Sensitivity analysis of European options in jump-diffusion models via the Malliavin calculus on the Wiener space

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Abstract

We present a Malliavin calculus approach to sensitivity analysis of European options in a jump-diffusion model. The lack of differentiability due to the presence of a jump component is tackled using partial differentials with respect to the (absolutely continuous) Gaussian part. The method appears to be particularly efficient to compute sensitivities with respect to the volatility parameter of the jump component.

Key words: Sensitivity analysis, Greeks, Malliavin calculus, jump-diffusion models, markets with jumps, Wiener space.
Classification: 90A09, 90A12, 90A60, 60H07, 60J75.

1 Introduction

Consider an option whose value is defined as the average discounted gain on an underlying asset S_t^ξ depending on a parameter ξ:

$$C_\xi = e^{-\int_0^T r(t)dt}\mathbf{E}\left[f(S_T^\xi)\right].$$

where f is called the payoff function, e.g. $f(x) = (x-K)^+$ for European options and $f = 1_{[K,+\infty[}$ for a binary option. Let Delta, Vega, Gamma, Rho and Theta be the Greek parameters measuring the sensitivity of option prices and defined by

$$\text{Delta} = \frac{\partial C}{\partial x}, \quad \text{Gamma} = \frac{\partial^2 C}{\partial x^2}, \quad \text{Rho} = \frac{\partial C}{\partial r}, \quad \text{Vega} = \frac{\partial C}{\partial \sigma}, \quad \text{Theta} = \frac{\partial C}{\partial T}.$$

Fast Monte-Carlo methods for the numerical computation of sensitivities in continuous markets have been developed in [6], [5], and also [2], using integration by parts formulas on probability spaces and differential tools of the Malliavin calculus on the Wiener space.

In [7], a Malliavin type gradient operator has been used for Asian options in a market with jumps driven by a standard Poisson process $(N_t)_{t\in\mathbb{R}_+}$. This gradient operator acts on smooth functionals $F = f(T_1, \ldots, T_n)$, of the Poisson process by differentiation with respect to the jump times $(T_k)_{k\geq 1}$ of the Poisson component $(N_t)_{t\in\mathbb{R}_+}$, as

$$D_w F = -\sum_{k=1}^{k=n} w(T_k)\partial_k f(T_1, \ldots, T_n),$$

where w is a functional parameter, cf. [9]. Functionals of the form

$$\int_0^T F(t, N_t)dt \tag{1.1}$$

do belong to the domain of D_w due to the smoothing effect of the integral. In particular it turned out in [7] that D_w can be applied to differentiate the value of an Asian option. However the L^2 domain of D_w does not contain the value N_T at time T of the Poisson process, excluding in particular European claims of the form $f(N_T)$ from this analysis. Ssee [1] for a different way to compute Greeks in jump models.

In this paper we consider a jump diffusion model driven by the sum of a Brownian motion and a jump process. Precisely, we consider an asset $(S_t)_{t\in\mathbb{R}_+}$ whose dynamics under the risk neutral probability are given by

$$\frac{dS_t}{S_t} = r(t)dt + \sigma_1(t)dW_t + \sigma_2(t)dX_t \tag{1.2}$$

where $(W_t)_{t\in\mathbb{R}_+}$ is a Brownian motion, $(X_t)_{t\in\mathbb{R}_+}$ is a jump process, $r(t)$ represents the interest rate, and $\sigma_1(t)$, $\sigma_2(t)$ are volatility parameters.

In order to compute these sensitivities we will use the integration by parts formula of Malliavin calculus, cf. Proposition 1 below. We will compute the Greeks corresponding to the sensitivity of European option prices with respect to parameters such as spot price x, interest rate r, or volatility σ. We use both the Malliavin calculus and finite differences, and compare the results obtained by each method. The formulas obtained are similar to the ones obtained for continuous markets, except in the case of derivation with respect to the volatility parameter of the jump component.

We proceed as follows. In Sections 2 and 3 we recall some basic notions of stochastic calculus and Malliavin calculus. The simulation graphs obtained show that the Malliavin method yields better numerical results in terms of accuracy and computation speed than the finite difference method.

This paper is based on [4], see also [3] for another approach to the computation of Greeks in jump models with respect to the Gaussian part of the process.

2 Stochastic calculus with jumps

In this section we recall some notions on stochastic calculus with respect to semimartingales with jumps, according to [10]. Given a complete filtered probability space $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}_+}, P\right)$, recall that a mapping $\tau : \Omega \to \mathbb{R}_+$ is a stopping time if $\{\tau \le t\} \in \mathcal{F}_t$, $\forall t \in \mathbb{R}_+$, and that $\mathcal{F}_\tau = \{A \in \mathcal{F} \ : \ A \cap \{\tau \le t\} \in \mathcal{F}_t, \ t \in \mathbb{R}_+\}$. Given $(X_t)_{t\in\mathbb{R}_+}$ a stochastic process, we define the linear operator $I_X : \mathbf{S} \to \mathbf{L^0}$ by

$$I_X(H) = H_0 X_0 + \sum_{i=1}^{n} H_i \left(X_{T_{i+1}} - X_{T_i} \right),$$

for $(H_t) \in \mathbf{S}$ a simple predictable process, i.e.

$$H_t = H_0 1_{\{0\}}(t) + \sum_{i=1}^{n} H_i 1_{(T_i, T_{i+1}]}(t) \tag{2.1}$$

where $0 = T_1 \le \cdots \le T_{n+1} < \infty$ is a sequence of stopping times and $H_i \in L^\infty(\Omega, \mathcal{F}_{T_i})$, $0 \le i \le n$. Note that this definition is independent of the choice of the representation of the simple process H. The set $\mathbf{S}$ of simple predictable processes is endowed with the topology of uniform convergence in (t, ω) denoted by $\mathbf{S_u}$. We denote by $\mathbf{L^0}$ the space of random variables endowed with the topology of convergence in probability.

Definition 1 *[10]*

A process (X_t) is called a total semimartingale *if (X_t) is càdlàg, adapted and if $I_X : \mathbf{S_u} \to \mathbf{L^0}$ is continuous, in the sense that if a sequence $(H^n)_{n\in\mathbb{N}}$ of simple predictable processes converges uniformly to H then $(I_X(H^n))_{n\in\mathbb{N}}$ converges in probability to $I_X(H)$.*

A process $(X_s)_{s\in\mathbb{R}_+}$ is called a semimartingale *if, for all $t \ge 0$, $(X_{s\wedge t})_{s\in\mathbb{R}_+}$ is a total semimartingale.*

Next, we recall the construction of the stochastic exponential.

Theorem 1 *[10] Let $(X_t)_{t\in\mathbb{R}_+}$ denote a semimartingale starting from $X_0 = 0$. Then there exists a (unique) semimartingale $(Z_t)_{t\in\mathbb{R}_+}$ which satisfies the equation*

$$Z_t = 1 + \int_0^t Z_{s^-} dX_s.$$

Moreover, $(Z_t)_{t\in\mathbb{R}_+}$ is given by

$$Z_t = \exp\left(X_t - \frac{1}{2}[X,X]_t\right) \prod_{0\le s\le t} (1+\Delta X_s) \exp\left(-\Delta X_s + \frac{1}{2}(\Delta X_s)^2\right), \quad t \in \mathbb{R}_+.$$

The next result (Girsanov theorem), often used to construct risk neutral probabilities, has been used for the computation of option sensitivities with respect to the interest rate.

