Stochastic Calculus of Variations for Martingales

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The framework of the stochastic calculus of variations on the standard Wiener and Poisson space is extended to certain martingales, consistently with other approaches. The method relies on changes of times for the gradient operators. We study the transfer of the structures of stochastic analysis induced by these time-changed operators, in particular the chaotic decompositions.

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1 Introduction

The stochastic calculus of variations for the Wiener process, initiated in Malliavin [12], aims to obtain conditions for the regularity of the density of Wiener functionals given by the values of diffusion processes. It also developed as an extension to anticipating processes of the Itô calculus, by means of the Skorohod integral, cf. Nualart-Pardoux [14], Üstünel [25]. In the case of point processes we can refer for instance to Bass-Cranston [1], Bichteler *et al.* [2], Bismut [3], for the absolute continuity of functionals of a Poisson measure, and to Carlen-Pardoux [5], Nualart-Vives [15], Privault [21] for the anticipative stochastic calculus for the standard Poisson process on the positive real line. The chaotic decompositions and the chaotic representation property played a central role in each of these approaches. The aim of this work is to extend those constructions to the case of martingales with a continuous part and a finite number of jump sizes which can be obtained by time changes from independent Wiener and Poisson processes. This approach is consistent with previous constructions and allows to unify them.

2 Stochastic calculus of variations on the standard Wiener and Poisson spaces

The aim of this section is to review some facts and definitions that will serve as starting points for the next sections. Let $(B_t)_{t\in\mathbb{R}_+}$ and $(N_t)_{t\in\mathbb{R}_+}$ be not necessarily independent Wiener and Poisson processes on the real line, defined on a probability space $(\Omega, \mathcal{F}, P)$, with associated filtrations $\left(\mathcal{F}_t^B\right)$ and $(\mathcal{F}_t^{\tilde{N}})$, $\mathcal{F} = \mathcal{F}_\infty^B \vee \mathcal{F}_\infty^{\tilde{N}}$, $\mathcal{F}^B = \mathcal{F}_\infty^B$ and $\mathcal{F}^{\tilde{N}} = \mathcal{F}_\infty^{\tilde{N}}$, where $\tilde{N}_t = N_t - t$ is the compensated Poisson process. Let $(h_k)_{k\in\mathbf{N}}$ be an orthonormal Hilbert basis of $L^2(\mathbf{R}_+)$, and let $(e_k)_{k\in\mathbf{N}}$ be the canonical basis of $l^2(\mathbf{N})$. We can define two collections of independent identically distributed Gaussian, resp. exponential random variables by $\xi_k = \int_0^\infty h_k(t)dB_t$ and $\tau_k = T_{k+1} - T_k$, $k \in \mathbf{N}$, where $T_k = \inf\{t \geq 0 \ : \ N_t = k\}$, $k \geq 1$, is the kth jump time of the Poisson process $(N_t)_{t\in\mathbf{R}_+}$, and $T_0 = 0$. Let $\mathcal{P}$ denote the set of functionals of the form

$$F = f(\xi_0, \ldots, \xi_n, \tau_0, \ldots, \tau_n)$$

where $f \in \mathcal{C}_b^\infty(\mathbf{R}^{2n+2})$, $n \in \mathbf{N}$. We know that $\mathcal{P}$ is dense in $L^p(\Omega, \mathcal{F}, P)$, $p \geq 1$. Gradient operators

$$\partial^B, \ \partial^{\tilde{N}} : L^2(\Omega) \to L^2(\Omega) \otimes l^2(\mathbf{N})$$

are defined as

$$\partial^B F = \sum_{k=0}^{k=n} e_k \partial_k f(\xi_0, \ldots, \xi_n, \tau_0, \ldots, \tau_n),$$

$$\partial^{\tilde{N}} F = \sum_{k=0}^{k=n} e_k \partial_{n+1+k} f(\xi_0, \ldots, \xi_n, \tau_0, \ldots, \tau_n), \quad F \in \mathcal{P}.$$

Such gradient operators can be expressed by an infinitesimal perturbation of the trajectories of $(B_t)_{t\in\mathbf{R}_+}$ and $(N_t)_{t\in\mathbf{R}_+}$. Let $F \in \mathcal{P}$ be $\mathcal{F}^B$-measurable, i.e. F is a Wiener functional. If we represent by $\omega = (\xi_k)_{k\in\mathbf{N}}$ a trajectory of the Wiener process, then

$$(\partial^B F, h)_{l^2(\mathbf{N})} = \frac{d}{d\varepsilon} F(\omega + \varepsilon h) \mid_{\varepsilon=0}, \quad h \in l^2(\mathbf{N}), \ F \in \mathcal{P}.$$

On the other hand, if $\omega = (\tau_k)_{k\in\mathbf{N}}$ denotes the sequence of interjump times of the Poisson process and if $F \in \mathcal{P}$ is $\mathcal{F}^{\tilde{N}}$-measurable, then

$$(\partial^{\tilde{N}} F, h)_{l^2(\mathbf{N})} = \frac{d}{d\varepsilon} F(\omega + \varepsilon h) \mid_{\varepsilon=0} .$$

Let

$$\mathcal{U} = \left\{ \sum_{k=0}^{k=n} u_k e_k \ : \ u_1, \ldots, u_n \in \mathcal{P}, \ n \in \mathbf{N} \right\},$$

and

$$\mathcal{U}^{\tilde{N}} = \left\{ \sum_{k=0}^{\infty} u_k e_k \in \mathcal{U} \ : \ u_k = 0 \ on \ \{\tau_k = 0\}, \ k \in \mathbf{N} \right\}.$$

