Laplace transform identities and measure-preserving transformations on the Lie-Wiener-Poisson spaces

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Abstract

Given a divergence operator δ on a probability space such that the law of $\delta(h)$ is infinitely divisible with characteristic exponent

$$h \mapsto -\frac{1}{2}\int_0^\infty h_t^2 dt, \quad \text{or} \quad \int_0^\infty (e^{ih(t)} - ih(t) - 1)dt, \qquad h \in L^2(\mathbb{R}_+), \qquad (0.1)$$

we derive a family of Laplace transform identities for the derivative $\partial E[e^{\lambda\delta(u)}]/\partial\lambda$ when u is a non-necessarily adapted process. These expressions are based on intrinsic geometric tools such as the Carleman-Fredholm determinant of a covariant derivative operator and the characteristic exponent (0.1), in a general framework that includes the Wiener space, the path space over a Lie group, and the Poisson space. We use these expressions for measure characterization and to prove the invariance of transformations having a quasi-nilpotent covariant derivative, for Gaussian and other infinitely divisible distributions.

Key words: Malliavin calculus, Skorohod integral, measure invariance, covariant derivatives, quasi-nilpotence, path space, Lie groups, Poisson space.
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1 Introduction

In this paper we work in the general framework of an arbitrary probability space $(\Omega, \mathcal{F}, \mu)$. We consider a linear space $\mathcal{S}$ dense in $L^2(\Omega, \mathcal{F}, \mu)$, and a closable linear operator

$$D : \mathcal{S} \longmapsto L^2(\Omega; H),$$

with closed domain $\mathrm{Dom}(D)$ containing $\mathcal{S}$, where $H = L^2(\mathbb{R}_+; \mathbb{R}^d)$ for some $d \geq 1$. We assume that

(H1) there exists a closable divergence (or Skorohod integral) operator

$$\delta : \mathcal{S} \otimes H \longmapsto L^2(\Omega),$$

acting on stochastic processes, adjoint of D, with the duality relation

$$E[\langle DF, u\rangle_H] = E[F\delta(u)], \qquad F \in \mathrm{Dom}(D), \quad u \in \mathrm{Dom}(\delta), \tag{1.1}$$

where $\mathrm{Dom}(\delta)$ is the domain of the closure of δ,

and we are interested in characterizing the distribution of $\delta(u)$ under a given choice of covariance derivative operator ∇ associated to D and δ, cf. (1.4) below.

The canonical example for this setting is when (Ω, μ) is the d-dimensional Wiener space with the Wiener measure μ, which is known to be invariant under random isometries whose Malliavin gradient D satisfies a quasi-nilpotence condition, cf. [19], [20], and Corollary 3.4 and Relation (3.10) below. This property is an anticipating extension of the classical invariance of Brownian motion under adapted isometries.

In addition to the Wiener space, the general framework of this paper covers both the Lie-Wiener space, for which the operators D and δ can be defined on the path space over a Lie group, cf. [5], [6], [18], and the discrete probability space of the Poisson process, cf. [2], [4], [7]. In those settings the distribution of $\delta(h)$ is given by

$$E[e^{i\delta(h)}] = \exp\left(\int_0^\infty \Psi(ih(t))dt\right), \qquad h \in H = L^2(\mathbb{R}_+; \mathbb{R}^d),$$

where the characteristic exponent Ψ is $\Psi(z) = \|z\|^2/2$ on the Lie-Wiener space, and

$$\Psi(z) = e^z - z - 1, \qquad z \in \mathbb{C}, \tag{1.2}$$

in the Poisson case with $d = 1$.

In order to state our main results we make the following additional assumptions.

(H2) The operator D satisfies the chain rule of derivation

$$D_t g(F) = g'(F) D_t F, \qquad t \in \mathbb{R}_+, \quad g \in \mathcal{C}_b^1(\mathbb{R}), \quad F \in \mathrm{Dom}(D), \tag{1.3}$$

where $D_t F = (DF)(t)$, $t \in \mathbb{R}_+$.

(H3) There exists a covariant derivative operator

$$\nabla : L^2(\Omega \times \mathbb{R}_+; \mathbb{R}^d) \longrightarrow L^2(\Omega \times \mathbb{R}_+; \mathbb{R}^d \otimes \mathbb{R}^d)$$

with domain $\mathrm{Dom}(\nabla)$ such that D, δ and ∇ satisfy the commutation relation

$$D_t \delta(u) = u_t + \delta(\nabla_t^\dagger u), \tag{1.4}$$

for $u \in \mathrm{Dom}(\nabla)$ such that $\nabla_t^\dagger u \in \mathrm{Dom}(\delta)$, $t \in \mathbb{R}_+$, where $\dagger$ denote matrix transposition in $\mathbb{R}^d \otimes \mathbb{R}^d$.

In this general framework we prove in Proposition 2.1 below the Laplace transform identity

$$\frac{\partial}{\partial \lambda} E[e^{\lambda \delta(u)}] = \lambda E[e^{\lambda \delta(u)} \langle (I - \lambda \nabla u)^{-1} u, u \rangle] + \lambda E\left[e^{\lambda \delta(u)} \langle \nabla^* u, D((I - \lambda \nabla u)^{-1} u) \rangle \right],$$

for λ in a neighborhood on 0, without any requirement on the probability measure μ. As a consequence of Proposition 2.1, we derive in Propositions 3.3, 4.2 and 5.1 below a family of Laplace transform identities of the form

$$\begin{aligned} \frac{\partial}{\partial \lambda} E[e^{\lambda \delta(u)}] = E\left[e^{\lambda \delta(u)} \langle \Psi'(\lambda u), u \rangle \right] + E\left[e^{\lambda \delta(u)} \left\langle (I - \lambda \nabla u)^{-1} u, D \int_0^\infty \Psi(\lambda u_t) dt \right\rangle \right] \\ + \lambda E\left[e^{\lambda \delta(u)} \langle \nabla^* u, D((I - \lambda \nabla u)^{-1} u) \rangle \right], \end{aligned} \tag{1.5}$$

which hold on both the Lie-Wiener and Poisson spaces. These identities are obtained inductively from the integration by parts (1.1), by removing all occurrences of the stochastic integral operator δ in factor of the exponential $e^{\lambda\delta(u)}$. We will study the relations between such identities and quasi-nilpotence and measure invariance in Corollaries 3.4, 4.3 and 4.5.

On the Lie-Wiener path spaces as well as on the Poisson space, Relation (1.5) involves a covariant derivative operator ∇, which appears in the commutation relation (1.4) of Condition **(H3)** above between D and δ, and the series

$$(I - \nabla u)^{-1} = \sum_{n=0}^{\infty} (\nabla u)^n, \qquad \|\nabla u\|_{L^2(\mathbb{R}_+^2)} < 1, \tag{1.6}$$

cf. (1.20) below for the definition of the operator $(\nabla u)^n$ on H. The proof of (1.5) relies on the relation

$$\langle (I - \nabla u)^{-1} v, u \rangle = \langle \Psi'(u), v \rangle + \left\langle (I - \nabla u)^{-1} v, D \int_0^\infty \Psi(u_t) dt \right\rangle, \tag{1.7}$$

$u \in \mathrm{Dom}(\nabla)$, $v \in H$, cf. Lemmas 3.7 and 4.6 below, which holds on both the path space and the Poisson space, respectively for $\Psi(z)$ of the form $\Psi(z) = \|z\|^2/2$ or (1.2).

Under the condition

$$\langle \nabla^* u, D((\nabla u)^n u) \rangle = 0, \qquad n \in \mathbb{N}, \tag{1.8}$$

Relation (1.5) reads

$$\frac{\partial}{\partial \lambda} E[e^{\lambda\delta(u)}] = E\left[e^{\lambda\delta(u)} \langle \Psi'(\lambda u), u \rangle\right] + E\left[e^{\lambda\delta(u)} \left\langle (I - \lambda\nabla u)^{-1} u, D \int_0^\infty \Psi(\lambda u_t) dt \right\rangle\right], \tag{1.9}$$

for λ in a neighborhood of zero. This is true in particular when $(u_t)_{t\in\mathbb{R}_+}$ is adapted with respect to the filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ generated by the underlying process, cf. Lemmas 3.5 and 4.4 below, in which case $\delta(u)$ is known to coincide with the forward Itô-Wiener, resp. Itô-Poisson, stochastic integral of $(u_t)_{t\in\mathbb{R}_+}$ as recalled in Sections 3 and 4.

In Corollaries 3.4 and 4.5 we apply (1.9) to obtain sufficient conditions for the invariance of Gaussian and infinitely divisible distributions on the Lie-Wiener path spaces and on the Poisson space. In particular, whenever the exponent $\int_0^\infty \Psi(\lambda u_t)dt$ is deterministic, $\lambda \in \mathbb{R}$, and ∇u satisfies (1.8), Relation (1.9) shows that we have

$$\begin{aligned}\frac{\partial}{\partial\lambda}E[e^{\lambda\delta(u)}] &= E\left[e^{\lambda\delta(u)}\right]\int_0^\infty \langle u_t, \Psi'(\lambda u_t)\rangle dt \\ &= E\left[e^{\lambda\delta(u)}\right]\frac{\partial}{\partial\lambda}\int_0^\infty \Psi(\lambda u_t)dt, \qquad \lambda \in \mathbb{R},\end{aligned}$$

which yields

$$E[e^{\delta(u)}] = \exp\left(\int_0^\infty \Psi(u_t)dt\right), \tag{1.10}$$

i.e. $\delta(u)$ is infinitely divisible with Lévy exponent $\int_0^\infty \Psi(u_t)dt$. Taking $u \in \bigcap_{p\geq 1} L^p(\mathbb{R}_+)$ to be a deterministic function, this also shows that the duality relation (1.1) in Hypothesis **(H1)** above and the definition of the gradient ∇ characterize the infinitely divisible law of $\delta(u)$.

In the Lie-Wiener case we also find the commutation relation

$$\langle \nabla^* v, (I-\nabla v)^{-1}Du - D((I-\nabla v)^{-1}u)\rangle = \langle (I-\nabla v)^{-1}u, D\log \det{}_2(I-\nabla v)\rangle, \tag{1.11}$$

cf. Lemma 3.6 below, where

$$\det{}_2(I-\nabla u) = \exp\left(-\sum_{n=2}^\infty \frac{1}{n}\mathrm{trace}\,(\nabla u)^n\right) \tag{1.12}$$

is the Carleman-Fredholm determinant of $I - \nabla u$, cf. e.g. Chapter 9 of [16].

In this case, Relations (1.11) and (1.12) allow us to rewrite (1.5) as

$$\begin{aligned}\frac{\partial}{\partial\lambda}E[e^{\lambda\delta(u)}] &= \lambda E\left[e^{\lambda\delta(u)}\langle u,u\rangle\right] + \frac{1}{2}\lambda^2 E\left[e^{\lambda\delta(u)}\left\langle (I-\lambda\nabla u)^{-1}u, D\langle u,u\rangle\right\rangle\right] \qquad (1.13)\\ &+\lambda E\left[e^{\lambda\delta(u)}\langle \nabla^* u, (I-\lambda\nabla u)^{-1}Du\rangle\right] - \lambda E\left[e^{\lambda\delta(u)}\langle (I-\lambda\nabla u)^{-1}u, D\log\det{}_2(I-\lambda\nabla u)\rangle\right],\end{aligned}$$

cf. Proposition 3.3, which becomes

$$\frac{\partial}{\partial\lambda}E[e^{\lambda\delta(u)}] = \lambda E\left[e^{\lambda\delta(u)}\langle u,u\rangle\right] + \frac{1}{2}\lambda^2 E\left[e^{\lambda\delta(u)}\left\langle (I-\lambda Du)^{-1}u, D\langle u,u\rangle\right\rangle\right] \tag{1.14}$$

$$-E\left[e^{\lambda\delta(u)}\frac{\partial}{\partial\lambda}\log\det{}_2(I-\lambda Du)\right]-\lambda E\left[e^{\lambda\delta(u)}\langle(I-\lambda Du)^{-1}u, D\log\det{}_2(I-\lambda Du)\rangle\right],$$

on the Wiener space, cf. Proposition 5.1, in which case we have $\nabla = D$.

As was noted in [21] the Carleman-Fredholm $\det_2(I-\lambda\nabla u)$ is equal to 1 when the trace

$$\text{trace}\,(\nabla u)^n = \int_0^\infty\cdots\int_0^\infty \langle \nabla^\dagger_{t_n} u_{t_1}, \nabla_{t_{n-1}} u_{t_n} \cdots \nabla_{t_1} u_{t_2}\rangle_{\mathbb{R}^d\otimes\mathbb{R}^d} dt_1\cdots dt_n \tag{1.15}$$

vanishes for all $n \geq 2$, and Condition (1.8) can be replaced by assuming quasi-nilpotence condition

$$\text{trace}\,(\nabla u)^n = \text{trace}\,(\nabla u)^{n-1} Du = 0, \qquad n \geq 2, \tag{1.16}$$

cf. Corollary 3.4.