Theorem 2 *[10] Let $(W_t)_{t\in\mathbb{R}_+}$ be a Brownian motion under historical probability P, let $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ be the filtration generated by $(W_t)_{t\in\mathbb{R}_+}$, let $(\theta_t)_{t\in\mathbb{R}_+}$ be an $\mathcal{F}_t$-adapted process, and let*

$$W_t^Q = \int_0^t \theta_s ds + W_t$$

and

$$M(t) = \exp\left(-\int_0^t \theta_s dW_s - \frac{1}{2}\int_0^t \theta_s^2 ds\right), \quad t \in \mathbb{R}_+,$$

and define the probability measure Q by

$$Q(F) = \int_F M(T) dP, \qquad F \in \mathcal{F}. \tag{2.2}$$

Then the process $W^Q(t)$ is a Brownian motion under Q and more generally we have,

$$\mathbf{E}_Q[F(W^Q)] = \mathbf{E}_P[F(W)], \qquad F \in L^1(\Omega, P).$$

3 Malliavin calculus on the Wiener space

In this section we recall the basics of Malliavin calculus, cf. e.g. [8], [11], in view of applications to sensitivity analysis. Let $(W_t)_{t\in\mathbb{R}_+}$ denote a d-dimensional Brownian motion, and let $\mathcal{C}$ denote the space of random variables F of the form

$$F = f\left(\int_0^\infty h_1(t) dW_t, \ldots, \int_0^\infty h_n(t) dW_t\right), \quad f \in \mathcal{S}(\mathbb{R}^n),$$

$h_1, \dots, h_n \in L^2(\mathbb{R}_+)$, where $\mathcal{S}(\mathbb{R}^n)$ is the space of rapidly decreasing $\mathcal{C}^\infty$ functions on $\mathbb{R}^n$. Given $F \in \mathcal{C}$, the gradient of F is the process $(D_t F)_{t\in\mathbb{R}_+}$ in $L^2(\Omega \times \mathbb{R}_+)$ defined by

$$D_t F = \sum_{i=1}^n \frac{\partial f}{\partial x_i} \left(\int_0^\infty h_1(t) dW_t, \dots, \int_0^\infty h_n(t) dW_t \right) h_i(t), \qquad t \in \mathbb{R}_+, \qquad a.s.$$

We also define the norm

$$\| F \|_{1,2} = \left(\mathbf{E}[F^2]\right)^{1/2} + \left(\mathbf{E}\left[\int_0^\infty |D_t F|^2 dt \right] \right)^{1/2}, \qquad F \in \mathcal{C},$$

and denote by $\mathbb{D}_{1,2}$ the completion of $\mathcal{C}$ with respect to the norm $\| \cdot \|_{1,2}$. The gradient operator D is a closed linear mapping defined on $\mathbb{D}_{1,2}$ and taking its values in $L^2(\Omega \times \mathbb{R}_+)$.

The gradient D has the derivation property, i.e. if $\phi : \mathbb{R}^n \longrightarrow \mathbb{R}$ is continuously differentiable with bounded partial derivatives, and $F = (F_1, \dots, F_n)$ a random vector whose components belong to $\mathbb{D}_{1,2}$, then $\phi(F) \in \mathbb{D}_{1,2}$ and:

$$D_t \phi(F) = \sum_{i=1}^n \frac{\partial \phi}{\partial x_i}(F) D_t F_i, \quad t \in \mathbb{R}_+, \quad a.s.$$

Next we recall other properties of the gradient D, cf. [8], [6].

Property 1 *Let $(X_t)_{t\in\mathbb{R}_+}$ be the solution to the stochastic differential equation*

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t,$$

where b and σ are continuously differentiable functions. Let $(Y_t)_{t\in\mathbb{R}_+}$ denote the first variation process defined by the stochastic differential equation:

$$dY_t = b'(X_t)Y_t dt + \sigma'(X_t)Y_t dW_t, \quad Y_0 = 1,$$

Then $X_t \in \mathbb{D}_{1,2}$, $t \in \mathbb{R}_+$, and its gradient is given by:

$$D_s X_t = Y_t Y_s^{-1} \sigma(X_s) 1_{\{s \le t\}}, \quad s \in \mathbb{R}_+, \quad a.s.,$$

hence if $f \in \mathcal{C}_1^b(\mathbb{R}^n)$ we have

$$D_s f(X_T) = \nabla f(X_T) Y_T Y_s^{-1} \sigma(X_s) 1_{\{s \le t\}}, \quad s \in \mathbb{R}_+, \quad a.s.$$

The gradient operator D admits an adjoint δ called divergence operator (or Skorohod integral), which satisfies the following integration by parts formula.

Property 2 *Let $u \in L^2(\Omega \times \mathbb{R}_+)$. Then $u \in \text{Dom }(\delta)$ if and only if for all $F \in \mathbb{D}_{1,2}$ we have*

$$\mathbf{E}[\langle DF, u\rangle_{\mathrm{H}}] = \mathbf{E}\left[\int_0^\infty D_t F u(t) dt\right] \leq K(u) \parallel F \parallel_{1,2},$$

where $K(u)$ is constant independent of $F \in \mathbb{D}_{1,2}$. If $u \in \text{Dom }(\delta)$, $\delta(u)$ is defined by the relation

$$\mathbf{E}[F\delta(u)] = \mathbf{E}[\langle DF, u\rangle_{\mathrm{H}}], \quad \forall F \in \mathbb{D}_{1,2}.$$

An important property of the divergence operator δ is that its domain Dom (δ) contains the adapted processes in $L^2(\Omega\times\mathbb{R}_+)$. Moreover, for such processes the action of δ coincides with that of Itô's stochastic integral.

Property 3 *For all adapted stochastic process $u \in L^2(\Omega \times \mathbb{R}_+)$ we have:*

$$\delta(u) = \int_0^\infty u(t) dW_t.$$

We have the following property.

Property 4 *Let $F \in \mathbb{D}_{1,2}$. For all $u \in \text{Dom }(\delta)$ such that $F\delta(u) - \int_0^T D_t F u(t) dt \in L^2(\Omega)$ we have*

$$\delta(Fu) = F\delta(u) - \int_0^T D_t F u(t) dt.$$

In the sequel we use the notation

$$D_w F = \int_0^\infty D_t F w(t) dt, \qquad F \in \mathbb{D}_{1,2}, \quad w \in L^2(\Omega \times \mathbb{R}_+).$$

The next proposition contains the main result used for the computation of sensitivities.

Proposition 1 *Let $(F^\xi)_\xi$ be a family of random variables, continuously differentiable in $\text{Dom }(D)$ with respect to ξ. Let $(w_t)_{t\in[0,T]}$ a process verifying $D_w F^\xi \neq 0$, a.s. on $\{\partial_\xi F^\xi \neq 0\}$, $\xi \in (a,b)$. We have*

$$\frac{\partial}{\partial \xi}\mathbf{E}\left[f\left(F^\xi\right)\right] = \mathbf{E}\left[f\left(F^\xi\right)\delta\left(w\frac{\partial_\xi F^\xi}{D_w F^\xi}\right)\right] \tag{3.1}$$

for all function f such that $f\left(F^\xi\right) \in L^2(\Omega)$, $\xi \in (a,b)$.

Proof. If $f \in \mathcal{C}_b^\infty(\mathbb{R})$, we have

$$\begin{aligned}
\frac{\partial}{\partial \xi}\mathbf{E}\left[f(F^\xi)\right] &= \mathbf{E}\left[f'\left(F^\xi\right)\partial_\xi F^\xi\right] \\
&= \mathbf{E}\left[\frac{D_w f(F^\xi)}{D_w F^\xi}\partial_\xi F^\xi\right] \\
&= \mathbf{E}\left[f\left(F^\xi\right)\delta\left(w\frac{\partial_\xi F^\xi}{D_w F^\xi}\right)\right].
\end{aligned}$$

The general case is obtained by approximation of f by functions in $\mathcal{C}_b^\infty(\mathbb{R})$. □

4 Sensitivity analysis

4.1 Market model

We consider a jump diffusion model in which the dynamics of the underlying asset price is given by

$$\frac{dS_t}{S_t} = r(t)dt + \sigma_1(t)dW_t + \sigma_2(t)dX_t, \quad S_0 = x, \tag{4.1}$$

where $r(t)$ denotes the interest rate, $(X_t)_{t\in\mathbb{R}_+}$ is a jump semimartingale, $(W_t)_{t\in\mathbb{R}_+}$ is an independent Brownian motion, and $\sigma_1(t)$, $\sigma_2(t)$ are volatility parameters, respectively relative to the continuous and jump components. The solution of (4.1) is given by:

$$\begin{aligned} S_t &= x \exp\left(\int_0^t r(s)ds + \int_0^t \sigma_1(s)dW_s + \int_0^t \sigma_2(s)dX_s - \frac{1}{2}\int_0^t \sigma_1^2(s)ds - \frac{1}{2}\int_0^t \sigma_2^2(s)d[X,X]_s\right) \\ &\quad \times \prod_{s=0}^{t} \left((1+\sigma_2(s)\Delta X_s)\exp\left(-\sigma_2(s)\Delta X_s + \frac{1}{2}(\sigma_2(s)\Delta X_s)^2\right)\right) \end{aligned}$$

with $\mu(t) = r(t) - \sigma_1^2(t)/2$. Note that if $(X_t)_{t\in\mathbb{R}_+}$ is a.s. of finite variation, i.e.

