Characterization of stochastic equilibrium controls by the Malliavin calculus

Jiang Yu Nguwi* Nicolas Privault†
Division of Mathematical Sciences
School of Physical and Mathematical Sciences
Nanyang Technological University
21 Nanyang Link, Singapore 637371

October 12, 2021

Abstract

We derive a characterization of equilibrium controls in continuous-time, time-inconsistent control (TIC) problems using the Malliavin calculus. For this, the classical duality analysis of adjoint BSDEs is replaced with the Malliavin integration by parts. This results into a necessary and sufficient maximum principle which is applied to a linear-quadratic TIC problem, recovering previous results obtained by duality analysis in the mean-variance case, and extending them to the linear-quadratic setting. We also show that our results apply beyond the linear-quadratic case by treating the generalized Merton problem.

Keywords: Stochastic maximum principle; spike perturbation; backward stochastic differential equation (BSDE); Malliavin calculus.
Mathematics Subject Classification (2010): 60H07, 93E20, 91G80, 91B70.

1 Introduction

Time-Inconsistent Control (TIC) problems can be formulated using pre-committed controls, in which case optimization is performed only at time 0, although the control attained in the infimum might not be "optimal" in the future, see e.g. Zhou and Li (2000), Buckdahn et al. (2011), Pham and Wei (2018).

*nguw0003@e.ntu.edu.sg
†nprivault@ntu.edu.sg

Stochastic maximum principles based on the characterizations of critical points of pre-committed controls using Malliavin integration by parts in time-inconsistent settings have been obtained in a number of recent works, starting in Meyer-Brandis et al. (2012). This involves applying the Malliavin calculus to express the Gâteaux derivative of the cost functional in terms of BSDE solutions and their Malliavin derivatives. Other related stochastic maximum principles have been established for different control problems using the Malliavin calculus in e.g. Øksendal and Sulem (2010), Øksendal and Sulem (2012), Wang et al. (2013), and Agram and Øksendal (2018).

Time inconsistency in stochastic control problems has also been dealt with via a game-theoretic approach via the construction of a time-consistent strategy. Such construction of time-consistent equilibrium controls has recently been the object of increased attention, with applications to the generalized Merton problem, economics problems with time-inconsistent preferences, and mean-variance portfolio selection with state-dependent risk aversion. In the case of deterministic TIC problems, Ekeland and Lazrak (2006) were the first to provide a characterization of equilibrium controls. This approach has been extended in Björk and Murgoci (2010) to a stochastic setting, via the derivation of an extended Hamilton-Jacobi-Bellman (HJB) system for the characterization of equilibrium controls. In Hu et al. (2012; 2017), this characterization has been achieved using a stochastic maximum principle of Pontryagin type in the linear quadratic case, see also Yong (2017) for the case of controlled mean-field SDEs. This idea has been extended in Djehiche and Huang (2016) to more general TIC problems with equilibrium controls.

In this paper, we replace the duality analysis of adjoint BSDEs used in the classical theory with the Malliavin integration by parts for the derivation of a necessary and sufficient maximum principle with equilibrium controls. For this, in Theorem 2.3 we express the variation of cost functions under spike perturbations using Malliavin integration by parts arguments. As a consequence of Theorem 2.3, we derive a necessary and sufficient condition in Corollary 2.4 by assuming that the feedback strategies are sufficiently regular. Our derivation differs from the approach of Meyer-Brandis et al. (2012) because equilibrium controls are defined by spike perturbations (see Definition 2.1) instead of the Gâteaux derivative, which has no clear connection to equilibrium controls.

Spike perturbations of optimal controls have also been considered in the Malliavin calculus in Agram and Øksendal (2018), however without involving equilibrium controls. In addition, the proofs in Agram and Øksendal (2018) use the duality analysis of adjoint BSDEs, and passing to the limit in the mean value theorem, which cannot be done without satisfying precise regularity conditions on the integrands.

Our main results are first applied to linear-quadratic problems in Proposition 3.1 which extends previous constructions of equilibrium controls obtained by duality analysis in the mean-variance case with state-dependent risk aversion, see § 4.1 of Hu et al. (2012). Moreover, to demonstrate that our results apply beyond the linear-quadratic setting, in Proposition 3.3 we deal with the generalized Merton problem, see § 6.2 in Yong (2012). Although our definition of equilibrium controls uses expectations instead of conditional means as in Hu et al. (2012) and Yong (2012), the equilibrium controls that we obtain coincide with theirs, see Remark 3.2 and Proposition 3.3.

Our optimality condition is more explicit than the one in Theorem 3.1 of Djehiche and Huang (2016), while allowing us to recover the results of § 4.1 and § 4.2.1 therein as a special case of Proposition 3.1, see Remark 3.2. Indeed, in duality analysis, the equilibrium control is characterized by a pair (p, q) solution to a linear BSDE, where q does not have an explicit representation, whereas in the Malliavin approach an expression q is explicitly provided, see (2.15b), thus allowing us to derive the closed-form equilibrium control in a more explicit manner.

This paper is organized as follows. In Section 2 we derive a maximum principle for the characterization of equilibrium control. In Section 3 we apply the maximum principle to a linear-quadratic problem and to the generalized Merton problem. Sections 4 and 5 contain regularity and boundedness results on the coefficients b, σ, h, g and their adjoint processes and Malliavin derivatives, for use in the proof of Theorem 2.3.

Malliavin integration by parts

We work on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ where $(\mathcal{F}_t)_{t\in[0,T]}$ is the filtration generated by a standard Brownian motion $(W_t)_{t\in[0,T]}$. Next, we recall two basic properties of the Malliavin derivative D_t which is defined on a dense domain $\mathbb{D}^{1,2}$ in $L^2(\mathbb{P})$, see e.g. Üstünel (1995), Nualart (2006) and references therein.

Lemma 1.1 *Let $F_1, \ldots, F_n \in \mathbb{D}^{1,2}$ and let $\psi : \mathbb{R}^n \to \mathbb{R}$ be a Lipschitz function of n variables.*

Then we have $\psi(F_1,\ldots,F_n) \in \mathbb{D}^{1,2}$, *and*

$$D_t\psi(F_1,\ldots,F_n) = \sum_{i=1}^{n} \frac{\partial \psi}{\partial x_i}(F_1,\ldots,F_n) D_t F_i. \tag{1.1}$$

In the next lemma, we let λ denote the Lebesgue measure on $[0,T]$.

Lemma 1.2 *Let* $(v_t)_{t\in[0,T]} \in L^2(\lambda \times \mathbb{P})$ *be a square-integrable* $(\mathcal{F}_t)_{t\in[0,T]}$*-adapted process, and let* $F \in \mathbb{D}^{1,2}$. *Then we have*

$$\mathbb{E}\left[F\int_0^T v_t dB_t\right] = \mathbb{E}\left[\int_0^T v_t D_t F dt\right]. \tag{1.2}$$

2 Stochastic maximum principle

We consider the stochastic control problem with cost functional

$$J(t,x,\varphi) = \mathbb{E}\left[g\big(X^{x,\varphi}_{t,T}, \mathbb{E}\big[X^{x,\varphi}_{t,T}\big]\big) + \int_t^T h\big(s, X^{x,\varphi}_{t,s}, \mathbb{E}\big[X^{x,\varphi}_{t,s}\big], \varphi(s, X^{x,\varphi}_{t,s})\big)ds\right],$$

where $(t,x) \in [0,T]\times\mathbb{R}$, $\varphi : [0,T]\times\mathbb{R} \to U$ is a sufficiently regular deterministic function, $U \subset \mathbb{R}$ is a control space, and $(X^{x,\varphi}_{t,s})_{s\in[t,T]}$ is solution of the stochastic differential equation (SDE)

$$\begin{cases} dX^{x,\varphi}_{t,s} = b\big(s, X^{x,\varphi}_{t,s}, \varphi\big(s, X^{x,\varphi}_{t,s}\big)\big)ds + \sigma\big(s, X^{x,\varphi}_{t,s}, \varphi\big(s, X^{x,\varphi}_{t,s}\big)\big)dW_s, \quad 0 \le t < s \le T, \\ X^{x,\varphi}_{t,t} = x. \end{cases} \tag{2.1}$$

At the expense of heavier notations, our results can be extended to the multidimensional case without essential difficulty, however, we prefer not to pursue such generality as the real-valued case is already notationally heavy. By abuse of notation, for $u = (u_t)_{t\in[0,T]}$ a U-valued $(\mathcal{F}_t)_{t\in[0,T]}$-adapted control process, we also let

$$J(t,x,u) = \mathbb{E}\left[g\big(X^{x,u}_{t,T}, \mathbb{E}\big[X^{x,u}_{t,T}\big]\big) + \int_t^T h\big(s, X^{x,u}_{t,s}, \mathbb{E}\big[X^{x,u}_{t,s}\big], u_s\big)ds\right], \tag{2.2}$$

with the SDE

$$\begin{cases} dX^{x,u}_{t,s} = b\big(s, X^{x,u}_{t,s}, u_s\big)ds + \sigma\big(s, X^{x,u}_{t,s}, u_s\big)dW_s, \quad 0 \le t < s \le T, \\ X^{x,u}_{t,t} = x. \end{cases} \tag{2.3}$$

Due to the presence of a mean-field term, problem (2.2)-(2.3) is time-inconsistent as the optimal control $\widehat{u}$ obtained at time t may not be optimal after time t. Following Ekeland and Lazrak (2006), a time-consistent control $\widehat{u}$ for this problem may be constructed by the following steps:

i) At current time t, assume that all the future-selves s with $s > t$ use the control $\widehat{u}_s$.

ii) Knowing this, it is optimal for the current-self t to also use $\widehat{u}_t$.

Given $(\widehat{u}_t)_{t\in[0,T]}$ a U-valued $(\mathcal{F}_t)_{t\in[0,T]}$-adapted control process and $(t,u) \in [0,T] \times U$, the "local" spike variation $u \otimes_{t,\varepsilon} \widehat{u}$ of $\widehat{u}$ is defined as

$$(u \otimes_{t,\varepsilon} \widehat{u})_s = \begin{cases} u, & 0 \le t \le s < t+\varepsilon, \\ \widehat{u}_s, & 0 \le t+\varepsilon \le s \le T, \end{cases} \tag{2.4}$$

and used for the next definition of equilibrium control, in which ∂_x refers to differentiation with respect to the state variable x. In the next definition we use the equilibrium controls of Björk and Murgoci (2010), and impose sufficient regularity on the feedback functions $\widehat{\varphi}$ to ensure the existence of solutions of the SDE (2.1), see e.g. (H3) in Yong (2012).

Definition 2.1 *A deterministic function $\widehat{\varphi} : [0,T] \times \mathbb{R} \to U$ is an equilibrium control for the problem (2.2)-(2.3) if $\widehat{\varphi}$ is differentiable with bounded derivatives, and both $\widehat{\varphi}$, $\partial_x \widehat{\varphi}$ are Lipschitz continuous in (t,x), and*

$$\lim_{\varepsilon \downarrow 0} \frac{J\big(t,x,u \otimes_{t,\varepsilon} \widehat{u}^{t,x}\big) - J\big(t,x,\widehat{u}^{t,x}\big)}{\varepsilon} \ge 0, \quad u \in U, \ x \in \mathbb{R}, \ a.e. \ t \in [0,T], \tag{2.5}$$

where $\widehat{u}^{t,x}_s := \widehat{\varphi}\big(s, X^{x,\widehat{\varphi}}_{t,s}\big)$, $0 \le t \le s \le T$.

We also make the following assumptions, in which ∂_y refers to differentiating with respect to mean field variable y.

Assumption 1 *i) The functions $g(x,y)$ and $h(t,x,y,u)$ in (2.2) admit Lipschitz continuous partial derivatives $\partial_x g$, $\partial_y g$, $\partial_x h$, $\partial_y h$, and continuous and bounded partial derivatives $\partial_x^i g$ and $\partial_x^i \partial_u^j h$ with $0 \le i \le 3$, $0 \le j \le 2$, $2 \le i+j \le 3$.*

ii) The functions $b(t,x,u)$ and $\sigma(t,x,u)$ in (2.3) are Lipschitz continuous, and admit continuous and bounded partial derivatives $\partial_x^i \partial_u^j b$ and $\partial_x^i \partial_u \sigma$ with $0 \le i \le 3$, $0 \le j \le 1$, $1 \le i+j \le 3$.

From now on, we let $\widehat{\varphi}$ denote the candidate equilibrium control in Definition 2.1 and let $\widehat{u}^{t,x}$ be the control process such that $\widehat{u}^{t,x}_s = \widehat{\varphi}(s, X^{x,\widehat{\varphi}}_{t,s})$. For $\psi \in \{b, \sigma, h, g\}$ and $A \in \{x, u, xx, xu, xxx, xxu, xuu\}$ we set the notation

$$\begin{aligned}
\partial_A \psi^{x,\widehat{\varphi}}_{t,s} &= \partial_A \psi\big(s, X^{x,\widehat{\varphi}}_{t,s}, \mathbb{E}\big[X^{x,\widehat{\varphi}}_{t,s}\big], \widehat{\varphi}(s, X^{x,\widehat{\varphi}}_{t,s})\big), \\
\partial_y \psi^{x,\widehat{\varphi}}_{t,s} &= \mathbb{E}\big[\partial_y \psi\big(s, X^{x,\widehat{\varphi}}_{t,s}, \mathbb{E}\big[X^{x,\widehat{\varphi}}_{t,s}\big], \widehat{\varphi}(s, X^{x,\widehat{\varphi}}_{t,s})\big)\big], \\
\delta \psi^{x,u,\widehat{\varphi}}_{t,s} &= \psi\big(s, X^{x,\widehat{\varphi}}_{t,s}, \mathbb{E}\big[X^{x,\widehat{\varphi}}_{t,s}\big], u\big) - \psi\big(s, X^{x,\widehat{\varphi}}_{t,s}, \mathbb{E}\big[X^{x,\widehat{\varphi}}_{t,s}\big], \widehat{\varphi}(s, X^{x,\widehat{\varphi}}_{t,s})\big), \\
\delta \partial_A \psi^{x,u,\widehat{\varphi}}_{t,s} &= \partial_A \psi\big(s, X^{x,\widehat{\varphi}}_{t,s}, \mathbb{E}\big[X^{x,\widehat{\varphi}}_{t,s}\big], u\big) - \partial_A \psi\big(s, X^{x,\widehat{\varphi}}_{t,s}, \mathbb{E}\big[X^{x,\widehat{\varphi}}_{t,s}\big], \widehat{\varphi}(s, X^{x,\widehat{\varphi}}_{t,s})\big).
\end{aligned}$$

Similarly, we let

$$\partial_x \widehat{\varphi}^{x,\widehat{\varphi}}_{t,s} = \partial_x \widehat{\varphi}\big(s, X^{x,\widehat{\varphi}}_{t,s}\big).$$

In addition, we set $b^{\widehat{\varphi}}(t,x) = b(t,x,\widehat{\varphi}(t,x))$ and $\sigma^{\widehat{\varphi}}(t,x) = \sigma(t,x,\widehat{\varphi}(t,x))$, $t \in [0,T]$. For $t \in [0,T)$ we let $\varepsilon > 0$ be small enough such that $t+\varepsilon < T$, and let $(y^{x,\varepsilon}_{t,s})_{s\in[t,T]}$ be the solution of the SDE

$$\begin{cases} dy^{x,\varepsilon}_{t,s} = \partial_x b^{x,\widehat{\varphi}}_{t,s} y^{x,\varepsilon}_{t,s} ds + \big(y^{x,\varepsilon}_{t,s}\partial_x\sigma^{x,\widehat{\varphi}}_{t,s} + \mathbb{1}_{[t,t+\varepsilon]}(s)\delta\sigma^{x,u,\widehat{\varphi}}_{t,s}\big)dW_s, \quad 0 \le t < s \le T, \\ y^{x,\varepsilon}_{t,t} = 0, \end{cases} \tag{2.7}$$

with

$$y^{x,\varepsilon}_{t,s} = y^{x,\varepsilon}_{t,t+\varepsilon} G^{t,x}_{t+\varepsilon,s}, \qquad s \in [t+\varepsilon, T]. \tag{2.8}$$

where

$$G^{t,x}_{t+\varepsilon,s} := \exp\left(\int_{t+\varepsilon}^s \left(\partial_x b^{x,\widehat{\varphi}}_{t,u} - \frac{1}{2}(\partial_x\sigma^{x,\widehat{\varphi}}_{t,u})^2\right)du + \int_{t+\varepsilon}^s \partial_x\sigma^{x,\widehat{\varphi}}_{t,u} dB_u\right), \quad s \in [t+\varepsilon, T]. \tag{2.9}$$

Let also $(z^{x,\varepsilon}_{t,s})_{s\in[t,T]}$ be the solutions of the SDE

$$\begin{cases} dz^{x,\varepsilon}_{t,s} = \left(z^{x,\varepsilon}_{t,s}\partial_x b^{x,\widehat{\varphi}}_{t,s} + \mathbb{1}_{[t,t+\varepsilon]}(s)\delta b^{x,u,\widehat{\varphi}}_{t,s} + \frac{1}{2}(y^{x,\varepsilon}_{t,s})^2\partial_{xx}b^{x,\widehat{\varphi}}_{t,s}\right)ds \\ \qquad\qquad + \left(z^{x,\varepsilon}_{t,s}\partial_x\sigma^{x,\widehat{\varphi}}_{t,s} + \mathbb{1}_{[t,t+\varepsilon]}(s) y^{x,\varepsilon}_{t,s}\delta\partial_x\sigma^{x,u,\widehat{\varphi}}_{t,s} + \frac{1}{2}(y^{x,\varepsilon}_{t,s})^2\partial_{xx}\sigma^{x,\widehat{\varphi}}_{t,s}\right)dW_s, \\ z^{x,\varepsilon}_{t,t} = 0, \end{cases} \tag{2.10}$$