The adjoint operators ∂_*^B, $\partial_*^{\tilde{N}}$ of ∂^B and $\partial^{\tilde{N}}$ are closable operators

$$\partial_*^B, \partial_*^{\tilde{N}} : L^2(\Omega) \otimes l^2(\mathbf{N}) \to L^2(\Omega)$$

such that

$$E[F\partial_*^B(u)] = E[(u, \partial^B F)_{l^2(\mathbf{N})}], \quad u \in \mathcal{U}, \ F \in \mathcal{P},$$

$$E[F\partial_*^{\tilde{N}}(u)] = E[(u, \partial^{\tilde{N}} F)_{l^2(\mathbf{N})}], \quad u \in \mathcal{U}^{\tilde{N}}, \ F \in \mathcal{P}.$$

At this stage it can be noted that the composition $\partial_*^B \partial^B$ gives the Ornstein-Uhlenbeck operator on the Wiener space, cf. Watanabe [26], and that the gradient ∂^B can be used to state a great number of results in Malliavin calculus that involve the Ornstein-Uhlenbeck operator and the norm of the gradient on the Wiener space. However this construction does not take into account an important property of the Gaussian measure, namely the fact that the adjoint of the gradient on Wiener space can be an extension of the Wiener stochastic integral, cf. Gaveau-Trauber [9], which can not be the case for ∂_*^B. An analog property exists on the Poisson space, and can not be verified by $\partial_*^{\tilde{N}}$ as it acts on discrete-time processes. Define two operators

$$i^B, i^{\tilde{N}} : l^2(\mathbf{N}) \to L^2(\mathbf{R}_+)$$

by $i^B(e_k) = h_k$ and $i^{\tilde{N}}(e_k) = -1_{]T_k, T_{k+1}]}$, $k \in \mathbf{N}$, and let

$$D^B = i^B \circ \partial^B, \quad D^{\tilde{N}} = i^{\tilde{N}} \circ \partial^{\tilde{N}}.$$

Let j^B and $j^{\tilde{N}}$ be the adjoint operators of i^B and $i^{\tilde{N}}$, a.s. defined as

$$(i^B(u), v)_{L^2(\mathbf{R}_+)} = (u, j^B(v))_{l^2(\mathbf{N})},$$

$$(i^{\tilde{N}}(u), v)_{L^2(\mathbf{R}_+)} = (u, j^{\tilde{N}}(v))_{l^2(\mathbf{N})},$$

$u \in L^2(\mathbf{R}_+), \ v \in l^2(\mathbf{N})$. An explicit description of j^B and $j^{\tilde{N}}$ is given as

$$j_k^B(u) = \int_0^\infty u(t) h_k(t) dt, \quad j_k^{\tilde{N}}(u) = \int_{T_k}^{T_{k+1}} u(t) dt, \quad k \in \mathbf{N}.$$

The introduction of i^B and j^B gives another interpretation of D^B as a derivative. For $F \in \mathcal{P}$ and $h \in L^2(\mathbf{R}_+)$, with $\omega = (\xi_k)_{k\in\mathbf{N}}$,

$$\begin{aligned}
&(D^B F, h)_{L^2(\mathbf{R}_+)} \\
&= (\partial^B F, j^B(h))_{l^2(\mathbf{N})} = \frac{d}{d\varepsilon} F(\omega + \varepsilon j^B(h)) \mid_{\varepsilon=0} \\
&= \frac{d}{d\varepsilon} F\left(\left(\int_0^\infty (h_k(t) + \varepsilon h(t)) dB_t \right)_{k\in\mathbf{N}} \right) \mid_{\varepsilon=0} \\
&= \frac{d}{d\varepsilon} F\left(B_\cdot + \varepsilon \int_0^\cdot h_s ds \right) \mid_{\varepsilon=0} .
\end{aligned}$$

Let

$$\mathcal{V} = i^B(\mathcal{U}) = \left\{ \sum_{k=0}^{k=n} u_k h_k \quad : \quad u_0, \ldots, u_n \in \mathcal{P}, n \in \mathbf{N} \right\},$$

and $\delta^B = \partial_*^B \circ j^B$, $\delta^{\tilde{N}} = \partial_*^{\tilde{N}} \circ j^{\tilde{N}}$. We have $j^{\tilde{N}}(\mathcal{V}) \subset \mathcal{U}^{\tilde{N}}$, hence $\delta^{\tilde{N}}$ is defined on $\mathcal{V}$.

Proposition 1 *If $u \in \mathcal{V}$, then for $X = B, \tilde{N}$,*

$$\delta^X(u) = \int_0^\infty u_t dX_t - \int_0^\infty D_t^X u_t dt.$$

Proof. cf. Nualart-Pardoux [14], Privault [21].

□

The adjoint operators $\delta^B = \partial_*^B \circ j^B$ and $\delta^{\tilde{N}} = \partial_*^{\tilde{N}} \circ j^{\tilde{N}}$ of D^B and $D^{\tilde{N}}$ extend respectively the Wiener and compensated Poisson stochastic integrals on the predictable square-integrable processes:

Proposition 2 *Let $u \in L^2(\Omega) \otimes L^2(\mathbf{R}_+)$. If u is $(\mathcal{F}_t^X)$-predictable, then*

$$\delta^X(u) = \int_0^\infty u(t) dX_t,$$

where $X = B, \tilde{N}$.

Proof. cf. Carlen-Pardoux [5], Nualart-Pardoux [14].

