Mixing of linear operators under infinitely divisible measures on Banach spaces

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Abstract

The mixing and ergodicity of Gaussian measures have been characterized in terms of their covariances, first for random sequences, and then in the framework of linear dynamics on Banach spaces. In this paper, we extend the latter results to the setting of infinitely divisible measures on Banach spaces, by deriving necessary and sufficient conditions for the strong and weak mixing of linear operators. Our approach relies on characterizations of mixing for infinitely divisible random sequences, and replaces the use of using covariance operators with codifference functionals and control measures on Banach spaces. Our results are then specialized in explicit form to α-stable measures, with examples of linear operators satisfying the required measure invariance conditions.

Keywords: Gaussian measures; Infinitely divisible measures; α-stable measures; Banach spaces; linear operator dynamics; weak mixing; strong mixing.
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1 Introduction

The study of mixing of Gaussian processes in connection with the spectral properties of unitary transformations started in [WA57], with the derivation of necessary and sufficient conditions in terms of Gaussian covariances. Characterizations of mixing and ergodicity for dynamical systems under a Gaussian measure have been obtained in [CFS82, Chapter 14, § 2, Theorems 1 and 2] on a space of real sequences, using spectral measures. On the Wiener

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space, the mixing of ergodicity of random isometries have been treated in [ÜZ00] using the Skorohod integral.

On the other hand, characterizations of mixing and ergodicity of linear operators on complex Banach spaces have been obtained in the framework of linear dynamics, see [BM09] for an introduction to the field and to its connections with the notion of hypercyclicity. We recall the following definition, see e.g. [BM09, Definition 5.23].

Definition 1.1 *A measure-preserving map T on a measure space $(X, \mathcal{B}, \mu)$ is strongly mixing if either of the two following equivalent conditions is satisfied:*

(i) $\lim_{n\to\infty} \mu(A \cap T^{-n}(B)) = \mu(A)\mu(B),\ A, B \in \mathcal{B},$

(ii) $\lim_{n\to\infty} \int_X f(z)g(T^n z)\mu(dz) = \int_X f(z)\mu(dz) \int_X g(z)\mu(dz),\ f, g \in L^2(X, \mu),$

and weakly mixing with respect to μ if either of the two following equivalent conditions is satisfied:

(i) $\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} |\mu(A \cap T^{-k}(B)) - \mu(A)\mu(B)| = 0,\ A, B \in \mathcal{B},$

(ii) $\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \left| \int_X f(z)g(T^k z)\mu(dz) - \int_X f(z)\mu(dz) \int_X g(z)\mu(dz) \right| = 0,\ f, g \in L^2(X, \mu).$

Given E a complex separable Banach space with continuous dual E^* and dual product $\langle x, x^* \rangle$ on $E \times E^*$, consider a Radon probability measure μ on E with characteristic functional

$$\widehat{\mu}(x^*) := \int_E e^{i \operatorname{Re}\langle z, x^* \rangle} \mu(dz), \qquad x^* \in E^*.$$

The measure μ on E is said to be Gaussian if for any $x^* \in E^*$ the random variable $x \mapsto \langle x, x^* \rangle$ has a complex symmetric Gaussian distribution on E, i.e. either it is almost surely 0 or $x \mapsto \operatorname{Re} \langle x, x^* \rangle$ and $x \mapsto \operatorname{Im} \langle x, x^* \rangle$ are independent and have centered Gaussian distributions with the same variance, see [BM09, Definitions 5.6-5.7]. Moreover, by [BM09, Theorem 5.9], for any Gaussian measure μ on E, the continuous conjugate-linear operator $R : E^* \to E$ defined as

$$\langle Rx^*, y^* \rangle = \int_E \overline{\langle z, x^* \rangle} \langle z, y^* \rangle \mu(dz), \qquad x^*, y^* \in E^*,$$

satisfies

$$\widehat{\mu}(x^*) = \exp\left(-\frac{1}{4} \langle Rx^*, x^* \rangle \right), \qquad x^* \in E^*,$$

see also [Lin86, p. 61], and we have

$$\begin{bmatrix} \mathbb{E}[\operatorname{Re}\langle X, x^* \rangle \operatorname{Re}\langle X, y^* \rangle] & \mathbb{E}[\operatorname{Re}\langle X, x^* \rangle \operatorname{Im}\langle X, y^* \rangle] \\ \mathbb{E}[\operatorname{Im}\langle X, x^* \rangle \operatorname{Re}\langle X, y^* \rangle] & \mathbb{E}[\operatorname{Im}\langle X, x^* \rangle \operatorname{Im}\langle X, y^* \rangle] \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \langle Rx^*, y^* \rangle & 0 \\ 0 & \langle Rx^*, y^* \rangle \end{bmatrix}, \tag{1.1}$$

$x^*, y^* \in E^*$.

Mixing criteria have been stated in [BM09, Theorem 5.24] for a continuous linear operator $T : E \to E$ invariant with respect to a Gaussian measure μ on E. Namely, a continuous linear operator T invariant with respect to a Gaussian measure μ on a separable Banach space E is strongly mixing (resp. weakly mixing) if and only if

$$\lim_{n\to\infty} \langle RT^{*n}x^*, y^* \rangle = 0, \quad x^*, y^* \in E^*, \quad \text{resp.} \quad \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \left| \langle RT^{*k}x^*, y^* \rangle \right| = 0, \quad x^*, y^* \in E^*. \tag{1.2}$$

In addition, sufficient conditions have been given on the eigenvectors of T so that it admits an invariant Gaussian measure with full support, see [Fly95] on Hilbert spaces, and [BM09, Proposition 5.27] on Banach spaces.

In the framework of stationary infinitely divisible stochastic processes, mixing criteria in a non-Gaussian setting have been obtained in [Mar70], [RZ96], [RZ97]. The ergodicity and mixing properties of Poisson random measures have also been considered by several authors, under deterministic transformations, see e.g. [Mar78], [Gra84], [Roy07], and [Pri16] for random transformations.

In this paper, we extend the mixing criteria of [BM09, Theorem 5.24] from Gaussian measures to the more general setting where μ is an infinitely divisible probability measure on the Banach space E, i.e. μ is such that for every $n \geq 1$ there exists another probability measure μ_n on E such that

$$\mu = \underbrace{\mu_n \star \cdots \star \mu_n}_{n \text{ times}}$$

see e.g. § 5.1 of [Lin86]. Recall that by e.g. [Lin86, Proposition 5.2.2], Gaussian measures are infinitely divisible.

Moreover, by e.g. [Ros87, § II.1], the characteristic functional of any infinitely divisible probability measure μ on E can be written as

$$\begin{aligned} &\int_E e^{i \operatorname{Re}\langle z, x^* \rangle} \mu(dz) \\ &= \exp\left(-\frac{1}{4} \langle Rx^*, x^* \rangle + \int_E \left(e^{i \operatorname{Re}\langle z, x^* \rangle} - 1 - ik(z) \operatorname{Re}\langle z, x^* \rangle \right) \lambda(dz) \right), \quad x^* \in E^*, \end{aligned} \tag{1.3}$$

where

- $R : E^* \to E$ is a symmetric and positive semidefinite covariance operator,
- $k(z)$ is a bounded measurable function on E such that $\lim_{z\to 0} k(z) = 1$ and $k(z) = O(1/\|z\|)$ as $\|z\|$ tends to infinity, called a truncation function, and
- λ is a Lévy measure, i.e. λ is a measure on E that satisfies $\lambda(\{0\}) = 0$ and

$$\int_E \min((\mathrm{Re}\,\langle z, x^*\rangle)^2, 1)\lambda(dz) < \infty, \qquad x^* \in E^*.$$

In addition, by [AR05, Theorem 4.1] specialized to the single time index $t = 1$, a random variable X with distribution μ on E admits the Lévy-Itô decomposition $X = X_g + X_p$, where X_g is an E-valued Gaussian random variable and X_p is a non-Gaussian infinitely divisible component. In terms of measures we have $\mu = \mu_g * \mu_p$, where μ_g (resp. μ_p) denotes the distribution of X_g (resp. X_p), see [Lin86, Theorem 5.7.3].

In Sections 4-6 we will focus on the case of Banach-valued stable random variables. Recall, see [Woy19, p. 6], [LT91, p. 124] [ST94, § 1.1 and 2.6], that a random vector X taking values in E is said to be α-stable, $\alpha \in (0, 2]$, if for any $a, b > 0$ there exists $z \in E$ such that

$$aX^{(1)} + bX^{(2)} = z + (a^\alpha + b^\alpha)^{1/\alpha} X,$$

where $X^{(1)}$, $X^{(2)}$ are independent copies of X. In the case X and $-X$ have same distribution, the random variable X is said to have a symmetric α-stable ($S\alpha S$) distribution. When $\alpha = 2$, X has a Gaussian distribution.

In Proposition 2.2, we obtain necessary and sufficient conditions for the mixing of a linear operator T on E invariant with respect to the infinitely divisible measure μ. For this, we will extend the covariances appearing in (1.2) into the codifference functionals

$$C_\mu^{=}(x^*, y^*) := \log \mathbb{E}\left[e^{i\,\mathrm{Re}\langle X, x^* - y^*\rangle}\right] - \log \mathbb{E}\left[e^{i\,\mathrm{Re}\langle X, x^*\rangle}\right] - \log \mathbb{E}\left[e^{-i\,\mathrm{Re}\langle X, y^*\rangle}\right], \tag{1.4}$$

and

$$C_\mu^{\neq}(x^*, y^*) := \log \mathbb{E}\left[e^{i\,\mathrm{Re}\langle X, x^*\rangle - i\,\mathrm{Im}\langle X, y^*\rangle}\right] - \log \mathbb{E}\left[e^{i\,\mathrm{Re}\langle X, x^*\rangle}\right] - \log \mathbb{E}\left[e^{-i\,\mathrm{Im}\langle X, y^*\rangle}\right], \tag{1.5}$$

$x^*, y^* \in E^*$. The codifference is a measure of bivariate dependence which, unlike the covariance, does not require the existence of a second moment which may not exist in general, as in the case of α-stable measures with parameter $\alpha \in (0,2)$. The codifference of μ can be rewritten from (1.3) as

$$C_\mu^{=}(x^*,y^*) = \frac{1}{2}\operatorname{Re}\langle Rx^*,y^*\rangle + \int_E \big(e^{i\operatorname{Re}\langle z,x^*\rangle}-1\big)\big(e^{-i\operatorname{Re}\langle z,y^*\rangle}-1\big)\lambda(dz), \tag{1.6}$$

and

$$C_\mu^{\neq}(x^*,y^*) = \frac{1}{2}\operatorname{Im}\langle Rx^*,y^*\rangle + \int_E \big(e^{i\operatorname{Re}\langle z,x^*\rangle}-1\big)\big(e^{-i\operatorname{Im}\langle z,y^*\rangle}-1\big)\lambda(dz), \tag{1.7}$$

$x^*,y^* \in E^*$, where X is an infinitely divisible random variable with distribution μ on E. Our proofs rely on the characterizations of strong and weak mixing properties of stationary infinitely divisible processes established in [RZ96, RZ97, FS13, PV19]. In Proposition 2.6, we obtain mixing criteria for linear operators on Hilbert spaces in terms of spectral measures using Wiener's theorem.

