Extended Mellin integral representations for the absolute value of the gamma function

Nicolas Privault
Division of Mathematical Sciences
School of Physical and Mathematical Sciences
Nanyang Technological University
SPMS-MAS-05-43, 21 Nanyang Link
Singapore 637371

January 19, 2018

Abstract

We derive Mellin integral representations in terms of Macdonald functions for the squared absolute value $s \mapsto |\Gamma(a+is)|^2$ of the gamma function and its Fourier transform when $a < 0$ is non-integer, generalizing known results in the case $a > 0$. This representation is based on a renormalization argument using modified Bessel functions of the second kind, and it applies to the representation of the solutions of a Fokker-Planck equation.

Key words: Gamma function; Mellin transform; Hartman-Watson distribution.
Mathematics Subject Classification (2000): 32A26, 33C10, 33B15.

1 Introduction

The classical Mellin integral representation, or Kontorovich-Lebedev transform, given by

$$\Gamma(z+is)\,\Gamma(z-is) = 4\int_0^\infty \left(\frac{x}{2}\right)^{2z} K_{2is}(x)\frac{dx}{x}, \tag{1.1}$$

holds for $s \in \mathbb{R}$ and $z \in \mathbb{C}$ such that $\Re(z) > 0$, see e.g. relation (26), page 331 of [6]. Here

$$K_w(y) = \int_0^\infty e^{-y\cosh x}\cosh(wx)dx, \qquad y > 0, \quad w \in \mathbb{C}, \tag{1.2}$$

is the modifed Bessel function of the second kind, or Macdonald function.

In this paper we extend the integral representation (1.1) to all $z \in \mathbb{C}$ such that $\Re(z)$ takes non-integer negative values, and we deduce related extensions for the Fourier transform of $s \mapsto \Gamma(a+is)$. In particular, when $\Re(z) \in (-1,0)$ and $s \in \mathbb{R}$, we show that

$$\Gamma(z+is)\Gamma(z-is) = 4\int_0^\infty \left(\frac{x}{2}\right)^{2z} K_{2is}(x)\left(1+\left(\frac{2}{x}\right)^{2z}\frac{4z}{\Gamma(1-2z)}K_{2z}(x)\right)\frac{dx}{x}. \tag{1.3}$$

More generally, when $z \in \mathbb{C}$ with $\Re(z) \in (-n-1,-n)$ and $s \in \mathbb{R}$, we have

$$\Gamma(z+is)\Gamma(z-is) = 4\int_0^\infty \left(\frac{x}{2}\right)^{2z} K_{2is}(x)\left(1+\left(\frac{2}{x}\right)^{2z}\sum_{k=0}^{n}\frac{4(k+z)}{k!\Gamma(1-k-2z)!}K_{2k+2z}(x)\right)\frac{dx}{x} \tag{1.4}$$

for any $n \in \mathbb{N} = \{0,1,2,\ldots\}$, see Proposition 3.2 below. In particular, the introduction of additional terms in (1.3)-(1.4) ensures the integrability lacking in (1.1) when $\Re(z) < 0$.

On the other hand, Ramanujan showed in [9] that for $a > 0$ the Fourier transform of $s \mapsto |\Gamma(a+is)|^2$ satisfies the relation

$$\int_{-\infty}^{\infty} e^{-i\xi s}\left|\Gamma(a+is)\right|^2 ds = \sqrt{\pi}\Gamma(a)\Gamma(a+1/2)(\cosh(\xi/2))^{-2a}. \tag{1.5}$$

This relation has been extended for all $a \in (-1,0)$ as an integral expression in Theorem 1.2 of [3].

As an application of (1.3)-(1.4) we deduce other integral expressions for (1.5), for all non-integer negative values of a. Namely when $a \in (-1,0)$ we derive the integral representation

$$\int_{-\infty}^{\infty} e^{-i\xi s}\left|\Gamma(a+is)\right|^2 ds \tag{1.6}$$

$$= \frac{2\pi}{2^{2a}}\int_0^\infty x^{2a-1}\left(1+\left(\frac{2}{x}\right)^{2a}\frac{4a}{\Gamma(1-2a)}K_{2a}(x)\right)e^{-x\cosh(\xi/2)}dx, \qquad \xi \in \mathbb{R},$$

for the Fourier transform of $s \mapsto |\Gamma(a+is)|^2$. More generally, when $a \in (-n-1, -n)$, we have

$$\int_{-\infty}^{\infty} e^{-i\xi s} \left|\Gamma\left(a+is\right)\right|^2 ds \tag{1.7}$$
$$= \frac{2\pi}{2^{2a}} \int_0^\infty x^{2a-1} \left(1 + \left(\frac{2}{x}\right)^{2a} \sum_{k=0}^{n} \frac{4(a+k)}{k!\Gamma(1-k-2a)} K_{2a+2k}(x)\right) e^{-x\cosh(\xi/2)} dx,$$

for any $\xi \in \mathbb{R}$ and $n \in \mathbb{N} = \{0, 1, 2, \ldots\}$, see Proposition 4.1 below.

