Moments of Markovian growth-collapse processes

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Abstract

We apply general moment identities for Poisson stochastic integrals with random integrands to the computation of the moments of Markovian growth-collapse processes. This extends existing formulas for mean and variance available in the literature to closed-form moment expressions of all orders. In comparison with other methods based on differential equations, our approach yields explicit summations in terms of the time parameter. We also treat the case of the associated embedded chain, and provide recursive codes in Maple and Mathematica for the computation of moments and cumulants of any order with arbitrary cut-off moment sequences and jump size functions.

Key words: Growth-collapse processes, Poisson shot noise, uniform cut-off rates, stochastic integrals with jumps, moments, cumulants.
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1 Introduction

Markovian growth-collapse processes, see Eliazar and Klafter (2004), are piecewise-deterministic Markov processes (Davis (1984)), that grow in between random jump times at which they may randomly crash. Growth-collapse processes are used in e.g. earth sciences and physics, and they have also been recently applied to the study of crypto-currencies, see Frolkova and Mandjes (2019).

Let $(N_t)_{t\in\mathbb{R}_+}$ denote a standard Poisson process with intensity $\lambda > 0$ and jump times $(T_k)_{k\geq 1}$, with $T_0 := 0$. The growth-collapse process $(X_t)_{t\in\mathbb{R}+}$ increases lineary as t and crashes at times T_k by the amount $(1 - Z_k)X_{T_k^-}$, i.e.,

$$X_{T_k} = Z_k X_{T_k^-}, \qquad k \geq 1,$$

where $X_{T_k^-}$ denotes the left limit of the process at time T_k and $(Z_k)_{k\geq 1}$ is an i.i.d. random sequence of cut-off rates on $[0, 1]$, independent of $(N_t)_{t\in\mathbb{R}_+}$.

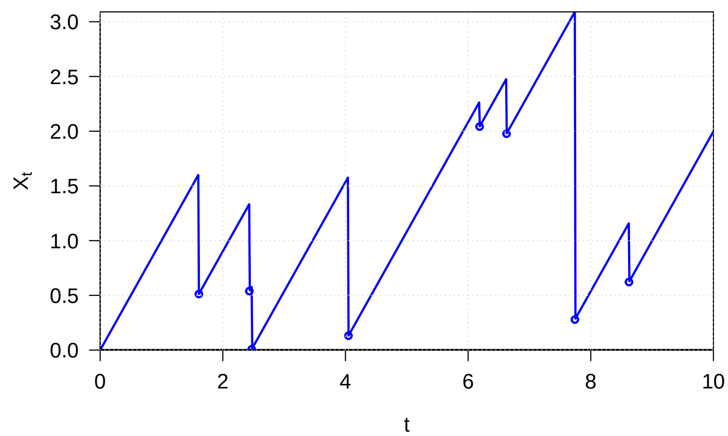


Figure 1: Sample path of a growth-collapse process.

In other words, $(X_t)_{t\in\mathbb{R}_+}$ solves the jump stochastic differential equation

$$dX_t = dt - (1 - Z_{N_t})X_{t^-}dN_t, \tag{1.1}$$

with $X_0 = 0$, or

$$X_t = t - \int_0^t (1 - Z_{N_s})X_{s^-}dN_s, \qquad t \geq 0,$$

with explicit solution

$$\begin{aligned} X_t &= t - \sum_{k=1}^{N_t} T_k(1 - Z_k) \prod_{l=k+1}^{N_t} Z_l \qquad (1.2) \\ &= t - \int_0^t s(1 - Z_{N_s}) \prod_{l=N_s+1}^{N_t} Z_l dN_s, \qquad t \geq 0. \end{aligned}$$

In particular, the process value after the nth collapse epoch is

$$X_{T_n} = \sum_{k=1}^{n} T_k \prod_{l=k}^{n} Z_l - \sum_{k=1}^{n-1} T_k \prod_{l=k+1}^{n} Z_l,$$

and its value before the nth collapse epoch is the left limit

$$X_{T_n^-} = \sum_{k=1}^{n} T_k \prod_{l=k}^{n-1} Z_l - \sum_{k=1}^{n-1} T_k \prod_{l=k+1}^{n-1} Z_l, \qquad n \geq 1,$$

see Figure 1.

The computation of moments of growth-collapse processes has been the object of several approaches, see Boxma et al. (2006) for the use of conditional distributions for the computation of mean and variance, and Daw and Pender (2020) for moment expressions of all orders using the solution of differential equations by matrix exponentials.

In this paper, we apply general moment identities written as sums over partitions for Poisson stochastic integrals with random integrands, see Privault (2009; 2012a;b; 2016), to the computation of the moments of growth-collapse processes. In particular, we obtain closed-form moment expressions in the case of uniformly distributed cut-off rates.

Given $(N_t)_{t\in\mathbb{R}_+}$ a standard Poisson process with intensity $\lambda > 0$ on $\mathbb{R}_+$, consider a process $(Y_t)_{t\in\mathbb{R}_+}$ of the form

$$Y_t = \int_0^t h(N_{s^-}) dN_s, \qquad t \geq 0,$$

where $h : \mathbb{N} \to \mathbb{R}$ is a deterministic function. As the left limit $h(N_{s^-})$ is predictable with respect to the filtration $(\mathcal{F}_s)_{s\in\mathbb{R}_+}$ generated by $(N_s)_{s\in\mathbb{R}_+}$ the mean of Y_t can be computed from the smoothing lemma, see e.g. Theorem 9.2.1 in Brémaud (1999), as

$$\mathbb{E}[Y_t] = \lambda \mathbb{E}\left[\int_0^t h(N_{s^-}) ds\right] = \lambda \int_0^t \mathbb{E}[h(N_{s^-})] ds, \qquad t \geq 0. \tag{1.3}$$

This calculation does not apply however to the process $h(N_s)$ which is only adapted and not predictable with respect to the filtration $(\mathcal{F}_s)_{s\in\mathbb{R}_+}$. In this case we may apply the Slivnyak-Mecke formula, see Slivnyak (1962), Mecke (1967) or Corollary 3.2.3 in Schneider and Weil (2008) and § 2.3.4 of Chiu et al. (2013), to obtain

$$\mathbb{E}[Y_t] = \lambda \mathbb{E}\left[\int_0^t \varepsilon_s^+ h(N_s) ds\right] = \lambda \mathbb{E}\left[\int_0^t h(1 + N_s) ds\right], \qquad t \geq 0,$$

where ε_s^+ denotes the operator that adds one jump at the location $s \geq 0$ to the Poisson process path.

As an example, the first moment of X_t given by (1.2) can be computed using the Slivnyak-Mecke formula as

$$\begin{aligned}
\mathbb{E}[X_t] &= t - \mathbb{E}\left[\int_0^t s(1-Z_{N_s}) \prod_{l=N_s+1}^{N_t} Z_l dN_s\right] \\
&= t - \lambda\mathbb{E}\left[\int_0^t \varepsilon_s^+\left(s(1-Z_{N_s}) \prod_{l=N_s+1}^{N_t} Z_l\right) ds\right] \\
&= t - \lambda\mathbb{E}\Bigg[\int_0^t s(1-Z_{1+N_s}) \prod_{l=2+N_s}^{1+N_t} Z_l ds\Bigg] \\
&= t - \lambda\mathbb{E}\Bigg[\int_0^t s \times (1-\mu_1)\mu_1^{N_t-N_s} ds\Bigg] \\
&= t - \lambda(1-\mu_1)\int_0^t s\mathrm{e}^{-\lambda(1-\mu_1)(t-s)} ds \\
&= \frac{1-\mathrm{e}^{-\lambda(1-\mu_1)t}}{\lambda(1-\mu_1)}, \qquad t \geq 0, \qquad (1.4)
\end{aligned}$$

where $\mu_1 = \mathbb{E}[Z_1]$, which extends Theorem 4 in Boxma et al. (2006) to non-uniform cut-off rates and yields the limiting behavior

$$\lim_{t\to\infty} \mathbb{E}[X_t] = \frac{1}{\lambda(1-\mu_1)},$$

provided that $\mu_1 < 1$.

In order to compute higher order moments in a more general setting, we will apply a nonlinear extension of the Slivnyak-Mecke identity, see Proposition 2.1 below, which allows us to express the moments of Poisson stochastic integrals as a sum of multiple integrals with respect to the intensity of the Poisson process over partitions. In Section 3 we consider the computation of moments of jump processes of the form

$$Y_t = \sum_{k=1}^{N_t} g(T_k, k, N_t) = \int_0^t g(s, N_s, N_t) dN_s,$$

where $(T_k)_{k\geq 1}$ denotes the sequence of jump times of the Poisson process $(N_t)_{t\in\mathbb{R}_+}$ and $g(s,k,m)$ is possibly random but independent of $(N_t)_{t\in\mathbb{R}_+}$, see Proposition 3.1 and its Corollary 3.2. Those identities are then specialized in Section 4 to the case of uniform cut-off distributions, for processes of the form

$$Y_t = \sum_{k=1}^{N_t} f_k(T_k)(1-U_k) \prod_{l=k+1}^{N_t} U_l, \qquad t \in \mathbb{R}_+,$$

where $(U_k)_{k\geq 1}$ is an i.i.d. uniform random sequence on $[0,1]$, independent of the standard Poisson process $(N_t)_{t\in\mathbb{R}_+}$, and $(f_k)_{k\geq 1}$ is a sequence of measurable functions on $\mathbb{R}_+$, see Corollary 4.1.

In particular, in Proposition 5.1 we derive the closed-form summation

$$\mathbb{E}[(X_t)^n] = \frac{(n+1)!}{\lambda^n} \sum_{k=0}^{n} (-1)^k (k+1)^{n-1} \binom{n}{k} \mathrm{e}^{-k\lambda t/(k+1)}, \qquad t \in \mathbb{R}_+, \tag{1.5}$$

for the moments of all orders $n \geq 0$ of the growth-collapse process

$$X_t = t - \sum_{k=1}^{N_t} T_k (1-U_k) \prod_{l=k+1}^{N_t} U_l, \qquad t \in \mathbb{R}_+,$$

where $(U_k)_{k\geq 1}$ is an i.i.d. uniform random sequence on $[0,1]$. This result extends Theorems 4 and 5 as well as Corollary 1 of Boxma et al. (2006) from mean and variance to higher moments of all orders, and provides a closed-form alternative to Corollary 4 in Daw and Pender (2020) which uses matrix exponentials. The expression (1.5) immediately yields the asymptotic moments

$$\lim_{t\to\infty} \mathbb{E}[(X_t)^n] = \frac{(n+1)!}{\lambda^n}, \qquad n \geq 1,$$

which recover the gamma stationary distribution of $(X_t)_{t\in\mathbb{R}_+}$ with shape parameter 2, see Theorem 3 in Boxma et al. (2006).

Finally, in Section 6 we show that our approach can also be applied to discrete-time embedded processes of the form

$$Y(m) = \sum_{k=1}^{m} g(T_k, k, m) = \int_0^{T_m} g(s, N_s, m) dN_s, \qquad m \geq 1,$$

see Corollaries 6.2-6.3, and to the embedded growth-collapse chain

$$X(m) = T_m - \sum_{k=1}^{m} T_k (1-U_k) \prod_{l=k+1}^{m} U_l, \qquad m \geq 1,$$

where $(U_k)_{k\geq 1}$ is an i.i.d. uniform random sequence on $[0,1]$, see Corollaries 6.4-6.5. This recovers Theorem 7 stated for mean and variance in Boxma et al. (2006), and provides moment expressions of all orders.

We proceed as follows. In Section 2 we review the derivation of moment identities for stochastic integrals using sums over partitions, and in Section 3 we apply them to the moments of jump processes driven by a Poisson process. Those expressions are then specialized as closed-form identities in Section 4 in the case of uniform cut-off distributions. The moments of growth-collapse processes are considered in Section 5, and the case of embedded chains is treated in Section 6.

2 Moment identities for Poisson stochastic integrals

In this section we review the computation of moments of Poisson stochastic integrals with random integrands using sums over partitions, see Proposition 3.1 in Privault (2012a). Consider a Poisson process $(N_t)_{t\in\mathbb{R}_+}$ constructed as $N_t = \omega([0,t])$, where $\omega(ds)$ is a Poisson random measure of intensity $\lambda(ds)$, with sequence $(T_k)_{k\geq 1}$ of jump times. For any $s_1,\ldots,s_k \in \mathbb{R}_+$, we let $\epsilon^+_{s_1}\cdots\epsilon^+_{s_k}$ denote the operator

$$(\epsilon^+_{s_1}\cdots\epsilon^+_{s_k}F)(\omega) = F(\omega\cup\{s_1,\ldots,s_k\})$$

acting on random variables F by addition of points at locations $s_1,\ldots,s_k$ to the point process $\omega(dx)$. For example, if F takes the form $F = f(N_{t_1},\ldots,N_{t_n})$, then we have

$$\epsilon^+_{s_1}\cdots\epsilon^+_{s_k}F = f\big(N_{t_1} + \#\{k \ : \ s_k \leq t_1\},\ldots,N_{t_n} + \#\{k \ : \ s_k \leq t_n\}\big).$$

The following moment identity, see Proposition 3.1 in Privault (2012a) and Theorem 1 in Privault (2016), uses sums over partitions $\{\pi_1,\ldots,\pi_k\}$ of $\{1,\ldots,n\}$, and applies to random integrands $u : \mathbb{R}_+\times\Omega \longrightarrow \mathbb{R}$.

