Sensitivity analysis and density estimation using the Malliavin calculus

Nicolas Privault

December 17, 2005

Consider $(F^{\zeta})_{\zeta}$ a family of random variables depending on a parameter ζ.

$$\frac{\partial}{\partial \zeta} \mathbf{E}\left[f\left(F^{\zeta}\right)\right] = \begin{cases} \mathbf{E}\left[f'\left(F^{\zeta}\right) \frac{\partial}{\partial \zeta} F^{\zeta}\right] \\ \simeq \dfrac{\mathbf{E}\left[f\left(F^{\zeta+h}\right)\right] - \mathbf{E}\left[f\left(F^{\zeta-h}\right)\right]}{2h} \end{cases}$$

Expectations can be computed by the Monte Carlo method:

$$E[F] \simeq \frac{F_1 + \cdots + F_n}{n},$$

where $F_1, \ldots, F_n$ is a random sample of F.

Application (1): sensitivity analysis in finance - Greeks

Price process:

$$\frac{dS_t^\zeta}{S_t^\zeta} = r(S_t^\zeta)dt + \sigma(S_t^\zeta)dM_t, \qquad S_0^\zeta = x.$$

$F^\zeta = S_T$ and $\zeta \in \{x, r, \sigma, T, ...\}$.

Payoff function:

- $f(x) = (x - K)^+$

- $f(x) = 1_{[K,\infty)}(x)$.

Option price:

$$E[f(S_T^\zeta)].$$

Greeks:

- $\zeta = x$: Delta
- $\zeta = \sigma$: Vega
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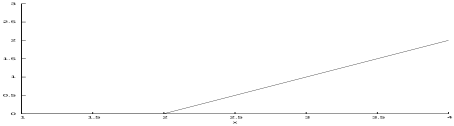

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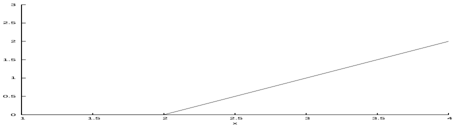

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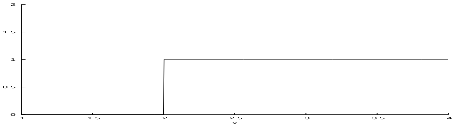

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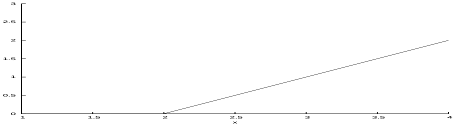

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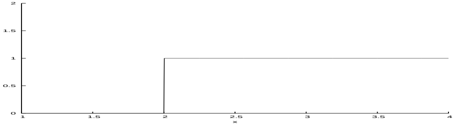

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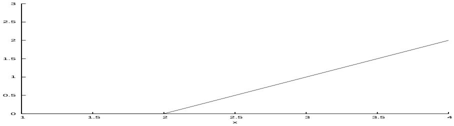

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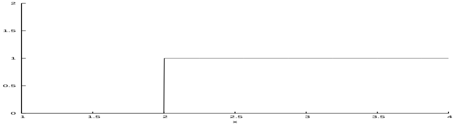

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Let $F^\zeta = F - \zeta$ and $f(x) = 1_{(-\infty,0]}$.

Density of F:

$$\begin{aligned}\phi_F(\xi) : &= \frac{\partial}{\partial\zeta} P(F \leq \zeta) = \frac{\partial}{\partial\zeta} \mathbf{E}\left[1_{(-\infty,0]}(F-\zeta)\right] = \frac{\partial}{\partial\zeta} \mathbf{E}\left[f\left(F^\zeta\right)\right] \\ &\simeq \frac{\mathbf{E}\left[f\left(F-(\zeta+h)\right)\right] - \mathbf{E}\left[f\left(F-(\zeta-h)\right)\right]}{2h} = \frac{\mathbf{E}\left[1_{[\zeta-h,\zeta+h]}(F)\right]}{2h}.\end{aligned}$$

Example: $F = \int_0^T e^{-rt} dN_t$.