$t < s \le T$, with

$$\begin{aligned} z^{x,\varepsilon}_{t,s} &= z^{x,\varepsilon}_{t,t+\varepsilon}G^{t,x}_{t+\varepsilon,s} + \frac{1}{2}\int_{t+\varepsilon}^s \big(y^{x,\varepsilon}_{t,r}\big)^2 G^{t,x}_{r,s}\partial_{xx}\sigma^{x,\widehat{\varphi}}_{t,r} dW_r \\ &\quad + \frac{1}{2}\int_{t+\varepsilon}^s \Big(\big(y^{x,\varepsilon}_{t,r}\big)^2\partial_{xx}b^{x,\widehat{\varphi}}_{t,r} - \big(y^{x,\varepsilon}_{t,r}\big)^2\partial_x\sigma^{x,\widehat{\varphi}}_{t,r}\partial_{xx}\sigma^{x,\widehat{\varphi}}_{t,r}\Big)G^{t,x}_{r,s}dr, \quad s \in [t+\varepsilon, T]. \end{aligned} \tag{2.11}$$

The characterization of equilibrium controls usually relies on an expansion of the form

$$J\big(t,x,u\otimes_{t,\varepsilon}\widehat{u}^{t,x}\big) - J\big(t,x,\widehat{u}^{t,x}\big)$$

$$= \mathbb{E}\left[y^{x,\varepsilon}_{t,T}\big(\partial_x g^{x,\widehat{\varphi}}_{t,T} + \partial_y g^{x,\widehat{\varphi}}_{t,T}\big) + \int_t^T y^{x,\varepsilon}_{t,s}\big(\partial_x h^{x,\widehat{\varphi}}_{t,s} + \partial_y h^{x,\widehat{\varphi}}_{t,s}\big)ds\right] \tag{2.12}$$

$$+ \mathbb{E}\left[z^{x,\varepsilon}_{t,T}\big(\partial_x g^{x,\widehat{\varphi}}_{t,T} + \partial_y g^{x,\widehat{\varphi}}_{t,T}\big) + \int_t^T z^{x,\varepsilon}_{t,s}\big(\partial_x h^{x,\widehat{\varphi}}_{t,s} + \partial_y h^{x,\widehat{\varphi}}_{t,s}\big)ds\right] \tag{2.13}$$

$$+ \frac{1}{2}\mathbb{E}\left[\big(y^{x,\varepsilon}_{t,T}\big)^2\partial_{xx}g^{x,\widehat{\varphi}}_{t,T} + \int_t^T \big(y^{x,\varepsilon}_{t,s}\big)^2\partial_{xx}h^{x,\widehat{\varphi}}_{t,s}ds\right] + \mathbb{E}\left[\int_t^{t+\varepsilon}\delta h^{x,u,\widehat{\varphi}}_{t,s}ds\right] + o(\varepsilon), \tag{2.14}$$

see Theorem 3.4.4 in Yong and Zhou (1999), where $y^{x,\varepsilon}_{t,s}$ is the first order approximation of $X^{x,u\otimes_{t,\varepsilon}\widehat{u}^{t,x}}_{t,s} - X^{x,\widehat{u}^{t,x}}_{t,s}$ while $z^{x,\varepsilon}_{t,s}$ is the second order approximation of $X^{x,u\otimes_{t,\varepsilon}\widehat{u}^{t,x}}_{t,s} - X^{x,\widehat{u}^{t,x}}_{t,s}$.

Duality analysis

In the classical theory, after obtaining (2.12)-(2.14), Itô's lemma is applied to (2.7)-(2.10) and to their corresponding adjoint BSDEs in order to represent $J\big(t,x,u\otimes_{t,\varepsilon}\widehat{u}^{t,x}\big)-J\big(t,x,\widehat{u}^{t,x}\big)$ using adjoint BSDE solutions. This step is known as the *duality analysis*, see Lemmas 4.5-4.6 in Chapter 3 of Yong and Zhou (1999).

Malliavin integration by parts

In Meyer-Brandis et al. (2012), the Malliavin calculus has been used in the framework of pre-committed controls in order to provide more explicit expressions as the adjoint BSDEs may not be completely solvable in closed form. In this paper, we apply this method in the setting of equilibrium controls using the Malliavin integration by parts of Lemma 1.2. In sequel, $C>0$ denotes a constant depending on T,p,x, and on the bounding constants in Assumption 1, that may vary line by line. The following result states a continuity property of Malliavin derivatives, which will be used in the proofs of Theorem 2.3 and of Propositions 3.1 and 3.3.

Lemma 2.2 *Let Assumption 1 hold, and assume that $\widehat{\varphi}$ is differentiable with bounded derivatives and both $\widehat{\varphi}$ and $\partial_x\widehat{\varphi}$ are Lipschitz continuous in (t,x). Define*

$$p^{x,\widehat{\varphi}}_{t,s}:=G^{t,x}_{s,T}\big(\partial_x g^{x,\widehat{\varphi}}_{t,T}+\partial_y g^{x,\widehat{\varphi}}_{t,T}\big)+\int_s^T G^{t,x}_{s,u}\big(\partial_x h^{x,\widehat{\varphi}}_{t,u}+\partial_y h^{x,\widehat{\varphi}}_{t,u}\big)du,\qquad 0\le t\le s\le T,\tag{2.15a}$$

$$q^{x,\widehat{\varphi}}_{t,s}:=D_s p^{x,\widehat{\varphi}}_{t,s},\qquad 0\le t<s\le T.\tag{2.15b}$$

Then, $s\mapsto q^{x,\widehat{\varphi}}_{t,s}$, $s>t$, is continuous in $\mathbb{L}^2(\mathbb{P})$. Consequently, as $s\searrow t$, $\big(q^{x,\widehat{\varphi}}_{t,s}\big)_{s>t}$ admits a limit denoted by $q^{x,\widehat{\varphi}}_{t,t}$ in $\mathbb{L}^2(\mathbb{P})$.

Proof. This follows from the bound

$$\mathbb{E}\big[|q^{x,\widehat{\varphi}}_{t,s_2}-q^{x,\widehat{\varphi}}_{t,s_1}|^2\big]=\mathbb{E}\big[|D_{s_2}p_{t,s_2}-D_{s_1}p_{t,s_1}|^2\big]\le C|s_2-s_1|,\qquad s_1\ge s_2>t,$$

which is a consequence of (5.10)-(5.11) and the triangle inequality. □

The process $\big(p^{x,\widehat{\varphi}}_{t,s}\big)_{s\in[t,T]}$ in (2.15a) is linked to the solution $(\tilde{p}_{t,s},\tilde{q}_{t,s})_{s\in[t,T]}$ of the linear BSDE appearing in classical duality analysis, see e.g. (3.1) in Djehiche and Huang (2016), by the relation $\tilde{p}_{t,s}=\mathbb{E}\big[p^{x,\widehat{\varphi}}_{t,s}\,\big|\,\mathcal{F}_s\big]$. In addition, we have

$$\tilde{q}_{t,s}=D_s\tilde{p}_{t,s}=\mathbb{E}\big[D_s p^{x,\widehat{\varphi}}_{t,s}\,\big|\,\mathcal{F}_s\big]=\mathbb{E}\big[q^{x,\widehat{\varphi}}_{t,s}\,\big|\,\mathcal{F}_s\big],\qquad a.e.\ \ s\in[t,T],$$

see e.g. Proposition 2.2 in Pardoux and Peng (1992). The following main result will be used to characterize equilibrium controls using the Malliavin calculus.

Theorem 2.3 *Let Assumption 1 hold, and assume that* $\widehat{\varphi}$ *is differentiable with bounded derivatives and both* $\widehat{\varphi}$ *and* $\partial_x\widehat{\varphi}$ *are Lipschitz continuous in* (t,x)*. Then we have*

$$\lim_{\varepsilon\downarrow 0}\frac{J\big(t,x,u\otimes_{t,\varepsilon}\widehat{u}^{t,x}\big)-J\big(t,x,\widehat{u}^{t,x}\big)}{\varepsilon}=\mathbb{E}\left[H(t,x,u)-H(t,x,\widehat{\varphi}(t,x))+\frac{1}{2}P^{x,\widehat{\varphi}}_{t,t}(\delta\sigma^{x,u,\widehat{\varphi}}_{t,t})^2\right],$$

$u\in U$, $x\in\mathbb{R}$, *a.e.* $t\in[0,T]$, *where*

$$\begin{aligned}
&H(t,x,u):=h(t,x,x,u)+p^{x,\widehat{\varphi}}_{t,t}b(t,x,u)+q^{x,\widehat{\varphi}}_{t,t}\sigma(t,x,u),\\
&P^{x,\widehat{\varphi}}_{t,s}:=\Gamma^{t,x}_{s,T}\partial_{xx}g^{x,\widehat{\varphi}}_{t,T} \qquad (2.16a)\\
&\qquad+\int_s^T\Gamma^{t,x}_{s,u}\big(\partial_{xx}h^{x,\widehat{\varphi}}_{t,u}+(\partial_{xx}b^{x,\widehat{\varphi}}_{t,u}-\partial_{xx}\sigma^{x,\widehat{\varphi}}_{t,u}\partial_x\sigma^{x,\widehat{\varphi}}_{t,u})p^{x,\widehat{\varphi}}_{t,u}+\partial_{xx}\sigma^{x,\widehat{\varphi}}_{t,u}f^{x,\widehat{\varphi}}_{t,u}\big)du,\\
&f^{x,\widehat{\varphi}}_{t,s}:=G^{t,x}_{s,T}D_s\partial_x g^{x,\widehat{\varphi}}_{t,T}+\int_s^T G^{t,x}_{s,u}D_s\partial_x h^{x,\widehat{\varphi}}_{t,u}du, \qquad (2.16b)\\
&\Gamma^{t,x}_{s,v}:=\exp\left(\int_s^v\Big(2\partial_x b^{x,\widehat{\varphi}}_{t,u}+\big(\partial_x\sigma^{x,\widehat{\varphi}}_{t,u}\big)^2-2\big(\partial_x\sigma^{x,\widehat{\varphi}}_{t,u}\big)^2\Big)du+2\int_s^v\partial_x\sigma^{x,\widehat{\varphi}}_{t,u}dB_u\right), \qquad (2.16c)
\end{aligned}$$

$0\le t\le s\le v\le T$.

Proof. The proof consists in rewriting (2.12)-(2.14) using the Malliavin integration by parts, see Lemma 1.2. Regarding (2.12), from (2.8) we have

$$\begin{aligned}
&\mathbb{E}\left[y^{x,\varepsilon}_{t,T}\big(\partial_x g^{x,\widehat{\varphi}}_{t,T}+\partial_y g^{x,\widehat{\varphi}}_{t,T}\big)+\int_t^T y^{x,\varepsilon}_{t,s}\big(\partial_x h^{x,\widehat{\varphi}}_{t,s}+\partial_y h^{x,\widehat{\varphi}}_{t,s}\big)ds\right]\\
&=\mathbb{E}\left[y^{x,\varepsilon}_{t,t+\varepsilon}\left(G^{t,x}_{t+\varepsilon,T}\big(\partial_x g^{x,\widehat{\varphi}}_{t,T}+\partial_y g^{x,\widehat{\varphi}}_{t,T}\big)+\int_{t+\varepsilon}^T G^{t,x}_{t+\varepsilon,s}\big(\partial_x h^{x,\widehat{\varphi}}_{t,s}+\partial_y h^{x,\widehat{\varphi}}_{t,s}\big)ds\right)+\int_t^{t+\varepsilon}y^{x,\varepsilon}_{t,s}\big(\partial_x h^{x,\widehat{\varphi}}_{t,s}+\partial_y h^{x,\widehat{\varphi}}_{t,s}\big)ds\right]\\
&=\mathbb{E}\left[y^{x,\varepsilon}_{t,t+\varepsilon}p^x_{t,t+\varepsilon}+\int_t^{t+\varepsilon}y^{x,\varepsilon}_{t,s}\big(\partial_x h^{x,\widehat{\varphi}}_{t,s}+\partial_y h^{x,\widehat{\varphi}}_{t,s}\big)ds\right]\\
&=\mathbb{E}\left[\int_t^{t+\varepsilon}\Big(y^{x,\varepsilon}_{t,s}p^x_{t,t+\varepsilon}\partial_x b^{x,\widehat{\varphi}}_{t,s}+\big(y^{x,\varepsilon}_{t,s}\partial_x\sigma^{x,\widehat{\varphi}}_{t,s}+\delta\sigma^{x,u,\widehat{\varphi}}_{t,s}\big)D_s p^x_{t,t+\varepsilon}+y^{x,\varepsilon}_{t,s}\big(\partial_x h^{x,\widehat{\varphi}}_{t,s}+\partial_y h^{x,\widehat{\varphi}}_{t,s}\big)\Big)ds\right]\\
&=\mathbb{E}\left[\int_t^{t+\varepsilon}\Big(y^{x,\varepsilon}_{t,s}\Big(\partial_x h^{x,\widehat{\varphi}}_{t,s}+\partial_y h^{x,\widehat{\varphi}}_{t,s}+p^x_{t,t+\varepsilon}\partial_x b^{x,\widehat{\varphi}}_{t,s}+D_s p^x_{t,t+\varepsilon}\partial_x\sigma^{x,\widehat{\varphi}}_{t,s}\Big)+\delta\sigma^{x,u,\widehat{\varphi}}_{t,s}D_s p^x_{t,t+\varepsilon}\Big)ds\right],
\end{aligned}$$

where the second equality is due to (2.15a) and the third equality follows from the SDE (2.7) and the integration by part formula of Lemma 1.2. Regarding (2.13), letting $Y^{x,\varepsilon}_{t,s}:=\big(y^{x,\varepsilon}_{t,s}\big)^2$ and using (2.11), we have

$$\mathbb{E}\left[z^{x,\varepsilon}_{t,T}\big(\partial_x g^{x,\widehat{\varphi}}_{t,T}+\partial_y g^{x,\widehat{\varphi}}_{t,T}\big)+\int_t^T z^{x,\varepsilon}_{t,s}\big(\partial_x h^{x,\widehat{\varphi}}_{t,s}+\partial_y h^{x,\widehat{\varphi}}_{t,s}\big)ds\right]$$

