Isoperimetric and related bounds on configuration spaces

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Abstract

Using finite difference operators, we define a notion of boundary and surface measure for configuration sets under Poisson measures. A Margulis-Russo type identity and a co-area formula are stated with applications to bounds on the probabilities of monotone sets of configurations and on related isoperimetric constants.

Key words: Configuration spaces, Poisson measures, surface measures, co-area formulas, isoperimetry.

1 Introduction

In this note we consider some isoperimetric problems on the space of configurations of a metric space X, equipped with a Poisson measure Π with finite intensity $\sigma(dx)$.

We introduce notions of inner and outer boundaries $\partial_{\text{in}}A$, $\partial_{\text{out}}A$, as well as symmetric boundary ∂A, and surface measure $\Pi_s(\partial A)$ valid for an arbitrary set A of configurations.

We then prove a Margulis-Russo type identity from which we deduce bounds on the probability $\Pi_\lambda(A)$ for monotone configuration sets A under the Poisson measure Π_λ of intensity $\lambda\sigma(dx)$, $\lambda > 0$. Thresholding results are obtained as a consequence.

We also prove a co-area formula which allows us to study related isoperimetric constants such as

$$h_2 = \inf_{0<\Pi(A)<1} \frac{\Pi_s(\partial A)}{\Pi(A)\Pi(A^c)}, \qquad\qquad h_\infty = \inf_{0<\Pi(A)<1} \frac{\Pi(\partial A)}{\Pi(A)\Pi(A^c)}.$$

The configuration space approach to Poisson measures allows one to consider functions of finite dimensional Poisson random vectors as particular cases, and in this situation our results recover and extend those obtained by Bobkov and Götze (1999).

We proceed as follows. In Section 2 we recall the definition of the Poisson measure and the properties of the finite differences gradient. We also extend the isoperimetric result of Bobkov and Götze (1999) to the setting of configuration spaces. In Section 3 we construct the inner and outer boundaries of subsets of configurations. The surface measures of such boundary sets are then defined by averaging the norms of finite difference gradients, and can be interpreted as the measures of inner and outer flows

through the boundary. We also prove a co-area formula and introduce the main associated isoperimetric constants. In Section 4 we present a notion of monotone set and state a Margulis-Russo type identity in the setting of configuration spaces. In Section 5 we deduce bounds on the probabilities of monotone sets in terms of the intensity parameter of the Poisson distribution and obtain a thresholding result as a consequence. Finally in Section 6 we prove some bounds on the isoperimetric constants using the co-area formula of Section 3.

2 Poisson measure and finite difference operators

Let X be a metric space with Borel σ-algebra $\mathcal{B}(X)$ and let σ be a finite and diffuse measure on X. Let Ω denote the configuration space over X, i.e. the set of Radon measures

$$\Omega = \left\{ \omega = \sum_{i=1}^{i=N} \delta_{x_i} \ : \ (x_i)_{i=1}^{i=N} \subset X, \ x_i \neq x_j, \ \forall i \neq j, \ N \in \mathbb{N} \cup \{\infty\} \right\},$$

where δ_x denotes the Dirac measure at $x \in X$. For convenience of notation, when ω contains n points, $n \geq 1$, we identify $\omega = \sum_{i=1}^{i=n} \delta_{x_i}$ with the set $\omega = \{x_1, \ldots, x_n\}$. Let $\mathcal{F}$ denote the σ-algebra generated by all mappings of the form $\omega \mapsto \omega(B)$, $B \in \mathcal{B}(X)$, and let Π denote the Poisson measure with intensity σ on Ω, defined via

$$\Pi(\{\omega \in \Omega \ : \ \omega(A_1) = k_1, \ldots, \omega(A_n) = k_n\}) = \exp\left(-\sum_{i=1}^{i=n} \sigma(A_i) \right) \prod_{j=1}^{j=n} \frac{\sigma(A_j)^{k_j}}{k_j!},$$

$k_1, \ldots, k_n \in \mathbb{N}$, on the σ-algebra $\mathcal{F}$ generated by the sets of the form

$$\{\omega \in \Omega \ : \ \omega(A_1) = k_1, \ldots, \omega(A_n) = k_n\},$$

for $k_1, \ldots, k_n \in \mathbb{N}$, and disjoint $A_1, \ldots, A_n \in \mathcal{B}(X)$.

Under Π, the expectation of any integrable random variable of the form

$$F(\omega) = f_0 \mathbf{1}_{\{|\omega|=0\}} + \sum_{n=1}^{\infty} \mathbf{1}_{\{|\omega|=n\}} f_n(x_1, \ldots, x_n), \tag{2.1}$$

where $f_0 \in \mathbb{R}$ and f_n is a symmetric measurable function on X^n, $n \geq 1$, is given by

$$E[F] = e^{-\sigma(X)} f_0 + e^{-\sigma(X)} \sum_{n=1}^{\infty} \frac{1}{n!} \int_X \cdots \int_X f_n(x_1, \ldots, x_n) \sigma(dx_1) \cdots \sigma(dx_n). \tag{2.2}$$

We will use the following finite difference operator.

