Moments of k-hop counts in the random-connection model

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Abstract

We derive moment identities for the stochastic integrals of multiparameter processes in a random-connection model based on a point process admitting a Papangelou intensity. Those identities are written using sums over partitions, and they reduce to sums over non-flat partition diagrams in case the multiparameter processes vanish on diagonals. As an application, we obtain general identities for the moments of k-hop counts in the random-connection model, which simplify the derivations available in the literature.

Key words: Point processes, moments, random-connection model, random graph, k-hops.

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1 Introduction

The random-connection model, see e.g. Chapter 6 of [12], is a classical model in continuum percolation. It consists in a random graph built on the vertices of a point process on $\mathbb{R}^d$, by adding edges between two distinct vertices x and y with probability $H(\|x-y\|)$. In the case of the Rayleigh fading $H_\beta(\|x-y\|) = \mathrm{e}^{-\beta\|x-y\|^2}$ with $x, y \in \mathbb{R}^2$, the mean value of the number $N_k^{x,y}$ of k-hop paths connecting $x \in \mathbb{R}^d$

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to $y \in \mathbb{R}^d$ has been computed in [9], together with the variance of 3-hop counts. However, this argument does not extend to $k \geq 3$ as the proof of the variance identity for 3-hop counts in [9] relies on the known Poisson distribution of the 2-hop count. As shown by [9], the knowledge of moments can provide accurate numerical estimates of the probability $P(N_k^{x,y} > 0)$ of at least one k-hop path, by expressing it as a series of factorial moments, and the need for a general theory of such moments has been pointed out therein.

On the other hand, moment identities for Poisson stochastic integrals with random integrands have been obtained in [18] based on moment identities for Skorohod's integral on the Poisson space, see [16, 17], and also [19] for a review. These moment identities have been extended to point processes with Papangelou intensities by [5], and to multiparameter processes by [2]. Factorial moments have also been computed by [4] for point processes with Papangelou intensities.

In this paper we derive closed-form expressions for the moments of the number of k-hop paths in the random-connection model. In Proposition 3.1 the moment of order n of the k-hop count is given as a sum over non-flat partitions of $\{1, \ldots, nk\}$ in a general random-connection model based on a point process admitting a Papangelou intensity. Those results are then specialized to the case of Poisson point processes, with an expression for the variance of the k-hop count given in Corollary 3.2 using a sum over integer sequences. Finally, in the case of Rayleigh fadings we show that some results of [9], such as the computation of variance for 3-hop counts, can be recovered via a shorter argument, see Corollary 5.3.

We proceed as follows. After presenting some background notation on point processes and Campbell measures, see [8], in Section 2 we review the derivation of moment identities for stochastic integrals using sums over partitions. In the multiparameter case we rewrite those identities for processes vanishing on diagonals, based on non-flat partition diagrams. In Section 3 we apply those results to the computation of the moments of k-hop counts in the random-connection model, and we specialize such computations to the case of Poisson point processes in Section 4. Section 5 is devoted to explicit computations in the case of Rayleigh fadings, which result into simpler derivations in comparison with the current literature on moments in the random-connection model.

Notation on point processes

Let X be a Polish space with Borel σ-algebra $\mathcal{B}(X)$, equipped with a σ-finite non-atomic measure $\lambda(dx)$. We let

$$\Omega^X = \big\{ \omega = \{x_i\}_{i \in I} \subset X \ : \ \#(A \cap \omega) < \infty \text{ for all compact } A \in \mathcal{B}(X) \big\}$$

denote the space of locally finite configurations on X, whose elements $\omega \in \Omega^X$ are identified with the Radon point measures $\omega = \sum_{x \in \omega} \epsilon_x$, where ϵ_x denotes the Dirac measure at $x \in X$. A point process is a probability measure P on Ω^X equipped with the σ-algebra $\mathcal{F}$ generated by the topology of vague convergence.

Point processes can be characterized by their Campbell measure C defined on $\mathcal{B}(X) \otimes \mathcal{F}$ by

$$C(A \times B) := \mathbb{E}\left[\int_A \mathbb{1}_B(\omega \setminus \{x\}) \, \omega(dx) \right], \quad A \in \mathcal{B}(X), \quad B \in \mathcal{F},$$

which satisfies the Georgii-Nguyen-Zessin [14] identity

$$\mathbb{E}\left[\int_X u(x;\omega)\omega(dx) \right] = \mathbb{E}\left[\int_{\Omega^X} \int_X u(x; \omega \cup x) C(dx, d\omega) \right], \tag{1.1}$$

for all measurable processes $u : X \times \Omega^X \to \mathbb{R}$ such that both sides of (1.1) make sense.

In the sequel we deal with point processes whose Campbell measure $C(dx, d\omega)$ is absolutely continuous with respect to $\lambda \otimes P$, i.e.

$$C(dx, d\omega) = c(x;\omega)\lambda(dx)P(d\omega),$$

where the density $c(x;\omega)$ is called the Papangelou density. We will also use the random measure $\hat{\lambda}^n(d\mathfrak{x}_n)$ defined on X^n by

$$\hat{\lambda}^n(d\mathfrak{x}_n) = \hat{c}(\mathfrak{x}_n;\omega)\lambda(dx_1)\cdots\lambda(dx_n),$$

where $\hat{c}(\mathfrak{x}_n;\omega)$ is the compound Campbell density $\hat{c} : \Omega_0^X \times \Omega^X \longrightarrow \mathbb{R}_+$ defined inductively on the set Ω_0^X of finite configurations in Ω^X by

$$\hat{c}(\{x_1,\ldots,x_n,y\};\omega) := c(y;\omega)\hat{c}(\{x_1,\ldots,x_n\};\omega\cup\{y\}), \qquad n \geq 0, \tag{1.2}$$

see Relation (1) in [5]. In particular, the Poisson point process with intensity $\lambda(dx)$ is a point process with Campbell measure $C = \lambda \otimes P$ and $c(x;\omega) = 1$, and in this case the identity (1.1) becomes the Slivnyak-Mecke formula [20], [11]. Determinantal point processes are examples of point processes with Papangelou intensities, see e.g. Theorem 2.6 in [6], and they can be used for the modeling of wireless networks with repulsion, see e.g. [7], [13], [10].

2 Moment identities

The moment of order $n \geq 1$ of a Poisson random variable Z_α with parameter $\alpha > 0$ is given by

$$\mathbb{E}[Z_\alpha^n] = \sum_{k=0}^n \alpha^k S(n,k), \qquad n \in \mathbb{N}, \tag{2.1}$$

where the Stirling number of the second kind $S(n,k)$ is the number of ways to partition a set of n objects into k non-empty subsets, see e.g. Proposition 3.1 of [3]. Regarding Poisson stochastic integrals of deterministic integrands, in [1] the moment formula

$$\mathbb{E}\left[\left(\int_X f(x)\omega(dx)\right)^n\right] = n! \sum_{\substack{r_1+2r_2+\cdots+nr_n=n \\ r_1,\ldots,r_n\geq 0}} \prod_{k=1}^n \left(\frac{1}{(k!)^{r_k} r_k!}\left(\int_X f^k(x)\lambda(dx)\right)^{r_k}\right) \tag{2.2}$$

has been proved for deterministic functions $f \in \bigcap_{p\geq 1} L^p(X,\lambda)$.

The identity (2.2) has been rewritten in the language of sums over partitions, and extended to Poisson stochastic integrals of random integrands in Proposition 3.1 of [18], and further extended to point processes admitting a Panpangelou intensity in Theorem 3.1 of [5], see also [4]. In the sequel, given $\mathfrak{z}_n = (z_1,\ldots,z_n) \in X^n$, we will use the shorthand notation $\varepsilon_{\mathfrak{z}_n}^+$ for the operator

$$(\varepsilon_{\mathfrak{z}_n}^+ F)(\omega) = F(\omega \cup \{z_1,\ldots,z_n\}), \qquad \omega \in \Omega,$$

where F is any random variable on Ω^X. Given $\rho = \{\rho_1, \ldots, \rho_k\} \in \Pi[n]$ a partition of $\{1, \ldots, n\}$ of size $|\rho| = k$, we let $|\rho_i|$ denote the cardinality of each block ρ_i, $i = 1, \ldots, k$.

Proposition 2.1 *Let $u : X \times \Omega^X \longrightarrow \mathbb{R}$ be a (measurable) process. For all $n \geq 1$ we have*

$$\mathbb{E}\left[\left(\int_X u(x;\omega)\omega(dx)\right)^n\right] = \sum_{\rho\in\Pi[n]} \mathbb{E}\left[\int_{X^{|\rho|}} \epsilon^+_{\mathfrak{z}_{|\rho|}} \prod_{l=1}^{|\rho|} u^{|\rho_l|}(z_l)\hat{\lambda}^{|\rho|}(d\mathfrak{z}_{|\rho|})\right],$$

where the sum runs over all partitions ρ of $\{1, \ldots, n\}$ with cardinality $|\rho|$.