In this way, and by a direct argument, (1.10) extends to the Lie-Wiener space the sufficient conditions found in [19] for the Skorohod integral $\delta(Rh)$ on the Wiener space to have a Gaussian law when $h \in H = L^2(\mathbb{R}_+, \mathbb{R}^d)$ and R is a random isometry of H with quasi-nilpotent gradient, cf. Theorem 2.1-b) of [19]. Such results hold in particular when the process $(u_t)_{t\in\mathbb{R}_+}$ is adapted with respect to the filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ by Lemmas 3.5 and 4.4, and we extend them to the Lie-Wiener space in Section 3. An example of anticipating process u satisfying (1.16) is also provided in (3.12) below on the Lie-Wiener space.

The results of this paper also admit various finite-dimensional interpretations. For such an interpretation, let us restrict ourselves to the 1-dimensional Wiener space, consider an orthonormal family $e = (e_1, \ldots, e_n)$ in $H = L^2(\mathbb{R}_+; \mathbb{R})$ and the sequence

$$X_k = \int_0^\infty e_k(t)dt, \qquad k = 1, \ldots, n,$$

of independent standard Gaussian random variables. We define u to be the process

$$u_t = \sum_{k=1}^n e_k(t) f_k(X_1, \ldots, X_n), \qquad t \in \mathbb{R}_+,$$

where $f_k \in \mathcal{C}_b^1(\mathbb{R}^n)$, $k = 1, \ldots, n$. In that case, from (3.2) below we have

$$D_s u_t = \sum_{k=1}^n \sum_{l=1}^n e_k(t) e_l(s) \partial_l f_k(X_1, \ldots, X_n) = \langle e(t), (\partial f) e^\dagger(s) \rangle_{\mathbb{R}^n}, \qquad s, t \in \mathbb{R}_+,$$

where

$$\partial f = (\partial_l f_k)_{k,l=1,\ldots,n}$$

denotes the usual matrix gradient of the column vector $f = (f_1, \ldots, f_n)^\dagger$ on $\mathbb{R}^n$. We assume in addition that $\partial_l f_k = 0$, $1 \leq k \leq l \leq n$, i.e. ∂f is strictly lower triangular and thus nilpotent. The divergence operator δ is then given by standard Gaussian integration by parts on $\mathbb{R}^n$ as

$$\delta(u) = \sum_{k=1}^n X_k f_k(X_1, \ldots, X_n) - \sum_{k=1}^n \partial_k f_k(X_1, \ldots, X_n) = \sum_{k=1}^n X_k f_k(X_1, \ldots, X_n).$$

In that case, (1.8) and (1.16) are satisfied and (1.9) reads, letting $\bar{x}_n = (x_1, \ldots, x_n)$,

$$\begin{aligned}
&\frac{\partial}{\partial \lambda} \int_{\mathbb{R}^n} \exp\left(-\frac{1}{2}(x_1^2 + \cdots + x_n^2) + \lambda \sum_{k=1}^n x_k f_k(\bar{x}_n) \right) dx_1 \cdots dx_n \\
&= \lambda \int_{\mathbb{R}^n} \left(\sum_{k=1}^n |f_k(\bar{x}_n)|^2 + \frac{\lambda}{2} \sum_{k,l=1}^n \sum_{p=0}^{n-1} \lambda^p ((\partial f)^p f)_k(\bar{x}_n) \partial_k f_l^2(\bar{x}_n) \right) \\
&\qquad \times \exp\left(-\frac{1}{2}(x_1^2 + \cdots + x_n^2) + \lambda \sum_{k=1}^n x_k f_k(\bar{x}_n) \right) dx_1 \cdots dx_n.
\end{aligned}$$

More complicated finite-dimensional identities can be obtained from (1.14) when ∂f is not quasi-nilpotent. On the other hand, simplifying to the extreme, if $n = 2$ and e.g. $f_1 = 0$ and $f_2(x_1, x_2) = x_1$, we explicitly recover the calculus result

$$\begin{aligned}
&\frac{\partial}{\partial \lambda} \int_{\mathbb{R}^2} \exp\left(\lambda x_1 x_2 - \frac{1}{2}(x_1^2 + x_2^2) \right) dx_1 dx_2 = 2\pi \frac{\partial}{\partial \lambda} \frac{1}{\sqrt{1 - \lambda^2}} \\
&= \lambda \int_{\mathbb{R}^2} x_1^2 \exp\left(\lambda x_1 x_2 - \frac{1}{2}(x_1^2 + x_2^2) \right) dx_1 dx_2 = \frac{2\pi\lambda}{(1 - \lambda^2)^{3/2}}, \qquad \lambda \in (-1, 1),
\end{aligned}$$

see Section 5 for the case of general quadratic Gaussian functionals in infinite dimensions.

The path space setting of Section 3 is less suitable for finite-dimensional examples as the Lie-group valued Brownian motion is inherently infinite-dimensional with respect to the underlying $\mathbb{R}^d$-valued Wiener process. To some extent, the same is true of Poisson stochastic integrals, as they naturally depend on an infinity of jump times.

Indeed, this geometric framework also covers the Poisson distribution in Section 4 via the use of a covariant derivative operator on the Poisson space, showing that the derivation property of the gradient operators is on the Lie-Wiener space is not characteristic of the Gaussianity of the underlying distribution. The results of this paper can also be applied to the computation of moments for the Itô-Wiener and Poisson stochastic integrals [14]. A different family of identities has been obtained for Hermite polynomials and stochastic exponentials in $\delta(u)$ in [13] on the Wiener space and in [15] on the path space, see also [10] for the use of finite difference operators on the Poisson space.

This paper is organized as follows. This section ends with a review of some notation on closable gradient and divergence operators and their associated commutation relations. In Section 2 we derive a general moment identity of the type (1.5), and in Section 3 we consider the setting of path spaces over Lie groups, which includes the Wiener space as a special case. In Section 4 we show that the general results of Section 2 also apply on the Poisson space. Finally in Section 5 we prove (1.14) and recover some classical Laplace identities for second order Wiener functionals in Proposition 5.2.

We close this introduction with some additional notation.

Notation

Given X a real separable Hilbert space, the definition of D is naturally extended to X-valued random variables by letting

$$DF = \sum_{k=1}^{n} x_i \otimes DF_i \tag{1.17}$$

for $F \in X \otimes \mathcal{S} \subset L^2(\Omega; X)$ of the form

$$F = \sum_{k=1}^{n} x_i \otimes F_i$$

$x_1, \ldots, x_n \in X$, $F_1, \ldots, F_n \in \mathcal{S}$. When D maps $\mathcal{S}$ to $\mathcal{S} \otimes H$, as on the Lie-Wiener space, iterations of this definition starting with $X = \mathbb{R}$, then $X = H$, and successively replacing X with $X \otimes H$ at each step, allow one to define

$$D^n : X \otimes \mathcal{S} \longmapsto L^2(\Omega; X \hat{\otimes} H^{\hat{\otimes} n})$$

for all $n \geq 1$, where $\hat{\otimes}$ denotes the completed symmetric tensor product of Hilbert spaces. In that case we let $\mathbb{D}_{p,k}(X)$ denote the completion of the space $X \otimes \mathcal{S}$ of X-valued random variables under the norm

$$\|u\|_{\mathbb{D}_{p,k}(X)} = \sum_{l=0}^{k} \|D^l u\|_{L^p(\Omega, X \hat{\otimes} H^{\hat{\otimes} l})}, \qquad p \geq 1, \tag{1.18}$$

with

$$\mathbb{D}_{\infty,k}(X) = \bigcap_{k \geq 1} \mathbb{D}_{p,k}(X),$$

and $\mathbb{D}_{p,k} = \mathbb{D}_{p,k}(\mathbb{R})$, $p \in [1, \infty]$, $k \geq 1$. Note that for all $p, q > 1$ such that $p^{-1} + q^{-1} = 1$ and $k \geq 1$, the gradient operator D is continuous from $\mathbb{D}_{p,k}(X)$ into $\mathbb{D}_{q,k-1}(X \hat{\otimes} H)$ and the Skorohod integral operator δ adjoint of D is continuous from $\mathbb{D}_{p,k}(H)$ into $\mathbb{D}_{q,k-1}$.

Given $u \in \mathbb{D}_{2,1}(H)$ we also identify

$$\nabla u = ((s,t) \longmapsto \nabla_t u_s)_{s,t \in \mathbb{R}_+} \in H \hat{\otimes} H$$

to the random operator

$$\begin{aligned} \nabla u : H &\longrightarrow H \\ u &\longmapsto (\nabla u) v = ((\nabla u) v_s)_{s \in \mathbb{R}_+}, \end{aligned}$$

almost surely defined by

$$(\nabla u) v_s := \int_0^\infty (\nabla_t u_s) v_t dt, \qquad s \in \mathbb{R}_+, \quad v \in H, \tag{1.19}$$

in which $a \otimes b \in X \hat{\otimes} H$ is identified to a linear operator from $a \otimes b : H \longmapsto X$ via

$$(a \otimes b)c = a\langle b, c\rangle_H, \qquad a \otimes b \in X \hat{\otimes} H, \quad c \in H.$$

More generally, for $u \in \mathbb{D}_{2,1}(H)$ and $v \in H$ we have

$$(\nabla u)^k v_s = \int_0^\infty \cdots \int_0^\infty (\nabla_{t_k} u_s \nabla_{t_{k-1}} u_{t_k} \cdots \nabla_{t_1} u_{t_2}) v_{t_1} dt_1 \cdots dt_k, \qquad s \in \mathbb{R}_+, \quad (1.20)$$

i.e.

$$(\nabla u)^k = \left((s,t) \longmapsto \int_0^\infty \cdots \int_0^\infty (\nabla_{t_k} u_s \nabla_{t_{k-1}} u_{t_k} \cdots \nabla_t u_{t_2}) dt_2 \cdots dt_k, \right)_{s,t \in \mathbb{R}_+} \in H \hat{\otimes} H,$$

$k \geq 1$. We also define

$$\nabla^* u := ((s,t) \longmapsto \nabla_s^\dagger u_t)_{s,t \in \mathbb{R}_+} \in H \hat{\otimes} H$$

where $\nabla_s^\dagger u_t$ denotes the transpose matrix of $\nabla_s u_t$ in $\mathbb{R}^d \otimes \mathbb{R}^d$, $s,t \in \mathbb{R}_+$, and we identify $\nabla^* u$ to the adjoint of ∇u on H which satisfies

$$\langle (\nabla u)v, h\rangle_H = \langle v, (\nabla^* u)h\rangle_H, \qquad v, h \in H,$$

and is given by

$$(\nabla^* u)v_s = \int_0^\infty (\nabla_s^\dagger u_t) v_t dt, \qquad s \in \mathbb{R}_+, \quad v \in L^2(W; H). \quad (1.21)$$

Although D is originally defined for scalar random variables, its definition extends pointwise to $u \in \mathbb{D}_{2,1}(H)$ by (1.17), i.e.

$$D(u) := ((s,t) \longmapsto D_t u_s)_{s,t \in \mathbb{R}_+} \in H \hat{\otimes} H, \quad (1.22)$$

and the operators Du and $D^* u$ are constructed in the same way as ∇u and $\nabla^* u$ in (1.19) and (1.21).

The commutation relation (1.4) shows that the Skorohod [17] isometry

$$E[\delta(u)^2] = E[\langle u, u\rangle_H] + E\left[\mathrm{trace}\,(\nabla u)^2\right], \qquad u \in \mathcal{U}, \quad (1.23)$$

holds, with

$$\mathrm{trace}\,(\nabla u)^k = \langle \nabla^* u, (\nabla u)^{k-1}\rangle, \qquad k \geq 2.$$

As will be recalled in Sections 3 and 4, such operators D, ∇, δ can be constructed in at least three instances, i.e. on the Wiener space, on the path space over a Lie group, and on the Poisson space for $k = 1$. In the sequel, all scalar products will be simply denoted by $\langle \cdot, \cdot \rangle$.

2 The general case

The results of this paper rely on the following general Laplace identity (2.1) for the Skorohod integral operator δ, obtained in Proposition 2.1 using the adjoint gradient D and the covariant derivative ∇ under Conditions **(H1)**, **(H2)** and **(H3)** above. Here we do not specify any underlying probability measure on Ω, so that the characteristic exponent $\Psi(z)$ plays no role in this section.