$$\sum_{0\le s\le t} |\Delta X_s| < \infty, \qquad a.s.,$$

then

$$\begin{aligned} S_t &= x \exp\left(\int_0^t \sigma_2(s)dX_s - \frac{1}{2}\int_0^t \sigma_1^2(s)ds - \sum_{0\le s\le t}\sigma_2(s)\Delta X_s\right) \\ &\quad \times \exp\left(\int_0^t r(s)ds + \int_0^t \sigma_1(s)dW_s\right)\prod_{0\le s\le t}(1+\sigma_2(s)\Delta X_s) \\ &= \exp\left(\int_0^t \sigma_2(s)dX_s^c + \int_0^t r(s)ds + \int_0^t \sigma_1(s)dW_s - \frac{1}{2}\int_0^t \sigma_1^2(s)ds\right)\prod_{0\le s\le t}(1+\sigma_2(s)\Delta X_s), \end{aligned}$$

where $(X_s^c)_{s\in\mathbb{R}_+}$ denotes the continuous part of $(X_s)_{s\in\mathbb{R}_+}$.

4.2 Delta

Delta represents the variation of the option price with respect to the initial price x of the underlying asset:

$$\text{Delta} = e^{-\int_0^T r(t)dt}\frac{\partial}{\partial x}\mathbf{E}\left[f(S_T^x)\right],$$

where $S_T^x = xS_T^1 = xS_T$. From Proposition 1,

$$\text{Delta} = e^{-\int_0^T r(t)dt}\mathbf{E}\left[f(S_T^x)\delta\left(w\frac{\partial_x S_T^x}{D_w S_T^x}\right)\right],$$

and from Property 1,

$$\begin{aligned} D_w S_T &= \int_0^\infty D_s S_T w(s) ds \\ &= \int_0^\infty S_T \sigma_1(s) 1_{\{s \le T\}} w(s) ds \\ &= S_T \int_0^T \sigma_1(s) w(s) ds. \end{aligned}$$

Hence

$$\begin{aligned} \text{Delta} &= e^{-\int_0^T r(t)dt} \mathbf{E}\left[f(S_T) \delta\left(\frac{w}{x \int_0^T \sigma_1(t) w(t) dt} \right) \right] \\ &= \frac{1}{x} \frac{e^{-\int_0^T r(t)dt}}{\int_0^T \sigma_1(t) w(t) dt} \mathbf{E}\left[f(S_T) \int_0^T w(t) dW_t \right]. \end{aligned} \tag{4.2}$$

4.3 Vega

Vega measures the sensitivity of the option price with respect to the volatility parameter.

4.3.1 Vega_1

In case of a derivation with respect to the volatility parameter σ_1 we consider the perturbed process $(S_t^\varepsilon)_{0 \le t \le T}$ given by:

$$dS_t^\varepsilon = r(t) S_t^\varepsilon dt + (\sigma_1(t) + \varepsilon \tilde{\sigma}_1(t)) S_t^\varepsilon dW_t + \sigma_2(t) S_t^\varepsilon dX_t, \tag{4.3}$$

where $\varepsilon > 0$ and $\tilde{\sigma}_1 : [0, T] \to \mathbb{R}$ is a bounded perturbation function. We have

$$\partial_\varepsilon S_T^\varepsilon = S_T^\varepsilon \left(\int_0^T \tilde{\sigma}_1(t) dW_t - \int_0^T \tilde{\sigma}_1(t)(\sigma_1(t) + \varepsilon \tilde{\sigma}_1(t)) dt \right).$$

Letting $C_\varepsilon = \mathbf{E}[f(S_T^\varepsilon)]$, from Proposition 1 we have

$$\begin{aligned} \text{Vega}_1 &= \frac{\partial C_\varepsilon}{\partial \varepsilon}\Big|_{\varepsilon=0} \\ &= \frac{e^{-\int_0^T r(t)dt}}{\int_0^T \sigma_1(t) w(t) dt} \mathbf{E}\left[f(S_T) \delta\left(w(\cdot) \left(\int_0^T \tilde{\sigma}_1(t) dW_t - \int_0^T \tilde{\sigma}_1 \sigma_1(t) dt \right) \right) \right]. \end{aligned}$$

As for the Delta, the obtained formula coincides with that of [6].

4.3.2 Vega$_2$

We choose here to differentiate with respect to the volatility σ_2 related to the jump process, in this case the formula we obtain is different from the one obtained for Vega$_1$. Consider the process $(S_t^\varepsilon)_{0\le t\le T}$ defined by:

$$dS_t^\varepsilon = r(t)S_t^\varepsilon dt + \sigma_1(t)S_t^\varepsilon dW_t + (\sigma_2(t)S_t^\varepsilon + \varepsilon\tilde{\sigma}_2(t))dX_t \tag{4.4}$$

where ε is a small real parameter and $\tilde{\sigma}_2 : [0,T] \to \mathbb{R}$ is a bounded perturbation function. Given the option price

$$C_\varepsilon = e^{-\int_0^T r(t)}\mathbf{E}\left[f(S_T^\varepsilon)\right], \quad S_0^\varepsilon = x,$$

we need to compute $\frac{\partial C_\varepsilon}{\partial \varepsilon}\big|_{\varepsilon=0}$. By Proposition 1 we have

$$\frac{\partial C_\varepsilon}{\partial \varepsilon} = e^{-\int_0^T r(t)}\mathbf{E}\left[f(S_T^\varepsilon)\delta\left(w\frac{\partial_\varepsilon S_T^\varepsilon}{D_w S_T^\varepsilon}\right)\right],$$

with

$$S_t^\varepsilon = A_t \exp\left(\int_0^t (\sigma_2(s)+\varepsilon\tilde{\sigma}_2(s))\, dX_s^c\right)\prod_{0\le s\le t}(1+(\sigma_2(s)+\varepsilon\tilde{\sigma}_2(s))\,\Delta X_s)$$

and

$$A_t = x\exp\left(\int_0^t \mu(s)ds + \int_0^t \sigma_1(s)dW_s\right).$$

Hence

$$\partial_\varepsilon S_T^\varepsilon = S_T^\varepsilon\left(\sum_{\substack{\Delta X_s\neq 0\\ 0\le s\le T}}\frac{\tilde{\sigma}_2(s)\Delta X_s}{1+(\sigma_2(s)+\varepsilon\tilde{\sigma}_2(s))\,\Delta X_s} + \int_0^t \tilde{\sigma}_2(s)dX_s^c\right).$$

We then obtain

$$\text{Vega}_2 = \frac{e^{-\int_0^T r(t)}}{\int_0^T \sigma_1(t)w(t)dt}\mathbf{E}\left[f(S_T)\int_0^T w(t)dW_t\left(\sum_{\substack{\Delta X_s\neq 0\\ 0\le s\le T}}\frac{\tilde{\sigma}_2(s)\Delta X_s}{1+\sigma_2(s)\Delta X_s} + \int_0^t \tilde{\sigma}_2(s)dX_s^c\right)\right].$$