In Section 3 we consider the case where the characteristic function of the infinitely divisible measure μ has the form

$$\int_E e^{i\operatorname{Re}\langle z,x^*\rangle}\mu(dz)$$
$$= \exp\left(-\frac{1}{4}\langle Rx^*,x^*\rangle + \int_E\int_{-\infty}^{\infty}\big(e^{iu\operatorname{Re}\langle z,x^*\rangle}-1-iuk(u)\operatorname{Re}\langle z,x^*\rangle\big)\rho(z,du)\xi(dz)\right), \quad x^*\in E^*,$$

where $\{\rho(s,\cdot)\}_{s\in E}$ is a family of Lévy measures on $\mathbb{R}$, ξ is a σ-finite measure called a control measure, and

$$k(u) = \mathbf{1}_{\{|u|<1\}} + \frac{1}{|u|}\mathbf{1}_{\{|u|\geq 1\}}, \qquad u\in\mathbb{R}.$$

In Theorem 3.2, we obtain necessary and sufficient conditions for the strong mixing of a linear operator $T: E\to E$ that leaves μ invariant, extending [BM09, Theorem 5.24] from the Gaussian to the infinitely divisible setting.

In Section 4, we specialize our results to the case where μ is an α-stable distribution parametrized by an index $\alpha\in(0,2)$, $\alpha\neq 1$, and a control measure ξ on the unit sphere $\mathcal{S}_E$ in E, see (4.1)-(4.2). In this stable setting, (1.6)-(1.7) become

$$C_\mu^{=}(x^*,y^*) = -\int_{\mathcal{S}_E}\big(|\operatorname{Re}\langle z,x^*-y^*\rangle|^\alpha - |\operatorname{Re}\langle z,x^*\rangle|^\alpha - |\operatorname{Re}\langle z,y^*\rangle|^\alpha\big)\xi(dz),$$

and

$$C_\mu^{\neq}(x^*,y^*) = -\int_{\mathcal{S}_E}\big(|\operatorname{Re}\langle z,x^*\rangle - \operatorname{Im}\langle z,y^*\rangle|^\alpha - |\operatorname{Re}\langle z,x^*\rangle|^\alpha - |\operatorname{Im}\langle z,y^*\rangle|^\alpha\big)\xi(dz),$$

$x^*, y^* \in E^*$. Note that although the Gaussian setting corresponds to $\alpha = 2$, it may not be recovered by letting $\alpha = 2$ in the identities defining the α-stable distribution, e.g. in (4.1).

When μ is symmetric α-stable ($S\alpha S$) and the Banach space E is of stable type $\alpha \in (0,2)$, see [Woy19, Definition 6.5.1], it follows from Corollary 4.1 that T is strongly mixing with respect to μ if and only if

$$\lim_{n\to\infty} \int_E |\operatorname{Re}\langle z, x^*\rangle|^{\alpha/2} |\operatorname{Re}\langle z, T^{*n}x^*\rangle|^{\alpha/2} \xi(dz) = 0$$

and

$$\lim_{n\to\infty} \int_E |\operatorname{Re}\langle z, x^*\rangle|^{\alpha/2} |\operatorname{Im}\langle z, T^{*n}x^*\rangle|^{\alpha/2} \xi(dz) = 0, \qquad x^* \in E^*,$$

with an example of operator T satisfying those conditions given in Proposition 4.2.

In Section 5 we construct a class of stable distributions on a Hilbert space H, and in Proposition 5.5 we provide sufficient conditions for their invariance under bounded invertible operators on H, with relevant examples, as required in Corollary 4.1. Finally, in Section 6 we consider the invariance of stable measures under σ-spanning operators. The role of the σ-spanning property in connection to invariance and mixing has been previously discussed in the Gaussian case in e.g. [BG06] in a Hilbert space setting, see also [BG07], and [BM16] for Banach spaces of cotype 2.

We proceed as follows. In Sections 2 and 3 we provide sufficient conditions for mixing in terms of codifference operators and control measures, respectively. The stable case is treated in Section 4 with an example of application constructed in terms of weighted shifts. Sufficient conditions for the existence of an invariant measure for T are given in the stable case in Sections 5 and 6.

2 Mixing in terms of codifferences

In the sequel, we let X denote a random variable with distribution μ on E. Our characterization of mixing in Proposition 2.2 below relies on the following transfer result as in the proof of [BM09, Theorem 5.24].

Lemma 2.1 *A bounded linear operator $T : E \to E$ is strongly mixing (resp. weakly mixing) with respect to μ if and only if the $\mathbb{R}^2$-valued process defined as*

$$X_n^{x^*} := \big(\operatorname{Re}\langle X, T^{*n}x^*\rangle, \operatorname{Im}\langle X, T^{*n}x^*\rangle\big), \qquad n \geq 0, \tag{2.1}$$

induced by X is strongly mixing (resp. weakly mixing) for every $x^* \in E^*$.

Proof. $\Leftarrow$) Let $x^* \in E^*$, and given two integer sequences $(n_1, \ldots, n_k)$ and $(m_1, \ldots, m_l)$ in $\mathbb{N}$, consider the continuous linear maps $\pi_{x^*,1} : E \to \mathbb{C}^k$ and $\pi_{x^*,2} : E \to \mathbb{C}^l$ defined by

$$\pi_{x^*,1}(x) = \left(\langle x, T^{*n_1} x^*\rangle, \ldots, \langle x, T^{*n_k} x^*\rangle\right), \quad \pi_{x^*,2}(x) = \left(\langle x, T^{*m_1} x^*\rangle, \ldots, \langle x, T^{*m_l} x^*\rangle\right),$$

$x \in E$, and the cylinder sets

$$A'_{x^*} := (\pi_{x^*,1})^{-1}(\widetilde{A}_{x^*}), \qquad B'_{x^*} := (\pi_{x^*,2})^{-1}(\widetilde{B}_{x^*}),$$

for $\widetilde{A}_{x^*} \subset \mathbb{C}^k$, $\widetilde{B}_{x^*} \subset \mathbb{C}^l$ Borel sets viewed as subsets of $\mathbb{R}^{2\times k}$ and $\mathbb{R}^{2\times l}$ respectively. If $(X_n^{x^*})_{n\in\mathbb{N}}$ is strongly mixing, we have

$$\begin{aligned}
\lim_{n\to\infty} \mu\big(A'_{x^*} \cap T^{-n} B'_{x^*}\big) &= \lim_{n\to\infty} \int_{\mathbb{R}^{\mathbb{N}}} \mathbb{1}_{A'_{x^*}}(z) \mathbb{1}_{B'_{x^*}}(T^n z)\mu(dz) \\
&= \lim_{n\to\infty} \int_E \mathbb{1}_{\widetilde{A}_{x^*}\times\widetilde{B}_{x^*}}(\pi_{x^*,1}(z), \pi_{x^*,2}(T^n z))\mu(dz) \\
&= \lim_{n\to\infty} \mathbb{E}\big[\mathbb{1}_{\widetilde{A}_{x^*}\times\widetilde{B}_{x^*}}(\pi_{x^*,1}(X), \pi_{x^*,2}T^n(X))\big] \\
&= \lim_{n\to\infty} \mathbb{E}\big[\mathbb{1}_{\widetilde{A}_{x^*}}(\pi_{x^*,1}(X))\mathbb{1}_{\widetilde{B}_{x^*}}(\pi_{x^*,2}(T^n X))\big] \\
&= \lim_{n\to\infty} \mathbb{E}\big[\mathbb{1}_{\widetilde{A}_{x^*}}(X_{n_1}, \ldots, X_{n_k})\mathbb{1}_{\widetilde{B}_{x^*}}(X_{m_1+n}, \ldots, X_{m_l+n})\big] \\
&= \mathbb{E}\big[\mathbb{1}_{\widetilde{A}_{x^*}}(X_{n_1}, \ldots, X_{n_k})\big]\mathbb{E}\big[\mathbb{1}_{\widetilde{B}_{x^*}}(X_{m_1}, \ldots, X_{m_l})\big] \\
&= \mathbb{E}\big[\mathbb{1}_{\widetilde{A}_{x^*}}(\pi_{x^*,1}(X))\big]\mathbb{E}\big[\mathbb{1}_{\widetilde{B}_{x^*}}(\pi_{x^*,2}(X))\big] \\
&= \int_E \mathbb{1}_{\widetilde{A}_{x^*}}(\pi_{x^*,1}(z))\mu(dz) \int_E \mathbb{1}_{\widetilde{B}_{x^*}}(\pi_{x^*,2}(z))\mu(dz) \\
&= \int_E \mathbb{1}_{A'_{x^*}}(z)\mu(dz) \int_E \mathbb{1}_{B'_{x^*}}(z)\mu(dz) \\
&= \mu\big(A'_{x^*}\big)\mu\big(B'_{x^*}\big),
\end{aligned}$$

and this relation extends to any Borel sets A'_{x^*}, B'_{x^*} by a monotone class argument. Next, if $(X_n^{x^*})_{n\in\mathbb{N}}$ is weakly mixing for every $x^* \in E^*$, then as above we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \Big| \mathbb{E}\big[\mathbb{1}_{\widetilde{A}_{x^*}}(\pi_{x^*,1}(X))\mathbb{1}_{\widetilde{B}_{x^*}}(\pi_{x^*,2}(T^k X))\big] - \mathbb{E}\big[\mathbb{1}_{\widetilde{A}_{x^*}}(\pi_{x^*,1}(X))\big]\mathbb{E}\big[\mathbb{1}_{\widetilde{B}_{x^*}}(\pi_{x^*,2}(X))\big] \Big| = 0,$$

and this relation also extends to Borel sets A'_{x^*}, B'_{x^*} by a monotone class argument, with

$$\begin{aligned}
&\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \big|\mu(A'_{x^*} \cap T^{-k} B'_{x^*}) - \mu(A'_{x^*})\mu(B'_{x^*})\big| \\
&= \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \left| \int_E \mathbb{1}_{A'_{x^*}}(z)\mathbb{1}_{B'_{x^*}}(T^k z)\mu(dz) - \int_E \mathbb{1}_{A'_{x^*}}(z)\mu(dz) \int_E \mathbb{1}_{B'_{x^*}}(z)\mu(dz) \right|
\end{aligned}$$

$$= 0.$$

$\Rightarrow$) Assume that T is strongly mixing with respect to μ. Letting $\mathcal{X} := (X_l^{x^*})_{l\in\mathbb{N}}$, for any $x^* \in E^*$ and $f, g \in L^2((\mathbb{R}^2)^{\mathbb{N}}, m)$, where m is the pushforward of μ by

$$z \mapsto (\mathrm{Re}\langle z, T^{*l}x^*\rangle, \mathrm{Im}\langle z, T^{*l}x^*\rangle)_{l\in\mathbb{N}},$$

denoting by S the shift operator by one time step, we have

$$\begin{aligned}
\lim_{n\to\infty} \mathbb{E}\left[f(\mathcal{X})g(S^n\mathcal{X})\right] &= \lim_{n\to\infty} \int_E f\big((\langle z, T^{*l}x^*\rangle)_{l\in\mathbb{N}}\big) g\big((\langle z, T^{*(n+l)}x^*\rangle)_{l\in\mathbb{N}}\big)\mu(dz) \\
&= \lim_{n\to\infty} \int_E f\big((\langle z, T^{*l}x^*\rangle)_{l\in\mathbb{N}}\big) g\big((\langle T^n z, T^{*l}x^*\rangle)_{l\in\mathbb{N}}\big)\mu(dz) \\
&= \int_E f\big((\langle z, T^{*l}x^*\rangle)_{l\in\mathbb{N}}\big)\mu(dz) \int_E g\big((\langle z, T^{*l}x^*\rangle)_{l\in\mathbb{N}}\big)\mu(dz) \\
&= \mathbb{E}[f(\mathcal{X})]\mathbb{E}[g(\mathcal{X})].
\end{aligned}$$