Proposition 2.1 *Let $(u_s(\omega))_{s\in[0,t]}$ denote a stochastic process indexed by $s\in[0,t]$. For any $n\geq 1$, we have*

$$\mathbb{E}\left[\left(\int_0^t u_s dN_s\right)^n\right]$$
$$= \sum_{k=1}^n \sum_{\pi_1\cup\cdots\cup\pi_k=\{1,\ldots,n\}} \mathbb{E}\left[\int_0^t\cdots\int_0^t \epsilon^+_{s_1}\cdots\epsilon^+_{s_k}\big(u^{|\pi_1|}(s_1,\omega)\cdots u^{|\pi_k|}(s_k,\omega)\big)\lambda(ds_1)\cdots\lambda(ds_k)\right],$$

where the power $|\pi_i|$ denotes the cardinality of the subset π_i and the above sum runs over all partitions $\pi_1,\ldots,\pi_k$ of $\{1,\ldots,n\}$.

In the sequel we will frequently use the equivalent combinatorial expressions

$$\sum_{k=1}^n \sum_{\pi_1\cup\ldots\cup\pi_k=\{1,\ldots,n\}} f_k(|\pi_1|,\ldots,|\pi_k|) \;=\; \sum_{k=1}^n \frac{n!}{k!} \sum_{\substack{p_1+\cdots+p_k=n\\ p_1,\ldots,p_k\geq 1}} \frac{f_k(p_1,\ldots,p_k)}{p_1!\cdots p_k!} \tag{2.1}$$

$$= \sum_{k=1}^{n} \frac{n!}{k!} \sum_{q_0=0<q_1<\cdots<q_k=n} \frac{f_k(q_1-q_0,\ldots,q_k-q_{k-1})}{(q_1-q_0)!\cdots(q_k-q_{k-1})!}$$

for f_k a function on $\mathbb{N}^k$, $k=1,\ldots,n$. In particular, for $x_1,\ldots,x_n \in \mathbb{R}$ and $f_k(p_1,\ldots,p_k) = x_{p_1}\cdots x_{p_k}$ this yields the Bell polynomial of order $n \geq 1$ as

$$\begin{aligned} B_n(x_1,\ldots,x_n) &= \sum_{k=1}^{n} \frac{n!}{k!} \sum_{\substack{p_1+\cdots+p_k=n \\ p_1\geq 1,\ldots,p_k\geq 1}} \frac{x_{p_1}\cdots x_{p_k}}{p_1!\cdots p_k!} \\ &= \sum_{k=1}^{n} \sum_{\pi_1\cup\cdots\cup\pi_k=\{1,\ldots,n\}} x_{|\pi_1|}\cdots x_{|\pi_k|} \qquad (2.2) \\ &= \sum_{k=1}^{n} B_{k,n}(x_1,\ldots,x_{n-k+1}), \end{aligned}$$

where $B_{k,n}$ is the partial Bell polynomial of order (k,n). We will also use the relation $\mathbb{E}[X^n] = B_n(\kappa_X^{(1)},\ldots,\kappa_X^{(n)})$ between the moments $\mathbb{E}[X^n]$ and the cumulants $\kappa_X^{(n)}$ of a random variable X, and the inversion relation

$$\begin{aligned} \kappa_n^X &= \sum_{k=1}^{n} (k-1)!(-1)^{k-1} \sum_{\pi_1\cup\cdots\cup\pi_k=\{1,\ldots,n\}} \mathbb{E}\big[X^{|\pi_1|}\big]\cdots\mathbb{E}\big[X^{|\pi_k|}\big] \qquad (2.3) \\ &= \sum_{k=1}^{n} (-1)^{k-1}(k-1)! B_{k,n}\big(\mathbb{E}[X],\ldots\mathbb{E}\big[X^{n-k+1}\big]\big), \qquad n \geq 1, \end{aligned}$$

see Theorem 1 of Lukacs (1955), or Leonov and Shiryaev (1959).

The case of shot noise processes

Before moving to the setting of Markovian growth-collapse processes, we use the case of Poisson shot noise processes as an illustration for the result of Proposition 2.1. Consider a shot noise process $(S_t)_{t\in\mathbb{R}_+}$ of the form

$$S_t = \sum_{k=1}^{N_t} h(T_k,t)J_k = \sum_{k=1}^{N_t} h(T_k,t)J_{N_{T_k}} = \int_0^t h(s,t)J_{N_s}dN_s, \qquad t \in \mathbb{R}_+,$$

where $(J_k)_{k\geq 0}$ is a sequence of i.i.d. random variables admitting moments of all orders, and $h(\cdot,\cdot)$ is a sufficiently integrable deterministic function. The next proposition provides a closed-form expression for the moments of shot noise processes using standard Bell polynomials, see also Corollary 2 in Daw and Pender (2020) for another expression using matrix exponentials in case $\lambda(ds) = \lambda ds$ for some rate $\lambda > 0$, and $h(s,t) = \mathrm{e}^{-\beta(t-s)}$ for some $\beta > 0$.

Proposition 2.2 *For any $n \geq 1$, we have*

$$\mathbb{E}[S_t^n] = B_n\left(\mathbb{E}[J_1]\int_0^t h(s,t)ds, \ldots, \mathbb{E}[J_1^n]\int_0^t h^n(s,t)ds\right), \tag{2.4}$$

where B_n is the Bell polynomial of order $n \geq 1$.

Proof. Taking $u_s(\omega) := J_{N_s} h(s,t)$, by Proposition 2.1 we have

$$\mathbb{E}[S_t^n] = \mathbb{E}\left[\left(\sum_{k=1}^{N_t} J_{N_{T_k}} h(T_k,t)\right)^n\right] = \sum_{l=1}^n n! \sum_{\pi_1\cup\cdots\cup\pi_l=\{1,\ldots,n\}}$$
$$\int_0^t\int_0^{s_l}\cdots\int_0^{s_2} h^{|\pi_1|}(s_1,t)\cdots h^{|\pi_l|}(s_l,t)\mathbb{E}\big[\epsilon_{s_1}^+\cdots\epsilon_{s_l}^+\big(J_{N_{s_1}}^{|\pi_1|}\cdots J_{N_{s_l}}^{|\pi_l|}\big)\big]\lambda(ds_1)\cdots\lambda(ds_l).$$

Next, we note that for $0 < s_1 < \cdots < s_l < t$ we have

$$\epsilon_{s_1}^+\cdots\epsilon_{s_l}^+ N_{s_i} = i + N_{s_i}, \qquad 1 \leq i \leq l \leq n, \tag{2.5}$$

hence

$$\begin{aligned}
\mathbb{E}\big[\epsilon_{s_1}^+\cdots\epsilon_{s_l}^+\big(J_{N_{s_1}}^{|\pi_1|}\cdots J_{N_{s_l}}^{|\pi_l|}\big)\big] &= \mathbb{E}\big[J_{1+N_{s_1}}^{|\pi_1|}J_{2+N_{s_2}}^{|\pi_2|}\cdots J_{l+N_{s_l}}^{|\pi_l|}\big]\\
&= \mathbb{E}\big[\mathbb{E}\big[J_{1+N_{s_1}}^{|\pi_1|}J_{2+N_{s_2}}^{|\pi_2|}\cdots J_{l+N_{s_l}}^{|\pi_l|}\,\big|\,N_{s_1},\ldots,N_{s_l}\big]\big]\\
&= \mathbb{E}\big[\mathbb{E}\big[J_{1+N_{s_1}}^{|\pi_1|}\,\big|\,N_{s_1},\ldots,N_{s_l}\big]\big]\times\cdots\times\mathbb{E}\big[\mathbb{E}\big[J_{l+N_{s_l}}^{|\pi_l|}\,\big|\,N_{s_1},\ldots,N_{s_l}\big]\big]\\
&= \mathbb{E}\big[\mathbb{E}\big[J_1^{|\pi_1|}\big]\times\cdots\times\mathbb{E}\big[J_1^{|\pi_l|}\big]\big]\\
&= \mathbb{E}\big[J_1^{|\pi_1|}\big]\mathbb{E}\big[J_1^{|\pi_2|}\big]\cdots\mathbb{E}\big[J_1^{|\pi_l|}\big]
\end{aligned}$$

and therefore

$$\mathbb{E}[S_t^n] = \sum_{l=1}^n \sum_{\pi_1\cup\cdots\cup\pi_l=\{1,\ldots,n\}} \mathbb{E}\big[J_1^{|\pi_1|}\big]\cdots\mathbb{E}\big[J_1^{|\pi_l|}\big]\int_0^t\cdots\int_0^t h^{|\pi_1|}(s_1,t)\cdots h^{|\pi_l|}(s_l,t)\lambda(ds_1)\cdots\lambda(ds_l).$$

which yields (2.4) from (2.2). □

We note that Proposition 2.2 is consistent with the Lévy-Khintchine formula for compound Poisson processes, as the Faà di Bruno formula, see e.g. §2 of Lukacs (1955), yields

$$\begin{aligned}
\mathbb{E}[\mathrm{e}^{\alpha S_t}] &= \sum_{n\geq 0}\frac{\alpha^n}{n!}\mathbb{E}[S_t^n]\\
&= \sum_{n\geq 0}\frac{\alpha^n}{n!}B_n\left(\mathbb{E}[J_1]\int_0^t h(s,t)ds, \ldots, \mathbb{E}[J_1^n]\int_0^t h^n(s,t)ds\right)
\end{aligned}$$

$$
\begin{aligned}
&= \exp\left(\sum_{n\geq 1}\frac{\alpha^n}{n!}\mathbb{E}\big[J_1^n\big]\int_0^t h^n(s,t)ds\right)\\
&= \exp\left(\int_0^t \big(\mathrm{e}^{\alpha h(s,t)J_1}-1\big)ds\right),
\end{aligned}
$$

which recovers the cumulants of S_t from the moments of J_1 as

$$
\kappa_{S_t}^{(n)} = \mathbb{E}\big[J_1^n\big]\int_0^t h^n(s,t)ds, \qquad n\geq 1, \quad t\geq 0.
$$

3 Moments of jump processes

From now on we assume that $(N_t)_{t\in\mathbb{R}_+}$ is a standard Poisson process with intensity $\lambda > 0$, and in this section we consider jump processes built as the anticipating Poisson integrals

$$
Y_t = \sum_{k=1}^{N_t} g(T_k,k,N_t) = \int_0^t g(s,N_s,N_t)dN_s, \qquad t\in\mathbb{R}_+. \tag{3.1}
$$

Proposition 3.1 *Let $(Y_t)_{t\in\mathbb{R}_+}$ be defined as in (3.1). For all $n\geq 1$, we have*

$$
\mathbb{E}[(Y_t)^n] = \sum_{k=1}^n \lambda^k \sum_{\pi_1\cup\ldots\cup\pi_k=\{1,\ldots,n\}} \int_0^t\cdots\int_0^t \mathbb{E}\left[\prod_{l=1}^k g^{|\pi_l|}(s_l, l+N_{s_l}, k+N_t)\right] ds_1\cdots ds_n,
$$

where the sum runs over all partitions $\pi_1,\ldots,\pi_k$ of $\{1,\ldots,n\}$.

Proof. Taking $u_s(\omega) := g(s,N_s,N_t)$, $0\leq s\leq t$, by Proposition 2.1 we have

$$
\begin{aligned}
\mathbb{E}[(Y_t)^n] &= \mathbb{E}\left[\left(\int_0^t g(s,N_s,N_t)dN_s\right)^n\right]\\
&= \sum_{k=1}^n \lambda^k \sum_{\pi_1\cup\ldots\cup\pi_k=\{1,\ldots,n\}} \int_0^t\cdots\int_0^t \mathbb{E}\left[\epsilon_{s_1}^+\cdots\epsilon_{s_n}^+\prod_{l=1}^k g^{|\pi_l|}(s_l,N_{s_l},N_t)\right] ds_1\cdots ds_n\\
&= \sum_{k=1}^n \lambda^k \sum_{\pi_1\cup\ldots\cup\pi_k=\{1,\ldots,n\}} \int_0^t\cdots\int_0^t \mathbb{E}\left[\prod_{l=1}^k g^{|\pi_l|}(s_l,l+N_{s_l},k+N_t)\right] ds_1\cdots ds_n,
\end{aligned}
$$

where we applied (2.5). □

Next, we specialize Proposition 3.1 to the case where the process $g(s,k,n)$ takes the form

$$
g(s,k,n) = f_k(s)(1-Z_k)\prod_{l=k+1}^n Z_l,
$$

where $(Z_k)_{k\geq 1}$ is an i.i.d. random sequence independent of the Poisson process $(N_t)_{t\in\mathbb{R}_+}$, i.e., we have

$$Y_t = \sum_{k=1}^{N_t} f_k(T_k)(1-Z_k) \prod_{l=k+1}^{N_t} Z_l, \qquad t \in \mathbb{R}_+. \tag{3.2}$$

The corresponding growth-collapse process can be written as

$$\begin{aligned} X_t &= f_{N_t}(t) - Y_t \\ &= f_{N_t}(t) - f_{N_t}(T_{N_t}) + \sum_{k=1}^{N_t}(f_k(T_k) - f_{k-1}(T_{k-1})) \prod_{l=k}^{N_t} Z_l, \end{aligned}$$

which corresponds to the process in Equation (1) of Kella (2009) by taking $I_n(u) := f_n(u + T_{n-1}) - f_{n-1}(T_{n-1})$ therein, $u \in [0, T_n)$, $n \geq 1$, with $f_0 := 0$.