$$
\begin{aligned}
&= \mathbb{E}\Bigg[\big(\partial_x g_{t,T}^{x,\widehat{\varphi}} + \partial_y g_{t,T}^{x,\widehat{\varphi}}\big)\Bigg(z_{t,t+\varepsilon}^{x,\varepsilon} G_{t+\varepsilon,T}^{t,x} + \frac{1}{2}\int_{t+\varepsilon}^{T} \big(Y_{t,s}^{x,\varepsilon}\partial_{xx} b_{t,s}^{x,\widehat{\varphi}} - Y_{t,s}^{x,\varepsilon}\partial_x\sigma_{t,s}^{x,\widehat{\varphi}}\partial_{xx}\sigma_{t,s}^{x,\widehat{\varphi}}\big) G_{s,T}^{t,x} ds \\
&\quad + \frac{1}{2}\int_{t+\varepsilon}^{T} Y_{t,s}^{x,\varepsilon} G_{s,T}^{t,x}\partial_{xx}\sigma_{t,s}^{x,\widehat{\varphi}} dW_s\Bigg) + \int_t^{t+\varepsilon} z_{t,s}^{x,\varepsilon}\big(\partial_x h_{t,s}^{x,\widehat{\varphi}} + \partial_y h_{t,s}^{x,\widehat{\varphi}}\big) ds \\
&\quad + \int_{t+\varepsilon}^{T}\big(\partial_x h_{t,s}^{x,\widehat{\varphi}} + \partial_y h_{t,s}^{x,\widehat{\varphi}}\big)\Bigg(z_{t,t+\varepsilon}^{x,\varepsilon} G_{t+\varepsilon,s}^{t,x} + \frac{1}{2}\int_{t+\varepsilon}^{s} Y_{t,r}^{x,\varepsilon} G_{r,s}^{t,x}\partial_{xx}\sigma_{t,r}^{x,\widehat{\varphi}} dW_r \\
&\quad + \frac{1}{2}\int_{t+\varepsilon}^{s}\big(Y_{t,r}^{x,\varepsilon}\partial_{xx} b_{t,r}^{x,\widehat{\varphi}} - Y_{t,r}^{x,\varepsilon}\partial_x\sigma_{t,r}^{x,\widehat{\varphi}}\partial_{xx}\sigma_{t,r}^{x,\widehat{\varphi}}\big) G_{r,s}^{t,x} dr\Bigg) ds\Bigg] \\
&= \mathbb{E}\Bigg[z_{t,t+\varepsilon}^{x,\varepsilon}\Bigg(G_{t+\varepsilon,T}^{t,x}\big(\partial_x g_{t,T}^{x,\widehat{\varphi}} + \partial_y g_{t,T}^{x,\widehat{\varphi}}\big) + \int_{t+\varepsilon}^{T} G_{t+\varepsilon,s}^{t,x}\big(\partial_x h_{t,s}^{x,\widehat{\varphi}} + \partial_y h_{t,s}^{x,\widehat{\varphi}}\big) ds\Bigg) + \int_t^{t+\varepsilon} z_{t,s}^{x,\varepsilon}\big(\partial_x h_{t,s}^{x,\widehat{\varphi}} + \partial_y h_{t,s}^{x,\widehat{\varphi}}\big) ds \\
&\quad + \frac{1}{2}\int_{t+\varepsilon}^{T}\big(Y_{t,s}^{x,\varepsilon}\partial_{xx} b_{t,s}^{x,\widehat{\varphi}} - Y_{t,s}^{x,\varepsilon}\partial_x\sigma_{t,s}^{x,\widehat{\varphi}}\partial_{xx}\sigma_{t,s}^{x,\widehat{\varphi}}\big)\Bigg(G_{s,T}^{t,x}\big(\partial_x g_{t,T}^{x,\widehat{\varphi}} + \partial_y g_{t,T}^{x,\widehat{\varphi}}\big) + \int_s^T G_{s,r}^{t,x}\big(\partial_x h_{t,r}^{x,\widehat{\varphi}} + \partial_y h_{t,r}^{x,\widehat{\varphi}}\big) dr\Bigg) ds \\
&\quad + \frac{1}{2}\int_{t+\varepsilon}^{T} Y_{t,s}^{x,\varepsilon}\partial_{xx}\sigma_{t,s}^{x,\widehat{\varphi}}\Bigg(G_{s,T}^{t,x} D_s\partial_x g_{t,T}^{x,\widehat{\varphi}} + \int_s^T G_{s,r}^{t,x} D_s\partial_x h_{t,r}^{x,\widehat{\varphi}} dr\Bigg) ds\Bigg] \\
&= \mathbb{E}\Bigg[z_{t,t+\varepsilon}^{x,\varepsilon} p_{t,t+\varepsilon}^{x} + \int_t^{t+\varepsilon} z_{t,s}^{x,\varepsilon}\big(\partial_x h_{t,s}^{x,\widehat{\varphi}} + \partial_y h_{t,s}^{x,\widehat{\varphi}}\big) ds \\
&\quad + \frac{1}{2}\int_{t+\varepsilon}^{T}\big(Y_{t,s}^{x,\varepsilon}\partial_{xx} b_{t,s}^{x,\widehat{\varphi}} - Y_{t,s}^{x,\varepsilon}\partial_x\sigma_{t,s}^{x,\widehat{\varphi}}\partial_{xx}\sigma_{t,s}^{x,\widehat{\varphi}}\big) p_{t,s}^{x,\widehat{\varphi}} ds + \frac{1}{2}\int_{t+\varepsilon}^{T} Y_{t,s}^{x,\varepsilon} f_{t,s}^{x,\widehat{\varphi}}\partial_{xx}\sigma_{t,s}^{x,\widehat{\varphi}} ds\Bigg] \\
&= \mathbb{E}\Bigg[\int_t^{t+\varepsilon} z_{t,s}^{x,\varepsilon}\Bigg(\big(\partial_x h_{t,s}^{x,\widehat{\varphi}} + \partial_y h_{t,s}^{x,\widehat{\varphi}} + p_{t,t+\varepsilon}^{x}\partial_x b_{t,s}^{x,\widehat{\varphi}} + D_s p_{t,t+\varepsilon}^{x}\partial_x\sigma_{t,s}^{x,\widehat{\varphi}}\big) \\
&\quad + \frac{1}{2} Y_{t,s}^{x,\varepsilon}\big(p_{t,t+\varepsilon}^{x}\partial_{xx} b_{t,s}^{x,\widehat{\varphi}} + D_s p_{t,t+\varepsilon}^{x}\partial_{xx}\sigma_{t,s}^{x,\widehat{\varphi}}\big) + y_{t,s}^{x,\varepsilon}\delta\partial_x\sigma_{t,s}^{x,u,\widehat{\varphi}} D_s p_{t,t+\varepsilon}^{x} + \delta b_{t,s}^{x,u,\widehat{\varphi}} p_{t,t+\varepsilon}^{x}\Bigg) ds \\
&\quad + \frac{1}{2}\int_{t+\varepsilon}^{T} Y_{t,s}^{x,\varepsilon} p_{t,s}^{x,\widehat{\varphi}}\big(\partial_{xx} b_{t,s}^{x,\widehat{\varphi}} - \partial_x\sigma_{t,s}^{x,\widehat{\varphi}}\partial_{xx}\sigma_{t,s}^{x,\widehat{\varphi}}\big) ds + \frac{1}{2}\int_{t+\varepsilon}^{T} Y_{t,s}^{x,\varepsilon} f_{t,s}^{x,\widehat{\varphi}}\partial_{xx}\sigma_{t,s}^{x,\widehat{\varphi}} ds\Bigg],
\end{aligned}
$$

where the first equality is due to the solution of linear SDE (2.10) and the second equality is due to the integration by part formula of Lemma 1.2. The third equality is due to (2.15a) and (2.16b), and the fourth equality follows from the SDE (2.10) and Lemma 1.2. Next, we consider the solution of the linear SDE

$$
\begin{cases}
dY_{t,s}^{x,\varepsilon} = & \big(2\partial_x b_{t,s}^{x,\widehat{\varphi}} Y_{t,s}^{x,\varepsilon} + \big(\partial_x\sigma_{t,s}^{x,\widehat{\varphi}}\big)^2 Y_{t,s}^{x,\varepsilon} + \mathbb{1}_{[t,t+\varepsilon]}(s)\big(\delta\sigma_{t,s}^{x,u,\widehat{\varphi}}\big)^2\big) ds \\
& + 2\mathbb{1}_{[t,t+\varepsilon]}(s) y_{t,s}^{x,\varepsilon}\partial_x\sigma_{t,s}^{x,\widehat{\varphi}}\delta\sigma_{t,s}^{x,u,\widehat{\varphi}} ds \\
& + \big(2\partial_x\sigma_{t,s}^{x,\widehat{\varphi}} Y_{t,s}^{x,\varepsilon} + \mathbb{1}_{[t,t+\varepsilon]}(s) y_{t,s}^{x,\varepsilon}\delta\sigma_{t,s}^{x,u,\widehat{\varphi}}\big) dW_s, \quad t < s \le T, \\
Y_{t,t}^{x,\varepsilon} = & 0,
\end{cases}
\tag{2.17}
$$

obtained from (2.7) by the application of Itô's lemma to $Y_{t,s}^{x,\varepsilon} := (y_{t,s}^{x,\varepsilon})^2$, with

$$
Y_{t,s}^{x,\varepsilon} = Y_{t,t+\varepsilon}^{x,\varepsilon}\Gamma_{t+\varepsilon,s}^{t,x}, \qquad s \in [t+\varepsilon, T]. \tag{2.18}
$$

Regarding the first term in (2.14), using (2.18) and (2.16a), we have

$$
\begin{aligned}
&\mathbb{E}\left[Y^{x,\varepsilon}_{t,T}\partial_{xx}g^{x,\widehat{\varphi}}_{t,T} + \int_t^T Y^{x,\varepsilon}_{t,s}\partial_{xx}h^{x,\widehat{\varphi}}_{t,s}ds\right] \\
&= \mathbb{E}\Big[Y^{x,\varepsilon}_{t,t+\varepsilon}\Gamma^{t,x}_{t+\varepsilon,T}\partial_{xx}g^{x,\widehat{\varphi}}_{t,T} + \int_t^{t+\varepsilon} Y^{x,\varepsilon}_{t,s}\partial_{xx}h^{x,\widehat{\varphi}}_{t,s}ds + Y^{x,\varepsilon}_{t,t+\varepsilon}\int_{t+\varepsilon}^T \Gamma^{t,x}_{t+\varepsilon,s}\partial_{xx}h^{x,\widehat{\varphi}}_{t,s}ds\Big] \\
&= \mathbb{E}\left[Y^{x,\varepsilon}_{t,t+\varepsilon}P^x_{t,t+\varepsilon} + \int_t^{t+\varepsilon} Y^{x,\varepsilon}_{t,s}\partial_{xx}h^{x,\widehat{\varphi}}_{t,s}ds\right] \\
&\quad - \mathbb{E}\left[\int_{t+\varepsilon}^T Y^{x,\varepsilon}_{t,s}\big(p^{x,\widehat{\varphi}}_{t,s}\big(\partial_{xx}b^{x,\widehat{\varphi}}_{t,s} - \partial_x\sigma^{x,\widehat{\varphi}}_{t,s}\partial_{xx}\sigma^{x,\widehat{\varphi}}_{t,s}\big) + f^{x,\widehat{\varphi}}_{t,s}\partial_{xx}\sigma^{x,\widehat{\varphi}}_{t,s}\big)ds\right] \\
&= \mathbb{E}\Bigg[\int_t^{t+\varepsilon}\Big(Y^{x,\varepsilon}_{t,s}\big(\big(2\partial_x b^{x,\widehat{\varphi}}_{t,s} + \big(\partial_x\sigma^{x,\widehat{\varphi}}_{t,s}\big)^2 + \partial_{xx}h^{x,\widehat{\varphi}}_{t,s}\big)P^x_{t,t+\varepsilon} + 2\partial_x\sigma^{x,\widehat{\varphi}}_{t,s}D_sP^x_{t,t+\varepsilon}\big) \\
&\qquad + y^{x,\varepsilon}_{t,s}\big(2\partial_x\sigma^{x,\widehat{\varphi}}_{t,s}\delta\sigma^{x,u,\widehat{\varphi}}_{t,s}P^x_{t,t+\varepsilon} + \delta\sigma^{x,u,\widehat{\varphi}}_{t,s}D_sP^x_{t,t+\varepsilon}\big) + \big(\delta\sigma^{x,u,\widehat{\varphi}}_{t,s}\big)^2P^x_{t,t+\varepsilon}\Big)ds\Bigg] \\
&\quad - \mathbb{E}\left[\int_{t+\varepsilon}^T Y^{x,\varepsilon}_{t,s}\big(p^{x,\widehat{\varphi}}_{t,s}\big(\partial_{xx}b^{x,\widehat{\varphi}}_{t,s} - \partial_x\sigma^{x,\widehat{\varphi}}_{t,s}\partial_{xx}\sigma^{x,\widehat{\varphi}}_{t,s}\big) + f^{x,\widehat{\varphi}}_{t,s}\partial_{xx}\sigma^{x,\widehat{\varphi}}_{t,s}\big)ds\right].
\end{aligned}
$$

The third equality follows from the SDE (2.17) and Lemma 1.2. Putting the above equalities together, we find

$$
\begin{aligned}
&J\big(t,x,u\otimes_{t,\varepsilon}\widehat{u}^{t,x}\big) - J\big(t,x,\widehat{u}^{t,x}\big) \\
&\quad= \mathbb{E}\left[\int_t^{t+\varepsilon}\big(\delta h^{x,u,\widehat{\varphi}}_{t,s} + \delta\sigma^{x,u,\widehat{\varphi}}_{t,s}D_sp^x_{t,t+\varepsilon} + \delta b^{x,u,\widehat{\varphi}}_{t,s}p^x_{t,t+\varepsilon} + (\delta\sigma^{x,u,\widehat{\varphi}}_{t,s})^2P^x_{t,t+\varepsilon}\big)ds\right] \\
&\qquad + \mathbb{E}\left[\int_t^{t+\varepsilon}\big(y^{x,\varepsilon}_{t,s}\Lambda^{1,\varepsilon}_s + z^{x,\varepsilon}_{t,s}\Lambda^{2,\varepsilon}_s + Y^{x,\varepsilon}_{t,s}\Lambda^{3,\varepsilon}_s\big)ds\right] + o(\varepsilon),
\end{aligned}
$$

where we let

$$
\begin{aligned}
\Lambda^{1,\varepsilon}_s &= \partial_x h^{x,\widehat{\varphi}}_{t,s} + \partial_y h^{x,\widehat{\varphi}}_{t,s} + \partial_x b^{x,\widehat{\varphi}}_{t,s}p^x_{t,t+\varepsilon} + \partial_x\sigma^{x,\widehat{\varphi}}_{t,s}D_sp^x_{t,t+\varepsilon} + \delta\partial_x\sigma^{x,u,\widehat{\varphi}}_{t,s}D_sp^x_{t,t+\varepsilon} + \partial_x\sigma^{x,\widehat{\varphi}}_{t,s}\delta\sigma^{x,u,\widehat{\varphi}}_{t,s}P^x_{t,t+\varepsilon} \\
&\quad + \frac{1}{2}\delta\sigma^{x,u,\widehat{\varphi}}_{t,s}D_sP^x_{t,t+\varepsilon}, \\
\Lambda^{2,\varepsilon}_s &= \partial_x h^{x,\widehat{\varphi}}_{t,s} + \partial_y h^{x,\widehat{\varphi}}_{t,s} + \partial_x b^{x,\widehat{\varphi}}_{t,s}p^x_{t,t+\varepsilon} + \partial_x\sigma^{x,\widehat{\varphi}}_{t,s}D_sp^x_{t,t+\varepsilon}, \\
\Lambda^{3,\varepsilon}_s &= \frac{1}{2}\partial_{xx}b^{x,\widehat{\varphi}}_{t,s}p^x_{t,t+\varepsilon} + \frac{1}{2}\partial_{xx}\sigma^{x,\widehat{\varphi}}_{t,s}D_sp^x_{t,t+\varepsilon} + \frac{1}{2}\partial_{xx}h^{x,\widehat{\varphi}}_{t,s} + \left(\partial_x b^{x,\widehat{\varphi}}_{t,s} + \frac{1}{2}\big(\partial_x\sigma^{x,\widehat{\varphi}}_{t,s}\big)^2\right)P^x_{t,t+\varepsilon} + \partial_x\sigma^{x,\widehat{\varphi}}_{t,s}D_sP^x_{t,t+\varepsilon}.
\end{aligned}
$$

Next, we prove the following convergence results:

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\mathbb{E}\left[\int_t^{t+\varepsilon}\delta h_{t,s}^{x,u,\widehat{\varphi}}ds\right]=\delta h_{t,t}^{x,u,\widehat{\varphi}} \tag{2.19a}$$

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\mathbb{E}\left[\int_t^{t+\varepsilon}\delta\sigma_{t,s}^{x,u,\widehat{\varphi}}D_s p_{t,t+\varepsilon}^x ds\right]=\mathbb{E}\big[\delta\sigma_{t,t}^{x,u,\widehat{\varphi}}q_{t,t}^{x,\widehat{\varphi}}\big], \tag{2.19b}$$

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\mathbb{E}\left[\int_t^{t+\varepsilon}\delta b_{t,s}^{x,u,\widehat{\varphi}}p_{t,t+\varepsilon}^x ds\right]=\mathbb{E}\big[\delta b_{t,t}^{x,u,\widehat{\varphi}}p_{t,t}^{x,\widehat{\varphi}}\big],$$

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\mathbb{E}\left[\int_t^{t+\varepsilon}\big(\delta\sigma_{t,s}^{x,u,\widehat{\varphi}}\big)^2 P_{t,t+\varepsilon}^x ds\right]=\big(\delta\sigma_{t,t}^{x,u,\widehat{\varphi}}\big)^2 P_{t,t}^{x,\widehat{\varphi}},$$

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\mathbb{E}\left[\int_t^{t+\varepsilon}y_{t,s}^{x,\varepsilon}\Lambda_s^{1,\varepsilon}+z_{t,s}^{x,\varepsilon}\Lambda_s^{2,\varepsilon}+Y_{t,s}^{x,\varepsilon}\Lambda_s^{3,\varepsilon}ds\right]=0. \tag{2.19c}$$

We will only show (2.19b) and

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\mathbb{E}\left[\int_t^{t+\varepsilon}y_{t,s}^{x,\varepsilon}\partial_x\sigma_{t,s}^{x,\widehat{\varphi}}D_s p_{t,t+\varepsilon}^x ds\right]=0, \tag{2.20}$$

as the remaining estimates admit similar proofs. Note that (2.20) is a part of (2.19c).
Proof of (2.19b). Using the continuous version $s\mapsto q_{t,s}^{x,\widehat{\varphi}}$ we have, as ε tends to zero,

$$\begin{aligned}
&\frac{1}{\varepsilon}\mathbb{E}\left[\int_t^{t+\varepsilon}\big|\delta\sigma_{t,s}^{x,u,\widehat{\varphi}}D_s p_{t,t+\varepsilon}^x-\delta\sigma_{t,t}^{x,u,\widehat{\varphi}}q_{t,t}^{x,\widehat{\varphi}}\big|ds\right]\\
&\le\frac{1}{\varepsilon}\int_t^{t+\varepsilon}\Big(\big(\mathbb{E}\big[\big|\delta\sigma_{t,s}^{x,u,\widehat{\varphi}}\big|^2\big]\mathbb{E}\big[\big|D_s p_{t,t+\varepsilon}^x-D_s p_{t,s}^x\big|^2\big]\big)^{1/2}\\
&\quad+\big(\mathbb{E}\big[\big|\delta\sigma_{t,s}^{x,u,\widehat{\varphi}}\big|^2\big]\mathbb{E}\big[\big|D_s p_{t,s}^x-q_{t,t}^{x,\widehat{\varphi}}\big|^2\big]\big)^{1/2}+\big(\mathbb{E}\big[\big|q_{t,t}^{x,\widehat{\varphi}}\big|^2\big]\mathbb{E}\big[\big|\delta\sigma_{t,s}^{x,u,\widehat{\varphi}}-\delta\sigma_{t,t}^{x,u,\widehat{\varphi}}\big|^2\big]\big)^{1/2}\Big)ds\\
&\le\frac{C}{\varepsilon}\int_t^{t+\varepsilon}\Big(\sqrt{(1+u^2)|\varepsilon|^2}+\sqrt{(1+u^2)\mathbb{E}\big[\big|D_s p_{t,s}^x-q_{t,t}^{x,\widehat{\varphi}}\big|^2\big]}+\sqrt{|s-t|^2}\Big)ds\\
&=o(1).
\end{aligned}$$

The first inequality is due to the triangle inequality and Hölder's inequality. The second inequality is due to Lemmas 4.2-4.3 and 5.3. The last equality is due to Lemma 2.2.
Proof of (2.20). We have

$$\begin{aligned}
\frac{1}{\varepsilon}\mathbb{E}\left[\int_t^{t+\varepsilon}y_{t,s}^{x,\varepsilon}\partial_x\sigma_{t,s}^{x,\widehat{\varphi}}D_s p_{t,t+\varepsilon}^x ds\right]&\le\frac{C}{\varepsilon}\int_t^{t+\varepsilon}\sqrt{\mathbb{E}\big[\big|y_{t,s}^{x,\varepsilon}\big|^2\big]\mathbb{E}\big[\big|D_s p_{t,t+\varepsilon}^x\big|^2\big]}ds\\
&\le\frac{C}{\sqrt{\varepsilon}}\int_t^{t+\varepsilon}ds\\
&=O(\sqrt{\varepsilon}).
\end{aligned}$$

The first inequality is due to Assumption 1 and Hölder's inequality. The second inequality is due to Lemmas 4.1 and 5.3. We conclude the proof by (2.19a)-(2.19c). □

The following characterization of equilibrium controls is a consequence of Theorem 2.3.