Proposition 2.1 *Let $u \in \mathbb{D}_{\infty,1}(H)$ such that for some $a > 0$ we have $E[e^{a|\delta(u)|}] < \infty$, and the power series (1.6) of $(I - \lambda \nabla u)^{-1} u$ converges in $\mathbb{D}_{2,1}(H)$ for all $\lambda \in (-a/2, a/2)$. Then we have*

$$\frac{\partial}{\partial \lambda} E[e^{\lambda \delta(u)}] = \lambda E[e^{\lambda \delta(u)} \langle (I - \lambda \nabla u)^{-1} u, u \rangle] + \lambda E\left[e^{\lambda \delta(u)} \langle \nabla^* u, D((I - \lambda \nabla u)^{-1} u) \rangle\right], \tag{2.1}$$

for all $\lambda \in (-a/2, a/2)$.

In Sections 3 and 4 we will describe the applications of Proposition 2.1 successively on the Lie-Wiener path space, on the Wiener space, and on the Poisson space. In order to prove Proposition 2.1 we will need the moment identity proved in the next Lemma 2.2.

Lemma 2.2 *For any $n \in \mathbb{N}$ and $F \in \mathbb{D}_{2,1}$, $u \in \mathbb{D}_{n+1,2}(H)$, $v \in \mathbb{D}_{n+1,1}(H)$, we have*

$$\begin{aligned} E[F\delta(u)^n \delta(v)] &= \sum_{k=1}^{n} \frac{n!}{(n-k)!} E\left[F\delta(u)^{n-k} \left(\langle (\nabla u)^{k-1} v, u \rangle + \langle \nabla^* u, D((\nabla u)^{k-1} v) \rangle\right)\right] \\ &\quad + \sum_{k=0}^{n} \frac{n!}{(n-k)!} E\left[\delta(u)^{n-k} \langle (\nabla u)^k v, DF \rangle\right]. \end{aligned} \tag{2.2}$$

Proof. We have $(\nabla u)^{k-1} v \in \mathbb{D}_{(n+1)/k,1}(H)$, $\delta(u) \in \mathbb{D}_{(n+1)/(n-k+1),1}$, and by Lemma 2.3 below we get

$$E\left[F\delta(u)^l \langle (\nabla u)^i v, D\delta(u) \rangle\right] - lE\left[F\delta(u)^{l-1} \langle (\nabla u)^{i+1} v, D\delta(u) \rangle\right]$$

$$
\begin{aligned}
&= E\left[F\delta(u)^l\langle(\nabla u)^i v, u\rangle\right] + E\left[F\delta(u)^l\langle(\nabla u)^i v, \delta(\nabla^* u)\rangle\right] \\
&\quad -lE\left[F\delta(u)^{l-1}\langle(\nabla u)^{i+1} v, u\rangle\right] - lE\left[F\delta(u)^{l-1}\langle(\nabla u)^{i+1} v, \delta(\nabla^* u)\rangle\right] \\
&= E\left[F\delta(u)^l\langle(\nabla u)^i v, u\rangle\right] + E[F\delta(u)^l\langle\nabla^* u, D((\nabla u)^i v)\rangle] + E[\delta(u)^l\langle(\nabla u)^{i+1} v, DF\rangle],
\end{aligned}
$$

and applying this formula to $l = n - k$ and $i = k - 1$ via a telescoping sum yields

$$
\begin{aligned}
&E[F\delta(u)^n\delta(v)] = E[F\langle v, D\delta(u)^n\rangle] + E[\delta(u)^n\langle v, DF\rangle] \\
&\quad = nE[F\delta(u)^{n-1}\langle v, D\delta(u)\rangle] + E[\delta(u)^n\langle v, DF\rangle] \\
&\quad = \sum_{k=1}^{n} \frac{n!}{(n-k)!}\left(E\left[F\delta(u)^{n-k}\langle(\nabla u)^{k-1} v, D\delta(u)\rangle\right] - (n-k)E\left[F\delta(u)^{n-k-1}\langle(\nabla u)^k v, D\delta(u)\rangle\right]\right) \\
&\quad\quad +E[\delta(u)^n\langle v, DF\rangle] \\
&\quad = \sum_{k=1}^{n} \frac{n!}{(n-k)!}\left(E\left[F\delta(u)^{n-k}\langle(\nabla u)^{k-1} v, u\rangle\right] + E\left[F\delta(u)^{n-k}\langle\nabla^* u, D((\nabla u)^{k-1} v)\rangle\right]\right) \\
&\quad\quad +\sum_{k=0}^{n} \frac{n!}{(n-k)!} E\left[\delta(u)^{n-k}\langle(\nabla u)^k v, DF\rangle\right].
\end{aligned}
$$

□

Lemma 2.2 coincides with the Skorohod isometry (1.23) when $n = 1$.

Proof of Proposition 2.1. We start by showing that for any $u, v \in \mathbb{D}_{\infty,1}(H)$ such that the power series of $(I - \nabla v)^{-1}u$ converges in $\mathbb{D}_{2,1}(H)$ and $E[e^{2|\delta(v)|}] < \infty$, we have

$$
E[\delta(u)e^{\delta(v)}] = E[e^{\delta(v)}\langle(I-\nabla v)^{-1}u, v\rangle] + E\left[e^{\delta(v)}\langle\nabla^* v, D((I-\nabla v)^{-1}u)\rangle\right]. \quad (2.3)
$$

Indeed, Lemma 2.2 shows that

$$
\begin{aligned}
E[\delta(u)e^{\delta(v)}] &= \sum_{n=0}^{\infty} \frac{1}{n!} E[\delta(v)^n\delta(u)] \\
&= \sum_{k=1}^{\infty}\sum_{n=k}^{\infty} \frac{1}{(n-k)!} E\left[\delta(v)^{n-k}\left(\langle(\nabla v)^{k-1}u, v\rangle + \langle\nabla^* v, D((\nabla v)^{k-1}u)\rangle\right)\right] \\
&= \sum_{k=1}^{\infty} E\left[e^{\delta(v)}\left(\langle(\nabla v)^{k-1}u, v\rangle + \langle\nabla^* v, D((\nabla v)^{k-1}u)\rangle\right)\right] \\
&= E[e^{\delta(v)}\langle(I-\nabla v)^{-1}u, v\rangle] + E\left[e^{\delta(v)}\langle\nabla^* v, D((I-\nabla v)^{-1}u)\rangle\right].
\end{aligned}
$$

Hence, applying (2.3) for $u = v$ we get

$$
\frac{\partial}{\partial\lambda} E[e^{\lambda\delta(u)}] = E[\delta(u)e^{\lambda\delta(u)}] \quad (2.4)
$$

$$= \lambda E[e^{\lambda\delta(u)}\langle (I-\lambda\nabla u)^{-1}u, u\rangle] + E\left[e^{\delta(u)}\langle \nabla^* u, D((I-\lambda\nabla u)^{-1}u)\rangle\right],$$

$\lambda \in (-a/2, a/2)$. □

Finally we prove the next Lemma 2.3 which has been used in the proof of Lemma 2.2 and extends Lemma 3.1 in [11] pages 120-121 to include a random variable $F \in \mathbb{D}_{2,1}$.

Lemma 2.3 *Let $F \in \mathbb{D}_{2,1}$, $u \in \mathbb{D}_{n+1,2}(H)$, and $v \in \mathbb{D}_{n+1,1}(H)$. For all $i, l \in \mathbb{N}$ we have*

$$E[F\delta(u)^l\langle (\nabla u)^i v, \delta(\nabla^* u)\rangle] - lE[F\delta(u)^{l-1}\langle (\nabla^* u)^{i+1} v, \delta(\nabla^* u)\rangle]$$
$$= lE[F\delta(u)^{l-1}\langle (\nabla u)^{i+1}v, u\rangle] + E[\delta(u)^l\langle (\nabla u)^{i+1}v, DF\rangle] + E[F\delta(u)^l\langle \nabla^* u, D((\nabla u)^i v)\rangle].$$

Proof. Using the duality (1.1) between D and δ, the chain rule of derivation (1.3) and the commutation relation (1.4), we have

$$\begin{aligned}
&E[F\delta(u)^l\langle (\nabla u)^i v, \delta(\nabla^* u)\rangle] - lE[F\delta(u)^{l-1}\langle (\nabla^* u)^{i+1} v, \delta(\nabla^* u)\rangle] \\
&= E[\langle \nabla^* u, D(F\delta(u)^l(\nabla u)^i v)\rangle] - lE[F\delta(u)^{l-1}\langle (\nabla^* u)^{i+1} v, \delta(\nabla^* u)\rangle] \\
&= lE[F\delta(u)^{l-1}\langle \nabla^* u, (\nabla u)^i v \otimes D\delta(u)\rangle] - lE[F\delta(u)^{l-1}\langle (\nabla^* u)^{i+1} v, \delta(\nabla^* u)\rangle] \\
&\quad + E[\delta(u)^l\langle \nabla^* u, D(F(\nabla u)^i v)\rangle] \\
&= lE[F\delta(u)^{l-1}\langle \nabla^* u, (\nabla u)^i v \otimes u\rangle] + lE[F\delta(u)^{l-1}\langle \nabla^* u, (\nabla u)^i v \otimes \delta(\nabla^* u)\rangle] \\
&\quad - lE[F\delta(u)^{l-1}\langle (\nabla^* u)^{i+1} v, \delta(\nabla^* u)\rangle] + E[\delta(u)^l\langle \nabla^* u, D(F(\nabla u)^i v)\rangle] \\
&= lE[F\delta(u)^{l-1}\langle (\nabla u)^{i+1} v, u\rangle] + E[\delta(u)^l\langle (\nabla u)^{i+1} v, DF\rangle] \\
&\quad + E[F\delta(u)^l\langle \nabla^* u, D((\nabla u)^i v)\rangle].
\end{aligned}$$

□

In the following sections we will reconsider Proposition 2.1 and its consequences in the Lie-Wiener and Poisson frameworks.

3 The path space case

Let G denote either $\mathbb{R}^d$ or a compact connected d-dimensional Lie group with associated Lie algebra $\mathcal{G}$ identified to $\mathbb{R}^d$ and equipped with an Ad-invariant scalar product on $\mathbb{R}^d \simeq \mathcal{G}$, also denoted by $\langle \cdot, \cdot \rangle$, with $H = L^2(\mathbb{R}_+; \mathcal{G})$. The commutator in $\mathcal{G}$ is denoted by $[\cdot,\cdot]$ and we let $\mathrm{ad}\,(u)v = [u,v]$, $u,v \in \mathcal{G}$, with $\mathrm{Ad}\, e^u = e^{\mathrm{ad}\, u}$, $u \in \mathcal{G}$. Here, $\Psi(x) = \|x\|^2/2$.

The Brownian motion $(\gamma(t))_{t\in\mathbb{R}_+}$ on G is constructed from a standard Brownian motion $(B_t)_{t\in\mathbb{R}_+}$ via the Stratonovich differential equation

$$\begin{cases} d\gamma(t) = \gamma(t) \odot dB_t \\ \\ \gamma(0) = \mathrm{e}, \end{cases}$$

where e is the identity element in G. Let $\mathbb{P}(\mathsf{G})$ denote the space of continuous G-valued paths starting at e, endowed with the image of the Wiener measure by the mapping $I : (B_t)_{t\in\mathbb{R}_+} \longmapsto (\gamma(t))_{t\in\mathbb{R}_+}$. Here we take

$$\mathcal{S} = \{F = f(\gamma(t_1),\dots,\gamma(t_n)) \quad : \quad f \in \mathcal{C}_b^\infty(\mathsf{G}^n)\},$$

and

$$\mathcal{U} = \mathcal{S} \otimes H = \left\{ \sum_{i=1}^n u_i F_i \quad : \quad F_i \in \mathcal{S},\ u_i \in L^2(\mathbb{R}_+;\mathcal{G}),\ i = 1,\dots,n,\ n \geq 1 \right\}.$$

Next is the definition of the right derivative operator D, which satisfies Condition **(H2)**.