Corollary 3.2 *Let $(Y_t)_{t\in\mathbb{R}_+}$ be defined as in (3.2) with $f_k(s) = f(s)$ independent of $k \geq 1$. For all $n \geq 1$, we have*

$$\mathbb{E}[(Y_t)^n] = n!\mathrm{e}^{\lambda t(\mu_n - 1)} \sum_{k=1}^{n} \lambda^k \sum_{q_0=0<q_1<\cdots<q_k=n} \tag{3.3}$$
$$\int_0^t \int_0^{s_k} \cdots \int_0^{s_2} \prod_{l=1}^{k} \left(\frac{f^{q_l - q_{l-1}}(s_l)}{(q_l - q_{l-1})!} C_{q_{l-1}, q_l - q_{l-1}} \mathrm{e}^{\lambda s_l(\mu_{q_{l-1}} - \mu_{q_l})} \right) ds_1 \cdots ds_k,$$

$t \in \mathbb{R}_+$, where

$$C_{p,q} := \mathbb{E}[Z^p(1-Z)^q] = \sum_{k=0}^{p} \binom{p}{k} (-1)^k \mu_{q+k} \quad \text{and} \quad \mu_p := C_{p,0} = \mathbb{E}[Z^p], \quad p, q \geq 0. \tag{3.4}$$

Proof. By Proposition 3.1, letting

$$W_{k,n} := (1 - Z_k) \prod_{l=k+1}^{n} Z_l, \qquad 1 \leq k \leq n,$$

for all $n \geq 1$ we have

$$\mathbb{E}[(Y_t)^n] = n! \sum_{k=1}^{n} \lambda^k \sum_{\substack{p_1+\cdots+p_k=n \\ p_1,\ldots,p_k\geq 1}} \tag{3.5}$$
$$\int_0^t \int_0^{s_k} \cdots \int_0^{s_2} \frac{f^{p_1}(s_1)\cdots f^{p_k}(s_k)}{p_1!\cdots p_k!} \mathbb{E}\left[\epsilon_{s_1}^+ \cdots \epsilon_{s_n}^+ \left(W_{N_{s_1},N_t}\right)^{p_1} \cdots \left(W_{N_{s_k},N_t}\right)^{p_k}\right] ds_1 \cdots ds_k.$$

Next, when $p_1 + \cdots + p_k = n$ and $0 \leq s_1 < \cdots < s_k \leq s_{k+1} := t$, we have

$$\mathbb{E}\left[\epsilon_{s_1}^+ \cdots \epsilon_{s_k}^+ \left(\left(W_{N_{s_1},N_t}\right)^{p_1} \cdots \left(W_{N_{s_k},N_t}\right)^{p_k}\right)\right] = \mathbb{E}\left[\left(W_{1+N_{s_1},k+N_t}\right)^{p_1} \cdots \left(W_{k+N_{s_k},k+N_t}\right)^{p_k}\right]$$

$$
\begin{aligned}
&= \mathbb{E}\left[\left((1-Z_{1+N_{s_1}})\prod_{l=2+N_{s_1}}^{k+N_t} Z_l\right)^{p_1}\cdots\left((1-Z_{k+N_{s_k}})\prod_{l=k+1+N_{s_k}}^{k+N_t} Z_l\right)^{p_k}\right] \\
&= \mathbb{E}\left[\prod_{l=1}^{k}\left((1-Z_{l+N_{s_l}})^{p_l}(Z_{l+N_{s_l}})^{p_1+\cdots+p_{l-1}}\prod_{p=1+N_{s_l}}^{N_{s_{l+1}}}(Z_{l+p})^{p_1+\cdots+p_l}\right)\right] \\
&= \mathbb{E}\left[\prod_{l=1}^{k}\left(C_{p_1+\cdots+p_{l-1},p_l}(\mu_{p_1+\cdots+p_l})^{N_{s_{l+1}}-N_{s_l}}\right)\right] \\
&= \prod_{l=1}^{k}\left(C_{p_1+\cdots+p_{l-1},p_l}\mathrm{e}^{\lambda(s_{l+1}-s_l)(\mu_{p_1+\cdots+p_l}-1)}\right) \\
&= \mathrm{e}^{\lambda t(\mu_n-1)}\prod_{l=1}^{k}\left(C_{p_1+\cdots+p_{l-1},p_l}\mathrm{e}^{\lambda s_l(\mu_{p_1+\cdots+p_{l-1}}-\mu_{p_1+\cdots+p_l})}\right),
\end{aligned}
$$

where we used (3.4) and the independence of the sequence $(Z_k)_{k\geq 1}$, which leads to (3.3) from (3.5). □

When $n=1$, Corollary 3.2 yields

$$
\mathbb{E}[Y_t] = \lambda(1-\mu_1)\int_0^t f(s)\mathrm{e}^{-\lambda(1-\mu_1)(t-s)}ds, \qquad t\in\mathbb{R}_+,
$$

which recovers (1.4) when $f(s)=s$, $s\in\mathbb{R}_+$. More generally, $\mathbb{E}[(Y_t)^n]$ can be computed for any $n\geq 1$ and any choice of integrable function $f(s)$ and moment sequence $\mu_n=\mathbb{E}[Z^n]$, $n\geq 0$, using the Maple and Mathematica codes 1 and 2 in the online appendix.

4 Uniform cut-off rates

In this section we assume that $(Y_t)_{t\in\mathbb{R}_+}$ takes the form

$$
Y_t = \sum_{k=1}^{N_t} f_k(T_k)(1-Z_k)\prod_{l=k+1}^{N_t} Z_l, \qquad t\in\mathbb{R}_+. \tag{4.1}
$$

where $(Z_k)_{k\geq 1}$ is an i.i.d. uniform random sequence on $[0,1]$. In this case we have $\mu_n = 1/(n+1)$ and $C_{p,q}$ is given by the beta function as $C_{p,q}=p!q!/(p+q+1)!$, hence we have

$$
\prod_{l=1}^{k} C_{p_1+\cdots+p_{l-1},p_l} = \prod_{l=1}^{k}\frac{(p_1+\cdots+p_{l-1})!p_l!}{(p_1+\cdots+p_l+1)!} = \frac{1}{n!}\prod_{l=1}^{k}\frac{p_l!}{p_1+\cdots+p_l+1} \tag{4.2}
$$

under the condition $p_1+\cdots+p_k=n$, which yields the next consequence of Corollary 3.2.

Corollary 4.1 *Let $(Y_t)_{t\in\mathbb{R}_+}$ be defined as in (4.1), where $(Z_k)_{k\geq 1}$ is an i.i.d. uniform random sequence on $[0,1]$. For any $n \geq 1$, we have*

$$\mathbb{E}[(Y_t)^n] \tag{4.3}$$
$$= \mathrm{e}^{-n\lambda t/(n+1)} \sum_{k=1}^{n} \lambda^k \sum_{q_0=0<q_1<\cdots<q_k=n} \int_0^t \int_0^{s_k} \cdots \int_0^{s_2} \prod_{l=1}^{k} \frac{f_l^{q_l-q_{l-1}}(s_l)\mathrm{e}^{\lambda s_l(1/(1+q_{l-1})-1/(1+q_l))}}{1+q_l} ds_1 \cdots ds_k,$$

$t \in \mathbb{R}_+$.

When $f(s) = s$, $s \in \mathbb{R}_+$, the relation

$$\int_0^{s_2} s_1^n \mathrm{e}^{\alpha s_1} ds_1 = 1 - \mathrm{e}^{\alpha s_2} \sum_{k=0}^{n} (-1)^k \frac{(\alpha s_2)^k}{k!}, \qquad s_2 \in \mathbb{R}_+, \quad n \geq 0, \tag{4.4}$$

can be used to compute the integrals in (4.3) by induction using the Maple command $\mathrm{MY}(t, \lambda, \mu, f, n)$, resp. the Mathematica command $\mathrm{MY}[t, \lambda, \mu, f, n]$ in the online appendix, by taking f $:= s \rightarrow s$ and mu $:= n \rightarrow 1/(n+1)$, resp. f$[s_] := s$ and mu$[n_] := 1/(n+1)$.

First moment of Y_t using $\mathrm{MY}(t, \lambda, \mu, f, 1)$

For $n = 1$, we have

$$\mathbb{E}[Y_t] = \frac{\lambda}{2}\mathrm{e}^{-\lambda t/2} \int_0^t s_1 \mathrm{e}^{\lambda s_1/2} ds_1 = t - \frac{2}{\lambda}(1 - \mathrm{e}^{-\lambda t/2}), \tag{4.5}$$

which is consistent with Theorem 4 in Boxma et al. (2006), with a shorter proof, see Figure 2.

Second moment of Y_t using $\mathrm{MY}(t, \lambda, \mu, f, 2)$

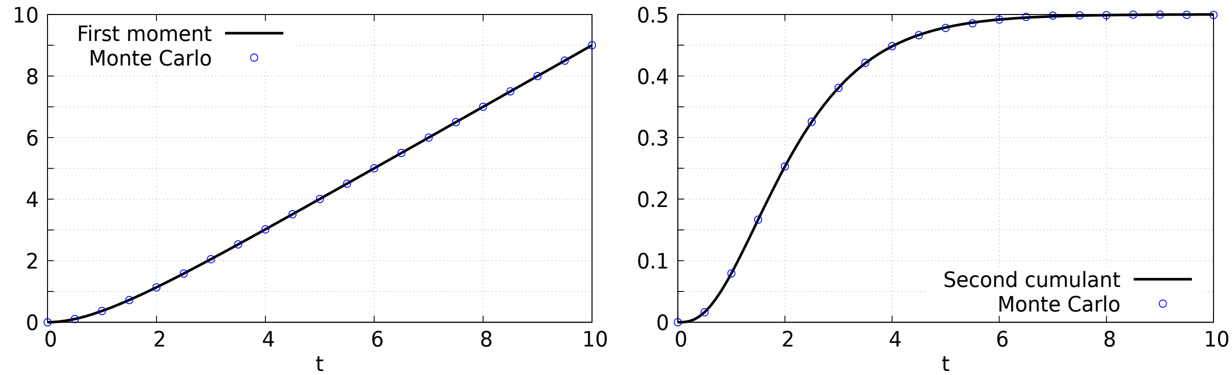
For $n = 2$, we have

$$\begin{aligned}
\mathbb{E}[(Y_t)^2] &= \frac{\lambda}{3}\mathrm{e}^{-2\lambda t/3} \int_0^t s_1^2 \mathrm{e}^{2\lambda s_1/3} ds_1 + \frac{\lambda^2}{6}\mathrm{e}^{-2\lambda t/3} \int_0^t s_2 \mathrm{e}^{\lambda s_2/6} \int_0^{s_2} s_1 \mathrm{e}^{\lambda s_1/2} ds_1 ds_2 \\
&= \frac{18}{\lambda^2}\mathrm{e}^{-2\lambda t/3} + \frac{4}{\lambda}\mathrm{e}^{-\lambda t/2} t - \frac{24}{\lambda^2}\mathrm{e}^{-\lambda t/2} + t^2 + \frac{6}{\lambda^2} - \frac{4}{\lambda} t, \qquad (4.6)
\end{aligned}$$

hence

$$\kappa^{(2)}(t) = \mathrm{Var}[Y_t] = \mathbb{E}[Y_t^2] - (\mathbb{E}[Y_t])^2 = \frac{2}{\lambda^2}\big(9\mathrm{e}^{-2\lambda t/3} - 2\mathrm{e}^{-\lambda t} - 8\mathrm{e}^{-\lambda t/2} + 1\big),$$

which recovers Theorem 5 in Boxma et al. (2006) with a shorter proof, see Figure 2. Figures 2 to 4 are plotted with 10 million Monte Carlo samples and $\lambda = 2$.



(a) First cumulant $\kappa^{(1)}(t)$. (b) Second cumulant $\kappa^{(2)}(t)$.

Figure 2: Mean and variance given by (4.5) and (4.6).

The subsequent integrals for higher order moments can be evaluated using Mathematica or based on the recurrence relation (4.4).

Third moment of Y_t using $\mathrm{MY}(t, \lambda, \mu, f, 3)$

For $n = 3$, we have

$$\begin{aligned}
\mathbb{E}[(Y_t)^3] &= \frac{\lambda}{4}\mathrm{e}^{-3\lambda t/4}\int_0^t s_1^3\mathrm{e}^{3\lambda s_1/4}ds_1 + \frac{\lambda^2}{8}\mathrm{e}^{-3\lambda t/4}\int_0^t s_2^2\mathrm{e}^{\lambda s_2/4}\int_0^{s_2} s_1\mathrm{e}^{\lambda s_1/2}ds_1ds_2 \\
&+ \frac{\lambda^2}{12}\mathrm{e}^{-3\lambda t/4}\int_0^t s_2\mathrm{e}^{\lambda s_2/12}\int_0^{s_2} s_1^2\mathrm{e}^{2\lambda s_1/3}ds_1ds_2 \\
&+ \frac{\lambda^3}{24}\mathrm{e}^{-3\lambda t/4}\int_0^t s_3\mathrm{e}^{\lambda s_3/12}\int_0^{s_3} s_2\mathrm{e}^{\lambda s_2/6}\int_0^{s_2} s_1\mathrm{e}^{\lambda s_1/2}ds_1ds_2ds_3 \\
&= \frac{\mathrm{e}^{-3\lambda t/4}}{\lambda^3}\big(384 + 54\mathrm{e}^{\lambda t/12}(\lambda t - 12) + 6\mathrm{e}^{\lambda t/4}(48 + \lambda t(\lambda t - 12)) + \mathrm{e}^{3\lambda t/4}(\lambda t(18 + \lambda t(\lambda t - 6)) - 24)\big),
\end{aligned} \tag{4.7}$$

see Figure 3 below.

