\xi(\Lambda_n))]^{-1/2}$.

*Proof.* For $n \geq 0$, let $(Z_k^{(n)})_{k\geq 0}$ be a sequence of independent $\{0,1\}$-valued random variables with $Z_k^{(n)} \sim \mathrm{Be}(\kappa_k^{(n)})$ and $(Y_k^{(n)})_{k\geq 0}$ defined by

$$Y_k^{(n)} = \frac{Z_k^{(n)} - \kappa_k^{(n)}}{\sqrt{\kappa_k^{(n)}(1-\kappa_k^{(n)})}}.$$

By Theorem 4.3 we have

$$\xi(\Lambda_n) \stackrel{d}{=} \sum_{k\geq 0} Z_k^{(n)} = \sum_{k\geq 0}\sqrt{\kappa_k^{(n)}(1-\kappa_k^{(n)})}Y_k^{(n)} + \sum_{k\geq 0}\kappa_k^{(n)},$$

where $\stackrel{d}{=}$ denotes the equality in distribution. Then

$$F_n = \frac{\xi(\Lambda_n) - \mathrm{E}_{\mu_K}[\xi(\Lambda_n)]}{\sqrt{\mathrm{Var}_{\mu_K}(\xi(\Lambda_n))}} \stackrel{d}{=} \sum_{k\geq 0}\alpha_k^{(n)}Y_k^{(n)},$$

where

$$\alpha_k^{(n)} = \frac{\sqrt{\kappa_k^{(n)}(1-\kappa_k^{(n)})}}{\sqrt{\sum_{k\geq 0}\kappa_k^{(n)}(1-\kappa_k^{(n)})}}.$$

We are going to apply Corollary 4.1. Clearly, for any $n \geq 0$, the sequence $(\alpha_k^{(n)})_{k\geq 0}$ is in $\ell^2(\mathbb{N})$. Moreover, for any $n \geq 0$,

$$\sum_{k\geq 0}\frac{|\alpha_k^{(n)}|^3}{\sqrt{\kappa_k^{(n)}(1-\kappa_k^{(n)})}} = \frac{1}{\sqrt{\mathrm{Var}_{\mu_K}(\xi(\Lambda_n))}} < \infty$$

and

$$\sum_{k\geq 0}\frac{|\alpha_k^{(n)}|^4}{\kappa_k^{(n)}(1-\kappa_k^{(n)})} = \frac{1}{\mathrm{Var}_{\mu_K}(\xi(\Lambda_n))} < \infty.$$

So condition (4.1) is satisfied. Moreover, a straightforward computation gives $B_1 = B_1^{(n)} = 0$, $B_2 = B_2^{(n)}$ and $B_3 = B_3^{(n)}$, and the proof is completed. □

**Example**

Let $E = \mathbb{C}$ and $\lambda$ the standard complex Gaussian measure on $\mathbb{C}$, i.e.

$$\lambda(\mathrm{d}z) = \frac{1}{\pi}\mathrm{e}^{-|z|^2}\,\mathrm{d}z,$$

where $\mathrm{d}z$ is the Lebesgue measure on $\mathbb{C}$. The Ginibre process $\mu_{\exp}$ is the determinantal process with exponential kernel $K(z,w) = \mathrm{e}^{-z\overline{w}}$, where $\overline{z}$ is the complex conjugate of $z \in \mathbb{C}$. Let $b(O,n)$ be the complex ball centered at the origin with radius $n$. By Theorem 1.3 in [21] we have

$$\begin{aligned}\mathrm{Var}_{\mu_{\exp}}(\xi(b(O,n))) &= \frac{n}{\pi}\int_0^{4n^2}(1-x/(4n^2))^{1/2}x^{-1/2}\mathrm{e}^{-x}\,\mathrm{d}x \\ &\sim \frac{n}{\sqrt{\pi}}, \quad \text{as } n \to \infty.\end{aligned}$$

So for the Ginibre process Corollary 4.4 provides a bound of order $n^{-1/2}$.

# 5 $n$th chaos bounds for the normal approximation

In this section we give explicit upper bounds for the constants $B_i$ and $C_i$, $i = 1,2,3$, involved in (3.5) and (3.14), when $F = J_n(f_n)$, $f_n \in \ell^2_{\mathfrak{s}}(\Delta_n)$. Our approach is based on the multiplication formula (5.3) below, which extends formula (2.11) in [14] (see the discussion after Proposition 5.1).

Given $f_n \in \ell^2_{\mathfrak{s}}(\Delta_n)$ and $g_m \in \ell^2_{\mathfrak{s}}(\Delta_m)$, the contraction $f_n \otimes_k^l g_m$, $0 \le l \le k$, is defined to be the function of $n+m-k-l$ variables

$$f_n \otimes_k^l g_m(a_{l+1},\dots,a_n,b_{k+1},\dots,b_m) := \varphi(a_{l+1})\cdots\varphi(a_k) f_n \star_k^l g_m(a_{l+1},\dots,a_n,b_{k+1},\dots,b_m),$$

where

$$\varphi(n) = \frac{q_n - p_n}{2\sqrt{p_n q_n}}, \quad n \in \mathbb{N} \tag{5.1}$$

(cf. the structure equation (2.1)) and

$$f_n \star_k^l g_m(a_{l+1},\dots,a_n,b_{k+1},\dots,b_m) := \sum_{a_1,\dots,a_l \in \mathbb{N}} f_n(a_1,\dots,a_n) g_m(a_1,\dots,a_k,b_{k+1},\dots,b_m)$$

is the contraction considered in [14] for the symmetric case, see p. 1707 therein. By convention, we define $\varphi(a_{l+1})\cdots\varphi(a_k) = 1$ if $l = k$ (even when $\varphi \equiv 0$). Denote by $f_n \widetilde{\otimes}_k^l g_m$, $0 \le l \le k$, the symmetrization of $f_n \otimes_k^l g_m$. Then, we shall consider the contraction

$$f_n \circ_k^l g_m(a_{l+1},\dots,a_n,b_{k+1},\dots,b_m) :=$$

$$= \mathbb{1}_{\Delta_{n+m-k-l}}(a_{l+1},\dots,a_n,b_{k+1},\dots,b_m)\widetilde{f_n \otimes_k^l g_m}(a_{l+1},\dots,a_n,b_{k+1},\dots,b_m). \qquad (5.2)$$

Note that in the symmetric case $p_n = q_n = 1/2$ we have $f_n \otimes_k^k g_m = f_n \star_k^k g_m$. However, $f_n \otimes_k^l g_m = 0$ if $l < k$ and so $f_n \star_k^l g_m \neq f_n \otimes_k^l g_m$ for $l < k$. The following multiplication formula holds.

**Proposition 5.1** *We have the chaos expansion*

$$J_n(f_n)J_m(g_m) = \sum_{s=0}^{2(n\wedge m)} J_{n+m-s}(h_{n,m,s}), \qquad (5.3)$$

*provided the functions*

$$h_{n,m,s} := \sum_{s\leq 2i\leq 2(s\wedge n\wedge m)} i!\binom{n}{i}\binom{m}{i}\binom{i}{s-i} f_n \circ_i^{s-i} g_m$$

*belong to* $\ell_{\mathfrak{s}}^2(\Delta_{n+m-s})$*,* $0 \leq s \leq 2(n\wedge m)$*. Here the symbol* $\sum_{s\leq 2i\leq 2(s\wedge n\wedge m)}$ *means that the sum is taken over all the integers* $i$ *in the interval* $[s/2, s\wedge n\wedge m]$*.*

Since it is not obvious that formula (5.3) extends the product formula (2.11) in [14], it is worthwhile to explain this point in detail. In the symmetric case $p_n = q_n = 1/2$, for any $f_n \in \ell_{\mathfrak{s}}^2(\Delta_n)$, $g_m \in \ell_{\mathfrak{s}}^2(\Delta_m)$, we have $f_n \circ_i^{s-i} g_m = 0$ if $s < 2i$, $0 \leq s \leq 2(n\wedge m)$. Therefore, for any fixed $0 \leq s \leq 2(n\wedge m)$ we have $h_{n,m,s} = 0$ if $s/2$ is not an integer and

$$h_{n,m,s} = (s/2)!\binom{n}{s/2}\binom{m}{s/2} f_n \circ_{s/2}^{s/2} g_m,$$

if $s/2$ is an integer. Note that if $s/2$ is an integer, we have

$$\begin{aligned}\|f_n \circ_{s/2}^{s/2} g_m\|_{\ell_{\mathfrak{s}}^2(\Delta_{n+m-s})} &= \|\widetilde{f_n \otimes_{s/2}^{s/2} g_m}\|_{\ell_{\mathfrak{s}}^2(\Delta_{n+m-s})}\\ &\leq \|f_n \otimes_{s/2}^{s/2} g_m\|_{\ell^2(\Delta_{n+m-s})},\end{aligned}$$

where we used the straightforward relation

$$\|\tilde{f}\|_{\ell^2(\mathbb{N})^{\otimes n}} \leq \|f\|_{\ell^2(\mathbb{N})^{\otimes n}}, \qquad (5.4)$$

being $\tilde{f}$ the symmetrization of $f$. Therefore, by Lemma 2.4(1) in [14] we have $f_n \circ_{s/2}^{s/2} g_m \in \ell_{\mathfrak{s}}^2(\Delta_{n+m-s})$, and so $h_{n,m,s} \in \ell_{\mathfrak{s}}^2(\Delta_{n+m-s})$, for any $0 \leq s \leq 2(n\wedge m)$. By (5.3) we have

$$J_n(f_n)J_m(g_m) = \sum_{s=0}^{2(n\wedge m)} J_{n+m-s}(h_{n,m,s})$$

$$= \sum_{s=0}^{2(n\wedge m)} (s/2)! \binom{n}{s/2} \binom{m}{s/2} J_{n+m-s}(f_n \circ_{s/2}^{s/2} g_m)$$
$$= \sum_{r=0}^{n\wedge m} r! \binom{n}{r} \binom{m}{r} J_{n+m-2r}(f_n \circ_r^r g_m),$$

which is exactly formula (2.11) in [14].

We conclude this part with the following lemma.

**Lemma 5.2** *For any $f_n \in \ell_{\mathfrak{s}}^2(\Delta_n)$, $g_m \in \ell_{\mathfrak{s}}^2(\Delta_m)$, we have*

$$\sum_{q\geq 0} f_n(*, q) \otimes_k^l g_m(*, q) = f_n \otimes_{k+1}^{l+1} g_m.$$

*Proof.* Note that

$$f_n(*, q) \otimes_k^l g_m(*, q)(a_{l+1}, \dots, a_{n-1}, b_{k+1}, \dots, b_{m-1})$$
$$= \varphi(a_{l+1}) \dots \varphi(a_k) \sum_{a_1,\dots,a_l \in \mathbb{N}} f_n(a_1, \dots, a_{n-1}, q) g_m(a_1, \dots, a_k, b_{k+1}, \dots, b_{m-1}, q),$$

and so summing up over $q \in \mathbb{N}$ we deduce

$$\sum_{q\geq 0} f_n(*, q) \otimes_k^l g_m(*, q)(a_{l+1}, \dots, a_{n-1}, b_{k+1}, \dots, b_{m-1})$$
$$= \varphi(a_{l+1}) \dots \varphi(a_k) \sum_{a_1,\dots,a_l,q \in \mathbb{N}} f_n(a_1, \dots, a_{n-1}, q) g_m(a_1, \dots, a_k, b_{k+1}, \dots, b_{m-1}, q)$$
$$= \varphi(a_{l+1}) \dots \varphi(a_k) f_n \star_{k+1}^{l+1} g_m(a_{l+1}, \dots, a_{n-1}, b_{k+1}, \dots, b_{m-1})$$
$$= f_n \otimes_{k+1}^{l+1} g_m(a_{l+1}, \dots, a_{n-1}, b_{k+1}, \dots, b_{m-1}).$$

□

## 5.1 Clark-Ocone bound

By e.g. Lemma 4.6 in [18], for the $n$th-chaos $J_n(f_n)$, $n \geq 2$, $f_n \in \ell_{\mathfrak{s}}^2(\Delta_n)$, we have

$$\mathrm{E}[J_n(f_n) \mid \mathcal{F}_k] = J_n(f_n \mathbb{1}_{[0,k]^n}), \qquad k \in \mathbb{N}.$$

Therefore

$$\mathrm{E}[D_k J_n(f_n) \mid \mathcal{F}_{k-1}] = n\mathrm{E}[J_{n-1}(f_n(*, k)) \mid \mathcal{F}_{k-1}] = n J_{n-1}(f_{n]k}), \tag{5.5}$$

where

$$f_{n]k}(*) := f_n(*, k) \mathbb{1}_{[0,k-1]^{n-1}}(*). \tag{5.6}$$

So by the isometric properties of discrete multiple stochastic integrals we have that the constants $B_i$ of Corollary 3.3 are equal, respectively, to

$$\tilde{B}_1 := |1 - n!\|f_n\|^2_{\ell^2_{\mathfrak{s}}(\Delta_n)}| + n^2\|\langle J_{n-1}(f_n(*,\cdot)), J_{n-1}(f_{n]\cdot}(*))\rangle_{\ell^2(\mathbb{N})} - \mathrm{E}[\langle J_{n-1}(f_n(*,\cdot)), J_{n-1}(f_{n]\cdot}(*))\rangle_{\ell^2(\mathbb{N})}]\|_{L^2(\Omega)}, \qquad (5.7)$$

$$\tilde{B}_2 : = n^3\sqrt{(n-1)!}\sum_{k\geq 0}\frac{|1-2p_k|}{\sqrt{p_kq_k}}\|f_n(*,k)\|_{\ell^2_{\mathfrak{s}}(\Delta_{n-1})}\sqrt{\mathrm{E}[|J_{n-1}(f_n(*,k))|^4]}, \qquad (5.8)$$

$$\tilde{B}_3 : = \frac{5n^4}{3}\sum_{k\geq 0}\frac{1}{p_kq_k}\mathrm{E}[|J_{n-1}(f_n(*,k))|^4]. \qquad (5.9)$$

In the proof of the next theorem we show that these constants can be bounded above by computable quantities.