Similarly, if T is weakly mixing with respect to μ, for any $f, g \in L^2((\mathbb{R}^2)^{\mathbb{N}}, m)$ and $x^* \in E^*$ we have

$$\begin{aligned}
&\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \left|\mathbb{E}\left[f(\mathcal{X})g(S^k\mathcal{X})\right] - \mathbb{E}\left(f(\mathcal{X})\right)\mathbb{E}\left(g(\mathcal{X})\right)\right| \\
&= \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \left|\int_E f\big((\langle z, T^{*l}x^*\rangle)_{l\in\mathbb{N}}\big) g\big((\langle z, T^{*(k+l)}x^*\rangle)_{l\in\mathbb{N}}\big)\mu(dz)\right. \\
&\qquad\qquad \left. - \int_E f\big((\langle z, T^{*l}x^*\rangle)_{l\in\mathbb{N}}\big)\mu(dz) \int_E g\big((\langle z, T^{*l}x^*\rangle)_{l\in\mathbb{N}}\big)\mu(dz)\right| \\
&= 0.
\end{aligned}$$

□

In Proposition 2.2 we characterize the mixing of a linear operator T on the complex separable Banach space E. The equality (2.2) below, see also (2.3), is a technical condition originating in [Mar70] and used in the proof of Theorem 1 of [RZ96] in order to ensure the characterization of mixing using codifferences. See [RZ96, page 282] for an example where this condition is necessary.

Proposition 2.2 *Let E be a complex Banach space, and assume that for every $x^* \in E^*$, the Lévy measure ν_{x^*} of $(\mathrm{Re}\langle X, x^*\rangle, \mathrm{Im}\langle X, x^*\rangle)$ satisfies*

$$\nu_{x^*}(\mathbb{R} \times 2\pi\mathbb{Z}) \quad \textit{and} \quad \nu_{x^*}(2\pi\mathbb{Z} \times \mathbb{R}) = 0. \tag{2.2}$$

Then, a bounded linear operator $T : E \to E$ that leaves invariant the infinitely divisible measure μ is strongly mixing, resp. weakly mixing, with respect to μ if and only if

$$\lim_{n\to\infty} C_\mu^{=}\big(x^*, T^{*n}x^*\big) = 0 \quad \textit{and} \quad \lim_{n\to\infty} C_\mu^{\neq}\big(x^*, T^{*n}x^*\big) = 0, \quad x^* \in E^*,$$

resp. for any $x^* \in E^*$ there exists a density one subset D_{x^*} of $\mathbb{N}$ such that

$$\lim_{\substack{n\to\infty\\ n\in D_{x^*}}} \frac{1}{n}\sum_{k=0}^{n-1} \big|C_\mu^{=}(x^*, T^{*k}x^*)\big| = 0 \quad \textit{and} \quad \lim_{\substack{n\to\infty\\ n\in D_{x^*}}} \frac{1}{n}\sum_{k=0}^{n-1} \big|C_\mu^{\neq}(x^*, T^{*k}x^*)\big| = 0.$$

Proof. Let $x^* \in E^*$. Since T leaves μ invariant, by (2.1) and (1.4)–(1.5) we have

$$\begin{aligned} C_\mu^{=}(x^*, T^{*n}x^*) &= \log \mathbb{E}\left[e^{i\,\mathrm{Re}\langle X, x^* - T^{*n}x^*\rangle}\right] - \log \mathbb{E}\left[e^{i\,\mathrm{Re}\langle X, x^*\rangle}\right] - \log \mathbb{E}\left[e^{-i\,\mathrm{Re}\langle X, T^{*n}x^*\rangle}\right] \\ &= \log\left(\mathbb{E}\left[e^{i\,\mathrm{Re}\langle X, x^* - T^{*n}x^*\rangle}\right] / \left(\mathbb{E}\left[e^{i\,\mathrm{Re}\langle X, x^*\rangle}\right]\mathbb{E}\left[e^{-i\,\mathrm{Re}\langle X, x^*\rangle}\right]\right)\right) \end{aligned}$$

and

$$\begin{aligned} C_\mu^{\neq}(x^*, T^{*n}x^*) &= \log \mathbb{E}\left[e^{i\,\mathrm{Re}\langle X, x^*\rangle - i\,\mathrm{Im}\langle X, T^{*n}x^*\rangle}\right] - \log \mathbb{E}\left[e^{i\,\mathrm{Re}\langle X, x^*\rangle}\right] - \log \mathbb{E}\left[e^{-i\,\mathrm{Im}\langle X, T^{*n}x^*\rangle}\right] \\ &= \log\left(\mathbb{E}\left[e^{i\,\mathrm{Re}\langle X, x^*\rangle - i\,\mathrm{Im}\langle X, T^{*n}x^*\rangle}\right] / \left(\mathbb{E}\left[e^{i\,\mathrm{Re}\langle X, x^*\rangle}\right]\mathbb{E}\left[e^{-i\,\mathrm{Im}\langle X, x^*\rangle}\right]\right)\right), \end{aligned}$$

$n \in \mathbb{N}$. Next, by [FS13, Theorem 2.1] applied to the time index $\mathbb{N}$ as in [PV19], $(X_n^{x^*})_{n\in\mathbb{N}}$ is strongly mixing if and only if

$$\lim_{n\to\infty} \mathbb{E}\big[e^{i\,\mathrm{Re}\langle X, T^{*n}x^*\rangle - i\,\mathrm{Re}\langle X, x^*\rangle}\big] = \mathbb{E}\big[e^{i\,\mathrm{Re}\langle X, x^*\rangle}\big]\mathbb{E}\big[e^{-i\,\mathrm{Re}\langle X, x^*\rangle}\big]$$

and

$$\lim_{n\to\infty} \mathbb{E}\big[e^{i\,\mathrm{Re}\langle X, T^{*n}x^*\rangle - i\,\mathrm{Im}\langle X, x^*\rangle}\big] = \mathbb{E}\big[e^{i\,\mathrm{Re}\langle X, x^*\rangle}\big]\mathbb{E}\big[e^{-i\,\mathrm{Im}\langle X, x^*\rangle}\big].$$

Similarly, by [PV19, Theorem 4.3], $(X_n^{x^*})_{n\in\mathbb{N}}$ is weakly mixing if and only if there exists a density one subset D_{x^*} of $\mathbb{N}$ such that

$$\lim_{\substack{n\to\infty\\ n\in D_{x^*}}} \mathbb{E}\big[e^{i\,\mathrm{Re}\langle x, T^{*n}x^*\rangle - i\,\mathrm{Re}\langle X, x^*\rangle}\big] = \mathbb{E}\big[e^{i\,\mathrm{Re}\langle X, x^*\rangle}\big]\mathbb{E}\big[e^{-i\,\mathrm{Re}\langle X, x^*\rangle}\big]$$

and

$$\lim_{\substack{n\to\infty\\ n\in D_{x^*}}} \mathbb{E}\big[e^{i\,\mathrm{Re}\langle x, T^{*n}x^*\rangle - i\,\mathrm{Im}\langle X, x^*\rangle}\big] = \mathbb{E}\big[e^{i\,\mathrm{Re}\langle X, x^*\rangle}\big]\mathbb{E}\big[e^{-i\,\mathrm{Im}\langle X, x^*\rangle}\big].$$

We conclude in both cases from Lemma 2.1. □

When the Banach space E is real we ignore the vanishing second component of the induced process $\big(X_n^{x^*}\big)_{n\in\mathbb{N}}$, and this process becomes $\mathbb{R}$-valued. In this case, following the same argument as in the proof of Proposition 2.2 by replacing the use of [FS13, Theorem 2.1] and [PV19, Theorem 4.3] with that of [RZ97, Proposition 4 and Theorem 2], we have the following result.

Proposition 2.3 *Let E be a real Banach space, and assume that for every $x^* \in E^*$, the Lévy measure ν_{x^*} of $\langle X, x^* \rangle$ satisfies*

$$\nu_{x^*}(2\pi\mathbb{Z}) = 0. \tag{2.3}$$

Then, a bounded linear operator $T : E \to E$ that leaves invariant the infinitely divisible measure μ is strongly mixing, resp. weakly mixing, with respect to μ if and only if

$$\lim_{n\to\infty} C_\mu^{=}\big(x^*, T^{*n}x^*\big) = 0, \quad \textit{resp.} \quad \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \big|C_\mu^{=}(x^*, T^{*k}x^*)\big| = 0, \quad x^* \in E^*.$$

In the case where μ is Gaussian with covariance operator R, (1.6)-(1.7) become

$$C_\mu^{=}(x^*, y^*) = \frac{1}{2} \operatorname{Re} \langle Rx^*, y^* \rangle = \frac{1}{2} \operatorname{Re} \langle Ry^*, x^* \rangle$$

and

$$C_\mu^{\neq}(x^*, y^*) = \frac{1}{2} \operatorname{Im} \langle Rx^*, y^* \rangle = -\frac{1}{2} \operatorname{Im} \langle Ry^*, x^* \rangle, \quad x^*, y^* \in E^*,$$

and Proposition 2.2 yields the following result, cf. [BM09, Theorem 5.24].

Corollary 2.4 *Let μ be a Gaussian measure on a complex Banach space E. Then, a bounded linear operator $T : E \to E$ is strongly mixing, resp. weakly mixing, with respect to μ if and only if*

$$\lim_{n\to\infty} \langle RT^{*n}x^*, x^* \rangle = 0, \quad x^* \in E^*,$$

resp. for any $x^* \in E^*$ there exists a density one subset D_{x^*} of $\mathbb{N}$ such that

$$\lim_{\substack{n\to\infty \\ n\in D_{x^*}}} \frac{1}{n} \sum_{k=0}^{n-1} \big|\big\langle RT^{*k}x^*, x^* \big\rangle\big| = 0.$$

We close this section with remarks on the characterization of mixing in real Hilbert spaces using spectral measures. The next proposition defines the spectral measure σ_{x^*} of x^* in H^* for H a Hilbert space. In what follows, we denote by $\mathbb{T}$ the unit circle in $\mathbb{C}$.