Fourth moment of Y_t using $\mathrm{MY}(t, \lambda, \mu, f, 4)$

For $n = 4$, we have

$$\begin{aligned}
\mathbb{E}[(Y_t)^4] &= \frac{\lambda}{5}\mathrm{e}^{-4\lambda t/5}\int_0^t s_1^4\mathrm{e}^{4\lambda s_1/5}ds_1 + \frac{\lambda^2}{10}\mathrm{e}^{-4\lambda t/5}\int_0^t s_2^3\mathrm{e}^{3\lambda s_2/10}\int_0^{s_2} s_1\mathrm{e}^{\lambda s_1/2}ds_1ds_2 \\
&+ \frac{\lambda^2}{20}\mathrm{e}^{-4\lambda t/5}\int_0^t s_2\mathrm{e}^{\lambda s_2/20}\int_0^{s_2} s_1^3\mathrm{e}^{3\lambda s_1/4}ds_1ds_2 + \frac{\lambda^2}{15}\mathrm{e}^{-4\lambda t/5}\int_0^t s_2^2\mathrm{e}^{2\lambda s_2/15}\int_0^{s_2} s_1^2\mathrm{e}^{2\lambda s_1/3}ds_1ds_2 \\
&+ \frac{\lambda^3}{30}\mathrm{e}^{-4\lambda t/5}\int_0^t s_3^2\mathrm{e}^{2\lambda s_3/15}\int_0^{s_3} s_2\mathrm{e}^{\lambda s_2/6}\int_0^{s_2} s_1\mathrm{e}^{\lambda s_1/2}ds_1ds_2ds_3
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{\lambda^3}{40}\mathrm{e}^{-4\lambda t/5} \int_0^t s_3 \mathrm{e}^{\lambda s_3/20} \int_0^{s_3} s_2^2 \mathrm{e}^{\lambda s_2/4} \int_0^{s_2} s_1 \mathrm{e}^{\lambda s_1/2} ds_1 ds_2 ds_3 \\
&+ \frac{\lambda^3}{60}\mathrm{e}^{-4\lambda t/5} \int_0^t s_3 \mathrm{e}^{\lambda s_1/20} \int_0^{s_3} s_2 \mathrm{e}^{\lambda s_1/12} \int_0^{s_2} s_1^2 \mathrm{e}^{2\lambda s_1/3} ds_1 ds_2 ds_3 \\
&+ \frac{\lambda^4}{120}\mathrm{e}^{-4\lambda t/5} \int_0^t s_4 \mathrm{e}^{\lambda s_4/20} \int_0^{s_4} s_3 \mathrm{e}^{\lambda s_3/12} \int_0^{s_3} s_2 \mathrm{e}^{\lambda s_2/6} \int_0^{s_2} s_1 \mathrm{e}^{\lambda s_1/2} ds_1 ds_2 ds_3 ds_4 \\
= &\frac{\mathrm{e}^{-4\lambda t/5}}{\lambda^4} \big(15000 + 1536\mathrm{e}^{\lambda t/20}(\lambda t - 20) + 108\mathrm{e}^{2\lambda t/15}(180 + \lambda t(\lambda t - 24)) \qquad (4.8) \\
&+ 8\mathrm{e}^{3\lambda t/10}(\lambda t(144 + \lambda t(\lambda t - 18)) - 480) + \mathrm{e}^{4\lambda t/5}(120 + \lambda t(\lambda t(36 + \lambda t(\lambda t - 8)) - 96))\big),
\end{aligned}
$$

see Figure 4 below.

5 Moments of growth-collapse processes

The moments of the growth-collapse process X_t defined as $X_t := f(t) - Y_t$ can be recovered from (4.3) with $f_l(s) = f(s)$, $l \geq 1$, and the binomial recursion

$$
\mathbb{E}[(X_t)^n] = \mathbb{E}[(f(t) - Y_t)^n] = (-1)^n \left(\mathbb{E}[(Y_t)^n] - \sum_{k=0}^{n-1} \binom{n}{k} (f(t))^{n-k} (-1)^k \mathbb{E}[(X_t)^k] \right), \quad (5.1)
$$

which is implemented in the Maple and Mathematica codes 3 and 4 in the online appendix.

In the remainder of this section we take $f(t) = t$ and consider the growth-collapse process $(X_t)_{t\in\mathbb{R}_+}$ of §4 in Boxma et al. (2006) defined as $X_t := t - Y_t$, $t \in \mathbb{R}_+$. The result of Corollary 4.1 clearly involves partition counts, however they may not be easy to identify in practice. This problem is solved using stochastic calculus and differential equation methods in the next proposition. Proposition 5.1 extends Theorems 4 and 5 as well as Corollary 1 of Boxma et al. (2006) from mean and variance to moments of all orders, see also Corollary 4 in Daw and Pender (2020) for an expression using matrix exponentials. It is also consistent with Theorem 3 of Boxma et al. (2006), which states that the stationary distribution of the Markovian growth-collapse process $(X_t)_{t\in\mathbb{R}_+}$ is the gamma distribution $\Gamma(2, \lambda)$ with shape parameter 2 and scaling parameter λ, and cumulants $\kappa^{(n)}(\infty) = 2(n-1)!/\lambda^n$, $n \geq 1$.

Proposition 5.1 *The moments of the growth-collapse process*

$$
X_t = t - \sum_{k=1}^{N_t} T_k(1 - U_k) \prod_{l=k+1}^{N_t} U_l, \qquad t \in \mathbb{R}_+,
$$

with uniform cut-off rates $(U_k)_{k\geq 1}$ *on* $[0,1]$*, are given by*

$$\mathbb{E}[(X_t)^n] = \frac{(n+1)!}{\lambda^n}\sum_{k=0}^n(-1)^k(k+1)^{n-1}\binom{n}{k}\mathrm{e}^{-k\lambda t/(k+1)}, \quad n\geq 0, \quad t\in\mathbb{R}_+.$$

As a consequence, the asymptotic moments of $(X_t)_{t\in\mathbb{R}_+}$ *are given by*

$$\lim_{t\to\infty}\mathbb{E}[(X_t)^n] = \frac{(n+1)!}{\lambda^n}, \qquad n\geq 1.$$

Before proving Proposition 5.1, we recover the first moments and cumulants of X_t from the expressions (4.5)-(4.8) and the identity (5.1). For this we may use the Maple commands $\mathrm{MX}(t,\lambda,\mu,f,n)$, $\mathrm{CX}(t,\lambda,\mu,f,n)$, resp. Mathematica commands $\mathrm{MX}[t,\lambda,\mu,f,n]$, $\mathrm{CX}[t,\lambda,\mu,f,n]$ in the online appendix, with f $:= s\to s$ and mu $:= n\to 1/(n+1)$, resp. f$[s_] := s$ and mu$[n_] := 1/(n+1)$.

First and second moments of X_t using $\mathrm{MX}(t,\lambda,\mu,f,1)$ **and** $\mathrm{MX}(t,\lambda,\mu,f,2)$

We find

$$\mathbb{E}[X_t] = -\mathbb{E}[Y_t]+t = \frac{\lambda}{2}\mathrm{e}^{-\lambda t/2}\int_0^t s_1\mathrm{e}^{\lambda s_1/2}ds_1 = \frac{2}{\lambda}(-\mathrm{e}^{-\lambda t/2}+1) \tag{5.2}$$

and

$$\mathbb{E}[(X_t)^2] = \mathbb{E}[(Y_t)^2]-t^2+2t\mathbb{E}[X_t] = \frac{3!}{\lambda^2}\left(3\mathrm{e}^{-2\lambda t/3}-4\mathrm{e}^{-\lambda t/2}+1\right), \tag{5.3}$$

see Theorems 4 and 5 of Boxma et al. (2006).

Third and fourth moments of X_t using $\mathrm{MX}(t,\lambda,\mu,f,3)$ **and** $\mathrm{MX}(t,\lambda,\mu,f,4)$

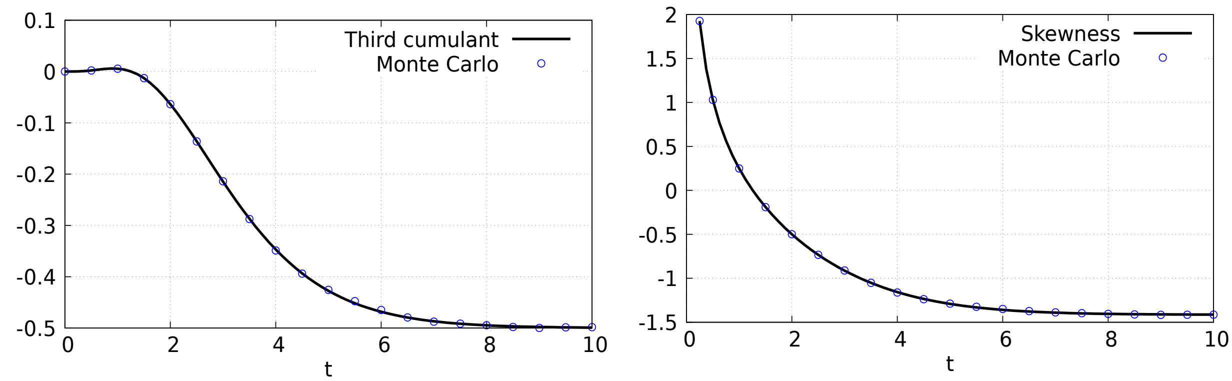
Next, from (4.7) we have

$$\begin{aligned}\mathbb{E}[(X_t)^3] &= -\mathbb{E}[(Y_t)^3]+t^3-3t^2\mathbb{E}[X_t]+3t\mathbb{E}[X_t^2]\\ &= \frac{4!}{\lambda^3}\big(-16\mathrm{e}^{-3\lambda t/4}+27\mathrm{e}^{-2\lambda t/3}-12\mathrm{e}^{-\lambda t/2}+1\big),\end{aligned} \tag{5.4}$$

and the third cumulant of Y_t is given by (2.3) as

$$\kappa^{(3)}(t) = 2\frac{2!}{\lambda^3}\big(-27\mathrm{e}^{-7\lambda t/6}+96\mathrm{e}^{-3\lambda t/4}-135\mathrm{e}^{-2\lambda t/3}+4\mathrm{e}^{-3\lambda t/2}+24\mathrm{e}^{-\lambda t}+39\mathrm{e}^{-\lambda t/2}-1\big), \tag{5.5}$$

see Figure 3.



(a) Third cumulant $\kappa^{(3)}(t)$. (b) Skewness $\kappa^{(3)}(t)/\big(\kappa^{(2)}(t)\big)^{3/2}$.

Figure 3: Third cumulant (5.5) and skewness of Y_t.

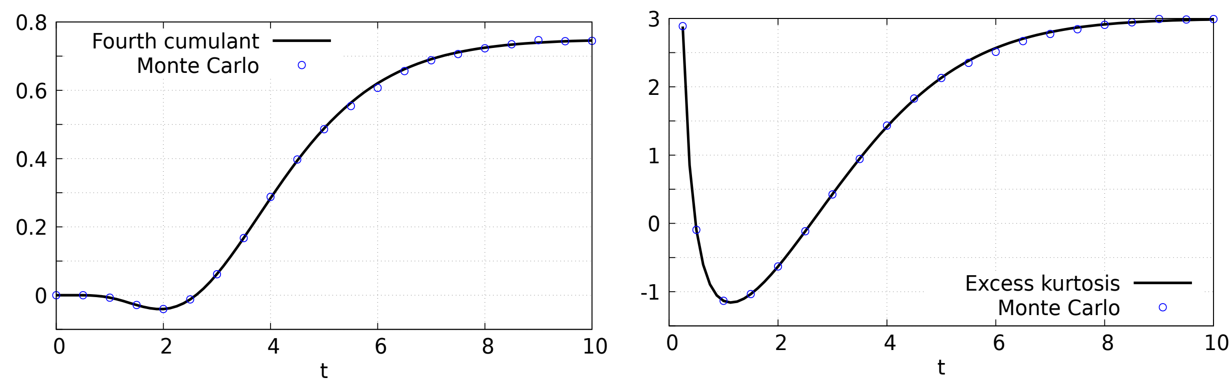
From (4.8) and (5.2)-(5.5) we have

$$\begin{aligned}\mathbb{E}[(X_t)^4] &= \mathbb{E}[(Y_t)^4] - t^4 + 4t^3\mathbb{E}[X_t] - 6t^2\mathbb{E}[(X_t)^2] + 4t\mathbb{E}[X_t^3] \\ &= \frac{5!}{\lambda^4}\big(125\mathrm{e}^{-4\lambda t/5} - 256\mathrm{e}^{-3\lambda t/4} + 162\mathrm{e}^{-2\lambda t/3} - 32\mathrm{e}^{-\lambda t/2} + 1\big), \end{aligned} \tag{5.6}$$

and the fourth cumulant of Y_t is given by (2.3) as

$$\begin{aligned}\kappa^{(4)}(t) &= 2\frac{3!}{\lambda^4}\big(- 8\mathrm{e}^{-2\lambda t} + 504\mathrm{e}^{-7\lambda t/6} + 1250\mathrm{e}^{-4\lambda t/5} - 256\mathrm{e}^{-5\lambda t/4} - 2304\mathrm{e}^{-3\lambda t/4} \\ &\quad +72\mathrm{e}^{-5\lambda t/3} - 81\mathrm{e}^{-4\lambda t/3} + 1206\mathrm{e}^{-2\lambda t/3} - 64\mathrm{e}^{-3\lambda t/2} - 168\mathrm{e}^{-\lambda t} - 152\mathrm{e}^{-\lambda t/2} + 1\big), \end{aligned} \tag{5.7}$$

see Figure 4.



(a) Fourth cumulant $\kappa^{(4)}(t)$. (b) Excess kurtosis $\kappa^{(4)}(t)/\big(\kappa^{(2)}(t)\big)^2$ of Y_t.

Figure 4: Fourth cumulant (5.7) and excess kurtosis.

