Deviation inequalities and the law of iterated logarithm on the path space over a loop group

Nicolas Privault * Xicheng Zhang * †

Abstract

A law of iterated logarithm (LIL) in small time and an asymptotic estimate of modulus of continuity are proved for Brownian motion on the loop group $\mathcal{L}(G)$ over a compact connected Lie group G. Upper bounds are obtained via infinite-dimensional deviation inequalities for functionals on the path space $\mathbb{P}(\mathcal{L}(G))$ on $\mathcal{L}(G)$, such as the supremum of Brownian motion on $\mathcal{L}(G)$, which are proved from the Clark-Ocone formula on $\mathbb{P}(\mathcal{L}(G))$. The lower bounds rely on analog finite-dimensional results that are proved separately on Riemannian path space.

Key words: Brownian motion, loop groups, Malliavin calculus, Clark-Ocone formula, deviation inequalities, sample path properties.

Classification: 60J65, 58J65, 60H07, 22E67, 60F10, 60G17.

1 Introduction

Using heat kernel estimates, Grigor'yan and Kelbert [10] have investigated the problem of escape rate of Brownian motion $\gamma(t)$ starting from m_0 on a stochastically complete, noncompact Riemannian manifold M with distance ρ^M, and proved the following law of iterated logarithm in large time:

$$\limsup_{t\to+\infty} \frac{\rho^M(\gamma(t), m_0)}{\sqrt{2t\log\log t}} = 1, \tag{1.1}$$

almost surely, when the Ricci curvature of M is nonnegative. Without noncompactness assumption on M, let now $\mathrm{Cut}(m_0)$ denote the cut-locus of $m_0 \in M$, and let

$$S_{m_0} = \inf\{t \geqslant 0 \ : \ \gamma(t) \in \mathrm{Cut}(m_0)\}.$$

*Laboratoire de Mathématiques, Université de La Rochelle, 17042 La Rochelle, France. nprivaul@univ-lr.fr

†Department of Mathematics, Huazhong University of Science and Technology, Wuhan, Hubei 430074, P.R.China. xczhang@hust.edu.cn. The second author is partially supported by NSF (No. 10301011) of China, Project 973. Support from a postdoctoral fellowship from Région Poitou-Charentes is also acknowledged.

From Itô's formula, cf. e.g. Hsu [13], page 89, we get

$$\rho^M(\gamma(t), m_0) = \beta(t) + \frac{1}{2}\int_0^t (\Delta^M \rho^M(\cdot, m_0))(\gamma(s))ds, \quad 0 \leqslant t \leqslant S_{m_0}, \tag{1.2}$$

where $\beta(t)$ is a one-dimensional standard Brownian motion and Δ^M denotes the Laplace-Beltrami operator on $(M, \langle\cdot,\cdot\rangle)$. Moreover, S_{m_0} is a.s. strictly positive because the exponential map at any point $m_0 \in M$ is a diffeomorphism in a neighborhood of m_0, hence any point in this neighborhood can be connected with m_0 by a unique geodesic. Therefore from (1.2) we get the law of iterated logarithm in small time:

$$\limsup_{t\downarrow 0} \frac{\rho^M(\gamma(t), m_0)}{\sqrt{2t\log\log t^{-1}}} = \limsup_{t\downarrow 0} \frac{\rho^M(\gamma(t\wedge S_{m_0}), m_0)}{\sqrt{2t\log\log t^{-1}}} = \limsup_{t\downarrow 0} \frac{\beta(t)}{\sqrt{2t\log\log t^{-1}}} = 1, \tag{1.3}$$

almost surely. It is well known that when $M = \mathbb{R}^d$, (1.1) is equivalent to (1.3) by time reversal. However, for Brownian motion on a Riemannian manifold M, relations (1.1) and (1.3) cannot be derived from each other in general, for instance (1.3) holds but (1.1) fails when M is compact. See also Blümlinger et al. [3] for a law of iterated logarithm in large time for Brownian motion on a compact connected Riemannian manifold without boundary, and Bendikov and Saloff-Coste [1] for recent analog results for Brownian motions on infinite dimensional tori.

In this paper we study related problems on the loop group over a compact connected Lie group G, using infinite-dimensional deviation inequalities for random functionals. We prove a general deviation result for functionals of Brownian motion on loop groups, including its supremum on a compact time interval, and deduce a law of iterated logarithm in small time and an asymptotic estimate for its modulus of continuity.

We proceed as follows. In Section 2 we recall the framework of stochastic analysis on the path space of a Riemannian manifold, and recover the upper bound of (1.1) by a proof different from the one of Grigor'yan and Kelbert [10].

In Section 3 we prove a preliminary result in the finite-dimensional case, i.e. we show that if the Ricci curvature of M is bounded below then the modulus of continuity of $\gamma(t)$ satisfies the inequality

$$\limsup_{\delta\downarrow 0} \sup_{0<s<t<T, t-s=\delta} \frac{\rho^M(\gamma(t), \gamma(s))}{\sqrt{2(t-s)\log(t-s)}} \leqslant 1, \qquad a.s.,$$

which becomes an equality if moreover the Ricci curvature of M is nonnegative. The proof of the upper bound relies on deviation inequalities for the supremum of Brownian motion on a Riemannian manifold, cf. e.g. Theorem 8.62 of Stroock [20] (see Theorem 2.2 below), while the lower bound is proved by applying classical arguments (as in e.g. Itô and McKean [14]) combined with stochastic analysis results on Riemannian manifolds (Hsu [13], Grigor'yan and Kelbert [10]).

In Section 4 we recall the Clark-Ocone formula (Theorem 4.8) on the path space

$$\mathbb{P}(\mathcal{L}(G)) = \{\gamma : [0,T] \mapsto \mathcal{L}(G) \text{ continuous } : \gamma(0) = e\}$$

over the based loop group $\mathcal{L}(G)$ on a connected compact Lie group G with unit e, cf. Fang [7], and in Section 5 a general deviation result (Theorem 5.4) is proved for functionals of Brownian motion over $\mathcal{L}(G)$. Note that this result could also be obtained using logarithmic Sobolev inequalities on the path space over loop groups, such as Theorem 6.4 of Driver and Lohrenz [5] (for the heat kernel measure on $\mathcal{L}(\mathcal{G})$) or Theorem 5.4 of Fang [7] (for functionals on $\mathbb{P}(\mathcal{L}(G))$), and the Herbst method, cf. § 2.3 of Ledoux [16], but our approach is somewhat more direct. In Section 6 we prove a tail estimate for Brownian motion $\gamma(t)$ on loop groups (Theorem 6.1) using arguments of Houdré and Privault [12], Ledoux [16], and the Clark-Ocone formula on loop groups. Using this estimate, in Section 7 we prove a law of iterated logarithm for $\gamma(t)$:

$$\limsup_{t\downarrow 0} \frac{\rho^{\mathcal{L}(G)}(\gamma(t), e)}{\sqrt{(t\log\log t^{-1})/2}} = 1, \qquad a.s.$$

In Section 8 we derive an asymptotic estimate for the modulus of continuity of $\gamma(t)$, i.e. for all $T > 0$ we prove the equality

$$\limsup_{\delta\downarrow 0} \sup_{0<s<t<T, t-s=\delta} \frac{\rho^{\mathcal{L}(G)}(\gamma(t), \gamma(s))}{\sqrt{((t-s)\log(t-s)^{-1})/2}} = 1, \qquad a.s. \tag{1.4}$$

Again, the proofs of upper bounds rely on the deviation inequality for the supremum of Brownian motion on loop groups (with Theorem 2.2 on Riemannian manifolds replaced by Theorem 6.1 on loop groups), while the lower bounds are proved as applications of the finite dimensional results obtained in Section 3.

2 Stochastic analysis and LIL on Riemannian path space

Let $(b(t))_{t\in[0,T]}$ denote the $\mathbb{R}^d$-valued Brownian motion on the Wiener space W, with Wiener measure μ and Cameron-Martin space H, generating the filtration $(\mathcal{F}_t)_{t\in[0,T]}$. Let $(M, \langle\cdot,\cdot\rangle_m)$ be a connected and complete d-dimensional Riemannian manifold, with Levi-Civita connection associated to the Riemannian metric $\langle\cdot,\cdot\rangle_m$, and let $\rho^M(\cdot,\cdot)$ denote the Riemannian distance on M. Let also $O(M)$ denote the bundle of orthogonal frames over M, namely

$$O(M) = \big\{(m,r) : r \text{ is a Euclidean isometry from } \mathbb{R}^d \text{ into } T_m(M),\ m \in M\big\}.$$

The Levi-Civita parallel transport defines d canonical horizontal vector fields $A_1, \ldots, A_d$ on $O(M)$, and the Stratonovich stochastic differential equation

$$\begin{cases} dr(t) = \sum_{i=1}^d A_i(r(t)) \circ db^i(t), & t \in [0,T], \\ r(0) = (m_0, r_0) \in O(M), \end{cases} \tag{2.1}$$

defines an $O(M)$-valued process $(r(t))_{t\in[0,T]}$ which is assumed to be non-explosive (this is the case in particular when the Ricci curvature of M is bounded below). Let π :

$O(M) \longrightarrow M$ be the canonical projection, let $\gamma(t) = \pi(r(t))$, $t \in [0,T]$, be the Brownian motion on M corresponding to the Laplace-Beltrami operator starting from a fixed point $m_0 \in M$, and defined on some probability space $(\Omega, \mathcal{F}, P)$. The Itô parallel transport along $(\gamma(t))_{t\in[0,T]}$ is defined as

$$\mathbf{t}^{\gamma}_{t\leftarrow 0} = r(t)r_0^{-1} : T_{m_0}M \simeq \mathbb{R}^d \longrightarrow T_{\gamma(t)}M, \quad t \in [0,T].$$

Let $\mathbb{P}(\mathbb{R}^d)$ denote the space of continuous $\mathbb{R}^d$-valued functions on $[0,T]$ vanishing at the origin, and let $\mathbb{P}_{m_0}(M)$ denote the set of continuous paths on M starting at m_0. Let

$$I : \mathbb{P}(\mathbb{R}^d) \longrightarrow \mathbb{P}_{m_0}(M); \quad (b(t))_{t\in[0,T]} \mapsto I(b) = (\gamma(t))_{t\in[0,T]}$$

be the Itô map, and let P denote the image measure on $\mathbb{P}_{m_0}(M)$ of the Wiener measure μ by I. Let Ω_r denote the curvature tensor of M, and let $\mathrm{Ric}_r : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ denote the Ricci tensor at the frame $r \in O(M)$.