Proposition 2.1 has also been extended, together with joint moment identities, to multiparameter processes $(u_{z_1,\ldots z_r})_{(z_1,\ldots z_r)\in X^r}$, see Theorem 3.1 of [2]. For this, let $\Pi[n \times r]$ denote the set of all partitions of the set

$$\Delta_{n\times r} := \{1, \ldots, n\} \times \{1, \ldots, r\} = \big\{(k, l) \ : \ k = 1, \ldots, n, \ l = 1, \ldots, r\big\},$$

identified to $\{1, \ldots, nr\}$, and let $\pi := (\pi_1, \ldots, \pi_n) \in \Pi[n \times r]$ denote the partition made of the n blocks $\pi_k := \{(k, 1), \ldots, (k, r)\}$ of size r, for $k = 1, \ldots, n$. Given $\rho = \{\rho_1, \ldots, \rho_m\}$ a partition of $\Delta_{n\times r}$, we let $\zeta^\rho : \Delta_{n\times r} \longrightarrow \{1, \ldots, m\}$ denote the mapping defined as

$$\zeta^\rho(k, l) = p \ \text{ if and only if } (k, l) \in \rho_p, \tag{2.3}$$

$k = 1, \ldots, n$, $l = 1, \ldots, r$, $p = 1, \ldots, m$. In other words, $\zeta^\rho(k, l)$ denotes the index p of the block $\rho_p \subset \Delta_{n\times r}$ to which (k, l) belongs.

Next, we restate Theorem 3.1 of [2] by noting that, in the same way as in Proposition 2.1, it extends to point processes admitting a Papangelou intensity using the arguments of [5], [4]. When $(u(z_1, \ldots, z_k;\omega))_{z_1,\ldots,z_k\in X}$ is a multiparameter process, we will write

$$\epsilon^+_{\mathfrak{z}_k} u(z_1, \ldots, z_k;\omega) := u\big(z_1, \ldots, z_k;\omega \cup \{z_1, \ldots, z_k\}\big), \quad \mathfrak{z}_n = (z_1, \ldots, z_n) \in X^n,$$

and in this case we may drop the variable $\omega \in \Omega^X$ by writing $\epsilon^+_{\mathfrak{z}_k} u(z_1, \ldots, z_k;\omega)$ instead of $\epsilon^+_{\mathfrak{z}_k} u(z_1, \ldots, z_k;\omega)$.

Proposition 2.2 *Let $u : X^r \times \Omega^X \longrightarrow \mathbb{R}$ be a (measurable) r-process. We have*

$$\mathbb{E}\left[\left(\int_{X^r} u(z_1,\ldots,z_r;\omega)\omega(dz_1)\cdots\omega(dz_r)\right)^n\right] = \sum_{\rho\in\Pi[n\times r]} \mathbb{E}\left[\int_{X^{|\rho|}} \varepsilon^+_{\mathfrak{z}_{|\rho|}} \prod_{k=1}^n u\big(z^\rho_{\pi_k}\big)\hat{\lambda}^{|\rho|}(d\mathfrak{z}_{|\rho|})\right] \tag{2.4}$$

with $z^\rho_{\pi_k} := (z_{\zeta^\rho(k,1)},\ldots,z_{\zeta^\rho(k,r)})$, $k=1,\ldots,n$.

Proof. The main change in the proof argument of [2] is to rewrite the proof of Lemma 2.1 therein by applying (1.2) recursively as in the proof of Theorem 3.1 of [5], while the main combinatorial argument remains identical. □

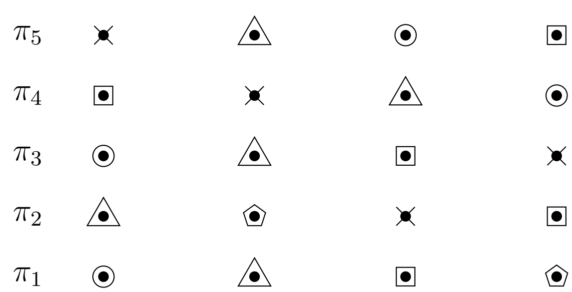
When $n = 1$, Proposition 2.2 yields a multivariate version of the Georgii-Nguyen-Zessin identity (1.1), i.e.

$$\mathbb{E}\left[\int_{X^r} u(z_1,\ldots,z_r;\omega)\omega(dz_1)\cdots\omega(dz_r)\right] = \sum_{\rho\in\Pi[1\times r]} \mathbb{E}\left[\int_{X^{|\rho|}} \varepsilon^+_{\mathfrak{z}_{|\rho|}} u(z_{\zeta^\rho(1,1)},\ldots,z_{\zeta^\rho(1,r)};\omega)\hat{\lambda}^{|\rho|}(d\mathfrak{z}_{|\rho|})\right].$$

Non-flat partitions

In the sequel we write $\nu \preceq \sigma$ when a partition $\nu \in \Pi[n\times r]$ is finer than another partition $\sigma \in \Pi[n\times r]$, i.e. when every block of ν is contained in a block of σ, and we let $\hat{0} := \{\{1,1\},\ldots,\{n,r\}\}$ denote the partition of $\Delta_{n\times r}$ made of singletons. We write $\rho \wedge \nu = \hat{0}$ when $\mu = \hat{0}$ is the only partition $\mu \in \Pi[n\times r]$ such that $\mu \preceq \nu$ and $\mu \preceq \rho$, i.e. $|\nu_k \cap \rho_l| \le 1$, $k = 1,\ldots,n$, $l = 1,\ldots,|\rho|$. In this case we say that the partition diagram $\Gamma(\nu,\rho)$ of ν and ρ is *non-flat*, see Chapter 4 of [15].

In the sequel, a partition $\rho \in \Pi[n\times r]$ is said to be *non-flat* if the partition diagram $\Gamma(\pi,\rho)$ of ρ and the partition π is *non-flat*, where $\pi := (\pi_1,\ldots,\pi_n) \in \Pi[n\times r]$ with $\pi_k := \{(k,1),\ldots,(k,r)\}$, $k = 1,\ldots,n$. The following figure shows an example of a non-flat partition



with $n = 5$, $r = 4$, and

$$\begin{aligned}
\triangle &= \{(1,2),(2,1),(2,2),(3,3),(4,2)\},\\
\bigcirc &= \{(1,1),(3,1),(4,4),(5,3)\},\\
\square &= \{(1,3),(2,4),(3,3),(4,1),(5,4)\},\\
\pentagon &= \{(1,4),(2,2)\},\\
\times &= \{(2,3),(3,4),(4,2),(5,1)\}\\
\pi_k &= \{(k,1),(k,2),(k,3),(k,4),(k,5)\}, \quad k = 1,2,3,4,5.
\end{aligned}$$

Processes vanishing on diagonals

The next consequence of Proposition 2.2 shows that when $u(z_1,\ldots,z_r;\omega)$ vanishes on the diagonals in X^r, the moments of $\int_{X^r} u(z_1,\ldots,z_r;\omega)\omega(dz_1)\cdots\omega(dz_r)$ reduce to sums over non-flat partition diagrams.

Proposition 2.3 *Assume that $u(z_1,\ldots,z_r;\omega) = 0$ whenever $z_i = z_j$, $1 \le i \neq j \le r$, $\omega \in \Omega^X$. Then we have*

$$\mathbb{E}\left[\left(\int_{X^r} u(z_1,\ldots,z_r;\omega)\omega(dz_1)\cdots\omega(dz_r)\right)^n\right] = \sum_{\substack{\rho\in\Pi[n\times r]\\ \rho\wedge\pi=\hat{0}}} \mathbb{E}\left[\int_{X^{|\rho|}} \epsilon^+_{\mathfrak{z}_{|\rho|}} \prod_{k=1}^n u\big(z^\rho_{\pi_k}\big)\hat{\lambda}^{|\rho|}(d\mathfrak{z}_{|\rho|})\right].$$

Proof. Assume that $u(z_1,\ldots,z_r;\omega)$ vanishes on diagonals, and let $\rho \in \Pi[n]$. Then, for any $z_1,\ldots,z_r \in X$ we have

$$\prod_{k=1}^n u\big(z^\rho_{\pi_k}\big) = \prod_{k=1}^n u\big(z_{\zeta^\rho(k,1)},\ldots,z_{\zeta^\rho(k,r)}\big) = 0$$

whenever $p := \zeta^\rho(k,a) = \zeta^\rho(k,b)$ for some $k \in \{1,\ldots,n\}$ and $a \neq b \in \{1,\ldots,r\}$. According to (2.3) this implies $(k,a) \in \rho_p$ and $(k,b) \in \rho_p$, therefore ρ is not a non-flat partition, and it should be excluded from the sum over $\Pi[n]$. $\square$