Definition 3.1 *For F of the form*

$$F = f(\gamma(t_1),\dots,\gamma(t_n)) \in \mathcal{S}, \qquad f \in \mathcal{C}_b^\infty(\mathsf{G}^n), \tag{3.1}$$

we let $DF \in L^2(\Omega \times \mathbb{R}_+;\mathcal{G}) \simeq L^2(\Omega; H)$ be defined by

$$\langle DF, v\rangle = \frac{d}{d\varepsilon} f\left(\gamma(t_1)e^{\varepsilon\int_0^{t_1} v_s ds},\dots,\gamma(t_n)e^{\varepsilon\int_0^{t_n} v_s ds}\right)_{|\varepsilon=0}, \quad v \in L^2(\mathbb{R}_+,\mathcal{G}).$$

For F of the form (3.1) we also have

$$D_t F = \sum_{i=1}^n \partial_i f(\gamma(t_1),\dots,\gamma(t_n))\mathbf{1}_{[0,t_i]}(t), \qquad t \geq 0. \tag{3.2}$$

The operator D is known to admit an adjoint δ that satisfies Condition **(H1)**, i.e.

$$E[F\delta(v)] = E[\langle DF, v\rangle], \quad F \in \mathcal{S},\ v \in L^2(\mathbb{R}_+;\mathcal{G}), \tag{3.3}$$

cf. e.g. [5]. The operator D is linked to the Malliavin derivative $\hat{D}$ with respect to the underlying linear Brownian motion $(B_t)_{t\in\mathbb{R}_+}$, cf. (5.1) below, and to its adjoint $\hat{\delta}$, via the relations

$$\langle h, DF\rangle = \langle h, \hat{D}F\rangle + \hat{\delta}\left(\int_0^{\cdot} \mathrm{ad}\,(h(s))ds\hat{D}.F\right), \qquad h \in H, \tag{3.4}$$

cf. e.g. Lemma 4.1 of [9], and

$$\delta(hF) = \hat{\delta}(hF) - \hat{\delta}\left(\int_0^{\cdot} \mathrm{ad}\,(h(s))ds\hat{D}.F\right), \qquad h \in H,$$

which follows from (3.4) by duality. When $(u_t)_{t\in\mathbb{R}_+}$ is square-integrable and adapted with respect to the Brownian filtration, $\delta(u)$ coincides with the Itô integral of $u \in L^2(\Omega; H)$ with respect to the underlying Brownian motion $(B_t)_{t\in\mathbb{R}_+}$, i.e.

$$\delta(u) = \int_0^\infty u_t dB_t.$$

Definition 3.2 *The operator* $\nabla : \mathbb{D}_{2,1}(H) \longmapsto L^2(\Omega; H\hat{\otimes}H)$ *is defined as*

$$\nabla_s u_t = D_s u_t + \mathbf{1}_{[0,t]}(s)\mathrm{ad}\,u_t \in \mathcal{G}\otimes\mathcal{G}, \qquad s,t \in \mathbb{R}_+, \tag{3.5}$$

$u \in \mathbb{D}_{2,1}(H)$.

In other words we have

$$\langle e_i \otimes e_j, \nabla_s(uF)(t)\rangle = \langle u_t, e_j\rangle\langle e_i, D_sF\rangle + \mathbf{1}_{[0,t]}(s)F\langle e_j, \mathrm{ad}\,(e_i)u_t\rangle,$$

$i,j = 1,\ldots,d$, where $(e_i)_{i=1,\ldots,d}$ is an orthonormal basis of $\mathcal{G}$ and $\mathrm{ad}\,u \in \mathcal{G}\otimes\mathcal{G}$, $u \in \mathcal{G}$, is the matrix

$$(\langle e_j, \mathrm{ad}\,(e_i)u\rangle)_{1\le i,j\le d} = (\langle e_j, [e_i, u]\rangle)_{1\le i,j\le d}.$$

The operator $\mathrm{ad}\,(u)$ is antisymmetric on $\mathcal{G}$ because $\langle\cdot,\cdot\rangle$ is Ad-invariant. By (3.5),

$$(\nabla u)v_t = \int_0^\infty (\nabla_s u_t)v_s ds, \qquad t \in \mathbb{R}_+,$$

is the covariant derivative of $u \in \mathcal{U}$ in the direction $v \in L^2(\mathbb{R}_+;\mathcal{G})$, with $\nabla_v u \in L^2(\mathbb{R}_+;\mathcal{G})$, cf. [5]. Note that if u_t is $\mathcal{F}_t$-measurable we have

$$\nabla_t u_t = D_s u_t + \mathbf{1}_{[0,t]}(s)\mathrm{ad}\,u_t = D_s u_t = 0, \qquad s > t. \tag{3.6}$$

It is known that D and ∇ satisfy Condition **(H3)** and the commutation relation (1.4), as well as the Skorohod isometry (1.23) as a consequence, cf. [5]. Proposition 3.3 below is a corollary of Proposition 2.1 and it yields (1.13) on the Lie-Wiener path space.

Proposition 3.3 *Let $u \in \mathbb{D}_{\infty,2}(H)$ such that $E[e^{a|\delta(u)|}] < \infty$ for some $a > 0$. We have*

$$\frac{\partial}{\partial\lambda}E[e^{\lambda\delta(u)}] = \lambda E[\langle u,u\rangle e^{\lambda\delta(u)}] + \frac{1}{2}\lambda^2 E[e^{\lambda\delta(u)}\langle (I-\lambda\nabla u)^{-1}u, D\langle u,u\rangle\rangle] \tag{3.7}$$
$$+\lambda E\left[e^{\lambda\delta(u)}\langle\nabla^* u,(I-\lambda\nabla u)^{-1}Du\rangle\right] - \lambda E\left[e^{\lambda\delta(u)}\langle (I-\lambda\nabla u)^{-1}u, D\log\det{}_2(I-\lambda\nabla u)\rangle\right],$$

for $\lambda \in (-a/2, a/2)$ such that $|\lambda| < \|\nabla u\|^{-1}_{\mathbb{D}_{\infty,1}(H)}$.

Proof. Let $u \in \mathbb{D}_{\infty,1}(H)$ and $v \in \mathbb{D}_{\infty,2}(H)$ such that $\|\nabla v\|_{\mathbb{D}_{\infty,1}(H)} < 1$, and $E[e^{2|\delta(u)|}] < \infty$. From Relation (2.3) above and Lemma 3.7 below we have

$$\begin{aligned} E[\delta(u)e^{\delta(v)}] &= E[e^{\delta(v)}\langle (I-\nabla v)^{-1}u, v\rangle] + E\left[e^{\delta(v)}\langle\nabla^* v, D((I-\nabla v)^{-1}u)\rangle\right] \\ &= E[\langle u,v\rangle e^{\delta(v)}] + \frac{1}{2}E[e^{\delta(v)}\langle (I-\nabla v)^{-1}u, D\langle v,v\rangle\rangle] + E\left[e^{\delta(v)}\langle\nabla^* v, D((I-\nabla v)^{-1}u)\rangle\right]. \end{aligned}$$

As a consequence of Lemma 3.6 below, this yields

$$E[\delta(u)e^{\delta(v)}] = E[\langle u,v\rangle e^{\delta(v)}] + \frac{1}{2}E[e^{\delta(v)}\langle (I-\nabla v)^{-1}u, D\langle v,v\rangle\rangle] \tag{3.8}$$
$$+E\left[e^{\delta(v)}\langle\nabla^* v,(I-\nabla v)^{-1}Du\rangle\right] - E\left[e^{\delta(v)}\langle (I-\nabla v)^{-1}u, D\log\det{}_2(I-\nabla v)\rangle\right].$$

Next, taking $v = \lambda u$ with $|\lambda| < \|\nabla u\|^{-1}_{\mathbb{D}_{\infty,1}(H)}$ in (3.8), we get

$$\begin{aligned} \frac{\partial}{\partial\lambda}E[e^{\lambda\delta(u)}] &= E[\delta(u)e^{\lambda\delta(u)}] \\ &= \lambda E[\langle u,u\rangle e^{\lambda\delta(u)}] + \frac{1}{2}\lambda^2 E[e^{\lambda\delta(u)}\langle (I-\lambda\nabla u)^{-1}u, D\langle u,u\rangle\rangle] \\ &\quad +\lambda E\left[e^{\lambda\delta(u)}\langle\nabla^* u,(I-\lambda\nabla u)^{-1}Du\rangle\right] \\ &\quad -\lambda E\left[e^{\lambda\delta(u)}\langle (I-\lambda\nabla u)^{-1}u, D\log\det{}_2(I-\lambda\nabla u)\rangle\right], \end{aligned}$$

which yields (3.7). □

When the operator $\nabla u : H \longmapsto H$ is quasi-nilpotent in the sense of (1.16), Proposition 3.3 shows that

$$\frac{\partial}{\partial\lambda}E[e^{\lambda\delta(u)}] = \lambda E[\langle u,u\rangle e^{\lambda\delta(u)}] + \frac{1}{2}\lambda^2 E[e^{\lambda\delta(u)}\langle (I-\lambda\nabla u)^{-1}u, D\langle u,u\rangle\rangle], \tag{3.9}$$

which is (1.9) with $\Psi(x) = \|x\|^2/2$.

In particular we have the following result, cf. Theorem 2.1-b) of [19] on the commutative Wiener space.

Corollary 3.4 *Let $n \geq 1$ and $u \in \mathbb{D}_{n+1,2}(H)$ such that $\langle u, u\rangle$ is deterministic and*

$$\mathrm{trace}\,(\nabla u)^k = \mathrm{trace}\,(\nabla u)^{k-1}(Du) = 0, \quad a.s., \quad k \geq 2. \tag{3.10}$$

Then $\delta(u)$ has a centered Gaussian distribution with variance $\langle u, u\rangle$.

Proof. Proposition 3.3 and Relation (3.9) show that when $\langle u, u\rangle$ is deterministic and $u \in \mathbb{D}_{2,1}(H)$,

$$\frac{\partial}{\partial\lambda}E[e^{\lambda\delta(u)}] = \lambda\langle u, u\rangle E[e^{\lambda\delta(u)}], \qquad \lambda \in \mathbb{R},$$

under Condition (3.10), which implies

$$E[e^{\lambda\delta(u)}] = e^{\lambda^2\langle u,u\rangle/2}, \qquad \lambda \in \mathbb{R},$$

from which the conclusion follows. □

Condition (3.10) holds in particular when the process $(u_t)_{t\in\mathbb{R}_+}$ is adapted, according to the next Lemma 3.5 which follows from (3.6).

Lemma 3.5 *Assume that the process $u \in \mathbb{D}_{2,1}(H)$ is adapted with respect to the Brownian filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$. Then we have*

$$\mathrm{trace}\,(\nabla u)^k = \mathrm{trace}\,(\nabla u)^{k-1}(Du) = 0, \qquad k \geq 2. \tag{3.11}$$

Proof. For almost all $t_1, \ldots, t_{k+1} \in \mathbb{R}_+$ there exists $i \in \{1, \ldots, k+1\}$ such that $t_i > t_{i+1 \bmod k+1}$, and (3.6) yields

$$\begin{aligned}
\nabla_{t_i} u_{t_{i+1 \bmod k+1}} &= D_{t_i} u_{t_{i+1 \bmod k+1}} + \mathbf{1}_{[0,t_{i+1 \bmod k+1}]}(t_i) \\
&= D_{t_i} u_{t_{i+1 \bmod k+1}} \\
&= 0,
\end{aligned}$$

since $u_{t_{i+1 \bmod k+1}}$ is $\mathcal{F}_{t_{i+1 \bmod k+1}}$-measurable because $(u_t)_{t\in\mathbb{R}_+}$ is $\mathcal{F}_t$-adapted, and this implies (3.11) by (1.15). □

Example

An anticipating example for Corollary 3.4 can be constructed by considering two orthonormal sequences $(e_k)_{k\in\mathbb{N}}$ and $(f_k)_{k\in\mathbb{N}}$ that are also mutually orthogonal in H, and such that the sequence $(e_k)_{k\in\mathbb{N}} \subset E$ is made of commuting elements in $\mathcal{G}$, while $(e_k)_{k\in\mathbb{N}}$ may not commute with $(f_k)_{k\in\mathbb{N}}$. We let

$$u = \sum_{k=0}^{\infty} A_k e_k \in \mathbb{D}_{2,1}(H) \tag{3.12}$$

where $(A_k)_{k\in\mathbb{N}}$ is a sequence of $\sigma(\delta(f_k)\ :\ k \in \mathbb{N})$-measurable scalar random variables, satisfying

$$\sum_{k=0}^{\infty} |A_k|^2 = 1, \qquad a.s.$$

Then we have $\|u\|_H = 1$, a.s.,

$$\begin{aligned} \nabla_{t_2} u_{t_3} &= \sum_{k=0}^{\infty} e_k(t_3) \otimes D_{t_2} A_k + \mathbf{1}_{[0,t_3]}(t_2) \sum_{k=0}^{\infty} A_k \mathrm{ad}\, e_k(t_3) \\ &= \sum_{k,l=0}^{\infty} \langle DA_k, f_l\rangle e_k(t_3) \otimes f_l(t_2) + \mathbf{1}_{[0,t_3]}(t_2) \sum_{k=0}^{\infty} A_k \mathrm{ad}\, e_k(t_3), \qquad t_2, t_3 \in \mathbb{R}_+, \end{aligned}$$

and

$$D_{t_1} u_{t_2} = \sum_{k=0}^{\infty} e_k(t_2) \otimes D_{t_1} A_k = \sum_{k,l=0}^{\infty} \langle DA_k, f_l\rangle e_k(t_2) \otimes f_l(t_1), \qquad t_1, t_2 \in \mathbb{R}_+,$$

hence

$$\begin{aligned} \nabla u_{t_3} \nabla_{t_1} u &= \int_0^\infty \nabla_{t_2} u_{t_3} \nabla_{t_1} u_{t_2} dt_2 \\ &= \int_0^\infty \left(D_{t_2} u_{t_3} + \mathbf{1}_{[0,t_3]}(t_2) \mathrm{ad}\, u_{t_3}\right) \left(D_{t_1} u_{t_2} + \mathbf{1}_{[0,t_2]}(t_1) \mathrm{ad}\, u_{t_2}\right) dt_2 \\ &= \int_0^\infty D_{t_2} u_{t_3} D_{t_1} u_{t_2} dt_2 \\ &= \sum_{p,q,k,l=0}^{\infty} \langle DA_k, f_l\rangle \langle DA_p, f_q\rangle \langle f_q, e_k\rangle e_p(t_3) \otimes f_l(t_1) \\ &= 0, \qquad t_1, t_3 \in \mathbb{R}_+, \end{aligned}$$

since $[u_{t_2}, u_{t_3}] = 0$, $t_2, t_3 \in \mathbb{R}_+$. Similarly we have $\nabla u_{t_3} D_{t_1} u = 0$, $t_1, t_3 \in \mathbb{R}_+$, and this shows that (3.11) holds.