A vector field along γ is a section process of the tangent bundle of M, i.e. a measurable map $Z_\gamma(\tau) \in T_{\gamma(\tau)}(M)$ such that $Z(0) = 0$. We say that Z is a Cameron-Martin vector field if the process $(z(t))_{t\in[0,T]} = (\mathbf{t}^{\gamma}_{0\leftarrow t}(Z(t)))_{t\in[0,T]}$ belongs to the Cameron-Martin space H on the Wiener space. For cylindrical functionals on the path space, namely for functionals F of the form

$$F(\gamma) = f(\gamma(t_1), \ldots, \gamma(t_n)),$$

with $f \in \mathcal{C}_b^\infty(M^n)$, $0 < t_1 < \cdots < t_n \leqslant T$, we consider the intrinsic gradient operator D defined as:

$$D_t F = \sum_{i=1}^{n} 1_{[0,t_i]}(t)\mathbf{t}^{\gamma}_{0\leftarrow t_i} \nabla_i^M f(\gamma(t_1), \ldots, \gamma(t_n)), \qquad t \in [0,T], \tag{2.2}$$

cf. e.g. [8]. The derivation with respect to a general Cameron-Martin vector field Z is defined as

$$D_Z F = \int_0^T D_{t,\alpha} F \cdot \dot{z}^\alpha(t) dt,$$

where $D_{t,\alpha} F = \langle \mathbf{t}^{\gamma}_{0\leftarrow t} D_t F, \varepsilon_\alpha \rangle$ with $(\varepsilon_1, \ldots, \varepsilon_d)$ a fixed orthonormal basis of T_{m_0}, and where we use Einstein's convention of summation. Next, we recall the integration by parts formula on path space $\mathbb{P}_{m_0}(M)$, cf. [2]:

Theorem 2.1 *Let F be a cylindrical functional on the path space $\mathbb{P}_{m_0}(M)$ and Z a Cameron-Martin vector field, adapted to the filtration generated by $(b(t))_{t\in[0,T]}$. We have*

$$\mathbb{E}[D_Z F] = \mathbb{E}\left[F \int_0^T \left(\dot{z}(t) + \frac{1}{2}\mathrm{Ric}_{r(t)} z(t) \right) db(t) \right]. \tag{2.3}$$

In order to establish the upper bound on the modulus of continuity of $\gamma(t)$ we will use the bound in Theorem 2.2 below, taken from Stroock [20] Relation (8.65), p. 199, which does not require a compactness assumption on M.

Theorem 2.2 *(Stroock [20]) Assume that the Ricci curvature of M is bounded below, i.e., for some $K \geqslant 0$,*

$$\langle \mathrm{Ric}_m X_m, X_m \rangle_m \geqslant -2K \langle X_m, X_m \rangle_m, \text{ for all } m \in M \text{ and } X_m \in T_m(M).$$

Then for any $\lambda \in (0,1)$ and $y > 0$ we have

$$P\left(\sup_{0 \leqslant t \leqslant T} \rho^M(\gamma(t), m_0) \geqslant y \right) \leqslant \frac{2}{\sqrt{1-\lambda}} \exp\left(-\frac{\lambda y^2}{2T} + \frac{\lambda(2d + Kd^2 T)}{1-\lambda} \right). \tag{2.4}$$

We close this section by recovering the upper bound in the law of iterated logarithm in large time (1.1) by a proof different from the one of Grigor'yan and Kelbert [10], the lower bound being proved in [10], p. 104. Assuming that the Ricci curvature of M is nonnegative, i.e., $K = 0$, let $\varphi(t) = 2t \log\log t$, fix two real numbers $\varepsilon \in (0,1)$, $\alpha > 1$ and let $\lambda = (1 + \varepsilon/2)^{-1} < 1$, so that

$$\lambda(1+\varepsilon) > 1.$$

Set also

$$A_n = \left\{ \sup_{0 \leqslant t \leqslant \alpha^n} \rho^M(\gamma(t), m_0) \geqslant \sqrt{(1+\varepsilon)\varphi(\alpha^n)} \right\}, \qquad n \geqslant 0.$$

Choosing $n > N$ sufficiently large, by Theorem 2.2 and the assumption $K = 0$ we have

$$P(A_n) \leqslant C_{\lambda,d} \exp\left(-\lambda(1+\varepsilon)\log\log \alpha^n\right) = C_{\lambda,d}(n \log \alpha)^{-\lambda(1+\varepsilon)},$$

where $C_{\lambda,d} = 2(1-\lambda)^{-\frac{1}{2}} \exp\left(\frac{2\lambda d}{1-\lambda}\right)$. Hence

$$\sum_{n=N}^{+\infty} P(A_n) < +\infty.$$

By the Borel-Cantelli Lemma there exists an event $\Omega_{\varepsilon,\alpha}$ with $P(\Omega_{\varepsilon,\alpha}) = 1$ such that for every $\omega \in \Omega_{\varepsilon,\alpha}$, when $n > N(\omega)$ is sufficiently large,

$$\sup_{0 \leqslant t \leqslant \alpha^n} \rho^M(\gamma(t), m_0) \leqslant \sqrt{(1+\varepsilon)\varphi(\alpha^n)}.$$

For $t > \alpha$, set $n = [\frac{\log t}{\log \alpha}] + 1$, then $t < \alpha^n < \alpha t$. Choose t (depending on α) large enough such that

$$\varphi(\alpha^n) \leqslant \varphi(\alpha t) \leqslant \alpha^2 \varphi(t).$$

We have

$$\rho^M(\gamma(t), m_0) \leqslant \sup_{0 \leqslant t \leqslant \alpha^n} \rho^M(\gamma(t), m_0) \leqslant \sqrt{(1+\varepsilon)\varphi(\alpha^n)} \leqslant \alpha\sqrt{(1+\varepsilon)\varphi(t)},$$

hence

$$\limsup_{t \to +\infty} \frac{\rho^M(\gamma(t), m_0)}{\sqrt{\varphi(t)}} \leqslant \alpha\sqrt{1+\varepsilon}.$$

Finally, letting $\varepsilon \downarrow 0$ and $\alpha \downarrow 1$ yields the desired result.

3 Modulus of continuity on Riemannian manifolds

In this section we prove a preliminary result on the modulus of continuity of Brownian motion in the finite-dimensional setting of Riemannian manifolds.

Theorem 3.1 *Let M be a complete Riemannian manifold and fix $T > 0$. If the Ricci curvature of M is bounded below, then*

$$\limsup_{\delta \downarrow 0} \sup_{0<s<t<T, t-s=\delta} \frac{\rho^M(\gamma(t), \gamma(s))}{\sqrt{-2(t-s)\log(t-s)}} \leqslant 1, \qquad a.s. \tag{3.1}$$

If the Ricci curvature of M is nonnegative, then

$$\limsup_{\delta \downarrow 0} \sup_{0<s<t<T, t-s=\delta} \frac{\rho^M(\gamma(t), \gamma(s))}{\sqrt{-2(t-s)\log(t-s)}} \geqslant 1, \qquad a.s. \tag{3.2}$$

Upper bound proof of Theorem 3.1

Let now $\varphi(t) = 2t \log t^{-1}$. By Theorem 2.2 and the Markov property, for any $s, t \in [0, T]$, $\lambda \in (0, 1)$, and $y > 0$, we have

$$P\left(\rho^M(\gamma(t), \gamma(s)) \geqslant y\right) \leqslant 2(1-\lambda)^{-\frac{1}{2}} \exp\left(-\frac{\lambda y^2}{2(t-s)} + \frac{\lambda(2d + Kd^2(t-s))}{1-\lambda}\right).$$

Let $\varepsilon > 0$ be fixed and choose $\alpha > 0$ such that

$$1 + \alpha < (1-\alpha)\lambda(1+\varepsilon),$$

where $0 < \lambda = (1 + \varepsilon/2)^{-1} < 1$. Set also

$$I_n = \{(i, j) \in \mathbb{N}^2 : 0 \leqslant i < j \leqslant 2^n, j - i \leqslant 2^{\alpha n}\}.$$

Then $\#(I_n) \leqslant 2^{(1+\alpha)n}$, and for $(i, j) \in I_n$ we have $0 < (j-i)2^{-n} \leqslant 2^{-n(1-\alpha)}$. Put

$$A_n = \left\{ \max_{(i,j)\in I_n} \frac{\rho^M(\gamma(i2^{-n}), \gamma(j2^{-n}))}{\sqrt{\varphi((j-i)2^{-n})}} \geqslant \sqrt{1+\varepsilon} \right\}.$$

Then for $n > N$ large enough, we have

$$\begin{aligned} P(A_n) &\leqslant C_{\lambda,d,K} \sum_{(i,j)\in I_n} \exp\left(-\lambda(1+\varepsilon)\log(((j-i)2^{-n})^{-1})\right) \\ &\leqslant C_{\lambda,d,K} 2^{(1+\alpha)n} \exp\left(-\lambda(1+\varepsilon)\log 2^{(1-\alpha)n}\right) \\ &\leqslant C_{\lambda,d,K} (2^{(1+\alpha)-(1-\alpha)\lambda(1+\varepsilon)})^n, \end{aligned}$$

where $C_{\lambda,d,K} = 2(1-\lambda)^{-\frac{1}{2}} \exp\left(\frac{\lambda(2d+Kd^2)}{1-\lambda}\right)$. Hence

$$\sum_{n=N}^{+\infty} P(A_n) < +\infty,$$

and the Borel-Cantelli Lemma implies that there is an event $\Omega_{\varepsilon,\alpha}$ with $P(\Omega_{\varepsilon,\alpha}) = 1$ such that for every $\omega \in \Omega_{\varepsilon,\alpha}$, and $n > N(\omega)$ sufficiently large,

$$\rho^M(\gamma(i2^{-n}), \gamma(j2^{-n})) \leqslant \sqrt{(1+\varepsilon)\varphi((j-i)2^{-n})}, \qquad (i,j) \in I_n.$$