**Theorem 5.3** *Let $n \geq 2$ be fixed and let $f_n \in \ell^2_{\mathfrak{s}}(\Delta_n)$. Assume that for any $k \in \mathbb{N}$ the functions*

$$h^{(k)}_{n-1,n-1,s} := \sum_{s\leq 2i\leq 2(s\wedge(n-1))} i!\binom{n-1}{i}^2\binom{i}{s-i}f_n(*,k)\circ_i^{s-i} f_{n]k}(*) \qquad (5.10)$$

*and*

$$\tilde{h}^{(k)}_{n-1,n-1,s} := \sum_{s\leq 2i\leq 2(s\wedge(n-1))} i!\binom{n-1}{i}^2\binom{i}{s-i}f_n(*,k)\circ_i^{s-i} f_n(*,k) \qquad (5.11)$$

*belong to $\ell^2_{\mathfrak{s}}(\Delta_{2n-2-s})$, $0 \leq s \leq 2n-2$, and that*

$$B_1 := |1 - n!\|f_n\|^2_{\ell^2_{\mathfrak{s}}(\Delta_n)}| + n^2\Bigg(\sum_{s=0}^{2n-3}(2n-2-s)!\sum_{s\leq\{2i_1,\,2i_2\}\leq 2(s\wedge(n-1))} i_1!i_2!\binom{n-1}{i_1}^2\binom{n-1}{i_2}^2\binom{i_1}{s-i_1}\binom{i_2}{s-i_2} \sum_{k\geq 0}\|f_n(*,k)\otimes_{i_1}^{s-i_1} f_{n]k}(*)\|_{\ell^2(\Delta_{2n-2-s})}\sum_{k\geq 0}\|f_n(*,k)\otimes_{i_2}^{s-i_2} f_{n]k}(*)\|_{\ell^2(\Delta_{2n-2-s})}\Bigg)^{1/2},$$

$$B_2 : = n^3\sqrt{(n-1)!}\sum_{k\geq 0}\frac{|1-2p_k|}{\sqrt{p_kq_k}}\|f_n(*,k)\|_{\ell^2_{\mathfrak{s}}(\Delta_{n-1})}\Bigg(\sum_{s=0}^{2n-2}(2n-2-s)! \times \sum_{s\leq\{2i_1,2i_2\}\leq 2(s\wedge(n-1))} i_1!i_2!\binom{n-1}{i_1}^2\binom{n-1}{i_2}^2\binom{i_1}{s-i_1}\binom{i_2}{s-i_2} \|f_n(*,k)\otimes_{i_1}^{s-i_1} f_n(*,k)\|_{\ell^2(\Delta_{2n-2-s})}\|f_n(*,k)\otimes_{i_2}^{s-i_2} f_n(*,k)\|_{\ell^2(\Delta_{2n-2-s})}\Bigg)^{1/2},$$

(5.12)

$$B_3 := \frac{5n^4}{3}\sum_{s=0}^{2n-2}(2n-2-s)! \times \sum_{s\le\{2i_1,\,2i_2\}\le 2(s\wedge(n-1))} i_1!i_2!\binom{n-1}{i_1}^2\binom{n-1}{i_2}^2\binom{i_1}{s-i_1}\binom{i_2}{s-i_2} \sum_{k\ge 0}\frac{1}{p_kq_k}\|f_n(*,k)\otimes_{i_1}^{s-i_1} f_n(*,k)\|_{\ell^2(\Delta_{2n-2-s})}\|f_n(*,k)\otimes_{i_2}^{s-i_2} f_n(*,k)\|_{\ell^2(\Delta_{2n-2-s})} \tag{5.13}$$

*are finite. Then for all* $h\in\mathcal{C}_b^2$ *we have*

$$|\mathrm{E}[h(J_n(f_n))]-\mathrm{E}[h(Z)]|\le B_1\min\{4\|h\|_\infty,\|h''\|_\infty\}+\|h'\|_\infty B_2+\|h''\|_\infty B_3.$$

*Proof.* The claim follows from Corollary 3.3 if we show that the constants $\tilde{B}_i$ defined by (5.7), (5.8) and (5.9) are bounded above by the constants $B_i$ defined in the statement, respectively.

*Step 1: Proof of* $\tilde{B}_1\le B_1$. By the hypotheses on the functions $h^{(k)}_{n-1,n-1,s}$ and the multiplication formula (5.3), we deduce

$$\begin{aligned} &J_{n-1}(f_n(*,k))J_{n-1}(f_{n]k})\\ &= \sum_{s=0}^{2n-2}\sum_{s\le 2i\le 2(s\wedge(n-1))} i!\binom{n-1}{i}^2\binom{i}{s-i}J_{2n-2-s}(f_n(*,k)\circ_i^{s-i} f_{n]k}(*))\\ &= (n-1)!f_n(*,k)\circ_{n-1}^{n-1} f_{n]k}(*)\\ &\quad+\sum_{s=0}^{2n-3}\sum_{s\le 2i\le 2(s\wedge(n-1))} i!\binom{n-1}{i}^2\binom{i}{s-i}J_{2n-2-s}(f_n(*,k)\circ_i^{s-i} f_{n]k}(*)). \end{aligned} \tag{5.14}$$

Since the constant $B_1$ in the statement is finite, we have

$$\sum_{k\ge 0}\|f_n(*,k)\otimes_i^{s-i} f_{n]k}(*)\|_{\ell^2(\Delta_{2n-2-s})}<\infty,$$

$0\le s\le 2n-3$, $s\le 2i\le 2(s\wedge(n-1))$. By (5.4) this in turn implies

$$\sum_{k\ge 0}\|f_n(*,k)\circ_i^{s-i} f_{n]k}(*)\|_{\ell^2_{\mathfrak{s}}(\Delta_{2n-2-s})}<\infty,$$

$0\le s\le 2n-3$, $s\le 2i\le 2(s\wedge(n-1))$, and so

$$\sum_{k\ge 0} f_n(*,k)\circ_i^{s-i} f_{n]k}(*)\in \ell^2_{\mathfrak{s}}(\Delta_{2n-2-s}),$$

$0 \le s \le 2n-3$, $s \le 2i \le 2(s \wedge (n-1))$ (it is worthwhile to note that one can not use Lemma 5.2 to express the infinite sum $\sum_{k\ge 0} f_n(*,k) \circ_i^{s-i} f_{n]k}(*)$ since the function of $n$ variables $f_{n]\cdot}(*)$ is not symmetric). Therefore, summing up over $k \ge 0$ in the equality (5.14), we get

$$\begin{aligned}
&\langle J_{n-1}(f_n(*,\cdot)), J_{n-1}(f_{n]\cdot}(*))\rangle_{\ell^2(\mathbb{N})} \\
&= (n-1)! \sum_{k\ge 0} f_n(*,k) \circ_{n-1}^{n-1} f_{n]k}(*) \\
&\quad + \sum_{s=0}^{2n-3} \sum_{s\le 2i\le 2(s\wedge(n-1))} i! \binom{n-1}{i}^2 \binom{i}{s-i} J_{2n-2-s}\left( \sum_{k\ge 0} f_n(*,k) \circ_i^{s-i} f_{n]k}(*) \right).
\end{aligned}$$

Taking the mean and noticing that discrete multiple stochastic integrals are centered, we have

$$\mathrm{E}[\langle J_{n-1}(f_n(*,\cdot)), J_{n-1}(f_{n]\cdot}(*))\rangle_{\ell^2(\mathbb{N})}] = (n-1)! \sum_{k\ge 0} f_n(*,k) \circ_{n-1}^{n-1} f_{n]k}(*),$$

and so

$$\begin{aligned}
&\langle J_{n-1}(f_n(*,\cdot)), J_{n-1}(f_{n]\cdot}(*))\rangle_{\ell^2(\mathbb{N})} - \mathrm{E}[\langle J_{n-1}(f_n(*,\cdot)), J_{n-1}(f_{n]\cdot}(*))\rangle_{\ell^2(\mathbb{N})}] \\
&= \sum_{s=0}^{2n-3} \sum_{s\le 2i\le 2(s\wedge(n-1))} i! \binom{n-1}{i}^2 \binom{i}{s-i} J_{2n-2-s}\left( \sum_{k\ge 0} f_n(*,k) \circ_i^{s-i} f_{n]k}(*) \right).
\end{aligned}$$

By means of the orthogonality and isometric properties of discrete multiple stochastic integrals, we have

$$\begin{aligned}
&\mathrm{E}\left[ \left( \sum_{s=0}^{2n-3} \sum_{s\le 2i\le 2(s\wedge(n-1))} i! \binom{n-1}{i}^2 \binom{i}{s-i} J_{2n-2-s}\left( \sum_{k\ge 0} f_n(*,k) \circ_i^{s-i} f_{n]k}(*) \right) \right)^2 \right] \\
&= \sum_{s=0}^{2n-3} \mathrm{E}\left[ \left( \sum_{s\le 2i\le 2(s\wedge(n-1))} i! \binom{n-1}{i}^2 \binom{i}{s-i} J_{2n-2-s}\left( \sum_{k\ge 0} f_n(*,k) \circ_i^{s-i} f_{n]k}(*) \right) \right)^2 \right] \\
&\quad + \sum_{\substack{s_1\ne s_2}}^{0,2n-3} \mathrm{E}\Bigg[ \left( \sum_{s_1\le 2i\le 2(s_1\wedge(n-1))} i! \binom{n-1}{i}^2 \binom{i}{s_1-i} J_{2n-2-s_1}\left( \sum_{k\ge 0} f_n(*,k) \circ_i^{s_1-i} f_{n]k}(*) \right) \right) \\
&\quad \times \left( \sum_{s_2\le 2i\le 2(s_2\wedge(n-1))} i! \binom{n-1}{i}^2 \binom{i}{s_2-i} J_{2n-2-s_2}\left( \sum_{k\ge 0} f_n(*,k) \circ_i^{s_2-i} f_{n]k}(*) \right) \right) \Bigg] \\
&= \sum_{s=0}^{2n-3} \mathrm{E}\left[ \left( \sum_{s\le 2i\le 2(s\wedge(n-1))} i! \binom{n-1}{i}^2 \binom{i}{s-i} J_{2n-2-s}\left( \sum_{k\ge 0} f_n(*,k) \circ_i^{s-i} f_{n]k}(*) \right) \right)^2 \right] \\
&= \sum_{s=0}^{2n-3} \sum_{s\le \{2i_1, 2i_2\}\le 2(s\wedge(n-1))} \mathrm{E}\Bigg[ i_1! \binom{n-1}{i_1}^2 \binom{i_1}{s-i_1} i_2! \binom{n-1}{i_2}^2 \binom{i_2}{s-i_2}
\end{aligned}$$

$$J_{2n-2-s}\left(\sum_{k\geq 0} f_n(*,k)\circ_{i_1}^{s-i_1} f_{n]k}(*)\right) J_{2n-2-s}\left(\sum_{k\geq 0} f_n(*,k)\circ_{i_2}^{s-i_2} f_{n]k}(*)\right)\Bigg]$$

$$= \sum_{s=0}^{2n-3}(2n-2-s)! \sum_{s\leq\{2i_1,\,2i_2\}\leq 2(s\wedge(n-1))} i_1!i_2!\binom{n-1}{i_1}^2\binom{n-1}{i_2}^2\binom{i_1}{s-i_1}\binom{i_2}{s-i_2}$$
$$\langle\sum_{k\geq 0} f_n(*,k)\circ_{i_1}^{s-i_1} f_{n]k}(*),\sum_{k\geq 0} f_n(*,k)\circ_{i_2}^{s-i_2} f_{n]k}(*)\rangle_{\ell_s^2(\Delta_{2n-2-s})}. \quad (5.15)$$

By the above relations and (5.7), we deduce

$$\tilde{B}_1 = |1-n!\|f_n\|^2_{\ell^2_s(\Delta_n)}|$$
$$+n^2\Bigg(\sum_{s=0}^{2n-3}(2n-2-s)! \sum_{s\leq\{2i_1,\,2i_2\}\leq 2(s\wedge(n-1))} i_1!i_2!\binom{n-1}{i_1}^2\binom{n-1}{i_2}^2\binom{i_1}{s-i_1}\binom{i_2}{s-i_2}$$
$$\langle\sum_{k\geq 0} f_n(*,k)\circ_{i_1}^{s-i_1} f_{n]k}(*),\sum_{k\geq 0} f_n(*,k)\circ_{i_2}^{s-i_2} f_{n]k}(*)\rangle_{\ell_s^2(\Delta_{2n-2-s})}\Bigg)^{1/2}. \quad (5.16)$$

Now, note that by the Cauchy-Schwarz inequality

$$|\langle f,g\rangle_{\ell^2(\mathbb{N})^{\otimes n}}|\leq\|f\|_{\ell^2(\mathbb{N})^{\otimes n}}\|g\|_{\ell^2(\mathbb{N})^{\otimes n}},\quad \text{for any } f,g\in\ell^2(\mathbb{N})^{\otimes n}. \quad (5.17)$$

By this relation, (5.4) and (5.16) we easily get $\tilde{B}_1\leq B_1$.

*Step 2: Proof of* $\tilde{B}_i\leq B_i$, $i=2,3$. By the hypotheses on the functions $\tilde{h}^{(k)}_{n-1,n-1,s}$ and the multiplication formula (5.3), we deduce

$$J_{n-1}(f_n(*,k))^2$$
$$= \sum_{s=0}^{2n-2}\sum_{s\leq 2i\leq 2(s\wedge(n-1))} i!\binom{n-1}{i}^2\binom{i}{s-i} J_{2n-2-s}(f_n(*,k)\circ_i^{s-i} f_n(*,k)).$$

By a similar computation as for (5.15), we have

$$\mathrm{E}[|J_{n-1}(f_n(*,k))|^4]$$
$$= \mathrm{E}\left[\left(\sum_{s=0}^{2n-2}\sum_{s\leq 2i\leq 2(s\wedge(n-1))} i!\binom{n-1}{i}^2\binom{i}{s-i} J_{2n-2-s}\left(f_n(*,k)\circ_i^{s-i} f_n(*,k)\right)\right)^2\right]$$
$$= \sum_{s=0}^{2n-2}(2n-2-s)! \sum_{s\leq\{2i_1,\,2i_2\}\leq 2(s\wedge(n-1))} i_1!i_2\binom{n-1}{i_1}^2\binom{n-1}{i_2}^2\binom{i_1}{s-i_1}\binom{i_2}{s-i_2}$$
$$\langle f_n(*,k)\circ_{i_1}^{s-i_1} f_n(*,k), f_n(*,k)\circ_{i_2}^{s-i_2} f_n(*,k)\rangle_{\ell_s^2(\Delta_{2n-2-s})}, \quad (5.18)$$

and so by (5.8) and (5.9) we deduce

$$\tilde{B}_2 = n^3\sqrt{(n-1)!}\sum_{k\geq 0}\frac{|1-2p_k|}{\sqrt{p_kq_k}}\|f_n(*,k)\|_{\ell_s^2(\Delta_{n-1})}\Bigg(\sum_{s=0}^{2n-2}(2n-2-s)!$$

$$\times \sum_{s\leq\{2i_1,\,2i_2\}\leq 2(s\wedge(n-1))} i_1!i_2!\binom{n-1}{i_1}^2\binom{n-1}{i_2}^2\binom{i_1}{s-i_1}\binom{i_2}{s-i_2}$$

$$\left.\langle f_n(*,k)\circ_{i_1}^{s-i_1} f_n(*,k), f_n(*,k)\circ_{i_2}^{s-i_2} f_n(*,k)\rangle_{\ell^2_{\mathfrak{s}}(\Delta_{2n-2-s})}\right)^{1/2} \tag{5.19}$$

and

$$\begin{aligned}
\tilde{B}_3 &= \frac{5n^4}{3}\sum_{s=0}^{2n-2}(2n-2-s)! \\
&\quad\times \sum_{s\leq\{2i_1,\,2i_2\}\leq 2(s\wedge(n-1))} i_1!i_2!\binom{n-1}{i_1}^2\binom{n-1}{i_2}^2\binom{i_1}{s-i_1}\binom{i_2}{s-i_2} \\
&\quad\sum_{k\geq 0}\frac{1}{p_kq_k}\langle f_n(*,k)\circ_{i_1}^{s-i_1} f_n(*,k), f_n(*,k)\circ_{i_2}^{s-i_2} f_n(*,k)\rangle_{\ell^2_{\mathfrak{s}}(\Delta_{2n-2-s})}.
\end{aligned} \tag{5.20}$$

The claim follows from the above equalities and relations (5.17) and (5.4).