□

Remark. The operators j^B and $j^{\tilde{N}}$ can also be used to give a unified formulation of the anticipating Girsanov theorems on the Wiener and Poisson spaces. Let $u : \Omega \to L^2(\mathbf{R}_+)$ be $\mathcal{F}^B$-measurable and satisfy the hypothesis of Th. 6.4. of Kusuoka [11], and denote again by $\omega = (\xi_k)_{k\in\mathbf{N}}$ a trajectory of the Wiener process. Then for any $\mathcal{F}^B$-measurable and bounded $f : \Omega \to \mathbf{R}$,

$$\begin{aligned} E[f] &= E\left[f\left(\omega + j^B(u)\right)\right. \\ &\quad \left.\times \det_2\left(I_{l^2(\mathbf{N})} + \partial^B j^B(u)\right)\exp\left(-\delta^B(u) - \frac{1}{2} \mid u \mid^2_{L^2(\mathbf{R}_+)}\right)\right], \end{aligned}$$

where $\det_2$ is the Carleman-Fredholm determinant. If $\omega = (\tau_k)_{k\in\mathbf{N}}$ denotes a discrete interjump times trajectory of the Poisson process, then for any $\mathcal{F}^{\tilde{N}}$-measurable and bounded $f : \Omega \to \mathbf{R}$,

$$E[f] = E\left[f\left(\omega + j^{\tilde{N}}(u)\right)\det_2\left(I_{l^2(\mathbf{N})} + \partial^{\tilde{N}} j^{\tilde{N}}(u)\right)\exp\left(-\delta^{\tilde{N}}(u)\right)\right],$$

provided that $j^{\tilde{N}}(u)$ is $\mathcal{F}^N$-measurable and satisfies the hypothesis of Th. 1 in Privault [19].

$\square$

If Y is either a compensated Poisson or Wiener process, define for $f_p \in L^2(\mathbf{R}_+)^{\circ p}$ symmetric and square-integrable

$$I_p^Y(f_p) = p! \int_0^\infty \int_0^{t_p^-} \cdots \int_0^{t_2^-} f_p(t_1, \ldots, t_p) dY(t_1) \cdots dY(t_p).$$

If $Y = B$, such integrals can be expressed with the Hermite polynomials $(H_n)_{n\in\mathbf{N}}$ as

$$I_n^B(h_{k_1}^{\circ n_1} \circ \cdots \circ h_{k_d}^{\circ n_d}) = \sqrt{n_1! \cdots n_d!} H_{n_1}(\xi_{k_1}) \cdots H_{n_d}(\xi_{k_d}),$$

$n_1 + \cdots + n_d = n$, $k_1 \neq \cdots \neq k_d$, where "$\circ$" denotes the symmetric tensor product. In case $Y = \tilde{N}$, the Charlier polynomials replace the Hermite polynomials, cf. Surgailis [24]. An annihilation operator $\nabla^Y : L^2(\Omega) \to L^2(\Omega) \otimes L^2(\mathbf{R}_+)$ is defined as

$$\nabla^Y I_n^Y(f_n) = n I_{n-1}^Y(f_n), \quad n \geq 1,$$

and $\nabla^Y c = 0$ if $c \in \mathbf{R}$. A creation operator $\nabla_*^Y : L^2(\Omega) \otimes L^2(\mathbf{R}_+) \to L^2(\Omega)$ is defined as

$$\nabla_*^Y I_n^Y(f_{n+1}) = I_{n+1}^Y(\hat{f}_{n+1}), \quad f_{n+1} \in L^2(\mathbf{R}_+)^{\circ n} \otimes L^2(\mathbf{R}_+),$$

where $\hat{f}_{n+1}$ is the symmetrization of the function f_{n+1} in its $n+1$ variables. The operator $L^Y = \delta^Y D^Y$ is a number operator, i.e.

$$L^Y I_n^Y(f_n) = n I_n^Y(f_n), \quad n \in \mathbf{N}, \; f_n \in L^2(\mathbf{R}_+)^{\circ n}.$$

Proposition 3 *The operator ∇_*^Y coincides with the stochastic integral with respect to Y on the predictable square-integrable processes.*
Proof. In the Gaussian case, $D^B = \nabla^B$, and this result is identical to Prop. 2. In the Poisson case, this can be found in Nualart-Vives [15].

$\square$

A consequence of Prop. 2 and 3 is the following formula, cf. Privault [21]. For $F \in Dom(D^{\tilde{N}}) \cap Dom(\nabla^{\tilde{N}})$,

$$E[D_t^{\tilde{N}} F \mid \mathcal{F}_{t^-}^{\tilde{N}}] = E[\nabla_t^{\tilde{N}} F \mid \mathcal{F}_{t^-}^{\tilde{N}}], \quad dt \otimes dP \ a.e.$$

For $Y = B, \tilde{N}$, the space $L^2(\Omega, \mathcal{F}_\infty^Y)$ admits the orthogonal decomposition

$$L^2(\Omega, \mathcal{F}_\infty^Y) = \bigoplus_{n \geq 0} C_n,$$

where $C_n = \{I_n^Y(f_n) \ : \ f_n \in L^2(\mathbf{R}_+)^{\circ n}\}$ is the nth order chaos, generated by the multiple stochastic integral I_n^Y. If $Y = B$, resp. $Y = \tilde{N}$, we refer to this decomposition as the Wiener-Hermite, resp. Poisson-Charlier chaotic decomposition. We now mention a transfer principle which allows to state on the Poisson space most results of the Malliavin calculus, using the operators $D^{\tilde{N}}$ and $\delta^{\tilde{N}}$. Assume until the end of this section that $(B_t)_{t \in \mathbf{R}_+}$ and $(N_t)_{t \in \mathbf{R}_+}$ are not independent, but linked by the following relation:

$$\tau_k = \frac{\xi_{2k}^2 + \xi_{2k+1}^2}{2}, \quad k \in \mathbf{N}, \tag{1}$$

which does not change anything to the fact that $(N_t)_{t \in \mathbf{R}_+}$ is a Poisson process, since the half sum of the squares of two Gaussian normal random variables is exponentially distributed. Then the following properties are satisfied, cf. Privault [20].
Proposition 4 *We have for $F \in \mathcal{P}$*

$$\delta^B D^B F = 2\delta^{\tilde{N}} D^{\tilde{N}} F, \quad and \quad \mid D^B F \mid_{L^2(\mathbf{R}_+)}^2 = 2 \mid D^{\tilde{N}} F \mid_{L^2(\mathbf{R}_+)}^2, \ a.s.$$

As a consequence, any result that uses the norm of D^B and the operator $\delta^B D^B$ can be stated on the Poisson space, using the norm of $D^{\tilde{N}}$ and $\delta^{\tilde{N}} D^{\tilde{N}}$.
Let us now write the Wiener-Hermite chaotic decomposition of $F \in \mathcal{P}$, F measurable with respect to $\mathcal{F}^{\tilde{N}}$ (i.e. F is a Poisson functional):

$$F = E[F] + \sum_{n \geq 1} I_n^B(f_n).$$

The Laguerre polynomials are defined as

$$L_n(x) = \sum_{k=0}^{k=n} C_n^k \frac{(-x)^k}{k!}, \quad x \in \mathbf{R}_+,$$

they are orthogonal for the exponential density.