4.4 Gamma

Delta is more sensitive to variations when the exercise price of an option is close to the spot price x. Gamma is used to evaluate the sensitivity of Delta with respect to the variations of x. We have

$$\text{Gamma} = e^{-\int_0^T r(t)dt}\frac{\partial^2}{\partial x^2}\mathbf{E}\left[f(S_T)\right]$$

$$
\begin{aligned}
&= e^{-\int_0^T r(t)dt} \frac{\partial}{\partial x} \mathbf{E}\left[f(S_T) \delta\left(\frac{w}{x\int_0^T \sigma_1(t) w(t) dt} \right) \right] \\
&= e^{-\int_0^T r(t)dt} \frac{\partial}{\partial x} \mathbf{E}\left[f(S_T) \frac{\delta(w)}{x\int_0^T \sigma_1(t) w(t) dt} \right] \\
&= \frac{e^{-\int_0^T r(t)dt}}{x\int_0^T \sigma_1(t) w(t) dt} \mathbf{E}\left[f(S_T) \delta\left(w \frac{\partial_x S_T}{D_w S_T} \delta(w) \right) \right] \\
&\quad - \frac{1}{x^2} \frac{e^{-\int_0^T r(t)dt}}{\int_0^T \sigma_1(t) w(t) dt} \mathbf{E}\left[f(S_T) \delta(w) \right],
\end{aligned}
$$

with

$$
\begin{aligned}
\delta\left(w \frac{\partial_x S_T}{D_w S_T} \frac{\delta(w)}{x\int_0^T \sigma_1(t) w(t) dt} \right) &= \delta\left(\frac{w\delta(w)}{\left(x\int_0^T \sigma_1(t) w(t) dt \right)^2} \right) \\
&= \frac{\delta(w)\delta(w) - \int_0^T D_t \delta(w) w(t) dt}{\left(x\int_0^T \sigma_1(t) w(t) dt \right)^2} \\
&= \frac{(\delta(w))^2 - 1}{\left(x\int_0^T \sigma_1(t) w(t) dt \right)^2}.
\end{aligned}
$$

From this we deduce

$$
\text{Gamma} = e^{-\int_0^T r(t)dt} \frac{\partial^2}{\partial x^2} \mathbf{E}\left[f(S_T) \right] = \tag{4.5}
$$

$$
\frac{e^{-\int_0^T r(t)dt}}{x^2} \mathbf{E}\left[f(S_T) \left(\left(\frac{\int_0^T w(t) dW_t}{\int_0^T \sigma_1(t) w(t) dt} \right)^2 - \frac{\int_0^T w(t) dW_t}{\left(\int_0^T \sigma_1(t) w(t) dt \right)^2} - \frac{\int_0^T w(t) dW_t}{\int_0^T \sigma_1(t) w(t) dt} \right) \right].
$$

4.5 Rho

Rho represents the sensitivity, i.e. the first derivative, of the option price with respect to the interest rate parameter. Consider the process $(S_t^\varepsilon)_{0\le t\le T}$ defined by:

$$
dS_t^\varepsilon = (r(t) + \varepsilon \tilde{r}(t)) S_t^\varepsilon dt + \sigma_1(t) S_t^\varepsilon dW_t + \sigma_2(t) S_t^\varepsilon dX_t \tag{4.6}
$$

where $\varepsilon > 0$ is small and $\tilde{r}$ is a bounded perturbation function from $[0, T]$ to $\mathbb{R}$. Given the option price

$$
C_\varepsilon = e^{-\int_0^T (r(t)+\varepsilon\tilde{r}(t))dt} \mathbf{E}\left[f(S_T^\varepsilon) \right],
$$

let

$$
\tilde{Z}_T^\varepsilon = \exp\left(-\varepsilon \int_0^T \tilde{r}(t) \sigma_1^{-1}(t) dW_t^\varepsilon + \frac{\varepsilon^2}{2} \int_0^T |\tilde{r}(t)\sigma_1^{-1}(t)|^2 dt \right),
$$

and define the probability measure Q^ε, equivalent to P, by $dQ^\varepsilon = \tilde{Z}_T^\varepsilon dP$. Then (4.6) reads

$$dS_t^\varepsilon = r(t)S_t^\varepsilon dt + \sigma_1(t)S_t^\varepsilon dW_t^\varepsilon + \sigma_2(t)S_t^\varepsilon dX_t, \tag{4.7}$$

with $dW_t^\varepsilon = dW_t + \varepsilon\tilde{r}(t)\sigma_1^{-1}(t)dt$. From Theorem 2 (Girsanov Theorem), the process $(W_t^\varepsilon)_{0\le t\le T}$ is a standard Brownian motion under the probability Q^ε. Hence $(W^\varepsilon)_{0\le t\le T}$ has same law as W, and $(S^\varepsilon)_{0\le t\le T}$ has same law as $(S_t)_{0\le t\le T}$, and we have

$$\begin{aligned} C_\varepsilon &= e^{-\int_0^T r(t)+\varepsilon\tilde{r}(t)dt}\mathbf{E}_P\left[f(S_T^\varepsilon)\right] \\ &= e^{-\int_0^T r(t)+\varepsilon\tilde{r}(t)dt}\mathbf{E}_{Q^\varepsilon}\left[\frac{1}{\tilde{Z}_T^\varepsilon}f(S_T^\varepsilon)\right] \\ &= e^{-\int_0^T r(t)+\varepsilon\tilde{r}(t)dt}\mathbf{E}_P\left[\frac{1}{Z_T^\varepsilon}f(S_T)\right], \end{aligned} \tag{4.8}$$

with $Z_T^\varepsilon = \exp\left(-\varepsilon\int_0^T \tilde{r}(t)\sigma_1^{-1}(t)dW_t + \frac{\varepsilon^2}{2}\int_0^T |\tilde{r}(t)\sigma_1^{-1}(t)|^2dt\right)$. By differentiation of (4.8) with respect to ε we get, after evaluation in $\varepsilon = 0$:

$$\text{Rho} = e^{-\int_0^T r(t)dt}\mathbf{E}\left[f(S_T)\int_0^T \frac{\tilde{r}(t)}{\sigma_1(t)}dW_t\right] - e^{-\int_0^T r(t)dt}\int_0^T \tilde{r}(t)dt\mathbf{E}\left[f(S_T)\right]. \tag{4.9}$$

5 Numerical simulations

In this section we present numerical simulations for the Greek parameters Delta, Vega, Gamma and Rho, for a European binary option with exercise price K. Expectations are computed numerically via the Monte Carlo method, i.e. using the approximation

$$\mathbf{E}[X] \simeq \frac{1}{N}\sum_{i=1}^N X(i)$$

due to the law of large numbers, given a large number N of independent samples $(X(i))_{i=1,\dots,N}$. We consider a simplified model where the parameters σ_1, σ_2, r are independent of time and where the jump process is taken to be a compensated Poisson process with intensity λ. Under these conditions the price of the underlying asset is given by:

$$S_t = x(1+\sigma_2)^{N_t}\exp\left(\left(\alpha - \frac{\sigma_1^2}{2}\right)t + \sigma_1 W_t\right),$$

with $\alpha = r - \sigma_2\lambda$, $t \in [0,T]$, and the value of the binary option is:

$$X_\xi = e^{-rT}\mathbf{E}\left[1_{[K,\infty[}(S_T^\xi)\right], \quad \xi = x, r, \sigma_1, \sigma_2.$$

For the Malliavin calculus approach we choose w_t to be the constant function $w(t) = 1/T$, $0 \le t \le T$, which yields

$$D_w S_T = S_T \int_0^T \sigma_1 w(s) ds = \sigma_1 S_T.$$

The intensity of the Poisson component is fixed equal to $\lambda = 1/10$. The following graphs allow to compare the efficiency of both methods.

5.1 Simulation of Delta

From (4.2), Delta is given by

$$\text{Delta} = \frac{e^{-rT}}{x\sigma_1 T} \mathbf{E}\left[f(S_T) W_T \right].$$

To approximate Delta by finite differences we use

$$\text{Delta} = \frac{C_{x(1+\varepsilon)} - C_{x(1-\varepsilon)}}{2x\varepsilon}.$$