Corollary 2.4 *Let Assumption 1 hold, and assume that $\widehat{\varphi}$ is differentiable with bounded derivatives and both $\widehat{\varphi}$ and $\partial_x\widehat{\varphi}$ are Lipschitz continuous in (t,x). Then $\widehat{\varphi}$ is an equilibrium control if and only if*

$$\mathbb{E}\left[H(t,x,u)-H(t,x,\widehat{\varphi}(t,x))+\frac{1}{2}P_{t,t}^{x,\widehat{\varphi}}(\delta\sigma_{t,t}^{x,u,\widehat{\varphi}})^2\right]\geq 0,\quad u\in U,\ x\in\mathbb{R},\ a.e.\ t\in[0,T].$$

We note that, by applying the same arguments, Theorem 2.3 and Corollary 2.4 can be stated for more general coefficients of the form $h\big(t,s,x,X_{t,s}^{x,\varphi},\mathbb{E}\big[X_{t,s}^{x,\varphi}\big],u_s\big)$ and $g\big(t,x,X_{t,T}^{x,\varphi},\mathbb{E}\big[X_{t,T}^{x,\varphi}\big]\big)$ in (2.2). This dependence on the initial time and state also makes the problem time-inconsistent, see e.g. Björk and Murgoci (2010).

3 Some applications

3.1 Application to a Linear-Quadratic TIC problem

We show that Corollary 2.4 can be applied to obtain the closed form solution of a time-inconsistent linear-quadratic problem which generalizes the usual mean-variance portfolio minimization problem of the form $\gamma\mathrm{Var}[\ \cdot\ ]-\mathbb{E}[\ \cdot\ ]$. For this, we consider the objective function

$$J(t,x,u)=\frac{G}{2}\mathbb{E}\big[\big(X_{t,T}^{x,\varphi}\big)^2\big]-\frac{h}{2}\big(\mathbb{E}\big[X_{t,T}^{x,\varphi}\big]\big)^2-(\mu_1x+\mu_2)\mathbb{E}\big[X_{t,T}^{x,\varphi}\big],\tag{3.1}$$

where $G\geq 0$, $\mu_1,\mu_2,h\in\mathbb{R}$, and $(X_{t,s}^{x,\varphi})_{s\in[t,t]}$ is the state process

$$\begin{cases} dX_{t,s}^{x,\varphi}=\big(A(s)X_{t,s}^{x,\varphi}+B(s)\varphi(s,X_{t,s}^{x,\varphi})+b(s)\big)ds+(C(s)\varphi(s,X_{t,s}^{x,\varphi})+\sigma(s))dW_s,\\ X_{t,t}^{x,\varphi}=x,\end{cases}\tag{3.2}$$

started at $x\in\mathbb{R}$ at time $t\in[0,T]$, where $A(\cdot)$, $B(\cdot)$, $C(\cdot)$, $b(\cdot)$, $\sigma(\cdot)$ are differentiable deterministic functions of time.

In the framework of pre-committed controls, the mean-variance case with constant risk aversion $\{b(\cdot)=\sigma(\cdot)=\mu_1=0,\ G=h\}$ was first solved in Zhou and Li (2000) by the embedding technique. It was later revisited in Meyer-Brandis et al. (2012) using the Malliavin calculus and in Pham and Wei (2018) using dynamic programming.

In the framework of equilibrium controls, this problem was first solved in Basak and Chabakauri (2010), resp. Björk et al. (2014), for constant risk aversion with $\{b(\cdot)=\sigma(\cdot)=$

$\mu_1 = 0,\ G = h\}$, resp. state-dependent risk aversion with $\{b(\cdot) = \sigma(\cdot) = \mu_2 = 0,\ G = h\}$. This framework has been later considered in Hu et al. (2012) and Djehiche and Huang (2016) using duality analysis.

In the following proposition we treat the more general linear-quadratic case by providing an explicit construction of equilibrium controls for (3.1)-(3.2) using the Malliavin calculus. For this, we use the formula for q given in (2.15b) which is not available in duality analysis where the equilibrium control is characterized by a pair (p, q) solution to a linear BSDE without explicit representation for q.

Proposition 3.1 *The equilibrium control $\widehat{\varphi}$ of (3.1) is given in linear feedback form as*

$$\widehat{\varphi}(t, y) = \alpha(t)y + \beta(t), \qquad (t, y) \in [0, T] \times \mathbb{R}, \tag{3.3}$$

where $\alpha(t)$ and $\beta(t)$ are the deterministic functions given by

$$\alpha(t) = -K_3(t) + \frac{e^{-\int_t^T K_2(u)K_3(u)du} K_1(t)}{1 + \int_t^T e^{-\int_s^T K_2(u)K_3(u)du} K_1(s)K_2(s)ds}, \tag{3.4}$$

$$\beta(t) = \left(\frac{K_4(T)}{K_6(T)} - \int_t^T \frac{K_4'(s)}{K_6(s)} e^{\int_s^T \frac{K_5(u) - K_6'(u)}{K_6(u)} du} ds \right) e^{-\int_t^T \frac{K_5(u) - K_6'(u)}{K_6(u)} du}, \quad t \in [0, T], \tag{3.5}$$

where

$$\begin{aligned}
K_1(t) &= \frac{\mu_1 B(t)}{G C^2(t)} \exp\left(-\int_t^T A(s)ds \right), \quad K_2(t) = B(t), \quad K_3(t) = \frac{(G-h)B(t)}{GC^2(t)}, \\
K_4(t) &= \mu_2 - G\sigma(t)\frac{C(t)}{B(t)} \exp\left(\int_t^T (A(s) + \alpha(s)B(s))ds \right) \\
&\quad - (G-h) \int_t^T b(s) \exp\left(\int_s^T (A(u) + \alpha(u)B(u))du \right) ds, \\
K_5(t) &= (G-h)B(t) \exp\left(\int_t^T (A(s) + \alpha(s)B(s))ds \right), \\
K_6(t) &= G\frac{C^2(t)}{B(t)} \exp\left(\int_t^T (A(s) + \alpha(s)B(s))ds \right).
\end{aligned}$$

Proof. By Corollary 2.4, $\widehat{\varphi}$ is an equilibrium control iff for $u \in U$, $x \in \mathbb{R}$, *a.e.* $t \in [0, T]$ we have

$$0 \le \mathbb{E}\Big[\big(p_{t,t}^{x,\widehat{\varphi}} B(t) + q_{t,t}^{x,\widehat{\varphi}} C(t) \big)(u - \widehat{\varphi}(t, x)) + \frac{1}{2} P_{t,t}^{x,\widehat{\varphi}} C^2(t)(u - \widehat{\varphi}(t, x))^2 \Big], \tag{3.6}$$

where for $s \in [t,T]$,

$$p^{x,\widehat{\varphi}}_{t,s} = \exp\left(\int_s^T A(u)du\right)\big(GX^{x,\widehat{\varphi}}_{t,T} - (\mu_1 x + \mu_2) - h\mathbb{E}\big[X^{x,\widehat{\varphi}}_{t,T}\big]\big), \tag{3.7}$$

and

$$q^{x,\widehat{\varphi}}_{t,s} = G\exp\left(\int_s^T A(u)du\right)D_s X^{x,\widehat{\varphi}}_{t,T}, \qquad P^{x,\widehat{\varphi}}_{t,t} = G\exp\left(\int_t^T A(s)ds\right), \tag{3.8}$$

see Lemma 2.2, where $D_t X^{x,\widehat{\varphi}}_{t,T}$ is defined from $q^{x,\widehat{\varphi}}_{t,t}$. Since $P^{x,\widehat{\varphi}}_{t,t}C^2(t)(u-\widehat{\varphi}(t,x))^2$ is positive and $(u-\widehat{\varphi}(t,x))$ is deterministic, $\mathbb{E}[p^{x,\widehat{\varphi}}_{t,t}B(t)+q^{x,\widehat{\varphi}}_{t,t}C(t)] = 0$ is a sufficient condition for (3.6) to hold, and using (3.7)-(3.8) it rewrites as

$$(G-h)B(t)\mathbb{E}\big[X^{x,\widehat{\varphi}}_{t,T}\big] - B(t)(\mu_1 x + \mu_2) + GC(t)\mathbb{E}\big[D_t X^{x,\widehat{\varphi}}_{t,T}\big] = 0. \tag{3.9}$$

Assuming the linear feedback form (3.3) for the equilibrium control $\widehat{\varphi}$, (3.2) becomes a linear SDE, and therefore we have

$$\begin{aligned}\mathbb{E}\big[X^{x,\varphi}_{t,T}\big] &= x\exp\left(\int_t^T (A(u)+\alpha(u)B(u))du\right) \\ &\quad + \int_t^T \exp\left(\int_s^T (A(u)+\alpha(u)B(u))du\right)(b(s)+\beta(s)B(s))ds.\end{aligned} \tag{3.10}$$

Similarly, from (3.2) and the feedback form (3.3) we have

$$\begin{aligned}D_s X^{x,\widehat{\varphi}}_{t,T} &= \Big(C(s)\big(\alpha(s)X^{x,\widehat{\varphi}}_{t,s} + \beta(s)\big) + \sigma(s)\Big) + \int_s^T (A(u)+\alpha(u)B(u))D_s X^{x,\widehat{\varphi}}_{t,u}du \\ &\quad + \int_s^T \alpha(u)C(u)D_s X^{x,\widehat{\varphi}}_{t,u}dW_u, \quad (t,s,x)\in[0,T]\times(t,T]\times\mathbb{R},\end{aligned}$$

which can be solved as

$$\begin{aligned}D_s X^{x,\widehat{\varphi}}_{t,T} &= \Big(C(s)\big(\alpha(s)X^{x,\widehat{\varphi}}_{t,s} + \beta(s)\big) + \sigma(s)\Big) \\ &\quad \times\exp\left(\int_s^T (A(u)+\alpha(u)B(u)) - \frac{1}{2}(\alpha(u)C(u))^2du + \int_s^T \alpha(u)C(u)dW_u\right),\end{aligned}$$

with

$$\mathbb{E}\big[D_s X^{x,\widehat{\varphi}}_{t,T}\big] = \Big(C(s)(\alpha(s)X^{x,\widehat{\varphi}}_{t,s} + \beta(s)) + \sigma(s)\Big)\exp\left(\int_s^T (A(u)+\alpha(u)B(u))du\right).$$

Due to the continuity in $s > t$ of all terms appearing in $D_s X^{x,\widehat{\varphi}}_{t,T}$, we have

$$\mathbb{E}\big[D_t X^{x,\widehat{\varphi}}_{t,T}\big] = (C(t)(\alpha(t)x+\beta(t))+\sigma(t))\exp\left(\int_t^T (A(u)+\alpha(u)B(u))du\right), \quad t\in[0,T]. \tag{3.11}$$

Putting (3.10) and (3.11) into (3.9) and by identification of the coefficients of x and the constant, we check that the functions $\alpha(t)$ and $\beta(t)$ should solve the integral equations

$$K_1(t) = \exp\left(\int_t^T \alpha(s)K_2(s)ds\right)(\alpha(t)+K_3(t)), \quad K_4(t) = \int_t^T K_5(s)\beta(s)ds + K_6(t)\beta(t). \tag{3.12}$$

From the first equation in (3.12), we have

$$\frac{K_1'(t)}{K_1(t)} = -\alpha(t)K_2(t) + \frac{\alpha'(t)+K_3'(t)}{\alpha(t)+K_3(t)},$$

hence

$$\frac{K_2(t)K_3(t) - K_1'(t)/K_1(t)}{\alpha(t)+K_3(t)} = K_2(t) - \frac{\alpha'(t)+K_3'(t)}{(\alpha(t)+K_3(t))^2}. \tag{3.13}$$

Therefore, letting $\Gamma(t) := 1/(\alpha(t)+K_3(t))$ and

$$\Theta(t) := \exp\left(\int_t^T K_2(s)K_3(s) - \frac{K_1'(s)}{K_1(s)}ds\right) = \frac{K_1(t)}{K_1(T)}\exp\left(\int_t^T K_2(s)K_3(s)ds\right),$$

(3.13) rewrites as

$$-\Theta(t)K_2(t) = \frac{d}{dt}(\Theta(t)\Gamma(t)), \quad \text{hence} \quad -\int_t^T \Theta(s)K_2(s)ds = \Gamma(T) - \Theta(t)\Gamma(t),$$

which yields

$$\Gamma(t) = \frac{K_1(T)}{K_1(t)}\Gamma(T)\exp\left(-\int_t^T K_2(s)K_3(s)ds\right) + \int_t^T \frac{K_1(s)K_2(s)}{K_1(t)}e^{\int_t^s K_2(u)K_3(u)du}ds,$$

and (3.4) after noting that $\alpha(T) = K_1(T) - K_3(T)$ in (3.12). Finally, we rewrite the second equation in (3.12) as

$$\frac{K_4'(t)}{K_6(t)} = -\beta(t)\frac{K_5(t)-K_6'(t)}{K_6(t)} + \beta'(t),$$

i.e.

$$\frac{K_4'(t)}{K_6(t)}\exp\left(\int_t^T \frac{K_5(u)-K_6'(u)}{K_6(u)}du\right) = \frac{d}{dt}\left(\beta(t)\exp\left(\int_t^T \frac{K_5(u)-K_6'(u)}{K_6(u)}du\right)\right),$$

or

$$\int_t^T \frac{K_4'(s)}{K_6(s)}\exp\left(\int_s^T \frac{K_5(u)-K_6'(u)}{K_6(u)}du\right)ds = \beta(T) - \beta(t)\exp\left(\int_t^T \frac{K_5(u)-K_6'(u)}{K_6(u)}du\right),$$

which yields (3.5) by noting that $\beta(T) = K_4(T)/K_6(T)$ in (3.12). $\square$

Remark 3.2 *In case $b(s) = \sigma(s) = 0$, $s \in [0,T]$, and $G = h = 1$, (3.3) yields the equilibrium control*

$$\widehat{\varphi}(t,y) = \frac{\theta(t)}{M(t)C(t)}(\mu_2 + \mu_1 y)\exp\left(\int_t^T A(s)ds\right),$$

where

$$\theta(t) := \frac{B(t)}{C(t)}, \quad M(t) := \exp\left(2\int_t^T A(s)ds\right)\left(1 + \mu_1 \int_t^T \exp\left(-\int_s^T A(u)du\right)\theta^2(s)ds\right),$$

which recovers the result of § 4.1 in Hu et al. (2012). Letting further $\mu_1 = 0$, resp. $\mu_2 = 0$, also recovers the result in § 4.1, resp. § 4.2.1, of Djehiche and Huang (2016).

3.2 Application to the generalized Merton problem

In this section we show that Corollary 2.4 applies beyond the framework of linear-quadratic problems by considering the nonlinear generalized Merton problem of minimizing the functional

$$J(t,x,\varphi) = -\mathbb{E}\left[\nu(t,T)(X_{t,T}^{x,\varphi})^\beta + \int_t^T \nu(t,s)\left(X_{t,s}^{x,\varphi}\varphi_c(s,X_{t,s}^{x,\varphi})\right)^\beta ds\right], \tag{3.14}$$

with the SDE

$$\begin{cases} dX_{t,s}^{x,\varphi} = X_{t,s}^{x,\varphi}\left((r + (\mu - r)\varphi_\pi(s,X_{t,s}^{x,\varphi}) - \varphi_c(s,X_{t,s}^{x,\varphi}))ds + \sigma\varphi_\pi(s,X_{t,s}^{x,\varphi})dW_s\right), \\ X_{t,t}^{x,\varphi} = x > 0, \end{cases} \tag{3.15}$$

where $\beta \in (0,1)$, r is the risk-free rate, μ is the rate of return of the risky asset, $\sigma > 0$ is the volatility of the risky asset, $\nu(t,s) > 0$ is a continuous discount function, φ_π is the proportion of portfolio invested in the risky asset, and $\varphi_c > 0$ is the proportion of portfolio consumed at time s.

The classical Merton problem with $\nu(t,s) = \mathrm{e}^{-\delta(s-t)}$ can be viewed as a time-consistent problem by minimizing

$$\hat{J}(t,x,\varphi) := \nu(0,t)J(t,x,\varphi) = -\mathbb{E}\left[\mathrm{e}^{-\delta T}\left(X_{t,T}^{x,\varphi}\right)^\beta + \int_t^T \mathrm{e}^{-\delta s}\left(X_{t,s}^{x,\varphi}\varphi_c\left(s,X_{t,s}^{x,\varphi}\right)\right)^\beta ds\right].$$

However, this approach fails in the general case where $\nu(t,s) = \nu(t,r)\nu(r,s)$ may not necessarily be true for all $0 \le t \le r \le s \le T$.

In the following proposition we apply Corollary 2.4 to recover the equilibrium controls in Ekeland and Pirvu (2008) and Yong (2012), see (6.20)-(6.21) therein. As in Ekeland and Pirvu (2008), the derivatives $\partial_x b$, $\partial_x \sigma$, $\partial_{xx} g$, and $\partial_{xx} h$ in (3.14)-(3.15) are not uniformly bounded.