Next we state and prove Lemma 3.6 which has been used in the proof of Proposition 3.3 and corresponds to the commutation relation (1.11).

Lemma 3.6 *Let $u \in \mathbb{D}_{\infty,1}(H)$ and $v \in \mathbb{D}_{\infty,2}(H)$ such that $\|\nabla v\|_{\mathbb{D}_{\infty,1}(H)} < 1$. Then we have*

$$\langle \nabla^* v, D((I - \nabla v)^{-1} u)\rangle = \langle \nabla^* v, (I - \nabla v)^{-1} Du\rangle - \langle (I - \nabla v)^{-1} u, D \log \det{}_2 (I - \nabla v)\rangle. \tag{3.13}$$

Proof. By the commutation relation $\nabla_s D_t = D_t \nabla_s$, $s, t \in \mathbb{R}_+$, for all $1 \leq k \leq n$ we have

$$\begin{aligned}
\langle \nabla^* u, D((\nabla u)^k v)\rangle &= \int_0^\infty \cdots \int_0^\infty \langle \nabla^\dagger_{t_k} u_{t_{k+1}}, D_{t_{k+1}} (\nabla_{t_{k-1}} u_{t_k} \cdots \nabla_{t_0} u_{t_1} v_{t_0})\rangle dt_0 \cdots dt_{k+1} \\
&= \int_0^\infty \cdots \int_0^\infty \langle \nabla^\dagger_{t_k} u_{t_{k+1}}, \nabla_{t_{k-1}} u_{t_k} \cdots \nabla_{t_0} u_{t_1} D_{t_{k+1}} v_{t_0}\rangle dt_0 \cdots dt_{k+1} \\
&\quad + \int_0^\infty \cdots \int_0^\infty \langle \nabla^\dagger_{t_k} u_{t_{k+1}}, D_{t_{k+1}} (\nabla_{t_{k-1}} u_{t_k} \cdots \nabla_{t_0} u_{t_1}) v_{t_0}\rangle dt_0 \cdots dt_{k+1} \\
&= \operatorname{trace}((\nabla u)^{k+1} Dv) + \sum_{i=0}^{k-1} \int_0^\infty \cdots \int_0^\infty \\
&\quad \langle \nabla^\dagger_{t_k} u_{t_{k+1}}, \nabla_{t_{k+1}} u_{t_{k+2}} \cdots \nabla_{t_{i+1}} u_{t_{i+2}} (\nabla_{t_i} D_{t_{k+1}} u_{t_{i+1}}) \nabla_{t_{i-1}} u_{t_i} \cdots \nabla_{t_0} u_{t_1} v_{t_0}\rangle dt_0 \cdots dt_{k+1} \\
&= \operatorname{trace}((\nabla u)^{k+1} Dv) + \sum_{i=0}^{k-1} \frac{1}{k+1-i} \int_0^\infty \cdots \int_0^\infty \\
&\quad \langle \nabla_{t_i} \langle \nabla^\dagger_{t_k} u_{t_{k+1}}, \nabla_{t_{k+1}} u_{t_{k+2}} \cdots \nabla_{t_{i+1}} u_{t_{i+2}} \nabla_{t_{k+1}} u_{t_{i+1}}\rangle, \nabla_{t_{i-1}} u_{t_i} \cdots \nabla_{t_0} u_{t_1} v_{t_0}\rangle dt_0 \cdots dt_{k+1} \\
&= \operatorname{trace}((\nabla u)^{k+1} Dv) + \sum_{i=0}^{k-1} \frac{1}{k+1-i} \langle (\nabla u)^i v, D \operatorname{trace}(\nabla u)^{k+1-i}\rangle,
\end{aligned}$$

which shows that

$$\langle \nabla^* u, D((\nabla u)^k v)\rangle = \operatorname{trace}((\nabla u)^{k+1} Dv) + \sum_{i=2}^{k+1} \frac{1}{i} \langle (\nabla u)^{k+1-i} v, D \operatorname{trace}(\nabla u)^i\rangle,$$

$k \in \mathbb{N}$. This yields

$$\langle \nabla^* v, D((I - \nabla v)^{-1} u)\rangle = \sum_{k=0}^{\infty} \langle \nabla^* v, D(\nabla v)^k u\rangle \tag{3.14}$$

$$
\begin{aligned}
&= \sum_{k=0}^{\infty} \mathrm{trace}\,((\nabla v)^{k+1} Du) + \sum_{k=0}^{\infty}\sum_{n=2}^{k+1} \frac{1}{n} \langle (\nabla v)^{k+1-n} u, D\,\mathrm{trace}\,(\nabla v)^n \rangle \\
&= \frac{1}{2} \langle (I - \nabla v)^{-1} u, D\langle v, v\rangle\rangle \\
&\quad + \langle \nabla^* v, (I-\nabla v)^{-1} Du\rangle + \sum_{n=2}^{\infty} \frac{1}{n} \langle (I-\nabla v)^{-1} u, D\,\mathrm{trace}\,(\nabla v)^n\rangle \\
&= \langle \nabla^* v, (I-\nabla v)^{-1} Du\rangle - \langle (I-\nabla v)^{-1} u, D \log \det\nolimits_2 (I - \nabla v)\rangle,
\end{aligned}
$$

where we used the relation

$$
D \log \det\nolimits_2 (I - \nabla v) = - \sum_{n=2}^{\infty} \frac{1}{n} D\,\mathrm{trace}\,(\nabla v)^n,
$$

that follows from (1.12). □

Next we prove Lemma 3.7 which has been used in the proof of Proposition 3.3, and corresponds to (1.7) on the path space with $\Psi(x) = \|x\|^2/2$.

Lemma 3.7 *For any $u \in \mathbb{D}_{2,1}(H)$ with $\|\nabla u\|_{L^2(\mathbb{R}_+^2)} < 1$ a.s., we have*

$$
\langle (I - \nabla u)^{-1} v, u\rangle = \langle u, v\rangle + \frac{1}{2} \langle (I - \nabla u)^{-1} v, D\langle u, u\rangle\rangle, \qquad v \in H.
$$

Proof. We first show that

$$
\langle (\nabla u) v, u\rangle = \frac{1}{2} \langle v, D\langle u, u\rangle\rangle, \qquad u \in \mathbb{D}_{2,1}(H), \quad v \in H. \tag{3.15}
$$

Indeed we have

$$
\begin{aligned}
(\nabla^* u) u_t &= \int_0^\infty (\nabla_t u_s)^\dagger u_s ds \\
&= \int_0^\infty (D_t u_s)^\dagger u_s ds + \int_0^\infty \mathbf{1}_{[0,s]}(t) (\mathrm{ad}\, u_s)^\dagger u_s ds \\
&= \int_0^\infty (D_t u_s)^\dagger u_s ds - \int_0^\infty \mathbf{1}_{[0,s]}(t) \mathrm{ad}\,(u_s) u_s ds \\
&= \int_0^\infty (D_t u_s)^\dagger u_s ds \\
&= (D^* u) u_t,
\end{aligned}
$$

hence by the relation

$$
D_t \langle u, u\rangle = \int_0^\infty D_t \langle u_s, u_s\rangle_{\mathbb{R}^d} ds
$$

$$\begin{aligned} &= 2\int_0^\infty (D_t^\dagger u_s) u_s ds \\ &= 2(D^* u) u_t, \end{aligned}$$

we get

$$(\nabla^* u) u_t = \frac{1}{2} D_t \langle u, u \rangle, \qquad t \in \mathbb{R}_+, \tag{3.16}$$

and by integration against $v(t)dt$ we find that

$$\langle (\nabla u) v, u \rangle = \langle (\nabla^* u) u, v \rangle = \langle (D^* u) u, v \rangle = \frac{1}{2} \langle v, D\langle u, u \rangle \rangle. \tag{3.17}$$

In addition, (3.15) easily extends to all powers of ∇u as

$$\langle (\nabla u)^n v, u \rangle = \frac{1}{2} \langle (\nabla u)^{n-1} v, D\langle u, u \rangle \rangle, \qquad n \geq 1. \tag{3.18}$$

Hence for any $u \in \mathbb{D}_{2,1}(H)$ such that $\|\nabla u\|_{\mathbb{D}_{\infty,1}(H)} < 1$ we have

$$\begin{aligned} \langle (I - \nabla u)^{-1} v, u \rangle &= \sum_{n=0}^\infty \langle (\nabla u)^n v, u \rangle \\ &= \langle u, v \rangle + \frac{1}{2} \sum_{n=1}^\infty \langle (\nabla u)^{n-1} v, D\langle u, u \rangle \rangle \\ &= \langle u, v \rangle + \frac{1}{2} \sum_{n=0}^\infty \langle (\nabla u)^n v, D\langle u, u \rangle \rangle \\ &= \langle u, v \rangle + \frac{1}{2} \langle (I - \nabla u)^{-1} v, D\langle u, u \rangle \rangle, \qquad v \in H. \end{aligned}$$

□

4 The Poisson case

Conditions for the Skorohod integral on path space to have a Gaussian distribution have been obtained from (2.2) in Section 3 and Corollary 3.4. In this section we show that the general framework of Section 3 also includes other infinitely divisible distributions as we apply it to the Poisson space on $\mathbb{R}_+$, with $\Psi(x) = e^x - x - 1$.

Let $(N_t)_{t\in\mathbb{R}_+}$ be a standard Poisson process with jump times $(T_k)_{k\geq 1}$, and $T_0 = 0$, generating a filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ on a probability space $(\Omega, \mathcal{F}_\infty, P)$. The gradient $\tilde{D}$ defined as

$$\tilde{D}_t F = -\sum_{k=1}^{n} \mathbf{1}_{[0,T_k]}(t) \frac{\partial f}{\partial x_k}(T_1, \ldots, T_n), \tag{4.1}$$

for

$$F \in \mathcal{S} := \{F = f(T_1, \ldots, T_n) \quad : \quad f \in \mathcal{C}_b^\infty(\mathbb{R}^n)\},$$

has the derivation property and therefore satisfies Condition **(H2)**. In addition, the operator $\tilde{D}$ has an adjoint $\tilde{\delta}$ that satisfies (1.1) and Condition **(H1)**, cf. [2], [4], [7], and § 7.2 of [12]. Moreover, $\tilde{\delta}$ coincides with the compensated Poisson stochastic integral on square-integrable adapted processes $(u_t)_{t\in\mathbb{R}_+}$, i.e.

$$\tilde{\delta}(u) = \int_0^\infty u_t d(N_t - t),$$

The next definition of the covariant derivative ∇ in the jump case, cf. [8], is the counterpart of Definition 3.2. Here we let

$$\mathcal{U} = \left\{ \sum_{i=1}^{n} u_i F_i \quad : \quad F_i \in \mathcal{S},\ u_i \in \mathcal{C}_c^1(\mathbb{R}_+),\ i = 1, \ldots, n,\ n \geq 1 \right\}.$$

Definition 4.1 *Let the operator $\tilde{\nabla}$ be defined as*

$$\tilde{\nabla}_s u_t := \tilde{D}_s u_t - \dot{u}_t \mathbf{1}_{[0,t]}(s), \qquad s, t \in \mathbb{R}_+, \quad u \in \mathcal{U}, \tag{4.2}$$

where $\dot{u}_t$ denotes the derivative of $t \longmapsto u_t$ with respect to t.