□

## 5.2 Semigroup bound

For the $n$th-chaos $J_n(f_n)$, $n\geq 2$, $f_n\in\ell^2_{\mathfrak{s}}(\Delta_n)$, we have

$$-D_kL^{-1}J_n(f_n) = n^{-1}D_kJ_n(f_n) = J_{n-1}(f_n(*,k)) \tag{5.21}$$

and the constants $C_i$ of Corollary 3.5 are equal, respectively, to

$$\begin{aligned}
\tilde{C}_1 := |1-n!\|f_n\|^2_{\ell^2_{\mathfrak{s}}(\Delta_n)}| + n\|&\langle J_{n-1}(f_n(*,\cdot)), J_{n-1}(f_n(*,\cdot))\rangle_{\ell^2(\mathbb{N})} \\
&- \mathrm{E}[\langle J_{n-1}(f_n(*,\cdot)), J_{n-1}(f_n(*,\cdot))\rangle_{\ell^2(\mathbb{N})}]\|_{L^2(\Omega)},
\end{aligned} \tag{5.22}$$

$$\tilde{C}_2 : \ = \ \tilde{B}_2, \quad \text{where } \tilde{B}_2 \text{ is defined by (5.8)}$$

$$\tilde{C}_3 : \ = \ \frac{\tilde{B}_3}{n}, \quad \text{where } \tilde{B}_3 \text{ is defined by (5.9).}$$

In the next theorem we show that these constants can be bounded above by computable quantities.

**Theorem 5.4** *Let $n \geq 2$ be fixed and let $f_n \in \ell^2_{\mathfrak{s}}(\Delta_n)$. Assume that for any $k \in \mathbb{N}$ the functions $\tilde{h}^{(k)}_{n-1,n-1,s}$ defined by* (5.11) *belong to $\ell^2_{\mathfrak{s}}(\Delta_{2n-2-s})$, $0 \leq s \leq 2n-2$, and that*

$$C_1 := |1 - n!\|f_n\|^2_{\ell^2_{\mathfrak{s}}(\Delta_n)}| + n\left(\sum_{s=0}^{2n-3}(2n-2-s)! \sum_{s\leq\{2i_1,\, 2i_2\}\leq 2(s\wedge(n-1))} i_1!i_2!\binom{n-1}{i_1}^2\binom{n-1}{i_2}^2\binom{i_1}{s-i_1}\binom{i_2}{s-i_2} \|f_n \otimes_{i_1+1}^{s-i_1+1} f_n\|_{\ell^2(\Delta_{2n-2-s})}\|f_n \otimes_{i_2+1}^{s-i_2+1} f_n\|_{\ell^2(\Delta_{2n-2-s})}\right)^{1/2},$$

*$C_2 := B_2$, where $B_2$ is defined by* (5.12)*, and $C_3 := B_3/n$ where $B_3$ is defined by* (5.13)*, are finite. Then for all $h \in \mathcal{C}^2_b$ we have*

$$|\mathrm{E}[h(J_n(f_n))] - \mathrm{E}[h(Z)]| \leq C_1 \min\{4\|h\|_\infty, \|h''\|_\infty\} + \|h'\|_\infty C_2 + \|h''\|_\infty C_3.$$

*Proof.* The claim follows from Corollary 3.5 if we show that the constant $\tilde{C}_1$ defined by (5.22) is bounded above by the constant $C_1$ defined in the statement (for the bounds $\tilde{C}_i \leq C_i$, $i = 2, 3$, see Step 2 of the proof of Theorem 5.3). Along a similar computation as in the Step 1 of the proof of Theorem 5.3, we have

$$\begin{aligned}
&\langle J_{n-1}(f_n(*,\cdot)), J_{n-1}(f_n(*,\cdot))\rangle_{\ell^2(\mathbb{N})} \\
&= (n-1)!\sum_{k\geq 0} f_n(*,k) \circ_{n-1}^{n-1} f_n(*,k) \\
&\quad + \sum_{s=0}^{2n-3} \sum_{s\leq 2i\leq 2(s\wedge(n-1))} i!\binom{n-1}{i}^2\binom{i}{s-i} J_{2n-2-s}\left(\sum_{k\geq 0} f_n(*,k) \circ_i^{s-i} f_n(*,k)\right) \\
&= (n-1)! f_n \circ_n^n f_n \\
&\quad + \sum_{s=0}^{2n-3} \sum_{s\leq 2i\leq 2(s\wedge(n-1))} i!\binom{n-1}{i}^2\binom{i}{s-i} J_{2n-2-s}\left(f_n \circ_{i+1}^{s-i+1} f_n\right),
\end{aligned} \tag{5.23}$$

where the latter equality follows from Lemma 5.2. By a similar computation as for (5.15), we have

$$\begin{aligned}
&\|\|J_{n-1}(f_n(*,\cdot))\|^2_{\ell^2(\mathbb{N})} - \mathrm{E}[\|J_{n-1}(f_n(*,\cdot))\|^2_{\ell^2(\mathbb{N})}]\|^2_{L^2(\Omega)} \\
&= \mathrm{E}\left[\left(\sum_{s=0}^{2n-3} \sum_{s\leq 2i\leq 2(s\wedge(n-1))} i!\binom{n-1}{i}^2\binom{i}{s-i} J_{2n-2-s}(f_n \circ_{i+1}^{s-i+1} f_n)\right)^2\right] \\
&= \sum_{s=0}^{2n-3}(2n-2-s)! \sum_{s\leq\{2i_1,\, 2i_2\}\leq 2(s\wedge(n-1))} i_1!i_2!\binom{n-1}{i_1}^2\binom{n-1}{i_2}^2
\end{aligned}$$

$$\binom{i_1}{s-i_1}\binom{i_2}{s-i_2}\langle f_n \circ_{i_1+1}^{s-i_1+1} f_n, f_n \circ_{i_2+1}^{s-i_2+1} f_n\rangle_{\ell^2_{\mathfrak{s}}(\Delta_{2n-2-s})}. \tag{5.24}$$

By this relation and (5.22) we deduce

$$\tilde{C}_1 = |1 - n!\|f_n\|^2_{\ell^2_{\mathfrak{s}}(\Delta_n)}| + n\Bigg(\sum_{s=0}^{2n-3}(2n-2-s)! \sum_{s\le\{2i_1,\,2i_2\}\le 2(s\wedge(n-1))} i_1!i_2!\binom{n-1}{i_1}^2\binom{n-1}{i_2}^2\binom{i_1}{s-i_1}\binom{i_2}{s-i_2} \langle f_n \circ_{i_1+1}^{s-i_1+1} f_n, f_n \circ_{i_2+1}^{s-i_2+1} f_n\rangle_{\ell^2_{\mathfrak{s}}(\Delta_{2n-2-s})}\Bigg)^{1/2}.$$

By this equality and (5.17) and (5.4) we finally have $\tilde{C}_1 \le C_1$. □

**Connection with Theorem 4.1 in [14]**

In this subsection we refine a little the bound given by Theorem 5.4 in order to strictly extend the bound provided by Theorem 4.1 in [14]. For the $n$th chaos $J_n(f_n)$, $n \ge 2$, $f_n \in \ell^2_{\mathfrak{s}}(\Delta_n)$, we have that the constants $C_i$ of Theorem 3.4 are equal, respectively, to

$$\tilde{C}_1 := \mathrm{E}\left[|1 - n\|J_{n-1}(f_n(*,\cdot))\|^2_{\ell^2(\mathbb{N})}|\right], \tag{5.25}$$

$$\tilde{C}_2 := n^2 \sum_{k\ge 0} \frac{|1-2p_k|}{\sqrt{p_k q_k}} \mathrm{E}[|J_{n-1}(f_n(*,k))|^3], \tag{5.26}$$

and

$$\tilde{C}_3 := \frac{\tilde{B}_3}{n}, \quad \text{where } \tilde{B}_3 \text{ is defined by (5.9)}$$

In the next theorem we show that these constants can be bounded above by computable quantities.

**Theorem 5.5** *Let $n \ge 2$ be fixed and let $f_n \in \ell^2_{\mathfrak{s}}(\Delta_n)$. Assume that for any $k \in \mathbb{N}$ the functions $\tilde{h}^{(k)}_{n-1,n-1,s}$ defined by (5.11) belong to $\ell^2_{\mathfrak{s}}(\Delta_{2n-2-s})$, $0 \le s \le 2n-2$, and that*

$$C_1 := \Bigg(|1 - n!\|f_n\|^2_{\ell^2_{\mathfrak{s}}(\Delta_n)}|^2 + n^2\sum_{s=0}^{2n-3}(2n-2-s)! \sum_{s\le\{2i_1,\,2i_2\}\le 2(s\wedge(n-1))} i_1!i_2!\binom{n-1}{i_1}^2\binom{n-1}{i_2}^2\binom{i_1}{s-i_1}\binom{i_2}{s-i_2} \|f_n \otimes_{i_1+1}^{s-i_1+1} f_n\|_{\ell^2(\Delta_{2n-2-s})}\|f_n \otimes_{i_2+1}^{s-i_2+1} f_n\|_{\ell^2(\Delta_{2n-2-s})}\Bigg)^{1/2},$$

$C_2 := B_2/n$, *where* $B_2$ *is defined by* (5.12) *and* $C_3 := B_3/n$, *where* $B_3$ *is defined by* (5.13) *are finite. Then for all* $h \in \mathcal{C}_b^2$ *we have*

$$|\mathrm{E}[h(J_n(f_n))] - \mathrm{E}[h(Z)]| \leq C_1 \min\{4\|h\|_\infty, \|h''\|_\infty\} + \|h'\|_\infty C_2 + \|h''\|_\infty C_3.$$

*Proof.* The claim follows from Theorem 3.4 if we show that the constants $\tilde{C}_i$, $i = 1, 2$, defined by (5.25) and (5.26) are bounded above by the constants $C_i$, $i = 1, 2$, defined in the statement, respectively (for the bound $\tilde{C}_3 \leq C_3$ see Step 2 of the proof of Theorem 5.3).

*Step 1: Proof of* $\tilde{C}_1 \leq C_1$. By the Cauchy-Schwarz inequality, (5.23) and (5.24) we have

$$\begin{aligned}
\tilde{C}_1 &\leq \mathrm{E}\left[|1 - n\|J_{n-1}(f_n(*,\cdot))\|^2_{\ell^2(\mathbb{N})}|^2\right]^{1/2} \\
&\leq \Bigg(|1 - n!\|f_n\|^2_{\ell^2_{\mathfrak{s}}(\Delta_n)}|^2 \\
&\quad + n^2 \mathrm{E}\left[\left(\sum_{s=0}^{2n-3} \sum_{s\leq 2i \leq 2(s\wedge(n-1))} i! \binom{n-1}{i}^2 \binom{i}{s-i} J_{2n-2-s}(f_n \circ_{i+1}^{s-i+1} f_n)\right)^2\right]\Bigg)^{1/2} \\
&= \Bigg(|1 - n!\|f_n\|^2_{\ell^2_{\mathfrak{s}}(\Delta_n)}|^2 \\
&\quad + n^2 \sum_{s=0}^{2n-3} (2n-2-s)! \sum_{s\leq\{2i_1,\, 2i_2\}\leq 2(s\wedge(n-1))} i_1! i_2! \binom{n-1}{i_1}^2 \binom{n-1}{i_2}^2 \binom{i_1}{s-i_1} \binom{i_2}{s-i_2} \\
&\qquad \langle f_n \circ_{i_1+1}^{s-i_1+1} f_n, f_n \circ_{i_2+1}^{s-i_2+1} f_n \rangle_{\ell^2_{\mathfrak{s}}(\Delta_{2n-2-s})}\Bigg)^{1/2}.
\end{aligned}$$

The claim follows from Relations (5.17) and (5.4).

*Step 2: Proof of* $\tilde{C}_2 \leq C_2$. By the Cauchy-Schwarz inequality we have

$$\mathrm{E}[|J_{n-1}(f_n(*,k))|^3] \leq (\mathrm{E}[|J_{n-1}(f_n(*,k))|^2])^{1/2} (\mathrm{E}[|J_{n-1}(f_n(*,k))|^4])^{1/2}.$$

By the isometry for discrete multiple stochastic integrals we have

$$\|J_{n-1}(f_n(*,k))\|_{L^2(\Omega)} = \sqrt{(n-1)!}\|f_n(*,k)\|_{\ell^2_{\mathfrak{s}}(\Delta_{n-1})}.$$

By the above relations and (5.18) we have $\tilde{C}_2 \leq \tilde{B}_2/n$, where $\tilde{B}_2$ is defined by (5.19). The claim follows from (5.17) and (5.4). $\square$

Since it is not obvious that the above theorem extends Theorem 4.1 in [14], it is worthwhile to explain this point in detail. Take $f_n \in \ell^2_{\mathfrak{s}}(\Delta_n)$, $n \geq 2$, and let $\tilde{h}^{(k)}_{n-1,n-1,s}$ be defined by (5.11). In the symmetric case $p_k = q_k = 1/2$, by the same arguments as those one after the statement of Proposition 5.1 we have that, for any fixed $0 \le s \le 2(n-1)$ and $k \in \mathbb{N}$, $\tilde{h}^{(k)}_{n-1,n-1,s} = 0$ if $s/2$ is not an integer and

$$\tilde{h}^{(k)}_{n-1,n-1,s} = (s/2)! \binom{n-1}{s/2}^2 f_n(*,k) \circ^{s/2}_{s/2} f_n(*,k)$$

otherwise. If $s/2$ is an integer we also have

$$\begin{aligned} \|f_n(*,k) \circ^{s/2}_{s/2} f_n(*,k)\|_{\ell^2_{\mathfrak{s}}(\Delta_{2n-2-s})} &\le \|f_n(*,k) \otimes^{s/2}_{s/2} f_n(*,k)\|_{\ell^2(\Delta_{2n-2-s})} \\ &= \|f_n(*,k) \star^{s/2}_{s/2} f_n(*,k)\|_{\ell^2(\Delta_{2n-2-s})} \\ &\le \|f_n(*,k)\|^2_{\ell^2_{\mathfrak{s}}(\Delta_{n-1})} < \infty, \end{aligned}$$

where the latter relation follows from Lemma 2.4(1) in [14]. So

$$\tilde{h}^{(k)}_{n-1,n-1,s} \in \ell^2_{\mathfrak{s}}(\Delta_{2n-2-s}).$$

In the symmetric case, by the definition of the contraction, for $0 \le s \le 2n-3$ and $s \le 2i \le 2(s \wedge (n-1))$, we have

$$\|f_n \otimes^{s-i+1}_{i+1} f_n\|_{\ell^2(\Delta_{2n-2-s})} = 0, \quad \text{if } s < 2i$$

and

$$\begin{aligned} \|f_n \otimes^{s-i+1}_{i+1} f_n\|_{\ell^2(\Delta_{2n-2-s})} &= \|f_n \star^{i+1}_{i+1} f_n\|_{\ell^2(\Delta_{2n-2-2i})} \\ &\le \|f_n\|^2_{\ell^2_{\mathfrak{s}}(\Delta_n)}, \quad \text{if } s = 2i \end{aligned}$$

where the latter relation follows from Lemma 2.4(1) in [14]. Consequently, the constant $C_1$ in the statement of Theorem 5.5 is finite and reduces to

$$\begin{aligned} C_1 &= \Bigg( |1 - n!\|f_n\|^2_{\ell^2_{\mathfrak{s}}(\Delta_n)}|^2 + n^2 \sum_{s=0}^{2(n-2)} \mathbb{1}\{s/2 \in \mathbb{N}\}(2n-2-s)! \left(\frac{s}{2}!\right)^2 \binom{n-1}{s/2}^4 \\ &\qquad \|f_n \star^{s/2+1}_{s/2+1} f_n\|^2_{\ell^2(\Delta_{2n-2-s})} \Bigg)^{1/2}, \\ &= \left( |1 - n!\|f_n\|^2_{\ell^2_{\mathfrak{s}}(\Delta_n)}|^2 + n^2 \sum_{s=1}^{n-1} (2n-2s)! \left[ (s-1)! \binom{n-1}{s-1}^2 \right]^2 \|f_n \star^s_s f_n\|^2_{\ell^2(\Delta_{2n-2s})} \right)^{1/2}. \end{aligned}$$