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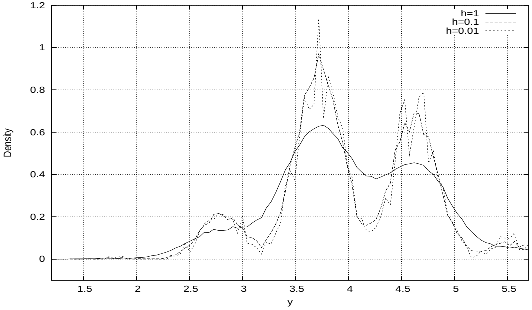

Integration by parts

1. D_w a derivation operator *acting on random variables*:

$$f'(F^\zeta) = \frac{D_w\left[f(F^\zeta)\right]}{D_w F^\zeta}.$$

2. D_w^* the adjoint of D_w:

$$\langle F, D_w G\rangle = E[FD_w G] = E[GD_w^* F] = \langle G, D_w^* F\rangle.$$

3. Main argument:

$$\begin{aligned}\frac{\partial}{\partial\zeta}\mathbf{E}\left[f(F^\zeta)\right] &= \mathbf{E}\left[f'\left(F^\zeta\right)\partial_\zeta F^\zeta\right]\\ &= \mathbf{E}\left[\frac{D_w[f(F^\zeta)]}{D_w F^\zeta}\partial_\zeta F^\zeta\right]\\ &= \mathbf{E}\left[f\left(F^\zeta\right)D_w^*\left(\frac{\partial_\zeta F^\zeta}{D_w F^\zeta}\right)\right].\end{aligned}$$

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- No differentiability assumption on f, compare with:

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Wiener case [FLL+99], [FLLL01], [Ben02]

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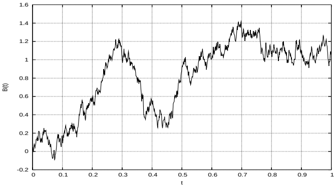

- $\frac{dS_t^\zeta}{S_t^\zeta} = \sigma(S_t^\zeta)dB_t + r(S_t^\zeta)dt$, $S_0^\zeta = x$.
- $w \in L^2(\mathbb{R}_+)$ and

$$D_w f(B_{t_1}, \dots, B_{t_n}) = \sum_{i=1}^n \int_0^{t_i} w_s ds \frac{\partial f}{\partial x_i}(B_{t_1}, \dots, B_{t_n}).$$

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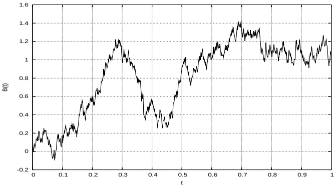

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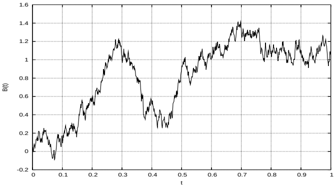

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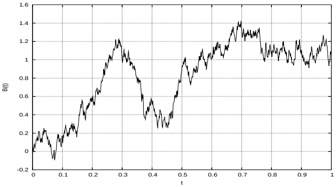

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In [FLL+99] the relations

$$f'(F^\zeta) = \frac{D_t\left[f(F^\zeta)\right]}{D_t F^\zeta}, \qquad 0 \le t \le T, \ a.s.,$$

and

$$D_t S_T^x = \frac{\partial_x S_T^x}{\partial_x S_t^x}\sigma(S_t^x), \qquad 0 \le t \le T, \ a.s.,$$