The integral representations (1.3)-(1.4) have applications to the solution of the Fokker-Planck type equation

$$\begin{cases} \dfrac{\partial U_p}{\partial t}(t,y) = y^2 \dfrac{\partial^2 U_p}{\partial y^2}(t,y) + y\dfrac{\partial U_p}{\partial y}(t,y) - (p^2+y^2)U_p(t,y), \qquad y, t > 0, \\ \\ U_p(0,y) = y^p, \end{cases} \tag{1.8}$$

which originates from statistical physics, see [10]. The solution of (1.8) admits the integral expression

$$U_p(t,y) = \frac{2}{\pi^2} \int_0^\infty u \sinh(\pi u) K_{iu}(y) e^{-(p^2+u^2)t} \int_0^\infty x^{p-1} K_{iu}(x) dx du \tag{1.9}$$

using the heat kernel of the operator $y^2 - y^2\partial^2/\partial y^2 - y\partial/\partial y$, see e.g. [12]. From the representations (1.1), (1.4) and the Fubini theorem, (1.9) can be rewritten as the single integral representation

$$\begin{aligned} U_p(s,y) &= \frac{2}{\pi^2} \int_0^\infty u \sinh(\pi u) K_{iu}(y) e^{-(p^2+u^2)s} \\ &\quad \times \int_0^\infty x^{p-1} K_{iu}(x) \left(1 + \sum_{k=0}^{n} \frac{2^{p+1}(p+2k)}{k!\Gamma(1-p-k)} K_{-p-2k}(x)\right) dx du \\ &\quad - e^{-p^2 s} \sum_{k=0}^{n} \frac{2^{p+1}(p+2k)}{k!\Gamma(1-p-k)!} K_{-p-2k}(y) \\ &= \frac{2^p}{2\pi^2} \int_0^\infty u \sinh(\pi u) e^{-(p^2+u^2)s} \left|\Gamma\left(\frac{p}{2} + i\frac{u}{2}\right)\right|^2 K_{iu}(y) du \\ &\quad - \sum_{k=0}^{n} e^{4((p+2k)^2-p^2)s} \frac{2^{p+1}(p+2k)}{k!\Gamma(1-p-k)!} K_{-p-2k}(y), \end{aligned}$$

where $p \in (-2n-2, 0]$, $n \in \mathbb{Z}$. The function $U_p(s, y)$ is the Laplace transform of the Hartman-Watson distribution which appears in the analysis of exponential functionals of Brownian motion [4] and in option pricing, cf. e.g. [5], [8].

The above expression of $U_p(s, y)$ has been originally obtained in [10] using spectral expansions, however the derivation presented here is much simpler since the argument used in [10] involves severe analytical difficulties in the computation of normalization constants via the use of Meijer functions, cf. [10] page 1641.

This paper is organized as follows. In Section 2 we start by proving some asymptotic expansion and integrability results that are needed for the proof of both (1.3) and (1.6). The proof of the integral representation (1.4) and its extension to $\Re(z) \in (-n-1, -n)$ for all $n \in \mathbb{N}$ are given in Section 3. Finally in Section 4 we derive the extension of the Fourier transform identity (1.6) to $\Re(z) \in (-n-1, -n)$ for all $n \in \mathbb{N}$ as a consequence of Proposition 3.2.

2 Asymptotic expansion and integrability

In this section we derive the asymptotic results needed for the proofs of (1.3)-(1.7) in Propositions 3.2 and 4.1 below. We will use the modified Bessel function of the first kind

$$I_z(x) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(k+1)\Gamma(k+z+1)} \left(\frac{x}{2}\right)^{z+2k}, \qquad x \in \mathbb{R}, \quad z \in \mathbb{C}. \tag{2.1}$$

Lemma 2.1 *For all $n \in \mathbb{N}$, $x \in \mathbb{R}$ and $z \in \mathbb{C}$ we have*

$$\left(\frac{2}{x}\right)^{2z} \sum_{k=0}^{n} \frac{k+z}{k!\Gamma(1-k-2z)} I_{2k+2z}(x) = \frac{\sin(2\pi z)}{2\pi} + \sum_{l=n+1}^{\infty} \frac{1}{l!} \left(\frac{x}{2}\right)^{2l} \sum_{k=0}^{n} \binom{l}{k} \frac{k+z}{\Gamma(1-2z-k)!\Gamma(k+l+2z+1)}. \tag{2.2}$$

Proof. From (2.1) we have

$$\left(\frac{2}{x}\right)^{2z} \sum_{k=0}^{n} \frac{k+z}{k!\Gamma(1-k-2z)} I_{2k+2z}(x) = \sum_{k=0}^{n} \frac{k+z}{k!\Gamma(1-k-2z)} \sum_{l=0}^{\infty} \frac{1}{l!\Gamma(2k+l+2z+1)} \left(\frac{x}{2}\right)^{2k+2l}$$

$$
\begin{aligned}
&= \sum_{k=0}^{n} \frac{k+z}{k!\Gamma(1-k-2z)} \sum_{l=k}^{\infty} \frac{1}{(l-k)!\Gamma(k+l+2z+1)} \left(\frac{x}{2}\right)^{2l} \\
&= \sum_{l=0}^{\infty} \left(\frac{x}{2}\right)^{2l} \sum_{k=0}^{\min(n,l)} \frac{k+z}{k!\Gamma(1-k-2z)(l-k)!\Gamma(k+l+2z+1)}. \qquad (2.3)
\end{aligned}
$$

