A LOGARITHMIC SOBOLEV INEQUALITY FOR AN INTERACTING SPIN SYSTEM UNDER A GEOMETRIC REFERENCE MEASURE

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Logarithmic Sobolev inequalities are an essential tool in the study of interacting particle systems, cf. e.g. [4], [5]. In this note we show that the logarithmic Sobolev inequality proved on the configuration space $\mathbb{N}^{\mathbb{Z}^d}$ under Poisson reference measures in [1] can be extended to geometric reference measures using the results of [2]. As a corollary we obtain a deviation estimate for an interacting particle system.

1. Logarithmic Sobolev inequality for the geometric distribution

Consider the forward and backward gradient operators

$$\mathrm{d}^+ f(k) = f(k+1) - f(k), \qquad \mathrm{d}^- f(k) = 1_{\{k\geq 1\}}(f(k-1) - f(k)), \quad k \in \mathbb{N},$$

and the Laplacian

$$\mathscr{L} = -\mathrm{d}_\pi^{+*}\mathrm{d}^+ = \mathrm{d}^+ + \frac{1}{p}\mathrm{d}^-$$

which generates a Markov process on $\mathbb{N}$ whose invariant measure is the geometric distribution π on $\mathbb{N}$ with parameter $p \in (0,1)$, i.e.

$$\pi(\{k\}) = (1-p)p^k, \qquad k \in \mathbb{N}.$$

Denote by $\mathbb{E}_\pi$ the expectation under π and by Ent_π the entropy under π, defined as

$$\mathrm{Ent}_\pi[f] = \mathbb{E}_\pi[f \log f] - \mathbb{E}_\pi[f] \log \mathbb{E}_\pi[f].$$

We recall the modified logarithmic Sobolev inequality proved in [2] for the geometric distribution π.

Theorem 1.1. *Let* $0 < c < -\log p$ *and let* $f : \mathbb{N} \to \mathbb{R}$ *such that* $|\mathrm{d}^+ f| \le c$. *We have*

$$\mathrm{Ent}_\pi \left[e^f \right] \le \frac{pe^c}{(1-p)(1-\sqrt{pe^c})} \mathbb{E}_\pi \left[|\mathrm{d}^+ f|^2 e^f \right]. \qquad (1.1)$$

In higher dimensions the multi-dimensional gradient is defined as

$$\mathrm{d}_i^+ f(k) = f(k + e_i) - f(k), \quad i = 1, \ldots, n,$$

where f is a function on $\mathbb{N}^n$, $k = (k_1, \ldots, k_n) \in \mathbb{N}^n$, $(e_1, \ldots, e_n)$ is the canonical basis of $\mathbb{R}^n$, and the gradient norm is

$$\|\mathrm{d}^+ f(k)\|^2 = \sum_{i=1}^n |\mathrm{d}_i^+ f(k)|^2 = \sum_{i=1}^n |f(k + e_i) - f(k)|^2. \qquad (1.2)$$

From the tensorization property of entropy, (1.1) still holds with respect to $\pi^{\otimes n}$ in any finite dimension n:

$$\mathrm{Ent}_{\pi^{\otimes n}} \left[e^f \right] \le \frac{pe^c}{(1-p)(1-\sqrt{pe^c})} \mathbb{E}_{\pi^{\otimes n}} \left[\|\mathrm{d}^+ f\|^2 e^f \right], \qquad (1.3)$$

provided $|d_i f| \le c$, $i = 1, \ldots, n$. As a consequence the following deviation inequality for functions of several variables under $\pi^{\otimes n}$ has been proved in [2] using (1.1) and the Herbst method.

Corollary 1.2. *Let* $0 < c < -\log p$ *and let* f *such that* $|\mathrm{d}_i^+ f| \le \beta$, $i = 1, \ldots, n$, *and* $\|\mathrm{d}^+ f\|^2 \le \alpha^2$ *for some* $\alpha, \beta > 0$. *Then for all* $r > 0$,

$$\pi^{\otimes n}(f - \mathbb{E}_{\pi^{\otimes n}}[f] \ge r) \le \exp\left(-\min\left(\frac{c^2 r^2}{4a_{p,c}\alpha^2\beta^2}, \frac{rc}{\beta} - \alpha^2 a_{p,c} \right) \right), \quad (1.4)$$

where

$$a_{p,c} = \frac{pe^c}{(1-p)(1-\sqrt{pe^c})}$$

denotes the logarithmic Sobolev constant in (1.1).

Our goal in the next section will be to extend these results to interacting spin systems under a geometric reference measure.