When $n = 1$, the first moment in Proposition 2.3 yields the Georgii-Nguyen-Zessin identity

$$\mathbb{E}\left[\int_{X^r} u(z_1,\ldots,z_r;\omega)\omega(dz_1)\cdots\omega(dz_r)\right] = \sum_{\substack{\rho\in\Pi[1\times r]\\ \rho\wedge\pi=\hat{0}}} \mathbb{E}\left[\int_{X^{|\rho|}} \epsilon^+_{\mathfrak{z}_{|\rho|}} u\big(z^\rho_{\pi_1}\big)\hat{\lambda}^{|\rho|}(d\mathfrak{z}_{|\rho|})\right]$$

$$= \mathbb{E}\left[\int_{X^r} \epsilon^+_{\mathfrak{z}_r} u(z_1,\ldots,z_r;\omega)\hat{\lambda}^r(d\mathfrak{z}_r)\right] \quad (2.5)$$

see Lemma IV.1 in [9] and Lemma 2.1 in [2] for different versions based on the Poisson point process. In the case of second moments, we find

$$\mathbb{E}\left[\left(\int_{X^r} u(z_1,\ldots,z_r;\omega)\omega(dz_1)\cdots\omega(dz_r)\right)^2\right] = \sum_{\substack{\rho\in\Pi[2\times r]\\ \rho\wedge\pi=\hat{0}}} \mathbb{E}\left[\int_{X^{|\rho|}} \epsilon^+_{\mathfrak{z}_{|\rho|}} u\big(z^\rho_{\pi_1}\big)u\big(z^\rho_{\pi_2}\big)\hat{\lambda}^{|\rho|}(d\mathfrak{z}_{|\rho|})\right],$$

and since the non-flat partitions in $\Pi[2\times r]$ are made of pairs and singletons, this identity can be rewritten as the following consequence of Proposition 2.3, in which for simplicity of notation we write $\pi_1 = \{1,\ldots,r\}$ and $\pi_2 = \{r+1,\ldots,2r\}$.

Corollary 2.4 *Assume that $u(z_1,\ldots,z_r;\omega) = 0$ whenever $z_i = z_j$, $1 \le i \ne j \le r$, $\omega \in \Omega^X$. Then the second moment of the integral of k-processes is given by*

$$\mathbb{E}\left[\left(\int_{X^r} u(z_1,\ldots,z_r;\omega)\omega(dz_1)\cdots\omega(dz_r)\right)^2\right]$$
$$= \sum_{A\subset\pi_1} \frac{1}{(r-|A|)!} \sum_{\gamma:\pi_2\to A\cup\{r+1,\ldots,2r-|A|\}} \mathbb{E}\left[\int_{X^{2r-|A|}} \epsilon^+_{\mathfrak{z}_{2r-|A|}} u\big(z_{\pi_1}\big)u\big(z_{\gamma(r+1)},\ldots,z_{\gamma(2r)}\big)\hat{\lambda}^{2r-|A|}(d\mathfrak{z}_{2r-|A|})\right],$$

where the above sum is over all bijections $\gamma : \pi_2 \to A \cup \{r+1,\ldots,2r-|A|\}$.

Proof. We express the partitions $\rho \in \Pi[n\times r]$ with non-flat diagrams $\Gamma(\pi,\rho)$ in Proposition 3.1 as the collections of pairs and singletons

$$\rho = \{i,\gamma(i)\}\}_{i\in A} \cup \{\{i\}\}_{i\in\pi_1, i\notin A} \cup \{\{i\}\}_{i\in\pi_2, i\notin\gamma(A)},$$

for all subsets $A \subset \pi_1 = \{1,\ldots,r\}$ and bijections $\gamma : \pi_2 \to A\cup\{r+1,\ldots,2r-|A|\}$.

□

In the case of 2-processes, Corollary 2.4 shows that

$$\mathbb{E}\left[\left(\int_{X^2} u(z_1,z_2;\omega)\omega(dz_1)\omega(dz_2)\right)^2\right] = \sum_{\substack{\rho\in\Pi[n\times 2]\\ \rho\wedge\pi=\hat{0}}} \mathbb{E}\left[\int_{X^{|\rho|}} \epsilon^+_{\mathfrak{z}_{|\rho|}} \prod_{k=1}^n u\big(z_{\zeta^\rho(k,1)}, z_{\zeta^\rho(k,2)}\big)\hat{\lambda}^{|\rho|}(d\mathfrak{z}_{|\rho|})\right]$$

$$= \sum_{\substack{A\subset\pi_1\\ \gamma:\{3,4\}\to A\cup\{3,\ldots,4-|A|\}}} \frac{1}{(r-|A|)!}\mathbb{E}\left[\int_{X^{4-|A|}} \epsilon^+_{\mathfrak{z}_{4-|A|}} u(z_1,z_2)u(z_{\gamma(3)},z_{\gamma(4)})\hat{\lambda}^{4-|A|}(d\mathfrak{z}_{4-|A|})\right]$$

$$
\begin{aligned}
= \ & \mathbb{E}\left[\int_{X^4} \epsilon^+_{\mathfrak{z}_4}(u(z_1,z_2)u(z_3,z_4))\hat{\lambda}^4(d\mathfrak{z}_4)\right] \\
& +\mathbb{E}\left[\int_{X^3} \epsilon^+_{\mathfrak{z}_3}(u(z_1,z_2)u(z_1,z_3))\hat{\lambda}^3(d\mathfrak{z}_3)\right] + \mathbb{E}\left[\int_{X^3} \epsilon^+_{\mathfrak{z}_3}(u(z_2,z_1)u(z_3,z_1))\hat{\lambda}^3(d\mathfrak{z}_3)\right] \\
& +\mathbb{E}\left[\int_{X^3} \epsilon^+_{\mathfrak{z}_3}(u(z_1,z_2)u(z_2,z_3))\hat{\lambda}^3(d\mathfrak{z}_3)\right] + \mathbb{E}\left[\int_{X^3} \epsilon^+_{\mathfrak{z}_3}(u(z_2,z_1)u(z_3,z_2))\hat{\lambda}^3(d\mathfrak{z}_3)\right] \\
& +\mathbb{E}\left[\int_{X^2} \epsilon^+_{\mathfrak{z}_2}(u(z_1,z_2)u(z_1,z_2))\hat{\lambda}^2(d\mathfrak{z}_2)\right] + \mathbb{E}\left[\int_{X^2} \epsilon^+_{\mathfrak{z}_2}(u(z_1,z_2)u(z_2,z_1))\hat{\lambda}^2(d\mathfrak{z}_2)\right].
\end{aligned}
$$

Similarly, in the case of 3-processes we find

$$
\begin{aligned}
& \mathbb{E}\left[\left(\int_{X^3} u(z_1,z_2,z_3;\omega)\omega(dz_1)\omega(dz_2)\omega(dz_3)\right)^2\right] \\
& = \sum_{\substack{A\subset\{1,2,3\} \\ \gamma:\{4,5,6\}\to A\cup\{4,\dots,6-|A|\}}} \frac{1}{(3-|A|)!}\mathbb{E}\left[\int_{X^5} \epsilon^+_{\epsilon^+_{\mathfrak{z}_5}} u(z_1,z_2,z_3)u\big(z_{\gamma(4)},z_{\gamma(5)},z_{\gamma(6)}\big)\hat{\lambda}^5(d\mathfrak{z}_5)\right] \\
& = \mathbb{E}\left[\int_{X^6} \epsilon^+_{\mathfrak{z}_6} u(z_1,z_2,z_3)u(z_4,z_5,z_6)\hat{\lambda}^6(d\mathfrak{z}_6)\right] \\
& \quad + \frac{1}{2}\sum_{\gamma:\{4,5,6\}\to\{1,5,6\}} \mathbb{E}\left[\int_{X^5} \epsilon^+_{\mathfrak{z}_5} u(z_1,z_2,z_3)u\big(z_{\gamma(4)},z_{\gamma(5)},z_{\gamma(6)}\big)\hat{\lambda}^5(d\mathfrak{z}_5)\right] \\
& \quad + \frac{1}{2}\sum_{\gamma:\{4,5,6\}\to\{2,5,6\}} \mathbb{E}\left[\int_{X^5} \epsilon^+_{\mathfrak{z}_5} u(z_1,z_2,z_3)u\big(z_{\gamma(4)},z_{\gamma(5)},z_{\gamma(6)}\big)\hat{\lambda}^5(d\mathfrak{z}_5)\right] \\
& \quad + \frac{1}{2}\sum_{\gamma:\{4,5,6\}\to\{3,5,6\}} \mathbb{E}\left[\int_{X^5} \epsilon^+_{\mathfrak{z}_5} u(z_1,z_2,z_3)u\big(z_{\gamma(4)},z_{\gamma(5)},z_{\gamma(6)}\big)\hat{\lambda}^5(d\mathfrak{z}_5)\right] \\
& \quad + \sum_{\gamma:\{4,5,6\}\to\{1,2,6\}} \mathbb{E}\left[\int_{X^4} \epsilon^+_{\mathfrak{z}_4} u(z_1,z_2,z_3)u\big(z_{\gamma(4)},z_{\gamma(5)},z_{\gamma(6)}\big)\hat{\lambda}^4(d\mathfrak{z}_4)\right] \\
& \quad + \sum_{\gamma:\{4,5,6\}\to\{1,3,6\}} \mathbb{E}\left[\int_{X^4} \epsilon^+_{\mathfrak{z}_4} u(z_1,z_2,z_3)u\big(z_{\gamma(4)},z_{\gamma(5)},z_{\gamma(6)}\big)\hat{\lambda}^4(d\mathfrak{z}_4)\right] \\
& \quad + \sum_{\gamma:\{4,5,6\}\to\{2,3,6\}} \mathbb{E}\left[\int_{X^4} \epsilon^+_{\mathfrak{z}_4} u(z_1,z_2,z_3)u\big(z_{\gamma(4)},z_{\gamma(5)},z_{\gamma(6)}\big)\hat{\lambda}^4(d\mathfrak{z}_4)\right] \\
& \quad + \sum_{\gamma:\{4,5,6\}\to\{1,2,3\}} \mathbb{E}\left[\int_{X^3} \epsilon^+_{\mathfrak{z}_3} z u(z_1,z_2,z_3)u\big(z_{\gamma(4)},z_{\gamma(5)},z_{\gamma(6)}\big)\hat{\lambda}^3(d\mathfrak{z}_3)\right].
\end{aligned}
$$