are used, cf. [Nua95]. This gives

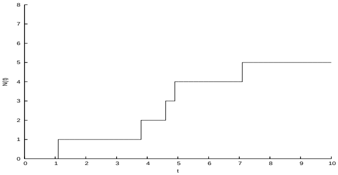
$$\partial_x S_T^x f'(S_T^x) = \frac{\partial_x S_t^x}{\sigma(S_t^x)} D_t f(S_T^x), \qquad 0 \le t \le T, \ a.s., \tag{2}$$

which implies if $\int_0^T w_s ds = 1$:

$$\begin{aligned} \text{Delta} &= \frac{\partial}{\partial x}\mathbf{E}\left[f\left(S_T^x\right)\right] = \mathbf{E}\left[\partial_x S_T^x f'\left(S_T^x\right)\right] = \mathbb{E}\left[\int_0^T w_t \frac{\partial_x S_t^x}{\sigma(S_t^x)} D_t f(S_T^x) dt\right] \\ &= \mathbb{E}\left[f(S_T^x)\delta\left(1_{[0,T]} w \frac{\partial_x S^x}{\sigma(S^x)}\right)\right] = \mathbb{E}\left[f(S_T^x)\int_0^T w_t \frac{\partial_x S_t^x}{\sigma(S_t^x)} dB_t\right] \\ &= \mathbf{E}\left[f\left(S_T^x\right)\frac{B_T}{\sigma x T}\right]. \end{aligned}$$

Poisson case

- $M_t = N_t$ is a Poisson process with jump times $T_1, T_2, T_3, \ldots,$



- $w \in \mathcal{C}_c^1((0,\infty))$, and

$$D_w F = -\sum_{n=1}^{\infty} 1_{\{N_T=n\}} \sum_{i=1}^{n} \frac{\partial f_n}{\partial x_i}(T_1, \ldots, T_n),$$

for F of the form

$$F = f_0 1_{\{N_T=0\}} + \sum_{n=1}^{\infty} 1_{\{N_T=n\}} f_n(T_1, \ldots, T_n).$$

- Recall that

$$E[F] = f_0 e^{-\lambda T} + e^{-\lambda T} \sum_{n=1}^{\infty} \frac{1}{n!} \int_0^T \cdots \int_0^T f_n(t_1, \ldots, t_n) dt_1 \cdots dt_n.$$

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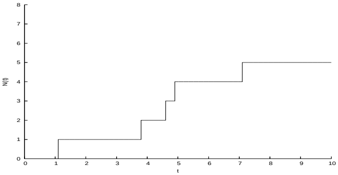

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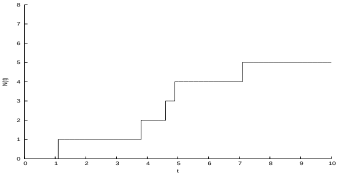

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By standard integration by parts we have, under the boundary condition $w(0) = w(T) = 0$:

$$
\begin{aligned}
E[D_w F] &= -e^{-\lambda T} \sum_{n=1}^{m} \frac{\lambda^n}{n!} \int_0^T \cdots \int_0^T \sum_{k=1}^{n} w(t_k) \partial_k f_n(t_1, \ldots, t_n) dt_1 \cdots dt_n \\
&= e^{-\lambda T} \sum_{n=1}^{m} \frac{\lambda^n}{n!} \int_0^T \cdots \int_0^T f_n(t_1, \ldots, t_n) \sum_{k=1}^{n} \dot{w}(t_k) dt_1 \cdots dt_n \\
&= E\left[F \sum_{k=1}^{k=N(T)} \dot{w}(T_k) \right] = E\left[F \int_0^T \dot{w}(t) dN(t) \right].
\end{aligned}
$$

Next, letting

$$
D_w^* G = G \int_0^T \dot{w}(t) dN(t) - D_w G
$$

we get

$$
\begin{aligned}
E[G D_w F] = E[D_w(FG) - F D_w G] &= E\left[F \left(G \int_0^T \dot{w}(t) dN(t) - D_w G \right) \right] \\
&= E[F D_w^* G].
\end{aligned}
$$