Definition 2.1 *For any $F : \Omega \longrightarrow \mathbb{R}$, let*

$$D_x F(\omega) = (F(\omega + \delta_x) - F(\omega)) \mathbf{1}_{\{x \in \omega^c\}}$$

for all $\omega \in \Omega$ and $x \in X$.

Let Π_λ, $\lambda > 0$, denote the Poisson measure of intensity $\lambda\sigma(dx)$ on Ω, and let E_λ denote the expectation with respect to Π_λ. The following variational formula is obtained by differentiation with respect to the intensity parameter λ of the Poisson measure.

Proposition 2.2 *Let $0 < a < b$ and $F : \Omega \to \mathbb{R}$ be such that $F \in L^1(\Pi_\lambda)$ and $DF \in L^1(\Pi_\lambda \otimes \sigma)$, for all $\lambda \in (a, b)$. Then,*

$$\frac{\partial}{\partial\lambda} E_\lambda[F] = E_\lambda\left[\int_X D_x F\sigma(dx)\right], \quad \lambda \in (a, b).$$

Proof. This result can be seen as an application of Remark 2.1 of Molchanov and Zuyev (2000), but we provide here a proof for completeness. From (2.2), we have for F of the form (2.1):

$$\frac{\partial}{\partial\lambda} E_\lambda[F] = -\sigma(X)E_\lambda[F]$$
$$+ e^{-\lambda\sigma(X)} \sum_{n=1}^{\infty} \frac{\lambda^{n-1}}{(n-1)!} \int_X \cdots \int_X f_n(x_1, \ldots, x_n)\sigma(dx_1)\cdots\sigma(dx_n)$$
$$= -\sigma(X)E_\lambda[F] + E_\lambda\left[\int_X F(\omega + \delta_x)\sigma(dx)\right]$$
$$= E_\lambda\left[\int_X D_x F(\omega)\sigma(dx)\right].$$

$\square$

We close this section with the following result which gives a version of isoperimetry on Poisson space which is independent of dimension and is akin to the result of Bobkov and Götze, 1999, p. 263. Let φ denote the standard Gaussian density, and let Φ denote its distribution function. Let $I(t) = \varphi(\Phi^{-1}(t))$, $0 \leq t \leq 1$ denote the Gaussian isoperimetric function, with the relations

$$I(t)I''(t) = -1 \qquad \text{and} \qquad I'(t) = -\Phi^{-1}(t), \quad t \in [0, 1].$$

Proposition 2.3 *For every integrable random variable $F : \Omega \to [0, 1]$ we have*

$$I(E[F]) \leq E\left[\sqrt{I(F)^2 + 2|DF|_{L^2(\sigma)}^2}\right]. \tag{2.3}$$

Proof. Let $A_1, \ldots, A_n$ a family of disjoint elements of $\mathcal{B}(X)$ and consider the vector

$$X_n(\omega) = (\omega(A_1), \ldots, \omega(A_n)), \qquad \omega \in \Omega, \tag{2.4}$$

of independent Poisson random variables with intensities $\sigma(A_1), \ldots, \sigma(A_n)$. For any cylindrical functional of the form $F = f \circ X_n$ we have

$$D_x F(\omega) = \sum_{k=1}^{k=n} \mathbf{1}_{A_k}(x)(f(X_n(\omega) + e_k) - f(X_n(\omega))),$$

$f : \mathbb{N}^n \to \mathbb{R}$, where $(e_k)_{1 \leq k \leq n}$ denotes the canonical basis of $\mathbb{R}^n$. For the cylindrical functional F, (2.3) follows by applying of Relation (3.13) in Bobkov and Götze (1999) and tensorization. The extension to general random variables can be done by martingale convergence, e.g. as in the proof of Theorem 3.4 of Wu (1998). $\quad\square$

In particular, (2.3) yields the inequality

$$\frac{1}{\sqrt{2}} I(\Pi(A)) \leq E[|D\mathbf{1}_A|_{L^2(\sigma)}], \qquad A \in \mathcal{F}, \tag{2.5}$$

since $I(\mathbf{1}_A) = 0$, Π-a.s.

3   Boundaries, surface measure and co-area formula

Let now

$$D_x^+ F(\omega) = \max(0, D_x F(\omega)) \quad \text{and} \quad D_x^- F(\omega) = -\min(0, D_x F(\omega)), \qquad x \in X.$$

The next definition is, in fact, independent of $p \in [1, \infty]$.