As far as the constant $C_2$ in the statement of Theorem 5.5 is concerned, in the symmetric case one clearly has

$$C_2 = 0.$$

Finally, consider the constant $C_3$ in the statement of Theorem 5.5. The following bound holds:

$$\begin{aligned}
C_3 \leq\ & \frac{20}{3} n^3 \sum_{s=0}^{2n-2} (2n-2-s)! \\
& \times \sum_{s\leq\{2i_1,\,2i_2\}\leq 2(s\wedge(n-1))} i_1! i_2! \binom{n-1}{i_1}^2 \binom{n-1}{i_2}^2 \binom{i_1}{s-i_1}\binom{i_2}{s-i_2} \\
& \sum_{k\geq 0} \|f_n(*,k)\otimes_{i_1}^{s-i_1} f_n(*,k)\|_{\ell^2(\mathbb{N})^{\otimes 2n-2-s}} \|f_n(*,k)\otimes_{i_2}^{s-i_2} f_n(*,k)\|_{\ell^2(\mathbb{N})^{\otimes 2n-2-s}} \\
=\ & \frac{20n^3}{3} \sum_{s=0}^{2n-2} \mathbb{1}\{s/2\in\mathbb{N}\}(2n-2-s)! \left(\frac{s}{2}!\right)^2 \binom{n-1}{s/2}^4 \\
& \qquad \times \sum_{k\geq 0} \|f_n(*,k)\star_{s/2}^{s/2} f_n(*,k)\|^2_{\ell^2(\mathbb{N})^{\otimes 2n-2-s}} \\
=\ & \frac{20n^3}{3} \sum_{s=1}^{n} (2n-2s)! \left[(s-1)!\binom{n-1}{s-1}^2\right]^2 \sum_{k\geq 0} \|f_n(*,k)\star_{s-1}^{s-1} f_n(*,k)\|^2_{\ell^2(\mathbb{N})^{\otimes 2n-2s}} \\
=\ & \frac{20n^3}{3} \sum_{s=1}^{n} (2n-2s)! \left[(s-1)!\binom{n-1}{s-1}^2\right]^2 \|f_n \star_s^{s-1} f_n\|^2_{\ell^2(\mathbb{N})^{\otimes 2n-2s+1}},
\end{aligned}$$

where the latter equality follows from Lemma 2.4(2) (relation (2.4)) in [14] and the constant $C_3$ is finite again by Lemma 2.4(1) in in [14]. We recovered the bound provided by Theorem 4.1 in [14].

## 5.3 Convergence to the normal distribution

The next theorems follow by Theorems 5.3 and 5.4, respectively.

**Theorem 5.6** *Let $n \geq 2$ be fixed and let $F_m = J_n(f_m)$, $m \geq 1$, be a sequence of discrete multiple stochastic integrals such that $f_m \in \ell^2_{\mathfrak{s}}(\Delta_n)$, for any $k \in \mathbb{N}$ the functions $h^{(k)}_{n-1,n-1,s}$ and $\tilde{h}^{(k)}_{n-1,n-1,s}$ defined by (5.10) and (5.11) with $f_m$ in place of $f_n$ belong to $\ell^2_{\mathfrak{s}}(\Delta_{2n-2-s})$, $0 \leq s \leq 2n-2$,*

$$n!\|f_m\|^2_{\ell_{\mathfrak{s}}(\Delta_n)} \to 1, \quad \text{as } m \to \infty \tag{5.27}$$

$$\sum_{k\geq 0} \|f_m(*,k)\otimes_i^{s-i} f_{m]k}(*)\|_{\ell^2(\Delta_{2n-2-s})} \to 0,$$
$$\text{as } m\to\infty, \text{ for any } 0 \leq s \leq 2n-3 \text{ and } s \leq 2i \leq 2(s\wedge(n-1)) \tag{5.28}$$

$$\sum_{k\geq 0} \frac{|1-2p_k|}{\sqrt{p_k q_k}} \|f_m(*,k)\|_{\ell^2_{\mathfrak{s}}(\Delta_{n-1})}$$

$$\sqrt{\|f_m(*,k) \otimes_{i_1}^{s-i_1} f_m(*,k)\|_{\ell^2(\Delta_{2n-2-s})} \|f_m(*,k) \otimes_{i_2}^{s-i_2} f_m(*,k)\|_{\ell^2(\Delta_{2n-2-s})}} \to 0,$$

$$\text{as } m \to \infty, \text{ for any } 0 \le s \le 2n-2 \text{ and } s \le \{2i_1, 2i_2\} \le 2(s \wedge (n-1)) \quad (5.29)$$

*and*

$$\sum_{k \ge 0} \frac{1}{p_k q_k} \|f_m(*,k) \otimes_{i_1}^{s-i_1} f_m(*,k)\|_{\ell^2(\Delta_{2n-2-s})} \|f_m(*,k) \otimes_{i_2}^{s-i_2} f_m(*,k)\|_{\ell^2(\Delta_{2n-2-s})} \to 0,$$

$$\text{as } m \to \infty, \text{ for any } 0 \le s \le 2n-2 \text{ and } s \le \{2i_1, 2i_2\} \le 2(s \wedge (n-1)). \quad (5.30)$$

*Then*

$$F_m \stackrel{Law}{\to} \mathcal{N}(0,1).$$

**Theorem 5.7** *Let $n \ge 2$ be fixed and let $F_m = J_n(f_m)$, $m \ge 1$, be a sequence of discrete multiple stochastic integrals such that $f_m \in \ell_{\mathfrak{s}}^2(\Delta_n)$, for any $k \in \mathbb{N}$ the functions $\tilde{h}_{n-1,n-1,s}^{(k)}$ defined by* (5.11) *with $f_m$ in place of $f_n$ belong to $\ell_{\mathfrak{s}}^2(\Delta_{2n-2-s})$, $0 \le s \le 2n-2$,*

$$\|f_m \otimes_{i+1}^{s-i+1} f_m\|_{\ell^2(\Delta_{2n-2-s})} \to 0,$$

$$\text{as } m \to \infty, \text{ for any } 0 \le s \le 2n-3 \text{ and } s \le 2i \le 2(s \wedge (n-1)) \quad (5.31)$$

*and* (5.27), (5.29) *and* (5.30) *hold. Then*

$$F_m \stackrel{Law}{\to} \mathcal{N}(0,1).$$

**Connection with Proposition 4.3 in [14]**

In this paragraph we explain the connection between Theorem 5.7, specialized in the symmetric case, and Proposition 4.3 in [14]. Take $f_m \in \ell_{\mathfrak{s}}^2(\Delta_n)$, $m \ge 1$, $n \ge 2$, and let $\tilde{h}_{n-1,n-1,s}^{(k)}$ be defined by (5.11) with $f_m$ in place of $f_n$. We already checked (after the proof of Theorem 5.5) that, in the symmetric case, one has $\tilde{h}_{n-1,n-1,s}^{(k)} \in \ell_{\mathfrak{s}}^2(\Delta_{2n-2-s})$, $k \in \mathbb{N}$, $0 \le s \le 2n-2$. Note that assumption (5.27) is explicitly required in Proposition 4.3 of [14] and, for $p_k = q_k = 1/2$, assumption (5.29) is automatically satisfied. In the symmetric case, conditions (5.30) and (5.31) read

$$\sum_{k \ge 0} \|f_m(*,k) \star_i^i f_m(*,k)\|_{\ell^2(\Delta_{2n-2-2i})}^2 \to 0, \quad \text{as } m \to \infty, \text{ for any } 0 \le i \le n-1 \quad (5.32)$$

and

$$\|f_m \star_{i+1}^{i+1} f_m\|_{\ell^2(\Delta_{2n-2-2i})} \to 0, \quad \text{as } m \to \infty, \text{ for any } 0 \le i \le n-2 \quad (5.33)$$

respectively. We are going to check that assumption (4.44) of Proposition 4.3 in [14], i.e.

$$\|f_m \star_r^r f_m\|_{\ell^2(\mathbb{N})^{\otimes 2n-2r}} \to 0, \quad \text{as } m \to \infty, \text{ for any } 1 \le r \le n-1 \tag{5.34}$$

implies (5.32) and (5.33). Clearly, (5.34) is equivalent to (5.33). Moreover, for any $0 \le i \le n-1$, by Lemma 2.4(2) (relation (2.4)) and Lemma 2.4(3) in [14], we have

$$\begin{aligned}\sum_{k\ge 0} \|f_m(*,k) \star_i^i f_m(*,k)\|^2_{\ell^2(\Delta_{2n-2-2i})} &\le \sum_{k\ge 0} \|f_m(*,k) \star_i^i f_m(*,k)\|^2_{\ell^2(\mathbb{N})^{\otimes 2n-2-2i}} \\ &\le \|f_m \star_n^{n-1} f_m\|_{\ell^2(\mathbb{N})} \|f_m\|^2_{\ell^2_{\mathfrak{s}}(\Delta_n)} \\ &\le \|f_m \star_{n-1}^{n-1} f_m\|_{\ell^2(\mathbb{N})^{\otimes 2}} \|f_m\|^2_{\ell^2_{\mathfrak{s}}(\Delta_n)}.\end{aligned}$$

So combining (5.27) and condition (5.34) with $r = n-1$ we have (5.32).

**Quadratic functionals**

In the next proposition we apply Theorem 5.7 with $n = 2$. In comparison with Proposition 4.3 of [14] we require an additional $\ell^4$ condition in the non-symmetric case.

**Proposition 5.8** *Assume that there exists some $\varepsilon > 0$ such that*

$$0 < \varepsilon < p_k < 1-\varepsilon, \qquad k \in \mathbb{N}, \tag{5.35}$$

*and consider a sequence $f_m \in \ell^2_{\mathfrak{s}}(\Delta_2)$ such that*

*a)* $\lim_{m\to\infty} \|f_m\|^2_{\ell^2_{\mathfrak{s}}(\Delta_2)} = 1/2$,

*b)* $\lim_{m\to\infty} \|f_m \star_1^1 f_m\|_{\ell^2(\mathbb{N})^{\otimes 2}} = 0$,

*c)* $\lim_{m\to\infty} \|f_m\|_{\ell^4_{\mathfrak{s}}(\Delta_2)} \to 0$.

*Then* $J_2(f_m) \overset{Law}{\to} \mathcal{N}(0,1)$ *as* $m \to \infty$.

*Proof.* We need to satisfy Conditions (5.27), (5.29), (5.30) and (5.31), in addition to an integrability check which is postponed to the end of this proof. First we note that (5.27) is Condition $a)$ above. Next we note that under (5.35), Conditions (5.29), (5.30) and (5.31) read

$$\sum_{k\ge 0} \|f_m(*,k)\|_{\ell^2_{\mathfrak{s}}(\Delta_1)} \|f_m(*,k) \otimes_0^0 f_m(*,k)\|_{\ell^2_{\mathfrak{s}}(\Delta_2)} \to 0,$$

$$\sum_{k\ge 0} \|f_m(*,k)\|_{\ell^2_{\mathfrak{s}}(\Delta_1)} \|f_m(*,k) \otimes_1^0 f_m(*,k)\|_{\ell^2_{\mathfrak{s}}(\Delta_1)} \to 0,$$

$$\sum_{k\geq 0} \|f_m(*,k)\|_{\ell^2_{\mathfrak{s}}(\Delta_1)} \|f_m(*,k) \otimes_1^1 f_m(*,k)\|_{\ell^2_{\mathfrak{s}}(\Delta_0)} \to 0,$$
$$\sum_{k\geq 0} \|f_m(*,k) \otimes_0^0 f_m(*,k)\|^2_{\ell^2_{\mathfrak{s}}(\Delta_2)} \to 0,$$
$$\sum_{k\geq 0} \|f_m(*,k) \otimes_1^0 f_m(*,k)\|^2_{\ell^2_{\mathfrak{s}}(\Delta_1)} \to 0,$$
$$\sum_{k\geq 0} \|f_m(*,k) \otimes_1^1 f_m(*,k)\|^2_{\ell^2_{\mathfrak{s}}(\Delta_0)} \to 0,$$

and

$$\|f_m \otimes_1^1 f_m\|_{\ell^2(\Delta_2)},\ \|f_m \otimes_2^1 f_m\|_{\ell^2(\Delta_1)} \to 0,$$

as $m \to \infty$. Using again (5.35), we have that the above conditions are implied by

$$\sum_{k\geq 0} \|f_m(*,k)\|_{\ell^2_{\mathfrak{s}}(\Delta_1)} \|f_m(*,k) \star_0^0 f_m(*,k)\|_{\ell^2_{\mathfrak{s}}(\Delta_2)} \to 0, \tag{5.36}$$
$$\sum_{k\geq 0} \|f_m(*,k)\|_{\ell^2_{\mathfrak{s}}(\Delta_1)} \|f_m(*,k) \star_1^0 f_m(*,k)\|_{\ell^2_{\mathfrak{s}}(\Delta_1)} \to 0, \tag{5.37}$$
$$\sum_{k\geq 0} \|f_m(*,k)\|_{\ell^2_{\mathfrak{s}}(\Delta_1)} \|f_m(*,k) \star_1^1 f_m(*,k)\|_{\ell^2_{\mathfrak{s}}(\Delta_0)} \to 0, \tag{5.38}$$
$$\sum_{k\geq 0} \|f_m(*,k) \star_0^0 f_m(*,k)\|^2_{\ell^2_{\mathfrak{s}}(\Delta_2)} \to 0, \tag{5.39}$$
$$\sum_{k\geq 0} \|f_m(*,k) \star_1^0 f_m(*,k)\|^2_{\ell^2_{\mathfrak{s}}(\Delta_1)} \to 0, \tag{5.40}$$
$$\sum_{k\geq 0} \|f_m(*,k) \star_1^1 f_m(*,k)\|^2_{\ell^2_{\mathfrak{s}}(\Delta_0)} \to 0, \tag{5.41}$$

and

$$\|f_m \star_1^1 f_m\|_{\ell^2(\Delta_2)},\ \|f_m \star_2^1 f_m\|_{\ell^2(\Delta_1)} \to 0, \tag{5.42}$$

as $m \to \infty$. Now by Lemma 2.4(3) of [14] we have

$$\|f_m \star_2^1 f_m\|_{\ell^2(\Delta_1)} \leq \|f_m \star_1^1 f_m\|_{\ell^2(\mathbb{N})^{\otimes 2}},$$

hence (5.42) is implied by Condition $b$) above. Next, by Lemma 2.4(2)-(3) in [14] we have

$$\begin{aligned}\sum_{k\geq 0} \|f_m(*,k) \star_0^0 f_m(*,k)\|^2_{\ell^2_{\mathfrak{s}}(\Delta_2)} &\leq \|f_m \star_2^1 f_m\|_{\ell^2(\mathbb{N})} \|f_m\|^2_{\ell^2(\mathbb{N})^{\otimes 2}} \\ &\leq \|f_m \star_1^1 f_m\|_{\ell^2(\mathbb{N})^{\otimes 2}} \|f_m\|^2_{\ell^2(\mathbb{N})^{\otimes 2}},\end{aligned} \tag{5.43}$$

hence Condition (5.39) is implied by Conditions $a$) and $b$) above. Similarly, by Lemma 2.4(2)-(3) in [14] we have

$$\sum_{k\geq 0}\|f_m(*,k)\star_1^1 f_m(*,k)\|^2_{\ell^2_{\mathfrak{s}}(\Delta_0)} \leq \|f_m\star_2^1 f_m\|_{\ell^2(\mathbb{N})}\|f_m\|^2_{\ell^2(\mathbb{N})^{\otimes 2}}$$
$$\leq \|f_m\star_1^1 f_m\|_{\ell^2(\mathbb{N})^{\otimes 2}}\|f_m\|^2_{\ell^2(\mathbb{N})^{\otimes 2}}, \tag{5.44}$$