3 Random-connection model

Two point process vertices $x \neq y$ are independently connected in the random-connection graph with the probability $H(x,y)$ given $\omega \in \Omega^X$, where $H : X \times X \longrightarrow [0,1]$. In particular, the 1-hop count $\mathbb{1}_{\{x \leftrightarrow y\}}$ is a Bernoulli random variable with parameter $H(x,y)$, and we have the relation

$$\mathbb{E}\left[\epsilon^+_{\mathfrak{z}_r} \prod_{i=0}^{r} \mathbb{1}_{\{z_i \leftrightarrow z_{i+1}\}}(\omega) \,\middle|\, \omega\right] = \prod_{i=0}^{r} H(z_i, z_{i+1})$$

for any subset $\{z_0, \ldots, z_{r+1}\}$ of distinct elements of X, where $\mathfrak{z}_r = \{z_1, \ldots, z_r\}$ and $x \leftrightarrow y$ means that $x \in X$ is connected to $y \in X$.

Given $x, y \in X$, the number of $(r+1)$-hop sequences $z_1, \ldots, z_r \in \omega$ of vertices connecting x to y in the random graph is given by the multiparameter stochastic integral

$$N^{x,y}_{r+1} = \int_{X^r} u(z_1, \ldots, z_r; \omega)\omega(dz_1)\cdots\omega(dz_r)$$

of the $\{0,1\}$-valued r-process

$$u(z_1, \ldots, z_r; \omega) := \mathbb{1}_{\{z_i \neq z_j,\ 1 \leq i < j \leq r\}} \mathbb{1}_{\{z_1, \ldots, z_r \in \omega\}} \prod_{i=0}^{r} \mathbb{1}_{\{z_i \leftrightarrow z_{i+1}\}}(\omega), \quad z_1, \ldots, z_r \in X, \tag{3.1}$$

which vanishes on the diagonals in X^r, with $z_0 := x$ and $z_{r+1} := y$. In addition, for any distinct $z_1, \ldots, z_r \in X$ and $u(z_1, \ldots, z_r; \omega)$ given by (3.1) we have

$$\mathbb{E}\big[\epsilon^+_{\mathfrak{z}_r} u(z_1, \ldots, z_r; \omega) \mid \omega\big] = \mathbb{E}\left[\epsilon^+_{\mathfrak{z}_r} \prod_{i=0}^{r} \mathbb{1}_{\{z_i \leftrightarrow z_{i+1}\}}(\omega) \,\middle|\, \omega\right] = \prod_{i=0}^{r} H(z_i, z_{i+1}), \tag{3.2}$$

therefore the first order moment of the $(r+1)$-hop count between $x \in X$ and $y \in X$ is given as

$$\mathbb{E}\left[\int_{X^r} u(z_1, \ldots, z_r; \omega)\omega(dz_1)\cdots\omega(dz_r)\right] = \mathbb{E}\left[\int_{X^r} \prod_{i=0}^{r} H(z_i, z_{i+1}) \hat{\lambda}^r(d\mathfrak{z}_r)\right], \tag{3.3}$$

see also Theorem II.1 of [9], as a consequence of the Georgii-Nguyen-Zessin identity (2.5).

In the next proposition we compute the moments of all orders of r-hop counts as sums over non-flat partition diagrams. The role of the powers $1/n^\rho_{l,i}$ in (3.4) is to ensure that all powers of $H(x,y)$ in (3.4) are equal to one, since all powers of $\mathbb{1}_{\{z\leftrightarrow z'\}}$ in (3.5) below are equal to $\mathbb{1}_{\{z\leftrightarrow z'\}}$.

Proposition 3.1 *The moment of order n of the $(r+1)$-hop count between $x \in X$ and $y \in X$ is given by*

$$\mathbb{E}\left[\left(N^{x,y}_{r+1}\right)^n\right] = \sum_{\substack{\rho\in\Pi[n\times r]\\ \rho\wedge\pi=\hat{0}}} \mathbb{E}\left[\int_{X^{|\rho|}} \prod_{l=1}^n \prod_{i=0}^r H^{1/n^\rho_{l,i}}(z_{\zeta^\rho(l,i)}, z_{\zeta^\rho(l,i+1)})\hat{\lambda}^{|\rho|}(d\mathfrak{z}_{|\rho|})\right], \qquad (3.4)$$

where $z_0 = x$, $z_{r+1} = y$, $\zeta^\rho(l,0) = 0$, $\zeta^\rho(l,r+1) = r+1$, and

$$n^\rho_{l,i} := \#\big\{(p,j) \in \{1,\ldots,n\}\times\{0,\ldots,r\} \ : \ \{\zeta^\rho(l,i),\zeta^\rho(l,i+1)\} = \{\zeta^\rho(p,j),\zeta^\rho(p,j+1)\}\big\},$$

$1 \le l \le n$, $0 \le i \le r$.

Proof. Since $u(z_1,\ldots,z_r;\omega)$ vanishes whenever $z_i = z_j$ for some $1 \le i < j \le r$, by Proposition 2.3 we have

$$\begin{aligned}
&\mathbb{E}\left[\left(\int_{X^r} u(z_1,\ldots,z_r;\omega)\omega(dz_1)\cdots\omega(dz_r)\right)^n\right] \\
&= \sum_{\substack{\rho\in\Pi[n\times r]\\ \rho\wedge\pi=\hat{0}}} \mathbb{E}\left[\int_{X^{|\rho|}} \prod_{l=1}^n \prod_{i=0}^r \mathbb{1}_{\{z_{\zeta^\rho(l,i)}\leftrightarrow z_{\zeta^\rho(l,i+1)}\}}\hat{\lambda}^{|\rho|}(d\mathfrak{z}_{|\rho|})\right] \qquad (3.5)\\
&= \sum_{\substack{\rho\in\Pi[n\times r]\\ \rho\wedge\pi=\hat{0}}} \mathbb{E}\left[\int_{X^{|\rho|}} \prod_{l=1}^n \prod_{i=0}^r H^{1/n^\rho_{l,i}}(z_{\zeta^\rho(l,i)}, z_{\zeta^\rho(l,i+1)})\hat{\lambda}^{|\rho|}(d\mathfrak{z}_{|\rho|})\right],
\end{aligned}$$

where we applied (3.2). □

As in Corollary 2.4 we have the following consequence of Proposition 3.1, which is obtained by expressing the partitions $\rho \in \Pi[n \times r]$ with non-flat diagrams $\Gamma(\pi,\sigma)$ as a collection of pairs and singletons.

Corollary 3.2 *The second moment of the $(r+1)$-hop count between $x \in X$ and $y \in X$ is given by*

$$\mathbb{E}\left[\left(N^{x,y}_{r+1}\right)^2\right]$$

$$= \sum_{\substack{A\subset\pi_1 \\ \gamma:\{1,\ldots,r\}\to A\cup\{r+1,\ldots,2r-|A|\}}} \frac{1}{(r-|A|)!}\mathbb{E}\left[\int_{X^{2r-|A|}} \prod_{i=0}^{r} H^{1/n^\gamma_{1,i}}(z_i, z_{i+1}) \prod_{j=0}^{r} H^{1/n^\gamma_{2,j}}(z_{\gamma(j)}, z_{\gamma(j+1)})\hat{\lambda}^{2r-|A|}(d\mathfrak{z}_{2r-|A|})\right],$$

where the above sum is over all bijections $\gamma : \{1, \ldots, r\} \to A \cup \{r+1, \ldots, 2r-|A|\}$ *with* $\gamma(0) := 0$, $\gamma(r+1) =: r+1$, $z_0 =: x$, *and* $z_{r+1} := y$, *and*

$$n^\gamma_{1,i} = \#\{j \in \{0, \ldots, r\} \ : \ \{i, i+1\} = \{\gamma(j), \gamma(j+1)\}\},$$

$$n^\gamma_{2,j} = \#\{i \in \{0, \ldots, r\} \ : \ (i, i+1) = (\gamma(j), \gamma(j+1))\},$$

$0 \leq i \leq r$.