Proposition 3.3 *The equilibrium controls of* (3.14) *are given by*

$$\widehat{\varphi}_\pi(t,y) = -\frac{\mu - r}{\sigma^2(\beta-1)} \tag{3.16}$$

and $\widehat{\varphi}_c(t,y) = \widehat{c}(t)$, $(t,y) \in [0,T] \times (0,\infty)$, *where* $\widehat{c}(t)$ *is the solution of the integral equation*

$$\begin{aligned}&\nu(t,t)\left(\widehat{c}(t)\right)^{\beta-1} \\ &= \nu(t,T)\exp\left(\lambda(T-t) - \beta\int_t^T \widehat{c}(u)du\right) + \int_t^T \nu(t,s)\exp\left(\lambda(s-t) - \beta\int_t^s \widehat{c}(u)du\right)ds,\end{aligned} \tag{3.17}$$

where $\lambda := \beta\left(2r\sigma^2(1-\beta) + (\mu-r)^2\right)/\left(2\sigma^2(1-\beta)\right)$.

Proof. Assuming that the equilibrium controls have the time-dependent form $\widehat{\varphi}_c(t,y) = \widehat{c}(t)$, $\widehat{\varphi}_\pi(t,y) = \widehat{\pi}(t)$, the solution $\left(X^{x,\varphi}_{t,s}\right)_{s\geq t}$ of (3.15) becomes a (positive) geometric Brownian motion, so that

$$X^{x,\varphi}_{t,s} = xG^{t,x}_{t,s} \quad \text{and} \quad D_r X^{x,\varphi}_{t,s} = X^{x,\varphi}_{t,r}\sigma\widehat{\pi}(r)G^{t,x}_{r,s} = X^{x,\varphi}_{t,s}\sigma\widehat{\pi}(r), \tag{3.18}$$

where

$$G^{t,x}_{t,s} := \exp\left(\int_t^s \left(r + (\mu-r)\widehat{\pi}(u) - \widehat{c}(u) - \frac{1}{2}(\sigma\widehat{\pi}(u))^2\right)du + \int_t^s \sigma\widehat{\pi}(u)dW_u\right),$$

$t < r \leq s \leq T$. By Corollary 2.4, $\widehat{\varphi}$ is an equilibrium control iff for $\pi \in \mathbb{R}$, $c > 0$, $x > 0$, and *a.e.* $t \in [0,T]$ we have

$$\begin{aligned}0 \leq \mathbb{E}\Big[&-\nu(t,t)x^\beta\left(c^\beta - \widehat{c}^\beta(t)\right) + xp^{x,\widehat{\varphi}}_{t,t}\left((\mu-r)(\pi - \widehat{\pi}(t)) - (c - \widehat{c}(t))\right) \\ &+ xq^{x,\widehat{\varphi}}_{t,t}\sigma\left(\pi - \widehat{\pi}(t)\right) + \frac{1}{2}P^{x,\widehat{\varphi}}_{t,t}(x\sigma)^2\left(\pi - \widehat{\pi}(t)\right)^2\Big],\end{aligned} \tag{3.19}$$

where, for $t \leq s \leq T$,

$$p^{x,\widehat{\varphi}}_{t,s} = -\beta G^{t,x}_{s,T}\nu(t,T)\left(X^{x,\widehat{\varphi}}_{t,T}\right)^{\beta-1} - \beta\int_s^T G^{t,x}_{s,u}\nu(t,u)\left(X^{x,\widehat{\varphi}}_{t,u}\right)^{\beta-1}du, \tag{3.20}$$

and, for $t < s \leq T$,

$$\begin{aligned}&q^{x,\widehat{\varphi}}_{t,s} = D_s p^{x,\widehat{\varphi}}_{t,s} = (\beta-1)\sigma\widehat{\pi}(s)p^{x,\widehat{\varphi}}_{t,s}, \qquad q^{x,\widehat{\varphi}}_{t,t} = \lim_{s\to t} q^{x,\widehat{\varphi}}_{t,s} = (\beta-1)\sigma\widehat{\pi}(t)p^{x,\widehat{\varphi}}_{t,t}, \\ &P^{x,\widehat{\varphi}}_{t,t} = -\beta(\beta-1)(G^{t,x}_{t,T})^2\nu(t,T)(X^{x,\widehat{\varphi}}_{t,T})^{\beta-2} - \beta(\beta-1)\int_t^T (G^{t,x}_{t,s})^2\nu(t,s)(X^{x,\widehat{\varphi}}_{t,s})^{\beta-2}ds,\end{aligned} \tag{3.21}$$

see Lemma 2.2. Next, we note that the conditions

$$\mathbb{E}\big[p^{x,\widehat{\varphi}}_{t,t}\big](\mu-r)+\mathbb{E}\big[q^{x,\widehat{\varphi}}_{t,t}\big]\sigma=0, \qquad -\nu(t,t)x^\beta\beta\widehat{c}^{\beta-1}(t)-\mathbb{E}\big[p^{x,\widehat{\varphi}}_{t,t}\big]x=0, \tag{3.22}$$

are sufficient for (3.19) to hold since $P^{x,\widehat{\varphi}}_{t,t}$ and $-p^{x,\widehat{\varphi}}_{t,t}$ are positive, $(\pi-\widehat{\pi}(t))$ and $(c-\widehat{c}(t))$ are deterministic, and the function $c\mapsto -\nu(t,t)x^\beta c^\beta-\mathbb{E}[p^{x,\widehat{\varphi}}_{t,t}]xc$ admits a unique minimizer $c^*>0$. Comparing (3.21) and the first equality in (3.22) yields (3.16). Finally, from (3.16), (3.18), and (3.20) we obtain

$$\begin{aligned}
&\mathbb{E}\big[p^{x,\widehat{\varphi}}_{t,t}\big]=\mathbb{E}\left[-\beta\left(G^{t,x}_{t,T}\right)^\beta\nu(t,T)x^{\beta-1}-\beta\int_t^T\left(G^{t,x}_{t,s}\right)^\beta\nu(t,s)x^{\beta-1}ds\right]\\
&=-\beta x^{\beta-1}\Bigg(\exp\Bigg(\beta\int_t^T\Big(r+(\mu-r)\widehat{\pi}(u)-\widehat{c}(u)+(\beta-1)\frac{1}{2}(\sigma\widehat{\pi}(u))^2\Big)du\Bigg)\nu(t,T)\\
&\quad+\int_t^T\exp\Bigg(\beta\int_t^s\Big(r+(\mu-r)\widehat{\pi}(u)-\widehat{c}(u)+(\beta-1)\frac{1}{2}(\sigma\widehat{\pi}(u))^2\Big)du\Bigg)\nu(t,s)\Bigg)\\
&=-\beta x^{\beta-1}\Bigg(\exp\Bigg(\lambda(T-t)-\beta\int_t^T\widehat{c}(u)du\Bigg)\nu(t,T)+\int_t^T\exp\Bigg(\lambda(s-t)-\beta\int_t^s\widehat{c}(u)du\Bigg)\nu(t,s)ds\Bigg),
\end{aligned} \tag{3.23}$$

and we conclude to (3.17) by putting (3.23) into the second term in (3.22). □

4 Solution estimates

Recall that we have used the notation $\widehat{u}^{t,x}_s=\widehat{\varphi}(s,X^{x,\widehat{u}^{t,x}}_{t,s})$ and $X^{x,\widehat{u}^{t,x}}_{t,s}=X^{x,\widehat{\varphi}}_{t,s}$, see (2.3) and (2.1). The estimates presented in Lemmas 4.1-4.3 have been used in the proof of Theorem 2.3.

Lemma 4.1 *Let Assumption 1 hold, and assume that $\widehat{\varphi}$ is differentiable with bounded derivatives and both $\widehat{\varphi}$ and $\partial_x\widehat{\varphi}$ are Lipschitz continuous in (t,x). Then for any $p\geq 1$, we have*

$$\begin{aligned}
\sup_{s\in[t,T]}\mathbb{E}\Big[\Big|X^{x,u\otimes_{t,\varepsilon}\widehat{u}^{t,x}}_{t,s}-X^{x,\widehat{u}^{t,x}}_{t,s}\Big|^{2p}\Big]&=O(\varepsilon^p)\\
\sup_{s\in[t,T]}\mathbb{E}\Big[\Big|y^{x,\varepsilon}_{t,s}\Big|^{2p}\Big]&=O(\varepsilon^p)\\
\sup_{s\in[t,T]}\mathbb{E}\Big[\Big|z^{x,\varepsilon}_{t,s}\Big|^{2p}\Big]&=O(\varepsilon^{2p})\\
\sup_{s\in[t,T]}\mathbb{E}\Big[\Big|X^{x,u\otimes_{t,\varepsilon}\widehat{u}^{t,x}}_{t,s}-X^{x,\widehat{u}^{t,x}}_{t,s}-y^{x,\varepsilon}_{t,s}\Big|^{2p}\Big]&=O(\varepsilon^{2p})\\
\sup_{s\in[t,T]}\mathbb{E}\Big[\Big|X^{x,u\otimes_{t,\varepsilon}\widehat{u}^{t,x}}_{t,s}-X^{x,\widehat{u}^{t,x}}_{t,s}-y^{x,\varepsilon}_{t,s}-z^{x,\varepsilon}_{t,s}\Big|^{2p}\Big]&=o(\varepsilon^{2p}),\qquad (t,x)\in[0,T]\times\mathbb{R}.
\end{aligned}$$

Proof. See Theorem 4.4 in Chapter 3 of Yong and Zhou (1999). □

Lemma 4.2 *Let Assumption 1 hold, and assume that $\widehat{\varphi}$ is differentiable with bounded derivatives and both $\widehat{\varphi}$ and $\partial_x\widehat{\varphi}$ are Lipschitz continuous in $(t,x)\in[0,T]\times\mathbb{R}$. Then the following inequalities hold for any $(t,x_1,x_2)\in[0,T]\times\mathbb{R}\times\mathbb{R}$, $s_1\geq s_2\geq t$, and $p\geq 1$:*

$$\mathbb{E}\bigg[\sup_{t\leq s\leq T}\big|X_{t,s}^{x_1,\widehat{\varphi}}\big|^p\bigg] \leq K_T(1+|x_1|^p), \tag{4.1}$$

$$\mathbb{E}\big[\big|X_{t,s_2}^{x_1,\widehat{\varphi}}-X_{t,s_1}^{x_1,\widehat{\varphi}}\big|^p\big] \leq K_T(1+|x_1|^p)|s_2-s_1|^{p/2}, \tag{4.2}$$

$$\mathbb{E}\bigg[\sup_{t\leq s\leq T}\big|X_{t,s}^{x_2,\widehat{\varphi}}-X_{t,s}^{x_1,\widehat{\varphi}}\big|^p\bigg] \leq K_T|x_2-x_1|^p, \tag{4.3}$$

where $K_T>0$ is a constant depending only on T.

Proof. Under Assumption 1 and using the fact that $\widehat{\varphi}$ is Lipschitz continuous in $(t,x)\in[0,T]\times\mathbb{R}$, the functions $b^{\widehat{\varphi}}(t,x)$ and $\sigma^{\widehat{\varphi}}(t,x)$ are Lipschitz continuous in (t,x). Then (4.1)-(4.3) follow from Theorem 1.6.3 in Yong and Zhou (1999). □

Lemma 4.3 *Let Assumption 1 hold, and assume that $\widehat{\varphi}$ is differentiable with bounded derivatives and both $\widehat{\varphi}$ and $\partial_x\widehat{\varphi}$ are Lipschitz continuous in (t,x). Then for any $(t,x,u)\in[0,T]\times\mathbb{R}\times U$ and $s_1\geq s_2\geq t$, we have the following estimates:*

$$\mathbb{E}\bigg[\sup_{t\leq s\leq T}\big|\partial_x g_{t,T}^{x,\widehat{\varphi}}\big|^p+\big|\partial_y g_{t,T}^{x,\widehat{\varphi}}\big|^p+\big|\partial_x h_{t,s}^{x,\widehat{\varphi}}\big|^p+\big|\partial_y h_{t,s}^{x,\widehat{\varphi}}\big|^p\bigg] \leq C, \tag{4.4}$$

$$\mathbb{E}\bigg[\sup_{t\leq s\leq T}\big|\delta b_{t,s}^{x,u,\widehat{\varphi}}\big|^p\bigg] \leq C(1+|u|^p), \tag{4.5}$$

$$\mathbb{E}\bigg[\sup_{t\leq s\leq T}\big|\delta \sigma_{t,s}^{x,u,\widehat{\varphi}}\big|^p\bigg] \leq C(1+|u|^p), \tag{4.6}$$

$$\mathbb{E}\bigg[\sup_{t\leq s\leq T}\big|\delta h_{t,s}^{x,u,\widehat{\varphi}}\big|^p\bigg] \leq C(1+|u|^p), \tag{4.7}$$

$$\mathbb{E}\big[\big|\delta b_{t,s_1}^{x,u,\widehat{\varphi}}-\delta b_{t,s_2}^{x,u,\widehat{\varphi}}\big] \leq C|s_1-s_2|^{p/2}, \tag{4.8}$$

$$\mathbb{E}\big[\big|\delta \sigma_{t,s_1}^{x,u,\widehat{\varphi}}-\delta \sigma_{t,s_2}^{x,u,\widehat{\varphi}}\big] \leq C|s_1-s_2|^{p/2}, \tag{4.9}$$

$$\mathbb{E}\big[\big|\delta h_{t,s_1}^{x,u,\widehat{\varphi}}-\delta h_{t,s_2}^{x,u,\widehat{\varphi}}\big] \leq C|s_1-s_2|^{p/2}. \tag{4.10}$$

Proof. Under Assumption 1 and using the fact that $\widehat{\varphi}$ is Lipschitz continuous in $(t,x)\in[0,T]\times\mathbb{R}$, the functions $b^{\widehat{\varphi}}(t,x)$ and $\sigma^{\widehat{\varphi}}(t,x)$ are Lipschitz continuous in (t,x). Then (4.1)-(4.3) follow from Theorem 1.6.3 in Yong and Zhou (1999). Regarding (4.4), by (4.1), Jensen's inequality and Assumption 1, we have

$$\mathbb{E}\bigg[\sup_{t\leq s\leq T}\big(\big|\partial_x g_{t,T}^{x,\widehat{\varphi}}\big|^p+\big|\partial_y g_{t,T}^{x,\widehat{\varphi}}\big|^p+\big|\partial_x h_{t,s}^{x,\widehat{\varphi}}\big|^p+\big|\partial_y h_{t,s}^{x,\widehat{\varphi}}\big|^p\big)\bigg]$$

$$\leq \quad CT^p + C\mathbb{E}\left[\sup_{t\leq s\leq T}\left(\left|X^{x,\widehat{\varphi}}_{t,s}\right|^p + \mathbb{E}\left[\left|X^{x,\widehat{\varphi}}_{t,s}\right|^p\right)\right]\right]$$

$$\leq \quad CT^p + C\mathbb{E}\bigg[\sup_{t\leq s\leq T}\left|X^{x,\widehat{\varphi}}_{t,s}\right|^p\bigg] \leq C.$$

We only show (4.5) and (4.8) because the arguments are similar for (4.6)-(4.7) and (4.9)-(4.10). Regarding (4.5), we have

$$\begin{aligned}
&\mathbb{E}\bigg[\sup_{t\leq s\leq T}\left|\delta b^{x,u,\widehat{\varphi}}_{t,s}\right|^p\bigg]\\
&\leq C\mathbb{E}\bigg[\sup_{t\leq s\leq T}\left|b^{\widehat{\varphi}}(s,X^{x,\widehat{\varphi}}_{t,s}) - b^{\widehat{\varphi}}(s,0)\right|^p + |b^{\widehat{\varphi}}(s,0)|^p + \left|b(s,X^{x,\widehat{\varphi}}_{t,s},u) - b(s,0,u)\right|^p + |b(s,0,u)|^p\bigg]\\
&\leq C\mathbb{E}\left[\sup_{t\leq s\leq T}\left(1 + \left|X^{x,\widehat{\varphi}}_{t,s}\right|^p + |u|^p\right)\right]\\
&\leq C(1+|x|^p+|u|^p)\\
&\leq C(1+|u|^p).
\end{aligned}$$

Regarding (4.8), we find

$$\begin{aligned}
\mathbb{E}\left[\left|\delta b^{x,u,\widehat{\varphi}}_{t,s_1} - \delta b^{x,u,\widehat{\varphi}}_{t,s_2}\right]\right. &\leq C|s_1-s_2|^p + C\mathbb{E}\left[\left|X^{x,\widehat{\varphi}}_{t,s_1} - X^{x,\widehat{\varphi}}_{t,s_2}\right|^p\right]\\
&\leq C(1+|x|^p)|s_1-s_2|^{p/2}\\
&\leq C|s_1-s_2|^{p/2}.
\end{aligned}$$

□

Lemma 4.4 *Let Assumption 1 hold, and assume that $\widehat{\varphi}$ is differentiable with bounded derivatives and both $\widehat{\varphi}$ and $\partial_x\widehat{\varphi}$ are Lipschitz continuous in (t,x). Then for any $(t,x)\in[0,T]\times\mathbb{R}$, $s_1\geq s_2\geq t$, $s\geq t_1\geq t_2\geq t$, and $p\geq 1$, we have the following estimates:*

$$\mathbb{E}\left[\sup_{t\leq s\leq T}\left|p^{x,\widehat{\varphi}}_{t,s}\right|^p\right] < \infty, \tag{4.11}$$

$$\mathbb{E}\left[\left|p^{x,\widehat{\varphi}}_{t,s_1} - p^{x,\widehat{\varphi}}_{t,s_2}\right|^p\right] \leq C|s_1-s_2|^{p/2}, \tag{4.12}$$

Proof. *Proof of* (4.11).