Proposition 2.5 *Let H be a real Hilbert space, and consider a bounded bijective linear operator $T : H \to H$. For every $x^* \in H^*$, $x \neq 0$, there exists a probability measure σ_{x^*} on $\mathbb{T}$ whose Fourier coefficients are given by*

$$\widehat{\sigma}_{x^*}(k) = -\frac{C_\mu^{=}(x^*, T^{*k}x^*)}{\log \left|\mathbb{E}\left[e^{iX_0}\right]\right|^2}, \qquad k \in \mathbb{Z}. \tag{2.4}$$

Proof. Following the steps of the proof of [RZ97, Proposition 2], we find that for any pairwise distinct sequence $(n_j)_{1\leq j\leq N}$ the matrix $(C_\mu^{=}(T^{*n_j}x^*, T^{*n_k}x^*))_{1\leq j,k\leq N}$ is nonnegative definite. Hence, the function $\widehat{\sigma}_{x^*}(k)$ is non-negative definite and the conclusion follows from Bochner's theorem, see e.g. [Rud91], since $\mathbb{T}$ is the Pontryagin dual of $\mathbb{Z}$. □

Recall that a complex measure σ on $\mathbb{T}$ is

1. *continuous* if $\sigma(\{\lambda\}) = 0$ for every $\lambda \in \mathbb{T}$, and

2. *Rajchman* if $\widehat{\sigma}(n) \to 0$ as $|n| \to \infty$,

where $\widehat{\sigma}(n)$ is the *nth* Fourier coefficient of σ.

Proposition 2.6 *Let μ be an infinitely divisible measure on a real Hilbert space H, and assume that for every $x^* \in H^*$, the Lévy measure of $X_0^{x^*} = \langle X, x^*\rangle$ has no atoms in $2\pi\mathbb{Z}$. Then, a bounded bijective linear operator $T : H \to H$ that leaves invariant the infinitely divisible measure μ is strongly mixing (resp. weakly mixing) with respect to μ if and only if σ_{x^*} is Rajchman for all $x^* \in H^*$ (resp. σ_{x^*} is continuous for all $x^* \in H^*$).*

Proof. *a*) If σ_{x^*} is Rajchman for all $x^* \in H^*$, then by (2.4) we have $\lim_{n\to\infty} C_\mu^{=}(x^*, T^{*n}x^*) = 0$, hence T is strongly mixing by Proposition 2.3. Conversely, if T is strongly mixing then by (2.4) and Proposition 2.3 we have

$$\lim_{n\to\infty} \widehat{\sigma}_{x^*}(n) = 0, \qquad x^* \in H^*.$$

Since

$$C_\mu^{=}(x^*, (T^{-1})^{*n}x^*) = C_\mu^{=}(T^{*n}x^*, x^*) = \overline{C_\mu^{=}(x^*, T^{*n}x^*)}, \qquad n \in \mathbb{N},$$

hence

$$\lim_{|n|\to\infty} \widehat{\sigma}_{x^*}(n) = 0, \qquad x^* \in H^*,$$

and therefore σ_{x^*} is Rajchman.

b) Suppose that σ_{x^*} is continuous for all $x^* \in H^*$. Then, by Proposition 2.5 and Wiener's theorem, see e.g. [BM09, Theorem 5.31], we have

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} C_\mu^{=}(x^*, T^{*k}x^*) = 0,$$

hence T is strongly mixing by Proposition 2.3. Conversely, if T is weakly mixing then by Propositions 2.3 and 2.5, for every $x^* \in H^*$ we have

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} |\widehat{\sigma}_{x^*}(k)| = 0,$$

hence σ_{x^*} is continuous by Wiener's theorem. □

3 Mixing in terms of control measures

In this section, we let Λ denote an infinitely divisible random measure on a complex Banach space E, defined by its characteristic function

$$\mathbb{E}\left[e^{it\Lambda(A)}\right] = \exp\left(-\frac{t^2}{4}\int_A \sigma^2(z)\xi(dz) + \int_A\int_{-\infty}^{\infty}\left(e^{iut} - 1 - ituk(u)\right)\rho(z,du)\xi(dz)\right)$$

for any measurable $A \subset E$ and $t \in \mathbb{R}$, see § 3 of [RZ96], where

- $\sigma^2 : E \to [0,\infty)$ is a measurable function,
- $\{\rho(s,\cdot)\}_{s\in E}$ is a family of Lévy measures on $\mathbb{R}$,
- ξ is a σ-finite measure called a control measure, and
- k is the truncation function

$$k(u) = \begin{cases} 1, & |u| < 1, \\ \dfrac{1}{|u|}, & \text{else}, \ u \in \mathbb{R}. \end{cases}$$

In addition, we assume as above that the Gaussian component of $\Lambda(A)$ either vanishes a.s., or has independent identically distributed Gaussian real and imaginary components. More generally, for any $f : E \to E$ a sufficiently integrable measurable function, the stochastic integral

$$\int_E f(z)\Lambda(dz)$$

has the characteristic functional

$$\mathbb{E}\left[e^{i\,\mathrm{Re}\left\langle\int_E f(z)\Lambda(dz),x^*\right\rangle}\right]$$

$$= \exp\left(-\frac{1}{4}\langle R_f x^*, x^*\rangle + \int_E\int_{-\infty}^{\infty}\left(e^{iu\,\mathrm{Re}\langle f(z),x^*\rangle} - 1 - iuk(u)\,\mathrm{Re}\,\langle f(z),x^*\rangle\right)\rho(z,du)\xi(dz)\right),$$

$x^* \in E^*$, where $R_f : E^* \to E$ is the covariance operator

$$\langle R_f x^*, y^*\rangle = \int_E \overline{\langle f(z), x^*\rangle}\,\langle f(z), y^*\rangle\,\sigma^2(z)\xi(dz), \qquad x^*, y^* \in E.$$

In particular, taking $f(z) = z$, the random variable

$$X := \int_E z\Lambda(dz),$$

has an infinitely divisible distribution with characteristic functional

$$\mathbb{E}\left[e^{i\operatorname{Re}\langle X,x^*\rangle}\right] = \exp\left(-\frac{1}{4}\langle Rx^*,x^*\rangle + \int_E\int_{-\infty}^{\infty}\left(e^{iu\operatorname{Re}\langle z,x^*\rangle} - 1 - iuk(u)\operatorname{Re}\langle z,x^*\rangle\right)\rho(z,du)\xi(dz)\right), \tag{3.1}$$

$x^* \in E^*$, where $R : E^* \to E$ is the covariance operator

$$\langle Rx^*, y^*\rangle = \int_E \overline{\langle z,x^*\rangle}\,\langle z,y^*\rangle\,\sigma^2(z)\xi(dz).$$

The following result is an extension of [RZ96, Theorem 4] to our setting. In what follows, we let

$$V(r,z) := \int_{-\infty}^{\infty}\min(|ru|,1)\rho(z,du), \qquad r \in \mathbb{R},\ z \in E.$$

Lemma 3.1 *Given $r \geq 1$, consider a family $(f_n)_{n\in\mathbb{N}}$ of $\mathbb{R}^r$-valued measurable functions on E. Then, the stationary $\mathbb{R}^r$-valued process $(Y_n)_{n\in\mathbb{N}}$ defined by*

$$Y_n = \int_E f_n(z)\Lambda(dz), \quad n \geq 0,$$

is strongly mixing if and only if

$$\lim_{n\to\infty}\left\|\int_E f_0(z)f_n(z)^\top\sigma^2(z)\xi(dz)\right\|_{\mathbb{R}^r\otimes\mathbb{R}^r} = 0 \tag{3.2}$$

and

$$\lim_{n\to\infty}\int_E V\big(\|f_0(z)\|_{\mathbb{R}^r}\,\|f_n(z)\|_{\mathbb{R}^r}, z\big)\xi(dz) = 0. \tag{3.3}$$

Proof. We use the argument of [PV19, Lemma 3.5]. The covariance of the Gaussian components of (Y_0, Y_n) is given by

$$\int_E f_0(z)f_n(z)^\top\sigma^2(z)\xi(dz), \qquad n \geq 0,$$

and from [FS13, Theorem 3.2] specialized to the discrete-time setting, the Lévy measure of the joint distribution of (Y_0, Y_n) is given by

$$\rho_{0n}(B) = \int_E\int_{\mathbb{R}} \mathbb{1}_B\left(f_0(z)u, f_n(z)u\right)\rho(z,du)\xi(dz)$$

for all Borel sets $B \subset \mathbb{R}^{2r}\backslash\{0\}$. Thus, we have

$$\int_{\mathbb{R}^{2r}}\min(\|a\|\|b\|,1)\rho_{0n}(da,db) = \int_E\int_{\mathbb{R}}\min(\|f_0(z)u\|_{\mathbb{R}^r}\,\|f_n(z)u\|_{\mathbb{R}^r},1)\rho(z,du)\xi(dz)$$

$$= \int_E \int_{\mathbb{R}} \min(u^2 \|f_0(z)\|_{\mathbb{R}^r} \|f_n(z)\|_{\mathbb{R}^r}, 1)\rho(z, du)\xi(dz)$$

$$= \int_{\mathbb{R}} V\big(\|f_0(z)\|_{\mathbb{R}^r} \|f_n(z)\|_{\mathbb{R}^r}, z\big)\xi(dz).$$

Since (Y_n) is stationary, [FS13, Corollary 2.5] (again specialized to the discrete-time case) shows that (3.2)-(3.3) is equivalent to the strong mixing of $(Y_n)_{n\in\mathbb{N}}$, as in the proof of [RZ96, Theorem 4]. □

Theorem 3.2 extends [BM09, Theorem 5.24] from the Gaussian to the infinitely divisible setting. When Λ is Gaussian, i.e. when $\rho(z, du) = 0$, we have $V(r, z) = 0$ and it recovers the strong mixing criterion of Corollary 2.4 from the vanishing of (3.5) and (3.6).

Theorem 3.2 *A bounded linear operator $T : E \to E$ that leaves invariant the infinitely divisible measure μ is strongly mixing with respect to μ if and only if for every $x^* \in E^*$ we have*

$$\lim_{n\to\infty} \langle RT^{*n}x^*, x^*\rangle = 0, \tag{3.4}$$

$$\lim_{n\to\infty} \int_E V\big(| \operatorname{Re}\langle z, x^*\rangle \operatorname{Re}\langle z, T^{*n}x^*\rangle |, z\big)\xi(dz) = 0, \tag{3.5}$$

and

$$\lim_{n\to\infty} \int_E V\big(| \operatorname{Re}\langle z, x^*\rangle \operatorname{Im}\langle z, T^{*n}x^*\rangle |, z\big)\xi(dz) = 0. \tag{3.6}$$

Proof. By Lemma 3.1 applied to the $\mathbb{R}^2$-valued functions

$$f_n(z) = \big(\operatorname{Re}\langle z, T^{*n}x^*\rangle, \operatorname{Im}\langle z, T^{*n}x^*\rangle\big), \qquad z \in E, \ n \in \mathbb{N},$$

the $\mathbb{R}^2$-valued process defined by

$$X_n^{x^*} := \left(\int_E \operatorname{Re}\langle z, T^{*n}x^*\rangle \Lambda(dz), \int_E \operatorname{Im}\langle z, T^{*n}x^*\rangle \Lambda(dz)\right), \qquad n \in \mathbb{N},$$

is strongly mixing if and only if for all $x^* \in E^*$ we have

$$\lim_{n\to\infty} \int_E \begin{bmatrix} \operatorname{Re}\langle z, x^*\rangle \operatorname{Re}\langle z, T^{*n}x^*\rangle & \operatorname{Re}\langle z, x^*\rangle \operatorname{Im}\langle z, T^{*n}x^*\rangle \\ \operatorname{Im}\langle z, x^*\rangle \operatorname{Re}\langle z, T^{*n}x^*\rangle & \operatorname{Im}\langle z, x^*\rangle \operatorname{Im}\langle z, T^{*n}x^*\rangle \end{bmatrix} \sigma^2(z)\xi(dz) = 0, \tag{3.7}$$

and

$$\lim_{n\to\infty} \int_E V(|\langle z, x^*\rangle \langle z, T^{*n}x^*\rangle |, z)\xi(dz) = 0. \tag{3.8}$$