- Price process: $S_t^\zeta = S_0^\zeta e^{rt}(1+\sigma)^{N_t}$.
- Asian options: $F^\zeta = \dfrac{1}{T}\displaystyle\int_0^T S_t^\zeta dt$.

$$W_\Delta = \frac{-1}{x\sigma}\left(1 - \frac{\int_0^T S_t^\zeta dt \int_0^T \dot{w}_t d\tilde{N}_t}{\int_0^T w_t S_{t-}^\zeta dN_t} + \frac{\int_0^T S_t^\zeta dt \int_0^T w_t\left(\dot{w}_t + rw_t\right) S_{t-}^\zeta dN_t}{\left(\int_0^T w_t S_{t-}^\zeta dN_t\right)^2}\right).$$

- Density of reserve processes: $F = \displaystyle\int_0^T e^{(T-t)r} dN_t$.

$$W_y = \frac{\int_0^T \dot{w}(t)dN(t) + \frac{\int_0^T e^{-rt}w(t)(rw(t)-\dot{w}(t))dN(t)}{\int_0^T w(t)e^{-rt}dN(t)}}{r\int_0^T w(t)e^{r(T-t)}dX(t)}.$$

Asian options and reserve processes [KP04], [PW04]

- Price process: $S_t^\zeta = S_0^\zeta e^{rt}(1+\sigma)^{N_t}$.
- Asian options: $F^\zeta = \dfrac{1}{T}\displaystyle\int_0^T S_t^\zeta \, dt$.

$$W_\Delta = \frac{-1}{x\sigma}\left(1 - \frac{\int_0^T S_t^\zeta dt \int_0^T \dot{w}_t d\tilde{N}_t}{\int_0^T w_t S_{t-}^\zeta \, dN_t} + \frac{\int_0^T S_t^\zeta dt \int_0^T w_t\left(\dot{w}_t + r w_t\right) S_{t-}^\zeta \, dN_t}{\left(\int_0^T w_t S_{t-}^\zeta \, dN_t\right)^2}\right).$$

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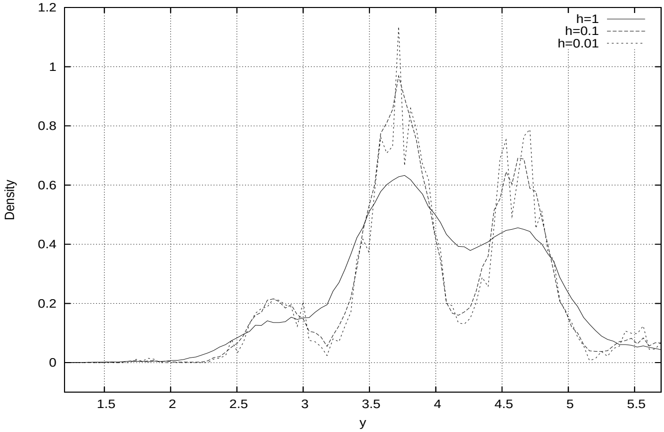
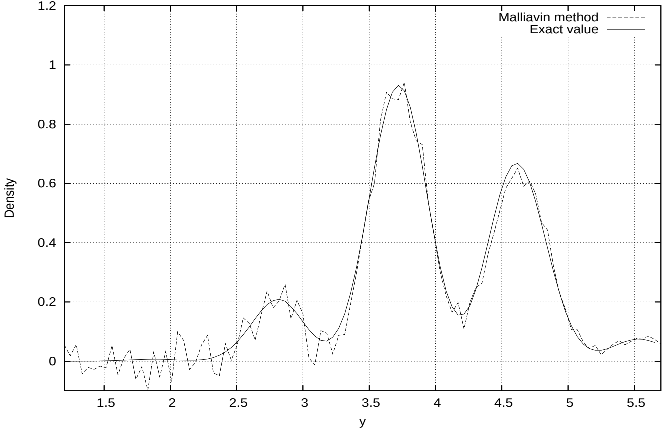
$$\phi_F(y) = \frac{\partial}{\partial y}\mathbf{E}\left[1_{(-\infty,0]}(F-y)\right] \simeq \frac{\mathbf{E}\left[1_{[y-h,y+h]}(F)\right]}{2h}.$$