2. Logarithmic Sobolev inequality for an interacting spin system

Given a bounded finite range interaction potential $\Phi = \{\Phi_R \ : \ R \subset \mathbb{Z}^d\}$, i.e.

$$\|\Phi\| = \sup_{k\in\mathbb{Z}^d} \sum_{R\ni k} \|\Phi_R\|_\infty < \infty,$$

let the Hamiltonian H_Λ be defined as

$$H_\Lambda(\eta) = \sum_{R\bigcap \Lambda\neq\emptyset} \Phi_R(\eta_R),$$

where η_R denotes the restriction of η to $\mathbb{N}^R$, $R \subset \mathbb{Z}^d$. The Gibbs measure π_Λ^ω on $\mathbb{N}^\Lambda$ associated to a $\mathbb{N}$-valued spin system on a finite lattice $\Lambda \subset \mathbb{Z}^d$ with boundary condition $\omega \in \mathbb{N}^{\mathbb{Z}^d\setminus\Lambda}$ is defined by its density with respect to $\pi_\Lambda := \pi^{\otimes\Lambda}$ as:

$$\frac{d\pi_\Lambda^\omega}{d\pi_\Lambda}(\sigma) = \frac{1}{Z_\Lambda^\omega} e^{-H_\Lambda^\omega(\sigma)}, \quad \sigma \in \mathbb{N}^\Lambda,$$

where π is the geometric reference distribution on $\mathbb{N}$, Z_Λ^ω is a normalization factor, and

$$H_\Lambda^\omega(\eta) = H_\Lambda(\eta_\Lambda\omega_{\Lambda^c}), \quad \eta \in \mathbb{N}^{\mathbb{Z}^d},$$

where $\eta_A\omega_B$ is defined as

$$(\eta_A\omega_B)_k = \eta_k 1_A(k) + \omega_k 1_B(k), \qquad k \in \mathbb{Z}^d,$$

whenever $\eta \in \mathbb{N}^A$, $\omega \in \mathbb{N}^B$, and $A, B \subset \mathbb{Z}^d$ are such that $A \cap B = \emptyset$. Let again

$$\mathrm{d}_k^+ f(\eta) = f(\eta + e_k) - f(\eta), \quad \text{and} \quad \mathrm{d}_k^- f(\eta) = 1_{\{\eta_k>0\}} \left(f(\eta - e_k) - f(\eta)\right),$$

$\eta \in \mathbb{N}^{\mathbb{Z}^d}$, $k \in \mathbb{Z}^d$, for every function $f : \mathbb{N}^{\mathbb{Z}^d} \to \mathbb{R}$, where $(e_k)_{k\in\mathbb{Z}^d}$, denotes the canonical basis $\{e_k = 1_{\{k\}} \ : \ k \in \mathbb{Z}^d\}$. Consider the Markov generator

$$\mathscr{L}_\Lambda^\omega f(\eta) = \sum_{k\in\Lambda} \left(c_\Lambda^\omega(k, \eta, +)\mathrm{d}_k^+ f(\eta) + c_\Lambda^\omega(k, \eta, -)\mathrm{d}_k^- f(\eta)\right),$$

where $c_\Lambda^\omega(k, \eta, \pm)$ are rate functions such that $\mathscr{L}_\Lambda^\omega$ is self-adjoint in $L^2(\pi_\Lambda^\omega)$, i.e.

$$c_\Lambda^\omega(k, \eta, +)\pi_\Lambda^\omega(\{\eta\}) = c_\Lambda^\omega(k, \eta + e_k, -)\pi_\Lambda^\omega(\{\eta + e_k\}), \quad k \in \Lambda, \ \eta \in \mathbb{N}^\Lambda,$$

$$c_\Lambda^\omega(k, \eta, -)\pi_\Lambda^\omega(\{\eta\}) = c_\Lambda^\omega(k, \eta - e_k, +)\pi_\Lambda^\omega(\{\eta - e_k\}), \quad k \in \Lambda, \ \eta \in \mathbb{N}^\Lambda,$$

$\eta_k > 0$, cf. [1]. We assume that there exists a constant $C > 0$ depending on $\|\Phi\|$ only, with

$$\frac{1}{C} \leq c_\Lambda^\omega(k, \eta, +) \leq C, \qquad \eta \in \mathbb{N}^\Lambda,\ \Lambda \subset \mathbb{Z}^d,\ k \in \Lambda. \tag{2.1}$$

For $f : \mathbb{N}^{\mathbb{Z}^d} \to \mathbb{R}$ we let:

$$\mathscr{E}_\Lambda^\omega(e^f) = \sum_{k \in \Lambda} \int_{\mathbb{N}^{\mathbb{Z}^d}} c_\Lambda^\omega(k, \sigma, +) e^{f(\sigma)} |\mathrm{d}_k^+ f(\sigma)|^2 d\pi_\Lambda^\omega(\sigma),$$

and

$$\mathscr{E}_\Lambda(e^f) = \sum_{k \in \Lambda} \int_{\mathbb{N}^{\mathbb{Z}^d}} e^{f(\sigma)} |\mathrm{d}_k^+ f(\sigma)|^2 d\pi_\Lambda(\sigma).$$