Now fix $\omega \in \Omega_{\varepsilon,\alpha}$ and $N(\omega)$ such that

$$2^{(n+1)\alpha-2} > 2, \qquad 2^{-n(1-\alpha)} < \exp(-1), \tag{3.3}$$

and

$$\sum_{m>n} \sqrt{\varphi(2^{-m})} = \sqrt{\varphi(2^{-n})} \sum_{m=1}^{\infty} \sqrt{2^{-m} \frac{\log 2^{n+m}}{\log 2^n}} < \varepsilon \sqrt{\varphi(2^{-(n+1)(1-\alpha)})} \tag{3.4}$$

for all $n > N(\omega)$. Given $0 \leqslant s < t \leqslant T$ such that $\delta = t - s < 2^{-(1-\alpha)N(\omega)}$, choose $n \geqslant N(\omega)$ such that

$$2^{-(n+1)(1-\alpha)} \leqslant \delta < 2^{-n(1-\alpha)}. \tag{3.5}$$

Consider the expansions:

$$s = i2^{-n} - 2^{-n_1} - 2^{-n_2} - \cdots, \quad n < n_1 < n_2 < \cdots,$$

$$t = j2^{-n} + 2^{-m_1} + 2^{-m_2} + \cdots, \quad n < m_1 < m_2 < \cdots.$$

Clearly we have

$$j - i \geqslant 2^n\delta - 2 \geqslant 2^n 2^{-(n+1)(1-\alpha)} - 2 = 2^{(n+1)\alpha-1} - 2 > 0,$$

and

$$j - i \leqslant 2^n\delta \leqslant 2^{\alpha n}.$$

Since $t \mapsto \gamma(t)$ is continuous and $\varphi(t)$ is increasing in $t \in (0, \exp(-1))$, by (3.3), (3.4) and (3.5) we have

$$\begin{aligned}
\rho^M(\gamma(s), \gamma(i2^{-n})) &\leqslant \sum_{m>n} \sqrt{(1+\varepsilon)\varphi(2^{-m})} \\
&\leqslant \sqrt{\varepsilon(1+\varepsilon)\varphi(2^{-(n+1)(1-\alpha)})} \\
&\leqslant \sqrt{\varepsilon(1+\varepsilon)\varphi(\delta)}.
\end{aligned}$$

Similarly,

$$\rho^M(\gamma(t), \gamma(j2^{-n})) \leqslant \sqrt{\varepsilon(1+\varepsilon)\varphi(\delta)}.$$

Therefore,

$$\begin{aligned}
\rho^M(\gamma(s), \gamma(t)) &\leqslant \rho^M(\gamma(s), \gamma(i2^{-n})) + \rho^M(\gamma(i2^{-n}), \gamma(j2^{-n})) + \rho^M(\gamma(t), \gamma(j2^{-n})) \\
&\leqslant 2\sqrt{\varepsilon(1+\varepsilon)\varphi(\delta)} + \sqrt{(1+\varepsilon)\varphi((j-i)2^{-n})} \\
&\leqslant \sqrt{1+\varepsilon}\left(1 + 2\sqrt{\varepsilon}\right)\sqrt{\varphi(\delta)}.
\end{aligned}$$

Finally, let $\varepsilon \downarrow 0$ to complete the proof.

Lower bound proof of Theorem 3.1

Under the assumption that the Ricci curvature of M is nonnegative, i.e. $K = 0$, we will use the following lemma due to Grigor'yan and Kelbert [10], Lemma 4.6, in order to obtain the lower bound.

Lemma 3.2 *([10]) For any $t > 0$, $\beta \in (0,1)$ and $y > \sqrt{t}$, under the assumption of nonnegative Ricci curvature, there is a constant $C_\beta > 0$ such that*

$$P(\rho^M(\gamma(t), m_0) \geqslant y) \geqslant C_\beta \exp\Big(-\frac{y^2}{2\beta t}\Big). \tag{3.6}$$

Let again $\varphi(t) = 2t \log t^{-1}$ and for $0 < \varepsilon < 1$, define

$$\begin{aligned} A_n^k &= \left\{\rho^M(\gamma(k2^{-n}), \gamma((k-1)2^{-n})) < (1-\varepsilon)\sqrt{\varphi(2^{-n})}\right\}, \qquad k = 1, \dots, 2^n, \\ A_n &= \left\{\max_{k \leqslant 2^n} \rho^M(\gamma(k2^{-n}), \gamma((k-1)2^{-n})) < (1-\varepsilon)\sqrt{\varphi(2^{-n})}\right\}. \end{aligned}$$

By the (conditional) independence of $\{A_n^k,\ k = 1, \dots, 2^n\}$, the inequality $1 - s \leqslant \exp(-s)$, and Lemma 3.2, we get for $1 - \varepsilon < \beta < 1$:

$$\begin{aligned} P(A_n) &\leqslant \left(1 - C_\beta \exp\left(-\frac{(1-\varepsilon)\varphi(2^{-n})}{2\beta 2^{-n}}\right)\right)^{2^n} \\ &= \left(1 - C_\beta 2^{-n(1-\varepsilon)/\beta}\right)^{2^n} \\ &\leqslant \exp\left(-2^{n(1-(1-\varepsilon)/\beta)}\right), \end{aligned}$$

hence

$$\sum_{n=1}^{\infty} P(A_n) < +\infty.$$

By the Borel-Cantelli Lemma we have

$$\begin{aligned} 1 &= P\left(\limsup_{n\to+\infty} \max_{1\leqslant k\leqslant 2^n} \frac{\rho^M(\gamma(k2^{-n}), \gamma((k-1)2^{-n}))}{\sqrt{\varphi(2^{-n})}} \geqslant 1 - \varepsilon\right) \\ &\leqslant P\left(\limsup_{\delta\downarrow 0} \sup_{0<s<t<T, t-s=\delta} \frac{\rho^M(\gamma(t), \gamma(s))}{\sqrt{\varphi(t-s)}} \geqslant 1 - \varepsilon\right). \end{aligned}$$

Letting $\varepsilon \downarrow 0$ gives the result and concludes the proof of Theorem 3.1.

4 Path spaces over loop groups

Let us review the construction of Brownian motion on loop groups. Let G be a connected compact Lie group with unit e and Lie algebra $\mathcal{G} = T_e G$ equipped with an Ad_G-invariant inner product $\langle\cdot,\cdot\rangle_{\mathcal{G}}$ also viewed as a bi-invariant Riemannian metric on G, and inducing the Riemannian distance ρ^G on $(G,\langle\cdot,\cdot\rangle_{\mathcal{G}})$. The based loop group

$$\mathcal{L}(G)=\{g:[0,1]\mapsto G \text{ continuous } \ : \ g(0)=g(1)=e\},$$

whose unit is also denoted by e, is endowed with the supremum distance

$$\rho^{\mathcal{L}(G)}(g_1,g_2)=\sup_{\tau\in[0,1]}\rho^G(g_1(\tau),g_2(\tau)), \qquad g_1,g_2\in\mathcal{L}(G).$$

Consider the loop group

$$\mathcal{L}(\mathcal{G})=\{x:[0,1]\mapsto \mathcal{G} \text{ continuous } \ : \ x(0)=x(1)=0\},$$

and denote by $\mathcal{L}'(\mathcal{G})$ the dual space of $\mathcal{L}(\mathcal{G})$, with pairing $\langle h,x\rangle$, $h\in\mathcal{L}'(\mathcal{G})$, $x\in\mathcal{L}(\mathcal{G})$. Let $H_0(\mathcal{G})$ denote the Cameron-Martin subspace of $\mathcal{L}(\mathcal{G})$:

$$H_0(\mathcal{G})=\left\{h\in\mathcal{L}(\mathcal{G}) \ : \ |h|^2_{H_0(\mathcal{G})}=\int_0^1\langle\dot h(\tau),\dot h(\tau)\rangle_{\mathcal{G}}d\tau<+\infty\right\},$$

with

$$\mathcal{L}'(\mathcal{G})\subset H_0'(\mathcal{G})\simeq H_0(\mathcal{G})\subset\mathcal{L}(\mathcal{G}).$$

In the sequel we fix a filtered probability space $(\Omega,(\mathcal{F}_t)_{t\geqslant 0},\mathcal{F},P)$ satisfying the usual hypotheses and a $\mathcal{L}(\mathcal{G})$-valued Brownian motion $(x(t))_{t\geqslant 0}$ with covariance operator $\langle\cdot,\cdot\rangle_{H_0(\mathcal{G})}$, i.e.

$$\mathbb{E}\left[\langle h,x(t)\rangle\langle k,x(s)\rangle\right]=(t\wedge s)\langle h,k\rangle_{H_0(\mathcal{G})}$$

for all $h,k\in\mathcal{L}'(G)$ and $s,t>0$. Letting

$$G(\tau,\tau')=\tau\wedge\tau'-\tau\tau', \qquad \tau,\tau'\in[0,1],$$

and defining $u_{a,\tau}\in\mathcal{L}'(G)$, $a\in\mathcal{G}$, $\tau\in[0,1]$, by

$$\langle u_{a,\tau},x\rangle=\langle x(\tau),a\rangle_{\mathcal{G}}, \qquad x\in\mathcal{L}(\mathcal{G}),$$

we have for $h\in H_0(\mathcal{G})$:

$$\langle u_{a,\tau},h\rangle=\langle h(\tau),a\rangle_{\mathcal{G}}=\int_0^1\frac{\partial}{\partial\theta}G(\tau,\theta)\langle\dot h(\theta),a\rangle_{\mathcal{G}}d\theta=\langle G(\tau,\cdot)a,h(\cdot)\rangle_{H_0(\mathcal{G})}.$$