Proof of Proposition 5.1. Applying the Itô formula with jumps to (1.1), we have

$$d(X_t)^n = nX_t^{n-1}dt + ((X_t)^n - (X_{t-})^n)dN_t$$

$$
\begin{aligned}
&= nX_t^{n-1}dt + ((X_{t^-} - (1-U_{N_t})X_{t^-}dN_t)^n - (X_{t-})^n)dN_t \\
&= nX_t^{n-1}dt + (X_{t^-})^n \sum_{k=1}^n \binom{n}{k} (U_{N_t}-1)^k dN_t \\
&= nX_t^{n-1}dt + (X_{t^-})^n((U_{N_t})^n - 1)dN_t,
\end{aligned}
$$

with $X_0 := 0$, hence

$$
X_t^n = n\int_0^t X_s^{n-1}ds + \int_0^t (X_{s^-})^n((U_{N_s})^n - 1)dN_s, \qquad t \in \mathbb{R}_+.
$$

Taking expectations on both sides of the above equality and using the smoothing lemma as in (1.3) yields

$$
\begin{aligned}
\mathbb{E}[X_t^n] &= n\int_0^t \mathbb{E}[X_s^{n-1}]ds + \lambda \int_0^t \mathbb{E}[(X_{s^-})^n]\mathbb{E}[(U_{N_s})^n - 1]ds \\
&= n\int_0^t \mathbb{E}[X_s^{n-1}]ds - \frac{\lambda n}{n+1}\int_0^t \mathbb{E}[(X_{s^-})^n]ds, \qquad t \in \mathbb{R}_+,
\end{aligned}
$$

which shows that the moments $\mathbb{E}[(X_t)^n]$ satisfy the differential equation

$$
\frac{d}{dt}\mathbb{E}[(X_t)^n] = n\mathbb{E}[(X_t)^{n-1}] - \frac{\lambda n}{n+1}\mathbb{E}[(X_t)^n], \qquad t \in \mathbb{R}_+, \tag{5.8}
$$

see also § 3 of Boxma et al. (2006) and § 3.5 of Daw and Pender (2020) for proofs using the infinitesimal generator of the Markov process $(X_t)_{t\in\mathbb{R}_+}$. Based on the intuition gained from (5.2)-(5.6), we now search for a solution of (5.8) of the form

$$
\mathbb{E}[(X_t)^n] = \frac{(n+1)!}{\lambda^n}\sum_{k=0}^n a_{k,n}\mathrm{e}^{-k\lambda t/(k+1)},
$$

which, by identification of terms, yields the recurrence relation

$$
a_{k,n} = \frac{n(k+1)}{n-k}a_{k,n-1}, \qquad 0 \le k < n,
$$

hence

$$
a_{k,n} = (k+1)^{n-k}\binom{n}{k}a_{k,k}, \qquad 0 \le k \le n.
$$

In addition, the initial condition $t=0$ requires

$$
\sum_{k=0}^n a_{k,n} = 0,
$$

hence

$$\sum_{k=0}^{n}(k+1)^{n-k}\binom{n}{k}a_{k,k}=0,$$

which is solved by taking $a_{k,k}=(-1)^k(k+1)^{k-1}$, due to the combinatorial relation

$$S(n,n+1)=\sum_{k=0}^{n}(-1)^{n-k}(k+1)^n\binom{n+1}{k+1}=(n+1)\sum_{k=0}^{n}(k+1)^{n-1}\binom{n}{k}(-1)^k=0,$$

which follows from the vanishing of the Stirling numbers of the second kind $S(n,n+1)$, see e.g. page 824 of Abramowitz and Stegun (1972). □

6 Embedded growth-collapse chain

In this section we show that Proposition 2.1 can also be used to compute the moments of all orders of the embedded chain

$$Y(m)=Y_{T_m}=\sum_{k=1}^{m}g(T_k,k,m)=\int_0^\infty g(s,N_s,m)\mathbf{1}_{[0,T_m]}(s)dN_s,\qquad m\geq 1. \tag{6.1}$$

Proposition 6.1 *Let $(Y(m))_{m\geq 1}$ be of the form (6.1). For any $n,m\geq 1$, we have*

$$\mathbb{E}\big[(Y(m))^n\big]= \tag{6.2}$$
$$n!\sum_{k=1}^{n}\lambda^k\sum_{0=q_0<q_1<\cdots<q_{k-1}<q_k=n}\mathbb{E}\left[\int_0^\infty\int_0^{s_k}\cdots\int_0^{s_2}\mathbf{1}_{\{N_{s_k^-}\leq m-k\}}\prod_{l=1}^{k}\frac{\big(g(s_l,l+N_{s_l},m)\big)^{q_l-q_{l-1}}}{(q_l-q_{l-1})!}ds_1\cdots ds_k\right].$$

Proof. Taking $u_s(\omega):=g(s,N_s,m)\mathbf{1}_{[0,T_m]}(s)$, $0\leq s\leq t$, by Proposition 2.1 and the identity $\{s\leq T_m\}=\{N_{s^-}<m\}$, $s>0$, we have

$$\mathbb{E}\left[\left(\sum_{k=1}^{m}g(T_k,k,m)\right)^n\right]=\mathbb{E}\left[\left(\int_0^\infty g(s,N_s,m)\mathbf{1}_{[0,T_m]}(s)dN_s\right)^n\right]$$
$$=\sum_{k=1}^{n}k!\lambda^k\sum_{\pi_1\cup\cdots\cup\pi_k=\{1,\ldots,n\}}\mathbb{E}\left[\int_0^\infty\int_0^{s_k}\cdots\int_0^{s_2}\epsilon_{s_1}^+\cdots\epsilon_{s_k}^+\prod_{l=1}^{k}\big(g(s_l,N_{s_l},m)\mathbf{1}_{\{N_{s_l^-}<m\}}\big)^{|\pi_l|}ds_1\cdots ds_k\right].$$

Next, since

$$\{N_{s_1^-}<m\}\subset\{1+N_{s_2^-}<m\}\subset\cdots\subset\{k-1+N_{s_l^-}<m\},$$

by (2.5) we have

$$\epsilon_{s_1}^+\cdots\epsilon_{s_k}^+\prod_{l=1}^{k}\big(g(s_l,N_{s_l},m)\mathbf{1}_{\{N_{s_l^-}<m\}}\big)^{|\pi_l|}=\prod_{l=1}^{k}\big(g(s_l,l+N_{s_l},m)\mathbf{1}_{\{l-1+N_{s_l^-}<m\}}\big)^{|\pi_l|}$$

$$= \mathbf{1}_{\{N_{s_k^-} \le m-k\}} \prod_{l=1}^{k} \big(g(s_l, l + N_{s_l}, m)\big)^{|\pi_l|},$$

which leads to (6.2). □

Next, we specialize Proposition 6.1 to the case where $g(s,k,m)$ takes the form

$$g(s,k,m) = f_{k,m}(s)(1 - Z_k) \prod_{l=k+1}^{m} Z_l, \tag{6.3}$$

where $f_{k,m}(s)$ is a deterministic function and $(Z_k)_{k\ge 1}$ is an i.i.d. random sequence independent of the Poisson process $(N_t)_{t\in\mathbb{R}_+}$, with moment sequence $\mu_n = \mathbb{E}[Z^n]$, $n \ge 0$. The corresponding embedded growth-collapse process can be written as

$$X(m) = f_{m,m}(T_m) - Y(m) = \sum_{k=1}^{m} (f_{k,m}(T_k) - f_{k-1,m}(T_{k-1})) \prod_{l=k}^{m} Z_l,$$

which corresponds to the wealth process in § 2 of Kella (2009) when $f_{k,m}(s) = f_k(s)$ is independent of m and $Y_k := f_k(T_k) - f_{k-1}(T_{k-1})$ therein, $k = 1, \ldots, m$, with $f_{0,m}(s) := 0$.

Corollary 6.2 *Let $(Y(m))_{m\ge 1}$ be defined as in (6.1) from (6.3), with $f_{k,m}(s) = f_m(s)$ independent of $k \ge 1$. For any $n, m \ge 1$, we have*

$$\mathbb{E}\big[(Y(m))^n\big] = n! \sum_{k=1}^{n} \sum_{i=0}^{m-k} \frac{\lambda^{k+i}}{i!} \mu_n^{m-i-k} \sum_{0=q_0<q_1<\cdots<q_{k-1}<q_k=n} \tag{6.4}$$

$$\prod_{l=1}^{k} \frac{C_{q_{l-1}, q_l - q_{l-1}}}{(q_l - q_{l-1})!} \int_0^\infty \mathrm{e}^{-\lambda s_k} \int_0^{s_k} \cdots \int_0^{s_2} \left(\sum_{l=0}^{k-1} \mu_{q_l} (s_{l+1} - s_l) \right)^i \prod_{l=1}^{k} f_m^{q_l - q_{l-1}}(s_l) ds_1 \cdots ds_k,$$

where we let $s_0 := 0$, $\sum_{i=1}^{0} = 0$, and $C_{p,q}$ is defined in (3.4).

Proof. By (2.1), (6.3) and Proposition 6.1 we have

$$\mathbb{E}\big[(Y(m))^n\big] = \sum_{k=1}^{n} k! \lambda^k \sum_{\pi_1 \cup \cdots \cup \pi_k = \{1,\ldots,n\}}$$

$$\mathbb{E}\left[\int_0^\infty \int_0^{s_k} \cdots \int_0^{s_2} \mathbf{1}_{\{N_{s_k^-} \le m-k\}} \prod_{l=1}^{k} \left(f_m(s_l)(1 - Z_{l+N_{s_l}}) \prod_{j=l+1+N_{s_l}}^{m} Z_j \right)^{|\pi_l|} ds_1 \cdots ds_k \right]$$

$$= \sum_{k=1}^{n} k! \lambda^k \sum_{\substack{p_1+\cdots+p_k=n \\ p_1,\ldots,p_k \ge 1}} \frac{n!}{p_1! \cdots p_k!} \prod_{l=1}^{k} f_m^{p_l}(s_l)$$

$$\int_0^\infty \int_0^{s_k} \cdots \int_0^{s_2} \mathbb{E}\left[\mathbf{1}_{\{N_{s_k^-} \le m-k\}} \prod_{l=1}^k \left((1-Z_{l+N_{s_l}})^{p_l} \prod_{j=l+1+N_{s_l}}^m Z_j^{p_l} \right) \right] ds_1 \cdots ds_k.$$

Next, when $N_{s_k^-} \le m-k$ we have

$$\begin{aligned}
&\prod_{l=1}^k \left((1-Z_{l+N_{s_l}})^{p_l} \prod_{j=l+1+N_{s_l}}^m Z_j^{p_l} \right) \\
&\quad = \left(\prod_{l=1}^k (1-Z_{l+N_{s_l}})^{p_l} \right) \left(\prod_{l=2}^k \prod_{j=l+N_{s_{l-1}}}^{l+N_{s_l}} Z_j^{p_1+\cdots+p_{l-1}} \right) \prod_{j=k+1+N_{s_k}}^m Z_j^{p_1+\cdots+p_k} \\
&\quad = \left(\prod_{l=1}^k (1-Z_{l+N_{s_l}})^{p_l} \right) \left(\prod_{l=2}^k Z_{l+N_{s_{l-1}}}^{p_1+\cdots+p_{l-1}} \right) \left(\prod_{l=2}^k \prod_{j=l+1+N_{s_{l-1}}}^{l+N_{s_l}} Z_j^{p_1+\cdots+p_{l-1}} \right) \prod_{j=k+1+N_{s_k}}^m Z_j^{p_1+\cdots+p_k} \\
&\quad = \left(\prod_{l=1}^k (1-Z_{l+N_{s_l}})^{p_l} Z_{l+N_{s_{l-1}}}^{p_1+\cdots+p_{l-1}} \right) \left(\prod_{l=1}^{k-1} \prod_{j=l+2+N_{s_l}}^{l+1+N_{s_{l+1}}} Z_j^{p_1+\cdots+p_l} \right) \prod_{j=k+1+N_{s_k}}^m Z_j^{p_1+\cdots+p_k},
\end{aligned}$$

which is a product of three independent random terms, whose expected value given $N_{s_1}, \ldots, N_{s_k}$ is

$$\left(\prod_{l=1}^k C_{p_1+\cdots+p_{l-1},p_l} \right) \mu_{p_1+\cdots+p_k}^{m-k-N_{s_k}} \prod_{l=1}^{k-1} \mu_{p_1+\cdots+p_l}^{N_{s_{l+1}}-N_{s_l}},$$

see (3.4). Therefore, using the fact that the jumps of $(N_s)_{s\in[0,s_k]}$ are uniformly distributed on $[0,s_k]$ given that $N_{s_k} = i$, we have

$$\begin{aligned}
&\mathbb{E}\left[\mathbf{1}_{\{N_{s_k^-} \le m-k\}} \prod_{l=1}^k \left((1-Z_{l+N_{s_l}})^{p_l} \prod_{j=l+1+N_{s_l}}^m Z_j^{p_l} \right) \right] \\
&\quad = \left(\prod_{l=1}^k C_{p_1+\cdots+p_{l-1},p_l} \right) \mathbb{E}\left[\sum_{i=1}^{m-k} \mathbf{1}_{\{N_{s_k^-}=i\}} \mu_{p_1+\cdots+p_k}^{m-k-N_{s_k}} \prod_{l=1}^{k-1} \mu_{p_1+\cdots+p_l}^{N_{s_{l+1}}-N_{s_l}} \right] \\
&\quad = \left(\prod_{l=1}^k C_{p_1+\cdots+p_{l-1},p_l} \right) \sum_{i=1}^{m-k} \mu_{p_1+\cdots+p_k}^{m-k-i} \mathbb{E}\left[\mathbf{1}_{\{N_{s_k^-}=i\}} \prod_{l=1}^{k-1} \mu_{p_1+\cdots+p_l}^{N_{s_{l+1}}-N_{s_l}} \right] \\
&\quad = \left(\prod_{l=1}^k C_{p_1+\cdots+p_{l-1},p_l} \right) \sum_{i=0}^{m-k} \mathbb{P}(N_{s_k^-} = i) \mu_n^{m-i-k} \left(\frac{\sum_{l=0}^{k-1} \mu_{p_1+\cdots+p_l}(s_{l+1}-s_l)}{s_k} \right)^i,
\end{aligned}$$

since, given $N_{s_k} = i$, the random vector $(N_{s_1}, N_{s_2} - N_{s_1}, \ldots, N_{s_k} - N_{s_{k-1}})$ is made of independent binomial random variables with maximum count i and respective probabilities

$s_1/s_k, (s_2 - s_1)/s_k, \ldots, (s_k - s_{k-1})/s_k$. This leads to

$$\mathbb{E}\big[(Y(m))^n\big] = n! \sum_{k=1}^{n} \sum_{\substack{p_1+\cdots+p_k=n \\ p_1,\ldots,p_k\geq 1}} \sum_{i=0}^{m-k} \lambda^{k+i} \frac{\mu_n^{m-i-k}}{i!} \prod_{l=1}^{k} \frac{C_{p_1+\cdots+p_{l-1},p_l}}{p_l!}$$

$$\times \int_0^\infty \mathrm{e}^{-\lambda s_k} \int_0^{s_k} \cdots \int_0^{s_2} \prod_{l=1}^{k} f_m^{p_l}(s_l) \left(\sum_{l=0}^{k-1} \mu_{p_1+\cdots+p_l}(s_{l+1} - s_l) \right)^i ds_1 \cdots ds_k.$$

□

The result of Corollary 6.2 can be implemented for any choice of integrable function $f(s)$ and moment sequence $\mu_n = \mathbb{E}[Z^n]$, $n \geq 0$, using the Maple and Mathematica codes 5 and 6 in the online appendix.