In other words, given a vector field $u \in \mathcal{U}$ of the form $u = \sum_{i=1}^{n} F_i h_i$ we have

$$\tilde{\nabla}_s u_t := \sum_{i=1}^{n} h_i(t) \tilde{D}_s F_i - F_i \dot{h}_i(t) \mathbf{1}_{[0,t]}(s), \qquad s, t \in \mathbb{R}_+,$$

and

$$(\tilde{\nabla} u) v_t = \int_0^\infty v_s \tilde{\nabla}_s u_t ds, \qquad t \in \mathbb{R}_+,$$

is the covariant derivative of $u \in \mathcal{U}$ in the direction $v \in L^2(\mathbb{R}_+)$, cf. [8]. The operator $\tilde{D}$ defines the Sobolev spaces $\tilde{\mathbb{D}}_{p,1}$ and $\tilde{\mathbb{D}}_{p,1}(H)$, $p \in [1, \infty]$, respectively by the Sobolev norms

$$\|F\|_{\tilde{\mathbb{D}}_{p,1}} = \|F\|_{L^p(\Omega)} + \|\tilde{D}F\|_{L^p(\Omega, H)}, \qquad F \in \mathcal{S},$$

and

$$\|u\|_{\tilde{\mathbb{D}}_{p,1}(H)} = \|u\|_{L^p(\Omega,H)} + \|\tilde{D}u\|_{L^p(\Omega,H\hat{\otimes}H)} + E\left[\left(\int_0^\infty t|\dot{u}_t|^2dt\right)^{p/2}\right]^{1/p},$$

$u \in \mathcal{U}$, with

$$\tilde{\mathbb{D}}_{\infty,1}(H) = \bigcap_{p\geq 1} \tilde{\mathbb{D}}_{p,1}(H).$$

In addition, the operators $\tilde{\nabla}$, $\tilde{\delta}$ and $\tilde{D}$ satisfy the Skorohod isometry (1.23) under the form

$$E[\tilde{\delta}(u)^2] = E[\langle u,u\rangle_H] + E\left[\int_0^\infty \int_0^\infty \tilde{\nabla}_s u_t \tilde{\nabla}_t u_s ds dt\right], \qquad u \in \tilde{\mathbb{D}}_{2,1}(H),$$

and the commutation relation

$$\tilde{D}_t\tilde{\delta}(u) = u_t + \tilde{\delta}(\tilde{\nabla}_t u), \qquad t \in \mathbb{R}_+,$$

which is the commutation relation (1.4) in Condition **(H3)**, for $u \in \tilde{\mathbb{D}}_{2,1}(H)$ such that $\tilde{\nabla}_t u \in \tilde{\mathbb{D}}_{2,1}(H)$, $t \in \mathbb{R}_+$, cf. Relation (3.6) and Proposition 3.3 in [8].

As a consequence of Proposition 2.1 we have the following result, which yields (1.5) in the Poisson case with $\Psi(x) = e^x - x - 1$.

Proposition 4.2 *Let $u \in \tilde{\mathbb{D}}_{\infty,1}(H)$ such that the power series $(I-\lambda\tilde{\nabla}u)^{-1}u$ converges in $\mathbb{D}_{2,1}(H)$, and*

$$\sum_{n=2}^\infty \frac{\lambda^n}{n!}\int_0^\infty \|u_t^n\|_{\mathbb{D}_{2,1}} dt < \infty, \qquad \lambda \in (-a/2,a/2),$$

and $E[e^{a|\tilde{\delta}(u)|}] < \infty$ for some $a > 0$. Then we have

$$\begin{aligned}\frac{\partial}{\partial\lambda}E[e^{\lambda\tilde{\delta}(u)}] &= E[e^{\lambda\tilde{\delta}(u)}\langle e^{\lambda u}-1,u\rangle] \\ &+E\left[e^{\lambda\tilde{\delta}(u)}\left\langle (I-\lambda\tilde{\nabla}u)^{-1}u, \tilde{D}\int_0^\infty (e^{\lambda u_t}-\lambda u_t - 1)dt\right\rangle\right] \\ &+\lambda E\left[e^{\lambda\tilde{\delta}(u)}\langle\tilde{\nabla}^* u, \tilde{D}((I-\lambda\tilde{\nabla}u)^{-1}u)\rangle\right],\end{aligned}$$

for $\lambda \in (-a/2,a/2)$ such that $|\lambda| < \|\tilde{\nabla}u\|^{-1}_{L^\infty(\Omega,H\hat{\otimes}H)}$.

Proof. We apply Proposition 2.1 and Lemma 4.6 below. □

As a consequence of Lemma 4.4 below, when $(u_t)_{t\in\mathbb{R}_+}$ is adapted with respect to the Poisson filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$, we have

$$\langle\tilde{\nabla}^* u, \tilde{D}((I-\lambda\tilde{\nabla}u)^{-1}u)\rangle = 0,$$

in which case Proposition 4.2 yields

$$\frac{\partial}{\partial\lambda}E[e^{\lambda\tilde{\delta}(u)}] = E[e^{\lambda\tilde{\delta}(u)}\langle e^{\lambda u}-1, u\rangle] + E\left[e^{\lambda\tilde{\delta}(u)}\left\langle (I-\lambda\tilde{\nabla}u)^{-1}u, \tilde{D}\int_0^\infty (e^{\lambda u_t}-\lambda u_t-1)dt\right\rangle\right], \tag{4.3}$$

which is (1.9) on the Poisson space.

The next consequence of Proposition 4.2 is the Poisson analog of Corollary 3.4. It applies in particular to adapted process by Lemma 4.4 below.

Corollary 4.3 *Let* $(u_t)_{t\in\mathbb{R}_+}$ *be a process in* $\tilde{\mathbb{D}}_{\infty,1}(H)$ *that satisfies Condition* (4.4), *i.e.* $\langle\tilde{\nabla}^* u, \tilde{D}((\tilde{\nabla}u)^k u)\rangle = 0$, $k \geq 1$, *and assume that* $\int_0^\infty (u_t)^i dt$ *is deterministic for all* $i \geq 1$ *and such that*

$$\sum_{n=2}^\infty \frac{\lambda^n}{n!}\int_0^\infty \|u_t^n\|_{\mathbb{D}_{2,1}} dt < \infty,$$

$\lambda \in (-a/2, a/2)$, *and* $E[e^{a|\tilde{\delta}(u)|}] < \infty$ *for some* $a > 0$. *Then* $\tilde{\delta}(u)$ *has an infinitely divisible distribution with cumulants* $\left\{0, \int_0^\infty (u_t)^i dt,\ i \geq 2\right\}$.

Proof. Proposition 4.2 and Relation (4.3) show that

$$\frac{\partial}{\partial\lambda}E[e^{\lambda\tilde{\delta}(u)}] = \langle e^{\lambda u}-1, u\rangle E[e^{\lambda\tilde{\delta}(u)}],$$

as n goes to infinity, which yields

$$E[e^{\lambda\tilde{\delta}(u)}] = \exp\left(\int_0^\infty (e^{\lambda u_t}-\lambda u_t-1)dt\right), \qquad \lambda \in (-a/2, a/2),$$

from which the conclusion follows. □

The next lemma is the Poisson analog of Lemma 3.5 on the Lie-Wiener space.

Lemma 4.4 *Let* $u, v \in \tilde{\mathbb{D}}_{\infty,1}(H)$ *be two processes adapted with respect to the Poisson filtration* $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$, *such that* $(\tilde{\nabla}u)^n u \in \mathbb{D}_{2,1}(H)$, $n \geq 1$. *Then we have*

$$\langle\tilde{\nabla}^* u, \tilde{D}((\tilde{\nabla}u)^k v)\rangle = 0, \qquad k \in \mathbb{N}. \tag{4.4}$$

Proof. The proof of this lemma differs from the argument of Lemma 3.6 in Section 3 due to the fact that here, $\tilde{D}_s u_t$ and $\tilde{\nabla}_s u_t$ defined by (4.1) and (4.2) no longer belong to $\tilde{\mathbb{D}}_{2,1}$, and $\tilde{D}$ does not commute with $\tilde{\nabla}$. We have

$$\tilde{\nabla}_s u_t = \tilde{D}_s u_t = 0, \qquad s \geq t,$$

since $(u_t)_{t\in\mathbb{R}_+}$ is $\mathcal{F}_t$-adapted. Hence for all $k \geq 1$ we have, with the convention $\int_a^b f(s)ds = 0$ for $a > b$,

$$\begin{aligned}
&\langle \tilde{\nabla}^* u, \tilde{D}((\tilde{\nabla} u)^{k-1} v)\rangle \\
&= \int_0^\infty \int_0^\infty \tilde{\nabla}_{t_{k-1}} u_{t_k} \tilde{D}_{t_k} \int_0^\infty \cdots \int_0^\infty v_{t_0} \tilde{\nabla}_{t_{k-2}} u_{t_{k-1}} \cdots \tilde{\nabla}_{t_0} u_{t_1} dt_0 dt_1 \cdots dt_k \\
&= \int_0^\infty \int_0^{t_k} \tilde{\nabla}_{t_{k-1}} u_{t_k} \int_0^{t_{k-1}} \cdots \int_0^{t_1} \tilde{D}_{t_k} v_{t_0} \tilde{\nabla}_{t_{k-2}} u_{t_{k-1}} \cdots \tilde{\nabla}_{t_0} u_{t_1} dt_0 dt_1 \cdots dt_k \\
&\quad + \int_0^\infty \int_0^{t_k} v_{t_0} \tilde{\nabla}_{t_{k-1}} u_{t_k} \tilde{D}_{t_k} \int_0^{t_{k-1}} \cdots \int_0^{t_1} \tilde{\nabla}_{t_{k-2}} u_{t_{k-1}} \cdots \tilde{\nabla}_{t_0} u_{t_1} dt_0 dt_1 \cdots dt_k \\
&= \int_0^\infty \int_0^{t_0} (\tilde{D}_{t_k} v_{t_0}) \int_{t_0}^{t_k} \cdots \int_{t_0}^{t_2} \tilde{\nabla}_{t_{k-1}} u_{t_k} \cdots \tilde{\nabla}_{t_0} u_{t_1} dt_1 \cdots dt_k dt_0 \\
&\quad + \int_0^\infty v_{t_0} \int_0^\infty \int_{t_0}^{t_k} \tilde{\nabla}_{t_{k-1}} u_{t_k} \tilde{D}_{t_k} \int_{t_0}^{t_k} \cdots \int_{t_0}^{t_2} \tilde{\nabla}_{t_{k-2}} u_{t_{k-1}} \cdots \tilde{\nabla}_{t_0} u_{t_1} dt_1 \cdots dt_k dt_0 \\
&= \int_0^\infty v_{t_0} \int_0^\infty \int_{t_0}^{t_k} \tilde{\nabla}_{t_{k-1}} u_{t_k} \tilde{D}_{t_k} \int_{t_0}^{t_k} \cdots \int_{t_0}^{t_2} \tilde{\nabla}_{t_{k-2}} u_{t_{k-1}} \cdots \tilde{\nabla}_{t_0} u_{t_1} dt_1 \cdots dt_k dt_0 \\
&= 0,
\end{aligned}$$

since

$$\int_{t_0}^{t_k} \cdots \int_{t_0}^{t_2} \tilde{\nabla}_{t_{k-2}} u_{t_{k-1}} \cdots \tilde{\nabla}_{t_0} u_{t_1} dt_1 \cdots dt_{k-1}$$

is $\mathcal{F}_{t_k}$-measurable, cf. e.g. Lemma 7.2.3 in [12]. □

Examples of processes satisfying the conditions of Corollary 4.3 can be constructed by composition of a function of $\mathbb{R}_+$ with an adapted process of measure-preserving transformations, as in the next consequence of Corollary 4.3, cf. also (4.5) below.

Corollary 4.5 *Let $T > 0$ and $\tau : [0,T] \longmapsto [0,T]$ be an adapted process of measure-preserving transformations, such that $\tau_t \in \tilde{\mathbb{D}}_{\infty,1}$, $t \in \mathbb{R}_+$, with*

$$\sum_{n=2}^{\infty} \frac{\lambda^n}{n!} \int_0^T \|\tau_t^n\|_{L^2(\Omega)} dt < \infty,$$

$\lambda \in (-a/2, a/2)$, for some $a > 0$. Then for all $f \in \mathcal{C}_c^1([0,T])$, $\tilde{\delta}(f \circ \tau)$ has same distribution as the Poisson stochastic integral $\tilde{\delta}(f)$.