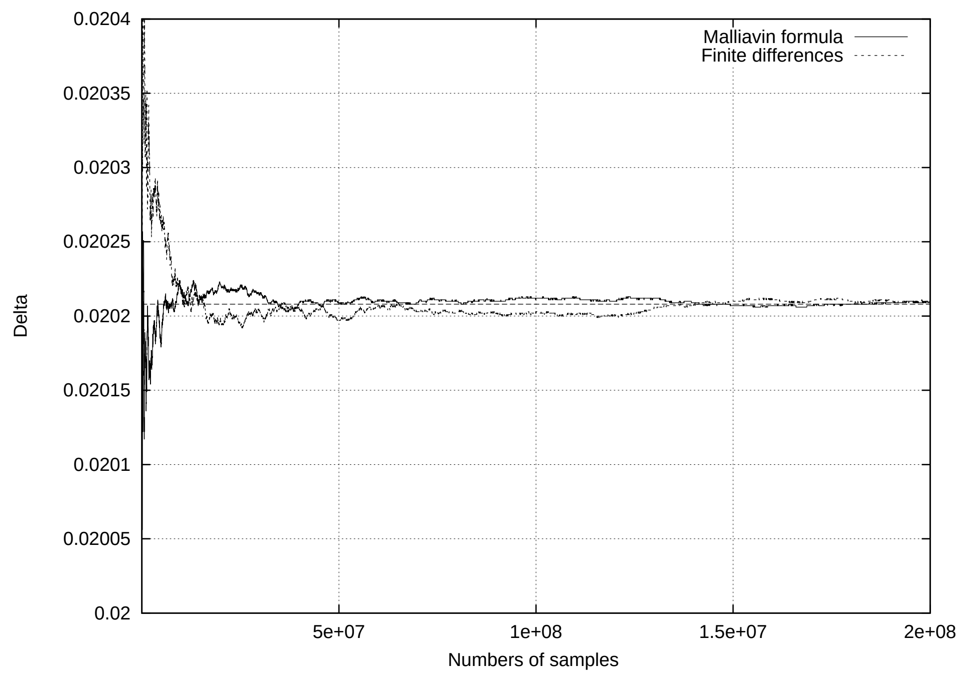

Figure 1: Delta with $N = 2 \times 10^8, x = 100, r = 0.2, \sigma_1 = 0.1, \sigma_2 = 0.1, T = 1, K = 120$ and $e = 0.01$.

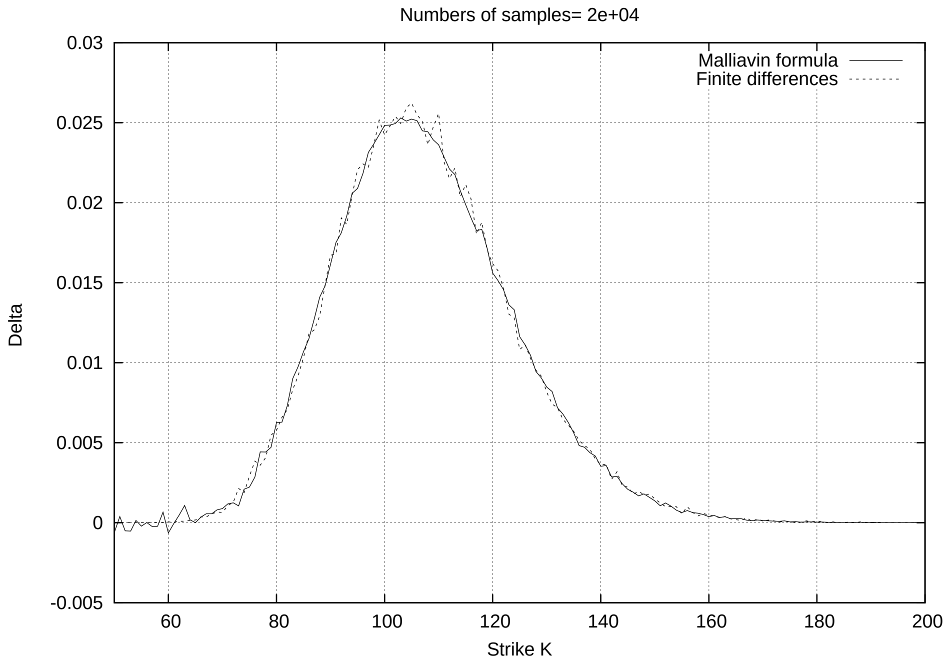

Figure 2: Delta as a function of K with $r = 0.05, \sigma_1 = 0.15, \sigma_2 = 0.01$.

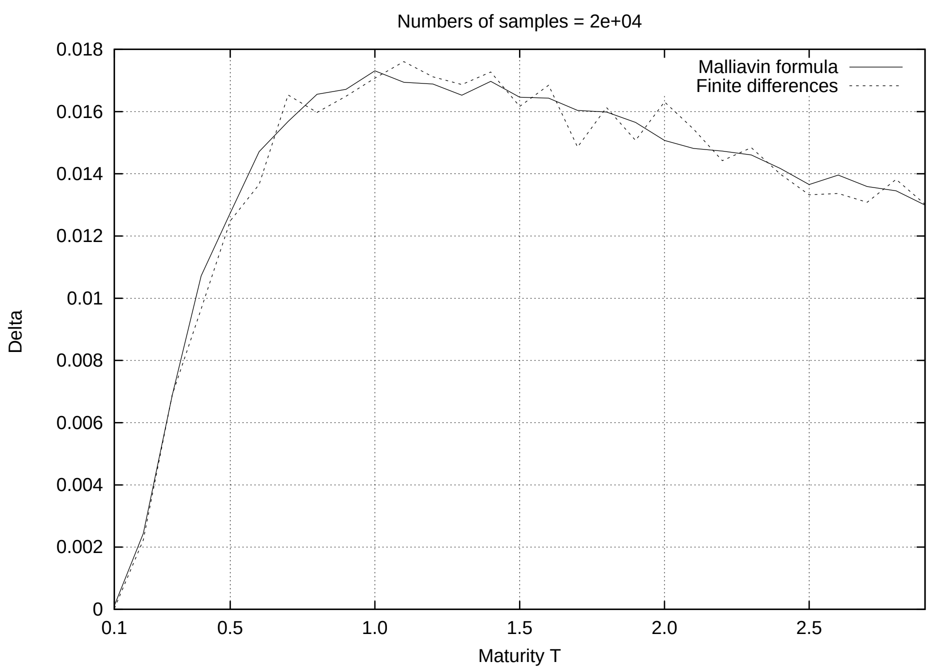

Figure 3: Delta as a function of T with $r = 0.05, \sigma_1 = 0.15, \sigma_2 = 0.01$.

5.2 Simulations of Vega

5.2.1 Vega_1

Vega_1 is defined by

$$\frac{\partial C_{\sigma_1}}{\partial \sigma_1} = e^{-rT}\mathbf{E}\left[f(S_T)\delta\left(w\frac{\partial_{\sigma_1} S_T}{D_w S_T}\right)\right]$$

We have

$$\partial_{\sigma_1} S_T = S_T\left(-\sigma_1 T + W_T\right)$$

and

$$\delta\left(w\frac{\partial_{\sigma_1} S_T}{D_w S_T}\right) = \delta\left(\frac{w}{\sigma_1}\left(-\sigma_1 T + W_T\right)\right) = \left(-T + \frac{W_T}{\sigma_1}\right)\frac{W_T}{T} - \frac{1}{\sigma_1},$$

hence

$$\text{Vega}_1 = e^{-rT}\mathbf{E}\left[f(S_T)\left(\frac{W_T^2}{T\sigma_1} - W_T - \frac{1}{\sigma_1}\right)\right].$$

By the finite difference method, Vega_1 is given by

$$\text{Vega}_1 = \frac{C_{\sigma_1(1+\varepsilon)} - C_{\sigma_1(1-\varepsilon)}}{2\sigma_1\varepsilon}$$

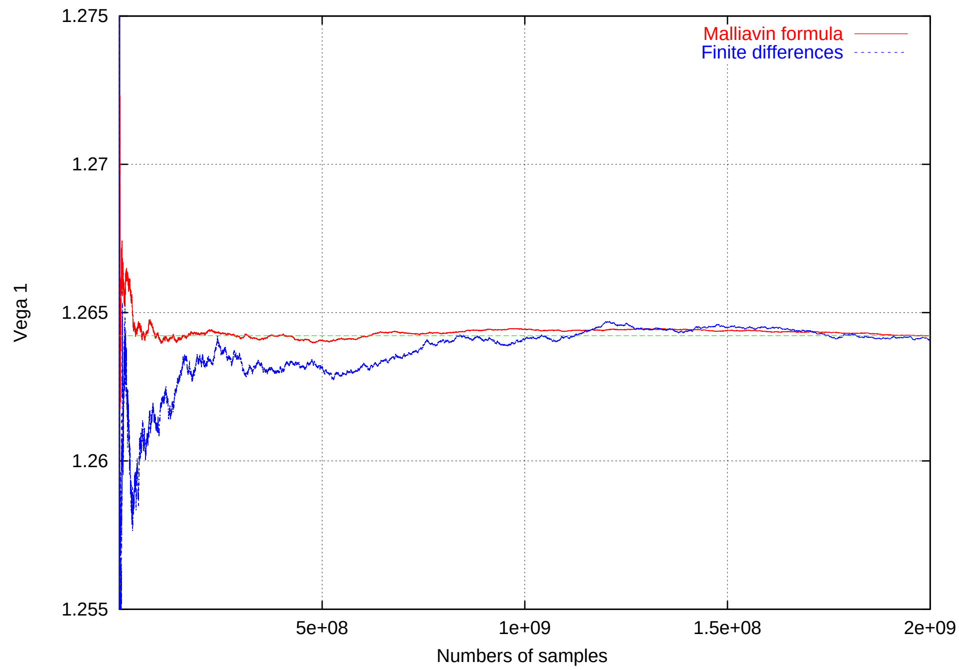

Figure 4: Vega_1 with $x = 100, r = 0.05, \sigma_1 = 0.15, \sigma_2 = 0.01, T = 1, K = 120$ and $e = 0.01$.