Next, using Euler's reflection formula

$$
\frac{1}{\Gamma(k+l+2z+1)} = -(-1)^{k+l}\frac{\sin(2\pi z)}{\pi}\Gamma(-2z-k-l), \qquad k,l \in \mathbb{N},
$$

cf. e.g. relation (6.1.17), page 256 of [1], we get

$$
\begin{aligned}
&\sum_{l=0}^{n} \left(\frac{x}{2}\right)^{2l} \sum_{k=0}^{l} \frac{k+z}{k!(l-k)!\Gamma(1-k-2z)\Gamma(k+l+2z+1)} \\
&= \frac{\sin(2\pi z)}{\pi} \sum_{l=0}^{n} \left(\frac{x}{2}\right)^{2l} \frac{(-1)^l}{l!} \sum_{k=0}^{l} (-1)^k \binom{l}{k} (k+z) \frac{\Gamma(-2z-k-l)}{\Gamma(1-2z-k)} \\
&= \frac{\sin(2\pi z)}{\pi} \sum_{l=0}^{n} \left(\frac{x}{2}\right)^{2l} \frac{(-1)^l}{l!} \frac{\Gamma(-2z-2l)}{\Gamma(1-2z-k)} \sum_{k=0}^{l} \binom{l}{k} (k+z)(-2z-2l)_{l-k}(2z)_k, \qquad (2.4)
\end{aligned}
$$

where

$$
(p)_k = p(p+1)\cdots(p+k-1)
$$

is the shifted factorial, $p \in \mathbb{C}$ and $k \geq 1$, with $(p)_0 = 1$. Next, for all $l \geq 1$ and $z \in \mathbb{C}$ we check that

$$
\begin{aligned}
&\sum_{k=0}^{l} \binom{l}{k} (k+z)(-2z-2l)_{l-k}(2z)_k \\
&= z\sum_{k=0}^{l} \binom{l}{k} (-2z-2l)_{l-k}(2z)_k - k\sum_{k=0}^{l} \binom{l}{k} (-2z-2l)_{l-k}(2z)_k \\
&= z\sum_{k=0}^{l} \binom{l}{k} (-2z-2l)_{l-k}(2z)_k - \sum_{k=1}^{l} \frac{l!}{(l-k)!(k-1)!} (-2z-2l)_{l-k}(2z)_k \\
&= z\sum_{k=0}^{l} \binom{l}{k} (-2z-2l)_{l-k}(2z)_k + 2zl\sum_{k=0}^{l-1} \binom{l-1}{k} (-2z-2l)_{l-1-k}(2z+1)_k \\
&= z(-2l)_l + 2zl(-2l+1)_{l-1} \\
&= 0, \qquad (2.5)
\end{aligned}
$$

where we used the Pfaff-Saalschütz binomial identity

$$(p+q)_l = \sum_{k=0}^{l} \binom{l}{k} (p)_k (q)_{l-k}, \qquad p,q \in \mathbb{C}, \quad l \in \mathbb{N},$$

see e.g. Theorem 2.2.6 and Remark 2.2.1 of [2]. As a consequence of (2.4) and (2.5) we get

$$\sum_{l=0}^{n} \left(\frac{x}{2}\right)^{2l} \sum_{k=0}^{l} \frac{k+z}{k!(l-k)!\Gamma(1-k-2z)\Gamma(k+l+2z+1)} = \frac{\sin(2\pi z)}{2\pi},$$

which allows us to rewrite (2.3) as

$$\begin{aligned} &\left(\frac{2}{x}\right)^{2z} \sum_{k=0}^{n} \frac{k+z}{k!\Gamma(1-k-2z)} I_{2k+2z}(x) \\ &= \frac{\sin(2\pi z)}{\pi} + \sum_{l=n+1}^{\infty} \left(\frac{x}{2}\right)^{2l} \sum_{k=0}^{n} \frac{k+z}{k!(l-k)!\Gamma(1-k-2z)!\Gamma(k+2z+l+1)}, \end{aligned}$$

which in turn shows (2.2). □

As a consequence of Lemma 2.1 we have the following estimates, which justify the existence of the integrals in (1.4)-(1.7). For example, (2.8) entails the estimate

$$x^{2a} + \frac{2^{2a+2}a}{\Gamma(1-2a)} K_{2a}(x) = o(x^{\varepsilon}), \qquad x \to 0, \tag{2.6}$$

for any $\varepsilon \in (0, 2+2a) \cap (0, -2a)$, when $a \in (-1, 0)$, which ensures the integrability as $x \to 0$ in (1.3) and (1.6).

Corollary 2.2 *Let $n \in \mathbb{N}$.*

(i) For all $z \in \mathbb{C}$ such that $\Re(z) > -n-1$ we have

$$\sum_{k=0}^{n} \frac{k+z}{k!\Gamma(1-k-2z)} I_{2k+2z}(x) = \frac{\sin(2\pi z)}{2\pi} \left(\frac{x}{2}\right)^{2z} + o(x^{\varepsilon}), \qquad x \to 0, \tag{2.7}$$

for all $\varepsilon \in (0, 2n+2+2\Re(z))$.