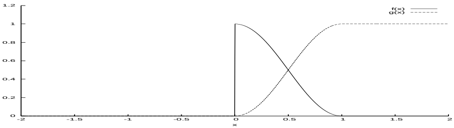
$$\phi_F(y) = \frac{\partial}{\partial y} \mathbf{E}\left[1_{(-\infty,0]}(F-y)\right] = -\mathbf{E}\left[1_{(-\infty,0]}(F-y) D_w^* \left(\frac{1}{D_w F}\right)\right].$$



Consider the decomposition

$$1_{[0,\infty)} = f + g,$$

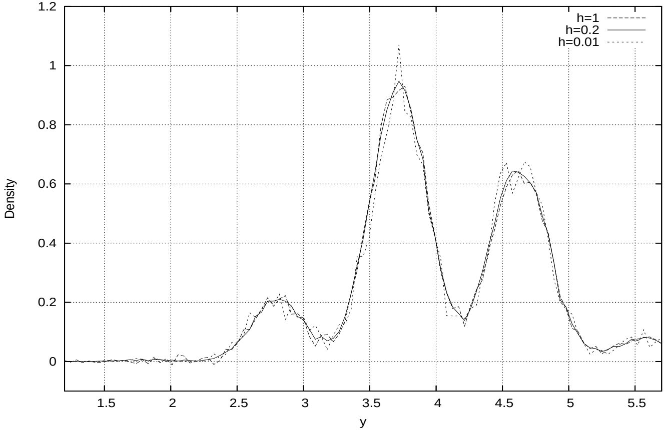
where g is $\mathcal{C}^1$:



We have

$$\begin{aligned} \frac{d}{dy} E[1_{[0,\infty)}(F-y)] &= \frac{d}{dy} E\left[f\left(\frac{F-y}{h} \right) \right] + \frac{d}{dy} E\left[g\left(\frac{F-y}{h} \right) \right] \\ &= E\left[D_w^*\left(\frac{1}{D_w F} \right) f\left(\frac{F-y}{h} \right) \right] + \frac{1}{h} E\left[1_{\{F>y\}} f'\left(\frac{F-y}{h} \right) \right]. \end{aligned}$$

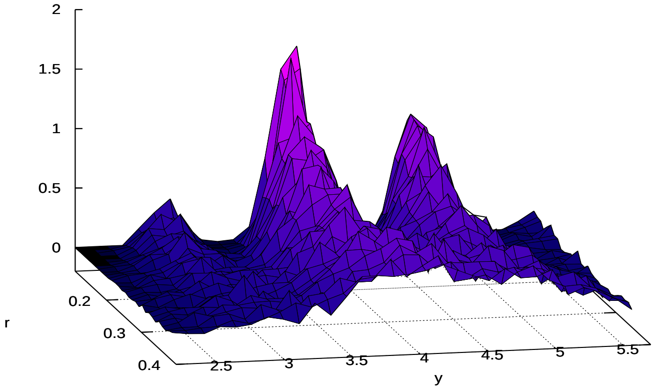
$$\phi_F(y) = -E\left[f\left(\frac{F-y}{h}\right)D_w^*\left(\frac{1}{D_wF}\right)\right] - \frac{1}{h}E\left[1_{\{F>y\}}f'\left(\frac{F-y}{h}\right)\right].$$



Optimization: $f(x) = e^{-x}$, $x \geq 0$, $h = \|W\|^{-1}_{L^2(\Omega)}$.