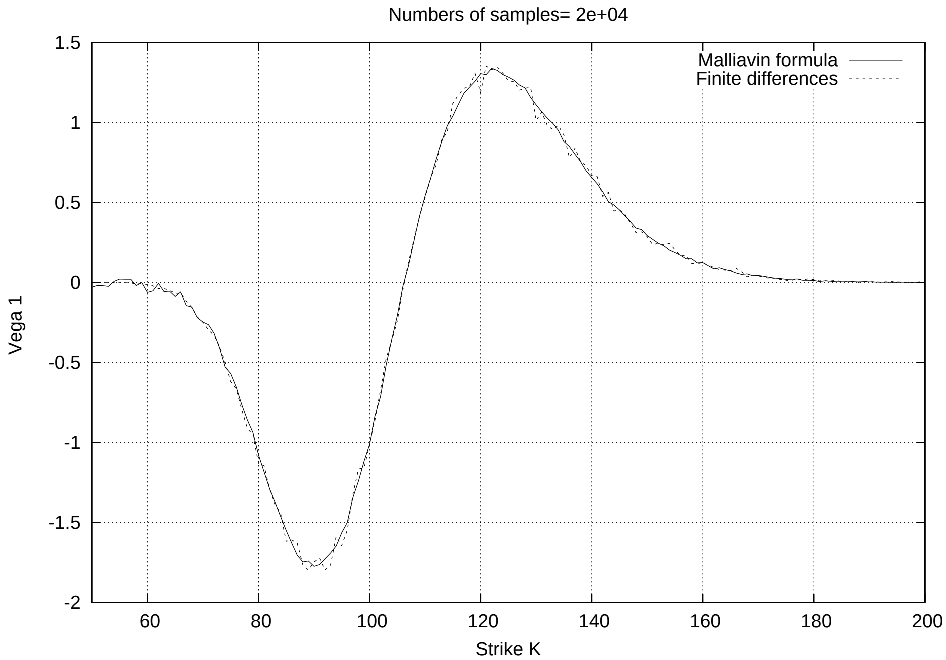

Figure 5: Vega_1 as a function of K $(r = 0.05, \sigma_1 = 0.15, \sigma_2 = 0.01)$.

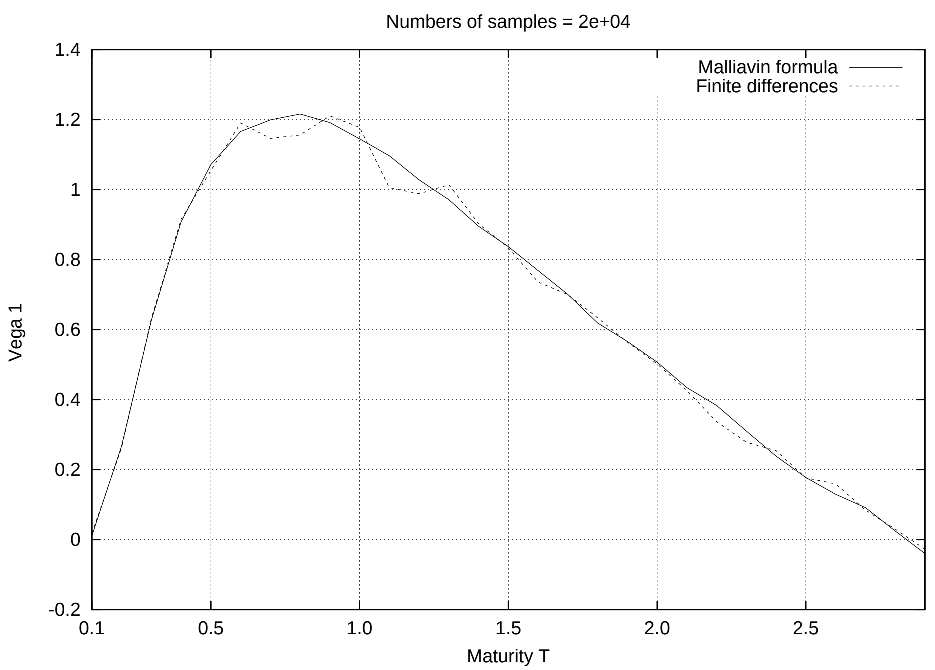

Figure 6: Vega_1 as a function of T $(r = 0.05, \sigma_1 = 0.15, \sigma_2 = 0.01)$.

5.2.2 Vega$_2$

Vega$_2$ is defined by

$$\frac{\partial C_{\sigma_2}}{\partial \sigma_2} = e^{-rT}\mathbf{E}\left[f(S_T)\delta\left(w\frac{\partial_{\sigma_2}S_T}{D_w S_T}\right)\right]$$

We have

$$\partial_{\sigma_2}S_T = S_T\left(\frac{N_T}{1+\sigma_2} - \lambda T\right)$$

and

$$\delta\left(w\frac{\partial_{\sigma_2}S_T}{D_w S_T}\right) = \delta\left(\frac{w}{\sigma_1}\left(\frac{N_T}{1+\sigma_2} - \lambda T\right)\right) = \frac{W_T}{\sigma_1 T}\left(\frac{N_T}{1+\sigma_2} - \lambda T\right),$$

hence

$$\text{Vega}_2 = e^{-rT}\mathbf{E}\left[f(S_T)\frac{W_T}{\sigma_1 T}\left(\frac{N_T}{1+\sigma_2} - \lambda T\right)\right].$$

By the finite difference method, Vega$_2$ is given by

$$\text{Vega}_2 = \frac{C_{\sigma_2(1+\varepsilon)} - C_{\sigma_2(1-\varepsilon)}}{2\sigma_2\varepsilon}$$

Note that for the numerical computation of Vega$_1$, the finite difference method performs similarly to the Malliavin method.

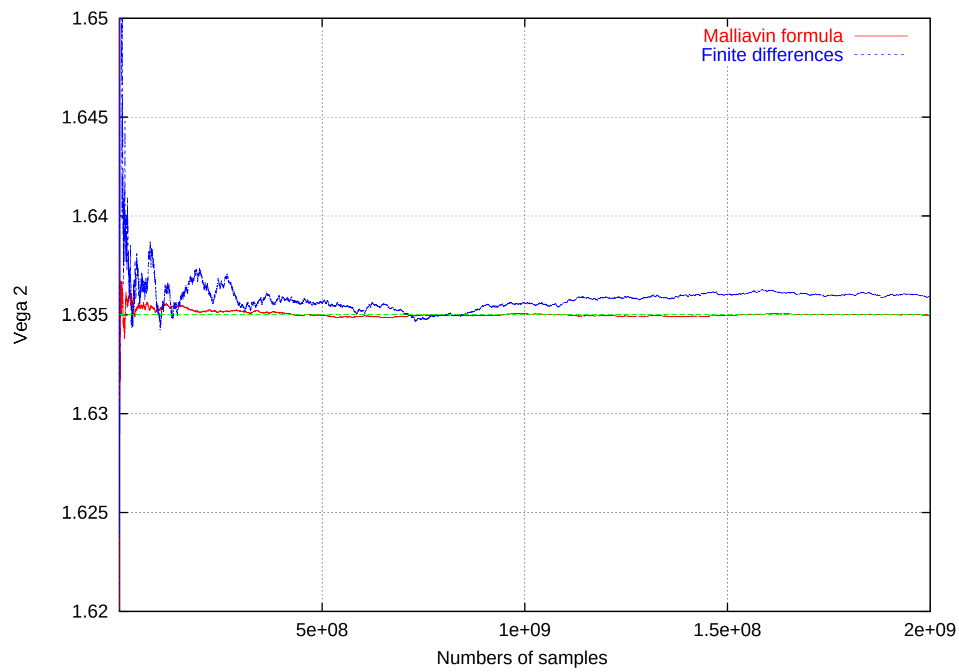

Figure 7: Vega$_2$ with $x = 100, r = 0.1, \sigma_1 = 0.2, \sigma_2 = 0.1, T = 1, K = 120$ and $e = 0.01$.