Uniform cut-off rates

The next result specializes Corollary 6.2 to the case of embedded growth processes with uniform cut-off rates.

Corollary 6.3 *Let $(Y(m))_{m\geq 1}$ be defined as in (6.3), with $f_{k,m}(s) = f_m(s)$ independent of $k \geq 1$, where $(Z_l)_{l\geq 1}$ an i.i.d. uniform random sequence on $[0,1]$. For any $n, m \geq 1$, we have*

$$\mathbb{E}\big[(Y(m))^n\big] = \sum_{k=1}^{n} \sum_{i=0}^{m-k} \frac{\lambda^{k+i}}{i!(n+1)^{m-i-k}} \tag{6.5}$$

$$\sum_{0=q_0<q_1<\cdots<q_{k-1}<q_k=n} \left(\prod_{l=1}^{k} \frac{1}{1+q_l} \right) \int_0^\infty \mathrm{e}^{-\lambda s_k} \int_0^{s_k} \cdots \int_0^{s_2} \left(\sum_{l=0}^{k-1} \frac{s_{l+1} - s_l}{1+q_l} \right)^i \prod_{l=1}^{k} f_m^{q_l - q_{l-1}}(s_l) ds_1 \cdots ds_k,$$

where we let $s_0 := 0$ and $\sum_{i=1}^{0} = 0$.

Proof. We rewrite the result of Corollary 6.2 as

$$\mathbb{E}\left[\left(\sum_{k=1}^{m} g(T_k, k, m) \right)^n \right] = n! \sum_{k=1}^{n} \sum_{\substack{p_1+\cdots+p_k=n \\ p_1,\ldots,p_k\geq 1}} \sum_{i=0}^{m-k} \lambda^{k+i} \frac{\mu_n^{m-i-k}}{i!} \prod_{l=1}^{k} \frac{(p_1 + \cdots + p_{l-1})!}{(p_1 + \cdots + p_l + 1)!}$$

$$\int_0^\infty \mathrm{e}^{-\lambda s_k} \int_0^{s_k} \cdots \int_0^{s_2} \prod_{l=1}^{k} f_m^{p_l}(s_l) \left(\sum_{l=0}^{k-1} \mu_{p_1+\cdots+p_l}(s_{l+1} - s_l) \right)^i ds_1 \cdots ds_k$$

and use the relations $\mu_n = 1/(n+1)$, $n \geq 0$, and

$$\prod_{l=1}^{k} \frac{(p_1 + \cdots + p_{l-1})!}{(p_1 + \cdots + p_l + 1)!} = \frac{1}{n!} \prod_{l=1}^{k} \frac{1}{1+q_l},$$

see (4.2), with $q_l := p_1 + \cdots + p_l$ and $q_k = n$, $1 \leq l \leq k \leq n$. □

The first moments of $Y(m)$ can be computed in closed form when $f_m(s) = s$, which corresponds to

$$Y(m) = Y_{T_m} = \sum_{k=1}^{m} T_k(1 - U_k) \prod_{l=k+1}^{m} U_l, \qquad m \geq 1,$$

where $(U_k)_{k\geq 1}$ is a uniform random sequence on $[0, 1]$. The next results can be recovered for any integer values of $m \geq 1$ by the Maple command $\mathrm{MYm}(\lambda, m, \mathrm{mu}, f, n)$, resp. Mathematica command $\mathrm{MYm}[\lambda, m, \mathrm{mu}, f, n]$ in the online appendix, by setting f $:= s \to s$ and mu $:= n \to 1/(n + 1)$, resp. f$[s_] := s$ and mu$[n_] := 1/(n + 1)$.

First moment of $Y(m)$ using $\mathrm{MYm}(\lambda, m, \mathrm{mu}, f, 1)$

For $n = 1$, Corollary 6.3 yields

$$\mathbb{E}\big[Y(m)\big] = \frac{1}{\lambda} \sum_{i=0}^{m-1} \frac{2^{i+1-m}}{i!} \int_0^\infty \mathrm{e}^{-s_1} s_1^i \frac{s_1}{2} ds_1 = \frac{2^{-m} + m - 1}{\lambda},$$

hence

$$\mathbb{E}\big[X(m)\big] = \mathbb{E}[T_m] - \mathbb{E}[Y(m)] = \frac{1 - 2^{-m}}{\lambda} \tag{6.6}$$

as in Theorem 7 in Boxma et al. (2006).

Second moment of $Y(m)$ using $\mathrm{MYm}(\lambda, m, \mathrm{mu}, f, 2)$

For $n = 2$, we find

$$\begin{aligned}
&\mathbb{E}\big[(Y(m))^2\big] \\
&= \frac{1}{\lambda^2} \sum_{i=0}^{m-1} \frac{1}{i!3^{m-1-i}} \int_0^\infty \mathrm{e}^{-s_1} s_1^i \frac{s_1^2}{3} ds_1 + \frac{1}{2\lambda^2} \sum_{i=0}^{m-2} \frac{1}{i!3^{m-2-i}} \int_0^\infty \mathrm{e}^{-s_2} \int_0^{s_2} \left(\frac{s_1 + s_2}{2}\right)^i s_1 \frac{s_2}{3} ds_1 ds_2 \\
&= \frac{1}{\lambda^2} \left(\frac{2}{3^m} + \frac{m-1}{2^{m-1}} - m + m^2\right).
\end{aligned}$$

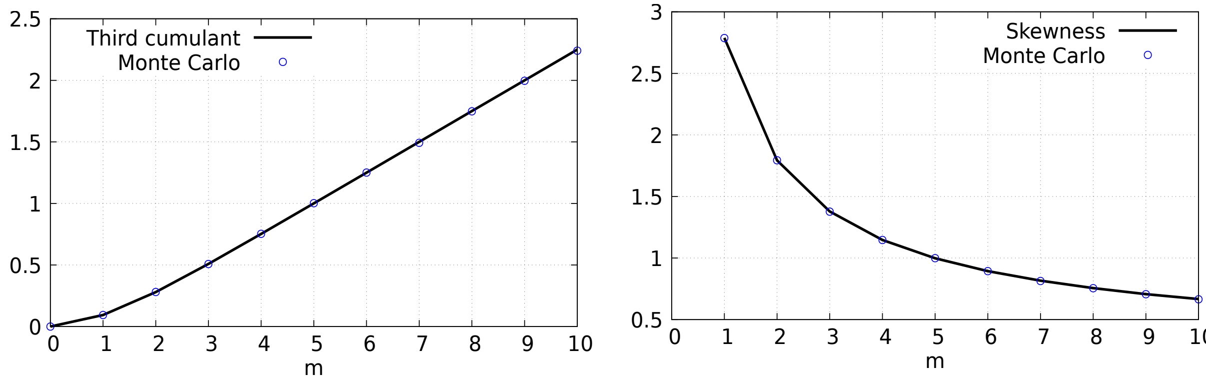
Third moment of $Y(m)$ using $\mathrm{MYm}(\lambda, m, \mathrm{mu}, f, 3)$

For $n = 3$, we have

$$\begin{aligned}
\mathbb{E}\big[Y(m)^3\big] &= \frac{1}{\lambda^3} \sum_{i=0}^{m-1} \frac{1}{i!4^{m-1-i}} \int_0^\infty \mathrm{e}^{-s_1} \frac{s_1^{i+3}}{4} ds_1 \\
&+ \frac{1}{\lambda^3} \sum_{i=0}^{m-2} \frac{1}{i!4^{m-2-i}} \int_0^\infty \mathrm{e}^{-s_2} \int_0^{s_2} \left(\left(\frac{s_1 + s_2}{2}\right)^i \frac{s_1}{2} \frac{s_2^2}{4} + \left(\frac{2s_1 + s_2}{3}\right)^i \frac{s_1^2}{3} \frac{s_2}{4} \right) ds_1 ds_2
\end{aligned}$$

$$+\frac{1}{\lambda^3}\sum_{i=0}^{m-3}\frac{1}{i!4^{m-3-i}}\int_0^\infty \mathrm{e}^{-s_3}\int_0^{s_3}\int_0^{s_2}\left(\frac{3s_1+s_2+2s_3}{6}\right)^i \frac{s_1}{2}\frac{s_2}{3}\frac{s_3}{4}ds_1ds_2ds_3. \tag{6.7}$$

Although the last partial summation (6.7) does not have a closed-form expression, it can easily be estimated using the Maple and Mathematica codes 5 and 6 in the online appendix, see Figures 5 and 6, which are plotted with 10 million Monte Carlo samples and $\lambda = 2$.



(a) Third cumulant of $Y(m)$.

(b) Skewness $\kappa^{(3)}(m)/\big(\kappa^{(2)}(m)\big)^{3/2}$.

Figure 5: Third cumulant and skewness of $Y(m)$.

Fourth moment of $Y(m)$ using $\mathrm{MYm}(\lambda, m, \mathrm{mu}, f, 4)$

For $n = 4$, we find

$$\mathbb{E}\big[Y(m)^4\big] = \frac{1}{\lambda^4}\sum_{i=0}^{m-1}\frac{1}{i!5^{m-1-i}}\int_0^\infty \mathrm{e}^{-s_1}\frac{s_1^{i+4}}{5}ds_1$$

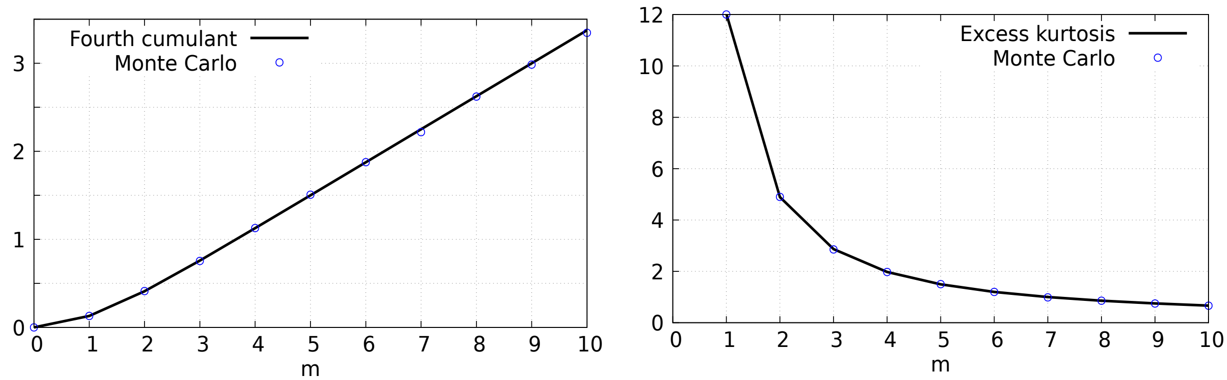
$$+\frac{1}{\lambda^4}\sum_{i=0}^{m-2}\frac{1}{i!5^{m-2-i}}\int_0^\infty \mathrm{e}^{-s_2}\int_0^{s_2}\left(\left(\frac{s_1+s_2}{2}\right)^i\frac{s_1}{2}\frac{s_2^3}{5}+\left(\frac{3s_1+s_2}{4}\right)^i\frac{s_1^3}{4}\frac{s_2}{5}+\left(\frac{2s_1+s_2}{3}\right)^i\frac{s_1^2}{3}\frac{s_2^2}{5}\right)ds_1ds_2$$

$$+\frac{1}{\lambda^4}\sum_{i=0}^{m-3}\frac{1}{i!5^{m-3-i}}\int_0^\infty \mathrm{e}^{-s_3}\int_0^{s_3}\int_0^{s_2}$$

$$\left(\left(\frac{8s_1+s_2+3s_3}{12}\right)^i\frac{s_1^2}{3}\frac{s_2}{4}\frac{s_3}{5}+\left(\frac{4s_1+2s_2+2s_3}{8}\right)^i\frac{s_1}{2}\frac{s_2^2}{4}\frac{s_3}{5}+\left(\frac{3s_1+s_2+2s_3}{6}\right)^i\frac{s_1}{2}\frac{s_2}{3}\frac{s_3^2}{5}\right)ds_1ds_2ds_3$$

$$+\frac{1}{\lambda^4}\sum_{i=0}^{m-4}\frac{1}{i!5^{m-4-i}}\int_0^\infty \mathrm{e}^{-s_4}\int_0^{s_4}\int_0^{s_3}\int_0^{s_2}\left(\frac{6s_1+2s_2+s_3+3s_4}{12}\right)^i\prod_{l=1}^{4}\frac{s_1}{2}\frac{s_2}{3}\frac{s_3}{4}\frac{s_4}{5}ds_1ds_2ds_3ds_4,$$

see Figure 6.