Next we consider the family of rectangles of the form

$$R = R(k, l_1, ..., l_d) = k + ([1, l_1] \times \cdots \times [1, l_d]) \cap \mathbb{Z}^d,$$

where $k \in \mathbb{Z}^d$ and $l_1, \ldots, l_d \in \mathbb{N}$, with

$$\mathrm{size}(R) = \max_{k=1,\ldots,d} l_k.$$

Let $\mathscr{R}_L$ denote the set of rectangles such that

$$\mathrm{size}(R) \leq L \quad \text{and} \quad \mathrm{size}(R) \leq 10 \min_{k=1,\ldots,d} l_k.$$

Definition 2.1. We say that π_Λ^ω satisfies the mixing condition if there exists constants C_1 and C_2, depending on d and $\|\Phi\|$ only, such that:

$$\sup_{\sigma,\omega} \left| \frac{\pi_\Lambda^\omega(\{\eta :\ \eta_A = \sigma_A\}) \pi_\Lambda^\omega(\{\eta :\ \eta_B = \sigma_B\})}{\pi_\Lambda^\omega(\{\eta \ :\ \eta_{A\cup B} = \sigma_{A \cup B}\})} - 1 \right| \leq C_1 e^{-C_2 d(A,B)}, \tag{2.2}$$

for all $L \geq 1$, $\Lambda \in \mathscr{R}_L$ and $A, B \subset \Lambda$ such that $A, B \in \mathscr{R}_L$ with $A \cap B = \emptyset$.

We refer to [1] and [4] for conditions on Φ under which (2.2) holds under a geometric reference measure.

Our goal is to prove the following logarithmic Sobolev inequality under the Gibbs measure π_Λ^ω.

Theorem 2.2. *Assume that the mixing condition (2.2) holds, and let* $0 < c < -\log p$. *Then there exists a constant* $\gamma_c > 0$, *independent of* Λ *and* ω, *such that*

$$\mathrm{Ent}_{\pi_\Lambda^\omega}\left[e^f\right] \leq \gamma_c \mathscr{E}_\Lambda^\omega(e^f), \tag{2.3}$$

for every $f : \mathbb{N}^{\mathbb{Z}^d} \to \mathbb{R}$ *such that* $\|\mathrm{d}^+ f\|_{l^\infty(\Lambda)} \leq c$, π_Λ*-a.e.*

In particular we have

$$\mathrm{Ent}_{\pi_\Lambda^\omega}\left[e^f\right] \le \gamma_c \left\| \sum_{k\in\Lambda} c_\Lambda^\omega(k,\cdot,+)|\mathrm{d}_k^+ f(\cdot)|^2 \right\|_{L^\infty(\pi_\Lambda)} \times \mathbb{E}_{\pi_\Lambda^\omega}\left[e^f\right],$$

which implies, as in Corollary 1.2, a deviation inequality under Gibbs measures.

Corollary 2.3. *Assume that the mixing condition (2.2) holds, and let* $0 < c < -\log p$. *Let* f *be such that* $\|\mathrm{d}^+ f\|_{l^\infty(\Lambda)} \le \beta$ *and*

$$\sum_{k\in\Lambda} c_\Lambda^\omega(k,\eta,+)|\mathrm{d}_k^+ f(\eta)|^2 \le \alpha^2, \quad \pi_\Lambda(d\eta) - a.e., \tag{2.4}$$

for some $\alpha, \beta > 0$. *Then for all* $r > 0$,

$$\pi_\Lambda^\omega\left(f - \mathbb{E}_{\pi_\Lambda^\omega}[f] \ge r\right) \le \exp\left(-\min\left(\frac{c^2 r^2}{4\gamma_c \alpha^2 \beta^2}, \frac{rc}{\beta} - \alpha^2\gamma_c\right)\right). \tag{2.5}$$

Due to Hypothesis (2.1), condition (2.4) can be replaced by

$$\|\mathrm{d}^+ f(\eta)\|_{l^2(\Lambda)}^2 \le C^{-1}\alpha^2, \quad \pi_\Lambda(d\eta) - a.e.$$