Moreover,

$$\begin{aligned}\mathbb{E}\left[\langle a,x(t,\tau_1)\rangle_{\mathcal{G}}\langle b,x(s,\tau_2)\rangle_{\mathcal{G}}\right] &= \mathbb{E}\left[\langle u_{a,\tau_1},x(t,\cdot)\rangle\langle u_{b,\tau_2},x(s,\cdot)\rangle\right]\\ &= (t\wedge s)\langle u_{a,\tau_1},u_{b,\tau_2}\rangle_{H_0(\mathcal{G})}\\ &= (t\wedge s)G(\tau_1,\tau_2)\langle a,b\rangle_{\mathcal{G}}, \end{aligned}\tag{4.1}$$

see e.g. Driver [4] and Fang [7], i.e. for all $\tau\in(0,1)$, $(x(t,\tau))_{t\in\mathbb{R}_+}$ is a $\mathcal{G}$-valued Brownian motion with variance $G(\tau,\tau)=\tau(1-\tau)$. The $\mathcal{L}(G)$-valued Brownian motion was constructed by Malliavin [17], see also Driver [4], as the unique $\mathcal{L}(G)$-valued continuous adapted process $(\gamma(t))_{t\in\mathbb{R}_+}$ such that for all $\tau \in (0,1)$, $\gamma(t,\tau) = \gamma(t)(\tau)$ satisfies the Stratonovich SDE with parameter τ:

$$d_t\gamma(t,\tau) = \gamma(t,\tau)\circ d_t x(t,\tau), \quad \gamma(0,\tau) = e. \tag{4.2}$$

Moreover, $(t,\tau) \mapsto \gamma(t,\tau)$ is continuous on $[0,\infty)\times[0,1]$. The law of $\gamma(T,\cdot)$ on $\mathcal{L}(G)$ is called the heat kernel measure on $\mathcal{L}(G)$. The Cameron-Martin space $H_0(\mathcal{G})$ is a Lie algebra under the bracket

$$[h,k](\tau) = [h(\tau),k(\tau)], \qquad \tau \in [0,1], \quad h,k \in H_0(\mathcal{G}),$$

and the metric on $H_0(\mathcal{G})$ determines the Levi-Civita connection

$$(\nabla_h k)(\tau) = \int_0^\tau [h(\theta),\dot{k}(\theta)]d\theta - \tau\int_0^1 [h(\theta),\dot{k}(\theta)]d\theta, \qquad \tau \in [0,1],$$

which satisfies

$$\langle \nabla_h k, z\rangle_{H_0(\mathcal{G})} = \frac{1}{2}\Big(\langle [h,k],z\rangle_{H_0(\mathcal{G})} - \langle [h,z],k\rangle_{H_0(\mathcal{G})} - \langle [k,z],h\rangle_{H_0(\mathcal{G})}\Big), \qquad h,k,z \in H_0(\mathcal{G}).$$

As pointed out by Freed [9] the curvature relative to the Levi-Civita connection is not trace class, nevertheless the trace of the curvature tensor can be computed for some suitable orthonormal bases of $H_0(\mathcal{G})$, and leads to the definition of the Ricci curvature on loop groups as a series converging in $H_0(\mathcal{G})$, cf. Freed [9], Driver and Lohrenz [5].

Definition 4.1 *Given $e_1,\ldots,e_d$ an orthonormal basis of $\mathcal{G}$ and $(c_n)_{n\in\mathbb{N}}$ an orthonormal basis of $H_0(\mathbb{R})$, let*

$$\mathrm{Ric}_{H_0(\mathcal{G})}h = \sum_{n=0}^{\infty}\sum_{i=1}^{d}([\nabla_h,\nabla_{c_n\otimes e_i}] - \nabla_{[h,c_n\otimes e_i]})(c_n\otimes e_i).$$

Let us recall the definition of the stochastic parallel transport on $\mathcal{L}(G)$ introduced by Driver [4].

Theorem 4.2 *Let $h_0 \in H_0(\mathcal{G})$. The equation*

$$dh(t) + \nabla_{\circ dx(t)}h(t) = 0, \quad h(0) = h_0, \tag{4.3}$$

admits a unique solution $h(t)$ in $H_0(\mathcal{G})$, such that

$$\langle h(t),h(t)\rangle_{H_0(\mathcal{G})} = \langle h_0,h_0\rangle_{H_0(\mathcal{G})}, \qquad t \geqslant 0.$$

Moreover, for all $t>0$, the mapping

$$\begin{aligned} U_t : &H_0(\mathcal{G}) \to H_0(\mathcal{G}) \\ &h_0 \quad \mapsto U_t h_0 = h(t) \end{aligned}$$

belongs P-a.s. to the group $O(H_0(\mathcal{G}))$ of unitary operators on $H_0(\mathcal{G})$.

For $h \in \mathcal{G}$ and $g \in G$, we define

$$(L_{g*}h)f = \frac{d}{d\varepsilon} f(g \exp(\varepsilon h))\Big|_{\varepsilon=0}, \qquad f \in \mathcal{C}^\infty(G),$$

and for convenience of notation gh denotes $L_{g*}h$. Let $\mathcal{F}C^\infty(\mathbb{P}(\mathcal{L}(G)))$ denote the set of cylinder functions

$$F : \mathbb{P}(\mathcal{L}(G)) \longrightarrow \mathbb{R}$$

of the form

$$F(\gamma) = f(\gamma(t_1, \tau_1), \ldots, \gamma(t_1, \tau_n), \ldots, \gamma(t_m, \tau_1), \ldots, \gamma(t_m, \tau_n)), \tag{4.4}$$

$f \in C^\infty(G^{n\times m})$, $0 < \tau_1 < \cdots < \tau_n < 1$, $0 \leqslant t_1 < t_2 < \cdots < t_m \leqslant T$.

Set also, for $F \in \mathcal{F}C^\infty(\mathbb{P}(\mathcal{L}(G)))$ as in (4.4):

$$(\nabla_i^{\mathcal{L}(G)} F)(\tau) = \sum_{j=1}^n G(\tau_j, \tau)\gamma^{-1}(t_i, \tau_j)\nabla_{ij}^G f(\gamma(t_1, \tau_1), \ldots, \gamma(t_1, \tau_n), \ldots, \gamma(t_m, \tau_1), \ldots, \gamma(t_m, \tau_n)),$$

$i = 1, \ldots, n$. Let us recall the definition of adapted vector fields on $\mathbb{P}(\mathcal{L}(G))$.

Definition 4.3 *The set of square-integrable adapted vector fields on $\mathbb{P}(\mathcal{L}(G))$ is defined as the set $\mathcal{A}_a(\mathbb{P}(\mathcal{L}(G)))$ of adapted $H_0(\mathcal{G})$-valued processes satisfying*

$$\mathbb{E}\left[\int_0^T |\dot{z}(t)|^2_{H_0(\mathcal{G})} dt\right] < +\infty,$$

where $\dot{z}(t,\tau) = \frac{d}{dt} z(t,\tau)$, $\tau \in [0,1]$.

Let $H(H_0(\mathcal{G}))$ denote the set of mappings $z : [0,T] \mapsto H_0(\mathcal{G})$ such that

$$\|z\|^2_{H(H_0(\mathcal{G}))} = \int_0^T |\dot{z}(t)|^2_{H_0(\mathcal{G})} dt < +\infty.$$

The intrinsic gradient operator $D : \mathcal{F}C^\infty(\mathbb{P}(\mathcal{L}(G))) \longrightarrow L^2([0,T], H_0(\mathcal{G}))$ is defined as

$$D_t F = \sum_{i=1}^m 1_{[0,t_i]}(t) U_{t_i}^* \nabla_i^{\mathcal{L}(G)} F, \qquad 0 \leqslant t \leqslant T,$$

with $F \in \mathcal{F}C^\infty(\mathbb{P}(\mathcal{L}(G)))$ written as in (4.4). Next, we introduce the damped gradient on $\mathbb{P}(\mathcal{L}(G))$. Consider the resolvent equation in the Banach space $\mathcal{B}(H_0(\mathcal{G}))$ of bounded operators (with the endomorphism norm) on $H_0(\mathcal{G})$:

$$\frac{dQ_{t,s}}{dt} = -\frac{1}{2}(U_t^{-1} \mathrm{Ric}_{H_0(\mathcal{G})} U_t) Q_{t,s}, \qquad t > s, \quad Q_{s,s} = \mathrm{Id}_{H_0(\mathcal{G})},$$

and let $Q^*_{t,s}$ (resp. U_t^*) denote the adjoint of $Q_{t,s}$ (resp. U_t), $0 \leqslant s \leqslant t$.

Definition 4.4 *The damped gradient operator* $\tilde{D} : \mathcal{F}C^\infty(\mathbb{P}(\mathcal{L}(G))) \to L^2([0,T], H_0(\mathcal{G}))$ *is defined as*

$$\tilde{D}_t F = \sum_{i=1}^m 1_{[0,t_i]}(t) Q^*_{t_i,t} U^*_{t_i} \nabla_i^{\mathcal{L}(G)} F, \qquad t \in \mathbb{R}_+,$$

with $F \in \mathcal{F}C^\infty(\mathbb{P}(\mathcal{L}(G)))$ *written as in* (4.4).