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t\leq s\leq T}\left|p^{x,\widehat{\varphi}}_{t,s}\right|^p\right]\\
&= \mathbb{E}\left[\sup_{t\leq s\leq T}\left|G^{t,x}_{s,T}\left(\partial_x g^{x,\widehat{\varphi}}_{t,T} + \partial_y g^{x,\widehat{\varphi}}_{t,T}\right) + \int_s^T G^{t,x}_{s,r}\left(\partial_x h^{x,\widehat{\varphi}}_{t,r} + \partial_y h^{x,\widehat{\varphi}}_{t,r}\right)dr\right|^p\right]\\
&\leq C\left(\mathbb{E}\left[\sup_{t\leq s\leq r\leq T}\left|G^{t,x}_{s,r}\right|^{2p}\right]\mathbb{E}\left[\left|\partial_x g^{x,\widehat{\varphi}}_{t,T}\right|^{2p} + \left|\partial_y g^{x,\widehat{\varphi}}_{t,T}\right|^{2p} + \sup_{t\leq s\leq T}\left(\left|\partial_x h^{x,\widehat{\varphi}}_{t,s}\right|^{2p} + \left|\partial_y h^{x,\widehat{\varphi}}_{t,s}\right|^{2p}\right)\right]\right)^{1/2}
\end{aligned}$$

$\leq C.$

The first inequality is due to Hölder's inequality. The second inequality is due to (4.4) and Lemma 4.5 below.

Proof of (4.12).

$$\begin{aligned}
&\mathbb{E}\big[\big|p^{x,\widehat{\varphi}}_{t,s_1} - p^{x,\widehat{\varphi}}_{t,s_1}\big|^p\big] = \mathbb{E}\Big[\Big|(G^{t,x}_{s_1,T} - G^{t,x}_{s_2,T})(\partial_x g^{x,\widehat{\varphi}}_{t,T} + \partial_y g^{x,\widehat{\varphi}}_{t,T})\\
&+ \int_{s_1}^{T} (G^{t,x}_{s_1,r} - G^{t,x}_{s_2,r})(\partial_x h^{x,\widehat{\varphi}}_{t,r} + \partial_y h^{x,\widehat{\varphi}}_{t,r})dr + \int_{s_2}^{s_1} G^{t,x}_{s_2,r}(\partial_x h^{x,\widehat{\varphi}}_{t,r} + \partial_y h^{x,\widehat{\varphi}}_{t,r})dr\Big|^p\Big]\\
&\leq C\big(\mathbb{E}\big[\big|(G^{t,x}_{s_1,T} - G^{t,x}_{s_2,T})\big|^{2p}\big]\mathbb{E}\big[\big|\partial_x g^{x,\widehat{\varphi}}_{t,T} + \partial_y g^{x,\widehat{\varphi}}_{t,T}\big|^{2p}\big]\big)^{1/2}\\
&\quad + C\int_{s_1}^{T} \big(\mathbb{E}\big[\big|(G^{t,x}_{s_1,r} - G^{t,x}_{s_2,r})\big|^{2p}\big]\mathbb{E}\big[\big|\partial_x h^{x,\widehat{\varphi}}_{t,r} + \partial_y h^{x,\widehat{\varphi}}_{t,r}\big|^{2p}\big]\big)^{1/2}dr\\
&\quad + C|s_1 - s_2|^{p-1}\int_{s_2}^{s_1} \big(\mathbb{E}\big[\big|G^{t,x}_{s_2,r}\big|^{2p}\big]\mathbb{E}\big[\big|\partial_x h^{x,\widehat{\varphi}}_{t,r} + \partial_y h^{x,\widehat{\varphi}}_{t,r}\big|^{2p}\big]\big)^{1/2}dr\\
&\leq C|s_1 - s_2|^{p/2}.
\end{aligned}$$

The first inequality is due to Hölder's inequality, and the second inequality follows from Lemma 4.5 below and (4.4). □

The next lemma deals with the boundedness and the continuity of (2.9) and (2.16c), which have been used in the proof of Lemma 5.3.

Lemma 4.5 *Let Assumption 1 hold. Then for any $(t,x) \in [0,T]\times\mathbb{R}$, $s_1 \geq s_2 \geq t$, $v_1 \geq v_2 \geq t$, and $p \geq 1$, we have*

$$\mathbb{E}\left[\sup_{t\leq s\leq k\leq T} \big|G^{t,x}_{s,v}\big|^p + \big|\Gamma^{t,x}_{s,v}\big|^p\right] \leq C$$

and

$$\mathbb{E}\big[\big|G^{t,x}_{s_1,v_1} - G^{t,x}_{s_1,v_2}\big|^p + \big|\Gamma^{t,x}_{s_1,v_1} - \Gamma^{t,x}_{s_1,v_2}\big|^p\big] \leq C\big(|s_1 - s_2|^{p/2} + |v_1 - v_2|^{p/2}\big).$$

Proof. We show only for the process G since the arguments for the process Γ are similar. Fix $(t,x) \in [0,T]\times\mathbb{R}$ and $p \geq 1$, and set $\rho_s := G^{t,x}_{t,s}$. Then, ρ satisfies the linear SDE

$$\begin{cases} d\rho_s = \rho_s\partial_x b^{x,\widehat{\varphi}}_{t,s}ds + \rho_s\partial_x\sigma^{x,\widehat{\varphi}}_{t,s}dW_s, \quad t \leq s \leq T,\\ \rho_t = 1, \end{cases}$$

and we have

$$\mathbb{E}\left[\sup_{t\leq r\leq T} |\rho_r|^p\right] \leq \mathbb{E}\left[\sup_{t\leq r\leq T} \left|1 + \int_t^r \rho_s\partial_x b^{x,\widehat{\varphi}}_{t,s}ds + \int_t^r \rho_s\partial_x\sigma^{x,\widehat{\varphi}}_{t,s}dW_s\right|^p\right]$$

$$\leq C + C\mathbb{E}\left[\int_t^T |\rho_s \partial_x b_{t,s}^{x,\widehat{\varphi}}|^p ds\right] + C\mathbb{E}\left[\int_t^T |\rho_s \partial_x \sigma_{t,s}^{x,\widehat{\varphi}}|^p ds\right]$$
$$\leq C + C\int_t^T \mathbb{E}\left[\sup_{t\leq r\leq s} |\rho_r|^p\right] ds.$$

The second inequality is Hölder's inequality, Theorem 1.7.2 in Mao (2007), and the third inequality holds by Assumption 1. Hence, by Gronwall's inequality, we have

$$\mathbb{E}\left[\sup_{t\leq r\leq T} |\rho_r|^p\right] \leq C.$$

Next, we show that ρ is continuous. Then for any $r_1 \geq r_2$ we have

$$\begin{aligned}
&\mathbb{E}[|\rho_{r_1} - \rho_{r_2}|^p] = \mathbb{E}\left[\left|\int_{r_2}^{r_1} \rho_s \partial_x b_{t,s}^{x,\widehat{\varphi}} ds + \int_{r_2}^{r_1} \rho_s \partial_x \sigma_{t,s}^{x,\widehat{\varphi}} dW_s\right|^p\right] \\
&\leq \ C|r_1 - r_2|^{p-1}\mathbb{E}\left[\int_{r_2}^{r_1} |\rho_s \partial_x b_{t,s}^{x,\widehat{\varphi}}|^p ds\right] + C|r_1 - r_2|^{p/2-1}\mathbb{E}\left[\int_{r_2}^{r_1} |\rho_s \partial_x \sigma_{t,s}^{x,\widehat{\varphi}}|^p ds\right] \\
&\leq \ C|r_1 - r_2|^{p/2}.
\end{aligned}$$

The first inequality is Hölder's inequality and Theorem 1.7.1 in Mao (2007). The third inequality is by Assumption 1 and the boundedness of ρ. Next, we denote $\gamma_s = 1/\rho_s$, where γ satisfies the following linear SDE:

$$\begin{cases}
d\gamma_s = \gamma_s\big(-\partial_x b_{t,s}^{x,\widehat{\varphi}}(+\partial_x \sigma_{t,s}^{x,\widehat{\varphi}})^2\big)ds - \gamma_s \partial_x \sigma_{t,s}^{x,\widehat{\varphi}} dW_s, \quad t \leq s \leq T, \\
\gamma_t = 1.
\end{cases}$$

By a similar argument, we obtain

$$\mathbb{E}\left[\sup_{t\leq r\leq T} |\gamma_r|^p\right] \leq C \quad \text{and} \quad \mathbb{E}[|\gamma_{r_1} - \gamma_{r_2}|^p] \leq C|r_1 - r_2|^{p/2}.$$

Noting $G_{s,v}^{t,x} = \gamma_s \rho_k$ and $G_{s_1,v_1}^{t,x} - G_{s_1,v_2}^{t,x} = \rho_{v_1}(\gamma_{s_1} - \gamma_{s_2}) + \gamma_{s_2}(\rho_{v_1} - \rho_{v_2})$, we complete the proof by Hölder's inequality. □

5 Malliavin estimates

The goal of this section is to prove Lemma 5.3 below on the regularity of $(P_{t,s}^{x,\widehat{\varphi}})_{s\in[t,T]}$ and of the Malliavin derivative of $(p_{t,s}^{x,\widehat{\varphi}})_{s\in[t,T]}$, which have been used in the proof of Theorem 2.3. Assume that $\widehat{\varphi}$ is differentiable with bounded derivatives and both $\widehat{\varphi}$ and $\partial_x \widehat{\varphi}$ are Lipschitz continuous in (t,x). Then by Theorem 2.2.1 and Exercise 2.2.1 in Nualart (2006), $X^{x,\widehat{\varphi}}_{t,s}$ belongs to $\mathbb{D}^{1,2}$ for all $(t,r,x)\in[0,T]^2\times\mathbb{R}$, $s\ge t$, and a version of $D_rX^{x,\widehat{\varphi}}_{t,s}$ is given by

$$D_rX^{x,\widehat{\varphi}}_{t,s}=\mathbb{1}_{(t,s]}(r)\sigma^{\widehat{\varphi}}\big(r,X^{x,\widehat{\varphi}}_{t,r}\big)\exp\left(\int_r^s\bar{\sigma}_u dW_u+\int_r^s\Big(\bar{b}_u-\frac{\bar{\sigma}^2_u}{2}\Big)du\right),$$

which solves the linear SDE

$$D_rX^{x,\widehat{\varphi}}_{t,s}=\sigma^{\widehat{\varphi}}(r,X^{x,\widehat{\varphi}}_{t,r})+\int_r^s\bar{b}_uD_rX^{x,\widehat{\varphi}}_{t,u}du+\int_r^s\bar{\sigma}_uD_rX^{x,\widehat{\varphi}}_{t,u}dW_u,\quad 0\le t<r\le s,$$

where $\bar{b}$ and $\bar{\sigma}$ are uniformly bounded and adapted processes, with the inequalities

$$\sup_{t\le r\le T}\mathbb{E}\bigg[\sup_{r\le s\le T}\big|D_rX^{x,\widehat{\varphi}}_{t,s}\big|^p\bigg]\le C,\qquad\sup_{t\le r_1,r_2\le T}\mathbb{E}\bigg[\sup_{r_1\wedge r_2\le s\le T}\big|D_{r_1}D_{r_2}X^{x,\widehat{\varphi}}_{t,s}\big|^p\bigg]\le C,\tag{5.1}$$

$t\in[0,T]$, $p\ge1$, see Lemma 2.2.2 and Theorem 2.2.2 in Nualart (2006). The next lemma deals with the properties of the Malliavin derivatives of the forward SDE, which have been used in the proof of Lemma 5.3.

Lemma 5.1 *Let Assumption 1 hold, and assume that $\widehat{\varphi}$ is differentiable with bounded derivatives and both $\widehat{\varphi}$ and $\partial_x\widehat{\varphi}$ are Lipschitz continuous in (t,x). Then for any $s_1\ge s_2\ge r>t$, $s\ge r_1\ge r_2>t$, and $p\ge1$, the following inequalities hold:*

$$\mathbb{E}\big[\big|D_rX^{x,\widehat{\varphi}}_{t,s_1}-D_rX^{x,\widehat{\varphi}}_{t,s_2}\big|^p\big]\le C|s_1-s_2|^{p/2},\tag{5.2}$$

$$\mathbb{E}\big[\big|D_{r_1}X^{x,\widehat{\varphi}}_{t,s}-D_{r_2}X^{x,\widehat{\varphi}}_{t,s}\big|^p\big]\le C|r_1-r_2|^{p/2}.\tag{5.3}$$

Proof. *Proof of* (5.2).

$$\begin{aligned}
&\mathbb{E}\big[\big|D_rX^{x,\widehat{\varphi}}_{t,s_1}-D_rX^{x,\widehat{\varphi}}_{t,s_2}\big|^p\big]=\mathbb{E}\left[\left|\int_{s_2}^{s_1}\bar{b}_uD_rX^{x,\widehat{\varphi}}_{t,u}du+\int_{s_2}^{s_1}\bar{\sigma}_uD_rX^{x,\widehat{\varphi}}_{t,u}dW_u\right|^p\right]\\
&\le C|s_1-s_2|^{p-1}\mathbb{E}\left[\int_{s_2}^{s_1}\big|\bar{b}_uD_rX^{x,\widehat{\varphi}}_{t,u}\big|^pdu\right]+C|s_1-s_2|^{p/2-1}\mathbb{E}\left[\int_{s_2}^{s_1}\big|\bar{\sigma}_uD_rX^{x,\widehat{\varphi}}_{t,u}\big|^pdu\right]\\
&\le C|s_1-s_2|^p+C|s_1-s_2|^{p/2}\\
&\le C|s_1-s_2|^{p/2}.
\end{aligned}$$

The first inequality is by Hölder's inequality and Theorem 1.7.1 in Mao (2007). The second inequality follows from the boundedness of $\bar{b}$ and $\bar{\sigma}$ and (5.1).

Proof of (5.3). We have

$$\mathbb{E}\big[\big|D_{r_1}X^{x,\widehat{\varphi}}_{t,s}-D_{r_2}X^{x,\widehat{\varphi}}_{t,s}\big|^p\big]$$

$$= \mathbb{E}\left[\left|\sigma^{\widehat{\varphi}}(r, X^{x,\widehat{\varphi}}_{t,r_1}) - \sigma^{\widehat{\varphi}}(r, X^{x,\widehat{\varphi}}_{t,r_2}) + \int_{r_2}^{r_1} \bar{b}_u D_{r_2} X^{x,\widehat{\varphi}}_{t,u} du + \int_{r_2}^{r_1} \bar{\sigma}_u D_{r_2} X^{x,\widehat{\varphi}}_{t,u} dW_u \right.\right.$$
$$\left.\left. + \int_{r_1}^{s} \bar{b}_u \big(D_{r_1} X^{x,\widehat{\varphi}}_{t,u} - D_{r_2} X^{x,\widehat{\varphi}}_{t,u}\big) du + \int_{r_1}^{s} \bar{\sigma}_u \big(D_{r_1} X^{x,\widehat{\varphi}}_{t,u} - D_{r_2} X^{x,\widehat{\varphi}}_{t,u}\big) dW_u \right|^p\right]$$
$$\leq C(1+|x|^p)|r_1 - r_2|^{p/2} + C\mathbb{E}\left[\int_{r_1}^{s} \left|D_{r_1} X^{x,\widehat{\varphi}}_{t,u} - D_{r_2} X^{x,\widehat{\varphi}}_{t,u}\right|^p du\right]$$
$$\leq C|r_1 - r_2|^{p/2} + C\int_{r_1}^{s} \mathbb{E}\big[\big|D_{r_1} X^{x,\widehat{\varphi}}_{t,u} - D_{r_2} X^{x,\widehat{\varphi}}_{t,u}\big|^p\big] du.$$

The first inequality is by (4.2), Hölder's inequality, Theorem 1.7.1 in Mao (2007) and a similar argument as in the proof of (5.2). We conclude the proof of (5.3) by Gronwall's inequality. □

The next lemma focuses on the properties of the Malliavin derivatives of (2.9) and (2.16c).

Lemma 5.2 *Let Assumption 1 hold, and assume that $\widehat{\varphi}$ is differentiable with bounded derivatives and both $\widehat{\varphi}$ and $\partial_x \widehat{\varphi}$ are Lipschitz continuous in (t,x). Then for any $(t,x) \in [0,T] \times \mathbb{R}$, $v \geq s \geq r \geq t$, we have*

$$D_r G^{t,x}_{s,v} = G^{t,x}_{s,v}\left(\int_s^v A_u D_r X^{x,\widehat{\varphi}}_{t,u} du + \int_s^v B_u D_r X^{x,\widehat{\varphi}}_{t,u} dW_u\right), \tag{5.4}$$
$$D_r \Gamma^{t,x}_{s,v} = \Gamma^{t,x}_{s,v}\left(2\int_s^v A_u D_r X^{x,\widehat{\varphi}}_{t,u} du + 2\int_s^v B_u D_r X^{x,\widehat{\varphi}}_{t,u} dW_u\right), \tag{5.5}$$

where A, B are the uniformly bounded processes

$$A_u := \partial_{xx} b^{x,\widehat{\varphi}}_{t,u} + \partial_{xu} b^{x,\widehat{\varphi}}_{t,u} \partial_x \widehat{\varphi}^{x,\widehat{\varphi}}_{t,u} - \partial_x \sigma^{x,\widehat{\varphi}}_{t,u} \partial_{xx} \sigma^{x,\widehat{\varphi}}_{t,u} - \partial_x \sigma^{x,\widehat{\varphi}}_{t,u} \partial_{xu} \sigma^{x,\widehat{\varphi}}_{t,u} \partial_x \widehat{\varphi}^{x,\widehat{\varphi}}_{t,u},$$
$$B_u := \partial_{xx} \sigma^{x,\widehat{\varphi}}_{t,u} + \partial_{xu} \sigma^{x,\widehat{\varphi}}_{t,u} \partial_x \widehat{\varphi}^{x,\widehat{\varphi}}_{t,u}.$$

In addition, for any $v \geq s_1 \geq s_2 \geq r > t$, $v \geq s \geq r_1 \geq r_2 > t$, and $p \geq 1$, the following inequalities holds:

$$\sup_{t \leq r \leq T} \mathbb{E}\left[\sup_{r \leq s \leq k \leq T} \left|D_r G^{t,x}_{s,v}\right|^p + \left|D_r \Gamma^{t,x}_{s,v}\right|^p\right] \leq C, \tag{5.6}$$
$$\mathbb{E}\left[\left|D_r G^{t,x}_{s_1,v} - D_r G^{t,x}_{s_2,v}\right|^p + \left|D_r \Gamma^{t,x}_{s_1,v} - D_r \Gamma^{t,x}_{s_2,v}\right|^p\right] \leq C|s_1 - s_2|^{p/2}, \tag{5.7}$$
$$\mathbb{E}\left[\left|D_{r_1} G^{t,x}_{s,v} - D_{r_2} G^{t,x}_{s,v}\right|^p + \left|D_{r_1} \Gamma^{t,x}_{s,v} - D_{r_2} \Gamma^{t,x}_{s,v}\right|^p\right] \leq C|r_1 - r_2|^{p/2}. \tag{5.8}$$

Proof. The proofs of (5.4) and (5.5) follow from Lemmas 1.1. Regarding (5.6)-(5.8), we consider only the process G since the arguments for Γ are similar.