From Lemma 2.1, it suffices to show (3.7) is equivalent to (3.4) and that (3.8) is equivalent to (3.5)-(3.6) in order to conclude the proof.

a) Since by (1.1) the real and imaginary parts of the Gaussian component of $X_n^{x^*}$ are independent for every $n \geq 0$, we have

$$\int_E \begin{bmatrix} \operatorname{Re}\langle z, x^*\rangle \operatorname{Re}\langle z, T^{*n}x^*\rangle & \operatorname{Re}\langle z, x^*\rangle \operatorname{Im}\langle z, T^{*n}x^*\rangle \\ \operatorname{Im}\langle z, x^*\rangle \operatorname{Re}\langle z, T^{*n}x^*\rangle & \operatorname{Im}\langle z, x^*\rangle \operatorname{Im}\langle z, T^{*n}x^*\rangle \end{bmatrix} \sigma^2(z)\xi(dz) = \frac{1}{2} \begin{bmatrix} \langle RT^{*n}x^*, x^*\rangle & 0 \\ 0 & \langle RT^{*n}x^*, x^*\rangle \end{bmatrix},$$

which shows that (3.7) is equivalent to (3.4).

b) Next, from the inequality

$$|\langle z, x^*\rangle \langle z, T^{*n}x^*\rangle| \geq |\operatorname{Re}\langle z, x^*\rangle \operatorname{Re}\langle z, T^{*n}x^*\rangle|,$$

we have

$$V(|\langle z, x^*\rangle \langle z, T^{*n}x^*\rangle|, z) \geq V\big(|\operatorname{Re}\langle z, x^*\rangle \operatorname{Re}\langle z, T^{*n}x^*\rangle|, z\big),$$

$z \in E$, hence (3.8) implies (3.5) and (3.6) similarly. On the other hand, from the relation $\operatorname{Im}\langle z, x^*\rangle = \operatorname{Re}\langle z, -ix^*\rangle$, (3.5)-(3.6) imply

$$\lim_{n\to\infty} \int_E V\big(|\operatorname{Im}\langle z, x^*\rangle \operatorname{Im}\langle z, T^{*n}x^*\rangle|, z\big)\xi(dz) = 0$$

and

$$\lim_{n\to\infty} \int_E V\big(|\operatorname{Im}\langle z, x^*\rangle \operatorname{Re}\langle z, T^{*n}x^*\rangle|, z\big)\xi(dz) = 0,$$

$x^* \in E^*$, hence the inequalities

$$\begin{aligned} |\langle z, x^*\rangle \langle z, T^{*n}x^*\rangle| \leq{}& |\operatorname{Re}\langle z, x^*\rangle \operatorname{Re}\langle z, T^{*n}x^*\rangle| + |\operatorname{Re}\langle z, x^*\rangle \operatorname{Im}\langle z, T^{*n}x^*\rangle| \\ &+ |\operatorname{Im}\langle z, x^*\rangle \operatorname{Im}\langle z, T^{*n}x^*\rangle| + |\operatorname{Im}\langle z, x^*\rangle \operatorname{Re}\langle z, T^{*n}x^*\rangle|, \end{aligned}$$

and

$$\min(1, a+b) \leq \min(1, a) + \min(1, b), \qquad a, b \geq 0,$$

yield (3.8). □

4 Stable case

In this section, we apply the results of Sections 2-3 to the case where μ is an α-stable distribution on E with parameter $\alpha \in (0,2)$, $\alpha \neq 1$. The characteristic functional (3.1) of an E-valued random variable X can be written by the Tortrat Theorem [Tor77] as

$$\mathbb{E}\left[e^{i\operatorname{Re}\langle X, x^*\rangle}\right] = \int_E e^{i\operatorname{Re}\langle z, x^*\rangle}\mu(dz) = \exp\left(-\int_{\mathcal{S}_E} |\operatorname{Re}\langle z, x^*\rangle|^\alpha \xi(dz)\right), \quad x^* \in E^*, \qquad (4.1)$$

where ξ is a finite measure concentrated on the unit sphere $\mathcal{S}_E$, see also [Woy19, p. 6] or [LT91, Corollary 5.5], [Lin86, Theorem 6.4.4 and Corollary 7.5.2]. On the other hand, by e.g. [Sat99, Lemma 14.11] or [PP16, Corollary 4.1], (4.1) can be rewritten when $\alpha \neq 1$ as

$$\mathbb{E}\left[e^{i\operatorname{Re}\langle X,x^*\rangle}\right] = \exp\left(c_\alpha \int_E \int_{-\infty}^{\infty} \left(e^{iu\operatorname{Re}\langle z,x^*\rangle} - 1 - iuk(u)\operatorname{Re}\langle z,x^*\rangle\right)\frac{du}{|u|^{1+\alpha}}\xi(dz)\right) \tag{4.2}$$

where $k(u)$ is a truncation function and $c_\alpha > 0$. Hence, in the framework of Section 3, the random variable $X = \int_E z\Lambda(dz)$ has an α-stable distribution when $\sigma^2 \equiv 0$ and $\rho(z, du)$ takes the form

$$\rho(z, du) = c_\alpha \mathbf{1}_{\{u>0\}}|u|^{-1-\alpha} + c_\alpha \mathbf{1}_{\{u<0\}}|u|^{-1-\alpha}, \qquad z \in E, \tag{4.3}$$

see the discussion following [RZ96, Theorem 4].

In addition, if E is a Banach space of stable type $\alpha \in (0, 2)$, by [Woy19, Remark 6.10.2], any α-stable random variable on E can be represented as the random integral $\int_{\mathcal{S}_E} z\Lambda(dz)$, with characteristic functional (3.1).

Corollary 4.1 *Assume that the Banach space E is of stable type $\alpha \in (0, 2)$, $\alpha \neq 1$, and let μ denote the α-stable distribution with characteristic functional (3.1), where $\rho(s, du)$ is given by (4.3). A bounded linear operator $T : E \to E$ that leaves μ invariant is strongly mixing with respect to μ if and only if for every $x^* \in E^*$ we have*

$$\lim_{n\to\infty} \int_E |\operatorname{Re}\langle z, x^*\rangle|^{\alpha/2} |\operatorname{Re}\langle z, T^{*n}x^*\rangle|^{\alpha/2}\xi(dz) = 0$$

and

$$\lim_{n\to\infty} \int_E |\operatorname{Re}\langle z, x^*\rangle|^{\alpha/2} |\operatorname{Im}\langle z, T^{*n}x^*\rangle|^{\alpha/2}\xi(dz) = 0, \qquad x^* \in E^*.$$

Proof. For all $u \in \mathbb{R}$ and $z \in E$, we have

$$\begin{aligned} V(u, z) &= \int_{-\infty}^{\infty} \min(|uy|, 1)\rho(z, dy) \\ &= 2c_\alpha \int_0^{\infty} \min(|uy|, 1)\frac{dy}{y^{\alpha+1}} \\ &= 2c_\alpha \left(\frac{1}{2-\alpha} + \frac{1}{\alpha}\right)|u|^\alpha, \end{aligned}$$

and we conclude from Theorem 3.2. □

Example

We construct an example of a strongly mixing operator T satisfying the conditions of Corollary 4.1 using weighted shifts as in § 5.5.2 of [BM09]. Let $p \geq 1$, and let $E := \ell^p(\mathbb{Z})$ be the Banach space of complex sequences $(z_i)_{i\in\mathbb{Z}}$ such that

$$\|z\|_p := \left(\sum_{j=-\infty}^{\infty} |z_j|^p \right)^{1/p} < \infty,$$

with canonical basis $(e_n)_{n\in\mathbb{Z}}$. Given $\alpha \in (1,2)$, consider a positive weight sequence $(\omega_n)_{n\in\mathbb{Z}}$ such that:

- there exist $c_1, c_2 \in (0,1)$ such that

$$\begin{cases} \omega_i \leq c_1, & i \geq 1, \\ \omega_i = 1, & i = -1, 0, \\ \omega_i \geq 1/c_2, & i \leq -2, \end{cases}$$

- $(\omega_{-i})_{i\geq 1}$ is strictly increasing,
- if $p = \alpha$, then in addition there exists $d \in (0,1)$ such that $\omega_i > d$ for all $i > 0$,

and let

$$k_n = \begin{cases} \prod_{0\leq i\leq n} \omega_i, & n \geq 0, \\ \prod_{n<i\leq 0} \frac{1}{\omega_i}, & n \leq -1, \end{cases}$$

with $k_{-1} = k_0 = \omega_0 = 1$. Given $(\theta_{1,n})_{n\in\mathbb{Z}}$ and $(\theta_{2,n})_{n\in\mathbb{Z}}$ two sequences of independent standard $S\alpha S$ random variables with characteristic function $e^{-|t|^\alpha}$, let

$$X := \sum_{n=-\infty}^{\infty} (\theta_{1,n} + i\theta_{2,n}) k_n e_n,$$

be represented in distribution as

$$X := \int_{\ell^p(\mathbb{Z})} z \Lambda(dz),$$

where Λ is the $S\alpha S$ random measure with control measure

$$\xi(dz) = \frac{1}{2} \sum_{n=-\infty}^{\infty} k_n^\alpha \big(\delta_{e_n}(dz) + \delta_{-e_n}(dz) + \delta_{ie_n}(dz) + \delta_{-ie_n}(dz) \big).$$

We observe that from Lemma 4.3 below, X is well-defined on $\ell^p(\mathbb{Z})$ since we have

$$k_n^\alpha \le \begin{cases} c_1^{\alpha n}, & n \ge 0, \\ 1, & n = -1, \\ c_2^{\alpha(|n|-2)}, & n \le -2, \end{cases}$$

hence $(k_n^\alpha)_{n\in\mathbb{Z}}$ is bounded by geometric sequences, and

$$\sum_{n=-\infty}^{\infty} \|k_n e_n\|_p^\alpha = \sum_{n=-\infty}^{\infty} k_n^\alpha < \infty,$$

so that the sequence $(k_n e_n)_{n\in\mathbb{Z}}$ can be used to define a measure. Moreover, we have

$$\begin{aligned}\mathbb{E}\Big[e^{i \operatorname{Re}\langle \int_E z\Lambda(dz), x^*\rangle}\Big] &= \exp\left(-\int_E |\operatorname{Re}\langle z, x^*\rangle|^\alpha \xi(dz)\right) \\ &= \exp\left(-\sum_{n=-\infty}^{\infty} k_n^\alpha |\operatorname{Re}\langle e_n, x^*\rangle|^\alpha + \sum_{n=-\infty}^{\infty} k_n^\alpha |\operatorname{Re}\langle ie_n, x^*\rangle|^\alpha\right),\end{aligned}$$

hence the distribution μ of X is α-stable with control measure $\xi(dz)$. As an application of Corollary 4.1, we show that T is strongly mixing with respect to the $S\alpha S$ distribution μ of X defined in (3.1) when $\alpha \in (1,2)$, and with respect to the Gaussian measure with covariance R given by $Re_n = k_n^2 e_n$, $n \in \mathbb{Z}$, when $\alpha = 2$.