hence Condition (5.41) is also implied by Conditions $a$) and $b$) above. Now, we have

$$\sum_{k\geq 0}\|f_m(*,k)\|_{\ell^2_{\mathfrak{s}}(\Delta_1)}\|f_m(*,k)\star_0^0 f_m(*,k)\|_{\ell^2_{\mathfrak{s}}(\Delta_2)}$$
$$\leq \left(\sum_{k\geq 0}\|f_m(*,k)\|^2_{\ell^2_{\mathfrak{s}}(\Delta_1)}\right)^{1/2}\left(\sum_{k\geq 0}\|f_m(*,k)\star_0^0 f_m(*,k)\|^2_{\ell^2_{\mathfrak{s}}(\Delta_2)}\right)^{1/2} \tag{5.45}$$
$$\leq \left(\sum_{k\geq 0}\|f_m(*,k)\|^2_{\ell^2_{\mathfrak{s}}(\Delta_1)}\right)^{1/2}\|f_m\star_1^1 f_m\|^{1/2}_{\ell^2(\mathbb{N})^{\otimes 2}}\|f_m\|_{\ell^2(\mathbb{N})^{\otimes 2}} \tag{5.46}$$
$$= \left(\sum_{k\geq 0} f_m(*,k)\star_1^1 f_m(*,k)\right)^{1/2}\|f_m\star_1^1 f_m\|^{1/2}_{\ell^2(\mathbb{N})^{\otimes 2}}\|f_m\|_{\ell^2(\mathbb{N})^{\otimes 2}} \tag{5.47}$$
$$= \left(f_m\star_2^2 f_m\right)^{1/2}\|f_m\star_1^1 f_m\|^{1/2}_{\ell^2(\mathbb{N})^{\otimes 2}}\|f_m\|_{\ell^2(\mathbb{N})^{\otimes 2}} \tag{5.48}$$
$$= \|f_m\|^2_{\ell^2_{\mathfrak{s}}(\Delta_2)}\|f_m\star_1^1 f_m\|^{1/2}_{\ell^2(\mathbb{N})^{\otimes 2}}, \tag{5.49}$$

where in (5.45) we used the Cauchy-Schwartz inequality, in (5.46) we used (5.43), in (5.47) we used the identity

$$f\star_n^n g = \langle f,g\rangle_{\ell^2(\mathbb{N})^{\otimes n}},\quad f,g\in\ell^2(\mathbb{N})^{\otimes n} \tag{5.50}$$

in (5.48) we used the equality

$$\sum_{k\geq 0} f_m(*,k)\star_1^1 f_m(*,k) = f_m\star_2^2 f_m,$$

and in (5.49) we used (5.50). Similarly, using (5.44) we have

$$\sum_{k\geq 0}\|f_m(*,k)\|_{\ell^2_{\mathfrak{s}}(\Delta_1)}\|f_m(*,k)\star_1^1 f_m(*,k)\|_{\ell^2_{\mathfrak{s}}(\Delta_0)} \leq \|f_m\|^2_{\ell^2_{\mathfrak{s}}(\Delta_2)}\|f_m\star_1^1 f_m\|^{1/2}_{\ell^2(\mathbb{N})^{\otimes 2}}. \tag{5.51}$$

So Conditions (5.36) and (5.38) are also implied by Conditions $a$) and $b$) above. By similar arguments as above, we have

$$\sum_{k\geq 0}\|f_m(*,k)\|_{\ell^2_{\mathfrak{s}}(\Delta_1)}\|f_m(*,k)\star_1^0 f_m(*,k)\|_{\ell^2_{\mathfrak{s}}(\Delta_1)}$$

$$\leq \|f_m\|_{\ell^2_{\mathfrak{s}}(\Delta_2)} \left( \sum_{k\geq 0} \|f_m(*,k) \star_1^0 f_m(*,k)\|^2_{\ell^2_{\mathfrak{s}}(\Delta_1)} \right)^{1/2}. \tag{5.52}$$

We note that

$$\sum_{k\geq 0} \|f_m(*,k) \star_1^0 f_m(*,k)\|^2_{\ell^2_{\mathfrak{s}}(\Delta_1)} \leq \sum_{k\geq 0} \sum_{a\geq 0} f_m^4(a,k) = \|f_m^2\|^2_{\ell^2_{\mathfrak{s}}(\Delta_2)}, \tag{5.53}$$

and so Condition $c$) above implies (5.37) and (5.40). Finally we note that for any $k \in \mathbb{N}$ the functions $\tilde{h}^{(k)}_{1,1,s}$, $0 \leq s \leq 2$, defined by (5.11) with $f_m$ in place of $f_n$, i.e.

$$\tilde{h}^{(k)}_{1,1,0} = f_m(*,k) \circ_0^0 f_m(*,k)$$

$$\tilde{h}^{(k)}_{1,1,1} = f_m(*,k) \circ_1^0 f_m(*,k)$$

and

$$\tilde{h}^{(k)}_{1,1,2} = f_m(*,k) \circ_1^1 f_m(*,k)$$

belong to $\ell^2_{\mathfrak{s}}(\Delta_2)$, $\ell^2_{\mathfrak{s}}(\Delta_1)$ and $\ell^2_{\mathfrak{s}}(\Delta_0)$, respectively. Indeed, this easily follows from (5.4), (5.35) and Lemma 2.4(1) of [14]. □

**Example**

A straightforward computation shows that examples of function sequences that satisfy the hypotheses of Proposition 5.8 include

$$f_m(k_1,k_2) = \frac{1}{m\sqrt{2}} \mathbb{1}_{[0,m]^2}(k_1,k_2), \qquad m \geq 1.$$

Note that the above example will also satisfy the hypotheses of Theorem 5.6 as well. More generally, any sequence of non-negative kernels satisfying the hypotheses of Proposition 5.8 will satisfy the hypotheses of Theorem 5.6. Indeed, under Condition (5.35), for non-negative kernels, Condition (5.28) is implied by (5.39), (5.40) and (5.41) which, as showed in the proof of Proposition 5.8, are in turn implied by Conditions $a$), $b$) and $c$). However, elementary computations have shown that, in general, it is difficult to compare the second addends of the constants $B_1$ and $C_1$ of Theorems 5.3 and 5.4, respectively. Consequently, in general, it is difficult to compare Conditions (5.28) and (5.31) of Theorems 5.6 and 5.7, respectively.

# 6 Poisson approximation of Bernoulli functionals

We recall that the total variation distance between the laws of two $\mathbb{N}$-valued random variables $Y_i$, $i = 1, 2$, is given by

$$d_{TV}(Y_1,Y_2) := \sup_{A\subseteq\mathbb{N}} |P(Y_1 \in A) - P(Y_2 \in A)|.$$

Of course, the topology induced by $d_{TV}$ on the class of all probability laws on $\mathbb{N}$ is strictly stronger than the topology induced by the convergence in distribution.

In this section we present two different upper bounds for $d_{TV}(F, \mathrm{Po}(\lambda))$, where $\mathrm{Po}(\lambda)$ is a Poisson random variable with mean $\lambda > 0$. The first one is obtained by using the covariance representation formula (2.15), while the second one is obtained by using the covariance representation formula (2.16).

Before proceeding further we recall some necessary background on the Chen-Stein method for the Poisson approximation and refer to [2] for more insight into this technique.

**Chen-Stein's method for Poisson approximation**

Given $A \subseteq \mathbb{N}$, it turns out that there exists a unique function $f_A : \mathbb{N} \to \mathbb{R}$ such that

$$\mathbb{1}_A(k) - P(\mathrm{Po}(\lambda) \in A) = \lambda f_A(k+1) - k f_A(k), \quad k \in \mathbb{N} \tag{6.1}$$

verifying the boundary condition $\Delta^2 f(0) = 0$. The above equation is called Chen-Stein's equation. Combining e.g. Theorem 2.3 in [9] and Theorem 1.3 in [6], we deduce that the function $f_A$ has the following properties:

$$\|f_A\|_\infty \leq \min\left(1, \sqrt{\frac{2}{\lambda \mathrm{e}}}\right), \quad \|\Delta f_A\|_\infty \leq \frac{1 - \mathrm{e}^{-\lambda}}{\lambda}, \quad \|\Delta^2 f_A\|_\infty \leq \frac{2 - 2\mathrm{e}^{-\lambda}}{\lambda^2}. \tag{6.2}$$

## 6.1 Clark-Ocone bound

**Theorem 6.1** *Let $F \in \mathrm{Dom}(D)$ be an $\mathbb{N}$-valued random variable with mean $\lambda$ and assume that*

$$b_1 := \mathrm{E}[|\langle \mathrm{E}[D_. F \,|\, \mathcal{F}_{.-1}], D_. F\rangle_{\ell^2(\mathbb{N})} - \lambda|],$$

*and*

$$\begin{aligned} b_2 \;:=\; & \mathrm{E}\left[\left\langle |\mathrm{E}[D_. F \,|\, \mathcal{F}_{.-1}]|, \left|\sqrt{\frac{q_.}{p_.}} D_. F \left(\frac{D_. F}{\sqrt{p_. q_.}} - 1\right)\right|\right\rangle_{\ell^2(\mathbb{N})}\right] \\ & + \mathrm{E}\left[\left\langle |\mathrm{E}[D_. F \,|\, \mathcal{F}_{.-1}]|, \left|\sqrt{\frac{p_.}{q_.}} D_. F \left(\frac{D_. F}{\sqrt{p_. q_.}} + 1\right)\right|\right\rangle_{\ell^2(\mathbb{N})}\right] \end{aligned}$$

*are finite. Then we have*

$$d_{TV}(F, \mathrm{Po}(\lambda)) \leq \frac{1 - \mathrm{e}^{-\lambda}}{\lambda} b_1 + \frac{1 - \mathrm{e}^{-\lambda}}{\lambda^2} b_2. \tag{6.3}$$

*Proof.* We start by checking the domain condition $f_A(F) \in \mathrm{Dom}(D)$. Assume $F_k^+ \geq F_k^-$. We note that

$$D_k f_A(F) = \sqrt{p_k q_k}(f_A(F_k^+) - f_A(F_k^-)) = \sqrt{p_k q_k} \sum_{h=1}^{F_k^+ - F_k^-} (f_A(F_k^+ - h + 1) - f_A(F_k^+ - h)),$$

and so $|D_k f_A(F)| \leq \|\Delta f_A\|_\infty \sqrt{p_k q_k}(F_k^+ - F_k^-)$. Similarly, if $F_k^+ < F_k^-$ then $|D_k f_A(F)| \leq \|\Delta f_A\|_\infty \sqrt{p_k q_k}(F_k^- - F_k^+)$. Consequently,

$$\mathrm{E}[\|Df_A(F)\|^2_{\ell^2(\mathbb{N})}] = \mathrm{E}\left[\sum_{k\geq 0} |D_k f_A(F)|^2\right] \leq \|\Delta f_A\|^2_\infty \mathrm{E}\left[\sum_{k\geq 0} |D_k F|^2\right] = \mathrm{E}[\|DF\|^2_{\ell^2(\mathbb{N})}],$$

and this latter quantity is finite since $F \in \mathrm{Dom}(D)$. The claimed domain condition follows. By the Chen-Stein equation (6.1), the covariance representation (2.15) and Proposition 2.2, we have

$$\begin{aligned} P(\mathrm{Po}(\lambda) \in A) - P(F \in A) &= \mathrm{E}[(F-\lambda) f_A(F) - \lambda(f_A(F+1) - f_A(F))] \\ &= \mathrm{E}\left[\sum_{k\geq 0} \mathrm{E}[D_k F \,|\, \mathcal{F}_{k-1}] D_k f_A(F) - \lambda \Delta f_A(F)\right] \\ &= \mathrm{E}\left[\sum_{k\geq 0} \mathrm{E}[D_k F \,|\, \mathcal{F}_{k-1}](\Delta f_A(F) D_k F + R_k^F(f_A)) - \lambda \Delta f_A(F)\right] \\ &= \mathrm{E}\left[\Delta f_A(F)(\langle \mathrm{E}[D_\cdot F \,|\, \mathcal{F}_{\cdot-1}], D_\cdot F\rangle_{\ell^2(\mathbb{N})} - \lambda)\right] + \mathrm{E}[\langle \mathrm{E}[D_\cdot F \,|\, \mathcal{F}_{\cdot-1}], R_\cdot^F(f_A)\rangle_{\ell^2(\mathbb{N})}]. \end{aligned}$$

The desired result follows by taking absolute values on both sides, as well as by applying the estimates (6.2) and (2.13), and noticing that the random variables $D_k F$ and $\mathrm{E}[D_k F \,|\, \mathcal{F}_{k-1}]$ are independent of $X_k$. □

**Corollary 6.2** *Let $F \in \mathrm{Dom}(D)$ be an $\mathbb{N}$-valued random variable with mean $\lambda$ and assume that*

$$b_1 := |\lambda - \mathrm{Var}(F)| + \|\langle D_\cdot F, \mathrm{E}[D_\cdot F \,|\, \mathcal{F}_{\cdot-1}]\rangle_{\ell^2(\mathbb{N})} - \mathrm{E}[\langle D_\cdot F, \mathrm{E}[D_\cdot F \,|\, \mathcal{F}_{\cdot-1}]\rangle_{\ell^2(\mathbb{N})}]\|_{L^2(\Omega)},$$

*and*

$$b_2 := \sum_{k\geq 0} \frac{1}{\sqrt{p_k q_k}} \mathrm{E}[|D_k F|^2] + \sum_{k\geq 0} \frac{1}{p_k q_k} \|D_k F\|_{L^2(\Omega)} \sqrt{\mathrm{E}[|D_k F|^4]} \qquad (6.4)$$

*are finite. Then we have*

$$d_{TV}(F, \mathrm{Po}(\lambda)) \leq \frac{1 - \mathrm{e}^{-\lambda}}{\lambda} b_1 + \frac{1 - \mathrm{e}^{-\lambda}}{\lambda^2} b_2.$$

*Proof.* We preliminary note that by the Clark-Ocone representation formula (2.15) one has

$$\mathrm{Var}(F) = \mathrm{E}[\langle D.F, \mathrm{E}[D.F \mid \mathcal{F}_{.-1}]\rangle_{\ell^2(\mathbb{N})}]. \tag{6.5}$$

By the Cauchy-Schwarz inequality and (6.5), we have

$$\begin{aligned}
&\mathrm{E}[|\lambda - \langle \mathrm{E}[D.F \mid \mathcal{F}_{.-1}], DF\rangle_{\ell^2(\mathbb{N})}|] \\
&\leq \|\lambda - \langle \mathrm{E}[D.F \mid \mathcal{F}_{.-1}], DF\rangle_{\ell^2(\mathbb{N})}\|_{L^2(\Omega)} \\
&\leq |\lambda - \mathrm{Var}(F)| + \|\langle D.F, \mathrm{E}[D.F \mid \mathcal{F}_{.-1}]\rangle_{\ell^2(\mathbb{N})} - \mathrm{E}[\langle D.F, \mathrm{E}[D.F \mid \mathcal{F}_{.-1}]\rangle_{\ell^2(\mathbb{N})}]\|_{L^2(\Omega)}.
\end{aligned}$$