Variance of 3-hop counts

When $n = 2$ and $r = 2$, Corollary 3.2 allows us to express the variance of the 3-hop count between $x \in X$ and $y \in X$ as follows:

$$\begin{aligned}
&\mathrm{Var}\,[N_3^{x,y}] \\
&= \sum_{\substack{\emptyset\neq A\subset\{1,2\} \\ \gamma:\{1,2\}\to A\cup\{3,4-|A|\}}} \frac{1}{(2-|A|)!}\mathbb{E}\left[\int_{X^{4-|A|}} \prod_{i=0}^{2} H^{1/n^\gamma_{1,i}}(z_i, z_{i+1}) \prod_{j=0}^{2} H^{1/n^\gamma_{2,j}}(z_{\gamma(j)}, z_{\gamma(j+1)})\hat{\lambda}^{4-|A|}(d\mathfrak{z}_{4-|A|})\right] \\
&= \sum_{\gamma:\{1,2\}\to\{1,4\}} \mathbb{E}\left[\int_{X^3} \prod_{i=0}^{2} H^{1/n^\gamma_{1,i}}(z_i, z_{i+1}) \prod_{j=0}^{2} H^{1/n^\gamma_{2,j}}(z_{\gamma(j)}, z_{\gamma(j+1)})\hat{\lambda}^{3}(dz_1, dz_2, dz_4)\right] \\
&+ \sum_{\gamma:\{1,2\}\to\{2,4\}} \mathbb{E}\left[\int_{X^3} \prod_{i=0}^{2} H^{1/n^\gamma_{1,i}}(z_i, z_{i+1}) \prod_{j=0}^{2} H^{1/n^\gamma_{2,j}}(z_{\gamma(j)}, z_{\gamma(j+1)})\hat{\lambda}^{3}(dz_1, dz_2, dz_4)\right] \\
&+ \sum_{\gamma:\{1,2\}\to\{1,2\}} \mathbb{E}\left[\int_{X^2} \prod_{i=0}^{2} H^{1/n^\gamma_{1,i}}(z_i, z_{i+1}) \prod_{j=0}^{2} H^{1/n^\gamma_{2,j}}(z_{\gamma(j)}, z_{\gamma(j+1)})\hat{\lambda}^{2}(dz_1, dz_2)\right].
\end{aligned}$$

Variance of 4-hop counts

When $r = 3$ and $n = 2$, Corollary 3.2 yields

$$\begin{aligned}
&\mathrm{Var}\,[N_4^{x,y}] \\
&= \sum_{\substack{\emptyset\neq A\subset\pi_1 \\ \gamma:\{1,\ldots,3\}\to A\cup\{4,\ldots,6-|A|\}}} \frac{1}{(3-|A|)!}\mathbb{E}\left[\int_{X^{6-|A|}} \prod_{i=0}^{3} H^{1/n^\gamma_{1,i}}(z_i, z_{i+1}) \prod_{j=0}^{3} H^{1/n^\gamma_{2,j}}(z_{\gamma(j)}, z_{\gamma(j+1)})\hat{\lambda}^{6-|A|}(d\mathfrak{z}_{6-|A|})\right]
\end{aligned}$$

$$= \frac{1}{2} \sum_{\gamma:\{1,\ldots,3\}\to\{1,5,6\}} \mathbb{E}\left[\int_{X^5} \prod_{i=0}^{3} H^{1/n^\gamma_{1,i}}(z_i, z_{i+1}) \prod_{j=0}^{3} H^{1/n^\gamma_{2,j}}(z_{\gamma(j)}, z_{\gamma(j+1)})\hat{\lambda}^5(dz_1, dz_2, dz_3, dz_5, dz_6)\right]$$

$$+ \frac{1}{2} \sum_{\gamma:\{1,\ldots,3\}\to\{2,5,6\}} \mathbb{E}\left[\int_{X^5} \prod_{i=0}^{3} H^{1/n^\gamma_{1,i}}(z_i, z_{i+1}) \prod_{j=0}^{3} H^{1/n^\gamma_{2,j}}(z_{\gamma(j)}, z_{\gamma(j+1)})\hat{\lambda}^5(dz_1, dz_2, dz_3, dz_5, dz_6)\right]$$

$$+ \frac{1}{2} \sum_{\gamma:\{1,\ldots,3\}\to\{3,5,6\}} \mathbb{E}\left[\int_{X^5} \prod_{i=0}^{3} H^{1/n^\gamma_{1,i}}(z_i, z_{i+1}) \prod_{j=0}^{3} H^{1/n^\gamma_{2,j}}(z_{\gamma(j)}, z_{\gamma(j+1)})\hat{\lambda}^5(dz_1, dz_2, dz_3, dz_5, dz_6)\right]$$

$$+ \sum_{\gamma:\{1,\ldots,3\}\to\{1,2,6\}} \mathbb{E}\left[\int_{X^4} \prod_{i=0}^{3} H^{1/n^\gamma_{1,i}}(z_i, z_{i+1}) \prod_{j=0}^{3} H^{1/n^\gamma_{2,j}}(z_{\gamma(j)}, z_{\gamma(j+1)})\hat{\lambda}^4(dz_1, dz_2, dz_3, dz_6)\right]$$

$$+ \sum_{\gamma:\{1,\ldots,3\}\to\{1,3,6\}} \mathbb{E}\left[\int_{X^4} \prod_{i=0}^{3} H^{1/n^\gamma_{1,i}}(z_i, z_{i+1}) \prod_{j=0}^{3} H^{1/n^\gamma_{2,j}}(z_{\gamma(j)}, z_{\gamma(j+1)})\hat{\lambda}^4(dz_1, dz_2, dz_3, dz_6)\right]$$

$$+ \sum_{\gamma:\{1,\ldots,3\}\to\{2,3,6\}} \mathbb{E}\left[\int_{X^4} \prod_{i=0}^{3} H^{1/n^\gamma_{1,i}}(z_i, z_{i+1}) \prod_{j=0}^{3} H^{1/n^\gamma_{2,j}}(z_{\gamma(j)}, z_{\gamma(j+1)})\hat{\lambda}^4(dz_1, dz_2, dz_3, dz_6)\right]$$

$$+ \sum_{\gamma:\{1,\ldots,3\}\to\{1,\ldots,3\}} \mathbb{E}\left[\int_{X^3} \prod_{i=0}^{3} H^{1/n^\gamma_{1,i}}(z_i, z_{i+1}) \prod_{j=0}^{3} H^{1/n^\gamma_{2,j}}(z_{\gamma(j)}, z_{\gamma(j+1)})\hat{\lambda}^3(dz_1, dz_2, dz_3)\right].$$

4 Poisson case

In this section and the next one, we work in the Poisson random-connection model, using a Poisson point process on $X = \mathbb{R}^d$ with intensity $\lambda(dx)$ on $\mathbb{R}^d$. We let

$$H^{(n)}(x_0, x_n) := \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \prod_{i=0}^{n-1} H(x_i, x_{i+1})\lambda(dx_1)\cdots\lambda(dx_{n-1}), \quad x_0, x_n \in \mathbb{R}^d, \quad n \geq 1. \tag{4.1}$$

The 2-hop count between $x \in \mathbb{R}^d$ and $y \in \mathbb{R}^d$ is given by the first order stochastic integral

$$\int_{\mathbb{R}^d} u(z;\omega)\omega(dz) = \int_{\mathbb{R}^d} \mathbb{1}_{\{x\leftrightarrow z_1\}}\mathbb{1}_{\{z_1\leftrightarrow y\}}(\omega)\omega(dz_1) = \int_{\mathbb{R}^d} \mathbb{1}_{\{x\leftrightarrow z_1\}}\mathbb{1}_{\{z_1\leftrightarrow y\}}\omega(dz_1),$$

and its moment of order n is

$$\mathbb{E}\left[\left(\int_{\mathbb{R}^d} u(z_1;\omega)\omega(dz_1)\right)^n\right] = \mathbb{E}\left[\left(\int_{\mathbb{R}^d} \mathbb{1}_{\{x\leftrightarrow z_1\}}\mathbb{1}_{\{z_1\leftrightarrow y\}}\omega(dz_1)\right)^n\right]$$

$$= \sum_{\rho\in\Pi[n\times 1]} \int_{X^{|\rho|}} \prod_{l=1}^{|\rho|} \big(H(x, z_l)H(z_l, y)\big)\lambda^{|\rho|}(dz_1, \ldots, dz_{|\rho|})$$

$$
\begin{aligned}
&= \sum_{k=1}^{n} S(n,k) \left(\int_{\mathbb{R}^d} H(x,z)H(z,y)\lambda(dz) \right)^k \\
&= \sum_{k=1}^{n} S(n,k) \big(H^{(2)}(x,y) \big)^k,
\end{aligned}
$$

therefore, from (2.1), the 2-hop count between $x \in \mathbb{R}^d$ and $y \in \mathbb{R}^d$ is a Poisson random variable with mean

$$
H^{(2)}(x,y) = \int_{\mathbb{R}^d} H(x,z)H(z,y)\lambda(dz).
$$

By (3.3), the first order moment of the r-hop count is given by

$$
H^{(r)}(x,y) = \int_{X^{r-1}} \prod_{i=0}^{r-1} H(z_i, z_{i+1}) \lambda^{r-1}(dz_1, \ldots dz_{r-1}).
$$