(ii) For all $z \in \mathbb{C}$ such that $-n-1 < \Re(z) < -n$ we have

$$1 + \left(\frac{2}{x}\right)^{2z} \sum_{k=0}^{n} \frac{4(z+k)}{k!\Gamma(1-k-2z)} K_{2k+2z}(x) = o(x^{\varepsilon - 2\Re(z)}), \qquad x \to 0, \tag{2.8}$$

for all $\varepsilon \in (0, 2n+2+2\Re(z)) \cap (0, -2n-2\Re(z))$.

Proof. (*i*) Relation (2.7) follows from Lemma 2.1. (*ii*) On the other hand, relation (2.1) with $-2\Re(z) - 2k \geq -2\Re(z) - 2n > \varepsilon > 0$ shows that

$$I_{-2z-2k}(x) = \sum_{l=0}^{\infty} \frac{1}{l!\Gamma(l - 2z - 2k + 1)} \left(\frac{x}{2}\right)^{-2z-2k+2l} = o(x^{\varepsilon}), \qquad x \to 0,$$

hence (2.7) and the identity

$$K_{2k+2z}(x) = \frac{\pi}{2\sin(2\pi z)}(I_{-2z-2k}(x) - I_{2k+2z}(x)), \qquad x \in \mathbb{R}, \tag{2.9}$$

allow us to conclude (2.8). □

The next integrability result is a consequence of Lemma 2.1 and will be useful for the proofs of Propositions 3.1, 3.2 and 4.1 below.

Lemma 2.3 *Let $n \in \mathbb{N}$.*

(i) For all $z \in \mathbb{C}$ such that $-n-1 < \Re(z)$ we have

$$\sup_{s\in\mathrm{R}} \int_0^{\infty} |K_{is}(x)| \left| \frac{\sin(2\pi z)}{\pi} - 2\left(\frac{2}{x}\right)^{2z} \sum_{k=0}^{n} \frac{k+z}{k!\Gamma(1-2z-k)} I_{2k+2z}(x) \right| \frac{dx}{x^{1-2z}} < \infty. \tag{2.10}$$

(ii) For all $z \in \mathbb{C}$ such that $-n-1 < \Re(z) < -n$ we have

$$\sup_{s\in\mathrm{R}} \int_0^{\infty} x^{2z-1} |K_{is}(x)| \left| 1 + \left(\frac{2}{x}\right)^{2z} \sum_{k=0}^{n} \frac{4(k+z)}{k!\Gamma(1-k-2z)} K_{2k+2z}(x) \right| dx < \infty. \tag{2.11}$$

Proof. (*i*) By relation (1.2), for all $\alpha > 0$ there exists a constant $c_\alpha > 0$ such that $\cosh x > c_\alpha x^\alpha$ for all $x > 0$, which shows that

$$|K_{is}(y)| \leq \int_0^{\infty} e^{-y\cosh x} dx \leq \int_0^{\infty} e^{-yc_\alpha x^\alpha} dx = \frac{\Gamma(1/\alpha)}{\alpha\,(c_\alpha)^{1/\alpha}} y^{-1/\alpha}, \qquad y > 0, \quad s \in \mathbb{R}.$$

Hence, using (2.7), for all $s \in \mathbb{R}$ and $\alpha > 1/\varepsilon$, we have

$$\begin{aligned} &\int_0^{1} |K_{is}(x)| \left| \frac{\sin(2\pi z)}{\pi} - \left(\frac{2}{x}\right)^{2z} \sum_{k=0}^{n} \frac{2z+2k}{k!\Gamma(1-k-2z)} I_{2k+2z}(x) \right| \frac{dx}{x^{1-2z}} \\ &\qquad \leq \; c\frac{\Gamma(1/\alpha)}{\alpha\,(c_\alpha)^{1/\alpha}} \int_0^{1} \frac{dx}{x^{1-\varepsilon+1/\alpha}} < \infty, \qquad s \in \mathbb{R}, \end{aligned}$$

for some constant $c > 0$. Next, the bound

$$|K_{is}(x)| \leq K_0(x), \qquad x > 0, \quad s \in \mathbb{R}, \tag{2.12}$$

that follows from relation (3.5) below and the equivalences

$$K_{is}(x) \simeq e^{-x}\sqrt{\frac{\pi}{2x}}, \qquad x \to \infty, \quad s \in \mathbb{R}, \tag{2.13}$$

and

$$I_p(x) \simeq \frac{e^x}{\sqrt{2\pi x}}, \qquad x \to \infty, \quad p \in \mathbb{C}, \tag{2.14}$$

show that

$$\sup_{s\in\mathbb{R}} \int_1^\infty |K_{is}(x)| \left| \frac{\sin(2\pi z)}{\pi} - 2\left(\frac{2}{x}\right)^{2z} \sum_{k=0}^n \frac{k+z}{k!\Gamma(1-2z-k)} I_{2k+2z}(x) \right| \frac{dx}{x^{1-2z}} < \infty,$$

which yields (2.10) for all $z \in \mathbb{C}$ such that $\Re(z) > -n-1$.