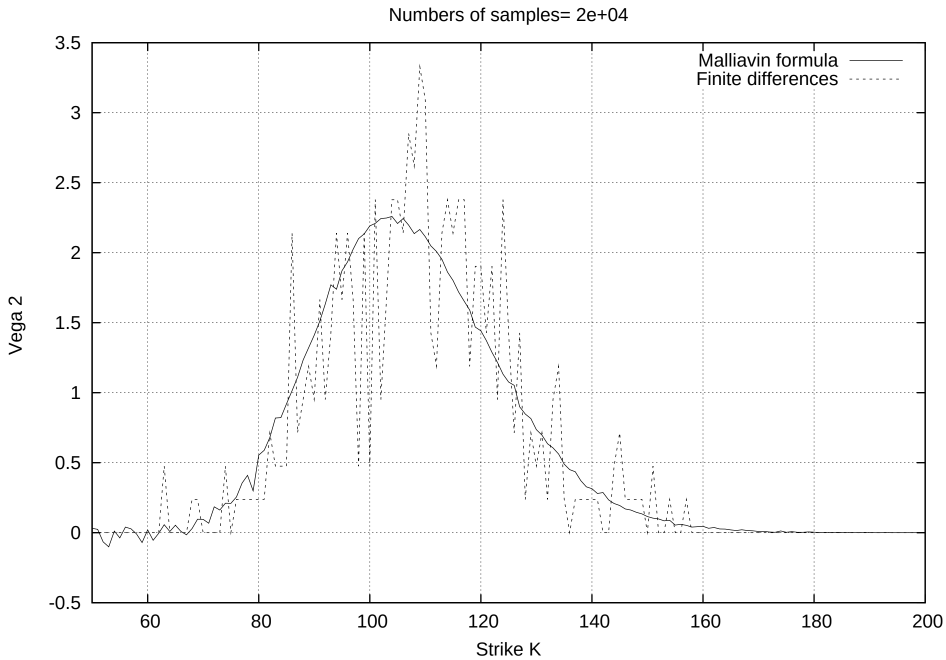

Figure 8: Vega_2 as a function of K with $r = 0.05, \sigma_1 = 0.15, \sigma_2 = 0.01$.

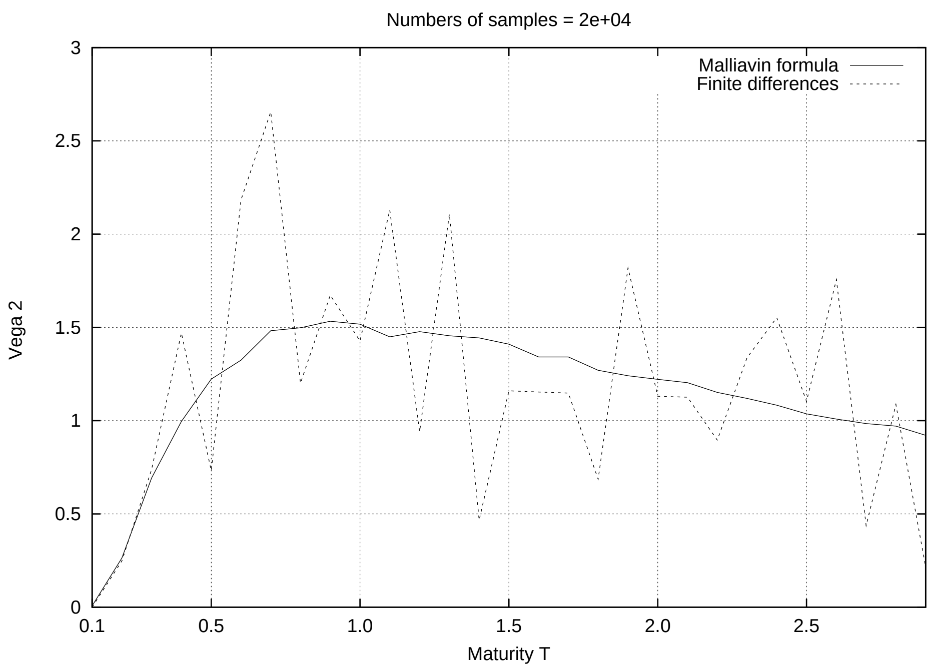

Figure 9: Vega_2 as a function of T with $r = 0.05$, $\sigma_1 = 0.15$, $\sigma_2 = 0.01$, $x = 100$.

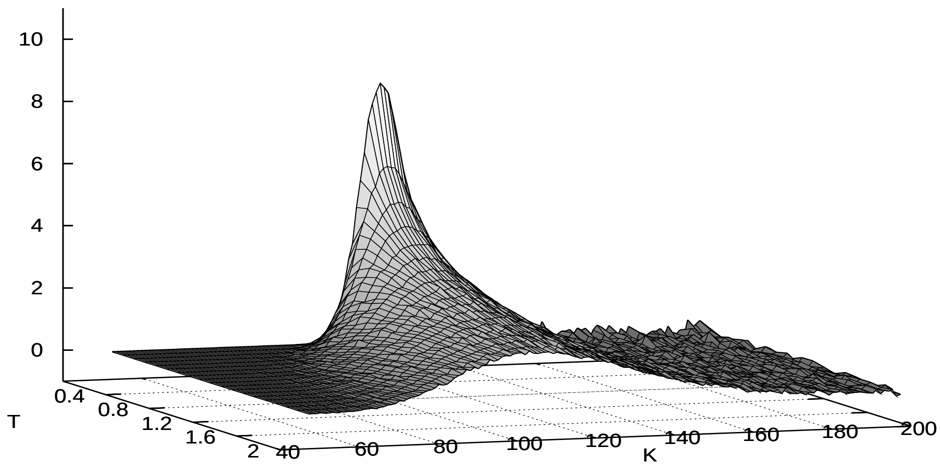

Figure 10: Vega_2 as a function of K and T: $x = 100$, $r = 0.05$, $\sigma_1 = 0.15$, $\sigma_2 = 0.01$, and $e = 0.01$ (Malliavin method, 10000 samples).

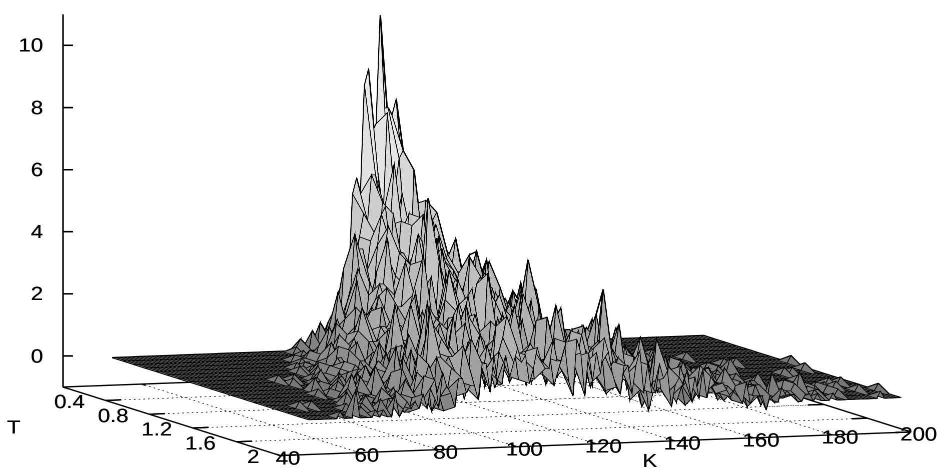

Figure 11: Vega_2 as a function of K and T: $x = 100$, $r = 0.05$, $\sigma_1 = 0.15$, $\sigma_2 = 0.01$, and $e = 0.01$ (finite differences, 10000 samples).