Corollary 4.1 *The variance of the r-hop count between $x \in \mathbb{R}^d$ and $y \in \mathbb{R}^d$ is given by*

$$
\begin{aligned}
&\mathrm{Var}\,[N_r^{x,y}] \\
&= \sum_{p=1}^{r-1} \sum_{\substack{1 \le k_1 < \cdots < k_p < r \\ 1 \le l_1 < \cdots < l_p < r}} \sum_{\sigma \in \Sigma[p]} \int_{X^p} \prod_{0 \le i \le p} H^{(k_{i+1}-k_i)}(z_i, z_{i+1}) \prod_{\substack{0 \le j \le p \\ l_{\sigma(j+1)} - l_{\sigma(j)} + k_{j+1} - k_j > 2 \\ \text{or } \{j, j+1\} \ne \{\sigma(j), \sigma(j+1)\}}} H^{(l_{\sigma(j+1)} - l_{\sigma(j)})}(z_{\sigma(j)}, z_{\sigma(j+1)}) \lambda^p(d\mathfrak{z}_p),
\end{aligned}
$$

with $k_0 = l_0 = 0$, $k_{p+1} = l_{p+1} = r$, $\sigma(0) = 0$, and $\sigma(r) = r$, where the above sum if over all permutations $\sigma \in \Sigma[p]$ of $\{1, \ldots, p\}$.

Proof. We rewrite the result of Corollary 3.2 by denoting the set $A \subset \pi_1$ as $A = \{k_1, \ldots, k_p\}$, for $1 \le k_1 < \cdots < k_p \le r-1$, and we identify $\gamma(A) \subset A \cup \{r+1, \ldots, 2r - |A|\}$ to $\{l_1, \ldots, l_p\}$, which requires a sum over the permutations of $\{1, \ldots, p\}$ since $1 \le l_1 < \cdots < l_p \le r-1$, where $1 \le p \le r-1$. In addition, the multiple integrals over contiguous index sets in A^c are evaluated using (4.1). □

Variance of 3-hop counts

When $n = 2$ and $r = 2$ Corollary 4.1 allows us to compute the variance of the 3-hop count between $x \in \mathbb{R}^d$ and $y \in \mathbb{R}^d$, as follows:

$$
\mathrm{Var}\,[N_3^{x,y}] \tag{4.2}
$$

$$= 2\int_{\mathrm{R}^d} H(x,z_1)H^{(2)}(z_1,y)H^{(2)}(z_1,y)\lambda(dz_1) + 2\int_{\mathrm{R}^d} H(x,z_1)H^{(2)}(x,z_1)H^{(2)}(z_1,y)H(z_1,y)\lambda(dz_1)$$
$$+ \int_{X^2} H(x,z_1)H(z_1,z_2)H(z_2,y)H(x,z_2)H(z_1,y)\lambda^2(dz_1,dz_2) + H^{(3)}(x,y).$$

By Corollary 4.1 the variance of 4-hop counts can be similarly computed explicitly as a sum of 33 terms, as follows:

$$\mathrm{Var}\,[N_4^{x,y}] = \int_{\mathrm{R}^d} H_\beta(x,z_1)H_\beta^{(3)}(z_1,y)H_\beta^{(3)}(z_1,y)\lambda(dz_1) \tag{4.3}$$
$$+ \int_{\mathrm{R}^d} H_\beta(x,z_1)H_\beta^{(3)}(z_1,y)H_\beta^{(2)}(x,z_1)H_\beta^{(2)}(z_1,y)\lambda(dz_1)$$
$$+ \int_{\mathrm{R}^d} H_\beta(x,z_1)H_\beta^{(3)}(x,z_1)H_\beta^{(3)}(z_1,y)H_\beta(z_1,y)\lambda(dz_1)$$
$$+ \int_{\mathrm{R}^d} H_\beta^{(2)}(x,z_2)H_\beta^{(2)}(z_2,y)H_\beta(x,z_2)H_\beta^{(3)}(z_2,y)\lambda(dz_2)$$
$$+ \int_{\mathrm{R}^d} H_\beta^{(2)}(x,z_2)H_\beta^{(2)}(x,z_2)H_\beta^{(2)}(z_2,y)H_\beta^{(2)}(z_2,y)\lambda(dz_2)$$
$$+ \int_{\mathrm{R}^d} H_\beta^{(2)}(x,z_2)H_\beta^{(2)}(z_2,y)H_\beta^{(3)}(x,z_2)H_\beta(z_2,y)\lambda(dz_2)$$
$$+ \int_{\mathrm{R}^d} H_\beta^{(3)}(x,z_3)H_\beta(z_3,y)H_\beta(x,z_3)H_\beta^{(3)}(z_3,y)\lambda(dz_3)$$
$$+ \int_{\mathrm{R}^d} H_\beta^{(3)}(x,z_3)H_\beta(z_3,y)H_\beta^{(2)}(x,z_3)H_\beta^{(2)}(z_3,y)\lambda(dz_3)$$
$$+ \int_{\mathrm{R}^d} H_\beta^{(3)}(x,z_3)H_\beta(z_3,y)H_\beta^{(3)}(x,z_3)\lambda(dz_3)$$
$$+ \int_{X^2} H_\beta(x,z_1)H_\beta(z_1,z_2)H_\beta^{(2)}(z_2,y)H_\beta^{(2)}(z_2,y)\lambda^2(dz_1,dz_2)$$
$$+ \int_{X^2} H_\beta(x,z_1)H_\beta(z_1,z_2)H_\beta^{(2)}(z_2,y)H_\beta(x,z_2)H_\beta^{(2)}(z_1,y)\lambda^2(dz_1,dz_2)$$
$$+ \int_{X^2} H_\beta(x,z_1)H_\beta(z_1,z_2)H_\beta^{(2)}(z_2,y)H_\beta^{(2)}(z_1,z_2)H_\beta(z_2,y)\lambda^2(dz_1,dz_2)$$
$$+ \int_{X^2} H_\beta(x,z_1)H_\beta(z_1,z_2)H_\beta^{(2)}(z_2,y)H_\beta(x,z_2)H_\beta^{(2)}(z_2,z_1)H_\beta(z_1,y)\lambda^2(dz_1,dz_2)$$
$$+ \int_{X^2} H_\beta(x,z_1)H_\beta(z_1,z_2)H_\beta^{(2)}(z_2,y)H_\beta^{(2)}(x,z_1)H_\beta(z_2,y)\lambda^2(dz_1,dz_2)$$
$$+ \int_{X^2} H_\beta(x,z_1)H_\beta(z_1,z_2)H_\beta(z_1,y)H_\beta^{(2)}(x,z_2)H_\beta^{(2)}(z_2,y)\lambda^2(dz_1,dz_2)$$
$$+ \int_{X^2} H_\beta(x,z_1)H_\beta^{(2)}(z_1,z_3)H_\beta(z_3,y)H_\beta(z_1,z_3)H_\beta^{(2)}(z_3,y)\lambda^2(dz_1,dz_3)$$