Note that for $F \in \mathcal{F}C^\infty(\mathbb{P}(\mathcal{L}(G)))$ and $z \in \mathcal{A}_a(\mathbb{P}(\mathcal{L}(G)))$, we have

$$\int_0^T \langle D_t F, \dot{z}(t)\rangle_{H_0(\mathcal{G})} dt = \frac{d}{d\varepsilon} F(\gamma \exp(\varepsilon U z))\Big|_{\varepsilon=0},$$

and

$$\int_0^T \langle \tilde{D}_t F, \dot{z}(t)\rangle_{H_0(\mathcal{G})} dt = \int_0^T \langle D_t F, \dot{\overline{Qz}}(t)\rangle_{H_0(\mathcal{G})} dt, \tag{4.5}$$

where $(Qz)(t) = \left(\int_0^\tau Q_{t,s}\dot{z}(s)ds\right)_{\tau \in \mathbb{R}_+}$, $t \in \mathbb{R}_+$, belongs to $\mathcal{A}_a(\mathbb{P}(\mathcal{L}(G)))$.

Lemma 4.5 *We have*

$$\tilde{D}_t F = D_t F + \int_t^T \frac{d}{ds} Q^*_{s,t} D_s F ds,$$

where $Q^*_{s,t}$ *denotes the adjoint of* $Q_{s,t}$.

Proof. For any $z \in H(H_0(\mathcal{G}))$ we have

$$\begin{aligned}
\int_0^T \langle \tilde{D}_t F, \dot{z}(t)\rangle_{H_0(\mathcal{G})} dt &= \int_0^T \left\langle D_t F, \dot{\overline{Qz}}(t)\right\rangle_{H_0(\mathcal{G})} dt \\
&= \int_0^T \langle D_t F, \dot{z}(t)\rangle_{H_0(\mathcal{G})} dt + \int_0^T \left\langle D_t F, \int_0^t \frac{d}{dt} Q_{t,s}\dot{z}(s) ds \right\rangle_{H_0(\mathcal{G})} dt \\
&= \int_0^T \langle D_t F, \dot{z}(t)\rangle_{H_0(\mathcal{G})} dt + \int_0^T \int_s^T \left\langle D_t F, \frac{d}{dt} Q_{t,s}\dot{z}(s) \right\rangle_{H_0(\mathcal{G})} dt ds \\
&= \int_0^T \langle D_t F, \dot{z}(t)\rangle_{H_0(\mathcal{G})} dt + \int_0^T \left\langle \int_t^T \frac{d}{ds} Q^*_{s,t} D_s F ds, \dot{z}(t) \right\rangle_{H_0(\mathcal{G})} dt,
\end{aligned}$$

which gives the result. $\square$

We recall the integration by parts formula of Driver [4] and Fang [6] on $\mathbb{P}(\mathcal{L}(G))$.

Theorem 4.6 *Let* $F \in \mathcal{F}C^\infty(\mathbb{P}(\mathcal{L}(G)))$ *with the form* (4.4), *and* $z \in \mathcal{A}_a(\mathbb{P}(\mathcal{L}(G)))$. *Then,*

$$\mathbb{E}[D_z F] = \mathbb{E}\left[F \int_0^T \langle U_t \dot{z}(t) + \frac{1}{2}\mathrm{Ric}_{H_0(\mathcal{G})} U_t z(t), dx(t)\rangle\right]. \tag{4.6}$$

The integration by parts formula for the damped gradient follows from Theorem 4.6:

Theorem 4.7 *Under the same assumptions as in Theorem 4.6 we have*

$$\mathbb{E}\left[\int_0^T \langle \tilde{D}_t F, \dot{z}(t)\rangle_{H_0(\mathcal{G})} dt\right] = \mathbb{E}\left[F \int_0^T \langle U_t \dot{z}(t), dx(t)\rangle_{H_0(\mathcal{G})}\right].$$

Proof. By definition of the damped gradient $\tilde{D}$ and (4.6) we have

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T \langle \tilde{D}_t F, \dot{z}(t)\rangle_{H_0(\mathcal{G})} dt\right] = \mathbb{E}\left[\int_0^T \langle D_t F, \dot{\overline{Qz}}(t)\rangle_{H_0(\mathcal{G})} dt\right] \\
&= \mathbb{E}\left[F \int_0^T \left\langle U_t \dot{\overline{Qz}}(t) + \frac{1}{2}\mathrm{Ric}_{H_0(\mathcal{G})} U_t (Qz)(t), dx(t)\right\rangle\right] \\
&= \mathbb{E}\left[F \int_0^T \left\langle U_t \dot{z}(t) - \frac{1}{2}\mathrm{Ric}_{H_0(\mathcal{G})} U_t \int_0^t Q_{\tau,s}\dot{z}(s)ds + \frac{1}{2}\mathrm{Ric}_{H_0(\mathcal{G})} U_t (Qz)(t), dx(t)\right\rangle\right] \\
&= \mathbb{E}\left[F \int_0^T \langle U_t \dot{z}(t), dx(t)\rangle\right].
\end{aligned}$$

□

From this formula we recover the Clark-Ocone formula originally obtained by Fang [7]:

Theorem 4.8 *([7]) We have for $F \in \mathcal{F}C^\infty(\mathbb{P}(\mathcal{L}(G)))$:*

$$F = \mathbb{E}[F] + \int_0^T \langle \mathbb{E}[U_t \tilde{D}_t F | \mathcal{F}_t], dx(t)\rangle.$$

Proof. By Lemma 4.1 of Fang [7], there is a unique $H_0(\mathcal{G})$-valued predictable process α_t such that

$$F(\gamma) = \mathbb{E}[F] + \int_0^T \langle \alpha_t, dx(t)\rangle.$$

Thus from Theorem 4.7 we have

$$\begin{aligned}
\mathbb{E}\left[\int_0^T \langle \tilde{D}_t F, \dot{z}(t)\rangle_{H_0(\mathcal{G})} dt\right] &= \mathbb{E}\left[F \int_0^T \langle U_t \dot{z}(t), dx(t)\rangle\right] \\
&= \mathbb{E}\left[\int_0^T \langle \alpha_t, dx(t)\rangle \int_0^T \langle U_t \dot{z}(t), dx(t)\rangle\right] \\
&= \mathbb{E}\left[\int_0^T \langle \alpha_t, U_t \dot{z}(t)\rangle_{H_0(\mathcal{G})} dt\right] \\
&= \mathbb{E}\left[\int_0^T \langle U^*(t)\alpha_t, \dot{z}(t)\rangle_{H_0(\mathcal{G})} dt\right],
\end{aligned}$$

which yields the desired representation. □

Theorem 4.7 also shows that the damped gradient $\tilde{D}$ is closable. We will denote its domain by $\mathrm{Dom}\,(\tilde{D})$.

5 Deviation inequalities for functionals on loop groups

Let $\mathrm{Cov}(F,G) = \mathbb{E}[FG] - \mathbb{E}[F]\mathbb{E}[G]$, for $F, G \in L^2(\mathbb{P}(\mathcal{L}(G)), \nu)$, where ν denotes the law of $(\gamma(t))_{t\in[0,T]}$ on $\mathbb{P}(\mathcal{L}(G))$. The following covariance identity is an immediate consequence of the Clark-Ocone formula (Theorem 4.8).

Proposition 5.1 *Let $F, G \in \mathrm{Dom}\,(\tilde{D})$, then*

$$\mathrm{Cov}(F,G) = \mathbb{E}\left[\int_0^T \left\langle \tilde{D}_t F, \mathbb{E}\,[\tilde{D}_t G | \mathcal{F}_t]\right\rangle_{H_0(\mathcal{G})} dt\right]. \tag{5.1}$$

Next, we apply this covariance representation to prove a concentration inequality on path spaces over loop groups, following the method used by Houdré and Privault [11] in the case of Riemannian manifolds. Let

$$\mathbf{H} = L^\infty(\mathbb{P}(\mathcal{L}(G)), L^2([0,T], H_0(\mathcal{G})))$$

and

$$\mathbb{H} = L^2([0,T], L^\infty(\mathbb{P}(\mathcal{L}(G)), H_0(\mathcal{G}))).$$

Lemma 5.2 *Let $F \in \mathrm{Dom}\,(\tilde{D})$. We have $\|\tilde{D}F\|_{\mathbf{H}} \leqslant \|\tilde{D}F\|_{\mathbb{H}}$ and*

$$\nu(F - \mathbb{E}\,[F] \geqslant y) \leqslant \exp\left(-\frac{y^2}{2\|\tilde{D}F\|_{\mathbb{H}}\|\tilde{D}F\|_{\mathbf{H}}}\right), \quad y \geqslant 0. \tag{5.2}$$

In particular, $\mathbb{E}\,[\exp(\lambda F^2)] < \infty$, for $\lambda < (2\|\tilde{D}F\|_{\mathbb{H}}\|\tilde{D}F\|_{\mathbf{H}})^{-1}$.

Proof. We first consider a bounded random variable $F \in \mathrm{Dom}\,(\tilde{D})$. The general case follows by approximating $F \in \mathrm{Dom}\,(\tilde{D})$ by the sequence $(\max(-n, \min(F, n)))_{n \geqslant 1}$. Let

$$\eta_F(t) = \mathbb{E}\,[\tilde{D}_t F | \mathcal{F}_t], \quad t \in [0,T].$$

Assuming first that $\mathbb{E}\,[F] = 0$, we have

$$\begin{aligned}
\mathbb{E}\,[F \exp(sF)] &= \mathbb{E}\left[\int_0^T \left\langle \tilde{D}_u \exp(sF), \eta_F(u)\right\rangle_{H_0(\mathcal{G})} du\right] \\
&= s\mathbb{E}\left[\exp(sF) \int_0^T \left\langle \tilde{D}_u F, \eta_F(u)\right\rangle_{H_0(\mathcal{G})} du\right] \\
&\leqslant s\mathbb{E}\left[\exp(sF)\|\tilde{D}F\|_{H(H_0(\mathcal{G}))}\ \|\eta_F\|_{H(H_0(\mathcal{G}))}\right] \\
&\leqslant s\|\tilde{D}F\|_{\mathbf{H}}\|\tilde{D}F\|_{\mathbb{H}}\mathbb{E}\,[\exp(sF)].
\end{aligned}$$

In the general case, letting $L(s) = \mathbb{E}\,[\exp(s(F - \mathbb{E}\,[F]))]$ we obtain:

$$\begin{aligned}
\log \mathbb{E}\,[\exp(t(F - \mathbb{E}\,[F]))] &= \int_0^t \frac{L'(s)}{L(s)} ds \leqslant \int_0^t \frac{\mathbb{E}\,[(F - \mathbb{E}\,[F]) \exp(s(F - \mathbb{E}\,[F]))]}{\mathbb{E}\,[\exp(s(F - \mathbb{E}\,[F]))]} ds \\
&= \frac{1}{2} t^2 \|\tilde{D}F\|_{\mathbf{H}}\|\tilde{D}F\|_{\mathbb{H}}, \quad 0 \leqslant t \leqslant T.
\end{aligned}$$

We now have for all $y \in \mathbb{R}_+$ and $t \in [0,T]$:

$$\nu(F - \mathbb{E}\,[F] \geqslant y) \leqslant \exp(-ty)\mathbb{E}\,\left[\exp(t(F - \mathbb{E}\,[F]))\right] \leqslant \exp\left(\frac{1}{2} t^2 \|\tilde{D}F\|_{\mathbf{H}}\|\tilde{D}F\|_{\mathbb{H}} - ty\right),$$

which yields (5.2) after minimization in $t \in [0,T]$. □

The following lemma allows us to compare the norms of the intrinsic and damped gradients.