(ii) Due to the equivalences (2.9) and (2.14) there also exists $c > 0$ such that

$$\left| 1 + \left(\frac{2}{x}\right)^{2z} \sum_{k=0}^n \frac{4(k+z)}{k!\Gamma(1-k-2z)} K_{2k+2z}(x) \right| \leq c x^{-2z-1/2} e^x, \tag{2.15}$$

for sufficiently large $x > 0$ and all $z \in \mathbb{C}$, and this yields (2.11) by replacing the use of (2.7) with that of (2.8) in the proof of part (i) above. □

3 Analytic continuation and integral representation

In the next proposition, using analytic continuation, we prove an integral representation formula that will be applied to the proof of (3.6) in Proposition 3.2 below.

Proposition 3.1 *For all $n \in \mathbb{N}$, $s \in \mathbb{R}$ and $z \in \mathbb{C}$ with $\Re(z) > -n-1$, $\Re(z) \notin -\mathbb{N}$, we have*

$$\Gamma(z+is)\Gamma(z-is) = \frac{2\pi}{\sin(2\pi z)} \sum_{k=0}^n \frac{k+z}{k!((z+k)^2+s^2)\Gamma(1-k-2z)} \tag{3.1}$$

$$+2^{2-2z} \int_0^\infty K_{2is}(x) \left(1 - \frac{\pi}{\sin(2\pi z)} \left(\frac{2}{x}\right)^{2z} \sum_{k=0}^n \frac{2z+2k}{k!\Gamma(1-k-2z)} I_{2k+2z}(x) \right) \frac{dx}{x^{1-2z}}.$$

Proof. Let $s \in \mathbb{R} \setminus \{0\}$. We will prove the equality

$$\sin(2\pi z)\Gamma(z+is)\overline{\Gamma(\bar{z}+is)} = \sin(2\pi z)\Gamma(z+is)\Gamma(z-is) \tag{3.2}$$

$$= \pi \sum_{k=0}^{n} \frac{2z+2k}{k!((z+k)^2+s^2)\Gamma(1-k-2z)}$$

$$+\frac{4}{2^{2z}} \int_0^\infty K_{2is}(x) \left(\sin(2\pi z) - \pi \left(\frac{2}{x}\right)^{2z} \sum_{k=0}^{n} \frac{2z+2k}{k!\Gamma(1-k-2z)} I_{2k+2z}(x) \right) \frac{dx}{x^{1-2z}}, \tag{3.3}$$

for all $z = a+ib \in \mathbb{C}$ with $a > -n-1$, in the following three steps.

Step 1. Analyticity. In (3.2), the function

$$\sin(2\pi z)\Gamma(z+is)\Gamma(z-is)$$

is analytic in $\{z \ : \ z+is \notin (-\mathbb{N}),\ z-is \notin (-\mathbb{N})\}$ and for each $k = 0, 1, \ldots, n$ the function $((z+k)^2+s^2)^{-1}$ is analytic in $z \in \mathbb{C} \setminus (-\mathbb{N})$. On the other hand, by Lemma 2.1 we can write the integrand in (3.2) as

$$x \mapsto \frac{K_{is}(x)}{x^{1-2z}} \left(\frac{\sin(2\pi z)}{\pi} - \left(\frac{2}{x}\right)^{2z} \sum_{k=0}^{n} \frac{2z+2k}{k!\Gamma(1-k-2z)} I_{2k+2z}(x) \right)$$

$$= -2K_{is}(x) \sum_{k=0}^{n} \frac{x^{2z-1}}{k!\Gamma(1-2z-k)} \sum_{l=n+1}^{\infty} \left(\frac{x}{2}\right)^{2l} \frac{k+z}{(l-k)!\Gamma(k+2z+l+1)}$$

$$= -2K_{is}(x) \sum_{k=0}^{n} \frac{x^{2z-1}}{k!\Gamma(1-2z-k)\Gamma(k+2z)} \sum_{l=n+1}^{\infty} \left(\frac{x}{2}\right)^{2l} \frac{k+z}{(l-k)!(k+2z+l)\cdots(k+2z)},$$

where for each $k = 0, 1, \ldots, n$ the partial derivatives of

$$z = a+ib \mapsto \sum_{l=n+1}^{\infty} \left(\frac{x}{2}\right)^{2l} \frac{k+z}{(l-k)!(k+l+2z)\cdots(k+2z)}$$

with respect to a and b are locally uniformly bounded by integrable functions of $x \in \mathbb{R}_+$ from the bounds (2.12) and (2.13), by the same arguments used in the proof of Lemma 2.3.

Hence we can exchange partial differentiation with respect to a and b with the integration in (3.3), showing that the Cauchy-Riemann conditions are satisfied by the integral (3.3) since all functions in the integrand are analytic in $z \in \mathbb{C}$. Consequently, all terms in (3.2) are analytic in $\{z \in \mathbb{C} : \Re(z) > -n-1, \ z+is \notin (-\mathbb{N}), \ z-is \notin (-\mathbb{N})\}$.