Moreover,

$$\begin{aligned}
& \mathrm{E}\left[\left\langle |\mathrm{E}[D.F \mid \mathcal{F}_{.-1}]|, \left|\sqrt{\frac{q.}{p.}}D.F\left(\frac{D.F}{\sqrt{p.q.}} - 1\right)\right|\right\rangle_{\ell^2(\mathbb{N})}\right] \\
& + \mathrm{E}\left[\left\langle |\mathrm{E}[D.F \mid \mathcal{F}_{.-1}]|, \left|\sqrt{\frac{p.}{q.}}D.F\left(\frac{D.F}{\sqrt{p.q.}} + 1\right)\right|\right\rangle_{\ell^2(\mathbb{N})}\right] \\
\leq\ & \mathrm{E}\left[\left\langle |\mathrm{E}[D.F \mid \mathcal{F}_{.-1}]|, \frac{|D.F|}{\sqrt{p.q.}}\left(1 + \frac{|D.F|}{\sqrt{p.q.}}\right)\right\rangle_{\ell^2(\mathbb{N})}\right] \\
\leq\ & \mathrm{E}\left[\left\langle |\mathrm{E}[D.F \mid \mathcal{F}_{.-1}]|, \frac{|D.F|}{\sqrt{p.q.}}\right\rangle_{\ell^2(\mathbb{N})}\right] + \mathrm{E}\left[\left\langle |\mathrm{E}[D.F \mid \mathcal{F}_{.-1}]|, \frac{|D.F|^2}{p.q.}\right\rangle_{\ell^2(\mathbb{N})}\right] \\
\leq\ & \mathrm{E}\left[\left\langle \mathrm{E}[|D.F| \mid \mathcal{F}_{.-1}], \frac{|D.F|}{\sqrt{p.q.}}\right\rangle_{\ell^2(\mathbb{N})}\right] + \mathrm{E}\left[\left\langle |\mathrm{E}[D.F \mid \mathcal{F}_{.-1}]|, \frac{|D.F|^2}{p.q.}\right\rangle_{\ell^2(\mathbb{N})}\right] \\
\leq\ & \sum_{k\geq 0} \frac{1}{\sqrt{p_k q_k}}\mathrm{E}[|D_kF|^2] + \sum_{k\geq 0} \frac{1}{p_k q_k}\mathrm{E}[|\mathrm{E}[D_kF \mid \mathcal{F}_{k-1}]||D_kF|^2] \\
\leq\ & \sum_{k\geq 0} \frac{1}{\sqrt{p_k q_k}}\mathrm{E}[|D_kF|^2] + \sum_{k\geq 0} \frac{1}{p_k q_k}\|\mathrm{E}[D_kF \mid \mathcal{F}_{k-1}]\|_{L^2(\Omega)}\sqrt{\mathrm{E}[|D_kF|^4]} \\
\leq\ & \sum_{k\geq 0} \frac{1}{\sqrt{p_k q_k}}\mathrm{E}[|D_kF|^2] + \sum_{k\geq 0} \frac{1}{p_k q_k}\|D_kF\|_{L^2(\Omega)}\sqrt{\mathrm{E}[|D_kF|^4]}.
\end{aligned}$$

The claim follows from Theorem 6.1. □

## 6.2 Semigroup bound

The next result is formally similar to Theorem 3.1 of [15]. More precisely, the first addend in the right-hand side of (6.6) coincides with the term in the right-hand side of relation (3.5) in [15] when replacing the finite difference operator on the Bernoulli space with the finite difference operator on the Poisson space. As for the second addend in the right-hand side of (6.6), although it has some similarities with the corresponding term in (3.6) of [15] (the expectations have the same multiplicative constant), the two terms remain different when replacing the finite difference operator on the Bernoulli space with the finite difference operator on the Poisson space.

**Theorem 6.3** *Let* $F \in \mathrm{Dom}(D)$ *be an* $\mathbb{N}$*-valued random variable with mean* $\lambda$ *and assume that*

$$c_1 := \mathrm{E}[|\lambda - \langle DF, -DL^{-1}F\rangle_{\ell^2(\mathbb{N})}|],$$

*and*

$$\begin{aligned} c_2 \;\; := \;\; & \mathrm{E}\left[\left\langle \left| \sqrt{\frac{q_\cdot}{p_\cdot}} D_\cdot F \left( \frac{D_\cdot F}{\sqrt{p_\cdot q_\cdot}} - 1 \right) \right|, |DL^{-1}F| \right\rangle_{\ell^2(\mathbb{N})}\right] \\ & + \mathrm{E}\left[\left\langle \left| \sqrt{\frac{p_\cdot}{q_\cdot}} D_\cdot F \left( \frac{D_\cdot F}{\sqrt{p_\cdot q_\cdot}} + 1 \right) \right|, |DL^{-1}F| \right\rangle_{\ell^2(\mathbb{N})}\right], \end{aligned}$$

*are finite. Then we have*

$$d_{TV}(F, \mathrm{Po}(\lambda)) \leq \frac{1 - \mathrm{e}^{-\lambda}}{\lambda} c_1 + \frac{1 - \mathrm{e}^{-\lambda}}{\lambda^2} c_2, \tag{6.6}$$

*Proof.* Although the proof is similar to that one of Theorem 6.1, we give the details since some points need a different justification. As in the proof of Theorem 6.1 one has $f_A(F) \in \mathrm{Dom}(D)$. By the Chen-Stein equation (6.1), the covariance representation (2.16) and Proposition 2.2, we have

$$\begin{aligned} P(\mathrm{Po}(\lambda) \in A) - P(F \in A) &= \mathrm{E}[(F - \lambda) f_A(F) - \lambda(f_A(F+1) - f_A(F))] \\ &= \mathrm{E}\left[\langle Df_A(F), -DL^{-1}F\rangle_{\ell^2(\mathbb{N})} - \lambda \Delta f_A(F)\right] \\ &= \mathrm{E}\left[\langle \Delta f_A(F) DF + R^F(f_A), -DL^{-1}F\rangle_{\ell^2(\mathbb{N})} - \lambda \Delta f_A(F)\right] \\ &= \mathrm{E}\left[\Delta f_A(F)(\langle -DL^{-1}F, DF\rangle_{\ell^2(\mathbb{N})} - \lambda)\right] \\ &\quad + \mathrm{E}[\langle -DL^{-1}F, R^F(f_A)\rangle_{\ell^2(\mathbb{N})}]. \end{aligned}$$

The desired result follows by taking absolute values on both sides, as well as by applying the estimates (6.2) and (2.13), and noticing that the random variables $D_kF$ and $D_kL^{-1}F$ are independent of $X_k$ (see Lemma 2.13 (1) in [14]). □

Note that, formally, the upper bound (6.3) may be obtained by (6.6) substituting the term $-D_kL^{-1}F$ in the definitions of $c_1$ and $c_2$ with $\mathrm{E}[D_kF \mid \mathcal{F}_{k-1}]$, and vice versa.

**Corollary 6.4** *Let $F \in \mathrm{Dom}(D)$ be an $\mathbb{N}$-valued random variable with mean $\lambda$ and assume that*

$$c_1 \;:=\; |\lambda - \mathrm{Var}(F)| + \|\langle D.F, -D.L^{-1}F\rangle_{\ell^2(\mathbb{N})} - \mathrm{E}[\langle D.F, -D.L^{-1}F\rangle_{\ell^2(\mathbb{N})}]\|_{L^2(\Omega)},$$

*and $c_2 := b_2$, where $b_2$ is defined by* (6.4)*, are finite. Then we have*

$$d_{TV}(F, \mathrm{Po}(\lambda)) \leq \frac{1-\mathrm{e}^{-\lambda}}{\lambda} c_1 + \frac{1-\mathrm{e}^{-\lambda}}{\lambda^2} c_2.$$

*Proof.* We preliminary note that by the covariance representation (2.16) it follows

$$\mathrm{Var}(F) = \mathrm{E}[\langle DF, -DL^{-1}F\rangle_{\ell^2(\mathbb{N})}]. \tag{6.7}$$

By the Cauchy-Schwarz inequality and (6.7), we have

$$\begin{aligned}
&\mathrm{E}[|\lambda - \langle -DL^{-1}F, DF\rangle_{\ell^2(\mathbb{N})}|] \\
&\leq \|\lambda - \langle -DL^{-1}F, DF\rangle_{\ell^2(\mathbb{N})}\|_{L^2(\Omega)} \\
&\leq |\lambda - \mathrm{Var}(F)| + \|\langle D.F, -DL^{-1}F\rangle_{\ell^2(\mathbb{N})} - \mathrm{E}[\langle D.F, -DL^{-1}F\rangle_{\ell^2(\mathbb{N})}]\|_{L^2(\Omega)}.
\end{aligned}$$

Moreover, using the inequality (3.19) we deduce

$$\begin{aligned}
& \mathrm{E}\left[\langle |DL^{-1}F|, \left|\sqrt{\frac{q.}{p.}} D.F \left(\frac{D.F}{\sqrt{p.q.}} - 1\right)\right|\rangle_{\ell^2(\mathbb{N})}\right] \\
& +\mathrm{E}\left[\langle |DL^{-1}F|, \left|\sqrt{\frac{p.}{q.}} D.F \left(\frac{D.F}{\sqrt{p.q.}} + 1\right)\right|\rangle_{\ell^2(\mathbb{N})}\right] \\
\leq\; & \mathrm{E}\left[\langle |DL^{-1}F|, \frac{|D.F|}{\sqrt{p.q.}}\left(1 + \frac{|D.F|}{\sqrt{p.q.}}\right)\rangle_{\ell^2(\mathbb{N})}\right] \\
\leq\; & \mathrm{E}\left[\langle |DL^{-1}F|, \frac{|D.F|}{\sqrt{p.q.}}\rangle_{\ell^2(\mathbb{N})}\right] + \mathrm{E}\left[\langle |DL^{-1}F|, \frac{|D.F|^2}{p.q.}\rangle_{\ell^2(\mathbb{N})}\right] \\
=\; & \sum_{k\geq 0} \frac{1}{\sqrt{p_k q_k}} \mathrm{E}[|D_k L^{-1}F||D_kF|] + \sum_{k\geq 0} \frac{1}{p_k q_k} \mathrm{E}[|D_k L^{-1}F||D_kF|^2] \\
\leq\; & \sum_{k\geq 0} \frac{1}{\sqrt{p_k q_k}} \mathrm{E}[|D_kF|^2] + \sum_{k\geq 0} \frac{1}{p_k q_k} \|D_k L^{-1}F\|_{L^2(\Omega)} \sqrt{\mathrm{E}[|D_kF|^4]} \\
\leq\; & \sum_{k\geq 0} \frac{1}{\sqrt{p_k q_k}} \mathrm{E}[|D_kF|^2] + \sum_{k\geq 0} \frac{1}{p_k q_k} \|D_kF\|_{L^2(\Omega)} \sqrt{\mathrm{E}[|D_kF|^4]}.
\end{aligned}$$

The claim follows from Theorem 6.3. □

# 7 First chaos bound for the Poisson approximation

In this section we specialize the results of Section 6 to (shifted) first order discrete stochastic integrals. As we shall see, the bounds (6.3) and (6.6) (and the corresponding assumptions) coincide on functionals of the form $F = \lambda + J_1(f_1)$, $f_1 \in \ell^2(\mathbb{N})$, although they differ for $F = \lambda + J_n(f_n)$, $f_n \in \ell^2_{\mathfrak{s}}(\Delta_n)$, $n \geq 2$.

**Corollary 7.1** *Assume that $\alpha = (\alpha_k)_{k\geq 0}$ is in $\ell^2(\mathbb{N})$ and such that*

$$F = \lambda + J_1(\alpha) = \lambda + \sum_{k\geq 0} \alpha_k Y_k$$

*is $\mathbb{N}$-valued. Assuming*

$$\sum_{k\geq 0} \sqrt{\frac{q_k}{p_k}} |\alpha_k|^2 \left| \frac{\alpha_k}{\sqrt{p_k q_k}} - 1 \right| < \infty$$

*and*

$$\sum_{k\geq 0} \sqrt{\frac{p_k}{q_k}} |\alpha_k|^2 \left| \frac{\alpha_k}{\sqrt{p_k q_k}} + 1 \right| < \infty,$$

*we have that the bound (6.3) (which in this case coincides with the bound (6.6)) holds for $F$ with*

$$b_1 = \left| \lambda - \sum_{k\geq 0} \alpha_k^2 \right|,$$

*and*

$$b_2 = \sum_{k\geq 0} \sqrt{\frac{q_k}{p_k}} |\alpha_k|^2 \left| \frac{\alpha_k}{\sqrt{p_k q_k}} - 1 \right| + \sum_{k\geq 0} \sqrt{\frac{p_k}{q_k}} |\alpha_k|^2 \left| \frac{\alpha_k}{\sqrt{p_k q_k}} + 1 \right|.$$

*Proof.* As in the proof of Corollary 4.1 we have $F \in \mathrm{Dom}(D)$ with $D_k F = \alpha_k$. The claim follows from e.g. Theorem 6.1. □

**Example**

Let $(Z_k)_{k\geq 0}$ be a sequence of independent and $\{0, 1\}$-valued random variables with $\mathrm{E}[Z_k] = p_k$ and define the random variables

$$Y_k = \frac{Z_k - p_k}{\sqrt{p_k q_k}} = \frac{q_k - p_k + X_k}{2\sqrt{p_k q_k}},$$

where $(X_k)_{k\geq 0}$ is a sequence of independent and $\{-1, 1\}$-valued random variables with $P(X_k = 1) = p_k$. Let $(\beta_k)_{k\geq 0} \subset \mathbb{N}$, assume

$$\lambda := \sum_{k\geq 0} p_k \beta_k < \infty, \quad \sum_{k\geq 0} p_k q_k \beta_k^2 < \infty$$

$$\sum_{k\geq 0} q_k\sqrt{p_kq_k}\beta_k^2|\beta_k-1|<\infty,\quad \sum_{k\geq 0} p_k\sqrt{p_kq_k}\beta_k^2(\beta_k+1)<\infty$$

and define $\alpha_k := \sqrt{p_kq_k}\beta_k$. We clearly have $\alpha=(\alpha_k)_{k\geq 0}\in\ell^2(\mathbb{N})$ and

$$F=\sum_{k\geq 0}\beta_k Z_k=\lambda+\sum_{k\geq 0}\alpha_k Y_k.$$

Note that, obviously, $F$ takes values in $\mathbb{N}$. Note also that

$$\sum_{k\geq 0}\sqrt{\frac{q_k}{p_k}}|\alpha_k|^2\left|\frac{\alpha_k}{\sqrt{p_kq_k}}-1\right|=\sum_{k\geq 0} q_k\sqrt{p_kq_k}\beta_k^2|\beta_k-1|,$$

and

$$\sum_{k\geq 0}\sqrt{\frac{p_k}{q_k}}|\alpha_k|^2\left|\frac{\alpha_k}{\sqrt{p_kq_k}}+1\right|=\sum_{k\geq 0} p_k\sqrt{p_kq_k}\beta_k^2(\beta_k+1).$$

So (with $\lambda$ as above) by Corollary 7.1 we have

$$d_{TV}(F,\mathrm{Po}(\lambda))\leq\frac{1-\mathrm{e}^{-\lambda}}{\lambda}\left|\sum_{k\geq 0}p_k\beta_k(1-\beta_kq_k)\right|+\frac{1-\mathrm{e}^{-\lambda}}{\lambda^2}\left(\sum_{k\geq 0}\sqrt{p_kq_k}\beta_k^2(|\beta_k-1|q_k+(\beta_k+1)p_k)\right).\tag{7.1}$$

We note that by the classical bound for independent Bernoulli random variables (see e.g. [2]) we have

$$d_{TV}\left(\sum_{k\geq 0}Z_k,\mathrm{Po}(\lambda)\right)\leq\frac{1-\mathrm{e}^{-\lambda}}{\lambda}\sum_{k\geq 0}p_k^2.\tag{7.2}$$

Although the inequality (7.1) with $\beta_k=1$ does not coincide with (7.2) (producing indeed a bigger upper bound), the bound in (7.1) holds, more generally, for sums of "weighted" Bernoulli random variables.