Density estimation - finite differences

$$F = \int_0^T e^{-rt} dN_t.$$



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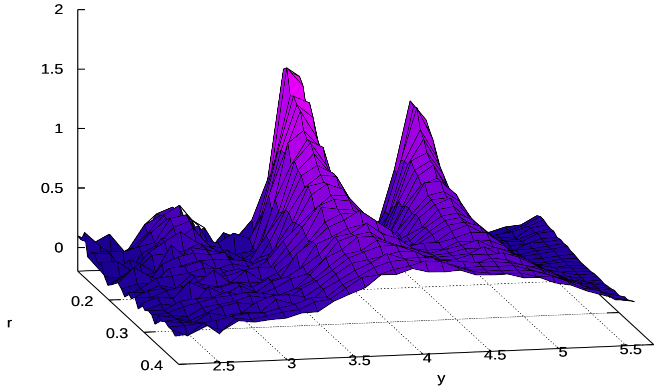

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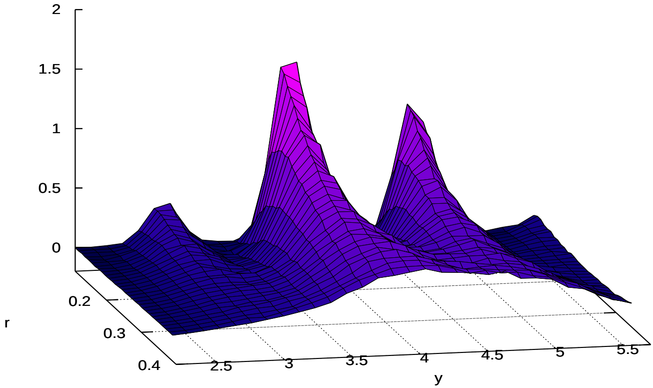

Jump diffusion model [DJ], [DP04], [BMM]

- $(B_t)_{t\in\mathbb{R}_+}$ a standard Brownian motion,
- $(N_t)_{t\in\mathbb{R}_+}$ is a Poisson process with intensity $\lambda > 0$,
- $(Z_k)_{k\geq 1}$ an i.i.d. sequence of random variables with probability distribution $\nu(dx)$,
- $(X_t)_{t\in\mathbb{R}_+}$ a compound Poisson process with Lévy measure $\mu(dy) = \lambda\nu(dx)$. and finite intensity λ:

$$X_t = \sum_{k=1}^{N_t} Z_k, \qquad t \in \mathbb{R}_+, \tag{3}$$

- $(S_t^x)_{t\in\mathbb{R}_+}$ a jump-diffusion price process:

$$\begin{cases} \dfrac{dS_t^x}{S_t^x} = r(S_t^x)dt + \sigma_1(S_t^x)dB_t + \sigma_2(S_{t-}^x)dX_t, \\ \\ S_0^x = x. \end{cases}$$

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$$\frac{dS_t^\zeta}{S_t^\zeta} = rdt + \sigma_1 dB_t + \sigma_2(dN_t - \lambda dt), \qquad S_0^\zeta = x.$$

- $w \in L^2(\mathbb{R}_+)$ and

$$D_w f(B_{t_1}, \ldots, B_{t_n}, T_1, \ldots, T_n) = \sum_{i=1}^n \int_0^{t_i} w_s ds \frac{\partial f}{\partial x_i}(B_{t_1}, \ldots, B_{t_n}, T_1, \ldots, T_n).$$

- Vega_2, *European options*:

$$\frac{\partial}{\partial \sigma_2} \mathbf{E}\left[f(S_T^{\sigma_2})\right] = \mathbf{E}\left[f(S_T^{\sigma_2}) \frac{B_T}{\sigma_1 T}\left(\frac{N_T}{1+\sigma_2} - \lambda T\right)\right].$$

- Derivation with respect to absolutely continuous jump amplitudes [BMM].