Denoting by Π denote the infinite volume Gibbs measure associated to π_Λ^ω, for some $r_0 > 0$ we get the Ruelle type bound:

$$\Pi(\{\eta \in \mathbb{N}^{\mathbb{Z}^d} : |\eta_\Lambda| \ge r|\Lambda|\}) \le \exp\left(-(cr - C\gamma_c)|\Lambda| + c\mathbb{E}_\Pi[|\eta_\Lambda|]\right), \quad r > r_0,$$

for all finite subset Λ of $\mathbb{Z}^d$, under the mixing condition (2.2). Indeed, it suffices to apply the uniform bound (2.5) with $f(\eta) = |\eta_\Lambda|$, $\alpha^2 = C|\Lambda|$, $\beta = 1$, and the compatibility condition

$$\Pi(E) = \int_{\mathbb{N}^{\mathbb{Z}^d}} \pi_\Lambda^\omega(E_\Lambda)\Pi(d\omega),$$

to $E = \{\eta \in \mathbb{N}^{\mathbb{Z}^d} : |\eta_\Lambda| \ge r|\Lambda|\}$, with

$$E_\Lambda := \{\eta \in \mathbb{N}^\Lambda : \eta_\Lambda \omega_{\Lambda^c} \in E\} = \{\eta \in \mathbb{N}^\Lambda : |\eta_\Lambda| \ge r|\Lambda|\}.$$

This shows in particular that Π satisfies the $(RPB)^1$ condition in [3].

3. Proof of Theorem 2.2

Recall that for $0 < c < -\log p$, by tensorization, Theorem 1.1 yields as in (1.3) the logarithmic Sobolev inequality

$$\mathrm{Ent}_{\pi_\Lambda}[e^f] \le s_c \mathscr{E}_\Lambda(e^f), \tag{3.1}$$

for all $f : \mathbb{N}^{\mathbb{Z}^d} \to \mathbb{R}$ such that $\|d^+ f\|_{l^\infty(\Lambda)} \leq c$, π_Λ-a.e., with an optimal constant $s_c \leq a_{p,c}$ which is independent of $\Lambda \subset \mathbb{Z}^d$. Let now $s_{\Lambda,\omega,c}$ denote the optimal constant in the inequality

$$\mathrm{Ent}_{\pi_\Lambda^\omega}[e^f] \leq s_{\Lambda,\omega,c} \mathscr{E}_\Lambda^\omega(e^f), \qquad \|d^+ f\|_{l^\infty(\Lambda)} \leq c.$$

Lemma 3.1. *For every $\Lambda \subset \mathbb{Z}^d$, there exists a constant $A := Ce^{4|\Lambda|\|\Phi\|} > 0$ depending only on $|\Lambda|$, c and independent of $\omega \in \mathbb{N}^{\mathbb{Z}^d \setminus \Lambda}$, such that*

$$\frac{s_c}{A} \leq s_{\Lambda,\omega,c} \leq A\, s_c.$$

Proof. We follow the proof of Proposition 3.1 in [1]. From (2.1) we obtain:

$$C^{-1} e^{-2|\Lambda|\|\Phi\|}\, \mathscr{E}_\Lambda(e^f) \leq \mathscr{E}_\Lambda^\omega(e^f) \leq Ce^{2|\Lambda|\|\Phi\|}\, \mathscr{E}_\Lambda(e^f). \tag{3.2}$$

From the relation

$$\mathrm{Ent}_\mu[f] = \min_{t>0} \mathbb{E}_\mu[f \log f - f \log t - f + t]$$

and the bound

$$e^{-2|\Lambda|\|\Phi\|} \leq \frac{d\pi_\Lambda^\omega}{d\pi_\Lambda} \leq e^{2|\Lambda|\|\Phi\|},$$

we have

$$e^{-2|\Lambda|\|\Phi\|} \mathrm{Ent}_{\pi_\Lambda}\left[e^f\right] \leq \mathrm{Ent}_{\pi_\Lambda^\omega}\left[e^f\right] \leq e^{2|\Lambda|\|\Phi\|} \mathrm{Ent}_{\pi_\Lambda}\left[e^f\right],$$

from which the conclusion follows using (3.1) and (3.2). □

Let for $L \geq 1$:

$$S_{L,c} := \sup_{R \in \mathscr{R}_L} \sup_{\omega \in \mathbb{N}^{\mathbb{Z}^d \setminus \Lambda}} s_{R,\omega,c} \leq C s_c e^{4|\Lambda|\|\Phi\|} < \infty,$$

which is finite by Lemma 3.1.

Proposition 3.2. *Assume the mixing condition (2.2) is satisfied. Then there exists a constant κ depending on $\|\Phi\|$, such that*

$$S_{2L,c} \leq \left(1 - \frac{\kappa}{\sqrt{L}}\right)^{-1} S_{L,c} \tag{3.3}$$

for L large enough.

Proof. The proof of this proposition is identical to that of Proposition 4.1, pp. 1970-1972 and Proposition 5.1, p. 1975 in [1], replacing the Dirichlet form used in [1] with $\mathscr{E}_\Lambda^\omega$. □

Finally, Theorem 2.2 is proved by taking $\gamma_c = \sup_L S_{L,c}$, which is finite from Proposition 3.2.

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