# 8 $n$th chaos bounds for the Poisson approximation

Throughout this section, for a fixed positive constant $\lambda>0$, we consider an $\mathbb{N}$-valued (shifted) $n$th chaos $F=\lambda+J_n(f_n)$, $f_n\in\ell^2_{\mathfrak{s}}(\Delta_n)$, $n\geq 2$.

## 8.1 Clark-Ocone bound

By (5.5) and the isometric properties of discrete multiple stochastic integrals we have that the constants $b_i$ of Corollary 6.2 are equal to

$$\tilde{b}_1 := |\lambda-n!\|f_n\|^2_{\ell^2_{\mathfrak{s}}(\Delta_n)}|+n^2\|\langle J_{n-1}(f_n(*,\cdot)),J_{n-1}(f_{n]\cdot}(*))\rangle_{\ell^2(\mathbb{N})}$$

$$- \mathrm{E}[\langle J_{n-1}(f_n(*,\cdot)), J_{n-1}(f_{n]\cdot}(*))\rangle_{\ell^2(\mathbb{N})}]\|_{L^2(\Omega)},$$

$$\tilde{b}_2 := n^2(n-1)! \sum_{k\geq 0} \frac{1}{\sqrt{p_k q_k}} \|f_n(*,k)\|^2_{\ell^2_{\mathfrak{s}}(\Delta_{n-1})} + n^3\sqrt{(n-1)!} \sum_{k\geq 0} \frac{1}{p_k q_k} \|f_n(*,k)\|_{\ell^2_{\mathfrak{s}}(\Delta_{n-1})} \sqrt{\mathrm{E}[|J_{n-1}(f_n(*,k))|^4]}. \tag{8.1}$$

The next theorem follows from the computations in the proof of Theorem 5.3.

**Theorem 8.1** *Let $n \geq 2$ be fixed and let $f_n \in \ell^2_{\mathfrak{s}}(\Delta_n)$. Assume that for any $k \in \mathbb{N}$ the functions $h^{(k)}_{n-1,n-1,s}$ and $\tilde{h}^{(k)}_{n-1,n-1,s}$ defined by* (5.10) *and* (5.11) *belong to $\ell^2_{\mathfrak{s}}(\Delta_{2n-2-s})$, $0 \leq s \leq 2n-2$, and that*

$$b_1 := |\lambda - n!\|f_n\|^2_{\ell^2_{\mathfrak{s}}(\Delta_n)}| + n^2 \Bigg( \sum_{s=0}^{2n-3} (2n-2-s)! \sum_{s\leq\{2i_1,\, 2i_2\}\leq 2(s\wedge(n-1))} i_1! i_2! \binom{n-1}{i_1}^2 \binom{n-1}{i_2}^2 \binom{i_1}{s-i_1}\binom{i_2}{s-i_2} \sum_{k\geq 0} \|f_n(*,k) \otimes_{i_1}^{s-i_1} f_{n]k}(*)\|_{\ell^2(\Delta_{2n-2-s})} \sum_{k\geq 0} \|f_n(*,k) \otimes_{i_2}^{s-i_2} f_{n]k}(*)\|_{\ell^2(\Delta_{2n-2-s})} \Bigg)^{1/2},$$

*and*

$$b_2 := n^2(n-1)! \sum_{k\geq 0} \frac{1}{\sqrt{p_k q_k}} \|f_n(*,k)\|^2_{\ell^2_{\mathfrak{s}}(\Delta_{n-1})} + n^3\sqrt{(n-1)!} \sum_{k\geq 0} \frac{1}{p_k q_k} \|f_n(*,k)\|_{\ell^2_{\mathfrak{s}}(\Delta_{n-1})} \Bigg( \sum_{s=0}^{2n-2} (2n-2-s)! \sum_{s\leq\{2i_1,\, 2i_2\}\leq 2(s\wedge(n-1))} i_1! i_2! \binom{n-1}{i_1}^2 \binom{n-1}{i_2}^2 \binom{i_1}{s-i_1}\binom{i_2}{s-i_2} \|f_n(*,k) \otimes_{i_1}^{s-i_1} f_n(*,k)\|_{\ell^2(\Delta_{2n-2-s})} \|f_n(*,k) \otimes_{i_2}^{s-i_2} f_n(*,k)\|_{\ell^2(\Delta_{2n-2-s})} \Bigg)^{1/2} \tag{8.2}$$

*are finite. Then we have*

$$d_{TV}(F, \mathrm{Po}(\lambda)) \leq \frac{1-\mathrm{e}^{-\lambda}}{\lambda} b_1 + \frac{1-\mathrm{e}^{-\lambda}}{\lambda^2} b_2.$$

## 8.2 Semigroup bound

By (5.21) and the isometric properties of discrete multiple stochastic integrals we have that the constants $c_i$ of Corollary 6.4 are equal to

$$\tilde{c}_1 := |\lambda - n!\|f_n\|^2_{\ell^2_{\mathfrak{s}}(\Delta_n)}| + n\|\langle J_{n-1}(f_n(*,\cdot)), J_{n-1}(f_n(*,\cdot))\rangle_{\ell^2(\mathbb{N})}$$

$$- \mathrm{E}[\langle J_{n-1}(f_n(*,\cdot)), J_{n-1}(f_n(*,\cdot))\rangle_{\ell^2(\mathbb{N})}]\|_{L^2(\Omega)},$$

$$\tilde{c}_2 := \tilde{b}_2, \quad \text{where } \tilde{b}_2 \text{ is defined by (8.1).}$$

Next theorem follows from the computations in the proof of Theorem 5.4.

**Theorem 8.2** *Let $n \geq 2$ be fixed. Assume that $f_n \in \ell^2_{\mathfrak{s}}(\Delta_n)$, that the functions $\tilde{h}_{n-1,n-1,s}$ defined by (5.11) belong to $\ell^2_{\mathfrak{s}}(\Delta_{2n-2-s})$, $0 \leq s \leq 2n-2$, and that*

$$c_1 := |\lambda - n!\|f_n\|^2_{\ell^2_{\mathfrak{s}}(\Delta_n)}| + n\Bigg(\sum_{s=0}^{2n-3}(2n-2-s)! \sum_{s\leq\{2i_1,\, 2i_2\}\leq 2(s\wedge(n-1))} i_1!i_2!\binom{n-1}{i_1}^2\binom{n-1}{i_2}^2\binom{i_1}{s-i_1}\binom{i_2}{s-i_2} \|f_n \otimes_{i_1+1}^{s-i_1+1} f_n\|_{\ell^2(\Delta_{2n-2-s})}\|f_n \otimes_{i_2+1}^{s-i_2+1} f_n\|_{\ell^2(\Delta_{2n-2-s})}\Bigg)^{1/2},$$

*and $c_2 := b_2$, where $b_2$ is defined by (8.2), are finite. Then we have*

$$d_{TV}(F, \mathrm{Po}(\lambda)) \leq \frac{1-\mathrm{e}^{-\lambda}}{\lambda}c_1 + \frac{1-\mathrm{e}^{-\lambda}}{\lambda^2}c_2.$$

**Quadratic functionals**

In the next proposition, we provide an explicit bound for $d_{TV}(\lambda + J_2(f), \mathrm{Po}(\lambda))$, $\lambda > 0$. The proof is omitted since it is a simple consequence of Theorem 8.2 with $n = 2$, Condition (5.35) and (5.49), (5.51), (5.52) and (5.53). We also note that the integrability condition required in Theorem 8.2 can be checked as in the proof of Proposition 5.8 from Lemma 2.4(1) in [14].

**Proposition 8.3** *Assume (5.35), $f, f^2 \in \ell^2_{\mathfrak{s}}(\Delta_2)$ and suppose that the shifted second chaos $\lambda + J_2(f)$ is $\mathbb{N}$-valued. Then we have*

$$d_{TV}(\lambda + J_2(f), \mathrm{Po}(\lambda)) \leq \frac{1-\mathrm{e}^{-\lambda}}{\lambda}d_1 + \frac{1-\mathrm{e}^{-\lambda}}{\lambda^2}d_2,$$

*where*

$$d_1 := |\lambda - 2\|f\|^2_{\ell^2_{\mathfrak{s}}(\Delta_2)}| + 2\sqrt{2}\|f \star_1^1 f\|_{\ell^2(\mathbb{N})^{\otimes 2}} + \frac{1-2\varepsilon}{\varepsilon}\|f \star_2^1 f\|_{\ell^2(\mathbb{N})},$$

*and*

$$d_2 := \frac{4}{\varepsilon}\|f\|^2_{\ell^2_{\mathfrak{s}}(\Delta_2)} + \frac{8}{\varepsilon^2}(\sqrt{2}+1)\|f\|^2_{\ell^2_{\mathfrak{s}}(\Delta_2)}\|f \star_1^1 f\|^{1/2}_{\ell^2(\mathbb{N})^{\otimes 2}} + \frac{4(1-2\varepsilon)}{\varepsilon^3}\|f\|_{\ell^2_{\mathfrak{s}}(\Delta_2)}\|f^2\|_{\ell^2_{\mathfrak{s}}(\Delta_2)}.$$

**Example**

Let $m \geq 2$ be a fixed integer, and suppose that $p_k = 1/m$, for any $k \in \mathbb{N}$. Define the function

$$f(k_1, k_2) = \frac{m-1}{m} \mathbb{1}_{\{k_1=0\}} \mathbb{1}_{[1,m]}(k_2) + \frac{m-1}{m} \mathbb{1}_{\{k_2=0\}} \mathbb{1}_{[1,m]}(k_1), \qquad k_1, k_2 \in \mathbb{N}$$

and let $\lambda$ be an integer bigger than or equal to $4m$. We are going to check that all the conditions of Proposition 8.3 are satisfied. Since (5.35) and the integrability of $f$ and $f^2$ are obvious, we only check that $\lambda + J_2(f)$ is $\mathbb{N}$-valued. Let $(Z_k)_{k\geq 0}$ be a sequence of independent and $\{0, 1\}$-valued random variables with $\mathrm{E}[Z_k] = p_k = 1/m$. We have

$$\begin{aligned}
J_2(f) &= \sum_{(k_1,k_2)\in\Delta_2} f(k_1, k_2) Y_{k_1} Y_{k_2} \\
&= \frac{2(m-1)}{m} Y_0 \sum_{k=1}^{m} Y_k \\
&= \frac{2(m-1)}{m} \left( \frac{mZ_0 - 1}{\sqrt{m-1}} \right) \sum_{k=1}^{m} \left( \frac{mZ_k - 1}{\sqrt{m-1}} \right) \\
&= \frac{2}{m} (mZ_0 - 1) \sum_{k=1}^{m} (mZ_k - 1) \\
&= 2(mZ_0 - 1) \left( \sum_{k=1}^{m} Z_k - 1 \right) \geq -4m.
\end{aligned}$$

So $J_2(f)$ is a $\mathbb{Z}$-valued random variable bounded below by $-4m$. Therefore, by the choice of $\lambda$, the shifted second chaos is $\mathbb{N}$-valued.

Again, the above example will also satisfy the hypotheses of Theorem 8.1 with $n = 2$, while it is difficult in general to compare the constants in Theorems 8.1 and 8.2.

# 9 Proof of the multiplication formula

In this section we prove the multiplication formula (5.3) for (possibly non-symmetric) discrete multiple stochastic integrals. In order to explain and prove such a formula we shall use the notion of continuous-time normal martingale.

### Continuous-time normal martingales

Given $\hat{\varphi} : \mathbb{R} \longrightarrow \mathbb{R}$ a deterministic function, let

$$i_t = \mathbb{1}_{\{\hat{\varphi}(t)=0\}}, \qquad j_t = 1 - i_t = \mathbb{1}_{\{\hat{\varphi}(t)\neq 0\}}, \qquad t \in \mathbb{R}_+,$$

and consider the martingale $(M_t)_{t\in\mathbb{R}_+}$ represented as

$$\mathrm{d}M_t = i_t \mathrm{d}B_t + \hat{\varphi}(t)(\mathrm{d}N_t - \lambda_t \mathrm{d}t), \qquad t \in \mathbb{R}_+, \quad M_0 = 0, \tag{9.1}$$

with $\lambda_t = (1-i_t)/\hat{\varphi}^2(t)$, $t \in \mathbb{R}_+$, where $(B_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion, and $(N_t)_{t\in\mathbb{R}_+}$ is a Poisson process independent of $(B_t)_{t\in\mathbb{R}_+}$, with intensity $\lambda_t$. Then the martingale $(M_t)_{t\in\mathbb{R}_+}$ has deterministic angle bracket $\langle M, M\rangle_t = t$ and it solves the structure equation

$$\mathrm{d}[M,M]_t = \mathrm{d}t + \hat{\varphi}(t)\mathrm{d}M_t, \qquad t \in \mathbb{R}_+, \tag{9.2}$$

cf. § 2.10 of [19]. Here $([M,M]_t)_{t\in\mathbb{R}_+}$ denotes the quadratic variation of $(M_t)_{t\in\mathbb{R}_+}$. Note that the continuous part of $(M_t)_{t\in\mathbb{R}_+}$ is given by $\mathrm{d}M_t^c = i_t \mathrm{d}M_t$ and the eventual jump of $(M_t)_{t\in\mathbb{R}_+}$ at time $t \in \mathbb{R}_+$ is given by $\Delta M_t = \hat{\varphi}(t)$ on $\{\Delta M_t \neq 0\}$, $t \in \mathbb{R}_+$, see [8], p. 70.

In the following, we denote by $L^2(\mathbb{R}_+^{\circ n})$ the subspace of $L^2(\mathbb{R}_+^n)$ made of symmetric functions in $n$ variables. The multiple stochastic integral $I_n(f_n)$ is defined by

$$I_n(f_n) = n! \int_0^\infty \int_0^{t_n^-} \cdots \int_0^{t_2^-} f_n(t_1, \ldots, t_n) \mathrm{d}M_{t_1} \cdots \mathrm{d}M_{t_n}, \qquad f_n \in L^2(\mathbb{R}_+^{\circ n}), \quad n \geq 1$$

and the following isometry formula holds

$$\mathrm{E}[I_n(f_n) I_m(g_m)] = n! \mathbb{1}_{\{n=m\}} \langle f_n, g_m \rangle_{L^2(\mathbb{R}_+^{\circ n})}, \tag{9.3}$$

where the symbol $\langle \cdot, \cdot \rangle_{L^2(\mathbb{R}_+^{\circ n})}$ denotes the usual inner product on $L^2(\mathbb{R}_+^{\circ n})$. For any $\hat{f}_n \in L^2(\mathbb{R}_+^{\circ n})$ and $\hat{g}_m \in L^2(\mathbb{R}_+^{\circ m})$ the contraction $\hat{f}_n \hat{\circ}_k^l \hat{g}_m$, $0 \leq l \leq k$, is defined to be the symmetrization of the function

$$\begin{aligned} &(x_{l+1}, \ldots, x_n, y_{k+1}, \ldots, y_m) \longmapsto \\ &\quad \hat{\varphi}(x_{l+1}) \cdots \hat{\varphi}(x_k) \int_{\mathbb{R}_+^l} \hat{f}_n(x_1, \ldots, x_n) \hat{g}_m(x_1, \ldots, x_k, y_{k+1}, \ldots, y_m) \, \mathrm{d}x_1 \cdots \mathrm{d}x_l \end{aligned} \tag{9.4}$$

in $n + m - k - l$ real variables. We recall the multiplication formula in the general context of normal martingales

$$I_n(\hat{f}_n) I_m(\hat{g}_m) = \sum_{s=0}^{2(n\wedge m)} I_{n+m-s}(\hat{h}_{n,m,s}), \tag{9.5}$$

cf. Proposition 2 of [17] or Proposition 4.5.6 of [19], provided the functions

$$\hat{h}_{n,m,s} := \sum_{s \leq 2i \leq 2(s\wedge n\wedge m)} i! \binom{n}{i} \binom{m}{i} \binom{i}{s-i} \hat{f}_n \hat{\circ}_i^{s-i} \hat{g}_m$$

belong to $L^2(\mathbb{R}_+^{\circ n+m-s})$, $0 \leq s \leq 2(n \wedge m)$, and we remark that (9.5) is of the same form as (5.3).