Proposition 5 *The space of Poisson functionals in the* $2n$*th Wiener chaos* C_{2n} *is the completion of the vector space generated by*

$$\left\{ \prod_{i=1}^{i=d} I_{2n_i}^B \left(\sum_{k=0}^{k=n_i} \left(\begin{array}{c} n_i \\ k \end{array} \right) h_{2k_i}^{\circ 2k} \circ h_{2k_i+1}^{\circ 2n_i - 2k} \right) \right. \\ \left. : \ n_1 + \cdots + n_d = \frac{n}{2}, \ k_1 \neq \cdots \neq k_d, \ d \in \mathbf{N} \right\},$$

and

$$C_{2n+1} \bigcap L^2\left(\Omega, \mathcal{F}^{\tilde{N}}\right) = \{0\}, \quad n \in \mathbf{N}.$$

For even n*, if* $I_n^B(f_n)$ *is a Poisson functional it can be represented in terms of multidimensional Laguerre polynomials as the linear combination*

$$I_n^B(f_n) = \sum_{\substack{n_1 + \cdots + n_p = n \\ k_1 \neq \cdots \neq k_p}} g(k_1, \ldots, k_p) L_{n_1}(\tau_{k_1}) \cdots L_{n_p}(\tau_{k_p}).$$

Proof. The proof relies on the following relation between the Laguerre and Hermite polynomials:

$$n! L_n\left(\frac{x^2 + y^2}{2}\right) = \frac{(-1)^n}{2^n} \sum_{k=0}^{k=n} \frac{n!}{k!(n-k)!} \sqrt{(2k)!(2n-2k)!} H_{2k}(x) H_{2n-2k}(y),$$

cf. Erdélyi[8], and the fact that the set $\{L_n(\tau_k) \ : \ k, n \in \mathbf{N}\}$ is total in $L^2(\Omega, \mathcal{F})$.

□

A consequence of this proposition is that the projection on the $2n$th order Wiener chaos of a Poisson functional can be represented as a discrete stochastic integral, defined below.

Definition 1 *We define* $J_n : l^2(\mathbf{N})^{\circ n} \to L^2\left(\Omega, \mathcal{F}^{\tilde{N}}\right)$ *as a linear functional with*

$$J_n(e_{k_1}^{\circ n_1} \circ \cdots \circ e_{k_p}^{\circ n_p}) = n_1! \cdots n_p! L_{n_1}(\tau_{k_1}) \cdots L_{n_p}(\tau_{k_p}), \tag{2}$$

$n_1 + \cdots + n_p = n$, $k_1 \neq \cdots \neq k_p$.

The maps $J_n : l^2(\mathbf{N})^{\circ n} \to L^2(\Omega, \mathcal{F}^{\tilde{N}}, P)$ are linear, bounded, and $J_n(g_n)$ is orthogonal to $J_m(g_m)$ for $n \neq m$. Moreover, each square-integrable Poisson functional has the decomposition

$$F = \sum_{n\in\mathbf{N}} J_n(g_n), \quad g_k \in l^2(\mathbf{N})^{\circ k}, \ k \in \mathbf{N}, \tag{3}$$

referred to as the Poisson-Laguerre chaotic decomposition. On the Wiener space, it is known that the notions of derivation defined by infinitesimal perturbations and by annihilation coincide, i.e. $D^B = \nabla^B$ and $\delta^B = \nabla^B_*$. The situation is different on the Poisson space, and more generally for point processes. The operators $D^{\tilde{N}}$ and $\delta^{\tilde{N}}$ are related to the Wiener-Hermite decomposition by the isometry property Prop. 4, and to the Poisson-Laguerre decomposition by the relations, cf. Privault [21]:

$$(\partial^{\tilde{N}} J_n(g_n), e_i)_{l^2(\mathbf{N})} = \sum_{k=0}^{k=n-1} \frac{n!}{k!} J_k(g_n(*, i \ldots, i)), \quad g_n \in l^2(\mathbf{N})^{\circ n},$$

$$\partial^{\tilde{N}}_* J_n(g_{n+1}) = J_{n+1}(\hat{g}_{n+1}) - n J_n(\hat{g}^1_{n+1}), \quad g_{n+1} \in l^2(\mathbf{N})^{\circ n} \otimes l^2(\mathbf{N})$$

where $\hat{g}$ denotes the symmetrization of g and g^1_{n+1} is the contraction defined as

$$g^1_{n+1}(k_1, \ldots, k_n) = \hat{g}_{n+1}(k_1, \ldots, k_n, k_n), \quad k_1, \ldots, k_n.$$