Definition 3.1   *The inner and outer boundaries of any $A \subset \Omega$ are defined as:*

$$\partial_{\text{in}} A = \{\omega \in A \; : \; |D^- \mathbf{1}_A(\omega)|_{L^p(\sigma)} > 0\},$$

and

$$\partial_{\text{out}} A = \{\omega \in A^c \; : \; |D^+ \mathbf{1}_A(\omega)|_{L^p(\sigma)} > 0\}.$$

In other words, the inner boundary $\partial_{\text{in}} A$ of A is made up of the $\omega \in A$ that have "at least" a neighbor in A^c, in the sense that

$$\sigma(\{x \in X \; : \; \omega + \delta_x \in A^c\}) > 0,$$

and the outer boundary $\partial_{\text{out}} A$ of A is made up of the $\omega \in A^c$ that have "at least" a neighbor in A, in the sense that

$$\sigma(\{x \in X \; : \; \omega + \delta_x \in A\}) > 0.$$

The symmetric boundary of A is defined as:

$$\partial A = \partial_{\text{in}} A \cup \partial_{\text{out}} A.$$

In the cylindrical case where $X_n : \Omega \to \mathbb{N}^n$ is defined by (2.4) we have the relation

$$\partial \{X_n \in B\} = \{X_n \in \partial B\}$$

where ∂B is the boundary of $B \subset \mathbb{N}^n$ defined in Bobkov and Götze, 1999, p. 263. For the set $\{\omega(B) \geq n\}$ we have

$$\partial_{\text{out}} \{\omega(B) \geq n\} = \{\omega(B) = n - 1\},$$

and for the set $\{\omega(B) \leq n\}$ we have

$$\partial_{\text{in}} \{\omega(B) \leq n\} = \{\omega(B) = n\}.$$

The surface measure of $\partial_{\text{in}} A$, resp. $\partial_{\text{out}} A$, is defined by averaging $|D^- \mathbf{1}_A(\omega)|_{L^2(\sigma)}$, resp. $|D^+ \mathbf{1}_A(\omega)|_{L^2(\sigma)}$ with respect to the Poisson measure $\Pi(d\omega)$.

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Definition 3.2 *For any $A \in \mathcal{F}$, let*

$$\Pi_s(\partial_{\mathrm{in}} A) = E[|D^- \mathbf{1}_A|_{L^2(\sigma)}],$$

and

$$\Pi_s(\partial_{\mathrm{out}} A) = E[|D^+ \mathbf{1}_A|_{L^2(\sigma)}].$$

The above quantities represent average numbers, with respect to the Poisson measure, of points in A, resp. A^c, which have a neighbor in A^c, resp. A. The surface measure of ∂A is then defined as

$$\Pi_s(\partial A) = \Pi_s(\partial_{\mathrm{in}} A) + \Pi_s(\partial_{\mathrm{out}} A).$$

We now turn to the statement of a co-area formula on configuration space.

Lemma 3.3 *For any sufficiently integrable $F : \Omega \to \mathbb{R}$ we have*

$$E[|D^- F|_{L^\infty(\sigma)}] = \int_{-\infty}^{+\infty} \Pi(\partial_{\mathrm{in}}\{F > t\}) dt$$

and

$$E[|D^+ F|_{L^\infty(\sigma)}] = \int_{-\infty}^{+\infty} \Pi(\partial_{\mathrm{out}}\{F > t\}) dt.$$

Proof. We follow the arguments used in Bobkov, Houdré and Tetali (2000) in the case of graphs. We have

$$|D^- F(\omega)|_{L^\infty(\sigma)} = \mathrm{ess\,sup}_{\sigma(dx)}(F(\omega) - F(\omega + \delta_x))^+ = F(\omega) - \mathrm{ess\,inf}_{\sigma(dx)} F(\omega + \delta_x),$$

hence

$$\begin{aligned}
E[|D^- F|_{L^\infty(\sigma)}] &= E[F] - E[\mathrm{ess\,inf}_{\sigma(dx)} F(\omega + \delta_x)] \\
&= \int_{-\infty}^{+\infty} \Pi(\{F > t\}) dt - \int_{-\infty}^{+\infty} \Pi(\mathrm{ess\,inf}_{\sigma(dx)} F(\omega + \delta_x) > t) dt \\
&= \int_{-\infty}^{+\infty} \Pi(\{F > t\}) dt - \int_{-\infty}^{+\infty} \Pi(\{\mathrm{ess\,inf}_{\sigma(dx)} F(\omega + \delta_x) > t \text{ and } F(\omega) > t\}) dt \\
&= \int_{-\infty}^{+\infty} \Pi(\{F(\omega) > t \text{ and } \sigma(\{x \in X \ : \ F(\omega + \delta_x) \le t\}) > 0\}) dt \\
&= \int_{-\infty}^{+\infty} \Pi(\{\omega \in \Omega \ : \ \sigma(\{x \in X \ : F(\omega) > t \text{ and } F(\omega + \delta_x) \le t\}) > 0\}) dt \\
&= \int_{-\infty}^{+\infty} \Pi(\{\omega \in \Omega \ : \ \sigma(\{x \in X \ : \mathbf{1}_{\{F(\omega) > t\}} - \mathbf{1}_{\{F(\omega + \delta_x) > t\}} = 1\}) > 0\}) dt \\
&= \int_{-\infty}^{+\infty} \Pi(\{\omega \in \Omega \ : \ |D^- \mathbf{1}_{\{F > t\}}|_{L^\infty(\sigma)} = 1\}) dt \\
&= \int_{-\infty}^{+\infty} E[|D^- \mathbf{1}_{\{F > t\}}|_{L^\infty(\sigma)}] dt.
\end{aligned}$$