$$+\int_{X^2} H_\beta(x,z_1)H_\beta^{(2)}(z_1,z_3)H_\beta^{(2)}(z_1,z_3)H_\beta(z_3,y)\lambda^2(dz_1,dz_3)$$
$$+\int_{X^2} H_\beta(x,z_1)H_\beta^{(2)}(z_1,z_3)H_\beta(z_3,y)H_\beta(x,z_3)H_\beta(z_3,z_1)H_\beta^{(2)}(z_1,y)\lambda^2(dz_1,dz_3)$$
$$+\int_{X^2} H_\beta(x,z_1)H_\beta^{(2)}(z_1,z_3)H_\beta(z_3,y)H_\beta(x,z_3)H_\beta^{(2)}(z_3,z_1)H_\beta(z_1,y)\lambda^2(dz_1,dz_3)$$
$$+\int_{X^2} H_\beta(x,z_1)H_\beta^{(2)}(z_1,z_3)H_\beta^{(2)}(x,z_1)H_\beta(z_1,z_3)H_\beta(z_3,y)\lambda^2(dz_1,dz_3)$$
$$+\int_{X^2} H_\beta(x,z_1)H_\beta^{(2)}(z_1,z_3)H_\beta(z_3,y)H_\beta^{(2)}(x,z_3)H_\beta(z_3,z_1)H_\beta(z_1,y)\lambda^2(dz_1,dz_3)$$
$$+\int_{X^2} H_\beta^{(2)}(x,z_2)H_\beta(z_2,z_3)H_\beta(z_3,y)H_\beta(x,z_2)H_\beta^{(2)}(z_3,y)\lambda^2(dz_2,dz_3)$$
$$+\int_{X^2} H_\beta^{(2)}(x,z_2)H_\beta(z_2,z_3)H_\beta(x,z_2)H_\beta^{(2)}(z_2,z_3)H_\beta(z_3,y)\lambda^2(dz_2,dz_3)$$
$$+\int_{X^2} H_\beta^{(2)}(x,z_2)H_\beta(z_2,z_3)H_\beta(z_3,y)H_\beta(x,z_3)H_\beta^{(2)}(z_2,y)\lambda^2(dz_2,dz_3)$$
$$+\int_{X^2} H_\beta^{(2)}(x,z_2)H_\beta(z_2,z_3)H_\beta(z_3,y)H_\beta(x,z_3)H_\beta^{(2)}(z_3,z_2)H_\beta(z_2,y)\lambda^2(dz_2,dz_3)$$
$$+\int_{X^2} H_\beta^{(2)}(x,z_2)H_\beta(z_2,z_3)H_\beta(z_3,y)H_\beta^{(2)}(x,z_2)\lambda^2(dz_2,dz_3)$$
$$+\int_{X^2} H_\beta^{(2)}(x,z_2)H_\beta(z_2,z_3)H_\beta(z_3,y)H_\beta^{(2)}(x,z_3)H_\beta(z_2,y)\lambda^2(dz_2,dz_3)$$
$$+H_\beta^{(4)}(x,y)+\int_{X^3} H_\beta(x,z_1)H_\beta(z_1,z_2)H_\beta(z_2,z_3)H_\beta(z_3,y)H_\beta(z_1,z_3)H_\beta(z_2,y)\lambda^3(dz_1,dz_2,dz_3)$$
$$+\int_{X^3} H_\beta(x,z_1)H_\beta(z_1,z_2)H_\beta(z_2,z_3)H_\beta(z_3,y)H_\beta(x,z_2)H_\beta(z_1,z_3)\lambda^3(dz_1,dz_2,dz_3)$$
$$+\int_{X^3} H_\beta(x,z_1)H_\beta(z_1,z_2)H_\beta(z_2,z_3)H_\beta(z_3,y)H_\beta(x,z_2)H_\beta(z_3,z_1)H_\beta(z_1,y)\lambda^3(dz_1,dz_2,dz_3)$$
$$+\int_{X^3} H_\beta(x,z_1)H_\beta(z_1,z_2)H_\beta(z_2,z_3)H_\beta(z_3,y)H_\beta(x,z_3)H_\beta(z_3,z_1)H_\beta(z_2,y)\lambda^3(dz_1,dz_2,dz_3)$$
$$+\int_{X^3} H_\beta(x,z_1)H_\beta(z_1,z_2)H_\beta(z_2,z_3)H_\beta(z_3,y)H_\beta(x,z_3)H_\beta(z_1,y)\lambda^3(dz_1,dz_2,dz_3).$$

5 Rayleigh fading

In this section we consider a Poisson point process on $X = \mathbb{R}^d$ with flat intensity $\lambda(dx) = \lambda dx$ on $\mathbb{R}^d$, $\lambda > 0$, and a Rayleigh fading function of the form

$$H_\beta(x,y) := \mathrm{e}^{-\beta\|x-y\|^2}, \qquad x,y \in \mathbb{R}^d, \quad \beta > 0.$$

Lemmas 5.1 and 5.2 can be used to evaluate the integrals appearing in Corollary 4.1 and in the variance (4.2) of 3-hop counts.

Lemma 5.1 *For all* $n \geq 1$, $y_1, \ldots, y_n \in \mathbb{R}^d$ *and* $\beta_1, \ldots, \beta_n > 0$ *we have*

$$\int_{\mathbb{R}^d} \prod_{i=1}^{n} H_{\beta_i}(x, y_i) dx = \left(\frac{\pi}{\beta_1 + \cdots + \beta_n} \right)^{d/2} \prod_{i=1}^{n-1} H_{\frac{\beta_{i+1}(\beta_1+\cdots+\beta_i)}{\beta_1+\cdots+\beta_{i+1}}} \left(y_{i+1}, \frac{\beta_1 y_1 + \cdots + \beta_i y_i}{\beta_1 + \cdots + \beta_i} \right).$$

Proof. We start by showing that for all $n \geq 1$ we have

$$\prod_{i=1}^{n} H_{\beta_i}(x, y_i) = H_{\beta_1+\cdots+\beta_n} \left(x, \frac{\beta_1 y_1 + \cdots + \beta_n y_n}{\beta_1 + \cdots + \beta_n} \right) \prod_{i=1}^{n-1} H_{\frac{\beta_{i+1}(\beta_1+\cdots+\beta_i)}{\beta_1+\cdots+\beta_{i+1}}} \left(y_{i+1}, \frac{\beta_1 y_1 + \cdots + \beta_i y_i}{\beta_1 + \cdots + \beta_i} \right). \tag{5.1}$$

Clearly, this relation holds for $n = 1$. In addition, at the rank $n = 2$ we have

$$\begin{aligned}
H_{\beta_1}(x, y_1) H_{\beta_2}(x, y_2) &= \mathrm{e}^{-\beta_1 \|y_1 - x\|^2} \mathrm{e}^{-\beta_2 \|x - y_2\|^2} \\
&= \mathrm{e}^{-\beta_1 \|y_1\|^2 - \beta_2 \|y_2\|^2 + 2\langle \beta_1 y_1 + \beta_2 y_2, x \rangle - (\beta_1 + \beta_2)\|x\|^2} \\
&= \mathrm{e}^{-\beta_1 \|y_1\|^2 - \beta_2 \|y_2\|^2 - (\beta_1+\beta_2)\|x - (\beta_1 y_1 + \beta_2 y_2)/(\beta_1+\beta_2)\|^2 + \|\beta_1 y_1 + \beta_2 y_2\|^2/(\beta_1+\beta_2)} \\
&= \mathrm{e}^{-(\beta_1+\beta_2)\|x - (\beta_1 y_1 + \beta_2 y_2)/(\beta_1+\beta_2)\|^2 - \beta_1 \beta_2 \|y_1 - y_2\|^2/(\beta_1+\beta_2)} \\
&= H_{\beta_1+\beta_2} \left(x, \frac{\beta_1 y_1 + \beta_2 y_2}{\beta_1 + \beta_2} \right) H_{\frac{\beta_1 \beta_2}{\beta_1+\beta_2}}(y_1, y_2),
\end{aligned}$$

Next, assuming that (5.1) holds at the rank $n \geq 1$, we have

$$\begin{aligned}
\prod_{i=1}^{n+1} H_{\beta_i}(x, y_i) &= H_{\beta_{n+1}}(x, y_{n+1}) H_{\beta_1+\cdots+\beta_n} \left(x, \frac{\beta_1 y_1 + \cdots + \beta_n y_n}{\beta_1 + \cdots + \beta_n} \right) \\
&\quad \times \prod_{i=1}^{n-1} H_{\frac{\beta_{i+1}(\beta_1+\cdots+\beta_i)}{\beta_1+\cdots+\beta_{i+1}}} \left(y_{i+1}, \frac{\beta_1 y_1 + \cdots + \beta_i y_i}{\beta_1 + \cdots + \beta_i} \right) \\
&= H_{\beta_1+\cdots+\beta_{n+1}} \left(x, \frac{\beta_1 y_1 + \cdots + \beta_{n+1} y_{n+1}}{\beta_1 + \cdots + \beta_n} \right) \prod_{i=1}^{n} H_{\frac{\beta_{i+1}(\beta_1+\cdots+\beta_i)}{\beta_1+\cdots+\beta_{i+1}}} \left(y_{i+1}, \frac{\beta_1 y_1 + \cdots + \beta_i y_i}{\beta_1 + \cdots + \beta_i} \right).
\end{aligned}$$