Step 2. Equality (3.2) holds for all $s \in \mathbb{R} \setminus \{0\}$ and $z = a + ib \in (0, \infty) + i\mathbb{R}$. This follows from the integral representation (1.1) which reads

$$\Gamma(z+is)\,\Gamma(z-is) = 4\int_0^\infty \left(\frac{x}{2}\right)^{2z} K_{2is}(x)\frac{dx}{x},$$

provided that $a > 0$, and from the Mellin transform

$$4\int_0^\infty K_{2is}(x) I_{2k+2z}(x)\frac{dx}{x} = \frac{1}{(z+k)^2 + s^2} \tag{3.4}$$

which is valid whenever $a + k > 0$, cf. e.g. relation (44) page 334 of [6].

Step 3. By analytic continuation the relation (3.2) extends to $\{z \in \mathbb{C} : \Re(z) > -n-1, \ z+is \notin (-\mathbb{N}), \ z-is \notin (-\mathbb{N})\}$ and we conclude by dividing (3.3) by $\sin(2\pi z)$ when $z \in \mathbb{C}$ with $\Re(z) > -n-1$ and $\Re(z) \notin -\mathbb{N}$. □

Note that in the above proof we could also have used the unique continuation principle for real analytic functions of a, see e.g. Corollary 1.2.3 of [7], however real analyticity requires to check the growth rate of partial derivatives, which would have been more delicate.

Relation (1.1) can be recovered from the integral representation

$$K_{is}(y) = \frac{1}{2}\left(\frac{y}{2}\right)^{is}\int_0^\infty x^{-is-1}e^{-x-y^2/(4x)}dx, \qquad s, y \in \mathbb{R}, \tag{3.5}$$

see [11] page 183, using the Fubini theorem, as follows:

$$\begin{aligned}\Gamma(z+is)\,\Gamma(z-is) &= \int_0^\infty x^{-2is-1}e^{-x}x^{z+is}\int_0^\infty y^{-1+z+2is}e^{-y}dydx \\ &= \int_0^\infty \left(\frac{t}{2}\right)^{2z-1+2is}\int_0^\infty x^{-2is-1}e^{-x-t^2/(4x)}dxdt\end{aligned}$$

$$= \quad 4\int_0^\infty \left(\frac{t}{2}\right)^{2z} K_{2is}(t)\frac{dt}{t}, \qquad s \in \mathbb{R},$$

where we applied the change of variable $y = t^2/(4x)$. However, this argument is valid only for $\Re(z) > 0$ due to integrability restrictions in the exchange of integrals.

We are now able to extend the above argument to all $\Re(z) \in (-n-1, -n)$, $n \in \mathbb{N}$, in order to prove the integral representation (3.6) which also implies (1.3).

Proposition 3.2 *For all $n \in \mathbb{N}$, $s \in \mathbb{R}$ and $z \in \mathbb{C}$ such that $-n-1 < \Re(z) < -n$ we have*

$$\begin{aligned} &\Gamma(z+is)\,\Gamma(z-is) \\ &= \; 2\int_0^\infty K_{2is}(x)\left(1+\left(\frac{2}{x}\right)^{2z}\sum_{k=0}^{n}\frac{4(k+z)}{k!\Gamma(1-k-2z)!}K_{2k+2z}(x)\right)\left(\frac{2}{x}\right)^{1-2z}dx. \end{aligned} \tag{3.6}$$

Proof. First we note that the integrability in (3.6) follows from the bound (2.10) above. Next, from relations (2.9), (3.1) and (3.4), with $-2\Re(z)-2k \geq -2\Re(z)-2n > 0$, we have

$$\begin{aligned} &2^{2z-2}\Gamma(z+is)\,\Gamma(z-is) \\ &= \frac{\pi}{\sin(2\pi z)}\int_0^\infty K_{2is}(x)\left(\frac{\sin(2\pi z)}{\pi}-\left(\frac{2}{x}\right)^{2z}\sum_{k=0}^{n}\frac{2z+2k}{k!\Gamma(1-k-2z)}I_{2k+2z}(x)\right)\frac{dx}{x^{1-2z}} \\ &\quad +\frac{2^{2z}\pi}{2\sin(2\pi z)}\sum_{k=0}^{n}\frac{k+z}{k!((k+z)^2+s^2)\Gamma(1-k-2z)} \\ &= \frac{\pi}{\sin(2\pi z)}\int_0^\infty K_{2is}(x)\left(\frac{\sin(2\pi z)}{\pi}-\left(\frac{2}{x}\right)^{2z}\sum_{k=0}^{n}\frac{2z+2k}{k!\Gamma(1-k-2z)}I_{2k+2z}(x)\right)\frac{dx}{x^{1-2z}} \\ &\quad +\frac{2^{z}\pi}{\sin(2\pi z)}\sum_{k=0}^{n}\frac{2z+2k}{k!\Gamma(1-k-2z)}\int_0^\infty K_{2is}(x)I_{-2z-2k}(x)\frac{dx}{x} \\ &= \int_0^\infty K_{2is}(x)\left(1+\left(\frac{2}{x}\right)^{2z}\sum_{k=0}^{n}\frac{4(k+z)}{k!\Gamma(1-k-2z)}K_{2k+2z}(x)\right)\frac{dx}{x^{1-2z}}. \end{aligned}$$

$\square$

4 Fourier transform of $|\Gamma(a+is)|^2$

We begin by proving an integral representation for the Fourier transform of

$$s \mapsto |\Gamma(a+is)|^2, \qquad a \in (-n-1,-n), \quad n \in \mathbb{N},$$

as a consequence of the integral representation (1.4) of Proposition 3.2.