3 A generalization to certain martingales by change of time

Consider a real square-integrable martingale on a filtered probability space $(\Omega, (\mathcal{F}_t), \mathcal{F}_\infty, P)$ with a continuous part X_0 and independent compensated sums $X_1, \ldots, X_d$ of jumps of sizes $z_1, \ldots, z_d \in \mathbf{R}$ respectively:

$$M = X_0 + \sum_{k=1}^{k=d} X_k - \nu_k.$$

Let $\nu_0 =< X_0 >$ denote the quadratic variation of X_0. Assume also that $(\mathcal{F}_t)$ is generated by M, that the processes $X_0, X_1, \ldots, X_d$ are independent, and that $\lim_{t\to\infty} \nu_k(t) = \infty$, $k = 0, \ldots, d$, P-a.s. We know, cf. Ikeda-Watanabe [10], that there is a Brownian motion B and $N^1, \ldots, N^d$ independent standard Poisson processes such that $X_0 = B_{\nu_0}$, $X^k = z_k N^k_{\nu_k}$, $k = 1, \ldots, d$. Let us call $(\mathcal{F}^0_t)$, $(\mathcal{F}^1_t), \ldots, (\mathcal{F}^d_t)$ the filtrations generated by $(B_t)_{t\in\mathbf{R}_+}$, $(N^1_t)_{t\in\mathbf{R}_+}, \ldots, (N^d_t)_{t\in\mathbf{R}_+}$. Denote by $\nu_0^{-1}, \cdots, \nu_d^{-1}$ the right-continuous inverses of $\nu_0, \ldots, \nu_d$:

$$\nu_k^{-1}(t) = \inf\{s \geq 0 \ : \ \nu_k(s) > t\}, \quad k = 0, \ldots, d.$$

We make the following hypothesis:
(**H1**) The processes $\nu_0^{-1}, \ldots, \nu_d^{-1}$ are respectively $(\mathcal{F}_t^0), \ldots, (\mathcal{F}_t^d)$-adapted.
(**H2**) The trajectories of $\nu_0^{-1}, \ldots, \nu_d^{-1}$ are continuous and strictly increasing.
Two consequences of those hypothesis are stated in the lemma below.
Lemma 1 *We have* $\mathcal{F}_\infty = \mathcal{F}_\infty^0 \vee \cdots \vee \mathcal{F}_\infty^d$, *and*

$$\mathcal{F}_{\nu_k^{-1}(t)} \subset \mathcal{F}_\infty^0 \vee \cdots \vee \mathcal{F}_\infty^{k-1} \vee \mathcal{F}_t^k \vee \mathcal{F}_\infty^{k+1} \vee \cdots \vee \mathcal{F}_\infty^d, \quad k = 0, \ldots, d.$$

Proof. It is sufficient to prove this result for $d = 0$. In this case, $M = B_{\nu_0}$ implies $\mathcal{F}_t \subset \mathcal{F}_{\nu_0(t)}^0$ since $(\mathcal{F}_t)$ is generated by M, and $\mathcal{F}_\infty = \mathcal{F}_\infty^0$. We also have $\mathcal{F}_{\nu_0^{-1}(t)} \subset \mathcal{F}_{\nu_0(\nu_0^{-1}(t))}^0$. If $A \in \mathcal{F}_{\nu_0(\nu_0^{-1}(t))}^0$, then $A = \bigcup_{s \in \mathbb{Q}} A \cap \{\nu_0^{-1}(t) \leq s \leq \nu_0^{-1}(u)\}$, $u > t$. Hence $A \in \mathcal{F}_t^0$ by right-continuity of the filtration $(\mathcal{F}_t^0)$. $\square$

We define an extension of the notion of Cameron-Martin space as

$$\mathcal{H} = \left\{ u : \mathbf{R}_+ \to \mathbf{R}^{d+1} \textit{ measurable } : | u |_H^2 = \sum_{k=0}^{k=d} \int_0^\infty u_k^2(t) d\nu_k(t) < \infty \right\},$$

and denote by H its equivalence classes for $| \cdot |_H$. Similarly, let

$$\mathcal{L}^2(M) = \left\{ u : \Omega \times \mathbf{R}_+ \to \mathbf{R}^{d+1} \textit{ measurable } : E[| u |_H^2] < \infty \right\},$$

and denote by $L^2(M)$ the equivalence classes of $\mathcal{L}^2(M)$. No adaptedness requirement is made on the elements of $L^2(M)$. We now turn to the definition of the gradient by change of time. If X is a point process with jumps of size $z \in \mathbf{R}$ and compensator ν, written as $X = N_\nu$, define $i^{X-\nu} : l^2(\mathbf{N}) \to L^2(\mathbf{R}_+)$ as

$$i_t^{X-\nu}(u) = z i_{\nu(t)}^{\tilde{N}}(u) \quad t \in \mathbf{R}_+.$$

If X is a continuous martingale with quadratic variation ν, written as $X = B_\nu$, let

$$i_t^X(u) = i_{\nu(t)}^B(u), \quad t \in \mathbf{R}_+.$$

If $M = M_0 + \cdots + M_d$ is a sum of $d+1$ independent martingales of the above type, define $i^M : l^2(\mathbf{N}, \mathbf{R}^{d+1}) \to L^2(\mathbf{R}_+, \mathbf{R}^{d+1})$ as

$$i_t^M(u) = \left(i_t^{M_1}(u_0), \ldots, i_t^{M_d}(u_d) \right),$$

$u = (u_0, \ldots, u_d)$, $t \in \mathbf{R}_+$. The operator i^M is easily extended to stochastic processes.
Definition 2 *Define an operator* $D^M : \mathcal{P} \to L^2(M)$ *by* $D^M = i^M \circ D$.

We notice that

$$| D^M F |_H^2 = | D^B F |_{L^2(\mathbf{R}_+)}^2 + \sum_{k=1}^{k=d} z_k^2 \, | D^{N^k} F |_{L^2(\mathbf{R}_+)}^2, \quad F \in \mathcal{P}, \tag{4}$$

hence $D^M : L^2(\Omega) \to L^2(M)$ is closable. We call $\mathbb{D}_{2,1}^M$ its domain.