Proof of (5.6).

$$\sup_{t \leq r \leq T} \mathbb{E}\left[\sup_{r \leq s \leq k \leq T} \left|D_r G^{t,x}_{s,v}\right|^p\right]$$

$$
\begin{aligned}
&\leq \sup_{t\leq r\leq T}\mathbb{E}\left[\sup_{r\leq s\leq k\leq T}\left(\left|G^{t,x}_{s,v}\right|^p\left|\int_s^v A_u D_r X^{x,\widehat{\varphi}}_{t,u}du+\int_s^v B_u D_r X^{x,\widehat{\varphi}}_{t,u}dW_u\right|^p\right)\right]\\
&\leq C\left(\mathbb{E}\left[\sup_{t\leq s\leq k\leq T}\left|G^{t,x}_{s,v}\right|^{2p}\right]\sup_{t\leq r\leq T}\mathbb{E}\left[\int_t^T\left(\left|A_u D_r X^{x,\widehat{\varphi}}_{t,u}\right|^{2p}+\left|B_u D_r X^{x,\widehat{\varphi}}_{t,u}\right|^{2p}\right)du\right]\right)^{1/2}.
\end{aligned}
$$

The first inequality is due to (5.4). The second inequality is due to Hölder's inequality and Theorem 1.7.2 in Mao (2007). Since A and B are uniformly bounded, we conclude by (5.1) and Lemma 4.5.

Proof of (5.7).

$$
\begin{aligned}
&\mathbb{E}\left[\left|D_r G^{t,x}_{s_1,v}-D_r G^{t,x}_{s_2,v}\right|^p\right]\\
&\leq\Bigg(\mathbb{E}\left[\left|G^{t,x}_{s_1,v}\right|^{2p}\right]\mathbb{E}\Bigg[\Bigg|\int_{s_1}^v A_u D_r X^{x,\widehat{\varphi}}_{t,u}du-\int_{s_2}^v A_u D_r X^{x,\widehat{\varphi}}_{t,u}du\\
&\qquad+\int_{s_1}^v B_u D_r X^{x,\widehat{\varphi}}_{t,u}dW_u-\int_{s_2}^v B_u D_r X^{x,\widehat{\varphi}}_{t,u}dW_u\Bigg|^{2p}\Bigg]\Bigg)^{1/2}\\
&\qquad+\left(\mathbb{E}\left[\left|G^{t,x}_{s_1,v}-G^{t,x}_{s_2,v}\right|^{2p}\right]\mathbb{E}\left[\left|\int_{s_2}^v A_u D_r X^{x,\widehat{\varphi}}_{t,u}du+\int_{s_2}^v B_u D_r X^{x,\widehat{\varphi}}_{t,u}dW_u\right|^{2p}\right.\right)^{1/2}\\
&\leq C\left(|s_1-s_2|^{2p-1}\mathbb{E}\left[\int_{s_2}^{s_1}\left|A_u D_r X^{x,\widehat{\varphi}}_{t,u}\right|^{2p}du\right]+|s_1-s_2|^{p-1}\mathbb{E}\left[\int_{s_2}^{s_1}\left|B_u D_r X^{x,\widehat{\varphi}}_{t,u}\right|^{2p}du\right]\right)^{1/2}\\
&\qquad+C\left(|s_1-s_2|^p\left(\mathbb{E}\left[\int_{s_2}^v\left|A_u D_r X^{x,\widehat{\varphi}}_{t,u}\right|^{2p}du\right]+\mathbb{E}\left[\int_{s_2}^v\left|B_u D_r X^{x,\widehat{\varphi}}_{t,u}\right|^{2p}du\right]\right)\right)^{1/2}\\
&\leq C|s_1-s_2|^p+C|s_1-s_2|^{p/2}\\
&\leq C|s_1-s_2|^{p/2}.
\end{aligned}
$$

The first inequality is due to (5.4), the triangle inequality, and Hölder's inequality. The second inequality is due to Lemma 4.5, Hölder's inequality, and Theorem 1.7.1 in Mao (2007). The third inequality follows from the boundedness of A and B and (5.1).

Proof of (5.8).

$$
\begin{aligned}
&\mathbb{E}\left[\left|D_{r_1}G^{t,x}_{s,v}-D_{r_2}G^{t,x}_{s,v}\right|^p\right]\\
&\leq\left(\mathbb{E}\left[\left|G^{t,x}_{s,v}\right|^{2p}\right]\mathbb{E}\left[\left|\int_s^v A_u\left(D_{r_1}X^{x,\widehat{\varphi}}_{t,u}-D_{r_2}X^{x,\widehat{\varphi}}_{t,u}\right)du\ +\int_s^v B_u\left(D_{r_1}X^{x,\widehat{\varphi}}_{t,u}-D_{r_2}X^{x,\widehat{\varphi}}_{t,u}\right)dW_u\right|^{2p}\right]\right)^{1/2}\\
&\leq C\left(\int_s^v\mathbb{E}\left[\left|D_{r_1}X^{x,\widehat{\varphi}}_{t,u}-D_{r_2}X^{x,\widehat{\varphi}}_{t,u}\right|^{2p}\right]du\right)^{1/2}\\
&\leq C|r_1-r_2|^{p/2}.
\end{aligned}
$$

The first inequality is due to (5.4), the triangle inequality and Hölder's inequality. The second inequality is due to the boundedness of A and B, Lemma 4.5, Hölder's inequality, and Theorem 1.7.1 in Mao (2007). The third inequality is due to (5.2). ☐

The next lemma focuses on the bounds and the continuity of $(p_{t,s}^{x,\widehat{\varphi}})_{s\in[t,T]}$, $(P_{t,s}^{x,\widehat{\varphi}})_{s\in[t,T]}$, and their Malliavin derivatives, which have been used in the proof of Theorem 2.3.

Lemma 5.3 *Let Assumption 1 hold, and assume that $\widehat{\varphi}$ is differentiable with bounded derivatives and both $\widehat{\varphi}$ and $\partial_x\widehat{\varphi}$ are Lipschitz continuous in (t,x). Then for any $(t,x)\in[0,T]\times\mathbb{R}$, $s_1\geq s_2\geq r>t$, $s\geq t_1\geq t_2>t$, and $p\geq 1$, we have the following estimates:*

$$\sup_{t\leq r\leq T}\mathbb{E}\left[\sup_{r\leq s\leq T}\left|D_r p_{t,s}^{x,\widehat{\varphi}}\right|^p\right]<\infty, \tag{5.9}$$

$$\mathbb{E}\left[\left|D_r p_{t,s_1}^{x,\widehat{\varphi}}-D_r p_{t,s_2}^{x,\widehat{\varphi}}\right|^p\right]\leq C|s_1-s_2|^{p/2}, \tag{5.10}$$

$$\mathbb{E}\left[\left|D_{t_1} p_{t,s}^{x,\widehat{\varphi}}-D_{t_2} p_{t,s}^{x,\widehat{\varphi}}\right|^p\right]\leq C|t_1-t_2|^{p/2}, \tag{5.11}$$

$$\sup_{t\leq s\leq T}\mathbb{E}\left[\left|f_{t,s}^{x,\widehat{\varphi}}\right|^p\right]+\sup_{t\leq r\leq T}\sup_{r\leq s\leq T}\mathbb{E}\left[\left|D_r f_{t,s}^{x,\widehat{\varphi}}\right|^p\right]<\infty, \tag{5.12}$$

$$\mathbb{E}\left[\left|P_{t,s_1}^{x,\widehat{\varphi}}-P_{t,s_2}^{x,\widehat{\varphi}}\right|^p\right]\leq C|s_1-s_2|^{p/2}, \tag{5.13}$$

$$\sup_{t\leq s\leq T}\mathbb{E}\left[\left|P_{t,s}^{x,\widehat{\varphi}}\right|^p\right]+\sup_{t\leq r\leq T}\sup_{r\leq s\leq T}\mathbb{E}\left[\left|D_r P_{t,s}^{x,\widehat{\varphi}}\right|^p\right]<\infty. \tag{5.14}$$

Proof. *Proof of* (5.9). By Lemma 1.1, we have

$$\begin{aligned} D_r p_{t,s}^{x,\widehat{\varphi}} &= \left(\partial_x g_{t,T}^{x,\widehat{\varphi}}+\partial_y g_{t,T}^{x,\widehat{\varphi}}\right)D_r G_{s,T}^{t,x}+G_{s,T}^{t,x}D_r X_{t,T}^{x,\widehat{\varphi}}\partial_{xx}g_{t,T}^{x,\widehat{\varphi}} \\ &\quad +\int_s^T\left(D_r G_{s,u}^{t,x}\left(\partial_x h_{t,u}^{x,\widehat{\varphi}}+\partial_y h_{t,u}^{x,\widehat{\varphi}}\right)+G_{s,u}^{t,x}D_r X_{t,u}^{x,\widehat{\varphi}}\left(\partial_{xx}h_{t,u}^{x,\widehat{\varphi}}+\partial_{xu}h_{t,u}^{x,\widehat{\varphi}}\partial_x\widehat{\varphi}_{t,u}^{x,\widehat{\varphi}}\right)\right)du. \end{aligned}$$

By Hölder's inequality, Assumption 1, and the boundedness of $\partial_x\widehat{\varphi}$ we have

$$\begin{aligned} \sup_{t\leq r\leq T}\mathbb{E}\left[\sup_{r\leq s\leq T}\left|D_r p_{t,s}^{x,\widehat{\varphi}}\right|^p\right] &\leq C\left(\mathbb{E}\left[\left|\partial_x g_{t,T}^{x,\widehat{\varphi}}\right|^{2p}+\left|\partial_y g_{t,T}^{x,\widehat{\varphi}}\right|^{2p}\right]\sup_{t\leq r\leq T}\mathbb{E}\left[\sup_{r\leq s\leq T}\left|D_r G_{s,T}^{t,x}\right|^{2p}\right]\right)^{1/2} \\ &\quad +C\left(\mathbb{E}\left[\sup_{t\leq s\leq T}\left|G_{s,T}^{t,x}\right|^{2p}\right]\sup_{t\leq r\leq T}\mathbb{E}\left[\sup_{r\leq s\leq T}\left|D_r X_{t,T}^{x,\widehat{\varphi}}\right|^{2p}\right]\right)^{1/2} \\ &\quad +C\int_t^T\left(\left(\mathbb{E}\left[\left|\partial_x h_{t,u}^{x,\widehat{\varphi}}\right|^{2p}+\left|\partial_y h_{t,u}^{x,\widehat{\varphi}}\right|^{2p}\right]\sup_{t\leq r\leq u}\mathbb{E}\left[\sup_{r\leq s\leq u}\left|D_r G_{s,u}^{t,x}\right|^{p}\right]\right)^{1/2}\right. \\ &\quad \left.+\left(\mathbb{E}\left[\sup_{t\leq s\leq u}\left|G_{s,u}^{t,x}\right|^{2p}\right]\sup_{t\leq r\leq u}\mathbb{E}\left[\sup_{r\leq s\leq u}\left|D_r X_{t,u}^{x,\widehat{\varphi}}\right|^{2p}\right]\right)^{1/2}\right)du. \end{aligned}$$

We can then conclude by Lemma 4.5, (4.4), (5.1), and (5.6).

Proof of (5.10). We have

$$
\begin{aligned}
\mathbb{E}\big[\big|D_r p^{x,\widehat{\varphi}}_{t,s_1} - D_r p^{x,\widehat{\varphi}}_{t,s_1}\big|^p\big] \le C\,\Big(&\big(\mathbb{E}\big[\big|D_r G^{t,x}_{s_1,T} - D_r G^{t,x}_{s_2,T}\big|^{2p}\big]\mathbb{E}\big[\big|\partial_x g^{x,\widehat{\varphi}}_{t,T}\big|^{2p} + \big|\partial_y g^{x,\widehat{\varphi}}_{t,T}\big|^{2p}\big]\big)^{1/2}\\
&+\big(\mathbb{E}\big[\big|G^{t,x}_{s_1,T} - G^{t,x}_{s_2,T}\big|^{2p}\big]\mathbb{E}\big[\big|D_r X^{x,\widehat{\varphi}}_{t,T}\big|^{2p}\big]\big)^{1/2}\\
&+\int_{s_1}^{T}\Big(\big(\mathbb{E}\big[\big|D_r G^{t,x}_{s_1,u} - D_r G^{t,x}_{s_2,u}\big|^{2p}\big]\mathbb{E}\big[\big|\partial_x h^{x,\widehat{\varphi}}_{t,u}\big|^{2p} + \big|\partial_y h^{x,\widehat{\varphi}}_{t,u}\big|^{2p}\big]\big)^{1/2}\\
&+\big(\mathbb{E}\big[\big|G^{t,x}_{s_1,u} - G^{t,x}_{s_2,u}\big|^{2p}\big]\mathbb{E}\big[\big|D_r X^{x,\widehat{\varphi}}_{t,u}\big|^{2p}\big]\big)^{1/2}\Big)du\\
&+|s_1 - s_2|^{p-1}\int_{s_2}^{s_1}\Big(\big(\mathbb{E}\big[\big|D_r G^{t,x}_{s_2,u}\big|^{2p}\big]\mathbb{E}\big[\big|\partial_x h^{x,\widehat{\varphi}}_{t,u}\big|^{2p} + \big|\partial_y h^{x,\widehat{\varphi}}_{t,u}\big|^{2p}\big]\big)^{1/2}\\
&+\big(\mathbb{E}\big[\big|G^{t,x}_{s_2,u}\big|^{2p}\big]\mathbb{E}\big[\big|D_r X^{x,\widehat{\varphi}}_{t,u}\big|^{2p}\big]\big)^{1/2}\Big)du\Big)\\
\le\ & C|s_1 - s_2|^{p/2}.
\end{aligned}
$$

The first inequality is due to Assumption 1, and the boundedness of $\partial_x \widehat{\varphi}$, the triangle inequality, and Hölder's inequality. The second inequality is due to Lemma 4.5, (4.4), (5.7), (5.1), and (5.6).

Proof of (5.11).

$$
\begin{aligned}
\mathbb{E}\big[\big|D_{t_1} p^{x,\widehat{\varphi}}_{t,s} - D_{t_2} p^{x,\widehat{\varphi}}_{t,s}\big|^p\big] &\le C\big(\mathbb{E}\big[\big|D_{t_1} G^{t,x}_{s,T} - D_{t_2} G^{t,x}_{s,T}\big|^{2p}\big]\mathbb{E}\big[\big(\big|\partial_x g^{x,\widehat{\varphi}}_{t,T}\big|^{2p} + \big|\partial_y g^{x,\widehat{\varphi}}_{t,T}\big|^{2p}\big)\big]\big)^{1/2}\\
&+ C\big(\mathbb{E}\big[\big|G^{t,x}_{s,T}\big|^{2p}\mathbb{E}\big[\big|D_{t_1} X^{x,\widehat{\varphi}}_{t,T} - D_{t_2} X^{x,\widehat{\varphi}}_{t,T}\big|^{2p}\big]\big]\big)^{1/2}\\
&+ C\int_s^T\Big(\big(\mathbb{E}\big[\big|D_{t_1} G^{t,x}_{s,u} - D_{t_2} G^{t,x}_{s,u}\big|^{2p}\big]\mathbb{E}\big[\big(\big|\partial_x h^{x,\widehat{\varphi}}_{t,u}\big|^{2p} + \big|\partial_y h^{x,\widehat{\varphi}}_{t,u}\big|^{2p}\big)\big]\big)^{1/2}\\
&\quad + \big(\mathbb{E}\big[\big|G^{t,x}_{s,u}\big|^{2p}\big]\mathbb{E}\big[\big|D_{t_1} X^{x,\widehat{\varphi}}_{t,u} - D_{t_2} X^{x,\widehat{\varphi}}_{t,u}\big|^{2p}\big]\big)^{1/2}\Big)du\\
&\le C|t_1 - t_2|^{p/2}.
\end{aligned}
$$

The first inequality is due to the boundedness of C, D, E and F, the triangle inequality, and Hölder's inequality. The second inequality is due to Lemma 4.5, (4.4), (5.8), (5.2).