The corresponding result with D^+ is proved similarly. $\qquad\square$

As a consequence of the previous lemma and since $D_x^- F D_x^+ F = 0$ we have:

$$
\begin{aligned}
& E[|D^- F|_{L^\infty(\sigma)}] + E[|D^+ F|_{L^\infty(\sigma)}] \\
& = \int_{-\infty}^\infty E[|D^- \mathbf{1}_{\{F>t\}}|_{L^\infty(\sigma)}] dt + \int_{-\infty}^\infty E[|D^+ \mathbf{1}_{\{F>t\}}|_{L^\infty(\sigma)}] dt \\
& = \int_{-\infty}^\infty E[|D^- \mathbf{1}_{\{F>t\}}|_{L^\infty(\sigma)} + |D^+ \mathbf{1}_{\{F>t\}}|_{L^\infty(\sigma)}] dt \\
& = \int_{-\infty}^\infty E[|D \mathbf{1}_{\{F>t\}}|_{L^\infty(\sigma)}] dt \\
& = \int_{-\infty}^\infty \Pi(\partial\{F>t\}) dt.
\end{aligned}
\tag{3.1}
$$

4   Monotone sets

Given $\omega \in \Omega$, let the set $\mathcal{N}_\omega$ of neighbors of ω be defined as

$$
\mathcal{N}_\omega = \{\omega + \delta_x \ : x \in \omega^c\}.
$$

Definition 4.1 *A subset A of Ω is called increasing if*

$$
\omega \in A \quad \Longrightarrow \quad \omega + \delta_x \in A, \quad \sigma(dx) - a.e.
\tag{4.1}
$$

It is called decreasing if

$$
\omega + \delta_x \in A \quad \Longrightarrow \quad \omega \in A, \quad \sigma(dx) - a.e.
\tag{4.2}
$$

For all $B \in \mathcal{B}(X)$ the sets $\{\omega(B) \geq n\}$, resp. $\{\omega(B) \leq n\}$, are examples of increasing, resp. decreasing, subsets of Ω. Another example of monotone set is given by

$$
\left\{\omega \in \Omega \ : \ \int_X f d\omega > K\right\}, \quad K \in \mathbb{R},
$$

which is increasing, resp. decreasing, if $f \geq 0$, resp. $f \leq 0$.

Furthermore we have the following.

Proposition 4.2 *A subset A of Ω is increasing if and only if one of the equivalent conditions*

$$
D_x \mathbf{1}_A \geq 0, \qquad D_x \mathbf{1}_A = D_x^+ \mathbf{1}_A, \qquad D_x^- \mathbf{1}_A = 0,
$$

holds, $\sigma(dx)$-a.e.

Similarly, a subset A of Ω is decreasing if and only of the conditions

$$
D_x \mathbf{1}_A \leq 0, \qquad D_x \mathbf{1}_A = -D_x^- \mathbf{1}_A, \qquad D_x^+ \mathbf{1}_A = 0,
$$

holds $\sigma(dx)$-a.e. Note that if $A \subset \Omega$ is increasing, resp. decreasing, then $\partial_{\text{in}} A = \emptyset$, resp. $\partial_{\text{out}} A = \emptyset$.

We now prove a Margulis-Russo type equality for monotone sets under Poisson measures, cf. Margulis (1974), Russo (1982) on other discrete spaces.

Lemma 4.3 *Let $A \in \mathcal{F}$ be a monotone and measurable subset of Ω.*

a) If A is increasing then

$$\frac{\partial}{\partial \lambda} \Pi_\lambda(A) = E_\lambda \left[\int_X D_x^+ \mathbf{1}_A \sigma(dx) \right], \qquad \lambda \in (a, b).$$

b) If A is decreasing then

$$\frac{\partial}{\partial \lambda} \Pi_\lambda(A) = -E_\lambda \left[\int_X D_x^- \mathbf{1}_A \sigma(dx) \right], \qquad \lambda \in (a, b).$$

Proof. Combine Proposition 2.2 with Proposition 4.2. □

Definition 4.4 *For any $A \in \mathcal{F}$, let*

$$\delta_A^- = \inf_{\partial_{\text{out}} A} |D^+ \mathbf{1}_A|_{L^1(\sigma)} \qquad and \qquad \delta_A^+ = \inf_{\partial_{\text{in}} A} |D^- \mathbf{1}_A|_{L^1(\sigma)}.$$

Note that when A is increasing, resp. decreasing, then δ_A^-, resp. δ_A^+, coincides with δ_A defined as

$$\delta_A = \inf_{\partial A} |D\mathbf{1}_A|_{L^1(\sigma)}.$$

For example, for the increasing set $\{\omega(B) \geq n\}$ we have

$$|D\mathbf{1}_{\{\omega(B) \geq n\}}|_{L^1(\sigma)} = \sigma(B) \mathbf{1}_{\{\omega(B) = n-1\}} = \sigma(B) \mathbf{1}_{\partial_{\text{out}} \{\omega(B) \geq n\}},$$

hence $\delta_{\{\omega(B) \geq n\}} = \sigma(B)$. For the decreasing set $\{\omega(B) \leq n\}$ we have

$$|D\mathbf{1}_{\{\omega(B) \leq n\}}|_{L^1(\sigma)} = \sigma(B) \mathbf{1}_{\{\omega(B) = n\}} = \sigma(B) \mathbf{1}_{\partial_{\text{in}} \{\omega(B) \leq n\}},$$

hence $\delta_{\{\omega(B) \leq n\}} = \sigma(B)$.