Remark. This definition is consistent with that of Bismut [3], Decreusefond [6]. For instance, if $M = X_1 - \nu_1$ and $F \in \mathcal{P}$ with $F = f(T_1, \ldots, T_n)$,

$$(D^M F, h)_{L^2(\mathbf{R}_+)} = - \sum_{k=1}^{k=n} \partial_k f(T_1, \ldots, T_n) \int_0^{\nu_1^{-1}(T_k)} h(t) d\nu_1(t),$$

which means that $(D^M F, h)_{L^2(\mathbf{R}_+)}$ is defined by perturbation of the kth jump time $\nu_1^{-1}(T_k)$ of X_1 into

$$\nu_1^{-1} \left(\int_0^{\nu_1^{-1}(T_k)} (1 + \varepsilon h(t)) d\nu_1(t) \right).$$

□

Let $j^M : L^2(\mathbf{R}_+, \mathbf{R}^{d+1}) \to l^2(\mathbf{N}, \mathbf{R}^{d+1})$ be the random dual operator of i^M with respect to $(\cdot,\cdot)_H$, satisfying

$$(j^M(u), v)_{l^2(\mathbf{N}, \mathbf{R}^d)} = (i(u), v)_H.$$

We have

$$j^M(u) = \left(j^B(u_0 \circ \nu_0^{-1}), j^{\tilde{N}_1}(u_1 \circ \nu_1^{-1}), \ldots, j^{\tilde{N}_d}(u_d \circ \nu_d^{-1}) \right).$$

The adjoint δ^M of D^M is defined below. Let

$$\begin{aligned} \mathcal{V}^M &= i^M(\mathcal{U}) \\ &= \left\{ \sum_{k=0}^{k=n} (u_k^0 h_k \circ \nu_0, \ldots, u_k^d h_k \circ \nu_d) \ : \ u_k^i \in \mathcal{P}, i = 0, \ldots, d, \quad n \in \mathbf{N} \right\}. \end{aligned}$$

Definition 3 *We define* $\delta^M : \mathcal{V}^M \to L^2(\Omega)$ *by*

$$\delta^M(u) = \delta^B \left(u_0 \circ \nu_0^{-1} \right) + \sum_{k=1}^{k=d} z_k \delta^{\tilde{N}^k} \left(u_k \circ \nu_k^{-1} \right).$$

It is clear that

$$E\left[F\delta^M(u)\right] = E\left[(D^M F, u)_H\right], \quad F \in \mathcal{P},\ u \in \mathcal{V}^M,$$

hence δ^M is closable and adjoint of D^M. Moreover, the operator δ^M coincides with the stochastic integral with respect to M on square-integrable predictable processes:

Proposition 6 *If $u = (u_0, \ldots, u_d) \in L^2(M)$ with u_k $(\mathcal{F}_t^k)$-predictable, $k = 0, \ldots, d$, then*

$$\delta^M(u) = \int_0^\infty u^0(t) dX_0(t) + \sum_{k=1}^{k=d} \int_0^\infty u_k(t) d(X_k(t) - \nu_k(t)).$$

Proof. In the case $d = 0$. If u_0 is $(\mathcal{F}_t^0)$-predictable, then $u_0 \circ \nu_0$ is $(\mathcal{F}_t^B)$-predictable and

$$\delta^M(u) = \delta^B(u_0 \circ \nu_0) = \int_0^\infty u_0(\nu_0(t)) dB_t = \int_0^\infty u_0(t) dM_t.$$

□

In particular, if $u \in L^2(M)$ is $(\mathcal{F}_t)$-predictable with

$$E\left[\sum_{k=0}^{k=d} \int_0^\infty u(t)^2 d\nu_k(t))\right] < \infty,$$

then

$$\delta^M(u) = \int_0^\infty u(t) dM_t.$$

Proposition 7 *In the anticipative case, we have for $u \in \mathcal{V}^M$:*

$$\begin{aligned}
\delta^M(u) = \\
& \int_0^\infty u_0(t) dX_0(t) + \sum_{k=1}^{k=d} \int_0^\infty u_k(t) d(X_k(t) - d\nu_k(t)) \\
& - \int_0^\infty D_t^{X_0}(u_0(\nu_0^{-1}(t))) d\nu_0(t) - \sum_{k=1}^{k=d} \int_0^\infty D_t^{X_k - \nu_k}(u_k(\nu_k^{-1}(t))) d\nu_k(t).
\end{aligned}$$

Proof. We apply Prop. 1.

□

For $f_{n_0,\ldots,n_d} \in L^2(\mathbf{R}_+)^{\circ n_0} \circ l^2(\mathrm{N})^{\circ n_1} \circ \cdots \circ l^2(\mathrm{N})^{\circ n_d}$, define the Wiener-Hermite-Laguerre integral $J_n(f_{n_0,\ldots,n_d})$ as

$$J_n(f_{n_0} \circ f_{n_1} \circ \cdots \circ f_{n_d}) = I^B_{n_0}(f_{n_0}) J^1_{n_1}(f_{n_1}) \cdots J^d_{n_d}(f_{n_d}).$$

Where $J^k_{n_k}(f_{n_k})$ is defined as in Def. 1, for the Poisson process $\tilde{N}_k$. Then any square-integrable functional on Ω has the decomposition

$$F = \sum_{n_0,\ldots,n_d \geq 0} J_n(f_{n_0,\ldots,n_d}),$$

$f_{n_0,\ldots,n_d} \in L^2(\mathbf{R}_+)^{\circ n_0} \circ l^2(\mathrm{N})^{\circ n_1} \circ \cdots \circ l^2(\mathrm{N})^{\circ n_d}$. Next, we notice that this decomposition is preserved by the operator $\delta^M D^M$:

Proposition 8 *We have $\delta^M D^M = \delta^B D^B + \sum_{k=1}^{k=d} z_k^2 \delta^{N^k} D^{N^k}$. Consequently,*

$$\delta^M D^M J_n(f_{n_0,\ldots,n_d}) = \left(\sum_{k=0}^{k=d} n_k z_k^2 \right) J_n(f_{n_0,\ldots,n_d}), \tag{5}$$

where $n = n_0 + \cdots + n_d$, and $f_{n_0,\ldots,n_d} \in L^2(\mathbf{R}_+)^{\circ n_0} \circ l^2(\mathrm{N})^{\circ n_1 + \cdots + n_d}$. (We let $z_0 = 1$).