As a consequence, we find

$$\begin{aligned}
\int_{\mathbb{R}^d} \prod_{i=1}^{n} H_{\beta_i}(x, y_i) dx &= \prod_{i=1}^{n-1} H_{\frac{\beta_{i+1}(\beta_1+\cdots+\beta_i)}{\beta_1+\cdots+\beta_{i+1}}} \left(y_{i+1}, \frac{\beta_1 y_1 + \cdots + \beta_i y_i}{\beta_1 + \cdots + \beta_i} \right) \\
&\quad \times \int_{\mathbb{R}^d} H_{\beta_1+\cdots+\beta_n} \left(x, \frac{\beta_1 y_1 + \cdots + \beta_n y_n}{\beta_1 + \cdots + \beta_n} \right) dx
\end{aligned}$$

$$= \left(\frac{\pi}{\beta_1+\cdots+\beta_n}\right)^{d/2} \prod_{i=1}^{n-1} H_{\frac{\beta_{i+1}(\beta_1+\cdots+\beta_i)}{\beta_1+\cdots+\beta_{i+1}}} \left(y_{i+1}, \frac{\beta_1 y_1+\cdots+\beta_i y_i}{\beta_1+\cdots+\beta_i}\right).$$

□

In particular, applying Lemma 5.1 for $n = 2$ yields

$$\begin{aligned}\int_{\mathbb{R}^d} H_{\beta_1}(y_1,x)H_{\beta_2}(x,y_2)dx &= \left(\frac{\pi}{\beta_1+\beta_2}\right)^{d/2} H_{\frac{\beta_1\beta_2}{\beta_1+\beta_2}}(y_1,y_2) \\ &= \left(\frac{\pi}{\beta_1+\beta_2}\right)^{d/2} e^{-\beta_1\beta_2\|y_1-y_2\|^2/(\beta_1+\beta_2)}, \quad y_1,y_2\in\mathbb{R}^d,\end{aligned} \tag{5.2}$$

and the 2-hop count between $x \in \mathbb{R}^d$ and $y \in \mathbb{R}^d$ is a Poisson random variable with mean

$$\begin{aligned}H_\beta^{(2)}(x,y) &= \lambda\int_{\mathbb{R}^d} H_\beta(x,z)H_\beta(z,y)dz \\ &= \lambda\left(\frac{\pi}{2\beta}\right)^{d/2} H_{\beta/2}(x,y) \\ &= \lambda\left(\frac{\pi}{2\beta}\right)^{d/2} e^{-\|x-y\|^2/2}.\end{aligned}$$

By an induction argument similar to that of Lemma 5.1, we obtain the following lemma.

Lemma 5.2 *For all $n \geq 1$, $x_0,\ldots,x_n \in \mathbb{R}^d$ and $\beta_1,\ldots,\beta_n > 0$ we have*

$$\begin{aligned}&\int_{\mathbb{R}^d}\cdots\int_{\mathbb{R}^d}\prod_{i=1}^n H_{\beta_i}(x_{i-1},x_i)dx_1\cdots dx_{n-1} \\ &= \left(\frac{\pi^{n-1}}{\sum_{i=1}^n \beta_1\cdots\beta_{i-1}\beta_{i+1}\cdots\beta_n}\right)^{d/2} H_{\frac{\beta_1\cdots\beta_n}{\sum_{i=1}^n \beta_1\cdots\beta_{i-1}\beta_{i+1}\cdots\beta_n}}(x_0,y_n).\end{aligned}$$

Proof. Clearly, the relation holds at the rank $n = 1$. Assuming that it holds at the rank $n \geq 1$ and using (5.2), we have

$$\begin{aligned}&\int_{\mathbb{R}^d}\cdots\int_{\mathbb{R}^d}\prod_{i=1}^{n+1} H_{\beta_i}(x_{i-1},x_i)dx_1\cdots dx_n \\ &= \int_{\mathbb{R}^d} H_{\beta_{n+1}}(x_n,x_{n+1})\int_{\mathbb{R}^d}\cdots\int_{\mathbb{R}^d}\prod_{i=1}^n H_{\beta_i}(x_{i-1},x_i)dx_1\cdots dx_n\end{aligned}$$

$$= \left(\frac{\pi^{n-1}}{\sum_{i=1}^n \beta_1 \cdots \beta_{i-1}\beta_{i+1}\cdots\beta_n} \right)^{d/2} \int_{\mathbb{R}^d} H_{\frac{\beta_1\cdots\beta_n}{\sum_{i=1}^n \beta_1\cdots\beta_{i-1}\beta_{i+1}\cdots\beta_n}}(x_0, x_n) H_{\beta_{n+1}}(x_n, x_{n+1}) dx_n$$

$$= \left(\frac{\pi^{n-1}}{\sum_{i=1}^n \beta_1 \cdots \beta_{i-1}\beta_{i+1}\cdots\beta_n} \right)^{d/2} \left(\frac{\pi}{\frac{\beta_1\cdots\beta_n}{\sum_{i=1}^n \beta_1\cdots\beta_{i-1}\beta_{i+1}\cdots\beta_n} + \beta_{n+1}} \right)^{d/2} H_{\frac{\beta_1\cdots\beta_{n+1}}{\sum_{i=1}^{n+1} \beta_1\cdots\beta_{i-1}\beta_{i+1}\cdots\beta_{n+1}}}(x_0, x_{n+1}).$$

□

In particular, the first order moment of the r-hop count between $x_0 \in \mathbb{R}^d$ and $x_r \in \mathbb{R}^d$ is given by

$$\begin{aligned} H_\beta^{(r)}(x_0, x_r) &= \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \prod_{i=0}^{r-1} H_\beta(x_i, x_{i+1}) \lambda(dx_1) \cdots \lambda(dx_{r-1}) \\ &= \lambda^{r-1} \left(\frac{\pi^{r-1}}{r\beta^{r-1}} \right)^{d/2} H_{\beta/r}(x, y) \\ &= \lambda^{r-1} \left(\frac{\pi^{r-1}}{r\beta^{r-1}} \right)^{d/2} e^{-\beta\|x-y\|^2/r}, \qquad x, y \in \mathbb{R}^d. \qquad (5.3) \end{aligned}$$

Variance of 3-hop counts

Corollary 4.1 and Lemma 5.2 allow us to recover Theorem II.3 of [9], for the variance of 3-hop counts by a shorter argument, while extending it from the plane $X = \mathbb{R}^2$ to $X = \mathbb{R}^d$.

Corollary 5.3 *The variance of the* 3*-hop count between* $x \in \mathbb{R}^d$ *and* $y \in \mathbb{R}^d$ *is given by*

$$\begin{aligned} \mathrm{Var}\,[N_3^{x,y}] &= 2\lambda^3 \left(\frac{\pi^3}{8\beta^3} \right)^{d/2} e^{-\beta\|x-y\|^2/2} + \lambda^2 \left(\frac{\pi^2}{3\beta^2} \right)^{d/2} e^{-\beta\|x-y\|^2/3} \\ &\quad + 2\lambda^3 \left(\frac{\pi^3}{12\beta^3} \right)^{d/2} e^{-3\beta\|x-y\|^2/4} + \lambda^2 \left(\frac{\pi^2}{8\beta^2} \right)^{d/2} e^{-\beta\|x-y\|^2}. \end{aligned}$$

Proof. By (5.3) and Lemma 5.2 we have

$$\int_{\mathbb{R}^d} H_\beta(x, z_1) H_\beta^{(2)}(z_1, y) H_\beta^{(2)}(z_1, y) \lambda(dz_1) = \lambda^2 \left(\frac{\pi^2}{4\beta^2} \right)^{d/2} \int_{\mathbb{R}^d} H_\beta(x, z_1) H_{\beta/2}^2(z_1, y) \lambda(dz_1)$$

$$= \lambda^3 \left(\frac{\pi^2}{4\beta^2} \right)^{d/2} \int_{\mathbb{R}^d} H_\beta(x, z_1) H_\beta(z_1, y) \lambda(dz_1) = \lambda^3 \left(\frac{\pi^3}{8\beta^3} \right)^{d/2} H_{\beta/2}(x, y),$$

$$\int_{\mathbb{R}^d} H_\beta(x, z_1) H_\beta^{(2)}(x, z_1) H_\beta^{(2)}(z_1, y) H_\beta(z_1, y) \lambda(dz_1)$$

$$= \lambda^2 \left(\frac{\pi^2}{4\beta^2}\right)^{d/2} \int_{\mathrm{R}^d} H_{3\beta/2}(z_1, y) H_{3\beta/2}(x, z_1) \lambda(dz_1) = \lambda^3 \left(\frac{\pi^3}{12\beta^3}\right)^{d/2} H_{3\beta/4}(x, y),$$

$$\int_{X^2} H_\beta(x, z_1) H_\beta(z_1, z_2) H_\beta(z_2, y) H_\beta(x, z_2) H_\beta(z_1, y) \lambda^2(dz_1, dz_2)$$
$$= \lambda \left(\frac{\pi}{3\beta}\right)^{d/2} H_\beta(x, y) \int_{\mathrm{R}^d} H_{2\beta/3}(z_2, (x+y)/2) H_{2\beta}(z_2, (x+y)/2) \lambda(dz_2)$$
$$= \lambda^2 \left(\frac{\pi^2}{8\beta^2}\right)^{d/2} H_\beta(x, y),$$

and we conclude by (4.2). □

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