Lemma 4.5 *Let $A \in \mathcal{F}$ be a monotone subset of Ω, let $p \in [1, \infty]$ and $1 = 1/p + 1/q$. If A is increasing, then*

$$\frac{\partial}{\partial \lambda} \Pi_\lambda(A) \geq \delta_A^{1/q} E_\lambda[|D\mathbf{1}_A|_{L^p(\sigma)}], \qquad \lambda > 0,$$

and if A is decreasing, then

$$\frac{\partial}{\partial \lambda} \Pi_\lambda(A) \leq -\delta_A^{1/q} E_\lambda[|D\mathbf{1}_A|_{L^p(\sigma)}], \qquad \lambda > 0.$$

Proof. Since A is monotone we have

$$E_\lambda[|D\mathbf{1}_A|_{L^p(\sigma)}] = E_\lambda[\mathbf{1}_{\{|D\mathbf{1}_A|_{L^\infty(\sigma)} > 0\}} |D\mathbf{1}_A|_{L^p(\sigma)}]$$
$$\leq (\Pi_\lambda(\{|D\mathbf{1}_A|_{L^\infty(\sigma)} > 0\}))^{1/q} E_\lambda[|D\mathbf{1}_A|_{L^p(\sigma)}^p]^{1/p}$$

$$\leq |\Pi_\lambda(\partial A)|^{1/q} E_\lambda[|D\mathbf{1}_A|_{L^1(\sigma)}]^{1/p}$$

$$\leq \frac{1}{\delta_A^{1/q}} E_\lambda[|D\mathbf{1}_A|_{L^1(\sigma)}].$$

Now in case A is increasing, from Proposition 4.3 we get

$$\frac{\partial}{\partial \lambda} \Pi_\lambda(A) = E_\lambda[|D^+\mathbf{1}_A|_{L^1(\sigma)}] = E_\lambda[|D\mathbf{1}_A|_{L^1(\sigma)}] \geq \delta_A^{1/q} E_\lambda[|D\mathbf{1}_A|_{L^p(\sigma)}].$$

The argument is similar in case A is decreasing. $\qquad\square$

Finally we introduce the main isoperimetric constants under the Poisson measure, see Houdré and Tetali (2004) for the case of graphs and Markov chains.

Definition 4.6 *For $p \in [1,\infty]$ let*

$$h_p = \inf_{0 < \Pi(A) < 1} \frac{E\left[|D\mathbf{1}_A|_{L^p(\sigma)}\right]}{\Pi(A)(1-\Pi(A))}, \quad h_p^+ = \inf_{\substack{0 < \Pi(A) < 1 \\ A \text{ decreasing}}} \frac{E\left[|D\mathbf{1}_A|_{L^p(\sigma)}\right]}{\Pi(A)(1-\Pi(A))}, \quad h_p^- = \inf_{\substack{0 < \Pi(A) < 1 \\ A \text{ increasing}}} \frac{E\left[|D\mathbf{1}_A|_{L^p(\sigma)}\right]}{\Pi(A)(1-\Pi(A))},$$

where the suprema and infima are taken over measurable sets $A \in \mathcal{F}$.

We have

$$h_2^+ = \inf_{\substack{0 < \Pi(A) < 1 \\ A \text{ decreasing}}} \frac{\Pi_s(\partial A)}{\Pi(A)\Pi(A^c)}, \quad h_2^- = \inf_{\substack{0 < \Pi(A) < 1 \\ A \text{ increasing}}} \frac{\Pi_s(\partial A)}{\Pi(A)\Pi(A^c)}, \quad h_2 = \inf_{0 < \Pi(A) < 1} \frac{\Pi_s(\partial A)}{\Pi(A)\Pi(A^c)}$$

and

$$h_\infty^+ = \inf_{\substack{0 < \Pi(A) < 1 \\ A \text{ decreasing}}} \frac{\Pi(\partial A)}{\Pi(A)\Pi(A^c)}, \quad h_\infty^- = \inf_{\substack{0 < \Pi(A) < 1 \\ A \text{ increasing}}} \frac{\Pi(\partial A)}{\Pi(A)\Pi(A^c)}, \quad h_\infty = \inf_{0 < \Pi(A) < 1} \frac{\Pi(\partial A)}{\Pi(A)\Pi(A^c)}.$$

5 Bounds on monotone sets and thresholding

As a consequence of the Margulis-Russo identity Proposition 4.3 we obtain bounds on $\Pi_\lambda(A)$ when A is a monotone set. In Proposition 5.1 below, $h_p^\pm$ and h_p denote the infima of the corresponding isoperimetric constants over the values of λ considered in each bound. Note that from results shown in Section 6, h_p and $h_p^\pm$ are lower bounded independently of $\lambda > 0$ for $p = 1, 2$, while h_∞ is of order λ^{-1} and $\lambda^{-1/2}$ under Π_λ as λ goes to 0 and infinity respectively.