For later purposes, we provide the relation between the contraction $\hat{f}_n \hat{\circ}_k^l \hat{g}_m$ and the one defined by (5.2). Given $f_n \in \ell^2_{\mathfrak{s}}(\Delta_n)$ and $g_m \in \ell^2_{\mathfrak{s}}(\Delta_m)$, we let

$$\hat{f}_n(x_1, \ldots, x_n) := \sum_{a_1,\ldots,a_n \in \mathbb{N}} f_n(a_1, \ldots, a_n) \mathbb{1}_{[a_1,a_1+1)}(x_1) \cdots \mathbb{1}_{[a_n,a_n+1)}(x_n), \tag{9.6}$$

$x_1, \ldots, x_n \in \mathbb{R}_+$, and

$$\hat{g}_m(y_1, \ldots, y_m) := \sum_{b_1,\ldots,b_m \in \mathbb{N}} g_m(b_1, \ldots, b_m) \mathbb{1}_{[b_1,b_1+1)}(y_1) \cdots \mathbb{1}_{[b_m,b_m+1)}(y_m), \tag{9.7}$$

$y_1, \ldots, y_m \in \mathbb{R}_+$, and

$$\hat{\varphi}(y) := \sum_{k \in \mathbb{N}} \varphi(k) \mathbb{1}_{[k,k+1)}(y), \qquad y \in \mathbb{R}_+ \tag{9.8}$$

where $\varphi$ is defined in (5.1). Then

$$f_n \circ_k^l g_m(a_1, \ldots, a_{n+m-k-l}) = \mathbb{1}_{\Delta_{n+m-k-l}}(a_1, \ldots, a_{n+m-k-l}) \hat{f}_n \hat{\circ}_k^l \hat{g}_m(a_1, \ldots, a_{n+m-k-l}). \tag{9.9}$$

**Proof of Proposition 5.1**

By the definition of the multiple stochastic integral, the contraction and the structure equation (2.1), for any $f_n \in \ell^2_{\mathfrak{s}}(\Delta_n)$ and $g \in \ell^2(\mathbb{N})$, we deduce

$$\begin{aligned}
J_n(f_n)J_1(g) &= \sum_{i_{n+1}=0}^{\infty} \sum_{i_1 \neq i_2 \neq \cdots \neq i_n} f_n(i_1, \ldots, i_n) g(i_{n+1}) Y_{i_1} \ldots Y_{i_n} Y_{i_{n+1}} \\
&= \sum_{i_1 \neq i_2 \neq \cdots \neq i_n \neq i_{n+1}} f_n(i_1, \ldots, i_n) g(i_{n+1}) Y_{i_1} \ldots Y_{i_n} Y_{i_{n+1}} \\
&\quad + n \sum_{i_1 \neq i_2 \neq \cdots \neq i_n} f_n(i_1, \ldots, i_n) g(i_n) Y_{i_1} \ldots Y_{i_{n-1}} Y_{i_n}^2 \\
&= \sum_{i_1 \neq i_2 \neq \cdots \neq i_n \neq i_{n+1}} f_n(i_1, \ldots, i_n) g(i_{n+1}) Y_{i_1} \ldots Y_{i_n} Y_{i_{n+1}} \\
&\quad + n \sum_{i_1 \neq i_2 \neq \cdots \neq i_n} \varphi(i_n) f_n(i_1, \ldots, i_n) g(i_n) Y_{i_1} \ldots Y_{i_n} \\
&\quad + n \sum_{i_1 \neq i_2 \neq \cdots \neq i_n} f_n(i_1, \ldots, i_n) g(i_n) Y_{i_1} \ldots Y_{i_{n-1}} \\
&= \sum_{i_1 \neq i_2 \neq \cdots \neq i_n \neq i_{n+1}} f_n(i_1, \ldots, i_n) g(i_{n+1}) Y_{i_1} \ldots Y_{i_n} Y_{i_{n+1}} \\
&\quad + n \sum_{i_1 \neq i_2 \neq \cdots \neq i_n} \varphi(i_n) f_n(i_1, \ldots, i_n) g(i_n) Y_{i_1} \ldots Y_{i_n}
\end{aligned}$$

$$+n\sum_{i_1\neq i_2\neq\cdots\neq i_{n-1}}\sum_{i_n=0}^{\infty} f_n(i_1,\dots,i_n)g(i_n)Y_{i_1}\dots Y_{i_{n-1}}$$
$$= J_{n+1}(f_n \circ_0^0 g) + nJ_n(f_n \circ_1^0 g) + nJ_{n-1}(f_n \circ_1^1 g), \tag{9.10}$$

which is exactly (5.3) with $g_m = g_1 = g$.

Next, consider $h_i(k) = \mathbb{1}_{\{d_i\}}(k)$, $i = 1,\dots,m$, $d_i \neq d_j$, $1 \leq i \neq j \leq m$, and let $g_m = h_1 \circ_0^0 \cdots \circ_0^0 h_m$, i.e. $J_m(g_m) = J_1(h_1)\cdots J_1(h_m)$. We shall show (5.3) by induction on $m = 1,\dots,n$. We already proved that (5.3) holds for $m = 1$. Next, assuming that (5.3) holds at the rank $m \in \{2,\dots,n-1\}$ we have

$$\begin{aligned}
J_n(f_n)J_{m+1}(g_{m+1}) &= J_n(f_n)J_m(g_m)J_1(h_{m+1})\\
&= \sum_{s=0}^{2m} J_{n+m-s}(h_{n,m,s})J_1(h_{m+1})\\
&= \sum_{s=0}^{2m} J_{n+m-s+1}(h_{n,m,s} \circ_0^0 h_{m+1}) + \sum_{s=0}^{2m}(n+m-s)J_{n+m-s}(h_{n,m,s} \circ_1^0 h_{m+1})\\
&\quad + \sum_{s=0}^{2m}(n+m-s)J_{n+m-s-1}(h_{n,m,s} \circ_1^1 h_{m+1})\\
&= \sum_{s=0}^{2m} J_{n+m-s+1}(h_{n,m,s} \circ_0^0 h_{m+1}) + \sum_{s=1}^{1+2m}(n+m+1-s)J_{n+m+1-s}(h_{n,m,s-1} \circ_1^0 h_{m+1})\\
&\quad + \sum_{s=2}^{2+2m}(n+m+2-s)J_{n+m+1-s}(h_{n,m,s-2} \circ_1^1 h_{m+1})\\
&= \sum_{s=0}^{2m+2} J_{n+m+1-s}(h_{n,m+1,s}),
\end{aligned}$$

since

$$\begin{aligned}
h_{n,m+1,s} = \mathbb{1}_{\{0\leq s\leq 2m\}}h_{n,m,s} \circ_0^0 h_{m+1} + \mathbb{1}_{\{1\leq s\leq 1+2m\}}(n+m+1-s)h_{n,m,s-1} \circ_1^0 h_{m+1} \quad (9.11)\\
+\mathbb{1}_{\{2\leq s\leq 2+2m\}}(n+m+2-s)h_{n,m,s-2} \circ_1^1 h_{m+1},
\end{aligned}$$

as follows from Lemma 9.1 below. We have shown that (9.5) holds for any $g_m$ of the form

$$g_m = \mathbb{1}_{\{d_1\}} \circ_0^0 \cdots \circ_0^0 \mathbb{1}_{\{d_m\}},$$

and the formula extends to all $g_m \in \ell^2_{\mathfrak{s}}(\Delta_m)$ by summation and linearity. The proof is completed.

**Lemma 9.1** *The identity* (9.11) *holds for* $g_m = h_1 \circ_0^0 \cdots \circ_0^0 h_m$ *and* $h_i(k) = \mathbb{1}_{\{d_i\}}(k)$, $i = 1, \ldots, m$.

*Proof.* Letting $\hat{h}_i(x) := \mathbb{1}_{[d_i, d_i+1)}(x)$, $i = 1, \ldots, m$, and $\hat{g}_m = \hat{h}_1 \circ_0^0 \cdots \circ_0^0 \hat{h}_m$, by (9.5) we have

$$I_n(\hat{f}_n) I_1(\hat{g}_1) = I_{n+1}(\hat{f}_n \hat{\circ}_0^0 \hat{g}_1) + n I_n(\hat{f}_n \hat{\circ}_1^0 \hat{g}_1) + n I_{n-1}(\hat{f}_n \hat{\circ}_1^1 \hat{g}_1). \tag{9.12}$$

By (9.5) and (9.12) it follows that

$$\begin{aligned}
&\sum_{s=0}^{2m+2} I_{n+m+1-s}(\hat{h}_{n,m+1,s}) = I_n(\hat{f}_n) I_{m+1}(\hat{g}_{m+1}) \\
&= I_n(\hat{f}_n) I_m(\hat{g}_m) I_1(\hat{h}_{m+1}) \\
&= \sum_{s=0}^{2m} I_{n+m-s}(\hat{h}_{n,m,s}) I_1(\hat{h}_{m+1}) \\
&= \sum_{s=0}^{2m} I_{n+m-s+1}(\hat{h}_{n,m,s} \hat{\circ}_0^0 \hat{h}_{m+1}) + \sum_{s=0}^{2m} (n+m-s) I_{n+m-s}(\hat{h}_{n,m,s} \hat{\circ}_1^0 \hat{h}_{m+1}) \\
&\quad + \sum_{s=0}^{2m} (n+m-s) I_{n+m-s-1}(\hat{h}_{n,m,s} \hat{\circ}_1^1 \hat{h}_{m+1}) \\
&= \sum_{s=0}^{2m} I_{n+m-s+1}(\hat{h}_{n,m,s} \hat{\circ}_0^0 \hat{h}_{m+1}) + \sum_{s=1}^{1+2m} (n+m+1-s) I_{n+m+1-s}(\hat{h}_{n,m,s-1} \hat{\circ}_1^0 \hat{h}_{m+1}) \\
&\quad + \sum_{s=2}^{2+2m} (n+m+2-s) I_{n+m+1-s}(\hat{h}_{n,m,s-2} \hat{\circ}_1^1 \hat{h}_{m+1}),
\end{aligned}$$

which, due to the isometry property of the multiple stochastic integrals $I_k$, shows that the identity (9.11) holds with $\hat{h}_{n,m,s}$ and $\hat{h}_{m+1}$ in place of $h_{n,m,s}$ and $h_{m+1}$, and with $\hat{\varphi}$ defined from (9.8). Using (9.9) we conclude that (9.11) holds for $h_{n,m,s}$ and $h_{m+1}$ as well, and in this case the identity holds for all non-diagonal terms while all functions in the relation vanish on the diagonals. □

### Acknowledgment

This work was initiated during a visit of GLT to the School of Physics and Mathematical Sciences of the Nanyang Technological University with the support of NTU MOE Tier 2 Grant No. M4020140.

# References

[1] A. D. Barbour. Stein's method for diffusion approximations. *Probab. Theory Related Fields*, 84(3):297–322, 1990.

[2] A.D. Barbour, L. Holst, and S. Janson. *Poisson approximation*, volume 2 of *Oxford Studies in Probability*. The Clarendon Press, Oxford University Press, New York, 1992. Oxford Science Publications.

[3] J. Blank, P. Exner, and M. Havlíček. *Hilbert space operators in quantum physics*. AIP Series in Computational and Applied Mathematical Physics. American Institute of Physics, New York, 1994.

[4] S. Chatterjee. A new method of normal approximation. *Ann. Probab.*, 36(4):1584–1610, 2008.

[5] O. Costin and J.L. Lebowitz. Gaussian fluctuation in random matrices. *Phys. Rev. Let.*, 75:69–72, 1995.

[6] F. Daly. Upper bounds for Stein-type operators. *Electron. J. Probab.*, 13:no. 20, 566–587, 2008.

[7] R.M. Dudley. *Real analysis and probability*, volume 74 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2002. Revised reprint of the 1989 original.

[8] M. Émery. On the Azéma martingales. In *Séminaire de Probabilités XXIII*, volume 1372 of *Lecture Notes in Mathematics*, pages 66–87. Springer Verlag, 1990.

[9] T. Erhardsson. Stein's method for Poisson and compound Poisson approximation. In *An introduction to Stein's method*, volume 4 of *Lect. Notes Ser. Inst. Math. Sci. Natl. Univ. Singap.*, pages 61–113. Singapore Univ. Press, Singapore, 2005.

[10] F. Götze. On the rate of convergence in the multivariate CLT. *Ann. Probab.*, 19(2):724–739, 1991.

[11] K. Krokowski, A. Reichenbachs, and C. Thaele. Discrete malliavin-stein method: Berry-esseen bounds for random graphs, point processes and percolation. Preprint arXiv:1208.4264, 3 Mar 2015.

[12] O. Macchi. The coincidence approach to stochastic point processes. *Advances in Appl. Probability*, 7:83–122, 1975.

[13] I. Nourdin and G. Peccati. Stein's method on Wiener chaos. *Probab. Theory Related Fields*, 145(1-2):75–118, 2009.

[14] I. Nourdin, G. Peccati, and G. Reinert. Stein's method and stochastic analysis of Rademacher functionals. *Electron. J. Probab.*, 15:no. 55, 1703–1742, 2010.

[15] G. Peccati. The Chen-Stein method for Poisson functionals. arXiv:1112.5051v3 [math.PR], 2011.

[16] G. Peccati, J. L. Solé, M. S. Taqqu, and F. Utzet. Stein's method and normal approximation of Poisson functionals. *Ann. Probab.*, 38(2):443–478, 2010.

[17] N. Privault. On the independence of multiple stochastic integrals with respect to a class of martingales. *C. R. Acad. Sci. Paris Sér. I Math.*, 323:515–520, 1996.

[18] N. Privault. Stochastic analysis of Bernoulli processes. *Probab. Surv.*, 5:435–483 (electronic), 2008.

[19] N. Privault. *Stochastic Analysis in Discrete and Continuous Settings*, volume 1982 of *Lecture Notes in Math.* Springer-Verlag, Berlin, 309 pp., 2009.

[20] N. Privault and G.L. Torrisi. Probability approximation by Clark-Ocone covariance representation. *Electron. J. Probab.*, 18:1–25, 2013.

[21] T. Shirai. Large deviations for the Fermion point process associated with the exponential kernel. *J. Stat. Phys.*, 123(3):615–629, 2006.

[22] A. Soshnikov. Determinantal random point fields. *Uspekhi Mat. Nauk*, 55(5(335)):107–160, 2000.

[23] A. Soshnikov. Gaussian limit for determinantal random point fields. *Ann. Probab.*, 30(1):171–187, 2002.

[24] C. Stein. A bound for the error in the normal approximation to the distribution of a sum of dependent random variables. In *Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability (Univ. California, Berkeley, Calif., 1970/1971), Vol. II: Probability theory*, pages 583–602, Berkeley, Calif., 1972. Univ. California Press.

[25] C. Stein. *Approximate computation of expectations.* Institute of Mathematical Statistics Lecture Notes—Monograph Series, 7. Institute of Mathematical Statistics, Hayward, CA, 1986.