Proof of (5.12). By Hölder's inequality and Assumption 1, we have

$$
\begin{aligned}
\sup_{t\le s\le T}\mathbb{E}\big[\big|f^{x,\widehat{\varphi}}_{t,s}\big|^p\big] \le C\sup_{t\le s\le T}&\big(\mathbb{E}\big[\big|G^{t,x}_{s,T}\big|^{2p}\big]\mathbb{E}\big[\big|D_s X^{x,\widehat{\varphi}}_{t,T}\big|^{2p}\big]\big)^{1/2}\\
&+ C\int_t^T\sup_{t\le s\le u}\big(\mathbb{E}\big[\big|G^{t,x}_{s,u}\big|^{2p}\big]\mathbb{E}\big[\big|D_s X^{x,\widehat{\varphi}}_{t,u}\big|^{2p}\big]\big)^{1/2}du.
\end{aligned}
$$

We conclude the first term in (5.12) by Lemma 4.5 and (5.1). Next, by Lemma 1.1 we have

$$
D_r f^{x,\widehat{\varphi}}_{t,s} = G_T D_r G^{t,x}_{s,T} D_s X^{x,\widehat{\varphi}}_{t,T} + H_T G^{t,x}_{s,T} D_s X^{x,\widehat{\varphi}}_{t,T} D_r X^{x,\widehat{\varphi}}_{t,T} + G_T G^{t,x}_{s,T} D_r D_s X^{x,\widehat{\varphi}}_{t,T}
$$

$$+\int_s^T \big(I_u D_r G^{t,x}_{s,u} D_s X^{x,\widehat{\varphi}}_{t,u} + J_u G^{t,x}_{s,u} D_s X^{x,\widehat{\varphi}}_{t,u} D_r X^{x,\widehat{\varphi}}_{t,u} + I_u G^{t,x}_{s,u} D_r D_s X^{x,\widehat{\varphi}}_{t,u}\big) du,$$

where G, H, I, and J are the uniformly bounded processes

$$\begin{aligned}
G_T &:= \partial_{xx} g^{x,\widehat{\varphi}}_{t,T},\\
H_T &:= \partial_{xxx} g^{x,\widehat{\varphi}}_{t,T},\\
I_u &:= \partial_{xx} h^{x,\widehat{\varphi}}_{t,u} + \partial_{xu} h^{x,\widehat{\varphi}}_{t,u} \partial_x \widehat{\varphi}^{x,\widehat{\varphi}}_{t,u},\\
J_u &:= \partial_{xxx} h^{x,\widehat{\varphi}}_{t,u} + \partial_{xxu} h^{x,\widehat{\varphi}}_{t,u} \partial_x \widehat{\varphi}^{x,\widehat{\varphi}}_{t,u} + \partial_{xux} h^{x,\widehat{\varphi}}_{t,u} \partial_x \widehat{\varphi}(u) + \partial_{xuu} h^{x,\widehat{\varphi}}_{t,u} (\partial_x \widehat{\varphi}^{x,\widehat{\varphi}}_{t,u})^2 + \partial_{xu} h^{x,\widehat{\varphi}}_{t,u} \bar{\varphi}_u,
\end{aligned}$$

where $\bar{\varphi}$ is a uniformly bounded process depending on $\widehat{\varphi}$. By Hölder's inequality and the boundedness of G, H, I and J we have

$$\begin{aligned}
\sup_{t\le r\le T}\sup_{r\le s\le T} \mathbb{E}\big[\big|D_r f^{x,\widehat{\varphi}}_{t,s}\big|^p\big] \;\le\;& C \sup_{t\le r\le T}\sup_{r\le s\le T} \big(\mathbb{E}\big[\big|D_r G^{t,x}_{s,T}\big|^{2p}\big]\mathbb{E}\big[\big|D_s X^{x,\widehat{\varphi}}_{t,T}\big|^{2p}\big]\big)^{1/2}\\
&+C \sup_{t\le r\le T}\sup_{r\le s\le T} \big(\mathbb{E}\big[\big|G^{t,x}_{s,T}\big|^{3p}\big]\mathbb{E}\big[\big|D_s X^{x,\widehat{\varphi}}_{t,T}\big|^{3p}\big]\mathbb{E}\big[\big|D_r X^{x,\widehat{\varphi}}_{t,T}\big|^{3p}\big]\big)^{1/3}\\
&+C \sup_{t\le r\le T}\sup_{r\le s\le T} \big(\mathbb{E}\big[\big|G^{t,x}_{s,T}\big|^{2p}\big]\mathbb{E}\big[\big|D_r D_s X^{x,\widehat{\varphi}}_{t,T}\big|^{2p}\big]\big)^{1/2}\\
&+C \int_t^T \Big(\sup_{t\le r\le u}\sup_{r\le s\le u} \big(\mathbb{E}\big[\big|D_r G^{t,x}_{s,u}\big|^{2p}\big]\mathbb{E}\big[\big|D_s X^{x,\widehat{\varphi}}_{t,u}\big|^{2p}\big]\big)^{1/2}\\
&+C \sup_{t\le r\le u}\sup_{r\le s\le u} \big(\mathbb{E}\big[\big|G^{t,x}_{s,u}\big|^{3p}\big]\mathbb{E}\big[\big|D_s X^{x,\widehat{\varphi}}_{t,u}\big|^{3p}\big]\mathbb{E}\big[\big|D_r X^{x,\widehat{\varphi}}_{t,u}\big|^{3p}\big]\big)^{1/3}\\
&+C \sup_{t\le r\le u}\sup_{r\le s\le u} \big(\mathbb{E}\big[\big|G^{t,x}_{s,u}\big|^{2p}\big]\mathbb{E}\big[\big|D_r D_s X^{x,\widehat{\varphi}}_{t,u}\big|^{2p}\big]\big)^{1/2}\Big) du.
\end{aligned}$$

We conclude the second term in (5.12) by Lemma 4.5, (5.1), and (5.6).
Proof of (5.13). By Hölder's inequality and Assumption 1, we have

$$\begin{aligned}
\mathbb{E}\big[\big|P^{x,\widehat{\varphi}}_{t,s_1} - P^{x,\widehat{\varphi}}_{t,s_1}\big|^p\big] \le C\Bigg(&\mathbb{E}\big[\big|\Gamma^{t,x}_{s_1,T} - \Gamma^{t,x}_{s_2,T}\big|^p\big]\\
&+ \int_{s_1}^T \big(\mathbb{E}\big[\big|\Gamma^{t,x}_{s_1,u} - \Gamma^{t,x}_{s_2,u}\big|^{2p}\big]\big(1+\mathbb{E}\big[\big|p^{x,\widehat{\varphi}}_{t,u}\big|^{2p}\big] + \mathbb{E}\big[\big|f^{x,\widehat{\varphi}}_{t,u}\big|^{2p}\big]\big)\big)^{1/2} du\\
&+ |s_1 - s_2|^{p-1} \int_{s_2}^{s_1} \big(\mathbb{E}\big[\big|\Gamma^{t,x}_{s_2,u}\big|^{2p}\big]\big(1+\mathbb{E}\big[\big|p^{x,\widehat{\varphi}}_{t,u}\big|^{2p}\big] + \mathbb{E}\big[\big|f^{x,\widehat{\varphi}}_{t,u}\big|^{2p}\big]\big)\big)^{1/2} du\Bigg),
\end{aligned}$$

and we conclude by Lemma 4.5, (5.9) and (5.12).
Proof of (5.14). By Hölder's inequality and Assumption 1 we have

$$\sup_{t\le s\le T} \mathbb{E}\big[\big|P^{x,\widehat{\varphi}}_{t,s}\big|^p\big] \le C \sup_{t\le s\le T} \mathbb{E}\big[\big|\Gamma^{t,x}_{s,T}\big|^p\big] + C\int_t^T \sup_{t\le s\le T} \mathbb{E}\big[\big|\Gamma^{t,x}_{s,u}\big|^{2p}\big]\big(1+\mathbb{E}\big[\big|p^{x,\widehat{\varphi}}_{t,u}\big|^{2p}\big] + \mathbb{E}\big[\big|f^{x,\widehat{\varphi}}_{t,u}\big|^{2p}\big]\big)^{1/2} du,$$

and we apply Lemma 4.5, (5.9) and (5.12). Next, by Lemma 1.1 we have

$$
\begin{aligned}
&D_r P_{t,s}^{x,\widehat{\varphi}} = D_r\Gamma_{s,T}^{t,x}\partial_{xx}g_{t,T}^{x,\widehat{\varphi}} + \Gamma_{s,T}^{t,x}D_rX_{t,T}^{x,\widehat{\varphi}}\partial_{xxx}g_{t,T}^{x,\widehat{\varphi}}\\
&+\int_s^T \Big(D_r\Gamma_{s,u}^{t,x}\big(\partial_{xx}h_{t,u}^{x,\widehat{\varphi}} + (\partial_{xx}b_{t,u}^{x,\widehat{\varphi}} - \partial_{xx}\sigma_{t,u}^{x,\widehat{\varphi}}\partial_x\sigma_{t,u}^{x,\widehat{\varphi}})p_{t,u}^{x,\widehat{\varphi}} + f_{t,u}^{x,\widehat{\varphi}}\partial_{xx}\sigma_{t,u}^{x,\widehat{\varphi}}\big)\\
&+\Gamma_{s,u}^{t,x}\big(\partial_{xxx}h_{t,u}^{x,\widehat{\varphi}} + (\partial_{xxx}b_{t,u}^{x,\widehat{\varphi}} - \partial_{xxx}\sigma_{t,u}^{x,\widehat{\varphi}}\partial_x\sigma_{t,u}^{x,\widehat{\varphi}} - \partial_{xx}\sigma_{t,u}^{x,\widehat{\varphi}}\partial_{xx}\sigma_{t,u}^{x,\widehat{\varphi}})p_{t,u}^{x,\widehat{\varphi}} + f_{t,u}^{x,\widehat{\varphi}}\partial_{xxx}\sigma_{t,u}^{x,\widehat{\varphi}}\big)D_rX_{t,u}^{x,\widehat{\varphi}}\\
&+\Gamma_{s,u}^{t,x}\big(\partial_{xxu}h_{t,u}^{x,\widehat{\varphi}} + (\partial_{xxu}b_{t,u}^{x,\widehat{\varphi}} - \partial_{xxu}\sigma_{t,u}^{x,\widehat{\varphi}}\partial_x\sigma_{t,u}^{x,\widehat{\varphi}} - \partial_{xx}\sigma_{t,u}^{x,\widehat{\varphi}}\partial_{xu}\sigma_{t,u}^{x,\widehat{\varphi}})p_{t,u}^{x,\widehat{\varphi}} + f_{t,u}^{x,\widehat{\varphi}}\partial_{xxu}\sigma_{t,u}^{x,\widehat{\varphi}}\big)D_rX_{t,u}^{x,\widehat{\varphi}}\partial_x\widehat{\varphi}_{t,u}^{x,\widehat{\varphi}}\\
&+\Gamma_{s,u}^{t,x}\big((\partial_{xx}b_{t,u}^{x,\widehat{\varphi}} - \partial_{xx}\sigma_{t,u}^{x,\widehat{\varphi}}\partial_x\sigma_{t,u}^{x,\widehat{\varphi}})D_rp_{t,u}^{x,\widehat{\varphi}} + D_rf_{t,u}^{x,\widehat{\varphi}}\partial_{xx}\sigma_{t,u}^{x,\widehat{\varphi}}\big)\Big)du.
\end{aligned}
$$

By Hölder's inequality and Assumption 1, we have

$$
\begin{aligned}
&\sup_{t\le r\le T}\sup_{r\le s\le T}\mathbb{E}\big[\big|D_rP_{t,s}^{x,\widehat{\varphi}}\big|^p\big]\\
&\le C\sup_{t\le r\le T}\sup_{r\le s\le T}\mathbb{E}\big[\big|D_r\Gamma_{s,T}^{t,x}\big|^p\big] + C\sup_{t\le r\le T}\sup_{r\le s\le T}\big(\mathbb{E}\big[\big|\Gamma_{s,T}^{t,x}\big|^{2p}\big]\mathbb{E}\big[\big|D_rX_{t,T}^{x,\widehat{\varphi}}\big|^{2p}\big]\big)^{1/2}\\
&\quad + C\int_t^T\Big(\sup_{t\le r\le u}\sup_{r\le s\le u}\big(\mathbb{E}\big[\big|D_r\Gamma_{s,u}^{t,x}\big|^{2p}\big]\big(1+\mathbb{E}\big[\big|p_{t,u}^{x,\widehat{\varphi}}\big|^{2p} + \mathbb{E}\big[\big|f_{t,u}^{x,\widehat{\varphi}}w\big|\big|^{2p}\big]\big)\big)^{1/2}\\
&\qquad + \sup_{t\le r\le u}\sup_{r\le s\le u}\big(\mathbb{E}\big[\big|\Gamma_{s,u}^{t,x}\big|^{3p}\big]\mathbb{E}\big[\big|D_rX_{t,u}^{x,\widehat{\varphi}}\big|^{3p}\big]\big(1+\mathbb{E}\big[\big|p_{t,u}^{x,\widehat{\varphi}}\big|^{3p}\big] + \mathbb{E}\big[\big|f_{t,u}^{x,\widehat{\varphi}}\big|^{3p}\big]\big)\big)^{1/3}\\
&\qquad + \sup_{t\le r\le u}\sup_{r\le s\le u}\mathbb{E}\big(\big[\big|\Gamma_{s,u}^{t,x}\big|^{2p}\big]\big(\mathbb{E}\big[\big|D_rp_{t,u}^{x,\widehat{\varphi}}\big|^{2p}\big] + \mathbb{E}\big[\big|D_rf_{t,u}^{x,\widehat{\varphi}}\big|^{2p}\big]\big)\big)^{1/2}\Big)du.
\end{aligned}
$$

We conclude by Lemma 4.5 and Relations (5.9), (5.12), (5.1), (5.6). □

References

[1] N. Agram and B. Øksendal. A Hida-Malliavin white noise calculus approach to optimal control. *Infinite Dimensional Analysis, Quantum Probability and Related Topics*, 21(03):1850014, 2018.

[2] S. Basak and G. Chabakauri. Dynamic mean-variance asset allocation. *The Review of Financial Studies*, 23(8):2970–3016, 2010.

[3] T. Björk and A. Murgoci. A general theory of Markovian time-inconsistent stochastic control problems. Available at SSRN: https://ssrn.com/abstract=1694759 or http://dx.doi.org/10.2139/ssrn.1694759, 2010.

[4] T. Björk, A. Murgoci, and X.Y. Zhou. Mean-variance portfolio optimization with state-dependent risk aversion. *Math. Finance*, 24(1):1–24, 2014.

[5] R. Buckdahn, B. Djehiche, and J. Li. A general stochastic maximum principle for SDEs of mean-field type. *Applied Mathematics & Optimization*, 64(2):197–216, 2011.

[6] B. Djehiche and M. Huang. A characterization of sub-game perfect equilibria for SDEs of mean-field type. *Dynamic Games and Applications*, 6(1):55–81, 2016.

[7] I. Ekeland and A. Lazrak. Being serious about non-commitment: subgame perfect equilibrium in continuous time. *Preprint arXiv:math/0604264*, 2006.

[8] I. Ekeland and T.A. Pirvu. Investment and consumption without commitment. *Math. Financ. Econ.*, 2(1):57–86, 2008.

[9] Y. Hu, H. Jin, and X.Y. Zhou. Time-inconsistent stochastic linear-quadratic control. *SIAM journal on Control and Optimization*, 50(3):1548–1572, 2012.

[10] Y. Hu, H. Jin, and X.Y. Zhou. Time-inconsistent stochastic linear-quadratic control: characterization and uniqueness of equilibrium. *SIAM Journal on Control and Optimization*, 55(2):1261–1279, 2017.

[11] X. Mao. *Stochastic differential equations and applications*. Elsevier, 2007.

[12] T. Meyer-Brandis, B. Øksendal, and X.Y. Zhou. A mean-field stochastic maximum principle via Malliavin calculus. *Stochastics An International Journal of Probability and Stochastic Processes*, 84(5-6): 643–666, 2012.

[13] D. Nualart. *The Malliavin calculus and related topics*, volume 1995. Springer, 2006.

[14] B. Øksendal and A. Sulem. Maximum principles for optimal control of forward-backward stochastic differential equations with jumps. *SIAM Journal on Control and Optimization*, 48(5):2945–2976, 2010.

[15] B. Øksendal and A. Sulem. Singular stochastic control and optimal stopping with partial information of Itô–Lévy processes. *SIAM Journal on Control and Optimization*, 50(4):2254–2287, 2012.

[16] É. Pardoux and S. Peng. Backward stochastic differential equations and quasilinear parabolic partial differential equations. In *Stochastic partial differential equations and their applications (Charlotte, NC, 1991)*, volume 176 of *Lecture Notes in Control and Inform. Sci.*, pages 200–217. Springer, Berlin, 1992.

[17] H. Pham and X. Wei. Bellman equation and viscosity solutions for mean-field stochastic control problem. *ESAIM: Control, Optimisation and Calculus of Variations*, 24(1):437–461, 2018.

[18] A.S. Üstünel. *An introduction to analysis on Wiener space*, volume 1610 of *Lecture Notes in Mathematics*. Springer Verlag, 1995.

[19] G. Wang, Z. Wu, and J. Xiong. Maximum principles for forward-backward stochastic control systems with correlated state and observation noises. *SIAM Journal on Control and Optimization*, 51(1):491–524, 2013.

[20] J. Yong. Time-inconsistent optimal control problems and the equilibrium HJB equation. *Mathematical Control and Related Fields*, 2(3):271–329, 2012.

[21] J. Yong. Linear-quadratic optimal control problems for mean-field stochastic differential equations–time-consistent solutions. *Transactions of the American Mathematical Society*, 369(8):5467–5523, 2017.

[22] J. Yong and X.Y. Zhou. *Stochastic controls*, volume 43 of *Applications of Mathematics (New York)*. Springer-Verlag, New York, 1999.

[23] X.Y. Zhou and D. Li. Continuous-time mean-variance portfolio selection: A stochastic LQ framework. *Appl. Math. Optim.*, 42(1):19–33, 2000.