Proposition 4.1 *Let $n \in \mathbb{N}$ and $a \in (-n-1,-n)$. For all $\xi \in \mathbb{R}$ we have*

$$\int_{-\infty}^{\infty} e^{-i\xi s} \left|\Gamma\left(a+is\right)\right|^2 ds \tag{4.1}$$
$$= \frac{2\pi}{2^{2a}} \int_0^\infty x^{2a-1} \left(1 + \left(\frac{2}{x}\right)^{2a} \sum_{k=0}^{n} \frac{4(a+k)}{k!\Gamma(1-k-2a)} K_{2a+2k}(x)\right) e^{-x\cosh(\xi/2)} dx.$$

Proof. This result can be informally deduced from (3.6) in Proposition 3.2 and the Fourier-Gelfand formula

$$\int_{-\infty}^{\infty} \cos(2sy) e^{-i\xi s} ds = \pi\left(\delta(\xi/2-y) + \delta(\xi/2+y)\right)$$

in distribution theory, where δ is the Dirac distribution at 0. However, with a view towards completeness, we provide a proof by approximation following the approach used in the proof of Theorem 1.1 of [3]. With the abbreviation

$$\Psi_n(x) := 1 + \left(\frac{2}{x}\right)^{2a} \sum_{k=0}^{n} \frac{4(a+k)}{k!\Gamma(1-k-2a)} K_{2a+2k}(x), \qquad x \in \mathbb{R}, \tag{4.2}$$

we rewrite (3.6) as

$$\left|\Gamma\left(a+is\right)\right|^2 = \frac{4}{2^{2a}} \int_0^\infty x^{2a-1} K_{2is}(x) \Psi_n(x) dx, \qquad s \in \mathbb{R},$$

for $a \in (-n-1,-n)$. Then for any $\varepsilon > 0$ we have

$$\int_{-\infty}^{\infty} e^{-i\xi s} \left|\Gamma\left(a+is\right)\right|^2 ds = \int_{-\infty}^{\infty} \lim_{\varepsilon\to 0} \left(e^{-2\varepsilon s^2 - i\xi s} \left|\Gamma\left(a+is\right)\right|^2\right) ds$$
$$= \lim_{\varepsilon\to 0} \int_{-\infty}^{\infty} e^{-2\varepsilon s^2 - i\xi s} \left|\Gamma\left(a+is\right)\right|^2 ds$$
$$= 2^{2-2a} \lim_{\varepsilon\to 0} \int_{-\infty}^{\infty} e^{-2\varepsilon s^2 - i\xi s} \int_0^\infty x^{2a-1} K_{2is}(x) \Psi_n(x) dx ds$$

$$= \quad 2^{2-2a} \lim_{\varepsilon\to 0} \int_0^\infty x^{2a-1}\Psi_n(x) \int_{-\infty}^\infty e^{-2\varepsilon s^2 - i\xi s} K_{2is}(x) ds dx, \tag{4.3}$$

where the exchange of limit follows from the fact that $s \mapsto |\Gamma(a+is)|^2$ is a rapidly decreasing function in the Schwartz class, and the last equality comes from (2.11), which ensures the integrability required for the exchange of integrals. Next, from relation (1.2) written as

$$K_{2is}(x) = \int_0^\infty e^{-x\cosh y} \cos(2sy) dy, \qquad x > 0, \quad s \in \mathbb{R},$$

we find

$$\begin{aligned}
\int_{-\infty}^\infty e^{-2\varepsilon s^2 - i\xi s} K_{2is}(x) ds &= \int_{-\infty}^\infty e^{-2\varepsilon s^2 - i\xi s} \int_0^\infty e^{-x\cosh y} \cos(2sy) dy ds \\
&= \int_{-\infty}^\infty e^{-x\cosh y} \int_0^\infty \cos(2sy) e^{-2\varepsilon s^2 - i\xi s} ds dy \\
&= \frac{1}{4}\sqrt{\frac{\pi}{2\varepsilon}} \int_{-\infty}^\infty e^{-x\cosh y} (e^{-\frac{1}{2\varepsilon}(y-\xi/2)^2} + e^{-\frac{1}{2\varepsilon}(y+\xi/2)^2}) dy, \qquad x > 0,
\end{aligned}$$