Lemma 5.3 *Let* $K = \|\mathrm{Ric}_{H_0(\mathcal{G})}\|_{\mathrm{op}} < +\infty$. *For all* $F \in \mathrm{Dom}\,(D)$ *we have*

$$\|\tilde{D}F\|^2_{L^2([0,T],H_0(\mathcal{G}))} \leqslant \exp(KT)\|DF\|^2_{L^2([0,T],H_0(\mathcal{G}))}, \qquad a.s.,$$

and

$$\|\tilde{D}F\|_{\mathbb{H}} \leqslant \|DF\|_{\mathbb{H}} + (\exp(KT/2)-1)\|DF\|_{\mathbf{H}}, \qquad a.s.$$

Proof. Note that from Driver [4] and Fang [6] we have $K = \|\mathrm{Ric}_{H_0(\mathcal{G})}\|_{\mathrm{op}} < +\infty$. By (4.5), we have

$$\begin{aligned}
|\tilde{D}_tF|^2_{H_0(\mathcal{G})} &\leqslant \left(|D_tF|_{H_0(\mathcal{G})} + \int_t^T \left\|\frac{d}{ds}Q^*_{t,s}\right\|_{\mathrm{op}} |D_sF|_{H_0(\mathcal{G})}ds\right)^2 \\
&\leqslant \left(|D_tF|_{H_0(\mathcal{G})} + \frac{1}{2}K\int_t^T \exp(K(s-t)/2)|D_tF|_{H_0(\mathcal{G})}ds\right)^2 \\
&\leqslant \left(|D_tF|_{H_0(\mathcal{G})} + \frac{1}{2}\sqrt{K(\exp(K(T-t))-1)}\|DF\|_{L^2([0,T],H_0(\mathcal{G}))}\right)^2 \\
&\leqslant |D_tF|^2_{H_0(\mathcal{G})} + \sqrt{K(\exp(K(T-t))-1)}|D_tF|_{H_0(\mathcal{G})}\|DF\|_{L^2([0,T],H_0(\mathcal{G}))} \\
&\quad + \frac{1}{4}K(\exp(K(T-t))-1)\|DF\|^2_{L^2([0,T],H_0(\mathcal{G}))},
\end{aligned}$$

which implies

$$\begin{aligned}
\|\tilde{D}F\|^2_{L^2([0,T],H_0(\mathcal{G}))} &\leqslant \left(1 + \frac{1}{2}\sqrt{\exp(KT)-KT-1}\right)^2 \|DF\|^2_{L^2([0,T],H_0(\mathcal{G}))} \\
&\leqslant \exp(KT)\|DF\|^2_{L^2([0,T],H_0(\mathcal{G}))},
\end{aligned}$$

and

$$\begin{aligned}
\|\tilde{D}F\|_{\mathbb{H}} &\leqslant \|DF\|_{\mathbb{H}} + \frac{1}{2}\sqrt{\exp(KT)-KT-1}\|DF\|_{\mathbf{H}} \\
&\leqslant \|DF\|_{\mathbb{H}} + (\exp(KT/2)-1)\|DF\|_{\mathbf{H}}.
\end{aligned}$$

□

As a consequence of Lemma 5.2 and Lemma 5.3 we have:

Theorem 5.4 *Given* $F \in \mathrm{Dom}\,(D)$ *we have for all* $x > 0$*:*

$$\begin{aligned}
\nu(F - \mathbb{E}[F] \geqslant x) &\leqslant \exp\left(-\frac{x^2}{2\exp(KT/2)(\|DF\|_{\mathbb{H}} + (\exp(KT/2)-1)\|DF\|_{\mathbf{H}})\|DF\|_{\mathbf{H}}}\right) \\
&\leqslant \exp\left(-\frac{x^2}{2\exp(KT)\|DF\|_{\mathbb{H}}\|DF\|_{\mathbf{H}}}\right).
\end{aligned}$$

6 Tail estimate for the supremum of Brownian motion on loop groups

Using the Clark-Ocone formula (Theorem 4.8) we now prove the following tail estimate for the supremum of $\gamma(t)$.

Theorem 6.1 *For all $T > 0$ and $y \geqslant 0$ we have*

$$P\left(\sup_{0\leqslant t\leqslant T} \rho^{\mathcal{L}(G)}(\gamma(t), e) \geqslant y + \mathbb{E}\left[\sup_{0\leqslant t\leqslant T} \rho^{\mathcal{L}(G)}(\gamma(t), e)\right]\right) \leqslant \exp\left(-\frac{2y^2}{T\exp(KT)}\right), \tag{6.1}$$

where $K = \|\mathrm{Ric}_{H_0(\mathcal{G})}\|_{\mathrm{op}}$.

Proof. First, recall that $K < +\infty$, cf. Driver [4] and Fang [6]. Let

$$f(g_{11}, \ldots, g_{1n}, \ldots, g_{m1}, \ldots, g_{mn}) = \max_{1\leqslant i\leqslant m} \max_{1\leqslant j\leqslant n} \rho^G(g_{ij}, e)$$

and

$$\begin{aligned} F(\gamma) &= f(\gamma(t_1, \tau_1), \ldots, \gamma(t_1, \tau_n), \ldots, \gamma(t_m, \tau_1), \ldots, \gamma(t_m, \tau_n)) \\ &= \max_{1\leqslant i\leqslant m} \max_{1\leqslant j\leqslant n} \rho^G(\gamma(t_i, \tau_j), e), \end{aligned}$$

where $\{(t_i, \tau_j)\ ,\ i = 1, \ldots, m,\ j = 1, \ldots, n\}$ denotes an arbitrary finite subset of $[0, T] \times [0, 1]$. Letting

$$A_{11} = \{\gamma \in \mathbb{P}(\mathcal{L}(G))\ :\ F(\gamma) = \rho^G(\gamma(t_1, \tau_1), e)\},$$

and

$$A_{ij} = \{\gamma \in \mathbb{P}(\mathcal{L}(G))\ :\ F(\gamma) = \rho^G(\gamma(t_i, \tau_j), e),\ F(\gamma) \neq \rho^G(\gamma(t_k, \tau_l), e),\ km + l < im + j\},$$

$1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant n$, we get as in Ledoux [16], p. 196, or Nualart [18], p. 92, a partition $\{A_{ij}\ :\ 1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant n\}$ of $\mathbb{P}(\mathcal{L}(G))$ such that from Rademacher's theorem,

$$|(\nabla^G_{ij} f)(\gamma(t_1, \tau_1), \ldots, \gamma(t_1, \tau_n), \ldots, \gamma(t_m, \tau_1), \ldots, \gamma(t_m, \tau_n))| \leqslant 1_{A_{ij}}(\gamma),$$

$1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant n$. We have $\|Q^*_{t_i,t}\|_{\mathrm{op}} \leqslant \exp(K(t_i - t)/2)$, thus, since $|G(\tau_j, \cdot)|_{H_0(\mathcal{G})} \leqslant 1/2$, we have from the definition of D:

$$\begin{aligned} |D_t F(\gamma)|_{H_0(\mathcal{G})} &\leqslant \sum_{i=1}^m 1_{[0,t_i]}(t) |U^*_{t_i} \nabla^{\mathcal{L}(G)}_i F|_{H_0(\mathcal{G})} \\ &\leqslant \sum_{i=1}^m 1_{(t_{i-1},t_i]}(t) \sum_{k=i}^m |\nabla^{\mathcal{L}(G)}_k F|_{H_0(\mathcal{G})} \\ &\leqslant \frac{1}{2} \sum_{i=1}^m 1_{(t_{i-1},t_i]}(t) \sum_{k=i}^m \sum_{j=1}^n 1_{A_k j} \\ &\leqslant \frac{1}{2}, \qquad t \geqslant 0. \end{aligned}$$

At this point we may conclude by an application of Theorem 5.4. An alternative argument consists in considering the continuous martingale

$$N_t = \int_0^t \langle \mathbb{E}\,[U_s \tilde{D}_s F | \mathcal{F}_s], dx(s) \rangle, \qquad 0 \leqslant t \leqslant T,$$

with angle bracket

$$\langle N \rangle_t = \int_0^t |\mathbb{E}\,[U_s \tilde{D}_s F | \mathcal{F}_s]|^2_{H_0(\mathcal{G})} ds \leqslant \int_0^t \mathbb{E}\,[|\tilde{D}_s F|^2_{H_0(\mathcal{G})} | \mathcal{F}_s] ds \leqslant t \exp(KT)/4 =: C_t.$$

By change of clock, $\beta_t = N_{\langle N \rangle_t^{-1}}$ is a standard Brownian motion with respect to $(\mathcal{F}_{\langle N \rangle_t^{-1}})_{t \in [0,T]}$. Moreover by Theorem 4.8 we have $F = \mathbb{E}\,[F] + N_T$, hence from the reflection principle of Brownian motion (e.g. Proposition 3.7 in Revuz and Yor [19]) we get

$$\begin{aligned} P\left(F \geqslant \mathbb{E}\,[F] + y\right) &= P(N_T \geqslant y) \\ &= P(\beta_{\langle N \rangle_T} \geqslant y) \\ &\leqslant P\Big(\sup_{0 \leqslant s \leqslant C_T} \beta_s \geqslant y \Big) \\ &= 2P(\beta_{C_T} \geqslant y) \\ &\leqslant \exp\Big(- \frac{y^2}{2C_T} \Big) \\ &= \exp\Big(- \frac{2y^2}{T \exp(KT)} \Big), \qquad y \geqslant 0, \end{aligned}$$

which yields (6.1) by monotone convergence as the mesh of the partition goes to zero.