Proposition 4.2 *Let $\alpha \in (1,2]$. The weighted forward shift operator $T : \ell^p(\mathbb{Z}) \to \ell^p(\mathbb{Z})$ defined as*

$$Te_n := \omega_{n+1} e_{n+1}, \qquad n \in \mathbb{Z},$$

leaves the $S\alpha S$ distribution μ invariant and is strongly mixing with respect to μ.

Proof. We first check that the $S\alpha S$ distribution μ of X is invariant with respect to T. Note that

- $k_n \omega_{n+1} = \omega_{n+1} \prod_{i=0}^{n} \omega_i = \prod_{i=0}^{n+1} \omega_i = k_{n+1}$, $n \ge 0$,
- $k_{-1}\omega_0 = 1 = k_0$,
- $k_n \omega_{n+1} = \prod_{i=n+1}^{0} \frac{\omega_{n+1}}{\omega_i} = \prod_{i=n+2}^{0} \frac{1}{\omega_i} = k_{n+1}$, $n \le -2$,

hence we have

$$TX = \sum_{n=-\infty}^{\infty} (\theta_{1,n} + i\theta_{2,n}) k_n T e_n = \sum_{n=-\infty}^{\infty} (\theta_{1,n} + i\theta_{2,n}) k_{n+1} e_{n+1} \stackrel{d}{=} X,$$

so that μ is invariant under T. Next, for any $x^* \in \ell^p(\mathbb{Z})^*$ and $n \geq 1$ we have

$$\begin{aligned}\int_{\ell^p(\mathbb{Z})} |\mathrm{Re}\,\langle z, x^*\rangle\,\mathrm{Re}\,\langle z, T^{*n}x^*\rangle|^{\alpha/2}\,\xi(dz) &= \sum_{i=-\infty}^{\infty} k_i^\alpha\,|\mathrm{Re}\,\langle e_i, x^*\rangle\,\mathrm{Re}\,\langle e_i, T^{*n}x^*\rangle|^{\alpha/2}\\ &= \sum_{i=-\infty}^{\infty} k_i^\alpha\,|\mathrm{Re}\,\langle e_i, x^*\rangle\,\mathrm{Re}\,\langle e_{i+n}, x^*\rangle\,\omega_{i+1}\ldots\omega_{i+n}|^{\alpha/2}\\ &\leq \|x^*\|_p^\alpha \sum_{i=-\infty}^{\infty} k_i^\alpha \prod_{j=i+1}^{i+n} \omega_j^{\alpha/2},\end{aligned}$$

where the bound

$$k_i^\alpha \prod_{j=i+1}^{i+n} \omega_j^{\alpha/2} \leq \begin{cases} k_i^\alpha, & i \geq 0\\ \displaystyle\prod_{j=i+1}^{0} \frac{1}{\omega_j^\alpha} \prod_{j=i+1}^{i+n} \omega_j^{\alpha/2} \leq k_i^\alpha, & -n < i \leq -1\\ \displaystyle\prod_{j=i+1}^{0} \frac{1}{\omega_j^{\alpha/2}} \prod_{j=i+n+1}^{0} \frac{1}{\omega_j^{\alpha/2}} \leq k_i^{\alpha/2}, & i \leq \min(-1, -n),\end{cases}$$

is uniform in $n \geq 1$ and $\sum k_i^{\alpha/2} < \infty$. Therefore, we have

$$\lim_{n\to\infty} \sum_{i=-\infty}^{\infty} k_i^\alpha \prod_{j=i+1}^{i+n} \omega_j^{\alpha/2} = \sum_{i=-\infty}^{\infty} \lim_{n\to\infty} k_i^\alpha \prod_{j=i+1}^{i+n} \omega_j^{\alpha/2} = 0,$$

which yields

$$\lim_{n\to\infty} \int_{\ell^p(\mathbb{Z})} |\,\mathrm{Re}\,\langle z, x^*\rangle\,|^{\alpha/2} |\,\mathrm{Re}\,\langle z, T^{*n}x^*\rangle\,|^{\alpha/2} \xi(dz) = 0.$$

By a similar argument, we have

$$\lim_{n\to\infty} \int_{\ell^p(\mathbb{Z})} |\,\mathrm{Re}\,\langle z, x^*\rangle\,|^{\alpha/2} |\,\mathrm{Im}\,\langle z, T^{*n}x^*\rangle\,|^{\alpha/2} \xi(dz) = 0,$$

and we conclude from Corollary 4.1. □

The next lemma has been used in the construction of the example of Proposition 4.2.

Lemma 4.3 *For any $\alpha \in (1, 2]$ and $p \geq 1$, the random variable X is almost surely $\ell^p(\mathbb{Z})$-valued.*

Proof. Since $\sum_{n=-\infty}^{\infty} k_n^\alpha < \infty$, if $\alpha \neq p$ we have $\sum_{n=-\infty}^{\infty} |\theta_{1,n} k_n|^p < \infty$ almost surely by in [Proposition (XXVI,3;3) page XXVI.9][Sch70]. If $\alpha = p$, $(\omega_n)_{n\in\mathbb{Z}}$ is bounded from below by some sufficiently small $d \in (0, 1)$ such that

$$1 + |\log k_n| = 1 + \left|\sum_{j=0}^{n} \log \omega_j\right| \leq 1 + n|\log d|, \quad n \geq 0,$$

so that

$$\begin{aligned}\sum_{n=1}^{\infty} k_n^{\alpha} (1 + |\log k_n|) &= \sum_{n=1}^{\infty} (k_n^{\alpha} + n k_n^{\alpha} |\log d|) \\ &= \sum_{n=1}^{\infty} k_n^{\alpha} + |\log d| \sum_{n=1}^{\infty} n k_n^{\alpha} \\ &\le (1 + |\log d|) \sum_{n=1}^{\infty} n (c_1^{\alpha})^n \\ &< \infty.\end{aligned}$$

On the other hand, we have

$$1 + |\log k_n| = 1 + \left| \sum_{n<j\le 0} \log \omega_j \right| \le 1 + n \log M, \qquad n \le -1,$$

where $M > 1$ is any upper bound of $(\omega_n)_{n\in\mathbb{Z}}$, so that

$$\sum_{n=-\infty}^{-1} k_n^{\alpha} (1 + |\log k_n|) = \sum_{n=-\infty}^{-1} (k_n^{\alpha} + n k_n^{\alpha} \log M) < \infty,$$

and we conclude again from [Sch70, Proposition (XXVI,3;3)] in the case $\alpha = p$, and the same argument holds also for $\sum_{n=-\infty}^{\infty} |\theta_{2,n} k_n|^p$. The almost sure convergence of X in $\ell^p(\mathbb{Z})$ then follows from the bound

$$\sum_{n=-\infty}^{\infty} |(\theta_{1,n} + i\theta_{2,n}) k_n|^p \le 2^{p-1} \left(\sum_{n=-\infty}^{\infty} |\theta_{1,n} k_n|^p + \sum_{n=-\infty}^{\infty} |\theta_{2,n} k_n|^p \right) < \infty.$$

□

5 Invariance of stable measures

In this section, we provide sufficient conditions for the invariance of a class of stable measures, as required in Corollary 4.1. The following result extends Proposition 3.1 of [BG07] to the stable setting.

Proposition 5.1 *Let E be a complex separable Banach space. Let $T : E \to E$ be a bounded linear operator such that the eigenvectors of T associated to unimodular eigenvalues span a dense subspace of E. Then for any $1 < \alpha < 2$, T admits a non-degenerate invariant $S\alpha S$ measure.*

Proof. Let $\alpha \in (1,2)$, $q := \alpha/(\alpha-1) > 1$, and let $(x_n)_{n\geq 0}$ be a (linearly independent) sequence of $\mathbb{T}$-eigenvectors with eigenvalues (λ_n), such that $\|x_n\| = 1/n$ and $|\lambda_n| = 1$, $n \geq 0$, and $\overline{\text{span}\{x_n : n \geq 0\}} = E$. Define the mapping $J : \ell^\alpha(\mathbb{N}) \to E$ by $J(e_n) = x_n$, $n \geq 0$, where (e_n) is the canonical basis of $\ell^\alpha(\mathbb{N})$, and let D denote the diagonal operator on $\ell^\alpha(\mathbb{N})$ defined by $D(e_n) = \lambda_n e_n$, $n \geq 0$. We note that J has dense range, and it is bounded on $\ell^\alpha(\mathbb{N})$ because for any $a = (a_n)_{n\geq 0} \in \ell^\alpha(\mathbb{N})$, we have, by Hölder's inequality,

$$\begin{aligned} \|Ja\| &= \left\| J \sum_{n=0}^{\infty} a_n e_n \right\| = \left\| \sum_{n=0}^{\infty} a_n x_n \right\| \leq \sum_{n=0}^{\infty} |a_n| \, \|x_n\| \\ &\leq \left(\sum_{n=0}^{\infty} |a_n|^\alpha \right)^{1/\alpha} \left(\sum_{n=0}^{\infty} \|x_n\|^q \right)^{1/q} = \|a\|_{\ell^\alpha(\mathbb{N})} \left(\sum_{n=0}^{\infty} n^{-q} \right)^{1/q}. \end{aligned}$$