hence by (4.3) we obtain

$$\begin{aligned}
&\int_{-\infty}^\infty e^{-i\xi s} |\Gamma(a+is)|^2 ds \\
&= 2^{-2a} \lim_{\varepsilon\to 0} \sqrt{\frac{\pi}{2\varepsilon}} \int_0^\infty x^{2a-1}\Psi_n(x) \int_{-\infty}^\infty e^{-x\cosh y} (e^{-\frac{1}{2\varepsilon}(y-\xi/2)^2} + e^{-\frac{1}{2\varepsilon}(y+\xi/2)^2}) dy dx \\
&= 2^{-2a} \lim_{\varepsilon\to 0} \sqrt{\frac{\pi}{2\varepsilon}} \int_0^\infty x^{2a-1}\Psi_n(x) \int_{-\infty}^\infty e^{-x\cosh(y+\xi/2) - \frac{1}{2\varepsilon}y^2} dy dx \\
&\quad + 2^{-2a} \lim_{\varepsilon\to 0} \sqrt{\frac{\pi}{2\varepsilon}} \int_0^\infty x^{2a-1}\Psi_n(x) \int_{-\infty}^\infty e^{-x\cosh(y-\xi/2) - \frac{1}{2\varepsilon}y^2} dy dx \\
&= 2^{-2a} \sqrt{\frac{\pi}{2}} \lim_{\varepsilon\to 0} \int_0^\infty x^{2a-1}\Psi_n(x) \int_{-\infty}^\infty e^{-x\cosh(z\sqrt{\varepsilon}+\xi/2) - z^2/2} dz dx \\
&\quad + 2^{-2a} \sqrt{\frac{\pi}{2}} \lim_{\varepsilon\to 0} \int_0^\infty x^{2a-1}\Psi_n(x) \int_{-\infty}^\infty e^{-x\cosh(z\sqrt{\varepsilon}-\xi/2) - z^2/2} dz dx \\
&= 2^{1-2a}\pi \int_0^\infty x^{2a-1}\Psi_n(x) e^{-x\cosh(\xi/2)} dx,
\end{aligned}$$

where the required integrability follows from the bounds (2.8) and (2.15). □

In case $a > 0$, $\Psi_{-1}(x)$ in (4.2) is identically equal to 1 and the proof of Proposition 4.1 also yields the Mellin transform

$$\int_{-\infty}^\infty e^{-i\xi s} |\Gamma(a+is)|^2 ds = \frac{2\pi}{2^{2a}} \int_0^\infty x^{2a-1} e^{-x\cosh(\xi/2)} dx = \frac{2\pi}{2^{2a}} (\cosh(\xi/2))^{-2a} \Gamma(2a),$$

which recovers (1.5), cf. also Theorem 1.1 in [3].

On the other hand, when $a = -1/2$ the Fourier transform of $s \mapsto |\Gamma(-1/2 + is)|^2$ can be explicitly computed as

$$\begin{aligned}\int_{-\infty}^{\infty} e^{-i\xi s} \left|\Gamma\left(-1/2 + is\right)\right|^2 ds &= 4\pi \int_{-\infty}^{\infty} \frac{e^{-i\xi s}}{(1+4s^2)\cosh(\pi s)} ds \\ &= 4\pi \log(1+e^{-\xi})\cosh(\xi/2) + 2\pi\xi e^{-\xi/2},\end{aligned}$$

see e.g. relation (22) page 32 of [6], whereas (4.1) yields

$$\int_{-\infty}^{\infty} e^{-i\xi s} \left|\Gamma\left(-1/2 + is\right)\right|^2 ds = 4\pi \int_0^{\infty} \left(\frac{1}{x} - K_{-1}(x)\right) e^{-x\cosh(\xi/2)} \frac{dx}{x},$$

where the integrability in 0 in the above integral follows from

$$\frac{1}{x} - K_{-1}(x) = o(x^{\varepsilon}), \qquad x \to 0,$$

for any $\varepsilon \in (0,1)$, cf. (2.6).

References

[1] M. Abramowitz and I.A. Stegun. *Handbook of mathematical functions with formulas, graphs, and mathematical tables*, volume 55. National Bureau of Standards Applied Mathematics Series, 1964.

[2] G. E. Andrews, R. Askey, and R. Roy. *Special functions*, volume 71 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1999.

[3] D. Chakrabarti and G.K. Srinivasan. On a remarkable formula of Ramanujan. *Arch. Math. (Basel)*, 99(2):125–135, 2012.

[4] A. Comtet, C. Monthus, and M. Yor. Exponential functionals of Brownian motion and disordered systems. *J. Appl. Probab.*, 35(2):255–271, 1998.

[5] M. Craddock. On an integral arising in mathematical finance. In R. Dieci, X.-Z. He, and C. Hommes, editors, *Nonlinear Economic Dynamics and Financial Modelling*, pages 355–370. Springer Verlag, 2014.

[6] A. Erdélyi, W. Magnus, F. Oberhettinger, and F.G. Tricomi. *Tables of integral transforms. Vols. I and II.* McGraw-Hill Book Company, Inc., New York-Toronto-London, 1954.

[7] S.G. Krantz and H.R. Parks. *A primer of real analytic functions.* Birkhäuser Boston, Inc., Boston, MA, second edition, 2002.

[8] C. Pintoux and N. Privault. The Dothan pricing model revisited. *Math. Finance*, 21:355–363, 2011.

[9] S. Ramanujan. Some definite integrals. *Messenger of Mathematics*, 44:10–18, 1915.

[10] A. Schenzle and H. Brand. Multiplicative stochastic processes in statistical physics. *Phys. Rev. A*, 20(4):1628–1647, 1979.

[11] G. N. Watson. *A treatise on the theory of Bessel functions.* Cambridge University Press, Cambridge, 1995. Reprint of the second (1944) edition.

[12] S. Yakubovich. A class of polynomials and discrete transformations associated with the Kontorovich-Lebedev operators. *Integral Transforms Spec. Funct.*, 20(7-8):551–567, 2009.