□

In view of applications it is important to state a deviation bound not involving the expectation of the supremum distance, as in Theorem 2.2 above.

Corollary 6.2 *For all $T \in (0,1)$ we have*

$$P\left(\sup_{0 \leqslant t \leqslant T} \rho^{\mathcal{L}(G)}(\gamma(t), e) \geqslant y \right) \leqslant \exp\left(- \frac{2\left(y - C_0\sqrt{T}\right)^2}{T \exp(KT)} \right), \qquad y \geqslant C_0 \sqrt{T}. \qquad (6.2)$$

Proof. This result follows from Theorem 6.1 above and from Lemma 6.3 below. □

The following lemma relies essentially on the compactness assumption made on G.

Lemma 6.3 *We have for $T \in (0,1)$:*

$$\mathbb{E}\left[\sup_{0 \leqslant t \leqslant T} \rho^{\mathcal{L}(G)}(\gamma(t), e) \right] \leqslant C_0 \sqrt{T}, \qquad (6.3)$$

where the constant C_0 is independent of T.

Proof. For fixed $\tau, \tau' \in [0,1]$, set $u(t) = \gamma(t,\tau)\gamma^{-1}(t,\tau')$, $t > 0$. Then $u(t)$ satisfies

$$du(t) = u(t) \circ dB_t,$$

where $B_t = \int_0^t \mathrm{Ad}_{\gamma(s,\tau')} \circ (d_s x(s,\tau) - d_s x(s,\tau'))$, $t \in \mathbb{R}_+$, is a Brownian motion on $\mathcal{G}$ with variance

$$F(\tau,\tau') = G(\tau,\tau) + G(\tau',\tau') - 2G(\tau,\tau'),$$

cf. Driver [4], pp. 486-488. For $f \in C^\infty(G)$ with $f(e) = 0$ we have

$$\begin{aligned} f(u(t)) &= \sum_{k=1}^d \int_0^t (\tilde{e}_k f)(u(s)) \circ dB_k(s) \\ &= \sum_{k=1}^d \int_0^t (\tilde{e}_k f)(u(s)) dB_k(s) + \frac{1}{2} F(\tau,\tau') \int_0^t (\Delta^G f)(u(s)) ds, \end{aligned}$$

where $\tilde{e}_k$ denotes the left invariant vector field induced by e_k, $B_k(t) = \langle e_k, B(t) \rangle_{\mathcal{G}}$ and $\Delta^G f = \sum_{k=1}^d \tilde{e}_k^2 f$. Therefore, by Doob's maximal inequality we have for some constant C independent of T:

$$\begin{aligned} \mathbb{E}\left[\sup_{0 \leqslant t \leqslant T} |f(u(t))|^4\right] &\leqslant C\left(F(\tau,\tau')\mathbb{E}\left[\int_0^T |f'(u(s))|^2 ds\right]\right)^2 \\ &\quad + C\left(F(\tau,\tau')\mathbb{E}\left[\int_0^T (\Delta^G f)(u(s)) ds\right]\right)^4 \\ &\leqslant C\Big(|TF(\tau,\tau')|^2 + |TF(\tau,\tau')|^4\Big) \\ &\leqslant C\Big(T^2|\tau-\tau'|^2 + T^4|\tau-\tau'|^4\Big) \\ &\leqslant C(T^2 + T^4)|\tau-\tau'|^2, \end{aligned}$$

where we used the bound $F(\tau,\tau') \leqslant 4|\tau - \tau'|$. As in Driver [4], p. 486, this implies

$$\mathbb{E}\left[\sup_{0 \leqslant t \leqslant T} |\rho^G(\gamma(t,\tau),\gamma(t,\tau'))|^4\right] \leqslant C(T^2 + T^4)|\tau-\tau'|^2.$$

A slight modification of the proof of the generalized Banach-valued Kolmogorov theorem, cf. Theorem 2.1, p. 26 of Revuz and Yor [19], or Theorem 3.23, p. 57 of [15], applied to the process $\tau \mapsto (\gamma(t,\tau))_{t \in [0,T]}$ with the distance $\sup_{t \in [0,T]} \rho^G(\gamma(t,\tau),\gamma(t,\tau'))$, $0 \leqslant \tau, \tau' \leqslant 1$, shows that for every $\alpha \in (0, 1/4)$,

$$\begin{aligned} \mathbb{E}\left[\sup_{0 \leqslant t \leqslant T} \rho^{\mathcal{L}(G)}(\gamma(t), e)\right] &\leqslant \left(\mathbb{E}\left[\sup_{0 \leqslant t \leqslant T} |\rho^{\mathcal{L}(G)}(\gamma(t), e)|^4\right]\right)^{1/4} \\ &\leqslant \left(\mathbb{E}\left[\sup_{0 \leqslant t \leqslant T,\ 0 \leqslant \tau < \tau' \leqslant 1} |\rho^G(\gamma(t,\tau),\gamma(t,\tau'))|^4\right]\right)^{1/4} \\ &\leqslant \left(\mathbb{E}\left[\sup_{0 \leqslant t \leqslant T,\ 0 \leqslant \tau < \tau' \leqslant 1} \frac{|\rho^G(\gamma(t,\tau),\gamma(t,\tau'))|^4}{|\tau-\tau'|^{4\alpha}}\right]\right)^{1/4} \\ &\leqslant C_1(T^2 + T^4)^{1/4}, \end{aligned}$$

where C_1 is a constant independent of $T \in (0,1)$. □

7 Law of iterated logarithm on loop groups

In this section, a law of iterated logarithm on loop groups is proved by combining the upper bound obtained from Theorem 6.1 with a finite dimensional argument for the lower bound.

Theorem 7.1 *We have*

$$\limsup_{t\downarrow 0}\frac{\rho^{\mathcal{L}(G)}(\gamma(t),e)}{\sqrt{(t\log\log t^{-1})/2}}=1,\qquad a.s.$$

Let $\varphi(t)=\left(t\log\log t^{-1}\right)/2$.

Upper bound proof of Theorem 7.1

Fix two positive real numbers $\varepsilon,\alpha\in(0,1)$, and set

$$A_n=\Big\{\sup_{0\leqslant t\leqslant\alpha^n}\rho^{\mathcal{L}(G)}(\gamma(t),e)\geqslant\sqrt{(1+\varepsilon)\varphi(\alpha^n)}\Big\},\qquad n\geqslant 1.$$

For sufficiently large $n>N$, we have from Corollary 6.2:

$$\begin{aligned}
P(A_n) &\leqslant \exp\left(-\frac{2\left(\sqrt{(1+\varepsilon)\varphi(\alpha^n)}-C_0\sqrt{\alpha^n}\right)^2}{\alpha^n\exp(K\alpha^n)}\right)\\
&= \exp\left(-\frac{\left(\sqrt{(1+\varepsilon)\log\log\alpha^{-n}}-\sqrt{2}C_0\right)^2}{\exp(K\alpha^n)}\right)\\
&= \exp\left(-\frac{(1+\varepsilon)\log\log\alpha^{-n}-2\sqrt{2}C_0\sqrt{(1+\varepsilon)\log\log\alpha^{-n}}+2C_0^2}{\exp(K\alpha^n)}\right)\\
&\leqslant \exp\left(-(1+\varepsilon/2)\log\log\alpha^{-n}\right)\\
&= \frac{1}{(n\log\alpha^{-1})^{1+\varepsilon/2}},
\end{aligned}$$

hence

$$\sum_{n=N}^{+\infty}P(A_n)<+\infty.$$

By the Borel-Cantelli Lemma there is an event $\Omega_{\varepsilon,\alpha}$ with $P(\Omega_{\varepsilon,\alpha})=1$ such that for every $\omega\in\Omega_{\varepsilon,\alpha}$, and $n>N(\omega)$ sufficiently large,

$$\sup_{0\leqslant t\leqslant\alpha^n}\rho^{\mathcal{L}(\mathcal{G})}(\gamma(t),m_0)\leqslant\sqrt{(1+\varepsilon)\varphi(\alpha^n)}.$$

For $0<t<\alpha$, set $n=[\frac{\log t}{\log\alpha}]$, then

$$t\leqslant\alpha^n\leqslant\frac{t}{\alpha},$$

and for small t we have

$$\varphi(\alpha^n) \leqslant \varphi(t/\alpha) \leqslant \alpha^{-1}\varphi(t).$$

Thus,

$$\rho^{\mathcal{L}(\mathcal{G})}(\gamma(t), m_0) \leqslant \sup_{0\leqslant t\leqslant \alpha^n} \rho^{\mathcal{L}(\mathcal{G})}(\gamma(t), m_0) \leqslant \sqrt{(1+\varepsilon)\varphi(\alpha^n)} \leqslant \sqrt{(1+\varepsilon)\alpha^{-1}\varphi(t)},$$

hence

$$\limsup_{t\downarrow 0} \frac{\rho^{\mathcal{L}(\mathcal{G})}(\gamma(t), m_0)}{\sqrt{\varphi(t)}} \leqslant \sqrt{(1+\varepsilon)\alpha^{-1}}.$$

Finally, letting $\varepsilon \downarrow 0$ and $\alpha \uparrow 1$ yields the desired result.