(a) Fourth cumulant of $Y(m)$. (b) Excess kurtosis $\kappa^{(4)}(t)/\big(\kappa^{(2)}(m)\big)^2$.

Figure 6: Fourth cumulant and excess kurtosis of $Y(m)$.

Embedded chain of the growth-collapse process

Finally, we consider the embedded chain of the growth-collapse process

$$\begin{aligned} X(m) &:= f_m(T_m) - Y(m) \\ &= f_m(T_m) - \sum_{k=1}^{m} g(T_k, k, m) \\ &= f_m(T_m) - \sum_{k=1}^{m} f_m(T_k)(1 - Z_k) \prod_{l=k+1}^{m} Z_l, \qquad m \geq 1, \end{aligned} \tag{6.8}$$

where $(Z_k)_{k\geq 1}$ is an i.i.d. random sequence independent with moment sequence $\mu_n = \mathbb{E}[Z^n]$, $n \geq 0$. This process can be obtained from (6.1) by replacing $g(s, m, m) = f_m(s)(1 - Z_m)$ with $g(s, m, m) = f_m(s) - f_m(s)(1 - Z_m) = f_m(s)Z_m$ therein. In Corollary 6.2, this amounts to modifying the last term of order $i = m - k$ or $N_{s_k} = m - k$ in (6.4), by changing the last term $C_{q_{k-1},q_k-q_{k-1}}$ of order $l = k$ in the product $\prod_{l=1}^{k} C_{q_{l-1},q_l-q_{l-1}}$ into $(-1)^{q_k-q_{k-1}}\mu_{q_k} = (-1)^{q_k-q_{k-1}}\mu_n$, yielding the next result.

Corollary 6.4 *Let $(X(m))_{m\geq 1}$ be defined as in (6.8). For any $n, m \geq 1$, we have*

$$\mathbb{E}\big[(X(m))^n\big] = n!(-1)^n \sum_{k=1}^{n} \sum_{i=0}^{m-k-1} \frac{\lambda^{k+i}}{i!} \mu_n^{m-i-k} \sum_{0=q_0<q_1<\cdots<q_{k-1}<q_k=n}$$

$$\int_0^\infty e^{-\lambda s_k} \int_0^{s_k} \cdots \int_0^{s_2} \left(\sum_{l=0}^{k-1} \mu_{q_l}(s_{l+1} - s_l) \right)^i \prod_{l=1}^{k} \frac{C_{q_{l-1},q_l-q_{l-1}} f_m^{q_l-q_{l-1}}(s_l)}{(q_l - q_{l-1})!} ds_1 \cdots ds_k$$

$$+\, n!(-1)^n \sum_{k=1}^{\min(n,m)} \frac{\lambda^m}{(m-k)!} \sum_{0=q_0<q_1<\cdots<q_{k-1}<q_k=n} (-1)^{q_k-q_{k-1}} \mu_n$$

$$\int_0^\infty \mathrm{e}^{-\lambda s_k} \frac{f_m^{n-q_{k-1}}(s_k)}{(n-q_{k-1})!} \int_0^{s_k} \cdots \int_0^{s_2} \left(\sum_{l=0}^{k-1} \mu_{q_l}(s_{l+1}-s_l) \right)^{m-k} \prod_{l=1}^{k-1} \frac{C_{q_{l-1},q_l-q_{l-1}} f_m^{q_l-q_{l-1}}(s_l)}{(q_l-q_{l-1})!} ds_1 \cdots ds_k,$$

where we let $s_0 := 0$ *and* $\sum_{i=1}^0 = 0$*.*

When $n = 1$, Corollary 6.4 yields the first moment

$$\mathbb{E}[X(m)] = -(1-\mu_1) \sum_{i=0}^{m-2} \frac{\lambda^{1+i}}{i!} \mu_1^{m-i} \int_0^\infty \mathrm{e}^{-\lambda s} s^i f_m(s) ds + \frac{\mu_1 \lambda^m}{(m-1)!} \int_0^\infty \mathrm{e}^{-\lambda s} f_m(s) s^{m-1} ds.$$

Taking $f_m(s) = s$, $s \in \mathbb{R}_+$, this shows that

$$\mathbb{E}[X(m)] = \frac{\mu_1}{\lambda} \left(\frac{1-\mu_1^m}{1-\mu_1} \right),$$

which extends (6.6) above and (52) in Boxma et al. (2006) to non-uniform cut-off rates. In the exponential case with $\mu_1 \in (-1, 1)$ this recovers the long range behavior of the first moment (see page 369 of Kella (2009), as well as § 4 of Boxma et al. (2011)) as m tends to infinity.

The moment $\mathbb{E}[(X(m))^n]$ can be computed from Corollary 6.3 for specific integer values $n, m \geq 1$ and for any choice of function $f(s)$ and moment sequence $\mu_n = \mathbb{E}[Z^n]$, $n \geq 0$, using the Maple and Mathematica codes 7 and 8 in the online appendix.

When $(U_k)_{k\geq 1}$ is an i.i.d. uniform random sequence on $[0, 1]$, computing the moments of

$$X(m) = T_m - \sum_{k=1}^m T_k (1-U_k) \prod_{l=k+1}^m U_l, \tag{6.9}$$

according to Corollary 6.3 means multiplying the product $\displaystyle\prod_{l=1}^k \frac{1}{1+q_l}$ for $i = m-k$ in (6.5) by

$$\frac{(-1)^{q_k-q_{k-1}} m_{q_k}}{C_{q_{k-1},q_k-q_{k-1}}} = \frac{(-1)^{q_k-q_{k-1}}(1+q_k)!}{(1+q_k)q_{k-1}!(q_k-q_{k-1})!} = (-1)^{q_k-q_{k-1}} \binom{n}{q_{k-1}},$$

as done in the next result.

Corollary 6.5 *Let* $(X(m))_{m\geq 1}$ *be defined as in* (6.9)*, with* $(U_k)_{k\geq 1}$ *an i.i.d. uniform random sequence on* $[0, 1]$*. For any* $n, m \geq 1$*, we have*

$$\mathbb{E}\big[(X(m))^n\big] = (-1)^n \sum_{k=1}^n \sum_{i=0}^{m-k-1} \frac{\lambda^{k+i}}{i!(n+1)^{m-i-k}} \sum_{0=q_0<q_1<\cdots<q_{k-1}<q_k=n}$$

$$\int_0^\infty e^{-\lambda s_k} \int_0^{s_k} \cdots \int_0^{s_2} \left(\sum_{l=0}^{k-1} \frac{s_{l+1} - s_l}{1 + q_l} \right)^i \prod_{l=1}^{k} \frac{f_m^{q_l - q_{l-1}}(s_l)}{1 + q_l} ds_1 \cdots ds_k$$

$$+(-1)^n \sum_{k=1}^{\min(n,m)} \frac{\lambda^m}{(m-k)!} \sum_{0=q_0<q_1<\cdots<q_{k-1}<q_k=n} (-1)^{q_k - q_{k-1}} \binom{n}{q_{k-1}}$$

$$\int_0^\infty e^{-\lambda s_k} \int_0^{s_k} \cdots \int_0^{s_2} \left(\sum_{l=0}^{k-1} \frac{s_{l+1} - s_l}{1 + q_l} \right)^{m-k} \prod_{l=1}^{k} \frac{f_m^{q_l - q_{l-1}}(s_l)}{1 + q_l} ds_1 \cdots ds_k,$$

where we let $s_0 := 0$ *and* $\sum_{i=1}^{0} = 0$*.*

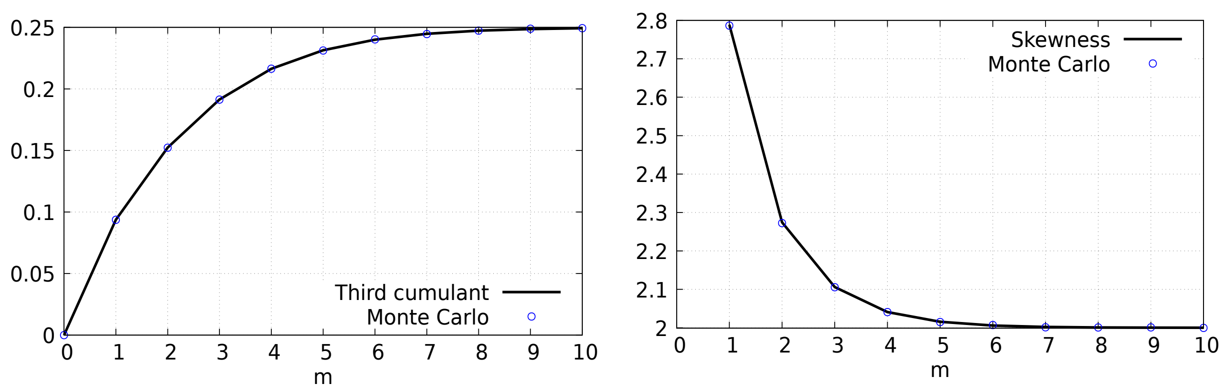
The following moments and cumulants can be computed from Corollary 6.5 for any specific integer values of $n, m \geq 1$ using the Maple commands $\mathrm{MXm}(\lambda, m, \mathrm{mu}, f, n)$, $\mathrm{CXm}(\lambda, m, \mathrm{mu}, f, n)$, resp. Mathematica commands $\mathrm{MXm}[\lambda, m, \mathrm{mu}, f, n]$, $\mathrm{CXm}[\lambda, m, \mathrm{mu}, f, n]$ in the online appendix, with $\mathrm{f} := s \to s$ and $\mathrm{mu} := n \to 1/(n+1)$, resp. $\mathrm{f}[s\_] := s$ and $\mathrm{mu}[n\_] := 1/(n+1)$. When $f_m(s) = s$, Corollary 6.5 shows that the second moment reads

$$\mathbb{E}\big[(X(m))^2\big] = \frac{1}{\lambda^2} \left(2 - 4 \left(\frac{1}{2}\right)^m + 2 \left(\frac{1}{3}\right)^m \right)$$

which yields

$$\mathrm{Var}[X(m)] = \frac{1}{\lambda^2} \left(1 - 2 \left(\frac{1}{2}\right)^m + 2 \left(\frac{1}{3}\right)^m - \left(\frac{1}{4}\right)^m \right), \qquad m \geq 0,$$

from (6.6), and recovers Theorem 7 in Boxma et al. (2006). Figures 7 to 8 are plotted with 100 million Monte Carlo samples and $\lambda = 2$.

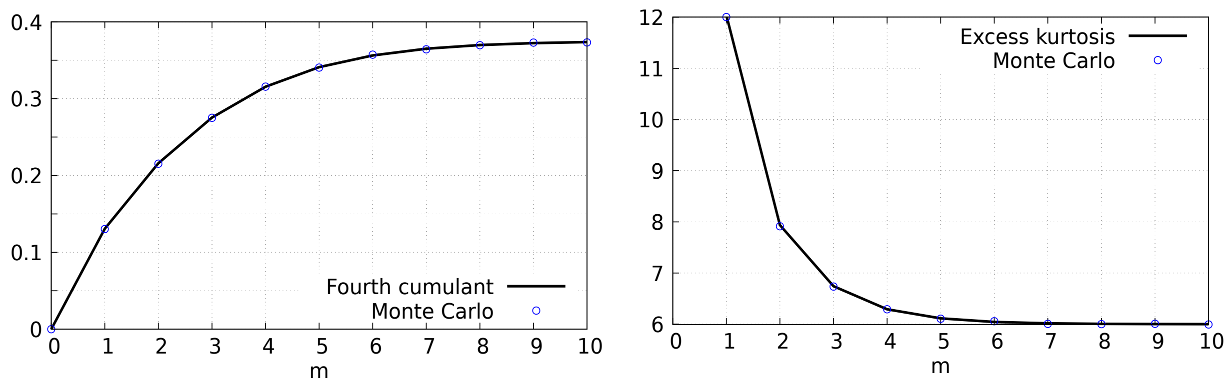


(a) Third cumulant of $X(m)$.

(b) Skewness $\kappa^{(3)}(m)/\big(\kappa^{(2)}(m)\big)^{3/2}$.

Figure 7: Third cumulant and skewness of $X(m)$.

In particular, the third and fourth cumulants of $X(m)$ can be obtained from Corollary 6.5 see Figures 7-8, along with Monte Carlo simulations used for confirmation.



(a) Fourth cumulant of $X(m)$.

(b) Excess kurtosis $\kappa^{(4)}(m)/\big(\kappa^{(2)}(m)\big)^2$.

Figure 8: Fourth cumulant and kurtosis of $X(m)$.