Proposition 5.1 *Let $A \in \mathcal{F}$ be a monotone subset of Ω and let $p, q \in [1,\infty]$ such that $1/p + 1/q = 1$.*

a) If A is increasing we have

$$\Pi_\theta(A) \geq \Pi_\lambda(A) e^{(\theta-\lambda)(1-\Pi_\theta(A))h_p^- \delta_A^{1/q}}, \qquad 0 < \lambda < \theta.$$

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b) In case A is decreasing we have

$$\Pi_\theta(A) \leq \Pi_\lambda(A) e^{-(\theta-\lambda)(1-\Pi_\lambda(A))h_p^+ \delta_A^{1/q}}, \qquad 0 < \lambda < \theta.$$

Proof. In case A is increasing we have

$$\frac{\partial}{\partial \lambda} \Pi_\lambda(A) \geq h_p^- \delta_A^{1/q} \Pi_\lambda(A)(1 - \Pi_\lambda(A)) \geq h_p^- \delta_A^{1/q} \Pi_\lambda(A)(1 - \Pi_\theta(A)), \qquad 0 < \lambda < \theta,$$

and in case A is decreasing we have

$$\frac{\partial}{\partial \theta} \Pi_\theta(A) \leq -h_p^+ \delta_A^{1/q} \Pi_\theta(A)(1 - \Pi_\theta(A)) \leq -h_p^+ \delta_A^{1/q} \Pi_\theta(A)(1 - \Pi_\lambda(A)), \qquad 0 < \lambda < \theta.$$

$\square$

An application of Lemma 4.5 for $p = 2$ yields the following bounds.

Proposition 5.2 *Let $A \in \mathcal{F}$ be a monotone subset of Ω.*

a) In case A is increasing we have

$$\Pi_\theta(A) \geq \Phi\left(\Phi^{-1}(\Pi_\lambda(A)) + \sqrt{2\delta_A}(\sqrt{\theta} - \sqrt{\lambda})\right), \qquad 0 < \lambda < \theta.$$

b) In case A is decreasing we have

$$\Pi_\theta(A) \leq \Phi\left(\Phi^{-1}(\Pi_\lambda(A)) - \sqrt{2\delta_A}(\sqrt{\theta} - \sqrt{\lambda})\right), \qquad 0 < \lambda < \theta.$$

Proof. Letting $f(\lambda) = \Pi_\lambda(A)$ and applying Lemma 4.5 with $p = 2$ and Proposition 2.3 we get from (2.5) for A increasing:

$$f'(\lambda) = E_\lambda[|D\mathbf{1}_A|_{L^1(\sigma)}] \geq \sqrt{\delta_A} E_\lambda[|D\mathbf{1}_A|_{L^2(\sigma)}] \geq \sqrt{\frac{\delta_A}{2\lambda}} I(f(\lambda)) = \frac{-\sqrt{\delta_A}}{\sqrt{2\lambda} I''(f(\lambda))}.$$

Hence for $\lambda < \theta$,

$$\Phi^{-1}(f(\lambda)) - \Phi^{-1}(f(\theta)) = I'(f(\theta)) - I'(f(\lambda)) = \int_\lambda^\theta I''(f(t)) f'(t) dt$$

$$\leq -\int_\lambda^\theta \frac{\sqrt{\delta_A}}{\sqrt{2t}} dt = \sqrt{2\delta_A}(\sqrt{\lambda} - \sqrt{\theta}).$$

The argument is similar in case A is decreasing. $\square$

In particular, for A increasing and $\Pi_\theta(A) = 1/2$ we have

$$\Pi_\lambda(A) \leq \Phi(-\sqrt{2\delta_A}(\sqrt{\theta} - \sqrt{\lambda})), \qquad 0 < \lambda < \theta,$$

whereas for $\Pi_\lambda(A) = 1/2$ we have

$$\Pi_\theta(A) \geq \Phi(\sqrt{2\delta_A}(\sqrt{\theta} - \sqrt{\lambda})), \qquad 0 < \lambda < \theta,$$

and if A is decreasing and $\Pi_\lambda(A) = 1/2$,

$$\Pi_\theta(A) \leq \Phi(-\sqrt{2\delta_A}(\sqrt{\theta} - \sqrt{\lambda})), \qquad 0 < \lambda < \theta.$$

whereas for $\Pi_\theta(A) = 1/2$,

$$\Pi_\lambda(A) \geq \Phi(\sqrt{2\delta_A}(\sqrt{\theta} - \sqrt{\lambda})), \qquad 0 < \lambda < \theta.$$