5.3 Simulations of Gamma

From (4.5) we have

$$\text{Gamma} = e^{-rT}\mathbf{E}\left[f(S_T)\frac{1}{x^2\sigma_1 T}\left(\frac{W_T^2}{\sigma_1 T} - \frac{1}{\sigma_1 T} - W_T\right)\right]$$

By the finite difference method, Gamma is given by

$$\text{Gamma} = \frac{C_{x(1+\varepsilon)} - C_x + C_{x(1-\varepsilon)}}{x^2\varepsilon^2}$$

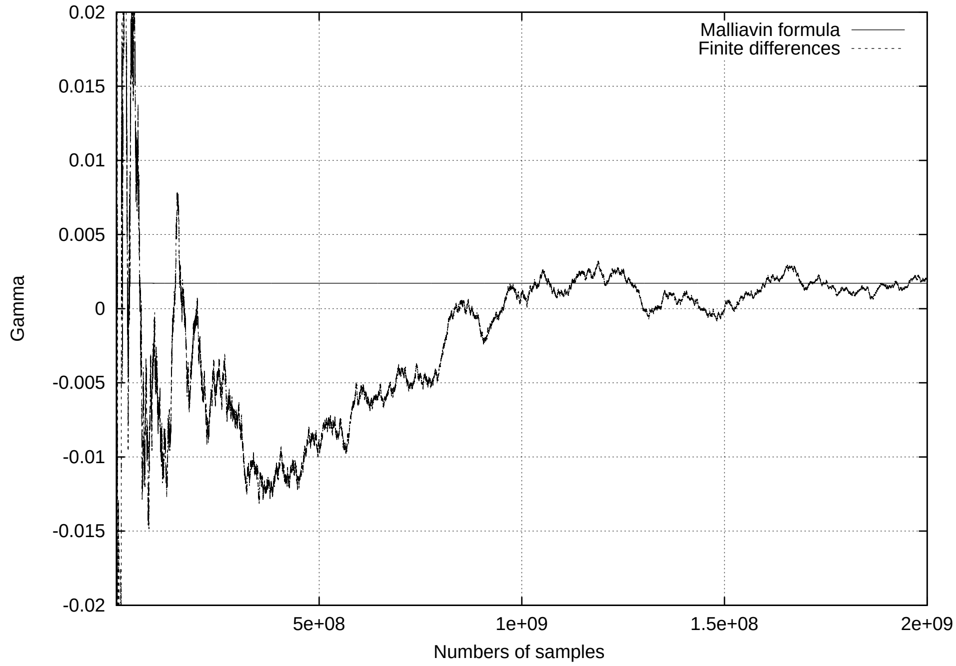

Figure 12: Gamma with $x = 100, r = 0.1, \sigma_1 = 0.2, \sigma_2 = 0.1, T = 1, K = 120$ and $e = 0.01$.

5.4 Simulations of Rho

From the equation (4.9) and assuming that the perturbation function $\tilde{r}$ is constant and equal to 1 we have

$$\begin{aligned} \text{Rho} &= \frac{e^{-rT}}{\sigma_1}\mathbf{E}\left[f(S_T)\int_0^T dW_t\right] - Te^{-rT}\mathbf{E}\left[f(S_T)\right] \\ &= e^{-rT}\mathbf{E}\left[f(S_T)\left(\frac{W_T}{\sigma_1} - T\right)\right] \end{aligned}$$

By the finite difference method, Rho is given by

$$\text{Rho} = \frac{C_{r(1+\varepsilon)} - C_{r(1-\varepsilon)}}{2\varepsilon r}$$

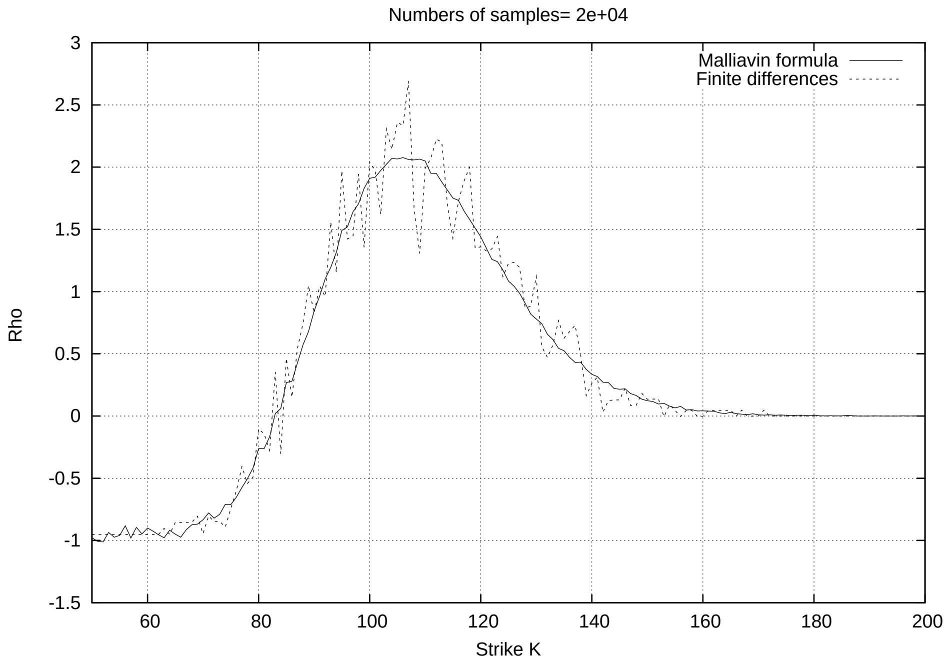

Figure 13: Rho as a function of K with $r = 0.05, \sigma_1 = 0.15, \sigma_2 = 0.01$.

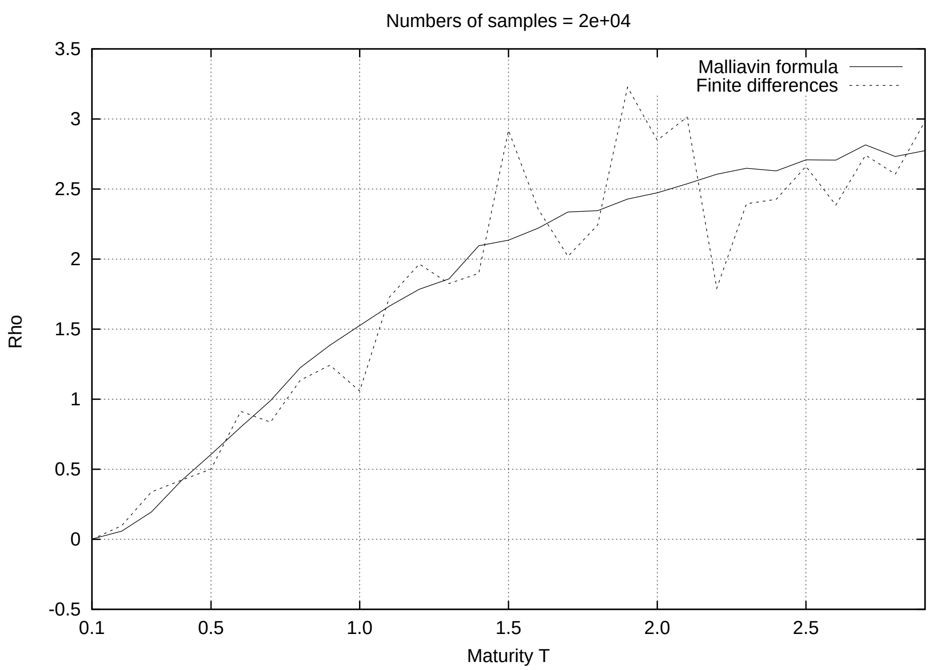

Figure 14: Rho as a function of T with $r = 0.05, \sigma_1 = 0.15, \sigma_2 = 0.01$.

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