4 Absolute continuity results

The aim of this section is to show that from Prop. 4, absolute continuity results can be obtained with D^M and δ^M as consequences of their Wiener space counterparts. Sobolev spaces are defined as follows. Let $\| \cdot \|_{p,k}$ be the norm defined as $\| F \|_{p,k} = \| (I + \mathcal{L}^M)^{k/2} F \|_p$, and denote by $\mathbb{D}^M_{p,k}$ the completion of $\mathcal{P}$ with respect to $\| \cdot \|_{p,k}$, and

$$\mathbb{D}^M_\infty = \bigcap_{p>1, k \in \mathbb{Z}} \mathbb{D}^M_{p,k}.$$

A number of results in stochastic analysis can be expressed with the operators δ^M and D^M. For the sake of simplicity, those results are stated in the case of $\mathbf{R}$-valued functionals, but can also be obtained in the finite dimensional vector-valued case.

Theorem 1 *(Meyer[13]) There exists two constants $A, B > 0$ such that for any $F \in \mathcal{P}$,*

$$A \, \| \, | D^M F |^2_H \|_p \leq \| F \|_{p,k} \leq B \, \| \, | D^M F |_H \|_p .$$

Theorem 2 *(Watanabe [26]) Let $F \in \mathbb{D}^M_\infty$ such that*

$$\mid D^M F \mid_H^{-1} \in \cap_{p>1} L^p(\Omega, \mathcal{F}).$$

Then F has a $\mathcal{C}^\infty$ density with respect to the Lebesgue measure.
Theorem 3 *(Bouleau-Hirsch [4]) Let $F \in \mathbb{D}^M_{2,1}$ such that $\mid D^M F \mid_H > 0$ a.s. Then F has a density with respect to the Lebesgue measure.*
Those results are directly obtained from their Wiener space counterparts, using the isometry property, cf. Prop. 4, and relations (4), (5) .

5 Clark formula and chaotic calculus

Functionals in $L^2(\Omega)$ have a Wiener-Hermite-Charlier chaotic decomposition. If $Y_1, \ldots, Y_p$ are compensated Poisson or Wiener processes, with Y_i independent of Y_j if $Y_i \neq Y_j$, $i, j \in \{1, \ldots, p\}$, define for $f_p \in L^2(\mathbf{R}_+)^{\circ p}$ symmetric and square-integrable

$$I_p^{Y_1,\ldots,Y_p}(f_p) = p! \int_0^\infty \int_0^{t_p^-} \cdots \int_0^{t_2^-} f_p(t_1, \ldots, t_p) dY_1(t_1) \cdots dY_p(t_p).$$

With the notation $Y_0 = B$, $Y_k(t) = N_k(t) - t$, $k = 1, \ldots, d$, any $F \in L^2(\Omega)$ can be written as

$$F = E[F] + \sum_{n\geq 1} \sum_{\theta \in \Theta_n} I_n^{Y_{\theta(1)},\ldots,Y_{\theta(n)}}(f_\theta),$$

where Θ_n is the set of all applications from $\{1, \ldots, n\}$ into $\{0, \ldots, d\}$, cf. Dellacherie *et al.* [7]. We can define an operator

$$\nabla^M : L^2(\Omega) \to L^2(\Omega) \otimes L^2(\mathbf{R}_+, \mathbf{R}^{d+1})$$

by $\nabla^M F = i^M \circ \nabla F$. The adjoint of this operator is given by

$$\nabla^M_*(u) = \nabla^B_*(u_0 \circ \nu_0^{-1}) + \sum_{i=1}^{i=d} z_i \nabla^{\tilde{N}^i}_*(u_i \circ \nu_i^{-1}).$$

This operator extends the stochastic integral with respect to M on the predictable processes in $L^2(M)$ as in Prop. 6, and the composition $\nabla^M_* \nabla^M$ satisfies

$$\nabla^M_* \nabla^M I_n^{Y_{\theta(1)},\ldots,Y_{\theta(n)}}(f_n) = \left(\sum_{k=0}^{k=n} z^2_{\theta(k)} \right) I_n^{Y_{\theta(1)},\ldots,Y_{\theta(n)}}(f_n).$$

The Clark formula can also be written with the operator D^X or ∇^X.

Proposition 9 *Let $F \in Dom(D^M)$. We have if $F \in \mathcal{F}^B_\infty$:*

$$F = E[F] + \int_0^\infty E[D_t^{X_0} F \mid \mathcal{F}_t] dX_0(t).$$

If F is $\mathcal{F}^k_\infty$ measurable,

$$F = E[F] + \int_0^\infty E[D_t^{X^k - \nu_k} F \mid \mathcal{F}_{t-}] d(X_k(t) - d\nu_k(t)).$$

If $F \in Dom(\nabla^M)$ and is $\mathcal{F}^k_\infty$-measurable,

$$F = E[F] + \int_0^\infty E[\nabla_t^{X^k} F \mid \mathcal{F}_{t-}] d(X_k(t) - d\nu_k(t)).$$

Proof. We do the proof for $d = 0$. Let us write the Clark formula on Wiener space, cf. Ocone [16]:

$$F = E[F] + \int_0^\infty E[D_t^B F \mid \mathcal{F}_t^0] dB_t,$$

and notice that by definition of the adapted projection, cf. Revuz-Yor [23],

$$E[D^B_{\nu_0(t)} F \mid \mathcal{F}^0_{\nu_0(t)}] = E[D^B_\cdot F \mid \mathcal{F}^0_\cdot](\nu_0(t)).$$

The same argument holds in case $d > 0$ for the predictable projection.