In Figure 1 below we compare the uppers bounds obtained in Propositions 5.1 and 5.2 for $\theta = 18$ and $\Pi_\theta(A) = 1/2$.
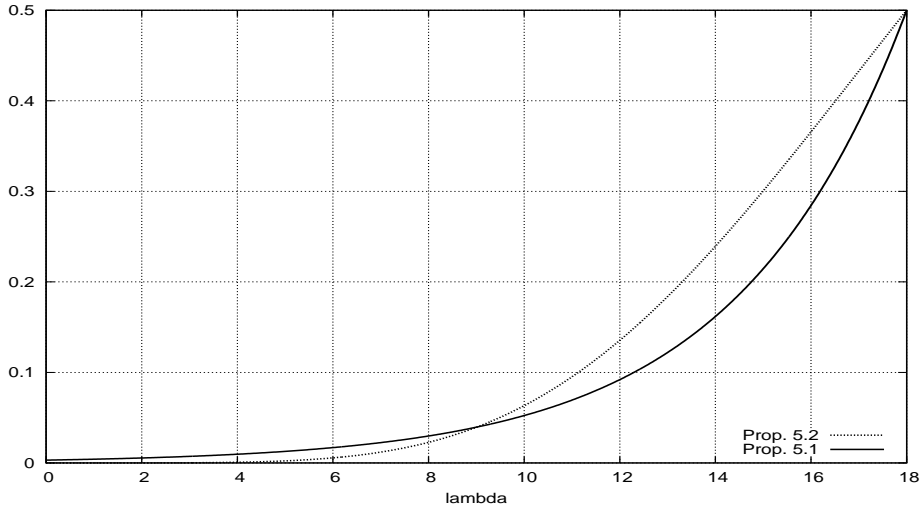

Fig. 1. Comparison of the upper bounds of Propositions 5.1 and 5.2.

As a consequence of Proposition 5.2 we have the following thresholding result, which can be seen as a Poisson space version of Russo's approximate zero-one law Russo (1982), Talagrand (1993), and extends Proposition 3.6 of Bobkov and Götze (1999) from finite dimensional Poisson vectors to the setting of configuration spaces.

Corollary 5.3 *Let $\varepsilon > 0$ and let A be a monotone subset of Ω such that*

$$0 < \varepsilon = \Pi_\lambda(A) < \Pi_\theta(A) = 1 - \varepsilon,$$

then

$$0 \leq \sqrt{\theta} - \sqrt{\lambda} \leq \frac{1}{\sqrt{2\delta_A}} \Phi^{-1}(1 - \varepsilon).$$

The above result can be interpreted by saying that the function $r \mapsto \Pi_{r^2}(A)$ can go from ε to $1 - \varepsilon$ on an interval of length at most $(2\delta_A)^{-1/2}\Phi^{-1}(1 - \varepsilon)$. This type of result has been first obtained in the framework of Bernoulli measures with parameter $p \in (0, 1)$ on $\{0, 1\}^n$, cf. Margulis (1974), Russo (1982), Talagrand (1993).

6   Bounds on isoperimetric constants

In this section we establish some bounds on the isoperimetric constants used in Proposition 5.1 of Section 5.

First, note that for all $p \in [1, \infty]$ we have the bounds $h_p \leq h_p^+ = h_p^-$ and

$$h_\infty \geq \frac{h_p}{(\sigma(X))^{1/p}} \qquad \text{and} \qquad h_\infty^\pm \geq \frac{h_p^\pm}{(\sigma(X))^{1/p}}, \qquad (6.1)$$

since $E[|D\mathbf{1}_A|_{L^p(\sigma)}] \leq (\sigma(X))^{1/p} E[|D\mathbf{1}_A|_{L^\infty(\sigma)}]$, $A \in \mathcal{F}$.

Let now the Poincaré constant λ_p, $p \in [1, \infty]$, be defined as

$$\lambda_p = \inf_{F \neq C} \frac{E\left[|DF|_{L^p(\sigma)}^2\right]}{\mathrm{Var}\,(F)},$$

where C denotes any constant function. Clearly, for all $p \in [1, \infty]$, we have

$$h_{p/2} \geq \lambda_p \qquad (6.2)$$

since

$$\lambda_p \Pi(A)(1 - \Pi(A)) = \lambda_p \mathrm{Var}\,(\mathbf{1}_A) \leq E[|D\mathbf{1}_A|_{L^p(\sigma)}^2] = E[|D\mathbf{1}_A|_{L^{p/2}(\sigma)}], \qquad A \in \mathcal{F}.$$

Lemma 6.1 *We have $\lambda_2 = 1$ and $\lambda_\infty = 1/\sigma(X)$.*

Proof. The fact that $\lambda_2 = 1$ is well-known, cf. e.g. Houdré and Privault (2003) for details, and it implies that

$$\mathrm{Var}\, F \leq E[|DF|_{L^2(\sigma)}^2] \leq \sigma(X) E[|DF|_{L^\infty(\sigma)}^2],$$

and so $\lambda_\infty \geq 1/\sigma(X)$. Letting $F(\omega) = \omega(X)$, we have $D_x F = 1$, $\sigma(dx)$-a.e., and so

$$\mathrm{Var}\,(F) = \sigma(X) = E[|DF|_{L^2(\sigma)}^2] = \sigma(X) E[|DF|_{L^\infty(\sigma)}^2],$$

which shows that $\lambda_\infty \leq 1/\sigma(X)$. $\qquad\qquad\square$

Proposition 6.2 *We have $h_2 \geq 1/\sqrt{\pi}$.*

Proof. From (2.5) we get

$$E[|D\mathbf{1}_A|_{L^2(\sigma)}] \geq \frac{1}{\sqrt{2}} I(\Pi(A)) \geq \frac{1}{\sqrt{\pi}} \Pi(A)(1 - \Pi(A)), \qquad A \in \mathcal{F}.$$

$\qquad\qquad\square$

The relation (6.3), below, improves on the lower bound $h_\infty \geq 1/\sqrt{\pi\sigma(X)}$, see (3.15) of Bobkov and Götze (1999) in the finite dimensional Poisson case.