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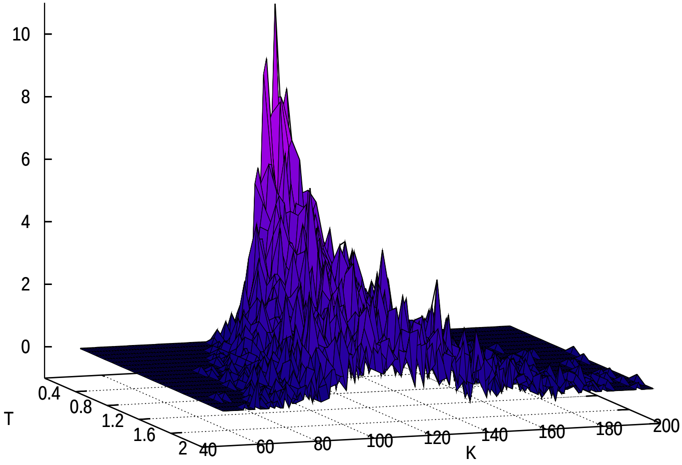
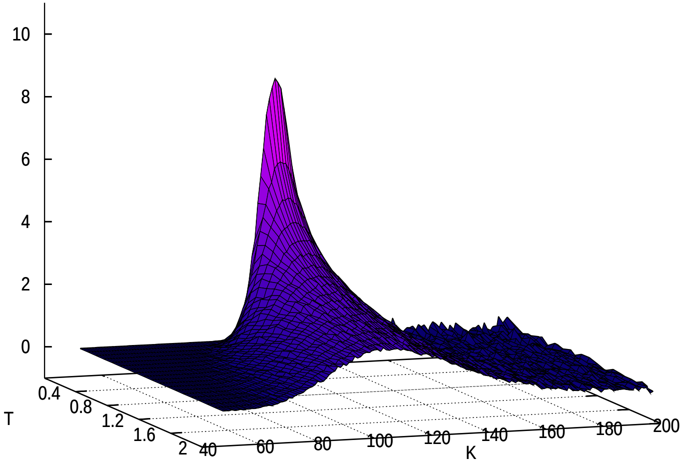
Vega_2 as a function of K and T (finite differences, 10e4 samples, h=0.01)



Vega_2 as a function of K and T (Malliavin method, 10e4 samples, h=0.01)



E. Benhamou.
Smart Monte Carlo: various tricks using Malliavin calculus.
Quant. Finance, 2(5):329–336, 2002.

M.-P. Bavouzet-Morel and M. Messaoud.
Computation of Greeks using Malliavin's calculus in jump type market models.
Preprint, 2004.

M.H.A. Davis and M.P. Johansson.
Malliavin Monte Carlo Greeks for jump diffusions.
Stochastic Processes and their Applications.

V. Debelley and N. Privault.
Sensitivity analysis of European options in jump diffusion models via the Malliavin calculus on Wiener space.
Preprint, 2004.

E. Fournié, J.M. Lasry, J. Lebuchoux, P.L. Lions, and N. Touzi.
Applications of Malliavin calculus to Monte Carlo methods in finance.
Finance and Stochastics, 3(4):391–412, 1999.

E. Fournié, J.M. Lasry, J. Lebuchoux, and P.L. Lions.
Applications of Malliavin calculus to Monte-Carlo methods in finance. II.
Finance and Stochastics, 5(2):201–236, 2001.

A. Kohatsu-Higa and R. Pettersson.
Variance reduction methods for simulation of densities on Wiener space.
SIAM J. Numer. Anal., 40(2):431–450, 2002.

Y. El Khatib and N. Privault.
Computations of Greeks in markets with jumps via the Malliavin calculus.
Finance and Stochastics, 4(2):161–179, 2004.

N. Privault and X. Wei.
A Malliavin calculus approach to sensitivity analysis in insurance.
Insurance Math. Econom., 35(3):679–690, 2004.