In addition to boundedness and injectivity, by [Rud91, Theorem 2.11] we have $J(\ell^p(\mathbb{N})) = E$ since J has dense range, hence J is a continuous linear bijection. Next, for $x \in \ell^\alpha(\mathbb{N})$ of the form $x := \sum_{j\geq 0} c_j e_j$, we have

$$TJx = T\left(\sum_{j\geq 0} c_j x_j \right) = \sum_{j\geq 0} \lambda_j c_j x_j$$

and

$$JDx = T\left(\sum_{j\geq 0} \lambda_j e_j \right) = \sum_{j\geq 0} \lambda_j c_j x_j,$$

hence $TJ = JD$. Let $(v_j)_{j\geq 0}$ denote an i.i.d. sequence of complex isotropic non-degenerate $S\alpha S$ random variables written as $v_j = v_j^{(1)} + i v_j^{(2)}$ where $v_j^{(1)}, v_j^{(2)}$ are $S\alpha S$, according to [ST94, Theorem 2.1.2 and Definition 2.6.1], and written as

$$v_j^{(1)} = s\theta_1, \qquad v_j^{(2)} = s\theta_2,$$

for a same constant s, where θ_1, θ_2 are standard $S\alpha S$ random variables. Since $\alpha \in (1,2)$, we have

$$\sum_{n=1}^{\infty} n^{-\alpha} (1 + \ln n) < \infty,$$

hence by [Sch70, Proposition XXVI, 3;3] we have $\sum_{j=1}^{\infty} |v_j|^\alpha \|x_j\| < \infty$ almost surely, hence

$$X := \sum_{j=1}^{\infty} \|x_j\| v_j e_j$$

belongs almost surely to $\ell^\alpha(\mathbb{N})$. We have $X \sim S\alpha S$ in E and $DX \overset{d}{=} X$, therefore, the distribution m of X is ($S\alpha S$) non-degenerate and invariant under D. Next, since $T^{-1} = JD^{-1}J^{-1}$, letting $\mu = J(m)$ denote the push-forward measure of m by J, defined as $\mu(B) = m(J^{-1}(B))$ for Borel sets $B \subset E$, we have

$$\begin{aligned}\mu(T^{-1}(B)) &= \mu(JD^{-1}J^{-1}(B)) \\ &= m(J^{-1}JD^{-1}J^{-1}(B)) \\ &= m(D^{-1}J^{-1}(B)) \\ &= m(J^{-1}(B)) \\ &= \mu(B),\end{aligned}$$

which proves the T-invariance of μ. Symmetry of μ holds because

$$\mu(-B) = m(J^{-1}(-B)) = m(-J^{-1}(B)) = m(J^{-1}(B)) = \mu(B).$$

Finally, by definition of symmetric α-stability, for two independent copies X_1, X_2 of X and any $a, b > 0$ we have $aX_1 + bX_2 \overset{d}{=} (a^\alpha + b^\alpha)^{1/\alpha}X$. From this, by linearity we have

$$aJ(X_1) + bJ(X_2) = J(aX_1 + bX_2) \overset{d}{=} J((a^\alpha + b^\alpha)^{1/\alpha}X) = (a^\alpha + b^\alpha)^{1/\alpha}J(X),$$

that is, $J(X)$ is again an α-stable random vector and we conclude that μ is a non-degenerate invariant $S\alpha S$ measure for T. □

In the remainder of this section, we consider the following class of α-stable distributions.

Definition 5.2 *Given H a Hilbert space and $\alpha \in (0, 2]$, let $\mu_{S,\xi}$ denote the α-stable distribution on H having a finite control measure*

$$m_{S,\xi}(dz) := |\langle Sz, z\rangle|^{\alpha/2}\,\xi(dz),$$

on the unit sphere $\mathcal{S}_H$ in H, where

- *ξ is a positive measure on $\mathcal{S}_H$, and*
- *$S : H \to H$ is a non-negative definite bounded operator.*

We note that by [Lin86, Corollary 7.5.2], every finite Radon measure on the unit sphere $\mathcal{S}_E$ of a Banach space E of stable type $\alpha \in (0, 2)$ is the control measure of an α-stable distribution on E, and since H is a separable Hilbert space, by [Woy19, Corollary 6.5.1-(ii)

and Proposition 7.1.1] it has stable type α for all $\alpha \in (0,2]$. Hence in the framework of (4.1), we have

$$\int_E e^{i\,\mathrm{Re}\langle z,x^*\rangle}\mu_{S,\xi}(dz) = \exp\left(-\int_{\mathcal{S}_E} |\mathrm{Re}\,\langle z,x^*\rangle|^\alpha\, |\langle Sz,z\rangle|^{\alpha/2}\,\xi(dz)\right), \quad x^* \in E^*.$$

We also note that if $\xi(dz)$ is symmetric then the measure $m_{S,\xi}(dz) = |\langle Sz,z\rangle|^{\alpha/2}\,\xi(dz)$ is also symmetric, hence $\mu_{S,\xi}$ is $S\alpha S$, see for instance [Lin86, Theorem 6.4.4]. The following proposition provides sufficient conditions for the invariance of $\mu_{S,\xi}$ under a linear operator T on H, as an extension of [BG06, Proposition 3.15] which deals with the Gaussian case.

Proposition 5.3 *Let T be a bounded invertible operator on H, and assume that*

a) $T^*Tx = \|Tx\|^2 x$, $\xi(dx)$-a.e.,

b) ξ is invariant by $x \mapsto Tx/\|Tx\|$ on H, and

c) $TST^* = S$.

Then, T admits $\mu_{S,\xi}$ as invariant α-stable measure.

Proof. Since T is invertible, we have that $Tx = \|Tx\|^2 T^{-1*}x$ for all $x \in H$, and $S = T^{-1}ST^{-1*}$. Hence, for any $y \in H$ we have

$$\begin{aligned}
\int_{\mathcal{S}_H} |\mathrm{Re}\,\langle z,T^*y\rangle|^\alpha\, |\langle Sz,z\rangle|^{\alpha/2}\,\xi(dz) &= \int_{\mathcal{S}_H} |\mathrm{Re}\,\langle Tz,y\rangle|^\alpha \left|\left\langle ST^{-1*}z, T^{-1*}z\right\rangle\right|^{\alpha/2}\xi(dz)\\
&= \int_{\mathcal{S}_H} \left|\mathrm{Re}\left\langle \frac{Tz}{\|Tz\|}, y\right\rangle\right|^\alpha \left|\left\langle S\|Tz\|T^{-1*}z, \|Tz\|T^{-1*}z\right\rangle\right|^{\alpha/2}\xi(dz)\\
&= \int_{\mathcal{S}_H} \left|\mathrm{Re}\left\langle \frac{Tz}{\|Tz\|}, y\right\rangle\right|^\alpha \left|\left\langle S\frac{Tz}{\|Tz\|}, \frac{Tz}{\|Tz\|}\right\rangle\right|^{\alpha/2}\xi(dz)\\
&= \int_{\mathcal{S}_H} |\mathrm{Re}\,\langle z,y\rangle|^\alpha\, |\langle Sz,z\rangle|^{\alpha/2}\,\xi(dz).
\end{aligned}$$

Hence, we have

$$\begin{aligned}
\widehat{\mu}(T^*y) &= \int_E e^{i\,\mathrm{Re}\langle z,T^*y\rangle}\mu(dz)\\
&= \exp\left(-\int_{\mathcal{S}_H} |\mathrm{Re}\,\langle z,T^*y\rangle|^\alpha\, |\langle Sz,z\rangle|^{\alpha/2}\,\xi(dz)\right)\\
&= \exp\left(-\int_{\mathcal{S}_H} |\mathrm{Re}\,\langle z,y\rangle|^\alpha\, |\langle Sz,z\rangle|^{\alpha/2}\,\xi(dz)\right)\\
&= \int_E e^{i\,\mathrm{Re}\langle z,y\rangle}\mu(dz), \quad y \in H^*,
\end{aligned}$$

so μ is invariant by T. □

As a consequence of Proposition 5.3, we have the following corollary.

Corollary 5.4 *Let T be a bounded invertible unitary operator on H, and assume that*

a) ξ is invariant by T, and

b) $TST^* = S$.

Then, T admits $\mu_{S,\xi}$ as invariant α-stable measure.

We also have the next non-degeneracy result.

Proposition 5.5 *Let T be a bounded invertible operator on H. In addition to the assumptions of Proposition 5.3, suppose that*

d) ξ is symmetric, and

e) $H = \overline{\operatorname{span}\operatorname{supp}(m)}$, where $m(dx) = |\langle Sx, x\rangle|^{\alpha/2}\,\xi(dx)$.

Then, T admits $\mu_{S,\xi}$ as a non-degenerate invariant symmetric α-stable measure.

Proof. If ξ is symmetric, then the measure $|\langle Sx, x\rangle|^{\alpha/2}\,\xi(dx)$ is also symmetric, hence μ is $S\alpha S$, see for instance [Lin86, Theorem 6.4.4]. To show non-degeneracy we note that the support of μ is the closed linear span $W := \overline{\operatorname{span}\operatorname{supp}(m)}$ of the support of m. Indeed, for all $y \in H$ we have

$$\begin{aligned}
\int_H e^{i\operatorname{Re}\langle z,y\rangle}\mu(dz) &= \exp\left(-\int_{W\cap\mathcal{S}_H} |\operatorname{Re}\langle z, y\rangle|^{\alpha}\, m(dz) - \int_{W^c\cap\mathcal{S}_H} |\operatorname{Re}\langle z, y\rangle|^{\alpha}\, m(dz)\right)\\
&= \exp\left(-\int_{W\cap\mathcal{S}_H} |\operatorname{Re}\langle z, y\rangle|^{\alpha}\, m(dz)\right)\\
&= \exp\left(-\int_{\mathcal{S}_W} |\operatorname{Re}\langle z, y\rangle|^{\alpha}\, m(dz)\right)\\
&= \int_W e^{i\operatorname{Re}\langle z,y\rangle}\mu_0(dz),
\end{aligned}$$

where the last step follows since W is a Banach space so the symmetric control measure m defines an α-stable probability measure μ_0 on W. By uniqueness of the symmetric control measure, it follows that μ and μ_0 must coincide on W. That is, $\mu(W) = 1$, so $\mu(W^c) = 0$, so that indeed $\operatorname{supp}(\mu) = \overline{\operatorname{span}\operatorname{supp}(m)}$. □

Next, we present an example of operator T satisfying the conditions of Propositions 5.3 and 5.5.

Example 5.6 *Let $\alpha \in (1,2)$, and consider a sequence $(\omega_n)_{n\in\mathbb{Z}}$ of positive weights satisfying*

$$\sum_{i=1}^{\infty} \omega_0^{\alpha} \cdots \omega_{i-1}^{\alpha} < \infty \qquad \text{and} \qquad \sum_{i=-\infty}^{-1} \omega_i^{-\alpha} \cdots \omega_{-1}^{-\alpha} < \infty. \tag{5.1}$$

Then, the forward shift operator T on $\ell^2(\mathbb{Z})$ defined by $T(e_n) = \omega_n e_{n+1}$, $n \in \mathbb{Z}$, admits the non-degenerate invariant $S\alpha S$ measure μ with control measure $|\langle Sz, z\rangle|^{\alpha/2}\,\xi(dz)$, where

$$\xi(dz) := \frac{c}{2} \sum_{n=-\infty}^{\infty} (\delta_{e_n}(dz) + \delta_{-e_n}(dz) + \delta_{ie_n}(dz) + \delta_{-ie_n}(dz)),$$

for some nonzero constant c, and S is the diagonal operator on $\ell^2(\mathbb{Z})$ with diagonal $(a_{nn})_{n\in\mathbb{Z}}$ given by

$$a_{nn} = \begin{cases} a_{00} \prod_{j=0}^{n-1} \omega_j^2, & n > 0, \\ a_{00} \prod_{j=n}^{-1} \omega_j^{-2}, & n < 0, \end{cases}$$

for some $a_{00} > 0$.