Lower bound proof of Theorem 7.1

Clearly, from (4.1), given any $\tau \in (0, 1)$, $(x(t, \tau))_{t\in\mathbb{R}_+}$ is a standard Brownian motion on $\mathcal{G}$ with respect to the modified inner product

$$\langle\cdot,\cdot\rangle_{\mathcal{G}}^{\tau} = \frac{1}{\tau(1-\tau)}\langle\cdot,\cdot\rangle_{\mathcal{G}},$$

hence $(\gamma(t, \tau))_{t\in\mathbb{R}_+}$ is also a Brownian motion on G. Let $\rho_\tau^G(\cdot,\cdot)$ denote the associated Riemannian distance, i.e.

$$\rho_\tau^G(\cdot,\cdot) = \frac{1}{\sqrt{\tau(1-\tau)}}\rho^G(\cdot,\cdot).$$

From the equality (1.3) applied to the Riemannian manifold $(G, \langle\cdot,\cdot\rangle_{\mathcal{G}}^{\tau})$, we get for $\tau = 1/2$:

$$P\left(\limsup_{t\downarrow 0} \frac{\rho^{\mathcal{L}(G)}(\gamma(t), e)}{\sqrt{\varphi(t)}} \geqslant 1\right) \geqslant P\left(\limsup_{t\downarrow 0} \frac{\rho_{1/2}^G(\gamma(t, 1/2), e)}{2\sqrt{\varphi(t)}} = 1\right) = 1.$$

The proof of Theorem 7.1 is complete.

8 Modulus of continuity on loop groups

In this section, an upper bound on the modulus of continuity of $\gamma(t)$ is obtained using Theorem 6.1, and the lower bound is proved using a result of Grigor'yan and Kelbert [10].

Theorem 8.1 *Let $T > 0$. We have*

$$\limsup_{\delta\downarrow 0} \sup_{0<s<t<T, t-s=\delta} \frac{\rho^{\mathcal{L}(G)}(\gamma(t), \gamma(s))}{\sqrt{((t-s)\log(t-s)^{-1})/2}} = 1, \qquad a.s. \tag{8.1}$$

Let now $\varphi(t) = (t\log t^{-1})/2$.

Upper bound proof of Theorem 8.1

We prove the upper bound of the modulus of continuity of $\gamma(t)$ following the classical method of Itô and McKean [14]. By (6.2) and the Markov property of $\gamma(t)$, for any $T > 0$ and $s, t \in [0, T]$, $0 < t - s < 1$ we have

$$P\left(\rho^{\mathcal{L}(G)}(\gamma(t), \gamma(s)) \geqslant y\right) \leqslant \exp\left(-\frac{2\left(y - C_0\sqrt{t-s}\right)^2}{(t-s)\exp(K(t-s))}\right), \qquad y > C_0\sqrt{t-s}. \quad (8.2)$$

Let $\alpha \in (0,1)$ and $\varepsilon > 0$ be fixed such that

$$1 + \alpha < (1-\alpha)(1+\varepsilon/2).$$

Set

$$I_n = \{(i,j) \in \mathbb{N}^2 \ : \ 0 \leqslant i < j \leqslant 2^n, \ j - i \leqslant 2^{\alpha n}\}.$$

Then $\# I_n \leqslant 2^{(1+\alpha)n}$, and for $(i,j) \in I_n$, we have $(j-i)2^{-n} \leqslant 2^{-n(1-\alpha)}$. Put

$$A_n = \left\{ \max_{(i,j)\in I_n} \frac{\rho^{\mathcal{L}(G)}(\gamma(i2^{-n}), \gamma(j2^{-n}))}{\sqrt{\varphi((j-i)2^{-n})}} \geqslant \sqrt{1+\varepsilon} \right\}.$$

Then for $n > N$ large enough, by (8.2) we have

$$\begin{aligned}
P(A_n) &\leqslant \sum_{(i,j)\in I_n} \exp\left(-\frac{\left(\sqrt{(1+\varepsilon)\log((j-i)2^{-n})^{-1}} - \sqrt{2}C_0\right)^2}{\exp(K(j-i)2^{-n})}\right) \\
&\leqslant 2^{(1+\alpha)n} \exp\left(-\frac{\left(\sqrt{(1+\varepsilon)\log 2^{(1-\alpha)n}} - \sqrt{2}C_0\right)^2}{\exp(K2^{-(1-\alpha)n})}\right) \\
&\leqslant 2^{(1+\alpha)n} \exp\left(-(1+\varepsilon/2)\log 2^{(1-\alpha)n}\right) \\
&\leqslant (2^{(1+\alpha)-(1-\alpha)(1+\varepsilon/2)})^n,
\end{aligned}$$

hence

$$\sum_{n=N}^{+\infty} P(A_n) < +\infty.$$

By the Borel-Cantelli Lemma and an argument similar to the one used in the upper bound proof of Theorem 3.1, we get

$$\rho^{\mathcal{L}(G)}(\gamma(s), \gamma(t)) \leqslant \left(2\sqrt{\varepsilon(1+\varepsilon)} + \sqrt{1+\varepsilon}\right)\sqrt{\varphi(\delta)}.$$

Finally, let $\varepsilon \downarrow 0$ to complete the proof.

Lower bound proof of Theorem 8.1

Let $\tau = 1/2$. Since the Lie algebra $\mathcal{G}$ is equipped with an Ad_G-invariant inner product, the Ricci curvature of $(G, \langle\cdot,\cdot\rangle_{\mathcal{G}})$ is nonnegative and we may apply the lower bound of Theorem 3.1 to the compact Riemannian manifold $(G, \langle\cdot,\cdot\rangle_{\mathcal{G}}^{1/2})$ to get

$$\begin{aligned} P\left(\limsup_{\delta\downarrow 0}\sup_{0<s<t<T, t-s=\delta}\frac{\rho^{\mathcal{L}(G)}(\gamma(t),\gamma(s))}{\sqrt{\varphi(t-s)}}\geqslant 1\right) \\ &\geqslant P\left(\limsup_{\delta\downarrow 0}\sup_{0<s<t<T, t-s=\delta}\frac{\rho^G_{1/2}(\gamma(t,1/2),\gamma(s,1/2))}{2\sqrt{\varphi(t-s)}}\geqslant 1\right) \\ &\geqslant 1. \end{aligned}$$

This completes the proof of Theorem 8.1.

References

[1] A. Bendikov and L. Saloff-Coste. On the sample paths of diagonal Brownian motions on the infinite dimensional torus. *Ann. Inst. Henri Poincaré, Probab. Stat.*, 40(2):227–254, 2004.

[2] J.M. Bismut. *Large deviations and the Malliavin calculus*, volume 45 of *Progress in Mathematics*. Birkhäuser, 1984.

[3] M. Blümlinger, M. Drmota, and R.F. Tichy. A uniform law of the iterated logarithm for Brownian motion on compact Riemannian manifolds. *Math. Z.*, 201(4):495–507, 1989.

[4] B.K. Driver. Integration by parts and quasi-invariance for heat kernel measures on loop groups. *J. Funct. Anal.*, 149(2):470–547, 1997.

[5] B.K. Driver and T. Lohrenz. Logarithmic Sobolev inequalities for pinned loop groups. *J. Funct. Anal.*, 140(2):381–448, 1996.

[6] S. Fang. Integration by parts for heat measures over loop groups. *J. Math. Pures Appl. (9)*, 78(9):877–894, 1999.

[7] S. Fang. Integration by parts formula and logarithmic Sobolev inequality on the path space over loop groups. *Ann. Probab.*, 27(2):664–683, 1999.

[8] S. Fang and P. Malliavin. Stochastic analysis on the path space of a Riemannian manifold. I: Markovian stochastic calculus. *J. Funct. Anal.*, 118(1):249–274, 1993.

[9] D.S. Freed. The geometry of loop groups. *J. Differential Geom.*, 28(2):223–276, 1988.

[10] A. Grigor'yan and M. Kelbert. Range of fluctuation of Brownian motion on a complete Riemannian manifold. *Ann. Probab.*, 26(1):78–111, 1998.

[11] C. Houdré and N. Privault. Concentration and deviation inequalities in infinite dimensions via covariance representations. *Bernoulli*, 8(6):697–720, 2002.

[12] C. Houdré and N. Privault. A concentration inequality on Riemannian path space. In *Stochastic inequalities and applications*, volume 56 of *Prog. Probab.*, pages 15–21. Birkhäuser, Basel, 2003.

[13] E.P. Hsu. *Stochastic analysis on manifolds*, volume 38 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2002.

[14] K. Itô and H.P. McKean. *Diffusion processes and their sample paths*. Springer-Verlag, Berlin, 1974. Die Grundlehren der mathematischen Wissenschaften, Band 125.

[15] O. Kallenberg. *Foundations of modern probability*. Probability and its Applications (New York). Springer-Verlag, New York, second edition, 2002.

[16] M. Ledoux. Concentration of measure and logarithmic Sobolev inequalities. In *Séminaire de Probabilités XXXIII*, volume 1709 of *Lecture Notes in Math.*, pages 120–216. Springer, 1999.

[17] P. Malliavin. Hypoellipticity in infinite dimensions. In *Diffusion processes and related problems in analysis, Vol. I (Evanston, IL, 1989)*, volume 22 of *Progr. Probab.*, pages 17–31. Birkhäuser Boston, Boston, MA, 1990.

[18] D. Nualart. *The Malliavin Calculus and Related Topics*. Probability and its Applications. Springer-Verlag, 1995.

[19] D. Revuz and M. Yor. *Continuous martingales and Brownian motion*, volume 293 of *Grundlehren der Mathematischen Wissenschaften*. Springer-Verlag, Berlin, third edition, 1999.

[20] D.W. Stroock. *An introduction to the analysis of paths on a Riemannian manifold*, volume 74 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2000.