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A Maple and Mathematica codes

```
m := proc(t, l, k, mu, c, f, q::list) local s; if k = 1 then return int(f(s)^q[1]*c(0, q[1])*exp(l*s*(1 -
     mu(q[1])))/q[1]!, s = 0 .. t); end if; return int(f(s)^(q[k] - q[k - 1])*c(q[k - 1], q[k] - q[k -
     1])*exp(l*s*(mu(q[k - 1]) - mu(q[k])))*m(s, l, k - 1, mu, c, f, q[1 .. k - 1])/(q[k] - q[k - 1])!, s
     = 0 .. t); end proc;
MY := proc(t, l, mu, f, n) local k, temp, c, p, pp, pp2; c := proc(p, qq) return sum(binomial(p,
     k0)*(-1)^k0*mu(qq + k0), k0 = 0 .. p); end proc; if n = 0 then return 1; end if; if n = 1 then
     return l*exp(l*t*(mu(n) - 1))*m(t, l, 1, mu, c, f, [1]); end if; temp := 0; pp2 :=
     combinat:-subsets({seq(1 .. n - 1)}); while not pp2[finished] do p := pp2[nextvalue](); k := nops(p)
     + 1; temp := temp + l^k*m(t, l, k, mu, c, f, [op(p), n]); end do; return n!*temp*exp(l*t*(mu(n) -
     1)); end proc;
CY := proc(t, l, mu, f, n) local tmp, z, k; tmp := 0; z := []; for k from n by -1 to 1 do z := [op(z),
     MY(t, l, mu, f, n - k + 1)]; tmp := tmp + (-1)^(k - 1)*(k - 1)!*IncompleteBellB(n, k, op(z)); end
     do; return tmp; end proc;
```

Maple code 1 for the computation of $\mathbb{E}[(Y_t)^n]$ in Corollary 3.2.

```
Needs["Combinatorica`"]
(*Multiple integrals*)m[t_, l_, k_, mu_, c_, f_, q__] := (Module[{s},
   If[k == 1, Return[Integrate[f[s]^q[[1]]*c[0, q[[1]]]*E^(l*s*(1 - mu[q[[1]]]))/q[[1]]!, {s, 0, t}]]];
   Return[Integrate[f[s]^(q[[k]] - q[[k - 1]])*c[q[[k - 1]], q[[k]] - q[[k - 1]]]*E^(l*s*(mu[q[[k - 1]]]
       - mu[q[[k]]]))*m[s, l, k - 1, mu, c, f, q[[1 ;; k - 1]]]/(q[[k]] - q[[k - 1]])!, {s, 0, t}]]]);
```

```
(*Moments*)MY[t_, l_, mu_, f_, n_] := (Module[{k, temp},
   c[p_, qq_] := (Sum[Binomial[p, k0]*(-1)^k0*mu[qq + k0], {k0, 0, p}]);
   If[n == 0, Return[1]];
   If[n == 1, Return[l*E^(l*t*(mu[n] - 1))*m[t, l, 1, mu, c, f, {1}]]];
   temp = 0; Do[k = 1 + Length[p]; temp += l^k*m[t, l, k, mu, c, f, Append[p, {n}]], {p,
       Subsets[Range[1, n - 1]]}];
   Return[n!*temp[[1]]*E^(l*t*(mu[n] - 1))]]);
(*Cumulants*)CY[t_, l_, mu_, f_, n_] := (Module[{tmp, z, k}, tmp = 0; z = {};
  For[k = n, k >= 1, k--, z = Append[z, MY[t, l, mu, f, n - k + 1]];
   tmp += (-1)^(k - 1)*(k - 1)!*BellY[n, k, z]]; tmp])
```

Mathematica code 2 for the computation of $\mathbb{E}[(Y_t)^n]$ in Corollary 3.2.

```
MX := proc(t, a, mu, f, n) local k, tmp; if n = 1 then return f(t) - MY(t, a, mu, f, 1); else tmp :=
    MY(t, a, mu, f, n); for k from 0 to n - 1 do tmp := tmp - binomial(n, k)*f(t)^(n - k)*(-1)^k*MX(t,
    a, mu, f, k); end do; return (-1)^n*tmp; end if; end proc;
```

Maple code 3 for the computation of $\mathbb{E}[(X_t)^n]$ by the recursion (5.1).

```
MX[t_, a_, mu_, f_, n_] := (If[n == 1, f[t] - MY[t, a, mu, f, 1], (-1)^
    n*(MY[t, a, mu, f, n] - Sum[Binomial[n, k]*f[t]^(n - k)*(-1)^k*MX[t, a, mu, f, k], {k, 0, n - 1}])])
```

Mathematica code 4 for the computation of $\mathbb{E}[(X_t)^n]$ by the recursion (5.1).

```
mm := proc(a, k, i, c, mu, f, l, q, s) if l = 1 then return c(0, q[2])*sum(mu(q[l0 + 1])*(s[l0 + 2] -
    s[l0 + 1]), l0 = 0 .. k - 1)^i*f(s[2])^q[2]/q[2]!; end if; return c(q[l], q[l + 1] - q[l])*int(f(s[l
    + 1])^(q[l + 1] - q[l])*mm(a, k, i, c, mu, f, l - 1, q, s)/(q[l + 1] - q[l])!, s[l] = 0 .. s[l +
    1]); end proc;
MYm := proc(l, m, mu, f, n) local u, c, temp, k, i, pt, p1; if n = 0 then return 1; end if; c := proc(p,
    qq) return sum(binomial(p, k0)*(-1)^k0*mu(qq + k0), k0 = 0 .. p); end proc; temp := 0; for k to n do
    for i from 0 to m - k do if k = 1 then assume(0 < l); temp := temp + l^(k + i)*mu(n)^(m - i -
    k)*int(exp(-l*s)*mm(l, 1, i, c, mu, f, 1, [0, n], [0, s]), s = 0 .. infinity)/i!; else pt :=
    combinat:-choose(n - 1, k - 1); u := array(1 .. k + 1); u[1] := 0; for p1 in pt do assume(0 < l);
    temp := temp + l^(k + i)*mu(n)^(m - i - k)*int(exp(-l*u[k + 1])*mm(l, k, i, c, mu, f, k, [0, op(p1),
    n], u), u[k + 1] = 0 .. infinity)/i!; end do; end if; end do; end do; interface(showassumed = 0);
    return n!*temp; end proc;
CYm := proc(l, m, mu, f, n) local tmp, z, k; tmp := 0; z := []; for k from n by -1 to 1 do z := [op(z),
    MYm(l, m, mu, f, n - k + 1)]; tmp := tmp + (-1)^(k - 1)*(k - 1)!*IncompleteBellB(n, k, op(z)); end
    do; return tmp; end proc;
```

Maple code 5 for the computation of $\mathbb{E}[(Y(m))^n]$ in Corollary 6.2.

```
(*Multiple integrals*)mm[a_, k_, i_, mu_, f_, ll_, qq__, ss__] := (CC[p_, q_] := (Module[{k0},
    Sum[Binomial[p, k0]*(-1)^k0*mu[q + k0], {k0, 0, p}]]);
  If[ll == 1, Return[CC[0, qq[[2]]]*(Sum[ mu[qq[[l0 + 1]]]*(ss[[l0 + 2]] - ss[[l0 + 1]]), {l0, 0, k -
      1}])^i*f[ss[[2]]]^qq[[2]]/qq[[2]]!]];
  Return[CC[qq[[ll]], qq[[ll + 1]] - qq[[ll]]]*Integrate[f[ss[[ll + 1]]]^(qq[[ll + 1]] - qq[[ll]])*
     mm[a, k, i, mu, f, ll - 1, qq, ss]/(qq[[ll + 1]] - qq[[ll]])!, {ss[[ll]], 0,
    ss[[ll + 1]]}]]);
(*Moments*)MYm[l_, m_, mu_, f_, n_] := (If[n == 0, Return[1]]; temp = 0;
  For[k = 1, k <= n, k++, For[i = 0, i <= m - k, i++,
    If[k == 1, temp += l^(k + i)*mu[n]^(m - i - k)*
      Integrate[ E^(-l*s)*mm[l, 1, i, mu, f, 1, {0, n}, {0, s}], {s, 0, Infinity}, Assumptions -> l >
         0]/i!, pt = Subsets[Range[1, n - 1], {k - 1}] //. {} -> Sequence[];
     sss = Array[s, k];
     Do[temp += l^(k + i)*mu[n]^(m - i - k)*
      Integrate[ E^(-l*sss[[k]])* mm[l, k, i, mu, f, k, Append[Prepend[p1, 0], n], Prepend[sss, 0]],
         {sss[[k]], 0, Infinity}, Assumptions -> l > 0]/i!, {p1, pt}]]]]; Return[n!*temp]);
(*Cumulants*)CYm[l_, m_, mu_, f_, n_] := (Module[{tmp, z, k}, tmp = 0; z = {};
  For[k = n, k >= 1, k--, z = Append[z, MYm[l, m, mu, f, n - k + 1]]; tmp += (-1)^(k - 1)*(k -
      1)!*BellY[n, k, z]]; tmp])
```

Mathematica code 6 for the computation of $\mathbb{E}[(Y(m))^n]$ in Corollary 6.2.

```
MXm := proc(l, m, mu, f, n) local k, temp, i, pt, c, p1, u; if n = 0 then return 1; end if; c := proc(p,
     qq) return sum(binomial(p, k0)*(-1)^k0*mu(qq + k0), k0 = 0 .. p); end proc; temp := 0; for k to n do
     for i from 0 to m - k - 1 do if k = 1 then assume(0 < l); temp := temp + l^(k + i)*mu(n)^(m - i -
     k)*int(exp(-l*s)*mm(l, 1, i, c, mu, f, 1, [0, n], [0, s]), s = 0 .. infinity)/i!; else pt :=
     combinat:-choose(n - 1, k - 1); u := array(1 .. k + 1); u[1] := 0; for p1 in pt do assume(0 < l);
     temp := temp + l^(k + i)*mu(n)^(m - i - k)*int(exp(-l*u[k + 1])*mm(l, k, i, c, mu, f, k, [0, op(p1),
     n], u), u[k + 1] = 0 .. infinity)/i!; end do; end if; end do; end do; for k to min(n, m) do if k = 1
     then assume(0 < l); temp := temp + l^m*(-1)^n*mu(n)*int(exp(-l*s)*mm(l, 1, m - 1, c, mu, f, 1, [0,
     n], [0, s]), s = 0 .. infinity)/((m - 1)!*c(0, n)); else pt := combinat:-choose(n - 1, k - 1); u :=
     array(1 .. k + 1); u[1] := 0; for p1 in pt do assume(0 < l); temp := temp + l^m*(-1)^(n - p1[k -
     1])*mu(n)*int(exp(-l*u[k + 1])*mm(l, k, m - k, c, mu, f, k, [0, op(p1), n], u), u[k + 1] = 0 ..
     infinity)/((m - k)!*c(p1[k - 1], n - p1[k - 1])); end do; end if; end do; interface(showassumed =
     0); return (-1)^n*n!*temp; end proc;
CXm := proc(l, m, mu, f, n) local tmp, z, k; tmp := 0; z := []; for k from n by -1 to 1 do z := [op(z),
     MXm(l, m, mu, f, n - k + 1)]; tmp := tmp + (-1)^(k - 1)*(k - 1)!*IncompleteBellB(n, k, op(z)); end
     do; return tmp; end proc;
```

Maple code 7 for the computation of $\mathbb{E}[(X(m))^n]$ in Corollary 6.4.

```
(*Moments*)MXm[a_, m_, mu_, f_, n_] := (Module[{k, temp, i, pt, sss}, If[n == 0, Return[1]]; temp = 0;
     For[k = 1, k <= n, k++, For[i = 0, i <= (m - k - 1), i++,
      If[k == 1, temp += a^(k + i)*mu[n]^(m - i - k)*Integrate[ E^(-a*s)*mm[a, 1, i, mu, f, 1, {0, n}, {0,
         s}], {s, 0, Infinity}, Assumptions -> a > 0]/i!,
       pt = Subsets[Range[1, n - 1], {k - 1}] //. {} -> Sequence[]; sss = Array[s, k];
       Do[temp += a^(k + i)*mu[n]^(m - i - k)*Integrate[ E^(-a*sss[[k]])*mm[a, k, i, mu, f, k,
            Append[Prepend[p1, 0], n],
             Prepend[sss, 0]], {sss[[k]], 0, Infinity}, Assumptions -> a > 0]/i!, {p1, pt}]]]];
    For[k = 1, k <= Min[n, m], k++,
     If[k == 1, temp += a^m*(-1)^n*mu[n]*Integrate[ E^(-a*s)*mm[a, 1, m - 1, mu, f, 1, {0, n}, {0, s}],
        {s, 0, Infinity}, Assumptions -> a > 0]/(m - 1)!/CC[0, n],
      pt = Subsets[Range[1, n - 1], {k - 1}] //. {} -> Sequence[]; sss = Array[s, k];
      Do[temp += a^m*(-1)^(n - p1[[k - 1]])*mu[n]*
         Integrate[ E^(-a*sss[[k]])*mm[a, k, m - k, mu, f, k, Append[Prepend[p1, 0], n], Prepend[sss, 0]],
            {sss[[k]], 0, Infinity}, Assumptions -> a > 0]/(m - k)!/
        CC[p1[[k - 1]], n - p1[[k - 1]]], {p1, pt}]]]; Return[(-1)^n*n!*temp]]);
(*Cumulants*)CXm[l_, m_, mu_, f_, n_] := (Module[{tmp, z, k}, tmp = 0; z = {}; For[k = n, k >= 1, k--, z
     = Append[z, MXm[l, m, mu, f, n - k + 1]]; tmp += (-1)^(k - 1)*(k - 1)!*BellY[n, k, z]]; tmp])
```

Mathematica code 8 for the computation of $\mathbb{E}[(X(m))^n]$ in Corollary 6.4.