Proposition 6.3 *We have*

$$\max\left(\frac{1}{\sqrt{\pi\sigma(X)}}, \frac{1}{\sigma(X)}\right) \leq h_\infty \leq \frac{8}{\sigma(X)} + \frac{8}{\sqrt{\sigma(X)}}. \qquad (6.3)$$

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Proof. First, note that (6.1) and Proposition 6.2 show that

$$h_\infty \geq h_2/\sqrt{\sigma(X)} \geq 1/\sqrt{\pi \sigma(X)}.$$

On the other hand, from (6.2) and Lemma 6.1 we have $h_\infty \geq \lambda_\infty = 1/\sigma(X)$.

We will conclude the proof of (6.3) by showing that

$$\lambda_\infty = \frac{1}{\sigma(X)} \geq \frac{(\sqrt{1 + h_\infty/2} - 1)^2}{4}. \tag{6.4}$$

In case F has a vanishing median $m(F) = 0$, from the co-area formula of Lemma 3.3 we have:

$$
\begin{aligned}
2E[|DF|_{L^\infty(\sigma)}] &\geq \int_{-\infty}^{+\infty} \Pi(\partial\{F > t\}) dt \\
&\geq \frac{h_\infty}{2} \int_{-\infty}^{+\infty} \min(\Pi(\{F > t\}), \Pi(\{F \leq t\})) dt \\
&= \frac{h_\infty}{2} E[F].
\end{aligned}
$$

Applying the above inequality to $(F^+)^2$ we have

$$
\begin{aligned}
\frac{h_\infty}{2} E[F^{+2}] &\leq 2E[|D(F^+)^2|_{L^\infty(\sigma)}] \\
&\leq 2E[\text{ess sup}_{\sigma(dx)} |F^+(\omega) - F^+(\omega + \delta_x)|(F^+(\omega) + F^+(\omega + \delta_x))] \\
&\leq 2E[\text{ess sup}_{\sigma(dx)} |F^+(\omega) - F^+(\omega + \delta_x)|(F^+(\omega + \delta_x) - F^+(\omega)) \\
&\quad + 2|F^+(\omega) - F^+(\omega + \delta_x)|F^+(\omega)] \\
&\leq 2E[\text{ess sup}_{\sigma(dx)} (F^+(\omega) - F^+(\omega + \delta_x))^2] \\
&\quad + 4E[\text{ess sup}_{\sigma(dx)} |F^+(\omega) - F^+(\omega + \delta_x)|F^+(\omega)] \\
&\leq 2E[\text{ess sup}_{\sigma(dx)} (F(\omega) - F(\omega + \delta_x))^2] \\
&\quad + 4E[\text{ess sup}_{\sigma(dx)} |F(\omega) - F(\omega + \delta_x)|F^+(\omega)].
\end{aligned}
$$

Similarly we have

$$
\begin{aligned}
\frac{h_\infty}{2} E[(F^-)^2] &\leq 2E[\text{ess sup}_{\sigma(dx)} (F(\omega) - F(\omega + \delta_x))^2] \\
&\quad + 4E[\text{ess sup}_{\sigma(dx)} |F(\omega) - F(\omega + \delta_x)|F^-(\omega)].
\end{aligned}
$$

Hence

$$
\begin{aligned}
\frac{h_\infty}{2} E[F^2] &\leq \frac{h_\infty}{2} E[(F^+)^2] + \frac{h_\infty}{2} E[(F^-)^2] \\
&\leq 4E[|DF|_{L^\infty(\sigma)}^2] + 4E[|DF|_{L^\infty(\sigma)}|F|]
\end{aligned}
$$

$$\leq 4E[|DF|^2_{L^\infty(\sigma)}] + 4E[|DF|^2_{L^\infty(\sigma)}]^{1/2}E[F^2]^{1/2},$$

which implies

$$\frac{(\sqrt{1+h_\infty/2}-1)^2}{4}\text{Var}\,F \leq \frac{(\sqrt{1+h_\infty/2}-1)^2}{4}E[F^2] \leq E[|DF|^2_{L^\infty(\sigma)}].$$

In the general case $(m(F) \neq 0)$, use the fact that $\text{Var}\,F \leq E[(F - m(F))^2]$. $\qquad\square$

Proposition 6.4 *We have*

$$1 \leq h_1 \leq 8 + 8\sqrt{\sigma(X)}. \tag{6.5}$$

Proof. From (6.2) and Lemma 6.1 we have $h_1 \geq \lambda_2 = 1$, and we conclude using (6.3) and (6.1) for $p = 1$. $\qquad\square$

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