Proof. We check that $f \circ \tau \in \tilde{\mathbb{D}}_{\infty,1}(H)$ by (1.3) and

$$\begin{aligned} \sum_{n=2}^{\infty} \frac{\lambda^n}{n!} \int_0^T \|f^n(\tau_t)\|_{\mathbb{D}_{2,1}} dt &= \sum_{n=2}^{\infty} \frac{\lambda^n}{n!} \|f'\|_\infty^n \int_0^T \|\tau_t^n\|_{L^2(\Omega)} dt \\ &\quad + \|f'\|_\infty \int_0^T \|\tilde{D}\tau_t\|_{\mathbb{D}_{2,1}} dt \sum_{n=2}^{\infty} \frac{\lambda^n}{n!} \|f\|_\infty^{n-1} \\ &< \infty, \end{aligned}$$

$\lambda \in (-a/2, a/2)$, hence Corollary 4.3 can be applied as the condition $E[e^{a|\tilde{\delta}(u)|}] < \infty$ follows from $|\tilde{\delta}(f \circ \tau)| \leq \|f\|_\infty (T + N_T)$. □

As a consequence of Corollary 4.5 the mapping $T_n \longmapsto \tau(T_n)$ preserves the Poisson measure, see e.g. Theorem 3.10.21 of [1] for the classical version of the Girsanov theorem for adapted transformations of jump processes.

As an example of a process to which Corollary 4.5 can be applied, we can take

$$\begin{aligned} \tau_t &= t\mathbf{1}_{[0,T_1]}(t) + (2t - T_1)\mathbf{1}_{(T_1, T_1/2+T/2]}(t) \\ &\quad + (2T + T_1 - 2t)\mathbf{1}_{(T_1/2+T/2, T]}(t) + t\mathbf{1}_{(T \vee T_1, \infty)}(t), \end{aligned} \tag{4.5}$$

$t \in \mathbb{R}$, for some $T > 0$.

The following Lemma 4.6 has been used in the proof of Proposition 4.2, is the analog of Lemma 3.7 in the Lie-Wiener case and corresponds to the Poisson space version of the general identity (1.7). We note that

$$\begin{aligned} (\tilde{\nabla}^* u) u_t &= \int_0^\infty u_s \tilde{\nabla}_t u_s ds \\ &= \int_0^\infty u_s \tilde{D}_t u_s ds - \int_t^\infty u_s \dot{u}_s ds \\ &= \frac{1}{2} u_t^2 + \frac{1}{2} \int_0^\infty \tilde{D}_t u_s^2 ds \\ &= \frac{1}{2} u_t^2 + \frac{1}{2} \tilde{D}_t \langle u, u \rangle, \end{aligned}$$

for all $u \in \tilde{\mathbb{D}}_{2,1}(H)$, which corresponds to (3.16) on the Lie-Wiener space, and implies

$$\langle (\tilde{\nabla} u) v, u \rangle = \frac{1}{2} \langle v, u^2 \rangle + \frac{1}{2} \langle v, D \langle u, u \rangle \rangle, \qquad v \in H,$$

provided $u \in L^2(\mathbb{R}_+) \cap L^4(\mathbb{R}_+)$ a.s., cf. (3.15) on the Lie-Wiener space. In the next lemma we show that this relation can be extended to all powers of $\tilde{\nabla} u$ as in (4.7) below, although the extension is more complex to obtain than (3.18) in the path space case.

Lemma 4.6 *Let $u \in \tilde{\mathbb{D}}_{2,1}(H)$ such that*

$$\sum_{n=2}^{\infty} \frac{1}{n!} \int_0^\infty \| u_t^n \|_{\tilde{\mathbb{D}}_{2,1}} dt < \infty,$$

and $\| \tilde{\nabla} u \|_{L^\infty(\Omega; H \hat{\otimes} H)} < 1$. We have

$$\langle (I - \tilde{\nabla} u)^{-1} v, u \rangle = \langle e^u - 1, v \rangle + \left\langle (I - \tilde{\nabla} u)^{-1} v, \tilde{D} \int_0^\infty (e^{u_t} - u_t - 1) dt \right\rangle, \tag{4.6}$$

$v \in H$.

Proof. We begin by showing that for all $n \in \mathbb{N}$ and $u \in \tilde{\mathbb{D}}_{2,1}(H)$ such that $u \in \bigcap_{k=1}^{2n+2} L^k(\mathbb{R}_+)$ a.s. we have

$$\langle (\tilde{\nabla} u)^n v, u \rangle = \frac{1}{(n+1)!} \int_0^\infty u_s^{n+1} v_s ds + \sum_{i=2}^{n+1} \frac{1}{i!} \left\langle (\tilde{\nabla} u)^{n+1-i} v, \tilde{D} \int_0^\infty u_t^i dt \right\rangle, \qquad v \in H. \tag{4.7}$$

For all $n \geq 1$ we have

$$(\tilde{\nabla}^* u)^n u_{t_0} = \int_0^\infty \cdots \int_0^\infty u_{t_n} \tilde{\nabla}_{t_0} u_{t_1} \tilde{\nabla}_{t_1} u_{t_2} \cdots \tilde{\nabla}_{t_{n-1}} u_{t_n} dt_1 \cdots dt_n, \tag{4.8}$$

and we will show by induction on $1 \leq k \leq n+1$ that we have

$$\begin{aligned} (\tilde{\nabla}^* u)^n u_{t_0} &= \sum_{i=2}^{k} \frac{1}{i!} \int_0^\infty \cdots \int_0^\infty \tilde{\nabla}_{t_0} u_{t_1} \cdots \tilde{\nabla}_{t_{n-i}} u_{t_{n+1-i}} \tilde{D}_{t_{n+1-i}} u_{t_{n+2-i}}^i dt_1 \cdots dt_{n-i} \\ &\quad + \frac{1}{k!} \int_0^\infty \cdots \int_0^\infty u_{t_{n+1-k}}^k \tilde{\nabla}_{t_0} u_{t_1} \cdots \tilde{\nabla}_{t_{n-k}} u_{t_{n+1-k}} dt_1 \cdots dt_{n+1-k}, \end{aligned} \tag{4.9}$$

which holds for $k = 1$ by (4.8), and yields the desired identity for $k = n+1$. Next, assuming that the identity (4.9) holds for some $k \in \{1, \ldots, n\}$, and using the relation

$$\tilde{\nabla}_{t_{n-k}} u_{t_{n+1-k}} = \tilde{D}_{t_{n-k}} u_{t_{n+1-k}} - \mathbf{1}_{[0, t_{n+1-k}]}(t_{n-k}) \dot{u}_{t_{n+1-k}}, \qquad t_{n-k}, t_{n+1-k} \in \mathbb{R}_+,$$

we have

$$
\begin{aligned}
(\tilde{\nabla}^* u)^n u_{t_0} = & \sum_{i=2}^{k} \frac{1}{i!} \int_0^\infty \cdots \int_0^\infty \tilde{\nabla}_{t_0} u_{t_1} \cdots \tilde{\nabla}_{t_{n-i}} u_{t_{n+1-i}} \tilde{D}_{t_{n+1-i}} u^i_{t_{n+2-i}} dt_1 \cdots dt_{n+2-i} \\
& + \frac{1}{k!} \int_0^\infty \cdots \int_0^\infty u^k_{t_{n+1-k}} \tilde{\nabla}_{t_0} u_{t_1} \cdots \tilde{\nabla}_{t_{n-k}} u_{t_{n+1-k}} dt_1 \cdots dt_{n+1-k} \\
= & \sum_{i=2}^{k} \frac{1}{i!} \int_0^\infty \cdots \int_0^\infty \tilde{\nabla}_{t_0} u_{t_1} \cdots \tilde{\nabla}_{t_{n-i}} u_{t_{n+1-i}} \tilde{D}_{t_{n+1-i}} u^i_{t_{n+2-i}} dt_1 \cdots dt_{n+2-i} \\
& + \frac{1}{k!} \int_0^\infty \cdots \int_0^\infty u^k_{t_{n+1-k}} \tilde{\nabla}_{t_0} u_{t_1} \cdots \tilde{\nabla}_{t_{n-k-1}} u_{t_{n-k}} \tilde{D}_{t_{n-k}} u_{t_{n+1-k}} dt_1 \cdots dt_{n+1-k} \\
& - \frac{1}{k!} \int_0^\infty \cdots \int_0^\infty \int_{t_{n-k}}^\infty \dot{u}_{t_{n+1-k}} u^k_{t_{n+1-k}} \tilde{\nabla}_{t_0} u_{t_1} \cdots \tilde{\nabla}_{t_{n-1-k}} u_{t_{n-k}} dt_1 \cdots dt_{n+1-k} \\
= & \sum_{i=2}^{k} \frac{1}{i!} \int_0^\infty \cdots \int_0^\infty \tilde{\nabla}_{t_0} u_{t_1} \cdots \tilde{\nabla}_{t_{n-i}} u_{t_{n+1-i}} \tilde{D}_{t_{n+1-i}} u^i_{t_{n+2-i}} dt_1 \cdots dt_{n+2-i} \\
& + \frac{1}{(k+1)!} \int_0^\infty \cdots \int_0^\infty \tilde{\nabla}_{t_0} u_{t_1} \cdots \tilde{\nabla}_{t_{n-k-1}} u_{t_{n-k}} \tilde{D}_{t_{n-k}} u^{k+1}_{t_{n+1-k}} dt_1 \cdots dt_{n+1-k} \\
& - \frac{1}{(k+1)!} \int_0^\infty \cdots \int_0^\infty \tilde{\nabla}_{t_0} u_{t_1} \cdots \tilde{\nabla}_{t_{n-k-1}} u_{t_{n-k}} \int_{t_{n-k}}^\infty (u^{k+1}_t)' dt dt_1 \cdots dt_{n-k} \\
= & \sum_{i=2}^{k+1} \frac{1}{i!} \int_0^\infty \cdots \int_0^\infty \tilde{\nabla}_{t_0} u_{t_1} \cdots \tilde{\nabla}_{t_{n-i}} u_{t_{n+1-i}} \tilde{D}_{t_{n+1-i}} u^i_{t_{n+2-i}} dt_1 \cdots dt_{n+2-i} \\
& + \frac{1}{(k+1)!} \int_0^\infty \cdots \int_0^\infty u^{k+1}_{t_{n-k}} \tilde{\nabla}_{t_0} u_{t_1} \cdots \tilde{\nabla}_{t_{n-k-1}} u_{t_{n-k}} dt_1 \cdots dt_{n-k} \\
= & \sum_{i=2}^{k+1} \frac{1}{i!} (\tilde{\nabla}^* u)^{n+1-i} \tilde{D}_t \int_0^\infty u^i_s ds + \frac{1}{(k+1)!} (\tilde{\nabla}^* u)^{n-k} u^{k+1}_t,
\end{aligned}
$$

which shows by induction for $k = n$ that

$$
(\tilde{\nabla}^* u)^n u_t = \frac{1}{(n+1)!} u^{n+1}_t + \sum_{i=2}^{n+1} \frac{1}{i!} (\tilde{\nabla}^* u)^{n+1-i} \tilde{D}_t \int_0^\infty u^i_s ds, \qquad t \in \mathbb{R}_+, \tag{4.10}
$$

and (4.7) follows by integration with respect to $t \in \mathbb{R}_+$.

Next, by (4.7), for all $u \in \tilde{\mathbb{D}}_{2,1}(H)$, $v \in H$ and $n \in \mathbb{N}$, by (4.10) we have

$$
\langle (I - \tilde{\nabla} u)^{-1} v, u \rangle = \sum_{n=0}^\infty \langle (\tilde{\nabla} u)^n v, u \rangle
$$

$$
\begin{aligned}
&= \sum_{n=0}^{\infty} \frac{1}{(n+1)!} \int_0^\infty u_s^{n+1} v_s ds + \sum_{n=0}^{\infty} \sum_{i=2}^{n+1} \frac{1}{i!} \left\langle (\tilde{\nabla} u)^{n+1-i} v, \tilde{D} \int_0^\infty u_t^i dt \right\rangle \\
&= \langle e^u - 1, v \rangle + \sum_{i=2}^{\infty} \frac{1}{i!} \left\langle (I - \tilde{\nabla} u)^{-1} v, \tilde{D} \int_0^\infty u_t^i dt \right\rangle \\
&= \langle e^u - 1, v \rangle + \left\langle (I - \tilde{\nabla} u)^{-1} v, \tilde{D} \int_0^\infty (e^{u_t} - u_t - 1) dt \right\rangle,
\end{aligned}
$$

which shows (4.6). □

We also have the following moment identity, which is the Poisson analog of Proposition 1 in [15], cf. also Lemma 1 of [10] for another version using finite difference operators.