Proof. First, we note that $(a_{nn})_{n\in\mathbb{Z}} \in \ell^{\alpha/2}(\mathbb{Z})$ from (5.1), hence S is bounded and positive. Next, we show that the measure

$$\begin{aligned} m(dz) :&= |\langle Sz, z\rangle|^{\alpha/2}\,\xi(dz) \\ &= \frac{c}{2} \sum_{n=-\infty}^{\infty} |\langle Se_n, e_n\rangle|^{\alpha/2} (\delta_{e_n}(dz) + \delta_{-e_n}(dz) + \delta_{ie_n}(dz) + \delta_{-ie_n}(dz)) \\ &= \frac{c}{2} \sum_{n=-\infty}^{\infty} a_{ii}^{\alpha/2} (\delta_{e_n}(dz) + \delta_{-e_n}(dz) + \delta_{ie_n}(dz) + \delta_{-ie_n}(dz)). \end{aligned}$$

is the control measure of an α-stable distribution. For this we note that, letting $(\theta_{1,n})_{n\in\mathbb{Z}}$ and $(\theta_{2,n})_{n\in\mathbb{Z}}$ denote two sequences of independent real-valued standard $S\alpha S$ random variables, the random series

$$c^{1/\alpha} \sum_{n=-\infty}^{\infty} \theta_{1,n} \sqrt{a_{nn}}\, e_n$$

is almost surely $\ell^2(\mathbb{Z})$-valued since by [Sch70, Proposition XXVI, 3;3] we have

$$\sum_{n=-\infty}^{\infty} \theta_{1,n}^2 |a_{nn}| < \infty$$

almost surely as $(\sqrt{a_{nn}})_{n\in\mathbb{Z}} \in \ell^{\alpha}(\mathbb{Z})$. Likewise,

$$c^{1/\alpha} \sum_{n=-\infty}^{\infty} i\theta_{2,n} \sqrt{a_{nn}}\, e_n$$

is almost surely $\ell^2(\mathbb{Z})$-valued, thus the random variable

$$X := c^{1/\alpha} \sum_{n=-\infty}^{\infty} (\theta_{1,n} + i\theta_{2,n})\sqrt{a_{nn}} e_n$$

is almost surely $\ell^2(\mathbb{Z})$-valued in the same way as Lemma 4.3. Furthermore, X is a $S\alpha S$ random variable, and by [LT91, Page 131] its control measure is m. Finally, we check that the conditions of Propositions 5.3 and 5.5 are satisfied.

a) For all $n \in \mathbb{Z}$ we have $T(\pm e_n)/\|T(\pm e_n)\| = \pm e_{n+1}$, hence $\xi(\{\pm e_n\}) = \xi(\{\widetilde{T}^{-1}(\pm e_n)\}) = c$, $n \in \mathbb{Z}$, where $\widetilde{T}$ denotes the mapping $x \mapsto Tx/\|Tx\|$. A similar argument holds for $\pm ie_n$. Furthermore, if $x \neq \pm e_n$ or $\pm ie_n$ for any $n \in \mathbb{Z}$ then $T(x)/\|T(x)\| \neq \pm e_n$ or $\pm ie_n$ for any $n \in \mathbb{Z}$, hence ξ is invariant by $x \mapsto T(x)/\|T(x)\|$ on H.

b) We have $Te_n = \omega_n e_{n+1}$ so that $\|Te_n\|^2 e_n = \omega_n^2 e_n$, $n \geq 0$. On the other hand, we have $T^*T = \mathrm{Diag}(\ldots, \omega_1^2, \omega_2^2, \ldots)$, so that $T^*Te_n = \omega_n^2 e_n$. Similar arguments hold for $-e_n$ and $\pm ie_n$, $n \geq 0$.

c) We note that $(a_{ij})_{i,j\in\mathbb{Z}}$ satisfies $a_{ij} = \omega_{i-1}\omega_{j-1}a_{i-1,j-1}$, $i, j \in \mathbb{Z}$, which reads

$$\begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \iddots \\ \cdots & \omega_0^2 a_{00} & \omega_0\omega_1 a_{01} & \omega_0\omega_2 a_{02} & \cdots \\ \cdots & \omega_1\omega_0 a_{10} & \omega_1^2 a_{11} & \omega_1\omega_2 a_{12} & \cdots \\ \cdots & \omega_2\omega_0 a_{20} & \omega_2\omega_1 a_{21} & \omega_2^2 a_{22} & \cdots \\ \iddots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} = \begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \iddots \\ \cdots & a_{11} & a_{12} & a_{13} & \cdots \\ \cdots & a_{21} & a_{22} & a_{23} & \cdots \\ \cdots & a_{31} & a_{32} & a_{33} & \cdots \\ \iddots & \vdots & \vdots & \cdots & \ddots \end{bmatrix},$$

and shows that $TST^* = S$.

The remaining points (d) and (e) hold by construction of ξ and m. $\square$

6 Invariance under σ-spanning operators

In this section, we provide sufficient conditions for invariance of stable measures in terms of eigenvalues and eigenvectors, in the framework of σ-spanning operators.

Definition 6.1 *Let σ be a probability measure on $\mathbb{T}$. A bounded linear operator $T : H \to H$ has a σ-spanning set of eigenvectors associated to unimodular eigenvalues if for every Borel subset $A \subset \mathbb{T}$ with $\sigma(A) = 1$, the eigenspaces* $\ker(T - \lambda I)$*, $\lambda \in A$, span a dense subspace of H. If such a σ exists for T, we say T is σ-spanning.*

If T is injective, from [BG06, Propositions 3.15, 3.18] there exists a bounded self-adjoint positive operator S of trace class such that $TST^* = S$. The next lemma provides sufficient conditions for (a)-(b) to be satisfied in Proposition 5.3. In the sequel, we write $\widetilde{T}$ to denote the mapping $x \mapsto Tx/\|Tx\|$.

Lemma 6.2 *Suppose that* $\ker T = \{0\}$, *and let* $(\lambda_i)_i$ *denote the singular values of* T. *Then,*

a) there exists a T^*T*-eigenbasis* $\{y_k\}_k$ *of* $\overline{\operatorname{span} \bigcup_k \ker(T^*T - \lambda_k I)}$ *such that*

$$T^*Ty_k = \|Ty_k\|^2 y_k, \qquad k \geq 1.$$

b) If in addition $\widetilde{T}(\{y_k\}_k) = \{y_k\}_k$, *then any constant measure* ξ *concentrated on* $\{-y_k, y_k\}_k$ *is invariant by* $x \mapsto Tx/\|Tx\|$.

Proof. $a)$ Since $\ker T = \{0\}$, T^*T is positive definite. Let $\{x_k\}_k$ be an eigenbasis of $\overline{\operatorname{span} \bigcup_k \ker(T^*T - \lambda_k I)}$. For every k there exists i_k such that $T^*Tx_k = \lambda_{i_k} x_k$, and for any $c_k \in \mathbb{C}$ such that $|c_k|^2 = \lambda_{i_k}/\|Tx_k\|^2$. Letting $y_k := c_k x_k$, we have

$$T^*Ty_k = T^*T(c_k x_k) = c_k \|T(c_k x_k)\|^2 x_k = \|Ty_k\|^2 y_k, \quad \forall k.$$

$b)$ Observe that $\widetilde{T}_{|\{y_k\}_k}$ is surjective by assumption, therefore it suffices to show that it is injective. If this was not the case we would have $\|Ty_j\| Ty_i = \|Ty_i\| Ty_j$ for some $y_i, y_j \in \{y_k\}_k$, $i \neq j$, which would imply $y_i = \|Ty_i\| y_j / \|Ty_j\|$ since $\ker T = \{0\}$, therefore contradicting the linear independence of $\{y_k\}_k$. The conclusion follows by repeating the above argument for $-y_k$, $\forall k$. □

We also note that Condition (b) in Proposition 5.3 is satisfied by σ-spanning operators. As a consequence of Propositions 5.3, 5.5 and Lemma 6.2, we obtain the following corollary.

Corollary 6.3 *Let* T *be a* σ*-spanning bounded invertible operator on* H, *and let* $(\lambda_i)_i$ *denote the singular values of* T. *Suppose that*

a) $H = \overline{\operatorname{span} \bigcup_k \ker(T^*T - \lambda_k I)}$,

b) H *admits a* T^*T*-eigenbasis* $\{y_k\}_k$, *such that*

c) $T^*Ty_k = \|Ty_k\|^2 y_k$, $\forall k$, *and* $\widetilde{T}(\{y_k\}_k) = \{y_k\}_k$.

Then, T *admits* $\mu_{S,\xi}$ *as symmetric invariant non-degenerate* α*-stable measure, where* ξ *is the constant measure concentrated on* $\{-y_k, y_k\}_k$.

Finally, we present an example of operator T satisfying the conditions of Corollary 6.3, as a specialization of Example 5.6.

Example 6.4 *In the framework of Example 5.6, take* $\omega_0 = 1$ *and*

$$\omega_n = \frac{1}{2},\ \omega_{-n} = 2, \qquad n \geq 1.$$

For any $\theta \in [0, 2\pi)$*, the* $\ell^2(\mathbb{Z})$*-vector*

$$v_\theta := \left(\ldots, \frac{1}{2^2} e^{-3i\theta}, \frac{1}{2} e^{-2i\theta}, e^{-i\theta}, 1, \frac{1}{2} e^{i\theta}, \frac{1}{2^2} e^{2i\theta}, \ldots \right)$$

is a T*-eigenvector with eigenvalue* $e^{i\theta}$*. Let* $A \subset \mathbb{T}$ *be a Borel set such that* $\sigma(A) = 1$*, where* σ *is the uniform probability measure on* $\mathbb{T}$*. For any vector* $z = (z_n)_{n \in \mathbb{Z}}$ *in* $\ell^2(\mathbb{Z})$ *such that* $\langle v_\theta, z \rangle = 0$*, i.e.*

$$f(\theta) := z_0 + \sum_{n=1}^{\infty} \frac{1}{2^n} e^{in\theta} z_n + \sum_{n=-\infty}^{-1} \frac{1}{2^{-n+1}} e^{in\theta} z_n = 0, \qquad e^{i\theta} \in A,$$

by Parseval's identity we have

$$|z_0|^2 + \sum_{n=1}^{\infty} \frac{1}{2^{2n}} |z_n|^2 + \sum_{n=-\infty}^{-1} \frac{1}{2^{-2n+2}} |z_n|^2 = \int_0^{2\pi} |f(\theta)|^2 d\theta = 0,$$

hence $z = 0$*. This shows that* $\operatorname{span}\{v_\theta : e^{i\theta} \in A\}$ *is dense in* $\ell^2(\mathbb{Z})$*, since its orthogonal complement is* $\{0\}$*, and therefore* T *is* σ*-spanning. Next, since* T^*T *is diagonal on* $\{e_n\}_{n \in \mathbb{Z}}$ *with eigenvalue equations*

$$T^*Te_n = \left\{ \begin{array}{ll} e_n/4, & n \geq 1 \\ e_0, & n = 0, \\ 4e_n, & n \leq -1 \end{array} \right\} = \|Te_n\|^2 e_n, \qquad n \in \mathbb{Z},$$

we have the singular values λ_j *of* T *take only the values* $\frac{1}{4}, 1, 4$*, so that*

$$\begin{aligned} \ell^2(\mathbb{Z}) &= \overline{\operatorname{span}\left(\ker\left(T^*T - \frac{1}{4} I \right) \cup \ker(T^*T - I) \cup \ker(T^*T - 4I) \right)} \\ &= \overline{\operatorname{span} \bigcup_{j \in \mathbb{Z}} \ker(T^*T - \lambda_j I)}, \end{aligned}$$

and also $\widetilde{T} e_n = e_{n+1}$*,* $n \in \mathbb{Z}$*, which yields* $\widetilde{T}(\{e_n\}_{n \in \mathbb{Z}}) = \{e_n\}_{n \in \mathbb{Z}}$*.*

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