□

Remark. Consider the space $\Omega = ([-1, 1], dx/2)^{\otimes\infty}$. Denote by θ_k the kth canonical projection, and by Y the process

$$Y_t = \sum_{k\geq 0} 1_{[2k+1+\theta_k, \infty[}, \quad t \in \mathbf{R}_+,$$

with compensator

$$d\nu(t) = \sum_{k\geq 0} \frac{1}{2k + 2 - t} 1_{[2k, 2k+1+\theta_k[}(t) dt.$$

The hypothesis H2 is not fulfilled. If, instead of $d\nu_t$ we use dt as a compensator, it can be shown, cf. Privault [18] that the definition of the gradient should be

$$\tilde{D}F = \sum_{k\geq 0} -((1 - \theta_k) 1_{]2k, 2k+1+\theta_k]} - (1 + \theta_k) 1_{]2k+1+\theta_k, 2k+2]}) \partial_k f(\theta_0, \ldots, \theta_n),$$

which is a variation of Def. 2. Let $\tilde{\delta}$ denote the adjoint of $\tilde{D}$ in $L^2(B) \otimes L^2(\mathbf{R}_+)$. The spectral decomposition of $\tilde{\delta}\tilde{D}$ is then given by the Legendre polynomials evaluated in the θ_k's, instead of the Laguerre or Hermite polynomials as in Prop. 8.

References

1. R.J. Bass and M. Cranston. Malliavin calculus for jump processes and applications to local time. *Annals of Probability* **17**, 507 (1984).
2. K. Bichteler, J.B. Gravereaux, and J. Jacod. *Malliavin Calculus for Processes with Jumps.* Stochastics Monographs. Gordon and Breach, 1987.
3. J.M. Bismut. Calcul des variations stochastique et processus de sauts. *Z. Warsch. Verw. Gebiete* **63**, 147 (1983).
4. N. Bouleau and F. Hirsch. *Dirichlet Forms and Analysis on Wiener Space.* de Gruyter, Berlin/New York, 1991.
5. E. Carlen and E. Pardoux. Differential calculus and integration by parts on Poisson space. In *Stochastics, Algebra and Analysis in Classical and Quantum Dynamics*, pages 63–73. Kluwer, 1990.
6. L. Decreusefond. Une autre formule d'intégration par parties. In P. Florchinger, editor, *Proceedings of a Second Workshop on Stochastic Analysis.* Birkäuser, 1994.
7. C. Dellacherie, B. Maisonneuve, and P.A. Meyer. *Probabilités et Potentiel*, volume 4. Hermann, 1992.
8. A. Erdélyi. *Higher Transcendental Functions*, volume 2. McGraw Hill, New York, 1953.
9. B. Gaveau and P. Trauber. L'intégrale stochastique comme opérateur de divergence dans l'espace fonctionnel. *Journal of Functional Analysis* **46**, 230 (1982).
10. N. Ikeda and S. Watanabe. *Stochastic Differential Equations and Diffusion Processes.* North-Holland, 1989.
11. S. Kusuoka. The nonlinear transformation of Wiener measure on Banach space and its absolute continuity. *Journal of the Faculty of Science of Tokyo University, Section IA, Mathematics* **29**, 567 (1982).
12. P. Malliavin. Stochastic calculus of variations and hypoelliptic operators. In *Intern. Symp. SDE. Kyoto*, pages 195–253, Tokyo, 1976. Kinokumiya.
13. P.A. Meyer. Transformations de Riesz pour les lois gaussiennes. In *Séminaire de Probabilité XVIII*, Lecture Notes in Mathematics, pages 179–193. Springer Verlag, 1984.
14. D. Nualart and E. Pardoux. Stochastic calculus with anticipative integrands. *Probability Theory and Related Fields* **78**, 535 (1988).
15. D. Nualart and J. Vives. Anticipative calculus for the Poisson process based on the Fock space. In *Séminaire de Probabilité de l'Université de Strasbourg XXIV*, volume 1426 of *Lecture Notes in Mathematics.*

Springer Verlag, 1990.
16. D. Ocone. A guide to the stochastic calculus of variations. In H. Korezlioglu and A.S. Üstünel, editors, *Stochastic Analysis and Related Topics*, volume 1316 of *Lecture Notes in Mathematics*. Springer-Verlag, 1988.
17. N. Privault. Calcul des variations stochastique pour les martingales. To appear in the Comptes Rendus de l'Académie des Sciences de Paris, Série I.
18. N. Privault. Calcul des variations stochastiques pour la mesure de densité uniforme. To appear in Potential Analysis.
19. N. Privault. Girsanov theorem for anticipative shifts on Poisson space. To appear in Probability Theory and Related Fields.
20. N. Privault. A transfer principle from Wiener to Poisson space and applications. To appear in the Journal of Functional Analysis.
21. N. Privault. Chaotic and variational calculus in discrete and continuous time for the Poisson process. *Stochastics and Stochastics Reports* **51**, 83 (1994).
22. N. Privault. Inégalités de Meyer sur l'espace de Poisson. *Comptes Rendus de l'Académie des Sciences de Paris, Série I* **318**, 559 (1994).
23. D. Revuz and M. Yor. *Continuous Martingales and Brownian Motion.* Springer-Verlag, 1994.
24. D. Surgailis. On multiple Poisson stochastic integrals and associated Markov semi-groups. *Probability and Mathematical Statistics* **3**, 217 (1984).
25. A.S. Üstünel. The Itô formula for anticipative processes with non-monotonous time via the Malliavin calculus. *Probability Theory and Related Fields* **79**, 249 (1988).
26. S. Watanabe. *Lectures on Stochastic Differential Equations and Malliavin Calculus.* Tata Institute of Fundamental Research, 1984.