Corollary 4.7 *For any $n \geq 1$, $u, v \in \tilde{\mathbb{D}}_{n+1,2}(H)$ and $F \in \tilde{\mathbb{D}}_{2,1}$ we have*

$$
\begin{aligned}
E[F \tilde{\delta}(u)^n \tilde{\delta}(v)] &= \sum_{k=1}^{n} \binom{n}{k} E\left[F \tilde{\delta}(u)^{n-k} \int_0^\infty u_s^k v_s ds \right] \\
&\quad + \sum_{k=2}^{n} \frac{n!}{(n-k)!} \sum_{i=2}^{k} \frac{1}{i!} E\left[F \tilde{\delta}(u)^{n-k} \left\langle (\tilde{\nabla} u)^{k-i} v, \tilde{D} \int_0^\infty u_s^i ds \right\rangle \right] \\
&\quad + \sum_{k=1}^{n} \frac{n!}{(n-k)!} E\left[F \tilde{\delta}(u)^{n-k} \langle \tilde{\nabla}^* u, \tilde{D}((\tilde{\nabla} u)^{k-1} v) \rangle \right] \\
&\quad + \sum_{k=0}^{n} \frac{n!}{(n-k)!} E\left[\tilde{\delta}(u)^{n-k} \langle (\tilde{\nabla} u)^k v, \tilde{D} F \rangle \right].
\end{aligned}
$$

Proof. This result is a consequence of Lemma 2.2 associated to Relation (4.10). □

5 The Wiener case

In this section we consider the case where $\mathsf{G} = \mathbb{R}^d$ and $(\gamma(t))_{t \in \mathbb{R}_+} = (B_t)_{t \in \mathbb{R}_+}$ is a standard $\mathbb{R}^d$-valued Brownian motion on the Wiener space $W = \mathcal{C}_0(\mathbb{R}_+, \mathbb{R}^d)$, in which case ∇ is equal to the Malliavin derivative $\hat{D}$ defined by

$$
\hat{D}_t F = \sum_{i=1}^{n} \mathbf{1}_{[0,t_i]}(t) \partial_i f(B_{t_1}, \ldots, B_{t_n}), \qquad t \in \mathbb{R}_+, \tag{5.1}
$$

for F of the form

$$F = f(B_{t_1}, \ldots, B_{t_n}), \tag{5.2}$$

$f \in \mathcal{C}_b^\infty(\mathbb{R}^n, X)$, $t_1, \ldots, t_n \in \mathbb{R}_+$, $n \geq 1$. Let $\hat{\delta}$ denote the Skorohod integral operator adjoint of $\hat{D}$, which coincides with the Itô integral of $u \in L^2(W; H)$ with respect to Brownian motion, i.e.

$$\hat{\delta}(u) = \int_0^\infty u_t dB_t,$$

when u is square-integrable and adapted with respect to the Brownian filtration. As a consequence of Proposition 3.3 we obtain the following derivation formula, which yields (1.14).

Proposition 5.1 *Let $u \in \mathbb{D}_{\infty,2}(H)$ such that $E[e^{a|\delta(u)|}] < \infty$ for some $a > 0$. We have*

$$\begin{aligned}
\frac{\partial}{\partial\lambda} E[e^{\lambda\hat{\delta}(u)}] &= \lambda E[\langle u, u\rangle e^{\lambda\hat{\delta}(u)}] + \frac{1}{2}\lambda^2 E[e^{\lambda\hat{\delta}(u)}\langle (I - \lambda\hat{D}u)^{-1}u, \hat{D}\langle u, u\rangle\rangle] \\
&\quad -\lambda E\left[e^{\lambda\hat{\delta}(u)} \frac{\partial}{\partial\lambda}\log\det{}_2(I - \lambda\hat{D}u)\right] \\
&\quad -\lambda E\left[e^{\lambda\hat{\delta}(u)}\langle (I - \lambda\hat{D}u)^{-1}u, \hat{D}\log\det{}_2(I - \lambda\hat{D}u)\rangle\right],
\end{aligned} \tag{5.3}$$

for $u \in \mathbb{D}_{\infty,2}(H)$ such that $E[e^{a|\delta(u)|}] < \infty$ for some $a > 0$ and $\lambda \in (-a/2, a/2)$ such that $|\lambda| < \|\hat{D}u\|^{-1}_{\mathbb{D}_{\infty,1}(H)}$,

Proof. We apply Proposition 3.3 with $\nabla = \hat{D}$, and use the equality

$$\begin{aligned}
\frac{\partial}{\partial\lambda}\log\det{}_2(I - \lambda\hat{D}u) &= -\sum_{n=2}^\infty \lambda^{n-1}\operatorname{trace}(\hat{D}u)^n \\
&= -\sum_{n=2}^\infty \lambda^{n-1}\langle \hat{D}^*u, (\hat{D}u)^{n-1}\rangle \\
&= -\lambda\langle \hat{D}^*u, (I - \lambda\hat{D}u)^{-1}\hat{D}u\rangle, \qquad \lambda \in (-a, a),
\end{aligned} \tag{5.4}$$

that follows from (1.12), □

Next we show how (5.3) can be used to recover some known results on the Laplace transform of second order Wiener functionals of the form

$$\hat{\delta}(\psi) + \hat{\delta}(\hat{\delta}(\phi))$$

where $\psi \in L^2(\mathbb{R}_+)$ and $\phi \in L^2(\mathbb{R}_+^2)$, cf. e.g. [3].

Proposition 5.2 *Let* $\psi \in L^2(\mathbb{R}_+)$ *and* $\phi \in L^2(\mathbb{R}_+^2)$ *such that* $\|\phi\|_{L^2(\mathbb{R}_+^2)} < 1$. *We have*

$$E[e^{\hat{\delta}(\psi)+\frac{1}{2}\hat{\delta}(\hat{\delta}(\phi))}] = \frac{1}{\sqrt{\det_2(I-\phi)}} e^{\frac{1}{2}\langle \psi,(I-\phi)^{-1}\psi\rangle}. \tag{5.5}$$

Proof. We let $u_t = \frac{1}{2}\hat{\delta}(\phi(\cdot,t))$, $t \in \mathbb{R}_+$, and we start by showing that

$$E[e^{\hat{\delta}(u)}] = \frac{1}{\sqrt{\det_2(I-2\hat{D}u)}}. \tag{5.6}$$

Since $\hat{D}u = \phi/2$ is deterministic, by Proposition 3.3, Relation (5.3) we have

$$\begin{aligned}
E[\hat{\delta}(u)e^{\lambda\hat{\delta}(u)}] &= \lambda E[\langle u,u\rangle e^{\lambda\hat{\delta}(u)}] + \frac{\lambda^2}{2} E[e^{\lambda\hat{\delta}(u)}\langle (I-\lambda\hat{D}u)^{-1}u, \hat{D}\langle u,u\rangle\rangle] \\
&\quad + \lambda E[e^{\lambda\hat{\delta}(u)}]\langle \hat{D}^*u, (I-\lambda\hat{D}u)^{-1}\hat{D}u\rangle \\
&= \lambda E[\langle (I-\lambda\hat{D}u)^{-1}(I-\lambda\hat{D}u)u,u\rangle e^{\lambda\hat{\delta}(u)}] + \lambda^2 E[e^{\lambda\hat{\delta}(u)}\langle (I-\lambda\hat{D}u)^{-1}u, (\hat{D}u)u\rangle] \\
&\quad + \lambda E[e^{\lambda\hat{\delta}(u)}]\langle \hat{D}^*u, (I-\lambda\hat{D}u)^{-1}\hat{D}u\rangle \\
&= \lambda E[\langle (I-\lambda\hat{D}u)^{-1}u,u\rangle e^{\lambda\hat{\delta}(u)}] + \lambda E[e^{\lambda\hat{\delta}(u)}]\langle \hat{D}^*u, (I-\lambda\hat{D}u)^{-1}\hat{D}u\rangle \\
&= \lambda E[\langle (I-\lambda\hat{D}u)^{-1}\hat{D}u, \hat{\delta}(u)\rangle e^{\lambda\hat{\delta}(u)}] + 2\lambda E[e^{\lambda\hat{\delta}(u)}]\langle (I-\lambda\hat{D}u)^{-1}\hat{D}u, \hat{D}u\rangle, \qquad (5.7)
\end{aligned}$$

since

$$u_s u_t = \hat{\delta}(\hat{D}u_s)\hat{\delta}(\hat{D}u_t) = \hat{\delta}(\hat{D}u_s\hat{\delta}(\hat{D}u_t)) + \langle \hat{D}u_s, \hat{D}u_t\rangle = \hat{D}u_s\hat{\delta}(u_t) + \langle \hat{D}u_s, \hat{D}u_t\rangle.$$

Hence by repeated application of (5.7) we get

$$\begin{aligned}
\frac{\partial}{\partial\lambda} E[e^{\lambda\hat{\delta}(u)}] &= E[\hat{\delta}(u)e^{\lambda\hat{\delta}(u)}] \\
&= 2\lambda E[e^{\lambda\hat{\delta}(u)}]\sum_{n=0}^{\infty}\langle \hat{D}^*u, ((I-\lambda\hat{D}u)^{-1}\hat{D}u)^n((I-\lambda\hat{D}u)^{-1}\hat{D}u)\rangle \\
&= 2\lambda E[e^{\lambda\hat{\delta}(u)}]\langle \hat{D}^*u, (I-2\lambda\hat{D}u)^{-1}\hat{D}u\rangle \\
&= -\frac{1}{2}\frac{\partial}{\partial\lambda}\log\det{}_2(I-2\lambda\hat{D}u),
\end{aligned}$$

and (5.6) holds. Next, since $\hat{D}u \in L^2(\mathbb{R}_+^2)$ is deterministic and $u = \hat{\delta}(\hat{D}u)$, from (3.14) we have, for $\psi \in L^2(\mathbb{R}_+)$,

$$E[\hat{\delta}(\psi)e^{\lambda\hat{\delta}(\psi)+\hat{\delta}(u)}] = E[\langle \lambda\psi + u, \psi\rangle e^{\lambda\hat{\delta}(\psi)+\hat{\delta}(u)}]$$

$$
\begin{aligned}
& +\frac{1}{2}E[e^{\lambda\hat\delta(\psi)+\hat\delta(u)}\langle (I-\hat D u)^{-1}\psi, \hat D\langle \lambda\psi+u,\lambda\psi+u\rangle\rangle] \\
= \ & \lambda\langle\psi,\psi\rangle E[e^{\lambda\hat\delta(\psi)+\hat\delta(u)}] + E[\langle u,\psi\rangle e^{\lambda\hat\delta(\psi)+\hat\delta(u)}] + \lambda E[e^{\lambda\hat\delta(\psi)+\hat\delta(u)}\langle (I-\hat D u)^{-1}\psi, \hat D\langle\psi,u\rangle\rangle] \\
& +\frac{1}{2}E[e^{\lambda\hat\delta(\psi)+\hat\delta(u)}\langle (I-\hat D u)^{-1}\psi, \hat D\langle u,u\rangle\rangle] \\
= \ & \lambda\langle\psi,\psi\rangle E[e^{\lambda\hat\delta(\psi)+\hat\delta(u)}] + \lambda\langle (I-\hat D u)^{-1}\psi, (\hat D u)\psi\rangle E[e^{\lambda\hat\delta(\psi)+\hat\delta(u)}] \\
& + E[\langle u,\psi\rangle e^{\lambda\hat\delta(\psi)+\hat\delta(u)}] + E[e^{\lambda\hat\delta(\psi)+\hat\delta(u)}\langle (I-\hat D u)^{-1}\psi, (\hat D u)u\rangle] \\
= \ & \lambda\langle\psi, (I-\hat D u)^{-1}\psi\rangle E[e^{\lambda\hat\delta(\psi)+\hat\delta(u)}] + E[e^{\lambda\hat\delta(\psi)+\hat\delta(u)}\langle (I-\hat D u)^{-1}\psi, u\rangle] \\
= \ & \lambda\langle\psi, (I-\hat D u)^{-1}\psi\rangle E[e^{\lambda\hat\delta(\psi)+\hat\delta(u)}] + E[e^{\lambda\hat\delta(\psi)+\hat\delta(u)}\langle (I-\hat D u)^{-1}\psi, \hat\delta(\hat D u)\rangle],
\end{aligned}
$$

hence by induction on $n \geq 1$,

$$
\begin{aligned}
\frac{\partial}{\partial\lambda}E[e^{\lambda\hat\delta(\psi)+\hat\delta(u)}] &= \lambda E[e^{\lambda\hat\delta(\psi)+\hat\delta(u)}]\sum_{n=0}^{\infty}(-1)^n\langle\psi, (I-\hat D u)^{-1}((I-\hat D u)^{-1}\hat D u)^n\psi\rangle \\
&= \lambda\langle\psi, (I-2\hat D u)^{-1}\psi\rangle E[e^{\lambda\hat\delta(\psi)+\hat\delta(u)}],
\end{aligned}
$$

which yields (5.5). □

Finally we remark that the formulas of Section 4 can be applied to the Skorohod integral $\hat\delta$ on the Wiener space when it is used to represent the Poisson stochastic integral $\tilde\delta(u)$ of a deterministic function by Proposition 6 of [7].

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